Ark. Mat., **54** (2016), 473–484
DOI: 10.1007/s11512-015-0228-9

On the Szegö kernel of Cartan–Hartogs domains

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Abstract. Inspired by the work of Z. Lu and G. Tian (Duke Math. J. 125:351–387, 2004) in the compact setting, in this paper we address the problem of studying the Szegö kernel of the disk bundle over a noncompact Kähler manifold. In particular we compute the Szegö kernel of the disk bundle over a Cartan–Hartogs domain based on a bounded symmetric domain. The main ingredients in our analysis are the fact that every Cartan–Hartogs domain can be viewed as an "iterated" disk bundle over its base and the ideas given in (Arezzo, Loi and Zuddas in Math. Z. 275:1207–1216, 2013) for the computation of the Szegö kernel of the disk bundle over an Hermitian symmetric space of compact type.

1. Introduction

Let (L,h) be a Hermitian line bundle over a Kähler manifold (M,ω) of complex dimension n such that $\mathrm{Ric}(h)=\omega$, where $\mathrm{Ric}(h)$ is a two-form on M whose local expression is given by:

$$\mathrm{Ric}(h) = -\frac{i}{2}\partial\bar{\partial}\log h\big(\sigma(x),\sigma(x)\big), \tag{1}$$

for a trivializing holomorphic section $\sigma:U\to L\setminus\{0\}$. In the (pre)quantum mechanics terminology the pair (L,h) is a *geometric quantization* of (M,ω) and L is called the *quantum line bundle*. For all integers $m>0$ consider the line bundle $(L^{\otimes m},h_m)$ over (M,ω) with $\mathrm{Ric}(h_m)=m\omega$. Let $\mathcal{H}_m$ be the complex Hilbert space consisting of the $L^{\otimes m}$'s global holomorphic sections bounded with respect to the norm generated by the L^2-product:

$$\langle s,t\rangle_m = \int_M h_m\big(s(x),t(x)\big)\frac{\omega^n}{n!}(x),$$

The first author was supported by Prin 2010/11—Varietà reali e complesse: geometria, topologia e analisi armonica—Italy; the third author was supported by the project FIRB "Geometria Differenziale e teoria geometrica delle funzioni". All the authors were supported by INdAM-GNSAGA—Gruppo Nazionale per le Strutture Algebriche, Geometriche e le loro Applicazioni.

for $s,t\in\mathcal{H}_m$. Note that if M is compact, then $\mathcal{H}_m=H^0(L^{\otimes m})$ is finite dimensional. Given an orthonormal basis $s^m=(s_0^m,\dots,s_{N_m}^m)$ of $\mathcal{H}_m$ (with $N_m+1=\dim\mathcal{H}_m\leq\infty$) with respect to $\langle\cdot,\cdot\rangle_m$, one can define a smooth and positive real valued function on M, called the *Kempf's distortion function*:

$$T_m(x):=\sum_{j=0}^{N_m}h_m\big(s_j^m(x),s_j^m(x)\big). \tag{2}$$

As suggested by the notation, it is not difficult to verify that this function depends only on the Kähler form $m\omega$ and not on the orthonormal basis chosen. When M is compact, G. Tian [28] and W. Ruan [27] solved a conjecture posed by Yau by proving that the metric g, associated to the form ω, is the C^∞-limit of Bergman metrics. Zelditch [35] generalized Tian–Ruan's theorem by proving the existence of a complete asymptotic expansion in the C^∞ category, namely

$$T_m(x)\sim\sum_{j=0}^{\infty}a_j(x)m^{n-j}, \tag{3}$$

where a_j, $j=0,1,\dots$, are smooth coefficients with $a_0(x)=1$, and for any nonnegative integers r,k the following estimate holds:

$$\left\|T_m(x)-\sum_{j=0}^{k}a_j(x)m^{n-j}\right\|_{C^r}\leq C_{k,r}m^{n-k-1}, \tag{4}$$

where $C_{k,r}$ is a constant depending on k, r and on the Kähler form ω, and $\|\cdot\|_{C^r}$ denotes the C^r norm in local coordinates. Notice that similar asymptotic expansions were obtained by D. Catlin [9] (see also [5]–[8], [25] and [26] for a deformation quantization procedure on Kähler manifolds based on these expansions).

Later on, Z. Lu [23], by means of Tian's peak section method, proved that each of the coefficients $a_j(x)$ in (3) is a polynomial of the curvature and its covariant derivatives at x of the metric g. Such polynomials can be found by finitely many algebraic operations. Furthermore, Z. Lu computes the first three coefficients a_1, a_2 and a_3 of this expansion (see also [18] and [19] for the computations of the coefficients a_j's through Calabi's diastasis function). The expansion (3) is called the *TYZ (Tian–Yau–Zelditch) expansion* and it is a key ingredient in the investigations of balanced metrics in Donaldson's terminology [10] (see also [2]). The reader is also referred to the recent paper [17] by C. Liu and Z. Lu for a more explicit exposition on the algorithm of the computation of the general terms and the generalization of the TYZ expansion. Notice that prescribing the values of the coefficients of the TYZ expansion gives rise to interesting elliptic PDEs as shown by Z. Lu and G. Tian [24].

The main result obtained in [24] is that *if the log–term of the Szegö kernel of the unit disk bundle over M vanishes then $a_k=0$, for all $k>n$*. Recall that the disk bundle over M is the strongly pseudoconvex domain $D\subset M$ defined by $D=\{v\in M\,|\,\rho(v)>0\}$ and we denote by $X=\partial D$ its boundary. Given the separable Hilbert space $\mathcal{H}^2(X)$ consisting of all holomorphic functions on D which are continuous on X and satisfy:

$$\int_X |f|^2\, d\nu<\infty,$$

where $d\nu=\alpha\wedge(d\alpha)^n$ and $\alpha=-i\partial\rho_{|X}=i\bar\partial\rho_{|X}$ is the contact form on X associated to the strongly pseudoconvex domain D (the 1-form α is defined on the smooth part of X), the Szegö kernel of D is defined by:

$$\mathcal{S}(v)=\sum_{j=1}^{+\infty} f_j(v)\overline{f_j(v)},\quad v\in D,$$

where $\{f_j\}_{j=1,\dots}$ is an orthonormal basis of $\mathcal{H}^2(X)$. A direct computation of the Szegö kernel could be in general very complicated. Although, when $D\subset M$ is a strongly pseudoconvex domain with smooth boundary, the following celebrated formula due to Fefferman (see [13] and also [4]) shows that there exist functions a and b continuous on $\bar D$ and with $a\neq 0$ on X, such that:

$$\mathcal{S}(v)=\frac{a(v)}{\rho(v)^{n+1}}+b(v)\log\rho(v). \tag{5}$$

The function $b(v)$ is called the *logarithmic term* (or *log–term*) of the Szegö kernel and one says that the log–term of the Szegö kernel of D vanishes if $b=0$.

Z. Lu has conjectured (private communication) that the converse of the above mentioned result is true:

Conjecture 1. (Lu) *Let (L,h) be a positive line bundle over a compact complex manifold (M,ω) of dimension n such that* $\mathrm{Ric}(h)=\omega$. *If the coefficients a_k of TYZ in (3) vanish for all $k>n$, then the log–term of the Szegö kernel of the unit disk bundle over M vanishes.*

In [16] (see also [22]) the authors address the problem of the existence of a TYZ expansion in the noncompact case and study its coefficients.

In this paper we study the analogous of the previous conjecture for an important family of noncompact Kähler manifolds called Cartan–Hartogs domains, defined as follows. Let $\Omega\subset\mathbb{C}^d$ be a bounded symmetric domain of genus $\gamma=$

$(r-1)a+b+2$, where r is the rank of Ω and a and b are its numerical invariants (see [1, p. 16]). Further denote by $N=N(z)$ its *generic norm*, namely,

$$N(z)=\big(V(\Omega)K(z,z)\big)^{-\frac{1}{\gamma}},$$

where $V(\Omega)$ is the total volume of Ω with respect to the Euclidean measure of $\mathbb{C}^d$ and $K(z,z)$ is its Bergman kernel (see e.g. [1] for more details). The *Cartan–Hartogs domain* $M_\Omega^{d_0}(\mu)$ *based on* Ω is the pseudoconvex domain of $\mathbb{C}^{d+d_0}$ defined by ($\mu>0$ is a fixed constant):

$$M_\Omega^{d_0}(\mu)=\big\{(z,w)\in\Omega\times\mathbb{C}^{d_0},\ \|w\|^2<N^\mu(z)\big\}. \tag{6}$$

It can be equipped with the natural Kähler form:

$$\omega_{d_0}=-\frac{i}{2}\partial\bar\partial\log\big(N^\mu(z)-\|w\|^2\big).$$

The Kähler manifold $(M_\Omega^{d_0}(\mu),\omega_{d_0})$ has been studied by several authors from different analytic and geometric points of view (see for example [14], [15], [20], [21], [29]–[32] and [34]). One can consider the trivial line bundle([1]) $L=M_\Omega^{d_0}(\mu)\times\mathbb{C}$ on $M_\Omega^{d_0}(\mu)$ endowed with the Hermitian metric:

$$h_{d_0}(z,w;\xi)=\big(N^\mu(z)-\|w\|^2\big)|\xi|^2,\quad (z,w)\in M_\Omega^{d_0}(\mu),\ \xi\in\mathbb{C}, \tag{7}$$

which satisfies $\mathrm{Ric}(h_{d_0})=\omega_{d_0}$ (cfr. (1)).

The main result about the TYZ expansion for Cartan–Hartogs domains is expressed by the following recent result in [15], which shows that in this case the expansion is indeed finite, namely it is a polynomial in m of degree $d+d_0=\dim M_\Omega^{d_0}(\mu)$ with computable (non-constant) coefficients.

Theorem 1.1. Feng-Tu *Let* $m>\max\{d+d_0,\frac{\gamma-1}{\mu}\}$, *then the Kempf's distorsion function associated to* $(M_\Omega^{d_0}(\mu),\omega_{d_0})$ *can be written as*:

$$T_m(z,w)=\frac{1}{\mu^d}\sum_{k=0}^{d}\frac{D^k\widetilde{X}(d)}{k!}\left(1-\frac{\|w\|^2}{N^\mu}\right)^{d-k}\frac{\Gamma(m-d+k)}{\Gamma(m-d-d_0)}, \tag{8}$$

([1]) Due to the contractibility and pseudoconvexity of $M_\Omega^{d_0}(\mu)$, *any* holomorphic line bundle over $M_\Omega^{d_0}(\mu)$ is holomorphically trivial.

with

$$D^k\widetilde{X}(d)=\sum_{j=0}^{k}\binom{k}{j}(-1)^j\prod_{l=1}^{r}\frac{\Gamma(\mu(d-j)-\gamma+2-(l+1)\frac{a}{2}+b+ra)}{\Gamma(\mu(d-j)-\gamma+1+(l-1)\frac{a}{2})}.$$

Formula (8) implies, in particular, that $a_k=0$ for $k>d+d_0$. Therefore it is natural to see if Conjecture 1 holds true in this (noncompact) case.(2)

Notice that the disk bundle of a Cartan–Hartogs $M_\Omega^{d_0}(\mu)$ is the Cartan–Hartogs domain $M_\Omega^{d_0+1}(\mu)$, whose the boundary of $M_\Omega^{d_0+1}(\mu)$ is not smooth being:

$$\partial M_\Omega^1(\mu)=\partial\Omega\cup\{(z,w)\in\Omega\times\mathbb{C}\,|\,|w|^2=N^\mu\}.$$

Thus, it does not make sense to speak of the log–term of the Szegö kernel, since formula (5) applies only when the domain involved has smooth boundary. Nevertheless, in order to consider the case of Cartan–Hartogs domain, we give the following definition (which in the smooth boundary case coincides with the standard one).

Definition 1.2. Let $D\subset M$ be a strongly pseudoconvex domain in a complex n-dimensional manifold M, let $X=\partial D$ be its boundary with defining function $\rho>0$, i.e. $D=\{v\in M\,|\,\rho(v)>0\}$. Assume that the points where X fails to be smooth are of measure zero. We say that *the log–term of the Szegö kernel of the disk bundle vanishes* if there exists a continuous function a on $\overline{D}$ with $a\neq 0$ on X, such that $\mathcal{S}(v)=\frac{a(v)}{\rho(v)^{n+1}}$.

The main result of this paper is the following:

Theorem 1.3. *The log–term of the Szegö kernel of the disk bundle over a Cartan–Hartogs domain vanishes.*

In the next section we compute the Szegö kernel of $(M_\Omega^{d_0}(\mu),\omega_{d_0})$ and prove Theorem 1.3.

2. Szegö kernel of Cartan–Hartogs domains

In the following lemma, needed in the proof of Theorem 1.3, we compute the volume form $\alpha\wedge(d\alpha)^d$ on the boundary $\partial M_\Omega^1(\mu)$ of $M_\Omega^1(\mu)$, namely a Cartan–Hartogs domain with $d_0=1$.

(2) Formula (8) is used by Feng and Tu to give a positive answer to a conjecture posed by the third author of the present paper in [32], namely they prove that if coefficient a_2 is constant, then $M_\Omega^{d_0}(\mu)$ is the complex hyperbolic space. This formula has been also used in [33] to study the Berezin quantization of $(M_\Omega^{d_0}(\mu),\omega_{d_0})$.

Lemma 2.1. *The volume form $\alpha\wedge(d\alpha)^d$ on the boundary $\partial M^1_\Omega(\mu)$ is given in polar coordinates (ρ,θ) by:*

$$\alpha\wedge(d\alpha)^d=\left(\frac{2\mu}{\gamma}\right)^d N^{\mu(d+1)-\gamma}\,d\theta_w\wedge\frac{\omega_0^d}{d!},$$

where $\frac{\omega_0^d}{d!}$ is the standard volume form of $\mathbb{C}^d$ and $\theta_w=\theta_{d+1}$.

Proof. By definition $\alpha=-i\partial\rho_{|\partial M^1_\Omega(\mu)}$, where $\rho=N^\mu-|w|^2>0$ is the defining function of $M^1_\Omega(\mu)$. Thus, we get:

$$\alpha=-i\left(\sum_{j=1}^d\partial_jN^\mu dz_j-\overline{w}\,dw\right).$$

Furthermore, by $d\alpha=(\partial+\bar\partial)\alpha=-i\bar\partial\partial\rho$, we get:

$$d\alpha=-i\left(\sum_{j,k=1}^d N^\mu_{j\bar k}dz_j\wedge d\bar z_k-dw\wedge d\overline{w}\right)=i\left(dw\wedge d\overline{w}-\sum_{j,k=1}^d N^\mu_{j\bar k}dz_j\wedge d\bar z_k\right),$$

$$(d\alpha)^d=i^d\left(\det\!\left(-N^\mu_{j\bar k}\right)d\xi+\sum_{s,q=1}^d(-1)^{s+q}\det\!\left(-N^\mu_{j\bar k}\right)_{s\bar q}d\zeta_{s\bar q}\right),$$

where we write $N^\mu_j=\partial N^\mu/\partial z_j$, $N^\mu_{\bar k}=\partial N^\mu/\partial\bar z_k$ and $N^\mu_{j\bar k}=\partial^2N^\mu/\partial z_j\partial\bar z_k$, we denote by $d\xi=dz_1\wedge d\bar z_1\wedge\ldots\wedge dz_d\wedge d\bar z_d$ and by $d\zeta_{\bar q}$ (resp. $d\zeta_{s\bar q}$) the form $d\xi$ where the term $d\bar z_q$ (resp. the terms dz_s, $d\bar z_q$) is replaced by $d\overline{w}$ (resp. dz_s with dw and $dz_{\bar q}$ with $d\overline{w}$). Further, we write $(-N^\mu_{j\bar k})_{s\bar q}$ for the matrix $(-N^\mu_{j\bar k})$ where the s-th row and the q-th column have been deleted. Thus, the volume form $\alpha\wedge(d\alpha)^d$ is given by:

$$\alpha\wedge(d\alpha)^d=-i^{d+1}\left(\sum_{s,q=1}^d(-1)^{s+q}N^\mu_s\det\!\left(-N^\mu_{j\bar k}\right)_{s\bar q}dz_s\wedge d\zeta_{s\bar q}\right.$$

$$\left.-\overline{w}\det\!\left(-N^\mu_{jk}\right)dw\wedge d\xi\right).\tag{9}$$

Observe first that:

$$dz_s\wedge d\zeta_{s\bar q}=-dw\wedge d\zeta_{\bar q}=dw\wedge d\overline{w}\wedge d\xi_{\bar q},$$

where $d\xi_{\bar q}$ is the form $d\xi$ where the term $d\bar z_q$ was deleted. Further, evaluating at the boundary, turning to polar coordinates (ρ,θ) and denoting ρ_{d+1} by ρ_w and θ_{d+1} by θ_w, from $\rho_w^2=N^\mu$ one has $2\rho_wd\rho_w=\sum_{j=1}^dN^\mu_{\bar j}e^{-i\theta_j}(d\rho_j-i\rho_j\,d\theta_j)$ and we get:

$$\overline{w}\,dw\wedge d\xi=\rho_w(d\rho_w+i\rho_w\,d\theta_w)\wedge d\xi=iN^\mu\,d\theta_w\wedge d\xi,\tag{10}$$

and

$$dw\wedge d\bar{w}=-2i\rho_w\, d\rho_w\wedge d\theta_w=-i\sum_{j=1}^{d}N^{\mu}_{\bar{j}}\, d\bar{z}_j\wedge d\theta_w,$$

which yields

$$dz_s\wedge d\zeta_{s\bar{q}}=-iN^{\mu}_{\bar{q}}\, d\bar{z}_q\wedge d\theta_w\wedge d\xi_{\bar{q}}=-iN^{\mu}_{\bar{q}}\, d\theta_w\wedge d\xi. \tag{11}$$

Substituting (10) and (11) into (9) we get:

$$\alpha\wedge(d\alpha)^d=i^d A\, d\theta_w\wedge d\xi=2^d A\, d\theta_w\wedge\frac{\omega_0^d}{d!},$$

where we used that $\frac{\omega_0^d}{d!}=(\frac{i}{2})^d\, d\xi$ and we set:

$$A=N^{\mu}\det\big([-N^{\mu}_{j\bar{k}}]\big)-\sum_{j,k=1}^{d}(-1)^{j+k}N^{\mu}_{j}N^{\mu}_{\bar{k}}\det\big([-N^{\mu}_{p\bar{q}}]\big)_{j\bar{k}}.$$

It remains to show that:

$$A=\left(\frac{\mu}{\gamma}\right)^{d}N^{\mu(d+1)-\gamma}. \tag{12}$$

In order to prove (12), consider the metric g_Ω of the domain Ω associated to ω_Ω defined by $(g_\Omega)_{j\bar{k}}=\frac{\partial^2\log(N^\mu)}{\partial z_j\partial\bar{z}_k}$. A direct computation gives:

$$\begin{aligned}\det(g_\Omega)&=\det\left(\left[\frac{N^{\mu}_{j}N^{\mu}_{\bar{k}}-N^{\mu}_{j\bar{k}}N^{\mu}}{N^{2\mu}}\right]\right)\\
&=\frac{1}{N^{2d\mu}}\det\big([N^{\mu}_{j}N^{\mu}_{\bar{k}}-N^{\mu}_{j\bar{k}}N^{\mu}]\big)\\
&=\frac{N^{\mu}_{1}\dots N^{\mu}_{d}}{N^{2d\mu}}\det\left(\left[N^{\mu}_{\bar{k}}-\frac{N^{\mu}_{j\bar{k}}N^{\mu}}{N^{\mu}_{j}}\right]\right)\\
&=\frac{\prod_{h=1}^{d}N^{\mu}_{h}N^{\mu}_{\bar{h}}}{N^{2d\mu}}\det\left([1]+\left[-\frac{N^{\mu}_{j\bar{k}}N^{\mu}}{N^{\mu}_{j}N^{\mu}_{\bar{k}}}\right]\right)\\
&=\frac{1}{N^{d\mu}}\det\big([-N^{\mu}_{j\bar{k}}]\big)-\frac{1}{N^{\mu(d+1)}}\sum_{j,k=1}^{d}(-1)^{j+k}N^{\mu}_{j}N^{\mu}_{\bar{k}}\det\big([-N^{\mu}_{p\bar{q}}]\big)_{j\bar{k}}\\
&=\frac{A}{N^{\mu(d+1)}}.\end{aligned}$$

Conclusion follows by:

$$\det(g_\Omega)=\left(\frac{\mu}{\gamma}\right)^d \det(g_B)=\left(\frac{\mu}{\gamma}\right)^d N^{-\gamma},$$

where $g_B=\frac{\mu}{\gamma}g_\Omega$ is the Bergman metric on Ω (whose determinant can be obtained easily by considering that it is Kähler–Einstein with Einstein constant -2). □

Proof of Theorem 1.3. Observe first that by an inflation principle (see e.g. Section 2.3 in [30]) we can assume without loss of generality $d_0=1$. In this case the defining function $\rho(z,w)=N^\mu(z)-|w|^2$ and

$$\partial M^1_\Omega(\mu)=\partial\Omega\cup\left\{(z,w)\in\Omega\times\mathbb{C}\,|\,|w|^2=N^\mu\right\}.$$

Observe that although $\partial M^1_\Omega(\mu)$ is smooth only when Ω is of rank 1 (i.e. when Ω is the complex hyperbolic space), the points where it fails to be smooth are of measure zero. The volume form $d\nu=\alpha\wedge(d\alpha)^d$ reads:

$$d\nu=\alpha\wedge(d\alpha)^d=\left(\frac{2\mu}{\gamma}\right)^d N^{\mu(d+1)-\gamma}\,d\theta_w\wedge\frac{\omega_0^d}{d!},\tag{13}$$

where $\frac{\omega_0^d}{d!}$ is the standard Lebesgue measure on $\mathbb{C}^d$ (ω_0 is the flat Kähler form on $\mathbb{C}^d$). In order to compute the Szegö kernel $\mathcal{S}_{M^1_\Omega(\mu)}$ of $M^1_\Omega(\mu)$ one needs to find an orthonormal basis of the separable Hilbert space $\mathcal{H}^2(\partial M^1_\Omega(\mu))$ (Hardy space) consisting of all holomorphic functions $\hat{s}$ on $M^1_\Omega(\mu)$, continuous on $\partial M^1_\Omega(\mu)$ and such that

$$\int_{\partial M^1_\Omega(\mu)}|\hat{s}|^2\,d\nu<\infty.$$

Consider the Hilbert space:

$$\mathrm{H}^2_m(\Omega)=\left\{s\in\mathrm{Hol}(\Omega)\,\middle|\,\int_\Omega N^{\mu m}\left|s(z)\right|^2\frac{\omega_\Omega^d}{d!}<\infty\right\},$$

(where $\omega_\Omega=\frac{\gamma}{\mu}\omega_B$ is the Kähler form in Ω given by $\omega_\Omega=-\frac{i}{2}\partial\bar\partial\log N^\mu$) and the map:

$$\wedge:\mathrm{H}^2_m(\Omega)\longrightarrow\mathcal{H}^2\left(\partial M^1_\Omega(\mu)\right):\quad s\longmapsto\hat{s}\tag{14}$$

defined by

$$\hat{s}(v)=2^{-\frac{d}{2}}N(z,z)^{-\frac{\mu(d+1)}{2}}w^m s(z),\quad v=(z,w)\in\partial M^1_\Omega(\mu).$$

Notice that the Hardy space $\mathcal{H}^2(\partial M^1_\Omega(\mu))$ admits a Fourier decomposition into irreducible factors with respect to the natural S^1-action, i.e.

$$\mathcal{H}^2\big(\partial M^1_\Omega(\mu)\big)=\bigoplus_{m=0}^{+\infty}\mathcal{H}^2_m\big(\partial M^1_\Omega(\mu)\big),$$

where $\mathcal{H}^2_m(\partial M^1_\Omega(\mu)){:=}\{\hat{s}\in\mathcal{H}^2(\partial M^1_\Omega(\mu))\,|\,\hat{s}(\lambda v)=\lambda^m\hat{s}(v)\}$ and $\lambda v{:=}(z,\lambda w)$, for $v=(z,w)$. Since

$$\frac{\omega^d_\Omega}{d!}=\left(\frac{\mu}{\gamma}\right)^d N^{-\gamma}\frac{\omega^d_0}{d!},$$

it is not hard to see that the map $\wedge$ defines an isometry between $\mathrm{H}^2_m(\Omega)$ and $\mathcal{H}^2_m(\partial M^1_\Omega(\mu))$. Thus, if we consider the orthogonal projection of the Szegö kernel on each $\mathcal{H}^2_m(\partial M^1_\Omega(\mu))$, we get:

$$\mathcal{S}_{M^1_\Omega(\mu)}(v)=\sum_{m=0}^{+\infty}\sum_{j=0}^{+\infty}\hat{s}^m_j(v)\overline{\hat{s}^m_j(v)}=2^{-d}N^{-\mu(d+1)}\sum_{m=0}^{+\infty}\sum_{j=0}^{+\infty}|w|^{2m}\big|s^m_j(z)\big|^2, \tag{15}$$

where s^m_j, $j{=}0,1,\ldots$ is an orthonormal basis of $\mathrm{H}^2_m(\Omega)$ and $\hat{s}^m_j{=}\wedge(s^m_j)$ is the corresponding orthonormal basis for $\mathcal{H}^2_m(\partial M^1_\Omega(\mu))$.

It is well-known (for a proof, see e.g. [11, p. 77] or [12, Chapter XIII.1]) that $\sum_{j=0}^\infty N^{\mu m}|s^m_j(z)|^2$ is a polynomial in m of degree $d{=}\dim\Omega$, hence it can be written as:

$$\sum_{j=0}^{\infty}N^{\mu m}\big|s^m_j(z)\big|^2=\sum_{l=0}^{d}b_l\binom{m+l}{l}$$

where b_l depends on the metric g_Ω associated to ω_Ω. Thus, this formula together with (15) yields:

$$\begin{aligned}\mathcal{S}_{M^1_\Omega(\mu)}(v)&=2^{-d}N^{-\mu(d+1)}\sum_{m=0}^{\infty}\sum_{l=0}^{d}|w|^{2m}N^{-\mu m}b_l\binom{m+l}{l}\\&=2^{-d}N^{-\mu(d+1)}\sum_{l=0}^{d}b_l\sum_{m=0}^{\infty}\binom{m+l}{l}\big(|w|^2N^{-\mu}\big)^m\\&=2^{-d}N^{-\mu(d+1)}\sum_{l=0}^{d}b_l\frac{1}{(1-|w|^2N^{-\mu})^{l+1}}.\end{aligned}$$

That is

$$\mathcal{S}_{M^1_\Omega(\mu)}(v) = 2^{-d} N^{-\mu(d+1)} \left[\frac{b_0 N^\mu}{(N^\mu - |w|^2)} + \ldots + \frac{b_d N^{\mu(d+1)}}{(N^\mu - |w|^2)^{d+1}} \right]$$
$$= 2^{-d} \frac{b_0 N^{-\mu d}(N^\mu - |w|^2)^d + \ldots + b_{d-1} N^{-\mu}(N^\mu - |w|^2)^2 + b_d}{(N^\mu - |w|^2)^{d+1}}.$$

Observe that in the above expression, all terms except $b_d = d! m^d$ vanish once evaluated at the boundary $\partial M^1_\Omega(\mu)$. The vanishing of the log–term of $\mathcal{S}_{M^1_\Omega(\mu)}$ (as in Definition 1.2) follows then by setting:

$$a(v) = 2^{-d}\left(b_0 N^{-\mu d}\left(N^\mu - |w|^2\right)^d + \ldots + b_{d-1} N^{-\mu}\left(N^\mu - |w|^2\right)^2 + b_d\right). \qquad \square$$

Remark 2.2. It is worth pointing out that in [14] it is shown that the log–term of the Szegö kernel of $M^{d_0}_\Omega(\mu) \subset \mathbb{C}^{d+d_0}$ vanishes (in the sense of our Definition 1.2) when the Szegö kernel is obtained using the standard volume form of $\mathbb{C}^{d+d_0}$ restricted to $\partial M^{d_0}_\Omega(\mu)$ instead of the volume form $d\nu = \alpha \wedge (d\alpha)^d$ used in this paper. The reader is referred also to [30] for the proof of the vanishing of the log–term of the Bergman kernel.

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Received October 8, 2014
in revised form February 12, 2015
published online October 23, 2015