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# Reducibility of invertible tuples to the principal component in commutative Banach algebras

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**Abstract.** Let $A$ be a complex, commutative unital Banach algebra. We introduce two notions of exponential reducibility of Banach algebra tuples and present an analogue to the Corach-Suárez result on the connection between reducibility in $A$ and in $C(M(A))$. Our methods are of an analytical nature. Necessary and sufficient geometric/topological conditions are given for reducibility (respectively reducibility to the principal component of $U_n(A)$) whenever the spectrum of $A$ is homeomorphic to a subset of $\mathbb{C}^n$.

## 1. Introduction

The concepts of stable ranges and reducibility of invertible tuples in rings originate from Hyman Bass's work [2] treating problems in algebraic $K$-theory. Later on, due to work of L. Vasershtein [29], these notions also turned out to be very important in the theory of function algebras and topology because of their intimate relations to extension problems. This direction has further been developed by Corach and Suárez, [4] and [5]. Function theorists have also been interested in this subject and mainly computed the stable ranks for various algebras of holomorphic functions. For example, P.W. Jones, D. Marshall and T. Wolff [9] determined the stable rank of the disk algebra $A(\mathbb{D})$, and Corach and Suárez [7] the one for the polydisk and ball algebras. The whole culminated in S. Treil's work on the stable rank for the algebra $H^\infty$ of bounded analytic functions on the unit disk [28]. Recent work includes investigations of stable ranks for real-symmetric function algebras (see for instance [17] and [15]). The subject of the present paper is linked to the theory developed by Corach and Suárez and provides a detailed analysis of the fine structure of the set $U_n(A)$ of invertible tuples within the realm of commutative Banach algebras. The main intention is the introduction of a new concept, the exponential reducibility of $n$-tuples, and to present a new view on the structure of the connected components of $U_n(A)$.

In contrast to the work of Corach and Suárez (and Lin), we use an analytic framework (and methods) instead of the powerful algebraic-topological setting. We think that this makes the theory accessible to a larger readership.

### 1.1. Notational background and scheme of the paper

Let $A$ be a commutative unital Banach algebra over $\mathbb{K}=\mathbb{R}$ or $\mathbb{K}=\mathbb{C}$, the identity element (or multiplicatively neutral element) being denoted by $\mathbf{1}$. Then the spectrum (= set of nonzero, multiplicative $\mathbb{K}$-linear functionals on $A$) of $A$ is denoted by $M(A)$, and the set of all $n\times n$-matrices over $A$ by $\mathcal{M}_n(A)$. If $f\in C(X,\mathbb{K})$, the space of all $\mathbb{K}$-valued continuous functions on the topological space $X$, then $Z(f):=\{x\in X: f(x)=0\}$. If $\boldsymbol{f}=(f_1,...,f_n)\in C(X,\mathbb{K}^n)$, then $Z(\boldsymbol{f}):=\bigcap_{j=1}^n Z(f_j)$ is the joint zero-set. Moreover, if $\boldsymbol{f}\in A^n$, then $|\boldsymbol{f}|=\sqrt{\sum_{j=1}^n |f_j|^2}$, $\langle \boldsymbol{f},\boldsymbol{g}\rangle:=\boldsymbol{f}\cdot\boldsymbol{g}:=\sum_{j=1}^n f_j g_j$ and, when viewed as an element in $A^n$, $\boldsymbol{e}_1:=(\mathbf{1},0,...,0)$. Finally, for $f\in C(X,\mathbb{K})$,

$$\|f\|_\infty = \|f\|_X = \sup\big\{\big|f(x)\big| : x\in X\big\}.$$

Let us begin with the pertinent definitions.

*Definition* 1.1.

- An $n$-tuple $(f_1,...,f_n)\in A^n$ is said to be *invertible* (or *unimodular*), if there exists $(x_1,...,x_n)\in A^n$ such that the Bézout equation $\sum_{j=1}^n x_j f_j=\mathbf{1}$ is satisfied. The set of all invertible $n$-tuples is denoted by $U_n(A)$. Note that $U_1(A)=A^{-1}$.
- An $(n+1)$-tuple $(f_1,...,f_n,g)\in U_{n+1}(A)$ is called *reducible* (in $A$) if there exists $(a_1,...,a_n)\in A^n$ such that $(f_1+a_1g,...,f_n+a_ng)\in U_n(A)$.
- The *Bass stable rank* of $A$, denoted by $\operatorname{bsr} A$, is the smallest integer $n$ such that every element in $U_{n+1}(A)$ is reducible. If no such $n$ exists, then $\operatorname{bsr} A=\infty$.

The following two results due to Corach and Suárez are the key to the theory of stable ranks.

**Lemma 1.2.** ([4, p. 636] *and* [6, p. 608]) *Let $A$ be a commutative, unital Banach algebra over $\mathbb{K}$. Then, for $g\in A$, the set*

$$R_n(g):=\big\{\boldsymbol{f}\in A^n : (\boldsymbol{f},g) \textit{ is reducible}\big\}$$

*is open-closed inside the open set*

$$I_n(g):=\big\{\boldsymbol{f}\in A^n : (\boldsymbol{f},g)\in U_{n+1}(A)\big\}.$$

*In particular, if* $\phi:[0,1]\to I_n(g)$ *is a continuous map and* $(\phi(0),g)$ *is reducible, then* $(\phi(1),g)$ *is reducible. Moreover,* $R_n(g)=gA^n+U_n(A)$.

The next assertion, which gives us a relation between reducibility in a Banach algebra $A$ and the associated uniform algebra $C(M(A))$ of all continuous complex-valued functions on the spectrum $M(A)$ of $A$, actually is one of the most important theorems in the theory of the Bass stable rank:

**Theorem 1.3.** (Corach-Suárez) ([5, p. 4]) *Let $A$ be a commutative unital complex Banach algebra and suppose that $(f_1,...,f_n,g)$ is an invertible $(n+1)$-tuple in $A$. Then $(f_1,...,f_n,g)$ is reducible in $A$ if and only if $(\hat{f}_1,...,\hat{f}_n,\hat{g})$ is reducible in $C(M(A))$.*

Here is now the scheme of the paper. In Section two we have a look at the principal components of $\mathcal{M}_n(A)$ and $U_n(A)$ and in Section three we give a connection between reducibility and the extension of invertible rows to invertible matrices in the principal component of $\mathcal{M}_n(A)$.

In the forth section of our paper we are concerned with the analogues of the results quoted above for our new notion of "reducibility of $(n+1)$-tuples in $A$ to the principal component of $U_n(A)$" (see below for the definition). In the fifth section we apply these results and give geometric/topological conditions under which $(n+1)$-tuples in $C(X,\mathbb{K})$ for $X\subseteq\mathbb{K}^n$ are reducible, respectively reducible to the principal component of $U_n(C(X,\mathbb{K}))$. Let us point out that due to Vasershtein's work, the Bass stable rank of $C(X,\mathbb{K})$ is less than or equal to $n+1$; hence every invertible $(n+2)$-tuple in $C(X,\mathbb{K})$ is reducible, but in general, not every tuple having length less than $n+1$ is reducible. In the sixth section we apply our results to the class of Euclidean Banach algebras. In Section 8 we give a simple proof of a result by V. Ya. Lin telling us that a left-invertible matrix $L$ over $A$ can be complemented to an invertible matrix over $A$ if and only if the matrix $\hat{L}$ of its Gelfand transforms can be complemented in the algebra $C(M(A))$.

## 2. The principal components of $\mathcal{M}_n(A)$ and $U_n(A)$

In this section we expose for the reader's convenience several results necessary to develop our theory, and whose proofs we could not locate in the literature (in particular for the case of real algebras). First, let us recall that if $A=(A,\|\cdot\|)$ is a commutative unital Banach algebra over $\mathbb{K}$, then the principal component $Exp\mathcal{M}_n(A)$ (= the connected component of the identity matrix $I_n$) of the group of invertible $n\times n$-matrices over $A$ is given by

$$Exp\mathcal{M}_n(A)=\{e^{M_1}...e^{M_k}:M_j\in\mathcal{M}_n(A)\}$$

(see [19, p. 201]).

Our description of the connected components of the set $U_n(A)$, viewed as a topological subspace of $A^n$, is based on the following classical result giving a relation between two invertible tuples that are close to each other. An elementary proof of that result (excepted the addendum) is given in [26].

**Theorem 2.1.** *Let $A=(A,\|\cdot\|)$ be a commutative unital Banach algebra over $\mathbb{K}$. Suppose that $\mathbf{f}=(f_1,...,f_n)$ is an invertible $n$-tuple in $A$. Then there exists $\varepsilon>0$ such that the following is true:*

*For each $\mathbf{g}=(g_1,...,g_n)\in A^n$ satisfying $\sum_{j=1}^n\|g_j-f_j\|<\varepsilon$ there is a matrix $H\in\mathcal{M}_n(A)$ such that*

$$\begin{pmatrix}g_1\\ \vdots\\ g_n\end{pmatrix}=(\exp H)\begin{pmatrix}f_1\\ \vdots\\ f_n\end{pmatrix}.$$

*In particular, $\mathbf{g}$ itself is an invertible $n$-tuple. Moreover, if $\boldsymbol{u}\cdot\boldsymbol{f}^t=\mathbf{1}$, then*

$$\boldsymbol{u}e^{-H}\cdot\boldsymbol{g}^t=\mathbf{1}.$$

*Addendum: if $f_n=g_n$, then $H$ can be chosen so that its last row is the zero vector and*

$$e^H=\begin{pmatrix}e^M & *\\ \mathbf{0}_{n-1} & \mathbf{1}\end{pmatrix}$$

*for some matrix $M\in\mathcal{M}_{n-1}(A)$.*

**Theorem 2.2.** *Let $A$ be a commutative unital Banach algebra over $\mathbb{K}$. If $\mathbf{f}=(f_1,...,f_n)\in U_n(A)$, then the connected component, $C(\mathbf{f})$, of $\mathbf{f}$ in $U_n(A)$ equals the set*

$$\mathbf{f}\cdot Exp\mathcal{M}_n(A).$$

*In particular, $C(\mathbf{f})$, is path-connected.*

*Proof.* Let $\mathfrak{C}=\mathbf{f}\cdot Exp\mathcal{M}_n(A)$; that is

$$\mathfrak{C}=\Big\{\mathbf{f}\cdot\Big(\prod_{j=1}^k\exp M_j\Big):M_j\in\mathcal{M}_n(A),k\in\mathbb{N}\Big\}.$$

We first note that $\mathfrak{C}$ is path-connected. To see this, let

$$\boldsymbol{g}=\boldsymbol{f}\cdot\exp(M_1)...\exp(M_k),$$

for some $M_j\in\mathcal{M}_n(A)$. Then the map $\phi:[0,1]\to U_n(A)$, given by

$$\phi(t):=\boldsymbol{f}\cdot\exp(tM_1)...\exp(tM_k),$$

is a continuous path joining $\boldsymbol{f}$ to $\boldsymbol{g}$ within $U_n(A)$.

We claim that $\mathfrak{C}$ is open and closed in $U_n(A)$. In fact, let $\mathbf{g}\in\mathfrak{C}$. According to Theorem 2.1, there is $\varepsilon>0$ so that for every $\mathbf{h}\in A^n$ with

$$\sum_{j=1}^{n}\|g_j-h_j\|<\varepsilon,$$

there is matrix $M\in\mathcal{M}_n(A)$ such that $\mathbf{h}^t=(\exp M)\mathbf{g}^t$. That is $\mathbf{h}=\mathbf{g}\cdot\exp M^t$. Therefore $\mathbf{h}\in\mathfrak{C}$. Hence $\boldsymbol{g}$ is an interior point of $\mathfrak{C}$. Thus $\mathfrak{C}$ is open in $A^n$. Since $U_n(A)$ itself is open in $A^n$, we conclude that $\mathfrak{C}$ is open in $U_n(A)$.

To show that $\mathfrak{C}$ is (relatively) closed in $U_n(A)$, we take a sequence $(\mathbf{g}_k)$ in $\mathfrak{C}$ that converges (in the product topology of $A^n$) to $\mathbf{g}\in U_n(A)$. Applying Theorem 2.1 again, there is $\varepsilon>0$ so that for every $\mathbf{h}\in A^n$ with $\sum_{j=1}^n\|g_j-h_j\|<\varepsilon$, there is matrix $M\in\mathcal{M}_n(A)$, depending on $\mathbf{h}$, such that $\mathbf{h}=\mathbf{g}\cdot\exp M^t$. This holds in particular for $\mathbf{h}=\mathbf{g}_k$, whenever $k$ is large. Thus $\mathbf{g}=\mathbf{g}_k\cdot\exp M_k$ for some $M_k\in\mathcal{M}_n(A)$, from which we conclude that $\mathbf{g}\in\mathfrak{C}$. Hence $\mathfrak{C}$ is closed in $U_n(A)$.

Being open-closed and connected now implies that $\mathfrak{C}$ is the maximal connected set containing itself. Hence, with $\boldsymbol{f}\in\mathfrak{C}$, we deduce that $C(\boldsymbol{f})=\mathfrak{C}$. □

We are now able to define the main object of this paper:

*Definition* 2.3. Let $A$ be a commutative unital Banach algebra over $\mathbb{K}$. Then the *principal component of* $U_n(A)$ is the connected component of $\boldsymbol{e}_1$ in $U_n(A)$ and is given by the set

$$\mathcal{P}\big(U_n(A)\big):=\boldsymbol{e}_1\cdot \mathit{Exp}\mathcal{M}_n(A).$$

*Remark* 2.4. • If $n=1$, then $U_1(A)=A^{-1}$ and

$$\mathcal{P}\big(U_n(A)\big)=\exp A:=\big\{e^a: a\in A\big\}.$$

• Let us note that in the representation $\boldsymbol{e}_1\cdot \mathit{Exp}\mathcal{M}_n(A)$ of the principal component of $U_n(A)$ any other "canonical" element $\boldsymbol{e}_j$, with

$$\boldsymbol{e}_j:=(0,...,0,\underbrace{\mathbf{1}}_{j},0,...,0),$$

is admissible, too. In fact, for $i \neq j$,

$$\gamma_{i,j}(t) := (1-t)\boldsymbol{e}_i + t\boldsymbol{e}_j$$

is a path in $U_n(A)$ joining $\boldsymbol{e}_i$ with $\boldsymbol{e}_j$, because

$$\big\langle (1-t)\boldsymbol{e}_i + t\boldsymbol{e}_j, \boldsymbol{e}_i + \boldsymbol{e}_j \big\rangle = \mathbf{1}.$$

Hence $\boldsymbol{e}_i$ and $\boldsymbol{e}_j$ belong to the same connected component of $U_n(A)$.

## 3. Extension of invertible rows to the principal component

An interesting connection between reducibility and extension of rows to invertible matrices (resp. to matrices in the principal component) is given in the following theorem (see for example [21, p. 311] and [20, p. 1129]). The additional property of being extendable to finite products of exponential matrices (hence to the principal component of $\mathcal{M}_n(A)$), seems not to have been considered in the literature before (as far as we know).

**Theorem 3.1.** *Let $A$ be a commutative unital Banach algebra over $\mathbb{K}$ with unit element $\mathbf{1}$. Suppose that $\boldsymbol{u} := (f_1, ..., f_n, g) \in U_{n+1}(A)$ is reducible. Then there is an invertible matrix $W \in \mathcal{M}_{n+1}(A)$ with determinant $\mathbf{1}$ and which is a finite product of exponential matrices such that $\boldsymbol{u}W = \boldsymbol{e}_1$. In other words, $\boldsymbol{u} \in \mathcal{P}(U_{n+1}(A))$. Moreover, if $M = W^{-1}$, then $\boldsymbol{u}$ is the first row of $M$.*

*Proof.* Let $\boldsymbol{f} = (f_1, ..., f_n)$. Since $\boldsymbol{u} = (\boldsymbol{f}, g)$ is reducible, there exists an $n$-tuple $\boldsymbol{x} = (x_1, ..., x_n) \in A^n$ with $\boldsymbol{f} + g\boldsymbol{x} \in U_n(A)$. Hence there is $\boldsymbol{y} = (y_1, ..., y_n) \in A^n$ such that (1)

$$\boldsymbol{y} \cdot (\boldsymbol{f} + g\boldsymbol{x})^t = \mathbf{1} - g.$$

Consider the matrices

$$W_1 = \begin{bmatrix} \mathbf{1} & & & & y_1 \\ 0 & \mathbf{1} & & & y_2 \\ \vdots & & \ddots & & \vdots \\ 0 & \dots & \dots & \mathbf{1} & y_n \\ x_1 & x_2 & \dots & x_n & \boldsymbol{y} \cdot \boldsymbol{x}^t + \mathbf{1} \end{bmatrix}$$

(1) Note that we do want the element $\mathbf{1} - g$ here on the right-hand side.

$$W_2 = \begin{bmatrix} \mathbf{1} & & & & \\ & \mathbf{1} & & & \\ & & \ddots & & \\ & & & & \\ -(f_1+gx_1) & \dots & -(f_n+gx_n) & \mathbf{1} \end{bmatrix}.$$

Since

$$W_1 = \begin{bmatrix} \mathbf{1} & 0 & \dots & \dots & 0 \\ 0 & \mathbf{1} & & & \vdots \\ \vdots & & \ddots & & \vdots \\ 0 & \dots & \dots & \mathbf{1} & 0 \\ x_1 & x_2 & \dots & x_n & \mathbf{1} \end{bmatrix} \cdot \begin{bmatrix} \mathbf{1} & 0 & \dots & & y_1 \\ 0 & \ddots & & & y_2 \\ \vdots & & \ddots & & \vdots \\ & & & \mathbf{1} & y_n \\ 0 & \dots & \dots & 0 & \mathbf{1} \end{bmatrix} =: M_1 M_2,$$

it is easy to see that $W_1$ and $W_2$ are invertible matrices in $\mathcal{M}_{n+1}(A)$ with determinant $\mathbf{1}$ satisfying ([2])

$$(f_1, \dots, f_n, g) W_1 W_2 = (0, \dots, 0, \mathbf{1}) \in A^{n+1}.$$

Let $W_3 = [\boldsymbol{e}_{n+1}^t | (-1)^n \boldsymbol{e}_1^t | \boldsymbol{e}_2^t | \dots | \boldsymbol{e}_n^t] \in \mathcal{M}_{n+1}(A)$; that is (when identifying $\mathbb{R}\cdot\mathbf{1}$ with $\mathbb{R}$),

$$W_3 = \begin{bmatrix} 0 & (-1)^n & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 \\ \vdots & \vdots & & \ddots & & \\ \vdots & \vdots & & & \ddots & \\ 0 & 0 & & & & 1 \\ 1 & 0 & \dots & \dots & \dots & 0 \end{bmatrix}.$$

Note that $\boldsymbol{e}_{n+1} W_3 = \boldsymbol{e}_1$ and $\det W_3 = 1$. If we put $W = W_1 W_2 W_3$, then $W$ is invertible in $\mathcal{M}_{n+1}(A)$, $\det W = \mathbf{1}$, and $\boldsymbol{u} W = \boldsymbol{e}_1$, where $\boldsymbol{e}_1 \in A^{n+1}$. Write $W^{-1} = \begin{bmatrix} \boldsymbol{w} \\ V \end{bmatrix}$, where $\boldsymbol{w}$ is the first row. It is easy to see that $\boldsymbol{u} = \boldsymbol{w}$. Thus

$$\begin{bmatrix} \boldsymbol{u} \\ V \end{bmatrix} W = I_{n+1}.$$

Hence the row $\boldsymbol{u}$ has been extended by $V$ to an invertible matrix $M := W^{-1}$.

([2]) The matrix multiplication here is preferably done from the left to the right: first multiply the one-row matrix with $W_1$, then go on.

Note that $M_1$ and $M_2$ in the decomposition $W_1 = M_1M_2$, as well as $W_2$, have the form $I_{n+1}+N$, where $N$ is a nilpotent matrix. Hence, $I_{n+1}+N=e^B$ for some $B\in\mathcal{M}_{n+1}(A)$ (just use an appropriate finite section of the power series expansion of the real logarithm $\log(1+x)$). Moreover, $W_3\in\mathcal{M}_{n+1}(\mathbb{R})$ and $\det W_3>0$. Thus, $W_3$ is a product of exponential matrices over $\mathbb{R}$ (3). Consequently, $W=W_1W_2W_3$ is a finite product of exponential matrices over $A$. $\square$

## 4. Reducibility to the principal component

*Definition* 4.1. Let $A$ be a commutative unital Banach algebra over $\mathbb{K}=\mathbb{R}$ or $\mathbb{K}=\mathbb{C}$. An invertible pair $(f,g)\in U_2(A)$ is said to be *reducible to the principal component* $\exp A$ *of* $A^{-1}$ if there exists $u,v\in A$ such that

$$f+ug=e^v.$$

It is clear that if $A^{-1}=U_1(A)$ is connected, then the notions of "reducibility of pairs" and "reducibility of pairs to the principal component" coincide. Our favourite example is the disk algebra $A(\mathbb{D})$. If $U_1(A)$ is disconnected, as it is the case for the algebra $C(\mathbb{T},\mathbb{C})$ for example, then for every $f\in A^{-1}\setminus\exp A$, the pair $(f,0)$ is reducible, but not reducible to the principal component of $A^{-1}$. This notion seems to have appeared for the first time in Laroco's work [10] in connection with the stable rank of $H^\infty$. Criteria for various function algebras have been established by the second author of this note in [22], [23], [24] and [25].

Now we generalize this notion to tuples, a fact that never before has been considered. We propose two different settings. Here is the first one (the second one will be dealt with in Section 7).

*Definition* 4.2. Let $A$ be a commutative unital Banach algebra over $\mathbb{K}$. An invertible $(n+1)$-tuple $(\boldsymbol{f},g)\in U_{n+1}(A)$ is said to be *reducible to the principal component of* $U_n(A)$ if there exists $\boldsymbol{h}\in A^n$ such that

$$\boldsymbol{f}+g\boldsymbol{h}\in\mathcal{P}\big(U_n(A)\big).$$

The following Proposition is pretty clear in the case of complex Banach algebras, since every permutation matrix $P\in\mathcal{M}_n(\mathbb{C})$ has a complex logarithm in $\mathcal{M}_n(\mathbb{C})$ (see [16]). So what does matter here, is that we consider real algebras, too.

(3) Actually, two exponentials will suffice; see [16].

**Proposition 4.3.** *Let $A$ be a commutative unital Banach algebra over $\mathbb{K}$ and let $(\boldsymbol{f}, g) \in U_{n+1}(A)$ be an invertible $(n+1)$-tuple in $A$ which is reducible to the principal component $\mathcal{P}(U_n(A))$ of $U_n(A)$. Suppose that $\tilde{\boldsymbol{f}}$ is a permutation of $\boldsymbol{f}$. Then also the tuple $(\tilde{\boldsymbol{f}}, g)$ is reducible to the principal component of $U_n(A)$.*

*Proof.* Without loss of generality, $n \geq 2$.

**Case 1.** Let $\boldsymbol{f}=(f_1, ..., f_n)$, $\tilde{\boldsymbol{f}}=(f_n, f_2, ..., f_{n-1}, f_1)$ and

$$S = \begin{pmatrix} 0 & \dots & & & \mathbf{1} \\ & \mathbf{1} & & & \\ & & \ddots & & \\ & & & \mathbf{1} & \\ \mathbf{1} & & & \dots & 0 \end{pmatrix}.$$

Note that $S=S^{-1}$ and $\det S=-\mathbf{1}$. The action of $S$ in $A \mapsto AS$ is to interchange the first and last column. Let $W \in M_n(\mathbb{R})$ be given by

$$W = \begin{pmatrix} 0 & \dots & & & 1 \\ (-1)^{n-1} & 0 & & & \\ & 1 & 0 & & \\ & & \ddots & \ddots & \\ 0 & & & 1 & 0 \end{pmatrix}.$$

Then $\det W=1$ and $\boldsymbol{e}_n=\boldsymbol{e}_1 W$. In particular $W \in Exp\mathcal{M}_n(\mathbb{R})$. Now, by assumption, $\boldsymbol{f}+g\boldsymbol{x}=\boldsymbol{e}_1 M$ for some $M \in Exp\mathcal{M}_n(A)$. Hence with $\tilde{x}:=xS$,

$$\begin{aligned} \tilde{\boldsymbol{f}}+g\tilde{\boldsymbol{x}} &= (\boldsymbol{f}+g\boldsymbol{x})S = \boldsymbol{e}_1 MS \\ &= (\boldsymbol{e}_n S)(MS) = \boldsymbol{e}_n(SMS) \\ &= (\boldsymbol{e}_1 W)(SMS) = \boldsymbol{e}_1(WSMS) \\ &\in \boldsymbol{e}_1 \cdot \mathcal{M}_n(A) = \mathcal{P}\big(U_n(A)\big), \end{aligned}$$

where we have used that $M \in Exp\mathcal{M}_n(A)$ if and only if $S^{-1}MS \in Exp\mathcal{M}_n(A)$ for every invertible matrix $S$; just observe that

$$S^{-1}\Big(\prod_{j=1}^{k} e^{M_j}\Big)S = \prod_{j=1}^{k} \big(S^{-1}e^{M_j}S\big) = \prod_{j=1}^{k} e^{S^{-1}M_j S}.$$

**Case 2.** Let $\tilde{\boldsymbol{f}}$ be an arbitrary permutation of $\boldsymbol{f}$. Hence $\tilde{\boldsymbol{f}}=\boldsymbol{f}P$ for some permutation matrix $P$. If $\det P>0$ then, $P=e^{P_1}e^{P_2}$ for some matrices $M_j \in \mathcal{M}_n(\mathbb{R})$

(see for example [16]). If $\det P<0$, then we aditionally interchange via $S$ the first coordinate with the last one in $\tilde{\boldsymbol{f}}$. Let us call this new $n$-tuple $\boldsymbol{F}$. Then $\boldsymbol{F}=\boldsymbol{f}Q$ for some permutation matrix $Q$ with $\det Q>0$, and again $Q=e^{Q_1}e^{Q_2}$ for some $Q_j\in M_n(\mathbb{R})$. Now, by assumption, there exists $\boldsymbol{x}\in A^n$ such that

$$\boldsymbol{f}+g\boldsymbol{x}=\boldsymbol{e}_1e^{M_1}...e^{M_k}$$

for some matrices $M_j\in\mathcal{M}_n(A)$. Hence, by multiplying at the right with $Q$,

$$\boldsymbol{f}Q+g\boldsymbol{x}Q=\boldsymbol{e}_1e^{M_1}...e^{M_k}e^{Q_1}e^{Q_2}.$$

Thus $(\boldsymbol{F},g)$ is reducible to the principal component of $U_n(A)$. The first case now implies that the same holds for $(\tilde{\boldsymbol{f}},g)$. $\square$

A sufficient condition for reducibility to the principal component is given in the following technical result:

**Lemma 4.4.** *Let $A$ be a commutative unital Banach algebra over $\mathbb{K}$. Suppose that $(\boldsymbol{f},g)\in U_{n+1}(A)$ and $n\geq 2$. Then $(\boldsymbol{f},g)$ is reducible to the principal component of $U_n(A)$ if there exist two vectors $\boldsymbol{x}\in A^n$ and $\boldsymbol{v}\in U_n(A)$ such that $\boldsymbol{v}$ is reducible itself with respect to some of its coordinates* ([4]) *and*

$$(\boldsymbol{f}+\boldsymbol{x}g)\cdot\boldsymbol{v}^t=\mathbf{1}.$$

*Proof.* Suppose that $i_0\neq n$. Then we interchange the $i_0$-th coordinate with the $n$-th coordinate in the three vectors $\boldsymbol{f},\boldsymbol{x}$ and $\boldsymbol{v}$ appearing here. The new vectors $\tilde{\boldsymbol{f}}$, $\widetilde{\boldsymbol{x}}$ and $\tilde{\boldsymbol{v}}$ still satisfy the Bézout equation

$$(\tilde{\boldsymbol{f}}+\widetilde{\boldsymbol{x}}g)\cdot\tilde{\boldsymbol{v}}^t=\mathbf{1}.$$

Since $(\tilde{\boldsymbol{f}},g)$ is reducible to the principal component of $U_n(A)$ if and only if $(\boldsymbol{f},g)$ does (Proposition 4.3), we may assume, right at the beginning, that $\boldsymbol{v}$ is reducible with respect to its last coordinate.

By Theorem 3.1, the reducibility of the row vector $\boldsymbol{v}$ implies the existence of a finite product $P_1$ of exponential matrices over $A$ such that $\boldsymbol{v}^t$ is the first column of a matrix $P_1\in Exp\mathcal{M}_n(A)$. Hence (as matricial products)

$$(\boldsymbol{f}+\boldsymbol{x}g)P_1=(\boldsymbol{f}+\boldsymbol{x}g)\big(\boldsymbol{v}^t|***\big)=(\mathbf{1},x_2,...,x_n)$$

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([4]) This means that there exists $i_0$ and $a_j\in A$, $(j=1,...,n-1)$, such that for $\boldsymbol{v}=(v_1,...,v_{i_0},...,v_n)$, the vector $(v_1+a_1v_{i_0},...,v_{i_0-1}+a_{i_0-1}v_{i_0},v_{i_0+1}+a_{i_0+1}v_{i_0},...,v_n+a_nv_{i_0})$ belongs to $U_{n-1}(A)$.

for some $x_j \in A$. If we let

$$P_2 = \begin{pmatrix} \mathbf{1} & -x_2 & \dots & -x_n \\ & \mathbf{1} & & \\ & & \ddots & \\ & & & \mathbf{1} \end{pmatrix},$$

then $P_2 \in Exp\mathcal{M}_n(A)$ (because it has the form $I_n+N$, where $N$ is nilpotent), and

$$(\boldsymbol{f}+\boldsymbol{x}g)P_1P_2 = (\mathbf{1}, x_2, \dots, x_n)P_2 = \boldsymbol{e}_1.$$

Hence $\boldsymbol{f}+\boldsymbol{x}g \in \boldsymbol{e}_1 \cdot Exp\mathcal{M}_n(A) = \mathcal{P}(U_n(A))$. $\square$

In order to study the reducibility to the principal component, we introduce a certain equivalence relation on the set of $n$-tuples, reminiscent of that in [4]. Corach and Suárez considered diagonal matrices $M$ all of whose diagonal entries were invertible elements in $A$: $\boldsymbol{f} \underset{a}{\overset{CS}{\sim}} \boldsymbol{g} \iff \boldsymbol{f}-\boldsymbol{g}M \in aA^n$ for such a matrix $M$. The equivalence classes of that relation, though, do not seem to be compatible with the connected components of $I_n(a)$; openness for example fails.

**Theorem 4.5.** *Let $A$ be a commutative unital Banach algebra over $\mathbb{K}$, $a \in A$, and consider the open set*

$$I_n(a) := \{\boldsymbol{f} \in A^n : (\boldsymbol{f}, a) \in U_{n+1}(A)\}.$$

*Given $\boldsymbol{f}, \boldsymbol{g} \in A^n$, define the relation*

$$\boldsymbol{f} \underset{a}{\overset{exp}{\sim}} \boldsymbol{g} \quad \Longleftrightarrow \quad \exists \boldsymbol{x} \in A^n, \exists B_1, \dots, B_k \in \mathcal{M}_n(A) : \boldsymbol{f} + a\boldsymbol{x} = \boldsymbol{g} e^{B_1} \dots e^{B_k}.$$

*Then*

(1) $\underset{a}{\overset{exp}{\sim}}$ *is an equivalence relation on $A^n$.*

(2) *If $\boldsymbol{f} \in I_n(a)$, then $[\boldsymbol{f}] \subseteq I_n(a)$, where*

$$[\boldsymbol{f}] := \{\boldsymbol{h} \in A^n : \boldsymbol{h} \underset{a}{\overset{exp}{\sim}} \boldsymbol{f}\}$$

*is the equivalence class associated with $\boldsymbol{f}$.*

(3) *If $\boldsymbol{f} \in I_n(a)$, then*

(3i) $[\boldsymbol{f}]$ *is open in $A^n$,*

(3ii) $[\boldsymbol{f}]$ *is a closed-open subset of $I_n(a)$,*

(3iii) $[\boldsymbol{f}]$ *is a (path)-connected set within $I_n(a)$.*

(4) *The connected components of $I_n(a)$ are the equivalence classes $[\boldsymbol{f}]$, where $\boldsymbol{f} \in I_n(A)$.*

*Proof.* (1) • $\overset{exp}{\underset{a}{\sim}}$ is reflexive: just take $\boldsymbol{x}=\boldsymbol{0}$ and $B_j=O$.

• $\overset{exp}{\underset{a}{\sim}}$ is symmetric: if $\boldsymbol{f}+a\boldsymbol{x}=\boldsymbol{g}e^{B_1}...e^{B_k}$, then

$$\boldsymbol{g}-a\big(\boldsymbol{x}e^{-B_k}...e^{-B_1}\big)=\boldsymbol{f}e^{-B_k}...e^{-B_1}.$$

• $\overset{exp}{\underset{a}{\sim}}$ is transitive (here we use that in the definition of the relation $\overset{exp}{\underset{a}{\sim}}$ products of exponential matrices appear; a single exponential matrix would not be sufficient): let $\boldsymbol{f}_1 \overset{exp}{\underset{a}{\sim}} \boldsymbol{f}_2$ and $\boldsymbol{f}_2 \overset{exp}{\underset{a}{\sim}} \boldsymbol{f}_3$, then there exist $\boldsymbol{x}_j\in A^n$ and $E_j\in Exp\mathcal{M}_n(A)$ such that

$$\boldsymbol{f}_1+a\boldsymbol{x}_1=\boldsymbol{f}_2E_1=(\boldsymbol{f}_3E_2-a\boldsymbol{x}_2)E_1.$$

Then

$$\boldsymbol{f}_1+a(\boldsymbol{x}_1+\boldsymbol{x}_2E_1)=\boldsymbol{f}_3E_2E_1.$$

Hence $\boldsymbol{f}_1 \overset{exp}{\underset{a}{\sim}} \boldsymbol{f}_3$.

(2) Let $\boldsymbol{f}\in I_n(a)$. Then there is $\boldsymbol{x}\in A^n$ such that $\boldsymbol{f}+a\boldsymbol{x}\in U_n(A)$. Now if $\tilde{\boldsymbol{f}}\in[\boldsymbol{f}]$ then,

$$\tilde{\boldsymbol{f}}+a\tilde{\boldsymbol{x}}=\boldsymbol{f}E$$

for some $E\in Exp\mathcal{M}_n(A)$. Hence

$$\tilde{\boldsymbol{f}}+a(\tilde{\boldsymbol{x}}+\boldsymbol{x}E)=(\boldsymbol{f}+a\boldsymbol{x})E\in U_n(A),$$

from which we conclude that $(\tilde{\boldsymbol{f}},a)\in U_{n+1}(A)$. In other words, $\tilde{\boldsymbol{f}}\in I_n(a)$. Thus $[\boldsymbol{f}]\subseteq I_n(a)$.

(3i) To show the openness of $[\boldsymbol{f}]$ whenever $\boldsymbol{f}\in I_n(a)$, let $\boldsymbol{h}\in[\boldsymbol{f}]$. By (2), $(\boldsymbol{h},a)\in U_{n+1}(A)$. We claim that there is $\varepsilon>0$ such that every $\boldsymbol{h}'\in A^n$ with $\|\boldsymbol{h}'-\boldsymbol{h}\|<\varepsilon$ is equivalent to $\boldsymbol{f}$. To see this, choose according to Theorem 2.1, $\varepsilon>0$ so small that $(\boldsymbol{h}',a)=(\boldsymbol{h},a)e^H$ for some $H\in\mathcal{M}_{n+1}(A)$. In particular $\boldsymbol{h}'\in I_n(a)$. By that same Theorem, $H$ may be chosen so that the last column of $H$ is zero and that

$$e^H=\begin{pmatrix} e^K & \boldsymbol{0}_n^t \\ \boldsymbol{x} & \boldsymbol{1}\end{pmatrix}$$

for some $K\in\mathcal{M}_n(A)$ and $\boldsymbol{x}\in A^n$. Since

$$\big(\boldsymbol{h}',a\big)=(\boldsymbol{h},a)\begin{pmatrix} e^K & \boldsymbol{0}_n^t \\ \boldsymbol{x} & \boldsymbol{1}\end{pmatrix},$$

we conclude that

$$\boldsymbol{h}'=\boldsymbol{h}e^K+a\boldsymbol{x}.$$

In other words, $\boldsymbol{h}'\in[\boldsymbol{h}]=[\boldsymbol{f}]$. Hence $[\boldsymbol{f}]$ is open in $A^n$.

(3ii) Let $(\boldsymbol{h}_j)$ be a sequence in $[\boldsymbol{f}]\subseteq I_n(a)$ converging to some $\boldsymbol{h}'\in I_n(a)$. As in the previous paragraph, if $n$ is sufficiently large, we may conclude that $\boldsymbol{h}'\in[\boldsymbol{h}_j]=[\boldsymbol{f}]$ for $j\geq j_0$. Hence $[\boldsymbol{f}]$ is (relatively) closed in $I_n(a)$. Furthermore, since $[\boldsymbol{f}]\subseteq I_n(a)$, we deduce from (3i) that $[\boldsymbol{f}]$ is also open in $I_n(a)$.

(3iii) Let $\tilde{\boldsymbol{f}}\in[\boldsymbol{f}]$; say $\tilde{\boldsymbol{f}}+a\boldsymbol{x}=\boldsymbol{f}e^{M_1}...e^{M_k}$ for some $\boldsymbol{x}\in A^n$ and $M_j\in\mathcal{M}_n(A)$. Then the map $H:[0,1]\to A^n$ given by

$$H(t)=\boldsymbol{f}e^{tM_1}...e^{tM_k}-ta\boldsymbol{x}$$

is a continuous path joining $\boldsymbol{f}$ with $\tilde{\boldsymbol{f}}$. By definition of $\overset{exp}{\underset{a}{\sim}}$, each $H(t)$ is equivalent to $\boldsymbol{f}$; that is $H(t)\in[\boldsymbol{f}]$. Thus $[\boldsymbol{f}]$ is path connected.

(4) This follows immediately from (3i)–(3iii). □

Here is the counterpart to Lemma 1.2.

**Theorem 4.6.** *Let $A$ be a commutative unital Banach algebra over $\mathbb{K}$. Then, for $g\in A$, the set*

$$\begin{aligned} R_n^{exp}(g) &:= \bigl\{\boldsymbol{f}\in A^n:(\boldsymbol{f},g) \text{ is reducible to the principal component of } U_n(A)\bigr\} \\ &= gA^n+\mathcal{P}\bigl(U_n(A)\bigr) \end{aligned}$$

*is open-closed inside $I_n(g)$. In particular, if $\boldsymbol{F}:[0,1]\to I_n(g)$ is a continuous map for which $(\boldsymbol{F}(0),g)$ is reducible to the principal component, then $(\boldsymbol{F}(1),g)$ is reducible to the principal component, too.*

*Proof.* We first note that, by definition, $R_n^{exp}(g)\subseteq I_n(g)$ and that the reducibility of $(\boldsymbol{f},g)\in U_{n+1}(A)$ to the principal component of $U_n(A)$ is equivalent to the assertion that $\boldsymbol{f}\overset{exp}{\underset{g}{\sim}}\boldsymbol{e}_1$. Thus

$$\boldsymbol{f}\in R_n^{exp}(g) \iff \boldsymbol{f}\in[\boldsymbol{e}_1] \iff [\boldsymbol{f}]=[\boldsymbol{e}_1]. \tag{4.1}$$

In other words, $R_n^{exp}(g)=[\boldsymbol{e}_1]$. The assertion then follows from Theorem 4.5.

Now if $\boldsymbol{F}:[0,1]\to I_n(g)$ is a curve in $I_n(g)$, then $C:=\boldsymbol{F}([0,1])$ is connected. Since $\boldsymbol{F}(0)\in R_n^{exp}(g)$, we deduce that $C\subseteq R_n^{exp}(g)$. □

The following corollaries are immediate (the second one is originally due to Corach and Suárez [4]).

**Corollary 4.7.** *Let $A$ be a commutative unital Banach algebra over $\mathbb{K}$ and $g\in A$. Then the following assertions are equivalent:*

(1) *Every invertible* $(n+1)$*-tuple* $(\boldsymbol{f},g)\in U_{n+1}(A)$ *is reducible to the principal component of* $U_n(A)$*; that is* $I_n(g)=R_n^{exp}(g)$.

(2) $I_n(g)$ *is connected.*

*Proof.* Just note that by equation (4.1), $R_n^{exp}(g)=[\boldsymbol{e}_1]$ and that $[\boldsymbol{e}_1]$ is a connected set which is contained in $I_n(g)$ for every $g\in A$. The result now follows from Theorem 4.5. $\square$

**Corollary 4.8.** *Let $A$ be a commutative unital Banach algebra over $\mathbb{K}$ and $g\in A$. Then the following assertions are equivalent:*

(1) $I_n(g)=R_n(g)$;

(2) *Each component of* $I_n(g)$ *meets* $U_n(A)$.

*Proof.* Since the connected components of $I_n(g)$ are the equivalence classes $[\boldsymbol{f}]$ for $\overset{exp}{\underset{g}{\sim}}$ with $\boldsymbol{f}\in I_n(g)$ (Theorem 4.5), we have the following equivalent assertions for a given $\boldsymbol{f}\in I_n(g)$:

(i) $[\boldsymbol{f}]\cap U_n(A)\neq\varnothing$,

(ii) there exists $\boldsymbol{u}\in U_n(A)$ such that $\boldsymbol{u}\overset{exp}{\underset{g}{\sim}}\boldsymbol{f}$,

(iii) there exists $\boldsymbol{u}\in U_n(A)$, $\boldsymbol{x}\in A^n$, and $B_1,\dots,B_k\in\mathcal{M}_n(A)$ such that

$$\boldsymbol{f}+g\boldsymbol{x}=\boldsymbol{u}e^{B_1}\dots e^{B_k},$$

(iv) $(\boldsymbol{f},g)$ is reducible. $\square$

Note that we actually proved a stronger result than stated, because the assertions (i)–(iv) are valid for each individual $\boldsymbol{f}$.

The following two Lemmas are very useful to check examples upon reducibility. They roughly say that the reducibility of $(\boldsymbol{f},g)$ depends only on the behaviour of the Gelfand transforms of the coordinates $f_j$ of $\boldsymbol{f}$ on the zero set of $\hat{g}$. We use the following notation: $\hat{\boldsymbol{f}}:=(\hat{f}_1,\dots,\hat{f}_n)$.

**Lemma 4.9.** *Let $A$ be a commutative unital Banach algebra over $\mathbb{C}$. Suppose that $(\boldsymbol{f},g)\in U_{n+1}(A)$. Let $E:=Z(\hat{g})$. If for some $\boldsymbol{u}\in U_n(A)$ and matrices $M_j\in\mathcal{M}_n(A)$*

$$\sup_{x\in E}\left|\hat{\boldsymbol{f}}(x)-\widehat{\boldsymbol{u}}(x)\exp\widehat{M_1}(x)\dots\exp\widehat{M_m}(x)\right|<\varepsilon,$$

*where $\varepsilon$ is sufficiently small, then $(\boldsymbol{f},g)$ is reducible in $A$.*

*Proof.* Let $\delta:=\min\{|\hat{\boldsymbol{f}}(x)|: x\in E\}$. Since $(\boldsymbol{f},g)\in U_n(A)$, we have $\delta>0$. Fix $\varepsilon\in\,]0,\delta/2]$, and let $\boldsymbol{b}:=\boldsymbol{u}\exp M_1 ... \exp M_m\in U_n(A)$ be chosen so that

$$\sup_{x\in E}\big|\hat{\boldsymbol{f}}(x)-\hat{\boldsymbol{b}}(x)\big|<\varepsilon.$$

Consider the path $\psi:[0,1]\to A^n$ given by

$$\psi(t)=(1-t)\boldsymbol{f}+t\boldsymbol{b}.$$

On $E$ we then have the following estimates:

$$\begin{aligned}\big|(1-t)\hat{\boldsymbol{f}}+t\hat{\boldsymbol{b}}\big| &= \big|t(\hat{\boldsymbol{b}}-\hat{\boldsymbol{f}})+\hat{\boldsymbol{f}}\big|\\ &\geq |\hat{\boldsymbol{f}}|-t|\hat{\boldsymbol{b}}-\hat{\boldsymbol{f}}|\\ &\geq \delta-\delta/2=\delta/2.\end{aligned}$$

Hence the tuples $(\psi(t),g)$ are invertible in $A$ for every $t$. Since for $t=1$, $\psi(1)=\boldsymbol{b}\in U_n(A)$, the tuple $(\psi(1),g)$ is reducible in $A$. By Lemma 1.2, $(\psi(0),g)$ then is reducible which in turn implies the reducibility of $(\boldsymbol{f},g)$. $\square$

**Lemma 4.10.** *Let $A$ be a commutative unital Banach algebra over $\mathbb{C}$. Suppose that $(\boldsymbol{f},g)\in U_{n+1}(A)$. Let $E:=Z(\hat{g})$. If for some matrices $M_j\in\mathcal{M}_n(A)$*

$$\sup_{x\in E}\big|\hat{\boldsymbol{f}}(x)-\boldsymbol{e}_1\cdot\exp\widehat{M}_1(x) ... \exp\widehat{M}_m(x)\big|<\varepsilon,$$

*where $\varepsilon$ is sufficiently small, then $(\boldsymbol{f},g)$ is reducible to the principal component $\mathcal{P}(U_n(A))$ of $U_n(A)$.*

*Proof.* Consider the path $\psi:[0,1]\to A^n$ given by

$$\psi(t)=(1-t)\boldsymbol{f}+t\boldsymbol{b},$$

where $\boldsymbol{b}:=\boldsymbol{e}_1\cdot\exp M_1 ... \exp M_m$. If

$$0<\varepsilon<(1/2)\min\big\{\big|\hat{\boldsymbol{f}}(x)\big|: x\in Z(\hat{g})\big\},$$

then $(\psi(t),g)\in U_{n+1}(A)$ for every $t\in[0,1]$. Now $(\psi(1),g)=(\boldsymbol{b},g)$ is reducible to the principal component of $U_n(A)$ since

$$\boldsymbol{b}+0\cdot g=\boldsymbol{e}_1\cdot\exp M_1 ... \exp M_m\in\mathcal{P}\big(U_n(A)\big).$$

Hence, by Theorem 4.6, $(\psi(0),g)=(\boldsymbol{f},g)$ is reducible to the principal component of $U_n(A)$, too. $\square$

We close this section with our main theorem, which is the analogue to the Corach-Suárez result Theorem 1.3 ([5]). It is based on the Arens-Novodvorski-Taylor theorem ([1], [18] and [27]), a version of which we recall here.

**Theorem 4.11.** (Arens-Novodvorski-Taylor) *Let $A$ be a commutative unital complex Banach algebra, $X:=M(A)$ its spectrum and $\mathcal{M}_n(A)$ the Banach algebra of $n\times n$ matrices over $A$.*

(1i) *Suppose that for some $M\in\mathcal{M}_n(A)$ there is $\underline{M}\in\mathcal{M}_n(C(X))$ such that $\widehat{M}=\exp\underline{M}$* (5)*. Then $M=\exp L_1\dots\exp L_m$ for some $L_j\in\mathcal{M}_n(A)$.*

(1ii) *Let $M\in\mathcal{M}_n(A)^{-1}$. If $\widehat{M}$ belongs to the principal component of $\mathcal{M}_n(C(X))^{-1}$, then $M$ already belongs to the principal component of $\mathcal{M}_n(A)^{-1}$.*

(2) *Let $\boldsymbol{f}\in U_n(C(X))$. Then there exist $\boldsymbol{g}\in U_n(A)$ and $\underline{G}_1,\dots,\underline{G}_m\in\mathcal{M}_n(C(X))$ such that $\boldsymbol{f}=\hat{\boldsymbol{g}}\exp\underline{G}_1\dots\exp\underline{G}_m$.*

(3) *Let $\boldsymbol{u}$ and $\boldsymbol{v}$ be in $U_n(A)$. Suppose that there are matrices $\underline{G}_j\in\mathcal{M}_n(C(X))$ such that $\widehat{\boldsymbol{u}}=\hat{\boldsymbol{v}}\exp\underline{G}_1\dots\exp\underline{G}_m$. Then $\boldsymbol{u}$ and $\boldsymbol{v}$ belong to the same connected component of $U_n(A)$.*

(4) *The Gelfand transform induces a group isomorphism between the quotient groups*

$$\mathcal{M}_n(A)^{-1}/Exp\mathcal{M}_n(A) \text{ and } \mathcal{M}_n\big(C(X)\big)^{-1}/Exp\mathcal{M}_n\big(C(X)\big).$$

Item (2), in particular, says that every connected component of $U_n(C(X))$ contains an element of the form $\hat{\boldsymbol{f}}:=(\hat{f}_1,\dots,\hat{f}_n)$, where $\boldsymbol{f}\in U_n(A)$. Moreover, (3) is equivalent to the assertion that if $\widehat{\boldsymbol{u}}$ and $\hat{\boldsymbol{v}}$ can be joined by a path in $U_n(C(X))$, then $\boldsymbol{u}$ and $\boldsymbol{v}$ can be joined by a path in $U_n(A)$. Item (4) also says that every element in $\mathcal{M}_n(C(X))^{-1}$ is homotopic in $\mathcal{M}_n(C(X))^{-1}$ to $\widehat{M}$ for some $M\in\mathcal{M}_n(A)^{-1}$.

**Theorem 4.12.** *Let $A$ be a commutative unital Banach algebra over $\mathbb{C}$. Given an invertible tuple $(\boldsymbol{f},g)\in U_{n+1}(A)$, the following assertions are equivalent:*

(1) *$(\boldsymbol{f},g)$ is reducible to the principal component of $U_n(A)$;*

(2) *$\hat{\boldsymbol{f}}|_{Z(\hat{g})}$ belongs to the principal component of $U_n(C(Z(\hat{g})))$.*

(3) *$(\hat{\boldsymbol{f}},\hat{g})$ is reducible to the principal component of $U_n(C(M(A)))$.*

*Proof.* (1) $\Longrightarrow$ (2) Let $E:=Z(\hat{g})$. By assumption there is $\boldsymbol{h}\in U_n(A)$ such that

$$\boldsymbol{u}:=\boldsymbol{f}+g\boldsymbol{h}\in\mathcal{P}\big(U_n(A)\big).$$

That is, there are matrices $M_j\in\mathcal{M}_n(A)$ such that

$$\boldsymbol{u}=\boldsymbol{e}_1\cdot\exp M_1\dots\exp M_m.$$

If we apply the Gelfand transform and restrict to $Z(\hat{g})$, then

$$\hat{\boldsymbol{f}}|_E=\boldsymbol{e}_1\cdot(\exp\widehat{M}_1\dots\exp\widehat{M}_m)|_E.$$

Hence $\hat{\boldsymbol{f}}|_E\in\mathcal{P}(U_n(C(E)))$.

(5) Here $\widehat{M}$ is the matrix whose entries are the Gelfand transforms of the entries of the matrix $M$.

(2) $\Longrightarrow$ (1) By assumption, there exist matrices $C_j \in \mathcal{M}_n(C(E))$ such that

$$\hat{\boldsymbol{f}}|_E = \boldsymbol{e}_1 \cdot \exp C_1 \ldots \exp C_k.$$

Let $A_E = \overline{\hat{A}|_E}^{\|\cdot\|_\infty}$ be the uniform closure of the restriction algebra $\hat{A}|_E$ in $C(E)$. Since $Z(g)$ is $A$-convex, $M(A_E)=E$ ([8]). Because $\hat{\boldsymbol{f}}|_E \in (A_E)^n$ belongs to the principal component $\mathcal{P}(U_n(C(E)))$ of $U_n(C(E))$, which is "generated" by $\boldsymbol{e}_1$, we conclude from the Arens-Novodvorski-Taylor Theorem 4.11(3) that $\hat{\boldsymbol{f}}|_E$ belongs to the same component of $U_n(A_E)$ as $\boldsymbol{e}_1$; namely the principal component $\mathcal{P}(U_n(A_E))$ of $U_n(A_E)$. Hence there are matrices $B_j \in \mathcal{M}_n(A_E)$ such that

$$\hat{\boldsymbol{f}}|_E = \boldsymbol{e}_1 \cdot \exp B_1 \ldots \exp B_m.$$

Now, we uniformly approximate on $E$ the matrices $B_j$ by matrices $\widehat{M_j}$ with $M_j \in \mathcal{M}_n(A)$; say

$$\sup_{x\in E} \left| \hat{\boldsymbol{f}}(x) - \boldsymbol{e}_1 \cdot \exp \widehat{M_1}(x) \ldots \exp \widehat{M_m}(x) \right| < \varepsilon$$

By Lemma 4.10, $(\boldsymbol{f}, g)$ is reducible to the principal component $\mathcal{P}(U_n(A))$ of $U_n(A)$.

(2) $\Longrightarrow$ (3) follows from Lemma 4.10 and (3) $\Longrightarrow$ (2) is clear. □

If $n=2$, then the previous result reads as follows:

**Corollary 4.13.** *Let $A$ be a commutative unital Banach algebra over $\mathbb{C}$. Given an invertible pair $(f,g) \in U_2(A)$, the following assertions are equivalent*:

(1) *There exist $a, h \in A$ such that $f + ag = e^h$;*

(2) *$\hat{f}|_{Z(\hat{g})} = e^v$ for some $v \in C(Z(\hat{g}))$.*

## 5. Reducibility in $C(X, \mathbb{K})$ with $X \subseteq \mathbb{K}^n$

In this section we study the reducibility in $C(X,\mathbb{K})$ with $X \subseteq \mathbb{K}^n$ in detail.

*Definition* 5.1.

(a) Let $K \subseteq \mathbb{R}^n$ be compact. A bounded connected component of $\mathbb{R}^n \setminus K$ is called a *hole* of $K$.

(b) Let $K, L$ be two compact sets in $\mathbb{R}^n$ with $K \subseteq L$. The pair $(K, L)$ is said to satisfy the *hole condition* if every hole of $K$ contains a hole of $L$.

The following concepts were introduced in $\mathbb{C}$ by the second author in [22].

*Definition* 5.2. Let $K\subseteq\mathbb{R}^n$ be compact and $g\in C(K,\mathbb{K})$. Then $g$ is said to satisfy the *boundary principle* if for every nonvoid open set $G$ in $\mathbb{R}^n$ with $G\subseteq K$ the following condition holds:

$$(B_1)\quad \text{If } g\equiv 0 \text{ on } \partial G, \text{ then } g\equiv 0 \text{ on } G.$$

**Proposition 5.3.** *Condition* $(B_1)$ *is equivalent to the following assertion*:

$(B_2)$ *If $G$ is an open set in $\mathbb{R}^n$ such that $G\subseteq K\setminus Z(g)$, then there exists*

$$x_0\in\partial G \textit{ such that } g(x_0)\neq 0.$$

*Proof.* Assume that $g$ satisfies $(B_1)$ and let $G\subseteq K\setminus Z(g)$ be open. Then $g$ cannot vanish identically on $\partial G$ since otherwise $(B_1)$ would imply that $g\equiv 0$ on $G$. A contradiction to the assumption that $G\cap Z(g)=\varnothing$. Hence $g$ satisfies $(B_2)$.

Conversely, let $g$ satisfy $(B_2)$. Suppose, to the contrary, that $g$ does not satisfy condition $(B_1)$. Then there is an open set $G$ in $\mathbb{R}^n$ with $G\subseteq K$, $g\equiv 0$ on $\partial G$, but such that $g$ does not vanish identically on $G$. Hence

$$U:=\{x\in G : g(x)\neq 0\}$$

is an open, nonvoid set in $\mathbb{R}^n$ which is contained in $K\setminus Z(g)$. But $\partial U\subseteq Z(g)$, because

$$\partial U=\overline{U}\setminus U\subseteq\overline{G}\setminus U\subseteq(G\setminus U)\cup\partial G\subseteq Z(g).$$

This contradicts condition $(B_2)$ for $U$. Hence such a set $G$ cannot exist and we deduce that $g$ has property $(B_1)$. $\square$

The following result gives an interesting connection between the hole condition (Definition 5.1) and the boundary principle (Definition 5.2).

**Theorem 5.4.** *Let $K\subseteq\mathbb{R}^n$ be compact and $g\in C(K,\mathbb{K})$. The following assertions are equivalent*:

(1) *$g$ satisfies the boundary principle* $(B_1)$.

(2) *$(Z(g),K)$ satisfies the hole condition*; *that is*, *every hole of $Z(g)$ contains a hole of $K$* ([6]).

*Proof.* (1) $\Longrightarrow$ (2) Suppose that there is a component $\Omega$ of $\mathbb{R}^n\setminus Z(g)$ with $\Omega\subseteq K$. Then $\Omega$ is open in $\mathbb{R}^n$ and $\partial\Omega\subseteq Z(g)$. Condition $(B_1)$ now implies that $g\equiv 0$ on $\Omega$ (note that by assumption $\Omega\subseteq K$). Hence $\Omega\subseteq Z(g)$. A contradiction.

([6]) Or which is the same, no hole of $Z(g)$ is entirely contained in $K$.

(2) $\Longrightarrow$ (1) We show that the equivalent condition $(B_2)$ is satisfied. So let $G$ be open in $\mathbb{R}^n$ and assume that $G \subseteq K \setminus Z(g)$. Let $\Omega$ be a component of $G$. Then $\Omega$ is bounded and open in $\mathbb{R}^n$. In view of achieving a contradiction, suppose that $g \equiv 0$ on $\partial\Omega$. Then $\partial\Omega \subseteq Z(g)$. Since $\Omega \cap Z(g) = \varnothing$, we deduce that $\Omega$ belongs to a connected component $C$ of $\mathbb{R}^n \setminus Z(g)$. If $\Omega$ is a proper subset of $C$, then a path connecting $z_0 \in \Omega$ and $w \in C \setminus \Omega$ within $C$ would pass through a boundary point $z_1$ of $\Omega$. But then $g(z_1) = 0$, contradicting $z_1 \in C$. Hence $\Omega = C$. Since $\Omega$ is bounded, we conclude that $\Omega$ is a hole of $Z(g)$. But $\Omega \subseteq G \subseteq K$; thus $(Z(g), K)$ cannot satisfy the hole condition (2). This is a contradiction. Hence there is $x_0 \in \partial\Omega$ such that $g(x_0) \neq 0$. Since $\partial\Omega \subseteq \partial G$ (note that $\Omega$ is supposed to be a component of $G$), we are done. $\square$

Let us emphasize that for $Z(g) \subseteq K \subseteq \mathbb{R}^n$ the pair $(Z(g), K)$ automatically satisfies the hole condition (and equivalently the boundary principle $(B_1)$) if $K^\circ = \varnothing$. An important class of zero-sets satisfying the equivalent conditions (1) and (2) is given in the following example:

*Example* 5.5. Let $K \subseteq \mathbb{R}^n$ be compact and $g \in C(K, \mathbb{K})$. Then $Z(g)$ has property $(B_1)$ if:

(i) $\mathbb{R}^n \setminus Z(g)$ is connected whenever $n \geq 2$;

(ii) $\mathbb{R} \setminus Z(g)$ has exactly two components whenever $n = 1$.

The next two theorems give nice geometric/topological conditions for reducibility of $n$-tuples, respectively for reducibility to the principal component of $U_n(C(X, \mathbb{K}))$. They generalize the corresponding facts for pairs developed by the second author in [22] for certain planar compacta.

**Theorem 5.6.** *Let $K \subseteq \mathbb{K}^n$ be compact and $g \in C(K, \mathbb{K})$. The following assertions are equivalent*:

(1) *$C(K, \mathbb{K})$ is $n$-stable at $g$, that is $(\boldsymbol{f}, g)$ is reducible for every $\boldsymbol{f} = (f_1, ..., f_n) \in C(K, \mathbb{K}^n)$ such that $(\boldsymbol{f}, g) \in U_{n+1}(C(K, \mathbb{K}))$.*

(2) *$\boldsymbol{f}|_{Z(g)}$ admits a zero-free extension to $K$ for all $\boldsymbol{f} \in C(K, \mathbb{K}^n)$ with $Z(\boldsymbol{f}) \cap Z(g) = \varnothing$.*

(3) *$g$ satisfies the boundary principle $(B_1)$.*

(4) *$(Z(g), K)$ satisfies the hole condition.*

*Proof.* (i) The equivalence of (1) with (2) is well-known (see [5] or [26]). The equivalence of (2) with (4) is [14, Theorem 5.6], provided we identify $(u_1 + iv_1, ..., u_n + iv_n)$ in the complex-valued case with the real-valued $(2n)$-tuple $(u_1, v_1, ..., u_n, v_n)$. The equivalence of (3) with (4) is Theorem 5.4. $\square$

**Theorem 5.7.** *Let $K \subseteq \mathbb{K}^n$ be compact and $g \in C(K,\mathbb{K})$. The following three assertions are equivalent:*

*(1) $(\boldsymbol{f}, g)$ is reducible to the principal component of $U_n(C(K,\mathbb{K}))$ for every $\boldsymbol{f} \in C(K,\mathbb{K}^n)$ such that $(\boldsymbol{f}, g) \in U_{n+1}(C(K,\mathbb{K}))$;*

*(2) there exist matrices $B_j \in \mathcal{M}_n(C(Z(g),\mathbb{K}))$ such that*

$$\boldsymbol{f}|_{Z(g)} = \boldsymbol{e}_1 \cdot e^{B_1} \dots e^{B_k}$$

*for every $\boldsymbol{f} \in C(K,\mathbb{K}^n)$ with $Z(\boldsymbol{f}) \cap Z(g) \neq \varnothing$;*

*(3) $Z(g)$ has no holes in $\mathbb{K}^n$.*

*Proof.* First we note that (1) and (2) are equivalent in view of Theorem 4.12. Since we have to deal here only with the special case $C(K,\mathbb{K})$, the following simple proof is available:

(1) $\Longrightarrow$ (2) If $\boldsymbol{f} + g\boldsymbol{h} \in \mathcal{P}(U_n(C(K,\mathbb{K})))$ for some $\boldsymbol{h} \in C(K,\mathbb{K}^n)$ then, by using the representation of the principal component,

$$\boldsymbol{f} + g\boldsymbol{h} = \boldsymbol{e}_1 \cdot e^{B_1} \dots e^{B_k}$$

for some matrices $B_j \in \mathcal{M}_n(C(K,\mathbb{K}))$. Restricting this identity to $Z(g)$ yields the assertion (2).

(2) $\Longrightarrow$ (1) This follows from Lemma 4.10.

(2) $\Longrightarrow$ (3) Suppose, to the contrary, that $Z(g)$ admits a bounded complementary component $G \subseteq \mathbb{K}^n$. Then $\partial G \subseteq Z(g)$. Let $\boldsymbol{a} \in G$ and $\boldsymbol{f}(\boldsymbol{z}) = \boldsymbol{z} - \boldsymbol{a}$, $\boldsymbol{z} \in Z(g)$. Since $(\boldsymbol{f}, g) \in U_{n+1}(C(Z(g),\mathbb{K}))$, there exist by hypothesis (2) matrices $B_j \in \mathcal{M}_n(C(Z(g),\mathbb{K}))$ such that

$$\boldsymbol{f}|_{Z(g)} = \boldsymbol{e}_1 \cdot e^{B_1} \dots e^{B_k}.$$

Extending via Tietze's result the matrices $B_j$ continuously to $\mathbb{K}^n$, would yield a zero-free extension of the $n$-tuple $(\boldsymbol{z} - \boldsymbol{a})|_{Z(g)}$ to $\overline{G}$. This contradicts a corollary to Brouwer's fixed point theorem (see [3, Chap. 4]).

(3) $\Longrightarrow$ (2) By a standard result in vector analysis (see for example [14, Corollary 5.8]), the connectedness of $\mathbb{K}^n \setminus Z(g)$ implies that the invertible tuple $\boldsymbol{f}|_{Z(g)} \in U_n(C(Z(g),\mathbb{K}))$ admits a zero-free extension $\boldsymbol{F}$ to $\mathbb{K}^n$. Let $\mathbf{B} \subseteq \mathbb{K}^n$ be a closed ball whose interior contains $X$. Note that $\mathbf{B}$ is a contractible Hausdorff space. Hence, the set $U_n(C(\mathbf{B},\mathbb{K}))$ is connected. Thus, $\boldsymbol{F}|_{\mathbf{B}} = \boldsymbol{e}_1 \cdot e^{B_1} \dots e^{B_k}$ for some matrices $B_j \in \mathcal{M}_n(C(\mathbf{B},\mathbb{K}))$. Restricting to $Z(g)$ yields the assertion (2). $\square$

## 6. Reducibility in Euclidean Banach algebras

Let us call a complex commutative unital Banach algebra $A$ a *Euclidean Banach algebra* if the spectrum $M(A)$ of $A$ is homeomorphic to a compact set in $\mathbb{C}^n$. This class of algebras includes every finitely generated Banach algebra over $\mathbb{C}$, for example the algebras

$$P(K)=\overline{\mathbb{C}[z_1,...,z_n]|_K}^{\|\cdot\|_K}$$

and certain algebras ([7]) of type

$$A(K)=\{f\in C(K,\mathbb{C}): f \text{ holomorphic in } K^\circ\},$$

$K\subseteq\mathbb{C}^n$ compact (for example $K=\overline{\mathbb{D}}^n$ or $K=\boldsymbol{B}_n$, the closed unit ball in $\mathbb{C}^n$). Using the Arens-Novodvorski-Taylor theorem we can now generalize Theorems 5.6 and 5.7 to Euclidean Banach algebras.

**Theorem 6.1.** *Let $A$ be a Euclidean Banach algebra with spectrum $X\subseteq\mathbb{C}^n$ and let $g\in A$. Then the assertions* (1)–(3), *respectively* (4)–(5), *are equivalent*:

(1) *$(\boldsymbol{f},g)$ is reducible for every $n$-tuple $\boldsymbol{f}\in A^n$ with $(\boldsymbol{f},g)\in U_{n+1}(A)$.*

(2) *$(Z(\hat g),X)$ satisfies the hole condition.*

(3) *$Z(\hat g)$ satisfies the boundary principle in $\mathbb{C}^n$.*

(4) *$(\boldsymbol{f},g)$ is reducible to the principal component of $U_n(A)$ for every $n$-tuple $\boldsymbol{f}\in A^n$ with $(\boldsymbol{f},g)\in U_{n+1}(A)$.*

(5) *$Z(\hat g)$ has no holes in $\mathbb{C}^n$.*

*Proof.* First we note that by Theorem 5.4, (2) and (3) are equivalent.

(1) $\Longrightarrow$ (2) (resp. (4) $\Longrightarrow$ (5)) By Theorem 5.6 (resp. Theorem 5.7) we need to show that for every $\boldsymbol{f}\in C(X,\mathbb{C}^n)$ with $Z(\boldsymbol{f})\cap Z(\hat g)=\varnothing$, the $(n+1)$-tuple $(\boldsymbol{f},\hat g)$ is reducible in $C(X,\mathbb{C})$ (resp. reducible to $\mathcal{P}(U_n(C(X,\mathbb{C})))$). Let $E:=Z(\hat g)$. Consider the algebra $B:=\overline{\hat A|_E}^{\|\cdot\|_\infty}$, that is the uniform closure of the restriction algebra $\hat A|_E$ in $C(E,\mathbb{C})$. Since $E$ is the zero set of the Gelfand transform of a function in $A$, $E$ is $A$-convex and so the spectrum, $M(B)$, of $B$ coincides with $E$ (see [8]). Note that $\boldsymbol{f}|_E\in U_n(C(E,\mathbb{C}))$. By the Arens-Novodvorski-Taylor Theorem 4.11 (2), there exist $\boldsymbol{h}\in U_n(B)$ and $\underline{G}_1,...,\underline{G}_m\in\mathcal{M}_n(C(E,\mathbb{C}))$ such that

$$\boldsymbol{f}|_E=\boldsymbol{h}\exp\underline{G}_1...\exp\underline{G}_m.$$

([7]) We do not know whether every algebra $A(K)$ is finitely generated.

In particular $|\boldsymbol{h}|\geq\delta>0$ on $E$. By Tietze's extension theorem, we may assume that $\underline{G}_j\in\mathcal{M}_n(C(X,\mathbb{C}))$. Using the definition of $B$, choose $\boldsymbol{a}\in A^n$ so that ([8])

$$\sup_{x\in E}\big|\boldsymbol{h}(x)-\hat{\boldsymbol{a}}(x)\big|<\frac{\varepsilon}{\|M\|_{HS}}, \tag{6.1}$$

where $M:=\exp\underline{G}_1\ldots\exp\underline{G}_m$ and where $\varepsilon$ is so small that $Z(\boldsymbol{a})\cap Z(\hat{g})=Z(\boldsymbol{a})\cap E=\varnothing$. Thus $(\boldsymbol{a},g)\in U_{n+1}(A)$. The hypothesis (1) (resp. (4)) implies that $(\boldsymbol{a},g)$ is reducible in $A$ (resp. reducible to the principal component of $U_n(A)$). That is, there is $\boldsymbol{x}\in A^n$ such that $\boldsymbol{u}:=\boldsymbol{a}+\boldsymbol{x}g\in U_n(A)$ (resp. $\boldsymbol{u}\in\mathcal{P}(U_n(A))$). In particular, $\widehat{\boldsymbol{u}}=\widehat{\boldsymbol{a}}$ on $E$. Since $|\boldsymbol{v}M|\leq|\boldsymbol{v}|\cdot\|M\|_{HS}$ for every vector $\boldsymbol{v}$, we have the following estimates on $E$:

$$\begin{aligned}|\boldsymbol{f}|_E-\widehat{\boldsymbol{u}}\exp\underline{G}_1\ldots\exp\underline{G}_m| &= |\boldsymbol{f}|_E-\hat{\boldsymbol{a}}\exp\underline{G}_1\ldots\exp\underline{G}_m|\\ &= \big|(\boldsymbol{h}-\hat{\boldsymbol{a}})\exp\underline{G}_1\ldots\exp\underline{G}_m\big|\\ &\leq|\boldsymbol{h}-\hat{\boldsymbol{a}}|\|M\|_{HS}<\varepsilon.\end{aligned}$$

Since $\widehat{\boldsymbol{u}}\in U_n(C(X,\mathbb{C}))$ (resp. $\widehat{\boldsymbol{u}}\in\mathcal{P}(U_n(C(X,\mathbb{C})))$), we deduce from Lemma 4.9 (resp. Lemma 4.10), applied to the algebra $C(X,\mathbb{C})$, that $(\boldsymbol{f},\hat{g})$ is reducible in $C(X,\mathbb{C})$ (resp. reducible to the principal component of $C(X,\mathbb{C})$) (whenever $\varepsilon>0$ is small).

(2) $\Longrightarrow$ (1) (resp. (5) $\Longrightarrow$ (4)) Let $(\boldsymbol{f},g)\in U_{n+1}(A)$. Hence $\hat{\boldsymbol{f}}$ and $\hat{g}$ have no common zeros on $X$. By Theorem 5.6, (resp. Theorem 5.7) hypothesis (2) (resp. (5)) and the assumption $M(A)=X$ imply that $(\hat{\boldsymbol{f}},\hat{g})$ is reducible in $C(X,\mathbb{C})$ (resp. reducible to the principal component of $U_n(C(X,\mathbb{C}))$). By the Corach-Suárez Theorem 1.3 (resp. Theorem 4.12) $(\boldsymbol{f},g)$ is reducible in $A$ (resp. reducible to the principal component of $U_n(A)$). □

## 7. Exponential reducibility II

Here we introduce our second notion of exponential reducibility.

*Definition* 7.1. Let $A$ be a commutative unital Banach algebra over $\mathbb{K}$ with identity element $\mathbf{1}$. Given

$$(\boldsymbol{a},g):=(a_1,\ldots,a_n,g)\in U_{n+1}(A),$$

we call $(\boldsymbol{a},g)$ *exponentially reducible* if there exists $x_j,b_j\in A$ such that

$$\sum_{j=1}^n e^{x_j}(a_j+b_jg)=\mathbf{1}.$$

([8]) Here $\|M\|_{HS}$ is the Hilbert-Schmidt norm.

**Observation 7.2.** *Let $A$ be a commutative unital Banach algebra over $\mathbb{K}$ such that*

(1) $\operatorname{bsr} A=1$,

(2) $U_1(A)$ *is connected.*

*Then every invertible pair $(a,g)\in A$ is exponentially reducible.*

*Proof.* By (1), $a+bg\in U_1(A)$ for some $b\in A$. Since $U_1(A)=\exp A$, we arrive at $a+bg=e^x$ for some $x\in A$. This is of course equivalent to say that $e^{-x}(a+bg)=\mathbf{1}$. $\square$

The following result gives a relation between exponential reducibility and reducibility to the principal component.

**Proposition 7.3.** *Let $A$ be a commutative unital Banach algebra over $\mathbb{K}$ and let $(\boldsymbol{a},g)\in U_{n+1}(A)$. Suppose that $(\boldsymbol{a},g)$ is exponentially reducible. Then $(\boldsymbol{a},g)$ is reducible to the principal component of $U_n(A)$.*

*Proof.* By assumption, $\sum_{j=1}^n e^{x_j}(a_j+b_jg)=\mathbf{1}$ for some $\boldsymbol{v}:=(e^{x_1},...,e^{x_n})\in U_n(A)$ and $\boldsymbol{b}:=(b_1,...,b_n)\in A^n$; that is

$$(\boldsymbol{a}+\boldsymbol{b}g)\cdot\boldsymbol{v}^t=\mathbf{1}.$$

Since $\boldsymbol{v}\in U_n(A)$ is reducible in $A$ (if $n\geq 2$), we deduce from Lemma 4.4 that

$$\boldsymbol{a}+\boldsymbol{b}g\in\boldsymbol{e}_1\cdot \mathit{Exp}\mathcal{M}_n(A)=\mathcal{P}\big(U_n(A)\big).$$

In other words, $(\boldsymbol{a},g)$ is reducible to the principal component of $U_n(A)$. The case $n=1$ is obvious. $\square$

*Remark.* Whereas for invertible pairs both notions coincide, exponential reducibility of tuples of length at least three is, in general, a much stronger requirement than reducibility to the principal component. As an example, we take the disk algebra $A(\mathbb{D})$. Let $(z,f)\in U_2(\mathbb{D})$ be an invertible pair that is not totally reducible (this means that there do not exist two invertible functions $u$ and $v$ in $A(\mathbb{D})$ such that $uz+vf=1$, see [13] for the existence). Then $(z,f,0)\in U_3(A(\mathbb{D}))$. Since $\operatorname{bsr} A(\mathbb{D})=1$, $U_2(A(\mathbb{D}))$ is connected ([5]), and so $U_2(A(\mathbb{D}))$ coincides with its principal component. In particular every invertible triple in $A(\mathbb{D})$ is reducible to the principal component of $U_2(A(\mathbb{D}))$. On the other hand, $(z,f,0)$ cannot be exponentially reducible, since otherwise

$$e^{a_1(z)}\big(z+b_1(z)\cdot 0\big)+e^{a_2(z)}\big(f(z)+b_2(z)\cdot 0\big)=1$$

for some functions $a_j,b_j\in A(\mathbb{D})$. But an equation of the form $e^{a_1(z)}z+e^{a_2(z)}f(z)=1$ is not possible because by our choice, $(z,f)$ is not totally reducible.

Examples of exponentially reducible tuples appeared in [10] (for pairs) and [12] (for tuples):

*Example* 7.4. Let $f_j \in H^\infty$ and let $b$ be an interpolating Blaschke product. Suppose that $(f_1, ..., f_n, b) \in U_{n+1}(H^\infty)$. Then $(f_1, ..., f_n, b)$ is exponentially reducible.

We do not know yet a characterization of the exponentially reducible tuples (in none of the standard algebras).

## 8. Complementing left-invertible matrices

Based on the Arens-Novodvorski-Taylor theorem, we conclude our paper by giving a simple analytic proof of a result by V. Ya. Lin [11, p. 127] concerning extension of left invertible matrices. Although this proof seems to be known among the specialists in the field, it never appeared explicitely in print (see also the footnote in [11, p. 127]). One may view this result as a companion result to Theorem 4.12 and Theorem 3.1.

**Theorem 8.1.** (Lin) *Let $A$ be a commutative unital Banach algebra over $\mathbb{C}$. Then a left-invertible matrix $L$ over $A$ can be complemented/extended to an invertible matrix over $A$ if and only if $\hat{L}$ ([9]) can be complemented in the algebra $C(M(A))$.*

*Proof.* It is sufficient to consider invertible rows (see [30, p. 345/346]). So let $\boldsymbol{a} \in U_n(A)$. Suppose that there exists an invertible matrix $M \in \mathcal{M}_n(C(M(A)))$ such that $\hat{\boldsymbol{a}} = \boldsymbol{e}_1 M$; that is, $\widehat{\boldsymbol{a}}$ is the first row of $M$.

By the Arens-Novodvorski-Taylor Theorem 4.11 (4), there is $Q \in \mathcal{M}_n(A)^{-1}$ such that $\widehat{Q}$ is homotopic in $M_n(C(M(A)))^{-1}$ to $M$. Hence, for some $G_j \in M_n(C(M(A)))$,

$$M = \widehat{Q} e^{G_1} ... e^{G_k},$$

and so

$$\hat{\boldsymbol{a}} = (\boldsymbol{e}_1 \widehat{Q}) e^{G_1} ... e^{G_k}.$$

Let $\boldsymbol{b}$ be the first row of $Q$; that is $\boldsymbol{b} = \boldsymbol{e}_1 Q$. Then $\hat{\boldsymbol{b}} = \boldsymbol{e}_1 \widehat{Q}$.

Since $\boldsymbol{b} \in U_n(A)$, we see that $\hat{\boldsymbol{a}}$ and $\hat{\boldsymbol{b}}$ belong to the same component of $U_n(C(M(A)))$. Hence, by another application of the Arens-Novodvorski Theorem 4.11(3), $\boldsymbol{a}$ and $\boldsymbol{b}$ belong to the same component of $U_n(A)$. Thus, there exist $H_j \in \mathcal{M}_n(A)$ such that

$$\boldsymbol{a} = \boldsymbol{b} e^{H_1} ... e^{H_k}.$$

([9]) Here $\hat{L} = (\widehat{a_{i,j}})$ is the matrix formed with the Gelfand-transforms of the entries of $L$.

Consequently,

$$\begin{aligned} \boldsymbol{a} &= (\boldsymbol{e}_1 Q) e^{H_1} \dots e^{H_k} \\ &= \boldsymbol{e}_1 R \end{aligned}$$

for some $R \in \mathcal{M}_n(A)^{-1}$. $\square$

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