Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2008, Article ID 672618, 13 pages
doi:10.1155/2008/672618

Research Article

Fixed Points and Stability of an Additive Functional Equation of n-Apollonius Type in C^*-Algebras

Fridoun Moradlou,[1] Hamid Vaezi,[1] and Choonkil Park[2]

[1] *Faculty of Mathematical Sciences, University of Tabriz, Tabriz, Iran*
[2] *Department of Mathematics, Hanyang University, Seoul 133–791, South Korea*

Correspondence should be addressed to Choonkil Park, baak@hanyang.ac.kr

Received 22 April 2008; Revised 11 June 2008; Accepted 16 July 2008

Recommended by John Rassias

Using the fixed point method, we prove the generalized Hyers-Ulam stability of C^*-algebra homomorphisms and of generalized derivations on C^*-algebras for the following functional equation of Apollonius type $\sum_{i=1}^{n} f(z - x_i) = -(1/n)\sum_{1\le i<j\le n} f(x_i + x_j) + nf(z - (1/n^2)\sum_{i=1}^{n} x_i)$.

1. Introduction and preliminaries

A classical question in the theory of functional equations is the following: "when is it true that a function, which approximately satisfies a functional equation $\mathcal{E}$, must be close to an exact solution of $\mathcal{E}$?" If the problem accepts a solution, we say that the equation $\mathcal{E}$ is stable. Such a problem was formulated by Ulam [1] in 1940 and solved in the next year for the Cauchy functional equation by Hyers [2]. It gave rise to the stability theory for functional equations. The result of Hyers was extended by Aoki [3] in 1950 by considering the unbounded Cauchy differences. In 1978, Rassias [4] proved that the additive mapping T, obtained by Hyers or Aoki, is linear if, in addition, for each $x \in E$, the mapping $f(tx)$ is continuous in $t \in \mathbb{R}$. Găvruţa [5] generalized the Rassias' result. Following the techniques of the proof of the corollary of Hyers [2], we observed that Hyers introduced (in 1941) the following Hyers continuity condition about the continuity of the mapping for each fixed point and then he proved homogeneity of degree one and, therefore, the famous linearity. This condition has been assumed further till now, through the complete Hyers direct method, in order to prove linearity for generalized Hyers-Ulam stability problem forms (see [6]). Beginning around 1980, the stability problems of several functional equations and approximate homomorphisms have been extensively investigated by a number of authors and there are many interesting results concerning this problem (see [7–21]).

Rassias [22], following the spirit of the innovative approach of Hyers [2], Aoki [3], and Rassias [4] for the unbounded Cauchy difference, proved a similar stability theorem in which he replaced the factor $\|x\|^p + \|y\|^p$ by $\|x\|^p \cdot \|y\|^q$ for $p, q \in \mathbb{R}$ with $p + q \neq 1$ (see also [23, 24] for a number of other new results).

In 2003, Cădariu and Radu applied the fixed-point method to the investigation of the Jensen functional equation [25] (see also [8, 26–30]). They could present a short and a simple proof (different of the "*direct method*," initiated by Hyers in 1941) for the generalized Hyers-Ulam stability of Jensen functional equation [25], for Cauchy functional equation [8], and for quadratic functional equation [26].

The following functional equation:

$$Q(x+y) + Q(x-y) = 2Q(x) + 2Q(y) \tag{1.1}$$

is called a *quadratic functional equation,* and every solution of (1.1) is said to be a *quadratic mapping*. Skof [31] proved the Hyers-Ulam stability of the quadratic functional equation (1.1) for mappings $f : E_1 \to E_2$, where E_1 is a normed space and E_2 is a Banach space. In [32], Czerwik proved the Hyers-Ulam stability of the quadratic functional equation (1.1). Borelli and Forti [33] generalized the stability result of the quadratic functional equation (1.1). Jun and Lee [34] proved the Hyers-Ulam stability of the Pexiderized quadratic equation

$$f(x+y) + g(x-y) = 2h(x) + 2k(y) \tag{1.2}$$

for mappings $f, g, h,$ and k. The stability problem of the quadratic equation has been extensively investigated by some mathematicians [35].

In an inner product space, the equality

$$\|z-x\|^2 + \|z-y\|^2 = \frac{1}{2}\|x-y\|^2 + 2\left\|z - \frac{x+y}{2}\right\|^2 \tag{1.3}$$

holds, then it is called the *Apollonius' identity.* The following functional equation, which was motivated by this equation,

$$Q(z-x) + Q(z-y) = \frac{1}{2}Q(x-y) + 2Q\left(z - \frac{x+y}{2}\right), \tag{1.4}$$

holds, then it is called *quadratic* (see [36]). For this reason, the functional equation (1.4) is called a *quadratic functional equation of Apollonius type,* and each solution of the functional equation (1.4) is said to be a *quadratic mapping of Apollonius type.* The quadratic functional equation and several other functional equations are useful to characterize inner product spaces [37].

In [36], Park and Rassias introduced and investigated a functional equation, which is called a *generalized Apollonius type quadratic functional equation.* In [38], Najati introduced and investigated a functional equation, which is called a *quadratic functional equation of n-Apollonius type.* Recently in [39], Park and Rassias introduced and investigated the following functional equation:

$$f(z-x) + f(z-y) = -\frac{1}{2}f(x+y) + 2f\left(z - \frac{x+y}{4}\right) \tag{1.5}$$

which is called an *Apollonius type additive functional equation,* and whose solution is called an *Apollonius type additive mapping.* In [40], Park introduced and investigated a functional equation, which is called a *generalized Apollonius-Jensen type additive functional equation* and whose solution is said to be a *generalized Apollonius-Jensen type additive mapping*.

In this paper, employing the above equality (1.5), for a fixed positive integer $n \geq 2$, we introduce a new functional equation, which is called an *additive functional equation of n-Apollonius type* and whose solution is said to be an *additive mapping of n-Apollonius type;*

$$\sum_{i=1}^{n} f(z - x_i) = -\frac{1}{n} \sum_{1 \leq i < j \leq n} f(x_i + x_j) + n f\left(z - \frac{1}{n^2} \sum_{i=1}^{n} x_i \right). \tag{1.6}$$

We will adopt the idea of Cădariu and Radu [8, 25, 28] to prove the generalized Hyers-Ulam stability results of C^*-algebra homomorphisms as well as to prove the generalized Ulam-Hyers stability of generalized derivations on C^*-algebra for additive functional equation of n-Apollonius type.

We recall two fundamental results in fixed-point theory.

Theorem 1.1 (see [25]). *Let (X, d) be a complete metric space and let $J : X \to X$ be strictly contractive, that is,*

$$d(Jx, Jy) \leq L f(x, y), \quad \forall x, y \in X \tag{1.7}$$

for some Lipschitz constant $L < 1$. Then, the following hold:

(1) *the mapping J has a unique fixed point $x^* = Jx^*$;*

(2) *the fixed point x^* is globally attractive, that is,*

$$\lim_{n \to \infty} J^n x = x^* \tag{1.8}$$

for any starting point $x \in X$;

(3) *one has the following estimation inequalities:*

$$\begin{aligned} d(J^n x, x^*) &\leq L^n d(x, x^*), \\ d(J^n x, x^*) &\leq \frac{1}{1 - L} d(J^n x, J^{n+1} x), \\ d(x, x^*) &\leq \frac{1}{1 - L} d(x, Jx) \end{aligned} \tag{1.9}$$

for all nonnegative integers n and all $x \in X$.

Let X be a set. A function $d : X \times X \to [0, \infty]$ is called a *generalized metric* on X if d satisfies the following:

(1) $d(x, y) = 0$ if and only if $x = y$;

(2) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(3) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

Theorem 1.2 (see [41]). *Let (X, d) be a complete generalized metric space and let $J : X \to X$ be a strictly contractive mapping with Lipschitz constant $L < 1$. Then for each given element $x \in X$, either*

$$d(J^n x, J^{n+1} x) = \infty \tag{1.10}$$

for all nonnegative integers n or there exists a positive integer n_0 such that the following hold:

(1) *$d(J^n x, J^{n+1} x) < \infty$ for all $n \geq n_0$;*

(2) *the sequence $\{J^n x\}$ converges to a fixed point y^* of J;*

(3) *y^* is the unique fixed point of J in the set $Y = \{y \in X \mid d(J^{n_0} x, y) < \infty\}$;*

(4) *$d(y, y^*) \leq (1/(1-L)) d(y, Jy)$ for all $y \in Y$.*

Throughout this paper, assume that A is a C^*-algebra with norm $\|\cdot\|_A$ and that B is a C^*-algebra with norm $\|\cdot\|_B$.

2. Stability of C^*-algebra homomorphisms

Lemma 2.1. *Let X and Y be real-vector spaces. A mapping $f : X \to Y$ satisfies* (1.6) *for all $x_1, \ldots, x_n, z$ if and only if the mapping f is additive.*

Proof. Letting $x_1 = \cdots = x_n = z = 0$ in (1.6), we get that $f(0) = 0$. Let j and k be fixed integers with $1 \leq j < k \leq n$. Setting $x_i = 0$ for all $1 \leq i \leq n$, $i \neq j, k$ in (1.6), we have

$$f(z - x_j) + f(z - x_k) + (n-2) f(z) = -\frac{1}{n} f(x_j + x_k) - \frac{n-2}{n}(f(x_j) + f(x_k)) + n f\left(z - \frac{1}{n^2}(x_j + x_k)\right) \tag{2.1}$$

for all $x_j, x_k, z \in X$. Replacing x_j by $-x_j$ and x_k by x_j in (2.1), respectively, we get

$$f(z + x_j) + f(z - x_j) = -\frac{n-2}{n}(f(-x_j) + f(x_j)) + 2f(z) \tag{2.2}$$

for all $x_j, z \in X$. Putting $z = 0$ in (2.2), we conclude that $f(-x_j) = -f(x_j)$ for all $x_j \in X$. This means that f is an odd function. Letting $x_k = z = 0$ in (2.1) and using the oddness of f, we obtain that

$$f\left(\frac{1}{n^2} x_j\right) = \frac{1}{n^2} f(x_j), \qquad f(n^2 x_j) = n^2 f(x_j) \tag{2.3}$$

for all $x_j \in X$. Letting $z = 0$ in (2.1), using the oddness of f and (2.3), we have

$$f(x_j + x_k) = f(x_j) + f(x_k) \tag{2.4}$$

for all $x_j, x_k \in X$. Therefore, $f : X \to Y$ is an additive mapping.

The converse is obviously true. □

For a given mapping $f : A \to B$ and for a fixed positive integer $n \geq 2$, we define

$$C_\mu f(z, x_1, \ldots, x_n) := \sum_{i=1}^{n} \mu f(z - x_i) + \frac{1}{n} \sum_{1 \leq i < j \leq n} f(\mu x_i + \mu x_j) - n f\left(\mu z - \frac{1}{n^2} \sum_{i=1}^{n} \mu x_i\right) \tag{2.5}$$

for all $\mu \in \mathbb{T}^1 := \{\nu \in \mathbb{C} : |\nu| = 1\}$ and all $z, x_1, \ldots, x_n \in A$.

We prove the generalized Hyers-Ulam stability of C^*-algebra homomorphisms for the functional equation $C_\mu f(z, x_1, \ldots, x_n) = 0$.

Theorem 2.2. *Let $f : A \to B$ be a mapping satisfying $f(0) = 0$ for which there exists a function $\varphi : A^{n+1} \to [0,\infty)$ such that*

$$\sum_{j=0}^{\infty}\left(\frac{n^2}{n^2-1}\right)^{2j}\varphi\left(\left(\frac{n^2-1}{n^2}\right)^{j}z,\left(\frac{n^2-1}{n^2}\right)^{j}x_1,\dots,\left(\frac{n^2-1}{n^2}\right)^{j}x_n\right)<\infty, \tag{2.6}$$

$$\|C_\mu f(z,x_1,\dots,x_n)\|_B \le \varphi(z,x_1,\dots,x_n), \tag{2.7}$$

$$\|f(xy)-f(x)f(y)\|_B \le \varphi(x,y,\underbrace{0,\dots,0}_{n-1\text{ times}}), \tag{2.8}$$

$$\|f(x^*)-f(x)^*\|_B \le \varphi(\underbrace{x,\dots,x}_{n+1\text{ times}}) \tag{2.9}$$

for all $\mu \in \mathbb{T}^1$ and all $x,y,z,x_1,\dots,x_n \in A$. If for some $1 \le j \le n$ there exists a Lipschitz constant $L < 1$ such that

$$\varphi(x,0,\dots,0,\underbrace{x}_{j\text{th}},0,\dots,0) \le \frac{n^2-1}{n^2}L\varphi\left(\frac{n^2}{n^2-1}x,0,\dots,0,\underbrace{\frac{n^2}{n^2-1}x}_{j\text{th}},0,\dots,0\right) \tag{2.10}$$

for all $x \in A$, then there exists a unique C^*-algebra homomorphism $H : A \to B$ such that

$$\|f(x)-H(x)\|_B \le \frac{n}{(n^2-1)\times(1-L)}\varphi(x,0,\dots,0,\underbrace{x}_{j\text{th}},0,\dots,0) \tag{2.11}$$

for all $x \in A$.

Proof. Consider the set

$$X := \{g : A \longrightarrow B,\ g(0) = 0\} \tag{2.12}$$

and introduce the *generalized metric* on X:

$$d(g,h) = \inf\left\{C \in \mathbb{R}_+ : \|g(x)-h(x)\|_B \le C\varphi(x,0,\dots,0,\underbrace{x}_{j\text{th}},0,\dots,0)\ \forall x \in A\right\}. \tag{2.13}$$

It is easy to show that (X,d) is complete.

For convenience, set

$$\varphi_j(x,y) := \varphi(x,0,\dots,0,\underbrace{y}_{j\text{th}},0,\dots,0) \tag{2.14}$$

for all $x,y \in A$ and all $1 \le j \le n$.

Now we consider the linear mapping $J : X \to X$ such that

$$Jg(x) := \frac{n}{\alpha}g\left(\frac{\alpha}{n}x\right) \tag{2.15}$$

for all $x \in A$, where $\alpha = (n^2-1)/n$.

For any $g, h \in X$, we have

$$\begin{aligned} d(g,h) < C &\Longrightarrow \|g(x) - h(x)\|_B \le C\varphi_j(x,x) \quad \forall x \in A \\ &\Longrightarrow \left\| \frac{n}{\alpha} g\left(\frac{\alpha}{n}x\right) - \frac{n}{\alpha} h\left(\frac{\alpha}{n}x\right) \right\|_B \le \frac{n}{\alpha} C\varphi_j\left(\frac{\alpha}{n}x, \frac{\alpha}{n}x\right) \\ &\Longrightarrow \left\| \frac{n}{\alpha} g\left(\frac{\alpha}{n}x\right) - \frac{n}{\alpha} h\left(\frac{\alpha}{n}x\right) \right\|_B \le LC\varphi_j(x,x) \\ &\Longrightarrow d(Jg, Jh) \le LC. \end{aligned} \tag{2.16}$$

Therefore, we see that

$$d(Jg, Jh) \le Ld(g,h), \quad \forall g, h \in A. \tag{2.17}$$

This means J is a strictly contractive self-mapping of X, with the Lipschitz constant L.

Letting $\mu = 1$, $z = x_j = x$, and for each $1 \le k \le n$ with $k \ne j$, $x_k = 0$ in (2.7), we get

$$\left\| \alpha f(x) - n f\left(\frac{\alpha}{n}x\right) \right\|_B \le \varphi_j(x,x) \tag{2.18}$$

for all $x \in A$. So

$$\left\| f(x) - \frac{n}{\alpha} f\left(\frac{\alpha}{n}x\right) \right\|_B \le \frac{1}{\alpha}\varphi_j(x,x) \tag{2.19}$$

for all $x \in A$. Hence $d(f, Jf) \le 1/\alpha$.

By Theorem 1.2, there exists a mapping $H : A \to B$ such that the following hold:

(1) H is a fixed point of J, that is,

$$H\left(\frac{\alpha}{n}x\right) = \frac{\alpha}{n} H(x) \tag{2.20}$$

for all $x \in A$; the mapping H is a unique fixed point of J in the set

$$Y = \{g \in X : d(f,g) < \infty\}; \tag{2.21}$$

and this implies that H is a unique mapping satisfying (2.20) such that there exists $C \in (0, \infty)$ satisfying

$$\|H(x) - f(x)\|_B \le C\varphi_j(x,x) \tag{2.22}$$

for all $x \in A$.

(2) $d(J^m f, H) \to 0$ as $m \to \infty$; and this implies the equality

$$\lim_{m\to\infty} \left(\frac{n}{\alpha}\right)^m f\left(\left(\frac{\alpha}{n}\right)^m x\right) = H(x) \tag{2.23}$$

for all $x \in A$;

(3) $d(f,H) \le (1/(1-L))d(f,Jf)$, which implies the inequality

$$d(f,H) \le \frac{1}{\alpha - \alpha L}; \tag{2.24}$$

and this implies that the inequality (2.11) holds.

It follows from (2.6), (2.7), and (2.23) that

$$\begin{aligned}
&\left\| \sum_{i=1}^{n} H(z-x_i) + \frac{1}{n}\sum_{1\le i<j\le n} H(x_i+x_j) - nH\left(z - \frac{1}{n^2}\sum_{i=1}^{n} x_i\right)\right\|_B \\
&\quad = \lim_{m\to\infty}\left(\frac{n}{\alpha}\right)^m \left\| \sum_{i=1}^{n} f\left(\left(\frac{\alpha}{n}\right)^m (z-x_i)\right) + \frac{1}{n}\sum_{1\le i<j\le n} f\left(\left(\frac{\alpha}{n}\right)^m (x_i+x_j)\right)\right. \\
&\qquad\qquad \left. - nf\left(\left(\frac{\alpha}{n}\right)^m z - \left(\frac{\alpha}{n}\right)^m \times \frac{1}{n^2}\sum_{i=1}^{n} x_i\right)\right\|_B \\
&\quad \le \lim_{m\to\infty}\left(\frac{n}{\alpha}\right)^m \varphi\left(\left(\frac{\alpha}{n}\right)^m z, \left(\frac{\alpha}{n}\right)^m x_1, \dots, \left(\frac{\alpha}{n}\right)^m x_n\right) \\
&\quad \le \lim_{m\to\infty}\left(\frac{n}{\alpha}\right)^{2m} \varphi\left(\left(\frac{\alpha}{n}\right)^m z, \left(\frac{\alpha}{n}\right)^m x_1, \dots, \left(\frac{\alpha}{n}\right)^m x_n\right) = 0
\end{aligned} \tag{2.25}$$

for all $x_1, \dots, x_n, z \in A$. So

$$\sum_{i=1}^{n} H(z-x_i) = -\frac{1}{n}\sum_{1\le i<j\le n} H(x_i+x_j) + nH\left(z - \frac{1}{n^2}\sum_{i=1}^{n} x_i\right) \tag{2.26}$$

for all $x_1, \dots, x_n, z \in A$. By Lemma 2.1, the mapping $H : A \to B$ is Cauchy additive, that is, $H(x+y) = H(x) + H(y)$ for all $x, y \in A$.

By a similar method to the proof of [14], one can show that the mapping $H : A \to B$ is $\mathbb{C}$-linear.

It follows from (2.8) that

$$\begin{aligned}
\|H(xy) - H(x)H(y)\|_B &= \lim_{m\to\infty}\left(\frac{n}{\alpha}\right)^{2m} \left\| f\left(\left(\frac{\alpha}{n}\right)^{2m} xy\right) - f\left(\left(\frac{\alpha}{n}\right)^m x\right) f\left(\left(\frac{\alpha}{n}\right)^m y\right)\right\|_B \\
&\le \lim_{m\to\infty}\left(\frac{n}{\alpha}\right)^{2m} \varphi\left(\left(\frac{\alpha}{n}\right)^m x, \left(\frac{\alpha}{n}\right)^m y, \underbrace{0, \dots, 0}_{n-1 \text{ times}}\right) = 0
\end{aligned} \tag{2.27}$$

for all $x, y \in A$. So

$$H(xy) = H(x)H(y) \tag{2.28}$$

for all $x, y \in A$.

It follows from (2.9) that

$$\begin{aligned} \|H(x^*) - H(x)^*\|_B &= \lim_{m\to\infty}\left(\frac{n}{\alpha}\right)^m \left\| f\left(\left(\frac{\alpha}{n}\right)^m x^*\right) - f\left(\left(\frac{\alpha}{n}\right)^m x\right)^* \right\|_B \\ &\leq \lim_{m\to\infty}\left(\frac{n}{\alpha}\right)^m \varphi\Bigg(\underbrace{\left(\frac{\alpha}{n}\right)^m x, \dots, \left(\frac{\alpha}{n}\right)^m x}_{n+1 \text{ times}}\Bigg) \\ &\leq \lim_{m\to\infty}\left(\frac{n}{\alpha}\right)^{2m} \varphi\Bigg(\underbrace{\left(\frac{\alpha}{n}\right)^m x, \dots, \left(\frac{\alpha}{n}\right)^m x}_{n+1 \text{ times}}\Bigg) = 0 \end{aligned} \tag{2.29}$$

for all $x \in A$. So

$$H(x^*) = H(x)^* \tag{2.30}$$

for all $x \in A$.

Thus $H : A \to B$ is a C^*-algebra homomorphism satisfying (2.11) as desired. □

Corollary 2.3. *Let $r > 2$ and θ be nonnegative real numbers, and let $f : A \to B$ be a mapping such that*

$$\|C_\mu f(z, x_1, \dots, x_n)\|_B \leq \theta\left(\|z\|_A^r + \sum_{i=1}^{n}\|x_i\|_A^r\right), \tag{2.31}$$

$$\|f(xy) - f(x)f(y)\|_B \leq \theta(\|x\|_A^r + \|y\|_A^r), \tag{2.32}$$

$$\|f(x^*) - f(x)^*\|_B \leq (n+1)\theta\|x\|_A^r \tag{2.33}$$

for all $\mu \in \mathbb{T}^1$ and all $x, y, z \in A$. Then there exists a unique C^*-algebra homomorphism $H : A \to B$ such that

$$\|f(x) - H(x)\|_B \leq \frac{2n(n^2-1)^{-r}\theta}{(n^2-1)^{1-r} - n^{2(1-r)}}\|x\|_A^r \tag{2.34}$$

for all $x \in A$.

Proof. The proof follows from Theorem 2.2 by taking

$$\varphi(z, x_1, \dots, x_n) := \theta\left(\|z\|_A^r + \sum_{i=1}^{n}\|x_i\|_A^r\right) \tag{2.35}$$

for all $x, y, z \in A$. It follows from (2.31) that $f(0) = 0$. We can choose $L = (n^2/(n^2-1))^{1-r}$ to get the desired result. □

Theorem 2.4. *Let $f : A \to B$ be a mapping satisfying $f(0) = 0$ for which there exists a function $\varphi : A^{n+1} \to [0, \infty)$ satisfying* (2.7), (2.8), *and* (2.9) *such that*

$$\sum_{j=0}^{\infty}\left(\frac{n^2-1}{n^2}\right)^j \varphi\left(\left(\frac{n^2}{n^2-1}\right)^j z, \left(\frac{n^2}{n^2-1}\right)^j x_1, \dots, \left(\frac{n^2}{n^2-1}\right)^j x_n\right) < \infty \tag{2.36}$$

for all $z, x_1, \ldots, x_n \in A$. If for some $1 \le j \le n$ there exists a Lipschitz constant $L < 1$ such that

$$\varphi(x,0,\ldots,0,\underbrace{x}_{j\text{th}},0,\ldots,0) \le \frac{n^2}{n^2-1} L\varphi\left(\frac{n^2-1}{n^2}x,0,\ldots,0,\underbrace{\frac{n^2-1}{n^2}x}_{j\text{th}},0,\ldots,0\right) \tag{2.37}$$

for all $x \in A$, then there exists a unique C^*-algebra homomorphism $H : A \to B$ such that

$$\|f(x) - H(x)\|_B \le \frac{nL}{(n^2-1)\times(1-L)}\varphi(x,0,\ldots,0,\underbrace{x}_{j\text{th}},0,\ldots,0) \tag{2.38}$$

for all $x \in A$.

Proof. Similar to proof of Theorem (2.2), we consider the linear mapping $J : X \to X$ such that

$$Jg(x) := \frac{\alpha}{n} g\left(\frac{n}{\alpha}x\right) \tag{2.39}$$

for all $x \in A$, where $\alpha = (n^2-1)/n$. We can conclude that J is a strictly contractive self mapping of X with the Lipschitz constant L.

It follows from (2.18) that

$$\left\| f(x) - \frac{\alpha}{n} f\left(\frac{n}{\alpha}x\right)\right\|_B \le \frac{1}{n}\varphi_j\left(\frac{n}{\alpha}x, \frac{n}{\alpha}x\right) \le \frac{L}{\alpha}\varphi_j(x,x) \tag{2.40}$$

for all $x \in A$. Hence, $d(f, Jf) \le (L/\alpha)$.

By Theorem 1.2, there exists a mapping $H : A \to B$ such that the following hold:

(1) H is a fixed point of J, that is,

$$H\left(\frac{n}{\alpha}x\right) = \frac{n}{\alpha}H(x) \tag{2.41}$$

for all $x \in A$; the mapping H is a unique fixed point of J in the set

$$Y = \{g \in X : d(f,g) < \infty\}; \tag{2.42}$$

and this implies that H is a unique mapping satisfying (2.41) such that there exists $C \in (0,\infty)$ satisfying

$$\|H(x) - f(x)\|_B \le C\varphi_j(x,x) \tag{2.43}$$

for all $x \in A$;

(2) $d(J^m f, H) \to 0$ as $m \to \infty$; and this implies the equality

$$\lim_{m\to\infty}\left(\frac{\alpha}{n}\right)^m f\left(\left(\frac{\alpha}{n}\right)^m x\right) = H(x) \tag{2.44}$$

for all $x \in A$;

(3) $d(f,H) \le (1/(1-L))d(f,Jf)$, which implies the inequality

$$d(f,H) \le \frac{L}{\alpha - \alpha L}, \tag{2.45}$$

which implies that the inequality (2.38) holds.

The rest of the proof is similar to the proof of Theorem 2.2. □

Corollary 2.5. *Let $r < 1$ and θ be nonnegative real numbers, and let $f : A \to B$ be a mapping satisfying* (2.31), (2.32), *and* (2.33). *Then there exists a unique C^*-algebra homomorphism $H : A \to B$ such that*

$$\|f(x) - H(x)\|_B \leq \frac{2n(n^2-1)^{r-2}L\theta}{(n^2-1)^{r-1} - n^{2(r-1)}}\|x\|_A^r \tag{2.46}$$

for all $x \in A$ and $L = (n^2/(n^2-1))^{r-1}$.

Proof. The proof follows from Theorem 2.4 by taking

$$\varphi(z, x_1, \dots, x_n) := \theta\left(\|z\|_A^r + \sum_{i=1}^{n}\|x_i\|_A^r\right) \tag{2.47}$$

for all $z, x_1, \dots, x_n \in A$. It follows from (2.31) that $f(0) = 0$. We can choose $L = (n^2/(n^2-1))^{r-1}$ to get the desired result. □

3. Stability of generalized derivations on C^*-algebras

For a given mapping $f : A \to A$ and for a fixed positive integer $n \geq 2$, we define

$$C_\mu f(z, x_1, \dots, x_n) := \sum_{i=1}^{n}\mu f(z - x_i) + \frac{1}{n}\sum_{1\leq i<j\leq n} f(\mu x_i + \mu x_j) - nf\left(\mu z - \frac{1}{n^2}\sum_{i=1}^{n}\mu x_i\right) \tag{3.1}$$

for all $\mu \in \mathbb{T}^1$ and all $z, x_1, \dots, x_n \in A$.

Definition 3.1 (see [42]). A generalized derivation $\delta : A \to A$ is involutive $\mathbb{C}$-linear and fulfills

$$\delta(xyz) = \delta(xy)z - x\delta(y)z + x\delta(yz)$$

for all $x, y, z \in A$.

We prove the generalized Hyers-Ulam stability of derivations on C^*-algebras for the functional equation $C_\mu f(z, x_1, \dots, x_n) = 0$.

Theorem 3.2. *Let $f : A \to A$ be a mapping satisfying $f(0) = 0$ for which there exists a function $\varphi : A^{n+1} \to [0, \infty)$ such that*

$$\sum_{j=0}^{\infty}\left(\frac{n^2}{n^2-1}\right)^{3j}\varphi\left(\left(\frac{n^2-1}{n^2}\right)^j z, \left(\frac{n^2-1}{n^2}\right)^j x_1, \dots, \left(\frac{n^2-1}{n^2}\right)^j x_n\right) < \infty, \tag{3.2}$$

$$\|C_\mu f(x_1, \dots, x_n, z)\|_A \leq \varphi(z, x_1, \dots, x_n), \tag{3.3}$$

$$\|f(xyz) - f(xy)z + xf(y)z - xf(yz)\|_A \leq \varphi(x, y, z, \underbrace{0, \dots, 0}_{n-2\text{ times}}), \tag{3.4}$$

$$\|f(x^*) - f(x)^*\|_A \leq \varphi(\underbrace{x, \dots, x}_{n+1\text{ times}}) \tag{3.5}$$

for all $\mu \in \mathbb{T}^1$ *and all* $x, y, z, x_1, \ldots, x_n \in A$. *If for some* $1 \le j \le n$ *there exists a Lipschitz constant* $L < 1$ *such that*

$$\varphi(x, 0, \ldots, 0, \underbrace{x}_{j\text{th}}, 0, \ldots, 0) \le \frac{n^2 - 1}{n^2} L\varphi\left(\frac{n^2}{n^2 - 1}x, 0, \ldots, 0, \underbrace{\frac{n^2}{n^2 - 1}x}_{j\text{th}}, 0, \ldots, 0\right) \tag{3.6}$$

for all $x \in A$, *then there exists a unique generalized derivation* $\delta : A \to A$ *such that*

$$\|f(x) - \delta(x)\|_A \le \frac{n}{(n^2 - 1) \times (1 - L)}\varphi(x, 0, \ldots, 0, \underbrace{x}_{j\text{th}}, 0, \ldots, 0) \tag{3.7}$$

for all $x \in A$.

Proof. By the same reasoning as in the proof of Theorem 2.2, there exists a unique involutive $\mathbb{C}$-linear mapping $\delta : A \to A$ satisfying (3.7). The mapping $\delta : A \to A$ is given by

$$\delta(x) = \left(\frac{n}{\alpha}\right)^m f\left(\left(\frac{n}{\alpha}\right)^m x\right) \tag{3.8}$$

for all $x \in A$.

It follows from (3.4) that

$$\begin{aligned}
&\|\delta(xyz) - \delta(xy)z + x\delta(y)z - x\delta(yz)\|_A \\
&\quad = \lim_{m\to\infty}\left(\frac{n}{\alpha}\right)^{3m}\left\| f\left(\left(\frac{\alpha}{n}\right)^{3m} xyz\right) - f\left(\left(\frac{\alpha}{n}\right)^{2m} xy\right)\cdot\left(\frac{\alpha}{n}\right)^m z \right. \\
&\qquad\qquad \left. + \left(\frac{\alpha}{n}\right)^m xf\left(\left(\frac{\alpha}{n}\right)^m y\right)\cdot\left(\frac{\alpha}{n}\right)^m z - \left(\frac{\alpha}{n}\right)^m xf\left(\left(\frac{\alpha}{n}\right)^{2m} yz\right)\right\|_A \\
&\quad \le \lim_{m\to\infty}\left(\frac{n}{\alpha}\right)^{3m}\varphi\left(\left(\frac{\alpha}{n}\right)^m x, \left(\frac{\alpha}{n}\right)^m y, \left(\frac{\alpha}{n}\right)^m z, \underbrace{0, \ldots, 0}_{n-2\text{ times}}\right) = 0
\end{aligned} \tag{3.9}$$

for all $x, y, z \in A$. So

$$\delta(xyz) = \delta(xy)z - x\delta(y)z + x\delta(yz) \tag{3.10}$$

for all $x, y, z \in A$. Thus $\delta : A \to A$ is a generalized derivation satisfying (3.7). □

Theorem 3.3. *Let* $f : A \to A$ *be a mapping satisfying* $f(0) = 0$ *for which there exists a function* $\varphi : A^{n+1} \to [0, \infty)$ *satisfying* (2.36),(3.3), (3.4) *and* (3.5) *for all* $x, y, z, x_1, \ldots, x_n \in A$. *If for some* $1 \le j \le n$ *there exists a Lipschitz constant* $L < 1$ *such that*

$$\varphi(x, 0, \ldots, 0, \underbrace{x}_{j\text{th}}, 0, \ldots, 0) \le \frac{n^2}{n^2 - 1} L\varphi\left(\frac{n^2 - 1}{n^2}x, 0, \ldots, 0, \underbrace{\frac{n^2 - 1}{n^2}x}_{j\text{th}}, 0, \ldots, 0\right) \tag{3.11}$$

for all $x \in A$, *then there exists a unique generalized derivation* $\delta : A \to A$ *such that*

$$\|f(x) - \delta(x)\|_B \le \frac{nL}{(n^2 - 1) \times (1 - L)}\varphi(x, 0, \ldots, 0, \underbrace{x}_{j\text{th}}, 0, \ldots, 0) \tag{3.12}$$

for all $x \in A$.

Proof. The proof is similar to the proofs of Theorems 2.4 and 3.2. □

Acknowledgments

This paper is based on final report of the research project of the Ph.D. thesis in University of Tabriz and the third author was supported by Grant no. F01-2006-000-10111-0 from the Korea Science and Engineering Foundation. The authors would like to thank the referees for a number of valuable suggestions regarding a previous version of this paper.

References

[1] S. M. Ulam, *A Collection of Mathematical Problems*, Interscience Tracts in Pure and Applied Mathematics, no. 8, Interscience Publishers, New York, NY, USA, 1960.
[2] D. H. Hyers, "On the stability of the linear functional equation," *Proceedings of the National Academy of Sciences of the United States of America*, vol. 27, pp. 222–224, 1941.
[3] T. Aoki, "On the stability of the linear transformation in Banach spaces," *Journal of the Mathematical Society of Japan*, vol. 2, pp. 64–66, 1950.
[4] Th. M. Rassias, "On the stability of the linear mapping in Banach spaces," *Proceedings of the American Mathematical Society*, vol. 72, no. 2, pp. 297–300, 1978.
[5] P. Găvruţa, "A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings," *Journal of Mathematical Analysis and Applications*, vol. 184, no. 3, pp. 431–436, 1994.
[6] L. Maligranda, "A result of Tosio Aoki about a generalization of Hyers-Ulam stability of additive functions—a question of priority ," *Aequationes Mathematicae*, vol. 75, no. 3, pp. 289–296, 2008.
[7] D. G. Bourgin, "Classes of transformations and bordering transformations," *Bulletin of the American Mathematical Society*, vol. 57, pp. 223–237, 1951.
[8] L. Cădariu and V. Radu, "On the stability of the Cauchy functional equation: a fixed point approach," *Grazer Mathematische Berichte*, vol. 346, pp. 43–52, 2004.
[9] H.-M. Kim and J. M. Rassias, "Generalization of Ulam stability problem for Euler-Lagrange quadratic mappings," *Journal of Mathematical Analysis and Applications*, vol. 336, no. 1, pp. 277–296, 2007.
[10] Y.-S. Lee and S.-Y. Chung, "Stability of an Euler-Lagrange-Rassias equation in the spaces of generalized functions," *Applied Mathematics Letters*, vol. 21, no. 7, pp. 694–700, 2008.
[11] P. Nakmahachalasint, "On the generalized Ulam-Gavruta-Rassias stability of mixed-type linear and Euler-Lagrange-Rassias functional equations," *International Journal of Mathematics and Mathematical Sciences*, vol. 2007, Article ID 63239, 10 pages, 2007.
[12] C.-G. Park, "On the stability of the linear mapping in Banach modules," *Journal of Mathematical Analysis and Applications*, vol. 275, no. 2, pp. 711–720, 2002.
[13] C.-G. Park, "On an approximate automorphism on a C^*-algebra," *Proceedings of the American Mathematical Society*, vol. 132, no. 6, pp. 1739–1745, 2004.
[14] C.-G. Park, "Homomorphisms between Poisson JC^*-algebras," *Bulletin of the Brazilian Mathematical Society*, vol. 36, no. 1, pp. 79–97, 2005.
[15] C. Park, "Fixed points and Hyers-Ulam-Rassias stability of Cauchy-Jensen functional equations in Banach algebras," *Fixed Point Theory and Applications*, vol. 2007, Article ID 50175, 15 pages, 2007.
[16] C.-G. Park, "Stability of an Euler-Lagrange-Rassias type additive mapping," *International Journal of Applied Mathematics & Statistics*, vol. 7, pp. 101–111, 2007.
[17] C. Park, "Generalized Hyers-Ulam stability of quadratic functional equations: a fixed point approach," *Fixed Point Theory and Applications*, vol. 2008, Article ID 493751, 9 pages, 2008.
[18] A. Pietrzyk, "Stability of the Euler-Lagrange-Rassias functional equation," *Demonstratio Mathematica*, vol. 39, no. 3, pp. 523–530, 2006.
[19] J. M. Rassias and M. J. Rassias, "Refined Ulam stability for Euler-Lagrange type mappings in Hilbert spaces," *International Journal of Applied Mathematics & Statistics*, vol. 7, pp. 126–132, 2007.
[20] Th. M. Rassias, "The problem of S. M. Ulam for approximately multiplicative mappings," *Journal of Mathematical Analysis and Applications*, vol. 246, no. 2, pp. 352–378, 2000.
[21] Th. M. Rassias, "On the stability of functional equations in Banach spaces," *Journal of Mathematical Analysis and Applications*, vol. 251, no. 1, pp. 264–284, 2000.
[22] J. M. Rassias, "On approximation of approximately linear mappings by linear mappings," *Bulletin des Sciences Mathématiques*, vol. 108, no. 4, pp. 445–446, 1984.
[23] J. M. Rassias, "On approximation of approximately linear mappings by linear mappings," *Journal of Functional Analysis*, vol. 46, no. 1, pp. 126–130, 1982.

[24] J. M. Rassias, "Solution of a problem of Ulam," *Journal of Approximation Theory*, vol. 57, no. 3, pp. 268–273, 1989.
[25] L. Cădariu and V. Radu, "Fixed points and the stability of Jensen's functional equation," *Journal of Inequalities in Pure and Applied Mathematics*, vol. 4, no. 1, article 4, 7 pages, 2003.
[26] L. Cădariu and V. Radu, "Fixed points and the stability of quadratic functional equations," *Analele Universităţii de Vest din Timişoara*, vol. 41, no. 1, pp. 25–48, 2003.
[27] S.-M. Jung and J. M. Rassias, "A fixed point approach to the stability of a functional equation of the spiral of Theodorus," *Fixed Point Theory and Applications*. In press.
[28] V. Radu, "The fixed point alternative and the stability of functional equations," *Fixed Point Theory*, vol. 4, no. 1, pp. 91–96, 2003.
[29] J. M. Rassias, "Alternative contraction principle and Ulam stability problem," *Mathematical Sciences Research Journal*, vol. 9, no. 7, pp. 190–199, 2005.
[30] J. M. Rassias, "Alternative contraction principle and alternative Jensen and Jensen type mappings," *International Journal of Applied Mathematics & Statistics*, vol. 4, pp. 1–10, 2006.
[31] F. Skof, "Local properties and approximation of operators," *Rendiconti del Seminario Matematico e Fisico di Milano*, vol. 53, pp. 113–129, 1983.
[32] S. Czerwik, "The stability of the quadratic functional equation," in *Stability of Mappings of Hyers-Ulam Type*, Th. M. Rassias and J. Tabor, Eds., Hadronic Press Collection of Original Articles, pp. 81–91, Hadronic Press, Palm Harbor, Fla, USA, 1994.
[33] C. Borelli and G. L. Forti, "On a general Hyers-Ulam stability result," *International Journal of Mathematics and Mathematical Sciences*, vol. 18, no. 2, pp. 229–236, 1995.
[34] K.-W. Jun and Y.-H. Lee, "On the Hyers-Ulam-Rassias stability of a Pexiderized quadratic inequality," *Mathematical Inequalities & Applications*, vol. 4, no. 1, pp. 93–118, 2001.
[35] G. H. Kim, "On the stability of the quadratic mapping in normed spaces," *International Journal of Mathematics and Mathematical Sciences*, vol. 25, no. 4, pp. 217–229, 2001.
[36] C.-G. Park and Th. M. Rassias, "Hyers-Ulam stability of a generalized Apollonius type quadratic mapping," *Journal of Mathematical Analysis and Applications*, vol. 322, no. 1, pp. 371–381, 2006.
[37] J. Aczél and J. Dhombres, *Functional Equations in Several Variables*, vol. 31 of *Encyclopedia of Mathematics and Its Applications*, Cambridge University Press, Cambridge, UK, 1989.
[38] A. Najati, "Hyers-Ulam stability of an n-Apollonius type quadratic mapping," *Bulletin of the Belgian Mathematical Society. Simon Stevin*, vol. 14, no. 4, pp. 755–774, 2007.
[39] C. Park and Th. M. Rassias, "Homomorphisms in C^*-ternary algebras and JB^*-triples," *Journal of Mathematical Analysis and Applications*, vol. 337, no. 1, pp. 13–20, 2008.
[40] C. Park, "Hyers-Ulam-Rassias stability of a generalized Apollonius-Jensen type additive mapping and isomorphisms between C^*-algebras," *Mathematische Nachrichten*, vol. 281, no. 3, pp. 402–411, 2008.
[41] J. B. Diaz and B. Margolis, "A fixed point theorem of the alternative, for contractions on a generalized complete metric space," *Bulletin of the American Mathematical Society*, vol. 74, pp. 305–309, 1968.
[42] P. Ara and M. Mathieu, *Local Multipliers of C^*-Algebras*, Springer Monographs in Mathematics, Springer, London, UK, 2003.