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Remark to my previous paper on a bilateral Tauberian theorem

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This remark is directly connected with paper [1] to which we refer for notations and theorems.

When $s-h$ is an integer the conditions of the Abelian theorem $A(-\infty,+\infty)$ require more from $\sigma(\lambda)$ than is obtained in the conclusion of the Tauberian theorem $T(-\infty,+\infty)$. This makes it natural to ask whether $T(-\infty,+\infty)$ can be sharpened so as to give not only $\sigma(\lambda)$, $\sigma(-\lambda)\in I^s$ for positive values of λ but also

$$I^{-s}(\sigma(\lambda)+\sigma(-\lambda))\in\omega^0 \quad \text{when } s-h=\text{odd integer}$$

and

$$I^{-s}(\sigma(\lambda)-\sigma(-\lambda))\in\omega^0 \quad \text{when } s-h=\text{even integer}.$$

Results of this kind were obtained in section 8 of [1] for the so-called non-exceptional cases. However, in the exceptional cases these results did not follow from the considerations in [1] but are, as will be seen, consequences of a lemma due to Hardy and Littlewood.

The main asymptotic condition in $T(-\infty,+\infty)$ is split into relations (8), (9) of [1]. Assume for instance that $s-h$ is an odd integer. According to the result of $T(-\infty,+\infty)$ we know that $\sigma(\lambda)$ and $\sigma(-\lambda)\in I^s$ so that also $S(\lambda)=\sigma(\lambda)+\sigma(-\lambda)\in I^s$ and $I^{-s/2}S(\sqrt{\Lambda})\in I^0$. Relation (9) of [1] can be written

$$\int_0^\infty \Lambda^{-k}(\Lambda+T)^{-1}dI^{-s/2}S(\sqrt{\Lambda})=T^{-1}P_2(T^{-1})+o(|T|^{-k-1}),$$

where $k=-\frac{1}{2}s+\frac{1}{2}(h-1)$ is a non-negative integer. In the way indicated in section 4 of [1] this relation can be replaced by

$$\int_0^\infty (\Lambda+T)^{-1}dI^{-s/2}S(\sqrt{\Lambda})=AT^{-1}+o(|T|^{-1}),$$

where A is a constant.

The lemma of Hardy and Littlewood reads (see [2], p. 198 for the simple proof)

If $t\int_0^\infty(\lambda+t)^{-1}d\varphi(\lambda)\to A$ when $t\to+\infty$ and if $\varphi\in I^0$ then $\varphi\in\omega^0$ and $\varphi(\lambda)-\varphi(0)\to A$ when $\lambda\to+\infty$.

Application of this lemma shows that $I^{-s/2}S(\sqrt{\Lambda})\in\omega^0$, i.e. $I^{-s}(\sigma(\lambda)+\sigma(-\lambda))\in\omega^0$ for $\lambda\to+\infty$, provided $s-h$ is an odd integer.

The case when $s-h$ is an even integer is similarly treated with the help of relation (8) in [1]. The result is, that in this case $I^{-s}(\sigma(\lambda)-\sigma(-\lambda)) \in \omega^0$ for $\lambda \to +\infty$.

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