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Embeddings through discrete sets of balls

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Abstract. We investigate whether a Stein manifold M which allows proper holomorphic embedding into $\mathbb{C}^n$ can be embedded in such a way that the image contains a given discrete set of points and in addition follow arbitrarily close to prescribed tangent directions in a neighbourhood of the discrete set. The infinitesimal version was proven by Forstnerič to be always possible. We give a general positive answer if the dimension of M is smaller than $n/2$ and construct counterexamples for all other dimensional relations. The obstruction we use in these examples is a certain hyperbolicity condition.

1. Introduction

It is a famous theorem of Remmert [15] that any Stein manifold admits a proper holomorphic embedding into affine N-space $\mathbb{C}^N$ of sufficiently high dimension N. The main theme of the present paper is the following question which was asked by Forstnerič in [9]:

> If M is a Stein manifold which admits a proper holomorphic embedding into $\mathbb{C}^n$ for some $n>1$, what other properties of the embedding can one prescribe?

In the above mentioned paper of Forstnerič, the main result states that there exist embeddings of M through any discrete subset of $\mathbb{C}^n$ with prescribed finite jets at the points of the discrete set. Strong tools for prescribing additional properties of embeddings are based on the Andersén–Lempert theory [2] developed in the 1990s.

We would like to mention that these sort of properties of an embedding are crucial for the constructions of non-straightenable embeddings of $\mathbb{C}$ into $\mathbb{C}^2$ (see [10]), in sharp contrast with the algebraic situation where the famous Abhyankar–Moh–Suzuki theorem states that any algebraic embedding of a line in a plane is always equivalent to a linear one [1], [17]. This was crucial for Derksen and the second author who constructed counterexamples to the holomorphic linearisation problem (they showed existence of non-linearisable holomorphic $\mathbb{C}^*$-actions on $\mathbb{C}^m$, $m\geq 4$, and, moreover, existence of such non-linearisable holomorphic actions for any compact Lie group K on $\mathbb{C}^n$ with n sufficiently large; see [5] and [6]). In the same way (using affine modifications) many interesting examples of manifolds with the density property can be constructed (see [14]).

The present paper is devoted to the following natural question in this context:

> Given a discrete set of points in $\mathbb{C}^n$ with prescribed tangent planes of dimension $\dim M$, is it possible to require an embedding of M to follow arbitrarily close to the tangent planes in some neighbourhood of the given points?

It turns out that there is not such a very general answer as in the case of Forstnerič's theorem. The situation is more subtle: in certain dimensions the intrinsic properties of the manifold M itself play an important role for the answer.

We do not have a complete answer to our question, but on one hand we have a positive general answer for $0<k<\frac{1}{2}n$ (see Theorem 3.5).

Theorem I. *Let $0<k<\frac{1}{2}n$. If X is a complex space of dimension k which admits a proper holomorphic embedding into $\mathbb{C}^n$, then for any discrete set D of k-dimensional balls in $\mathbb{C}^n$ and any $\varepsilon\in\mathbb{R}_+^\infty$, there exists a proper holomorphic embedding $F\colon X\hookrightarrow\mathbb{C}^n$ such that $F(X)$ contains an ε-perturbation of D.*

On the other hand, for $\frac{1}{2}n\le k<n$ we are able to construct counterexamples to the corresponding result by using a hyperbolicity obstruction: For $\frac{1}{2}n\le k<n$ there exists a discrete set of k-dimensional balls in $\mathbb{C}^n$ such that no embedding $F\colon\mathbb{C}^k\hookrightarrow\mathbb{C}^n$ maps $\mathbb{C}^k$ through small perturbations of the balls (see Proposition 4.5). In fact, we are able to prove the following (see Theorem 4.7):

Theorem II. *For $\frac{1}{2}n\le k<n$ there exist a discrete set D of k-dimensional balls in $\mathbb{C}^n$ and $\varepsilon\in\mathbb{R}_+^\infty$ such that if a Stein manifold X admits a proper holomorphic embedding into $\mathbb{C}^n$ which contains an ε-perturbation of D, then X is $(n-k)$-Eisenman hyperbolic.*

We would like to state separately the following special case of Theorem II (see Corollary 4.8).

Corollary I. *There exist a discrete set D of discs in $\mathbb{C}^2$ and $\varepsilon\in\mathbb{R}_+^\infty$ such that no ε-perturbation of D can be contained in the image of a proper holomorphic embedding of $\mathbb{C}$ or $\mathbb{C}^*$.*

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2. k-balls in $\mathbb{C}^n$

Throughout this paper B_n will always denote the open unit ball in $\mathbb{C}^n$ centred at the origin. For $0<k<n$, we say that $D\subset\mathbb{C}^n$ is a *k-ball* centred at $p\in\mathbb{C}^n$ if $D=B\cap V$, where B is an open ball in $\mathbb{C}^n$ centred at p and $V\subset\mathbb{C}^n$ is a k-dimensional affine plane through p. We say that a sequence of k-balls D_j in $\mathbb{C}^n$ is *discrete* if their closures are pairwise disjoint and each set of the type $\{p_j\in D_j; j\in\mathbb{N}\}$ forms a discrete subset of $\mathbb{C}^n$. Often we will use the phrase "discrete set D of k-balls in $\mathbb{C}^n$" when referring to a discrete sequence D_j of k-balls in $\mathbb{C}^n$.

Let $\varepsilon>0$ and let $D\subset\mathbb{C}^n$ be a k-ball centred at p, i.e., $D=B\cap V$ as above. We say that a set $D'\subset\mathbb{C}^n$ is an *ε-perturbation of the k-ball D* if $D'=\{z+F(z); z\in D\}$, where $F\colon D\to\mathbb{C}^n$ is a holomorphic map such that $\|F\|<\varepsilon$ and the image of F is contained in the orthogonal complement of the linear subspace $V-p=\{z\in\mathbb{C}^n; z+p\in V\}$.

Given a discrete set $D=\{D_j\}_{j=1}^{\infty}$ of k-balls in $\mathbb{C}^n$ and a sequence ε_j of positive real numbers, denoted $\varepsilon\in\mathbb{R}_+^{\infty}$, where $\varepsilon=(\varepsilon_1,\varepsilon_2,...)$, we say that D' is an *ε-perturbation of D* if D' is a union of ε_j-perturbations D_j' of D_j.

Let $M\subset\mathbb{C}^n$ be an analytic subspace of dimension k, and assume that M contains an ε-perturbation D' of some k-ball $D\subset\mathbb{C}^n$. We will need the fact that deformations of M, which are small near D', will still contain a perturbation of D. By definition, D' is the graph of a holomorphic function f over D. If $\varphi\colon\mathbb{C}^n\to\mathbb{C}^n$ is a holomorphic map such that $\varphi\circ f$ is close enough to f in $\mathcal{C}^1$-norm, then it follows that the image of $\varphi\circ f$ contains a graph over a slightly smaller ball $\widetilde{D}\subset D$. Hence, deforming M by some holomorphic map φ for which the $\mathcal{C}^1$-norm $\|\varphi-\mathrm{id}\|_{\mathcal{C}^1}$ is small enough near D', we are assured that $\varphi(M)$ still contains a perturbation of D. Since φ deforms D' by at most $\nu=\sup_{z\in D'}|\varphi(z)|$, it follows that $\varphi(M)$ contains an $(\varepsilon+\nu)$-perturbation of D.

3. Theorem I

The statement of Theorem I is essentially contained in the following result.

Proposition 3.1. *Let $0<k<\frac{1}{2}n$. For any discrete set D of k-balls in $\mathbb{C}^n$, $\varepsilon\in\mathbb{R}_+^{\infty}$, and analytic subspace $M\subset\mathbb{C}^n$ of dimension k, there exist an open set Ω, $M\subset\Omega\subset\mathbb{C}^n$, and a biholomorphic map $\Phi\colon\Omega\to\mathbb{C}^n$, such that $\Phi(M)$ contains an ε-perturbation of D.*

To prove the proposition we make small modifications of the methods used by Forstnerič to prove the main theorem in [9]. The methods being inductive, we first perturb the discrete set of k-balls in order to find an exhaustion of the target space $\mathbb{C}^n$ by compact polynomially convex sets in such a way that a new k-ball is added in each step of the exhaustion (see Lemma 3.2). Using this exhaustion, we can follow the methods in [9], modifying the inductive step to guarantee the statement of the proposition (see Lemma 3.3).

3.1. The two lemmata

The following two lemmata are essential in the construction of Φ in Proposition 3.1.

Lemma 3.2. *Let $0<k<\frac{1}{2}n$. Given a discrete set D of k-balls and $\varepsilon\in\mathbb{R}_+^\infty$, there exist an ε-perturbation D' of D and compact polynomially convex sets $X_j\subset\mathbb{C}^n$ such that*

(i) *the sets X_j exhaust $\mathbb{C}^n$;*

(ii) *$\overline{D}_l'\subset X_j$ for $l\leq j$;*

(iii) *$\overline{D}_l'\cap X_j=\varnothing$ for $l>j$.*

Furthermore, there are pairwise disjoint analytic subsets P_j' (biholomorphic to $\mathbb{C}^k$) in $\mathbb{C}^n$ with $D_j'\subset P_j'$ and $P_j'\cap X_l=\varnothing$, $l<j$.

Proof. Let P_j be the k-dimensional affine plane in $\mathbb{C}^n$ containing the k-ball D_j. First we will define perturbations P_j' of the affine planes P_j. If possible, let k_j be the largest positive integer such that

$$D_j\cap 2k_jB_n=\varnothing \quad \text{and} \quad P_j\cap k_jB_n\neq\varnothing.$$

For j so that no such k_j exists, we set $P_j'=P_j$. Otherwise, let π_j be the orthogonal projection onto P_j and observe that D_j and $\pi_j(k_jB_n)$ are two open disjoint balls in P_j sharing no boundary points. As a result, their union is Runge in P_j and we can find a holomorphic function f_j on P_j such that

$$|f_j(z)|<\frac{\varepsilon_j}{2}, \quad z\in D_j,$$

and

$$|f_j(z)|>2k_j, \quad z\in\pi_j(k_jB_n).$$

Choose a vector $v\in\mathbb{C}^n$ orthogonal to P_j with $|v|=1$ and define a shear $\sigma\in\mathrm{Aut}(\mathbb{C}^n)$ by

$$z\longmapsto z+f\circ\pi_j(z)v.$$

Set $P_j'=\sigma(P_j)$ and observe that, since f is small enough on D_j, the set $\sigma(D_j)$ is an $\varepsilon_j/2$-perturbation of D_j.

Assume that the sets $P'_1, \ldots, P'_m$ are pairwise disjoint and let $N=\bigcup_{j=1}^m P'_j$. Consider deformations of P'_{m+1} given by translations in $\mathbb{C}^n$. Since almost every deformation of P'_{m+1} makes it transversal to N, and since $2k<n$, this implies that almost every such deformation of P'_{m+1} do not intersect N. Hence, if we choose such a deformation τ_{m+1} small enough, we can guarantee the additional property that $\tau_{m+1}(D'_{m+1})$ is an ε_m-perturbation of D_{m+1}. For this reason we may assume that the sets P'_j are pairwise disjoint, and each P'_j contains an ε_j-perturbation D'_j of D_j.

Now we have pairwise disjoint sets P'_j containing ε_j-perturbations of D_j. Furthermore, any compact set in $\mathbb{C}^n$ intersects at most finitely many sets P'_j. It follows that the disjoint union

$$P=\bigcup_{j=1}^{\infty} P'_j$$

is closed and constitutes an analytic subset of $\mathbb{C}^n$. Define a sequence C_j of non-negative real numbers inductively so that

$$C_1=0 \quad \text{and} \quad \min_{z\in \overline{D}'_{j+1}} |z|^2+C_{j+1} > \max_{z\in \overline{D}'_j} |z|^2+C_j, \ j\geq 1.$$

Then, since P is an analytic set in $\mathbb{C}^n$, there is a holomorphic function F on $\mathbb{C}^n$ such that $F|_{P'_j}=C_j$. Let $\rho\colon \mathbb{C}^n\to\mathbb{R}$ be given by

$$\rho(z)=|z|^2+|F(z)|.$$

Observe that ρ is a continuous strictly plurisubharmonic exhaustion function. The proof is finished by defining sublevel sets of ρ. Choose positive real numbers R_j in such a way that

$$R_j < \max_{z\in \overline{D}'_{j+1}} \rho(z) < R_{j+1}$$

and define $X_j=\{z\in\mathbb{C}^n;\rho(z)\leq R_j\}$. $\square$

Let D be a discrete set of k-balls in $\mathbb{C}^n$ and $\varepsilon\in\mathbb{R}_+^\infty$. Apply Lemma 3.2 to find an ε-perturbation D' of D together with an exhaustion X_j. Moreover, assume that $M\subset\mathbb{C}^n$ is an analytic subset of dimension k and $K\subset M$ a compact set such that $M\cap X_m\subset K$. Set $K_m=X_m\cup K$. Observe that, by the same methods used to prove Lemmata 5.4 and 5.6 in [9], it follows that K_m and $K_m\cup\overline{D}'_{m+1}$ are compact and polynomially convex. We describe the modification of the inductive step in the following lemma.

Lemma 3.3. *We use the above notation and assume that* $0<k<\frac{1}{2}n$, $\nu_1>0$. *If* $M\cap D'_{m+1}=\varnothing$ *and* M *contains a* ν_1*-perturbation* $\widetilde{D}_j$ *of* D_j, $1\le j\le m$, *with* $\widetilde{D}_j\subset \operatorname{Int} K_m$, *then for all* $\nu_2>0$ *there exists* $\varphi\in\operatorname{Aut}(\mathbb{C}^n)$ *such that*

(i) $\varphi(M)$ *contains a* $(\nu_1+\nu_2)$*-perturbation of* D_j, $1\le j\le m$;

(ii) $\varphi(M)$ *contains an* $(\varepsilon_{m+1}+\nu_2)$*-perturbation of* D_{m+1};

(iii) $\varphi(M)\cap D'_{m+2}=\varnothing$;

(iv) $|\varphi(z)-z|<\nu_2$ *for all* $z\in X_m$.

Proof. Let p_j be the centre of $\widetilde{D}_j$, $1\le j\le m$, i.e., the point in $\widetilde{D}_j$ corresponding to the centre of D_j, and let p be the centre of D'_{m+1}. Choose a point $q\in(\operatorname{Reg} M)\setminus K_m$ such that $\dim_q M=\dim M=k$. For any $\nu_3>0$, Proposition 1.1 in [9] assures that there exists $\varphi_1\in\operatorname{Aut}(\mathbb{C}^n)$ with the following properties:

(1) $\varphi_1(q)=p$;

(2) $T_p\varphi_1(M)=T_pD'_{m+1}$;

(3) $\varphi_1(z)=z+O(|z-p_j|^2)$ as $z\to p_j$, $1\le j\le m$;

(4) $|\varphi_1(z)-z|<\nu_3$ for all $z\in K_m$.

We let $\nu_3>0$ be small enough for (4) to ensure that $\varphi_1(M)$ contains $(\nu_1+\nu_2/4)$-perturbations of $D_1, ..., D_m$.

It is enough to find a Runge neighbourhood $R=R_1\cup R_2$ of $K_m\cup\overline{D}'_{m+1}$ and a vector field Z on R such that

(a) $K_m\subset R_1$, $\overline{D}'_{m+1}\subset R_2$, and $R_1\cap R_2=\varnothing$;

(b) $Z|_{R_1}=0$;

(c) the time-one flow of Z on a neighbourhood U of p, with U relatively compact in R_2, deforms $\overline{U}$ inside R_2 such that a part of $\varphi_1(M)\cap U$ is stretched onto a small enough perturbation of D_{m+1}.

Indeed, if this is achieved we can approximate Z uniformly on $K_m\cup\overline{U}$ by an entire vector field in $\mathbb{C}^n$. Therefore, the flow of the vector field Z can be uniformly approximated on $K_m\cup\overline{U}$ by a holomorphic automorphism φ_2 of $\mathbb{C}^n$ according to Lemma 1.4 in [11]. Making the approximations good enough, (b) assures that $\varphi_2\circ\varphi_1|_{K_m}$ is close enough to the identity. Moreover, (b) and (c) assures that $\varphi_2\circ\varphi_1(M)$ contains $(\nu_1+\nu_2/2)$-perturbations of $D_1, ..., D_m$ and an $(\varepsilon_{m+1}+\nu_2/2)$-perturbation of D_{m+1}. If $\varphi_2\circ\varphi_1(M)$ does not intersect D'_{m+2}, we let $\varphi=\varphi_2\circ\varphi_1$.

If $\varphi_2\circ\varphi_1(M)$ intersects D'_{m+2} we use small deformations of $\varphi_2\circ\varphi_1$ to avoid intersection. In general we know that small deformations, e.g. translations, of $\varphi_2\circ\varphi_1$ makes its image transversal to D'_{m+2}. Since $2k<n$, this means that the image does not intersect D'_{m+2}. Hence, it is enough to choose a small enough deformation $\varphi_3\in\operatorname{Aut}(\mathbb{C}^n)$ and define $\varphi=\varphi_3\circ\varphi_2\circ\varphi_1$.

Let us now construct the Runge neighbourhood R together with the desired vector field Z as above. Consider the analytic subspace P'_{m+1} from Lemma 3.2

which contains D'_{m+1} and for which there exists $\psi\in\mathrm{Aut}(\mathbb{C}^n)$ such that $\psi(P'_{m+1})=\mathbb{C}^k\times\{0\}$. Moreover, we may assume that $\psi(D'_{m+1})=B_k\times\{0\}$. Since $K_m\cup D'_{m+1}$ is polynomially convex it admits a Stein neighbourhood basis, and we can choose a Runge neighbourhood $R=R_1\cup R_2$ of $K_m\cup D'_{m+1}$ satisfying (a) above. In fact, we may assume that $\psi(R_2)=sB_k\times tB_{n-k}$ for some $s>1$ close enough to one and $t>0$ close enough to zero.

Use the coordinates $(w',w'')\in\mathbb{C}^k\times\mathbb{C}^{n-k}$ and define the vector field Y on $\psi(R_2)$ by $Y(w',w'')=(\mu w',-\nu w'')$, $\mu,\nu>0$. Moreover, the time-t flow F_t of Y in $\psi(R_2)$ is given by

$$F_t(w',w'')=(e^{\mu t}w',e^{-\nu t}w'').$$

We consider the time-one flow F_1 of Y. For each large enough choice of μ and ν we can find $r>0$ such that

$$rB_n\subset\psi(R_2)\quad\text{and}\quad \overline{B}_k\times\{0\}\subset F_1(rB_n)\subset\psi(R_2).$$

Additionally, we can increase μ and ν and decrease r so that the time-one flow of $\psi\circ\varphi_1(M)\cap rB_n$ contains a perturbation of $\overline{B}_k\times\{0\}$. This is possible since (2) above implies that $\psi\circ\varphi_1(M)$ is tangent to $\overline{B}_k\times\{0\}$ at the origin. Hence, it is possible to choose μ, ν, and r in order to make the resulting perturbation arbitrarily small.

For our purpose, and in order to satisfy (b) above, we define the vector field Z on $R=R_1\cup R_2$ by letting $Z|_{R_1}=0$ and $Z|_{R_2}=\psi^{-1}\circ Y$. We use the freedom of choice of the parameters μ, ν, and r to guarantee that the resulting time-one flow of Z in $R_2\cup\psi^{-1}(rB_n)$ stretches $\varphi_1(M)\cap\psi^{-1}(rB_n)$ so that it contains a small enough perturbation of D_{m+1}. Hence (c) above is satisfied with $U=\psi^{-1}(rB_n)$, and the proof is finished. □

Remark 3.4. By a more careful analysis it is possible to assure that $\varphi(M)$ passes through the midpoints of the balls D_j and the tangent space of M at these points coincides with that of the balls. The best way of keeping this property in the inductive step is using the Andersén–Lempert theorem for the geometric structure of holomorphic vector fields on $\mathbb{C}^n$ vanishing to order at least 2 at a finite set of points.

3.2. Proof of Theorem I

Using the two lemmata above we can now prove Proposition 3.1, and in turn guarantee the statement of Theorem I.

Proof of Proposition 3.1. Lemma 3.2 furnishes an $\varepsilon/4$-perturbation D' of D. Let p_j be the centre of D'_j, i.e., the point in D'_j corresponding to the centre of D_j. We may assume that $0\in X_1\cap M$ and choose $a\in X_1\setminus D'$ arbitrarily small to guarantee that the map $\Phi_0\in\operatorname{Aut}(\mathbb{C}^n)$, defined by $\Phi_0(z)=z+a$, satisfies $\Phi_0(M)\cap D'_1=\varnothing$. Choose $r>0$ small enough so that $X_0=rB_n\subset\operatorname{Int}X_1$.

Let $\tilde{\varepsilon}\in\mathbb{R}^\infty_+$ be chosen such that $0<\tilde{\varepsilon}<\varepsilon$,

$$\tilde{\varepsilon}_{j+1}\le\frac{\tilde{\varepsilon}_j}{2},\quad\text{and}\quad 0<\tilde{\varepsilon}_j<\operatorname{dist}(X_{j-1},\mathbb{C}^n\setminus X_j),\quad j>0.$$

Suppose that m is a positive integer, and set

$$\nu_j=\frac{\varepsilon_j}{2}+\sum_{j<l<m}\frac{\tilde{\varepsilon}_l}{2},\quad 0<j<m.$$

Assume that we have constructed a map $\Phi_{m-1}\in\operatorname{Aut}(\mathbb{C}^n)$ in such a way that, for $M_{m-1}=\Phi_{m-1}(M)$, the following holds true:

(a) M_{m-1} contains an ν_j-perturbation of D_j, $0<j<m$;

(b) $M_{m-1}\cap D'_m=\varnothing$.

Let $\rho_m>\max\{m,\rho_{m-1}\}$ be such that for $z\in M$ we have

$$\Phi_{m-1}(z)\in\mathbb{C}^n\setminus X_m,\quad |z|\ge\rho_m,$$

and set

$$K_m=\Phi_{m-1}(M\cap\rho_mB_n)\cup X_{m-1}.$$

As in the prerequisites of Lemma 3.3 the sets K_m and $K_m\cup\overline{D}'_m$ are compact and polynomially convex.

Observe that there are no obstructions to have separate parameters ν_1^j defining the scale of deformations of D_j in Lemma 3.3 (replacing each occurrence of ν_1 with ν_1^j). Hence, by Lemma 3.3 there exists $\Psi_m\in\operatorname{Aut}(\mathbb{C}^n)$ such that

(1) $\Psi_m(M_{m-1})$ contains a $(\nu_j+\tilde{\varepsilon}_m/2)$-perturbation of D_j, $0<j<m$;

(2) $\Psi_m(M_{m-1})$ contains an $\varepsilon_m/2$-perturbation of D_m;

(3) $\Psi_m(M_{m-1})\cap D'_{m+1}=\varnothing$;

(4) $|\Psi_m(z)-z|<\tilde{\varepsilon}_m$ for all $z\in K_m$.

It follows that $\Phi_m=\Psi_m\circ\Phi_{m-1}$ satisfies (a) and (b) above with m replaced by $m+1$. By Theorem 4.1 in [9], the sequence Φ_m of automorphisms of $\mathbb{C}^n$ converges on an open set $\Omega\subset\mathbb{C}^n$ to a biholomorphic map $\Phi\colon\Omega\to\mathbb{C}^n$. Furthermore, since $\rho_mB_n\cap M$ is an exhaustion of M by compact sets, (4) above guarantees that Φ_m converges on M, i.e., $M\subset\Omega$.

To see that Φ has the needed properties, it remains to show that $\Phi(M)$ contains an ε-perturbation of D. Let m be a positive integer. We know that $\Phi_m(M_{m-1})$ contains an $\varepsilon_m/2$-perturbation T_m of D_m. Furthermore,

$$\Phi = \lim_{j\to\infty} \Psi_j \circ \Psi_{j-1} \circ \ldots \circ \Psi_{m+1} \circ \Phi_m,$$

and each map Ψ_j deforms T_m by at most $\tilde{\varepsilon}_j/2$, $j>m$. Recall that $\tilde{\varepsilon}_{j+1}<\tilde{\varepsilon}_j/2$ for all positive integers j. Hence, the total deformation of T_m is at most

$$\sum_{j=1}^{\infty} \frac{\tilde{\varepsilon}_{j+m}}{2} \leq \sum_{j=1}^{\infty} 2^{-j-1}\tilde{\varepsilon}_m = \frac{\tilde{\varepsilon}_m}{2}.$$

Thus, $\Phi(M)$ contains a perturbation of D_m which is an $\tilde{\varepsilon}_m/2$-perturbation of T_m. Since T_m itself is an $\varepsilon_m/2$-perturbation of D_m, and $\tilde{\varepsilon}_m<\varepsilon_m$, we get that $\Phi(M)$ contains an ε_m-perturbation of D_m. This finishes the proof. □

Let X be a complex space of dimension k, where $0<k<\frac{1}{2}n$, which admits a proper holomorphic embedding $\Psi\colon X\hookrightarrow\mathbb{C}^n$. Composing Ψ with the map Φ from Proposition 3.1, we get the following result.

Theorem 3.5. *Let $0<k<\frac{1}{2}n$. If X is a complex space of dimension k which admits a proper holomorphic embedding into $\mathbb{C}^n$, then for any discrete set D of k-balls in $\mathbb{C}^n$ and any $\varepsilon\in\mathbb{R}_+^\infty$, there exists a proper holomorphic embedding $F\colon X\hookrightarrow\mathbb{C}^n$ such that $F(X)$ contains an ε-perturbation of D.*

The case $X=\mathbb{C}^k$ deserves to be stated separately.

Corollary 3.6. *Let $0<k<\frac{1}{2}n$. Then for any discrete set D of k-balls in $\mathbb{C}^n$ and any $\varepsilon\in\mathbb{R}_+^\infty$, there exists a proper holomorphic embedding $G\colon \mathbb{C}^k\hookrightarrow\mathbb{C}^n$ such that $G(\mathbb{C}^k)$ contains an ε-perturbation of D.*

4. Theorem II

In order to construct counterexamples for the corresponding statement of Corollary 3.6 in the case $\frac{1}{2}n\leq k<n$, we use Eisenman hyperbolicity. We can always choose a discrete set of k-balls in $\mathbb{C}^n$ in such a way that the complement of any small enough perturbation is $(n-k)$-Eisenman hyperbolic (see Proposition 4.3). Making more careful choices of balls, we get a certain kind of global lower bound for the Eisenman norm in the complement, i.e., a global lower bound of Ω_k^M as defined in (4.1). We use this bound to prove Theorem II.

4.1. The Eisenman norm

Let M be a complex manifold of dimension n. We will use the following notation: TM is the holomorphic tangent bundle of M; T_pM is the holomorphic tangent space at the point $p\in M$; $\Lambda^k T_pM$ (resp. $\Lambda^k TM$) is the kth exterior power of T_pM (resp. TM); $D_p^k M$ (resp. $D^k M$) is the set of decomposable elements of $\Lambda^k T_pM$ (resp. $\Lambda^k TM$).

Given any Hermitian metric $\langle\,\cdot\,,\cdot\,\rangle$ on TM, it can be extended to a Hermitian metric on $\Lambda^k TM$ by defining it pointwise as $\langle u,v\rangle=\det(\langle u_i,v_j\rangle)_{i,j=1}^k$ for decomposable elements $u=u_1\wedge\ldots\wedge u_k$ and $v=v_1\wedge\ldots\wedge v_k$ of $\Lambda^k T_pM$ and then extend this definition linearly to arbitrary elements of $\Lambda^k TM$. Below we will always use $\langle\,\cdot\,,\cdot\,\rangle$ to denote the complex Euclidean metric and $\|u\|^2=\langle u,u\rangle$.

The Kobayashi–Royden norm of a vector in the tangent bundle of a complex manifold has a natural extension to higher dimensions. This extension was first introduced by Eisenman in [7]. The following definition can be found in [12].

Definition 4.1. For $p\in M$ and $u\in D_p^k M$, $1\le k\le n$, the *k-Eisenman norm* of u is given by

$$E_k^M(p,u)=\inf\{\|v\|^2\,;v\in D_0^k B_k \text{ and there is an } F\in\mathcal{O}(B_k,M) \text{ with } F(0)=p \text{ and } F_*(v)=u\}.$$

Equivalently, for $R>0$ we can define the *k-Eisenman norm* by

$$E_k^M(p,u)=\inf\{R^{-2k}\,;\text{there is an } F\in\mathcal{O}(RB_k,M) \text{ with } F(0)=p \text{ and } F_*(e)=u\},$$

where $e\in D_0^k B_k$ is the unit element $e=\partial/\partial z_1\wedge\ldots\wedge\partial/\partial z_k$.

Although we have not formally defined a norm above, we still call it a norm based on how it is used intuitively.

Definition 4.2. A complex manifold M is *k-Eisenman hyperbolic at a point* $p\in M$ if $E_k^M(p,u)>0$ for all non-zero $u\in D_p^k M$, and M is *k-Eisenman hyperbolic* if it is k-Eisenman hyperbolic at each point of M.

Whenever the complex manifold M is an open subset of a complex Euclidean space $\mathbb{C}^n$ for some n, we may compare the k-Eisenman norm to the extended Euclidean norm induced from $\mathbb{C}^n$. We consider a pointwise lower bound given by

$$\Omega_k^M(p)=\inf_{\substack{u\in\Lambda^k T_pM\\ u\neq 0}}\frac{E_k^M(p,u)}{\|u\|^2}=\inf_{\substack{u\in D_p^k M\\ \|u\|=1}}E_k^M(p,u). \tag{4.1}$$

In the case $M=B_n$, $0<k<n$, and $p\in B_n$ we get

$$\Omega_k^{B_n}(p)=(1-|p|^2)^{-k}. \tag{4.2}$$

For details see Example 1.2 in [3].

4.2. k-balls and Eisenman hyperbolicity

The following proposition generalises a result of Kaliman (see [13], where $k=0$ and the balls are just points).

Proposition 4.3. *Let $0<k<n$. Then there exist a discrete set D of k-balls in $\mathbb{C}^n$ and $\varepsilon\in\mathbb{R}_+^\infty$ such that $\mathbb{C}^n\setminus D'$ is $(n-k)$-Eisenman hyperbolic for any ε-perturbation D' of D. Furthermore, the k-balls in D can be chosen in such a way that there exists $K>0$ for which $\Omega_{n-k}^{\mathbb{C}^n\setminus D'}\geq K$.*

The proposition is merely a simple observation, and the method for choosing balls can be found in [9]. The whole construction goes back to the work of Rosay and Rudin in [16] and was earlier used by Fornæss and Buzzard in [4] (with $n=2$ and $k=1$). The authors of the present paper also used k-balls with hyperbolic complement in [3]. The crucial point in proving Theorem II is the fact that we can choose k-balls D with a positive global lower bound for $\Omega_{n-k}^{\mathbb{C}^n\setminus D'}$.

Proof. Let $\alpha\subset\{1,...,n\}$ be a set with $m=\dim X$ elements. We will consider α as a strictly increasing multiindex and also write $\alpha=(\alpha_1,...,\alpha_m)$. Moreover, $\alpha'=\{1,...,n\}\setminus\alpha$ is considered as the complementary (strictly increasing) multiindex. Given α, we let $\mathbb{C}_\alpha$ be the subspace of $\mathbb{C}^n$ spanned by the coordinates $z_{\alpha_1},...,z_{\alpha_m}$ and $\mathbb{C}_{\alpha'}$ is defined similarly. We use π_α to denote the natural projection $\pi_\alpha\colon\mathbb{C}^n\to\mathbb{C}_{\alpha'}$.

Let r_j be any strictly increasing sequence of positive real numbers which diverges and set $r_0=0$. Consider a fixed positive integer j. For each α we choose an open non-empty spherical shell S_α with respect to the origin such that

$$S_\alpha\subset r_{j+1}B_n\setminus r_j\overline{B}_n$$

and

$$\overline{S}_\alpha\cap\overline{S}_\beta=\varnothing,\quad \alpha\neq\beta.$$

In each shell S_α we choose a countable dense subset $\{q_{\alpha,l}\}_{l=1}^\infty$ such that $\pi_\alpha|_{\{q_{\alpha,l}\}}$ is injective. Let $B_{\alpha,l}$ be the largest open ball in $\mathbb{C}_\alpha$, centred at the origin, such that

$$q_{\alpha,l}+2B_{\alpha,l}\subset S_\alpha.$$

Given a positive integer $l_0=l_0(j)$ (which is specified in the lemma below), we define

$$\Delta_{j,\alpha}=\bigcup_{l=1}^{l_0(j)}(q_{\alpha,l}+B_{\alpha,l}) \quad \text{and} \quad \Delta_j=\bigcup_{\alpha}\Delta_{j,\alpha}.$$

Let $d=n-k$ and choose a strictly decreasing real sequence $\{\nu_m\}_{m=1}^{\infty}$ such that $0<\nu_m<1$ and

$$1<2\prod_{m=1}^{\infty}(1-\nu_m)^d. \tag{4.3}$$

Given a map F we write JF to denote the determinant of the Jacobian of F. The following is a small modification of Lemma 5.6 in [9].

Lemma 4.4. *There is an integer $l_0(j)$ sufficiently large and a real number $\delta_j>0$ sufficiently small such that the set Δ_j satisfies the following property: If $F\colon B_d\to r_{j+2}B_n$ is any holomorphic map for which*

(i) $|F(0)|\le r_j$;

(ii) $\max_\alpha |J(\pi_\alpha\circ F)(0)|\ge 1/j$;

(iii) *$F(B_d)$ avoids a δ_j-perturbation $\Delta'_j\subset\mathbb{C}^n$ of Δ_j;*

then $F((1-\nu_j)B_d)\subset r_{j+1}B_n$.

From this point on we consider $l_0=l_0(j)$ and $\delta_j>0$ to be chosen in accordance to Lemma 4.4. By choosing k-balls like this for each j we get a discrete set D of k-balls in $\mathbb{C}^n$. Let us enumerate this set, i.e., $D=\{D_l\}_{l=1}^{\infty}$, and set $\varepsilon_l=\delta_j$ if D_l is one of the k-balls in the finite set Δ_j of k-balls.

Let D' be any ε-perturbation of D and set $P=\mathbb{C}^n\setminus D'$. Suppose that $p\in P$ and $0\ne u\in D_p^dP$ are arbitrarily chosen. To prove that P is d-Eisenman hyperbolic we need to show that

$$0<E_d^P(u,p)=\inf\{\|v\|^2\,;v\in D_0^dB_d,\text{ and there is an }F\in\mathcal{O}(B_d,P)\text{ with }F(0)=p\text{ and }F_*v=u\}.$$

Choose F as stated above, and note that for some $0\ne c\in\mathbb{C}$ we have

$$v=c\frac{\partial}{\partial w_1}\wedge\ldots\wedge\frac{\partial}{\partial w_d}.$$

Expressing F_*v in terms of the global coordinates $z_1,\ldots,z_n$ inherited from $\mathbb{C}^n$, we get

$$F_*\left(c\frac{\partial}{\partial w_1}\wedge\ldots\wedge\frac{\partial}{\partial w_d}\right)=c\sum_{\alpha}J(\pi_\alpha\circ F)(0)\frac{\partial}{\partial z^{\alpha'}}, \tag{4.4}$$

where

$$\frac{\partial}{\partial z^{\alpha'}}=\frac{\partial}{\partial z_{\alpha'_1}}\wedge\ldots\wedge\frac{\partial}{\partial z_{\alpha'_d}}.$$

Set $j_0=\min\{j\in\mathbb{N};|p|<r_j\}$. We choose $0<\nu<1$ and an integer $N>j_0$ such that

$$1<2(1-\nu)^d \tag{4.5}$$

and $F((1-\nu)B_d)\subset r_{N+1}B_n$. Let $l=0$ and consider

$$\widetilde{F}_l\colon B_d\longrightarrow r_{N-l+1}B_n,\quad \widetilde{F}_l(z)=F((1-\nu)z).$$

In order to apply Lemma 4.4 to the map $\widetilde{F}_l$, we need to assure that

(1) $|\widetilde{F}_l(0)|\le r_{N-l-1}$;

(2) $\max_\alpha|J(\pi_\alpha\circ\widetilde{F}_l)(0)|\ge 1/(N-l-1)$;

(3) the image of $\widetilde{F}_l$ avoids a δ_{N-l-1}-perturbation of Δ_{N-l-1}.

Note that (1) is true since $N-1\ge j_0$. Furthermore, since $\varepsilon_j=\delta_{N-l-1}$ whenever $D_j\subset\Delta_{N-l-1}$, (3) holds true by the assumptions on F. However, we cannot assure (2).

Assume that (2) is true. Then by Lemma 4.4 (with F replaced by $\widetilde{F}_l$ and j replaced by $N-l-1$) we get $\widetilde{F}_l((1-\nu_{N-l-1})B_d)\subset r_{N-l}B_n$. Let

$$\widetilde{F}_{l+1}\colon B_d\longrightarrow r_{N-l}B_n,\quad \widetilde{F}_{l+1}(w)=\widetilde{F}_l((1-\nu_{N-l-1})w).$$

Defining $\widetilde{F}_{l+1}$ in terms of $\widetilde{F}_l$ can be done repeatedly for increasing l as long as $l<N-j_0$ and (2) holds true, but (1) is no longer true for $l\ge N-j_0$. We continue to assume that (2) holds for increasing l, as long as $0\le l<N-j_0$. As $l=N-j_0-1$ we get the map $\widetilde{F}_{N-j_0}\colon B_d\to r_{j_0+1}B_n$ given by

$$\widetilde{F}_{N-j_0}(w)=\widetilde{F}_{N-j_0-1}((1-\nu_{j_0})w)=\ldots=F((1-\nu)(1-\nu_{N-1})\ldots(1-\nu_{j_0})w).$$

Consider the automorphism $\varphi(z)=z/r_{j_0+1}$ of $\mathbb{C}^n$. Set $\tilde{p}=\varphi(p)=p/r_{j_0+1}$, $\tilde{u}=\varphi_*u$ and define the map $G\colon B_d\to B_n$ by

$$G(z)=\varphi\circ\widetilde{F}_{N-j_0}(w)=\frac{\widetilde{F}_{N-j_0}(w)}{r_{j_0+1}}=\frac{F((1-\nu)(1-\nu_{N-1})\ldots(1-\nu_{j_0})w)}{r_{j_0+1}}.$$

According to Definition 4.1, the d-Eisenman norm $E_d^{B_n}(\tilde{p},\tilde{u})$ of $\tilde{u}$ at $\tilde{p}$ in B_n is given by

$$E_d^{B_n}(\tilde{p},\tilde{u})=\inf\{\|v\|^2; v\in D_0^dB_n \text{ and there is an } H\in\mathcal{O}(B_d,B_n) \text{ with } H(0)=\tilde{p} \text{ and } H_*v=\tilde{u}\}.$$

Since $G\in\mathcal{O}(B_d,B_n)$, $G(0)=\tilde{p}$, and $G_*\tilde{v}=\tilde{u}$ for

$$\tilde{v}=c(1-\nu)^{-d}(1-\nu_{N-1})^{-d}...(1-\nu_{j_0})^{-d}\frac{\partial}{\partial w_1}\wedge...\wedge\frac{\partial}{\partial w_d}\in D_0^d B_d,$$

it follows from (4.1) and (4.2) that

$$\|\tilde{v}\|^2\geq E_d^{B_n}(\tilde{p},\tilde{u})\geq\Omega_d^{B_n}(\tilde{p})\|\tilde{u}\|^2=(1-|\tilde{p}|^2)^{-d}\|\tilde{u}\|^2. \tag{4.6}$$

Using that $0\leq r_{j_0-1}/r_{j_0+1}\leq|\tilde{p}|<1$ and $\|\tilde{u}\|^2=r_{j_0+1}^{-2d}\|u\|^2$ together with (4.3), (4.5), and (4.6), we get the estimate

$$\begin{aligned}\|v\|^2&=(1-\nu)^{2d}(1-\nu_{N-1})^{2d}...(1-\nu_{j_0})^{2d}\|\tilde{v}\|^2\\&\geq\tfrac{1}{16}(1-|\tilde{p}|^2)^{-d}\|\tilde{u}\|^2\geq\tfrac{1}{16}(r_{j_0+1}^2-r_{j_0-1}^2)^{-d}\|u\|^2,\end{aligned} \tag{4.7}$$

which gives a lower bound for $\|v\|^2$ in the case that (2) holds true for $0\leq l<N-j_0$.

We now turn to the case when, during the process of making repeated use of Lemma 4.4, we get to an $l<N-j_0-1$ for which (2) does not hold true. For such an l we thus have

$$\max_\alpha|J(\pi_\alpha\circ\widetilde{F}_l)(0)|<\frac{1}{N-l-1},$$

and it follows from (4.3) and (4.5) that

$$\max_\alpha|J(\pi_\alpha\circ F)(0)|\leq 4\max_\alpha|J(\pi_\alpha\circ\widetilde{F}_l)(0)|<\max_\alpha|J(\pi_\alpha\circ F)(0)|<\frac{4}{N-l-1}\leq 4.$$

In view of (4.4) we get

$$\|u\|^2=\|F_*v\|^2\leq|c|^2\sum_\alpha|J(\pi_\alpha\circ F)(0)|^2\leq|c|^2\max_\alpha|J(\pi_\alpha\circ F)(0)|^2\binom{n}{k}\leq 4|c|^2\binom{n}{k},$$

and since $\|v\|^2=|c|^2$ it follows that

$$\|v\|^2\geq\frac{\|u\|^2}{4\binom{n}{k}}. \tag{4.8}$$

It follows from (4.7) and (4.8) that for any holomorphic map $F\colon B_d\to\mathbb{C}^n\setminus D'$ such that $F(0)=p$ and $F_*v=u$ we get the bound

$$\|v\|^2\geq K_p\|u\|^2,$$

where

$$K_p=\min\left\{\frac{(r_{j_0+1}^2-r_{j_0-1}^2)^{-d}}{16},\frac{1}{4\binom{n}{k}}\right\}$$

depends only on the point $p\in P$. Hence,

$$E_d^P(p,u) \geq K_p\|u\|^2 > 0.$$

If the sequence r_j is chosen in such a way that $r_{j+2}^2 - r_j^2 \leq C$ for some $C>0$, e.g., $r_j=\sqrt{j}$, we see that K_p can be chosen independent of p by setting

$$K = \min\left\{\frac{C^{-d}}{16}, \frac{1}{4\binom{n}{k}}\right\}.$$

In such a situation we get the same lower bound K at each point p, i.e.,

$$\Omega_d^P(p) = \inf_{0\neq u\in D_p^d P} \frac{E_d^P(p,u)}{\|u\|^2} \geq K > 0. \quad \square$$

4.3. Proof of Theorem II

We will now make use of the "global lower bound" for the Eisenman norm which is assured for certain choices of k-balls according to Proposition 4.3. Assuming the existence of a proper holomorphic embedding whose image contains an ε-perturbation of the balls D, it follows from Lemma 4.4 that the complement of the embedding enjoys the same hyperbolic property as the complement of the balls. This allows us to prove the following result.

Proposition 4.5. *For $\frac{1}{2}n \leq k < n$ there exists a discrete set of k-dimensional balls in $\mathbb{C}^n$ for which one cannot find a proper holomorphic embedding of $\mathbb{C}^k$ into $\mathbb{C}^n$ containing small perturbations of the k-balls.*

It is crucial to use the fact that there is a neighbourhood of the embedded space which is biholomorphic to a neighbourhood of $\mathbb{C}^k\times\{0\}$ in $\mathbb{C}^n$ (identifying the image of the embedding with $\mathbb{C}^k\times\{0\}$). Since $2k\geq n$, i.e., $n-k\leq k$, this enables us to put large $(n-k)$-dimensional balls in the complement of the embedding by including the balls into $\mathbb{C}^k\times\{0\}$ and then translate them out of $\mathbb{C}^k\times\{0\}$ (within the normal bundle). This means that as a point in the complement approaches the embedding, we may put larger and larger $(n-k)$-dimensional balls through it. Considering the Eisenman norm (in the complement of the embedding), this has the effect that the pointwise lower bound of the Eisenman norm in Section 4.1 must tend to zero as one approaches the embedding, i.e., there cannot exist a global positive lower bound in the sense studied earlier. Hence we get a contradiction, if the k-balls are chosen in accordance with Proposition 4.3. We will now give the details of the proof.

Proof. Let $d=n-k$. By Proposition 4.3 there exist a discrete set D of k-balls, $\varepsilon\in\mathbb{R}_+^\infty$, and $K>0$ such that for any ε-perturbation D' of D, the complex manifold $\mathbb{C}^n\setminus D'$ is d-Eisenman hyperbolic and $\Omega_d^{\mathbb{C}^n\setminus D'}\geq K$. For a contradiction, assume that there exists an embedding $F\colon\mathbb{C}^k\hookrightarrow\mathbb{C}^n$ such that the image of F contains an ε-perturbation D' of D.

We need the following result.

Theorem 4.6. ([8, Hilfssatz 11]) *Let X and Y be Stein manifolds, $\Phi\colon X\to Y$ be a holomorphic embedding, and identify X with the zero section of the normal bundle of Φ. Then there exist a neighbourhood U of X in the normal bundle and a biholomorphic map $\widetilde{\Phi}\colon U\to Y$ such that $\widetilde{\Phi}|_X=\Phi$.*

We can now apply the theorem to find a neighbourhood U of the zero section in the normal bundle of our map $F\colon\mathbb{C}^k\hookrightarrow\mathbb{C}^n$ such that there is a map $\widetilde{F}\colon U\to\mathbb{C}^n$ which is biholomorphic onto its image $V\supset F(\mathbb{C}^k)$ in $\mathbb{C}^n$ and which satisfies $\widetilde{F}|_{\mathbb{C}^k}=F$. Since by Grauert's Oka principle any vector bundle over a complex Euclidean space is trivial, the normal bundle of F is trivial. Choose a direction v in the normal bundle of F which is orthogonal to the zero section.

For positive integers j, consider jB_d to be included in the zero section of the normal bundle of F using the standard inclusion of $\mathbb{C}^d$ into $\mathbb{C}^k$ (which can be done since $d=n-k\leq k$). Choose a positive real number c_j such that the translation c_jv+jB_d (considered in the normal bundle) is a subset of U. If needed, we modify the sequence c_j to guarantee that $c_j\to 0$ as $j\to\infty$.

Set $R_j=\widetilde{F}(c_jv+j\overline{B}_d)\subset\mathbb{C}^n\setminus F(\mathbb{C}^k)$ and define the map

$$F_j\colon jB_d\longrightarrow\mathbb{C}^n\setminus F(\mathbb{C}^k),\quad F_j(w)=\widetilde{F}(c_jv+w).$$

Let $u=\partial/\partial w_1\wedge\ldots\wedge\partial/\partial w_d\in D_0^d(jB_d)$ and define $p_j=F_j(0)$ and $u_j=(F_j)_*u$. Then we have $\|u_j\|\to\|F_*u\|>0$. Choose j_0 such that $\|u_j\|>\|F_*u\|/2$ whenever $j>j_0$. Since $\Omega_d^{\mathbb{C}^n\setminus F(\mathbb{C}^k)}\geq K$, we get

$$E_d^{\mathbb{C}^n\setminus F(\mathbb{C}^k)}(p_j,u_j)\geq K\|u_j\|^2>\frac{K}{4}\|F_*u\|^2=:C,\quad j>j_0.$$

On the other hand, by the definition of the Eisenman norm, we get

$$E_d^{\mathbb{C}^n\setminus F(\mathbb{C}^k)}(p_j,u_j)\leq j^{-2d},$$

which gives a contradiction when $j>j_0$ is large enough to ensure $j^{-2d}\leq C$. □

Assume that X is a Stein manifold, $\dim X=k$, $\frac{1}{2}n\leq k<n$, and assume that X is not $(n-k)$-Eisenman hyperbolic, i.e., there is a point $p\in X$ and a non-zero $u\in D_p^{n-k}X$ such that $E_{n-k}^X(p,u)=0$.

Let us choose a discrete set of k-balls as in the proof of the proposition above, and assume that there is a proper holomorphic embedding $\Phi\colon X\hookrightarrow\mathbb{C}^n$ such that its image contains a small perturbation of the balls. Set $q=\Phi(p)$ and $v=\Phi_*u$. Then it follows from Definition 4.1 that there for each positive integer j exists $F_j\colon jB_{n-k}\to\Phi(X)$ such that

$$G_j(0)=q \quad\text{and}\quad (G_j)_*\left(\frac{\partial}{\partial w_1}\wedge\ldots\wedge\frac{\partial}{\partial w_{n-k}}\right)=v.$$

Furthermore, there is a neighbourhood U of $\Phi(X)$ in $\mathbb{C}^n$ which is biholomorphic to a neighbourhood V of the zero section of the normal bundle N of $\Phi(X)$ in $\mathbb{C}^n$ (identifying $\Phi(X)$ with the zero section). Hence, given F_j as above, we may consider the pullback bundle F_j^*N, for which the diagram

$$\begin{array}{ccc} F_j^*N & \xrightarrow{\widetilde{F}_j} & N \\ {\scriptstyle\tilde{\pi}}\big\downarrow & & \big\downarrow{\scriptstyle\pi} \\ jB_{n-k} & \xrightarrow[F_j]{} & \Phi(X) \end{array}$$

commutes. Observe that the pullback bundle F_j^*N is trivial. Hence, for any j, we may choose a direction e_j in the bundle F_j^*N and $\varepsilon_j>0$ such that the translation of the zero section in F_j^*N by $\varepsilon_j e_j$ is a section s in the complement of the zero section in F_j^*N. Choosing ε_j small enough guarantees that $\widetilde{F}_j\circ s$ is a section in the complement of the zero section in N over the set $F_j(jB_{n-k})$ and that the section is contained in V. In this way we get an induced map $G_j\colon jB_{n-k}\to U\setminus\Phi(X)$ corresponding to a small translation of the image of F_j into the complement of $\Phi(X)$.

Next, we define $q_j=G_j(0)$ and $v_j=(G_j)_*\big(\partial/\partial w_1\wedge\ldots\wedge\partial/\partial w_{n-k}\big)$. Observe that we may assume that e_j and ε_j are chosen in such a way that $q_j\to q$ and $v_j\to v$ as $j\to\infty$. Now we get the same kind of contradiction as in the proof of Proposition 4.5. Hence, we have proved the following result.

Theorem 4.7. *For $\frac{1}{2}n\le k<n$ there exist a discrete set D of k-dimensional balls in $\mathbb{C}^n$ and $\varepsilon\in\mathbb{R}_+^\infty$ such that if a Stein manifold X admits a proper holomorphic embedding into $\mathbb{C}^n$ which contains an ε-perturbation of D, then X is $(n-k)$-Eisenman hyperbolic.*

We find the following special case worthwhile to be stated separately.

Corollary 4.8. *There exist a discrete set D of discs in $\mathbb{C}^2$ and $\varepsilon\in\mathbb{R}_+^\infty$ such that no ε-perturbation of D can be contained in the image of a proper holomorphic embedding of $\mathbb{C}$ or $\mathbb{C}^*$.*

We see that, for $\frac{1}{2}n<k<n$, there are at least two obstructions for embedding a Stein manifold X of dimension k through perturbations of k-balls in $\mathbb{C}^n$. If $n<[3k/2]+1$, there might not exist proper holomorphic embeddings of X into $\mathbb{C}^n$. On the other hand, if it does, Theorem 4.7 states that non-hyperbolicity of X is an obstruction. Hence, we formulate the following question.

Open problem. Let $\frac{1}{2}n<k<n$ and let X be an $(n-k)$-Eisenman hyperbolic Stein manifold of dimension k which admits a proper holomorphic embedding into $\mathbb{C}^n$. Given any discrete set D of k-balls in $\mathbb{C}^n$ and any $\varepsilon\in\mathbb{R}_+^\infty$, does there exist such an embedding for which the image contains an ε-perturbation of D?

More concretely. Given any discrete set D of discs in $\mathbb{C}^2$ and any $\varepsilon\in\mathbb{R}_+^\infty$, does there exist a proper holomorphic embedding of the unit disc into $\mathbb{C}^2$ such that the image contains an ε-perturbation of D?

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