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Research Article

A Theorem of Galambos-Bojanić-Seneta Type

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In the theorems of Galambos-Bojanić-Seneta's type, the asymptotic behavior of the functions $c_{[x]}$, $x \geq 1$, for $x \to +\infty$, is investigated by the asymptotic behavior of the given sequence of positive numbers (c_n), as $n \to +\infty$ and vice versa. The main result of this paper is one theorem of such a type for sequences of positive numbers (c_n) which satisfy an asymptotic condition of the Karamata type $\underline{\lim}_{n\to\infty} c_{[\lambda n]}/c_n > 1$, for $\lambda > 1$.

1. Introduction

A function $f : [a, +\infty) \mapsto (0, +\infty)$ $(a > 0)$ is called *$\mathcal{O}$-regularly varying in the sense of Karamata* (see [1]) if it is measurable and if for every $\lambda > 0$,

$$\overline{k}_f(\lambda) := \overline{\lim_{x\to+\infty}} \frac{f(\lambda x)}{f(x)} < +\infty. \tag{1.1}$$

Function $\overline{k}_f(\lambda)$ $(\lambda > 0)$ is called *the index function* of f, and ORV_f is the class of all $\mathcal{O}$-regularly varying functions defined on some interval $[a, +\infty)$.

A function $f \in \mathrm{ORV}_f$ is called *$\mathcal{O}$-regularly varying in the Schmidt sense* (see [2, 3]) if

$$\overline{\lim_{\lambda\to 1}} \overline{k}_f(\lambda) = 1. \tag{1.2}$$

$\mathcal{O}$-regularly varying functions in the Schmidt sense form the functional class IRV_f and $\mathrm{IRV}_f \subsetneq \mathrm{ORV}_f$ (see [3]). They represent an important object in the analysis of divergent processes (see [4–9]). In particular, we have that the class RV_f of regularly varying functions in the Karamata sense satisfies $\mathrm{RV}_f \subsetneq \mathrm{IRV}_f$ (see [3], and some of its applications can be found in [10]).

A function $f \in \mathrm{IRV}_f$ is called *regularly varying in Karamata sense* if $\overline{k}_f(\lambda) = \lambda^\rho$ for every $\lambda > 0$ and a fixed $\rho \in \mathbb{R}$. If $\rho = 0$, then f is called *slowly varying in the Karamata sense* , and all such functions form the class SV_f. We have that $\mathrm{SV}_f \subsetneq \mathrm{RV}_f$ (see [10]).

A sequence of positive numbers (c_n) is called *$\mathcal{O}$-regularly varying in the Karamata sense* (i.e., it belongs to the class ORV_s), if

$$\overline{k}_c(\lambda) = \overline{\lim_{n\to+\infty}} \frac{c_{[\lambda n]}}{c_n} < +\infty, \tag{1.3}$$

for every $\lambda > 0$.

A sequence $(c_n) \in \mathrm{ORV}_s$ is called *$\mathcal{O}$-regularly varying in the Schmidt sense* (i.e., it belongs to the class IRV_s), if

$$\overline{\lim_{\lambda\to 1}}\, \overline{k}_c(\lambda) = 1. \tag{1.4}$$

The classes of sequences ORV_s and IRV_s have an important place in the qualitative analysis of sequential divergent processes (see, e.g., [11–14]). Asymptotic properties of sequences (1.3) and (1.4) are very important in the Theory of Tauberian theorems (see [7, 15]).

The class of regularly varying sequences in the Karamata sense RV_s and similarly the class of slowly varying sequences in the Karamata sense SV_s are defined analogously to the classes RV_f and SV_f. They are fundamenatal in the theory of sequentional regular variability in general (see [16]).

Next, let (c_n) be a strictly increasing, unbounded sequence of positive numbers. Then

$$\delta_c(x) = \max\{n \in \mathbb{N} \mid c_n \le x\}, \tag{1.5}$$

for $x \ge c_1$, is the *numerical function* of the sequence (c_n) (see, e.g., [17]).

In the sequel, let $\sim$ be the strong asymptotic equivalence of sequences and functions, and let (p_n) be the sequence of prime numbers in the increasing order. Since $p_n \sim n \ln n$, $n \to +\infty$ ($(p_n) \in \mathrm{IRV}_s$) and since $\delta_p(x) \sim (x/\ln x)$, $x \to +\infty$, ($\delta_p \in \mathrm{IRV}_f$) (see, e.g., [17]), the next question seems to be natural:

> *what is the largest proper subcclass of the class of all strictly increasing, unbounded sequences from IRV_s, such that the numerical function of every one of its elements belongs to IRV_f?*

The next example shows that this question has some sense.

Example 1.1. Define $c_1 = \ln 2/2$ and $c_n = \ln n$, for $n \ge 2$. Then (c_n) is a strictly increasing, unbounded sequence of positive numbers. Since $\ln x$, $x \ge 2$ belongs to the functional class SV_f (see, e.g., [10]), by a result from [18], we have that $(c_n) \in \mathrm{SV}_s$. Hence, $(c_n) \in \mathrm{IRV}_s$. Next, since $\delta_c(x) \sim h^{-1}(x)$, $x \to +\infty$, where $h(x)$, $x \ge 1$, is continuous and strictly increasing, and $h(n) = c_n$ ($n \in N$) (see, e.g., [17]), we can assume that $h(x) = \ln x$ for $x \ge 2$, while for $x \in [1,2)$ we can suppose that h is linear and continuous on $[1,2]$ such that $h(1) = \ln 2/2$.

Therefore, $\delta_c(x) \sim e^x$, as $x \to +\infty$, so that δ_c belongs to de Haan class of rapidly varying functions with index $+\infty$ (the class $R_{\infty,f}$) (see, e.g., [19]). Hence, if $\lambda > 1$ we have

$$\lim_{x\to+\infty} \frac{\delta_c(\lambda x)}{\delta_c(x)} = +\infty, \tag{1.6}$$

so that δ_c does not belong to IRV_f.

Knowing of asymptotic characteristics of a considered sequence and of its numerical function can be of a great importance in many constructions of the asymptotic analysis (see, e.g., [17]).

Next, we say that a function $f : [a, +\infty) \mapsto (0, +\infty)$, $a > 0$, belongs to the class ARV_f if it is measurable and for every $\lambda > 1$ we have

$$\underline{k}_f(\lambda) = \varliminf_{x\to+\infty} \frac{f(\lambda x)}{f(x)} > 1. \tag{1.7}$$

The function $\underline{k}_f(\lambda)$, $\lambda > 0$, is the *auxiliary index function* of the function $f(x)$, $x \geq a$.

Condition (1.7) is equivalent with assumption that there exists an $x_0 = x_0(\lambda) \geq a$ and $c(\lambda) > 1$ for $\lambda > 1$, so that for every $\lambda > 1$ and every $x \geq x_0$ it holds

$$f(\lambda x) \geq c(\lambda) \cdot f(x). \tag{1.8}$$

The class ARV_f contains (as proper subclasses) the class of all regularly varying functions in the Karamata sense whose index of variability is positive as well as the class of all rapidly varying functions in de Haan sense whose index of variabilty is $+\infty$, but it does not contain any slowly varying function in the Kararamata sense.

We also have that $\mathrm{ARV}_f \cap \mathrm{IRV}_f \neq \varnothing$ and $\mathrm{ARV}_f \Delta \mathrm{IRV}_f \neq \varnothing$. Besides, the class ARV_f considered in the space of the so-called φ-functions (see, e.g., [8]) is also an essential object of the asymptotic and the functional analysis (see, e.g., [20]).

Next, let ARV_s be the class of all positive numbers (c_n) such that for every $\lambda > 1$ we have

$$\underline{k}_c(\lambda) = \varliminf_{n\to+\infty} \frac{c_{[\lambda n]}}{c_n} > 1. \tag{1.9}$$

The function $\underline{k}_c(\lambda)$, $\lambda > 0$, is called the *auxiliary index function* of the sequence (c_n).

The above condition is equivalent with fact that there is an $n_0 = n_0(\lambda) \in N$ and a function $c(\lambda) > 1$, $\lambda > 1$, such that for every $\lambda > 1$ and for every $n \geq n_0$ we have

$$c_{[\lambda n]} \geq c(\lambda) \cdot c_n. \tag{1.10}$$

The class ARV_s contains (as proper subclasses) the class of all regularly varying sequences in the Karamata sense whose index of variability is positive as well as the class of all rapidly varying sequences in de Haan sense whose index is $+\infty$, but does not contain any slowly varying sequence in the Karamata sense (see [21, 22]).

We also have that $\mathrm{ARV}_s \cap \mathrm{IRV}_s \neq \varnothing$ and $\mathrm{ARV}_s \Delta \mathrm{IRV}_s \neq \varnothing$.

2. Main Results

The next theorem is a theorem of Galambos-Bojanic-Seneta type (see [16, 18]) for classes ARV_s and ARV_f. The analogous theorems for regularly varying sequences and functions in the Karamata sense, $\mathcal{O}$-regularly varying sequences and functions in the Karamata sense, sequences from the class IRV_s and functions from the class IRV_f, rapidly varying sequences and functions in de Haan sense with index $+\infty$, the Seneta sequences and functions (see, e.g., [23]) can be found, respectively, in [13, 16, 24–27].

Theorem 2.1. *Let (c_n) be a sequence of positive numbers. Then the next assertions are equivalent as follows:*

(a) $(c_n) \in ARV_s$,

(b) $f(x) = c_{[x]}$, $x \geq 1$, *belongs to the class* ARV_f.

Proof. (a) $\Rightarrow$ (b) Let (c_n) be a sequence of positive numbers and assume that $(c_n) \in \mathrm{ARV}_s$, thus that $\underline{\lim}_{n\to+\infty}(c_{[\lambda n]}/c_n) > 1$ for every $\lambda > 1$. If $\lambda > 1$ is arbitrary fixed number, then $\underline{k}_c(\alpha) > 1$ for every $\alpha \in (1,\lambda)$. For arbitrary $\alpha \in (1,\lambda)$ define $n_\alpha \in N$ in the following way: $n_\alpha = 1$ if $c_{[\alpha n]}/c_n > 1$ for every $n \in N$, and $n_\alpha = 1 + \max\{n \in N \mid c_{[\alpha n]}/c_n \leq 1\}$ else. One can easily see that $1 \leq n_\alpha < +\infty$ for every considered α.

Next, define a sequence of sets (A_k) by $A_k = \{\alpha \in (1,\lambda) \mid n_\alpha > k\}$ $(k \in N)$. Then this sequences is nonincreasing, thus $A_{k+1} \subseteq A_k$ $(k \in N)$ and $\bigcap_{k=1}^{\infty} A_k = \emptyset$. We shall show that not all subsets A_k $(k \in N)$ are dense in $(1,\lambda)$. If $\alpha \in A_k$ for a fixed $k \in N$, then $c_{[(n_\alpha-1)\alpha]}/c_{n_\alpha-1} \leq 1$, and there is a $\delta_\alpha > 0$ such that $c_{[(n_\alpha-1)t]}/c_{n_\alpha-1} = c_{[(n_\alpha-1)\alpha]}/c_{n_\alpha-1} \leq 1$, for every $t \in [\alpha,\alpha+\delta_\alpha) \subsetneq (1,\lambda)$. Hence, every $t \in (\alpha,\alpha+\delta_\alpha)$ belongs to A_k, since $n_t \geq (n_\alpha - 1) + 1 > k$. This gives that $(\alpha,\alpha+\delta_\alpha) \subseteq A_k$ if $\alpha \in A_k$. Assuming now that a set A_k is dense in $(1,\lambda)$, we get that the set $\operatorname{Int} A_k$ is also dense in $(1,\lambda)$. If else, we assume that all sets A_k $(k \in N)$ are dense in $(1,\lambda)$, we find that $(\operatorname{Int} A_k)$ is a sequence of open dense subsets of the set $(1,\lambda)$ of the second category. Then we get that the set $\bigcap_{k=1}^{\infty} A_k$ is dense in $(1,\lambda)$, so it must be nonempty, which is a contradiction. Hence, we conclude that there is an $n_0 \in N$, so that the set A_{n_0} is not dense in $(1,\lambda)$. Hence, there is an intervals $[A,B] \subsetneq (1,\lambda)$ $(A < B)$ such that $[A,B] \subseteq (1,\lambda) \setminus A_{n_0} = \{\alpha \in (1,\lambda) \mid n_\alpha \leq n_0\}$.

Therefore, for every $\alpha \in [A,B]$ we have $n_\alpha \leq n_0$. Hence, for every $n \geq n_0 \geq n_\alpha$ and every $\alpha \in [A,B]$ we have $c_{[\alpha n]}/c_n > 1$. Consequently, for any $\lambda \in (1,+\infty)$ and all sufficiently large $x \geq x_0$ we have that $c_{[\lambda x]}/c_{[x]} = (c_{[t[\eta[x]]]}/c_{[\eta[x]]}) \cdot (c_{[\eta[x]]}/c_{[x]})$, where $t = t(x) \in [A,B]$ and $\eta = 2\lambda/(A+B)$. Since $\eta > 1$, we get $\underline{\lim}_{x\to+\infty} c_{[\lambda x]}/c_{[x]} \geq \underline{k}_c(\eta) > 1$, so that $f(x) = c_{[x]}$ $(x \geq 1)$ belongs to the class ARV_f.

Since (b) $\Rightarrow$ (a) is immediate, we completed the proof. □

The above theorem provides (analogously, as in cases given before Theorem 2.1) a unique development of the theory of sequences from the class ARV_s and theory of the functions from the class ARV_f. Thus, Theorem 2.1 can be used to interpret all asymptotic behaviors of functions from the class ARV_f (some of them are given in [28]) as behavior of sequences from the class ARV_s and vice versa.

Corollary 2.2. *Let (c_n) be a strictly increasing unbounded sequence of positive numbers. Then,*

(a) $(c_n) \in ARV_s$ *if and only if* $\delta_c(x)$ $(x \geq c_1) \in IRV_f$;

(b) $(c_n) \in IRV_s$ *if and only if* $\delta_c(x)$ $(x \geq c_1) \in ARV_f$.

Proof. (a) Let (c_n) be a strictly increasing unbounded sequence of positive numbers, and assume that $(c_n) \in \mathrm{ARV}_s$. Then by Theorem 2.1, $f(x) = c_{[x]}$, $x \geq 1$, belongs to ARV_f. f is nondecreasing and unbounded for $x \geq 1$. Let $f^{\leftarrow}(x) = \inf\{y \geq 1 \mid f(y) > x\}$, $x \geq c_1$, be the generalized inverse (see [1]) of f. It is correctly defined nondecreasing and unbounded function for $x \geq c_1$. It is also stepwise and right continuous. We also have that $\delta_c(x) = f^{\leftarrow}(x) - 1$ for $x \geq c_1$.

According to [22] we have that function $f^{\leftarrow}(x)$, $x \geq c_1$, belongs to the class IRV_f.

Since $f^{\leftarrow}$ is nondecreasing and unbounded, we get $\lim_{x\to+\infty}(\delta_c(x)/f^{\leftarrow}(x)) = 1$, so that $\delta_c(x)$, $x \geq c_1$, belongs to IRV_f.

Next, let (c_n) be a strictly increasing unbounded sequence of positive numbers, and let $\delta_c(x)$, $x \geq c_1$, belong to IRV_f. Besides, let $f(x) = c_{[x]}$, $x \geq 1$. Since $f^{\leftarrow}(x) = \delta_c(x) + 1$ for $x \geq c_1$, we find that $f^{\leftarrow} \in \mathrm{IRV}_f$. According to [28] we have that function $f(x)$, $x \geq 1$, belongs to the class ARV_f. So by Theorem 2.1 we get that $(c_n) \in \mathrm{ARV}_c$.

(b) Now, assume that (c_n) is a strictly increasing unbounded sequence of positive numbers and $(c_n) \in \mathrm{IRV}_s$. Then by [13], $f(x) = c_{[x]}$, $x \geq 1$, belongs to IRV_f. Analogously to (a), then $\delta_c(x) = f^{\leftarrow}(x) - 1$, $x \geq c_1$. According to [29] (or [28]) we have that function $f^{\leftarrow}(x)$, $x \geq c_1$, belongs to the class ARV_f, and consequently $\delta_c \in \mathrm{ARV}_f$.

Next, let (c_n) be a strictly increasing unbounded sequence of positive numbers, and assume that $\delta_c \in \mathrm{ARV}_f$. Besides, let $f(x) = c_{[x]}$, $x \geq 1$. Since $f^{\leftarrow}(x) = \delta_c(x) + 1$, for $x \geq c_1$, then $f^{\leftarrow} \in \mathrm{ARV}_f$. According to [29] (or [28]) we have that function $f(x)$, $x \geq 1$, belongs to the class IRV_f. According to [13] the sequence (c_n) $(c_n = f(n),\ n \in N)$ belongs to IRV_s. □

Let $K_{c,s}^{*,i}$ be the class of all strictly increasing unbounded sequences from the class $\mathrm{IRV}_s \cap \mathrm{ARV}_s$ (see [8]). This class contains (as a proper subclass) all strictly increasing unbounded regularly varying sequences in the Karamata sense whose index of variability is positive, and it does not contain any sequence from the class SV_s, nor from the class $R_{\infty,s}$.

The next statement gives the answer to the question from the introduction of this paper. It is a corollary of Corollary 2.2 (and, indirectly, of Theorem 2.1).

Corollary 2.3. *The class $K_{c,s}^{*,i}$ is the largest proper subclass of the class of strictly increasing unbounded sequences from the class IRV_s, such that the numerical function of any its element belongs to the class IRV_f.*

Proof. Let (c_n) be a strictly increasing unbounded sequence of positive numbers from the class $\mathrm{IRV}_s \cap \mathrm{ARV}_s$. Then by Corollaries 2.2(a) and 2.2(b), $\delta_c \in \mathrm{IRV}_f \cap \mathrm{ARV}_f$, thus $\delta_c \in \mathrm{IRV}_f$. Next, assume that (c_n) is a strictly increasing unbounded sequence of positive numbers from the class $\mathrm{IRV}_s \setminus \mathrm{ARV}_s$. Then by Corollary 2.2(b), $\delta_c \in \mathrm{ARV}_f$, and by Corollary 2.2(a) $\delta_c \notin \mathrm{IRV}_f$.

This completes the proof. □

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