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On some expansions of stable distribution functions

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1. Introduction

The function e^{-t^α} for any fixed value α in the interval $0 < \alpha < 1$ admits a unique representation.

$$e^{-t^\alpha} = \int_0^\infty e^{-xt} G'_\alpha(t)\, dx, \qquad 0 \leqq t \leqq \infty \tag{1}$$

where $G_\alpha(x)$ is a stable d. f. (distribution function) with $G_\alpha(0) = 0$.[1] P. HUMBERT[2] has formally given the expansion

$$G'_\alpha(x) = -\frac{1}{\pi} \sum_{k=1}^{\infty} \frac{(-1)^k}{k!} (\sin \pi \alpha k) \frac{\Gamma(\alpha k + 1)}{x^{\alpha k + 1}} \tag{2}$$

for $0 < \alpha < 1$, $x < 0$, which has later been rigorously proved by H. POLLARD.[3]
From (1) follows that $G_\alpha(x)$ has the characteristic function

$$\gamma_\alpha(t) = e^{-|t|^\alpha \left(\cos \frac{\pi \alpha}{2} - i \sin \frac{\pi \alpha}{2} \operatorname{sgn} t\right)}$$

(sgn: read signum). Now owing to P. LÉVY[4] the characteristic function of a stable d.f., when suitably normalized can be written in the form.

$$\gamma_{\alpha\beta}(t) = e^{-|t|^\alpha (\cos \beta - i \sin \beta \operatorname{sgn} t)}, \tag{3}$$

where

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[1] S. BOCHNER, Completely monotone functions of the Laplace operator for torus and sphere, Duke Math. J. vol. 3, 1937.
P. LÉVY, Théorie de l'addition des variables aléatoires, Gauthier-Willars (1937) pp. 94—97, 198—204.
Compare also H. POLLARD, The representation of e^{-x^λ} as a Laplace integral, Bull. Amer. Math. Soc. vol. 52 (1946) pp. 908—910.
[2] P. HUMBERT, Nouvelles correspondances symboliques, Bull. Soc. Math. France vol. 69 (1945) pp. 121—129.
[3] H. POLLARD loc. cit.
[4] P. LÉVY loc. cit.

$$\cos\ \beta \geq 0, \quad \left|\sin\beta\cos\frac{\pi\alpha}{2}\right| \leq \cos\beta\sin\frac{\pi\alpha}{2}$$

$0 < \alpha \leq 2$. We omit the uninteresting case $\cos\beta = 0$. Then we can give a generalization of Humbert's expansion corresponding to (3) in the form

$$G'_{\alpha\beta}(x) = -\frac{1}{\pi}\sum_{k=1}^{\infty}\frac{(-1)^k}{k!}\frac{\Gamma(\alpha k+1)}{x|x|^{\alpha k}}\sin k\left(\frac{\alpha\pi}{2}+\beta-\alpha\arg x\right) \tag{4}$$

for $0 < \alpha < 1$ ($\arg x = \pi$ for $x < 0$.) The series in the right side of (4) is divergent for $\alpha \geq 1$. However, we can prove that the partial sum of the n first terms in (4) for every n is an asymptotic expansion in the case $1 \leq \alpha < 2$, i.e. the remainder term has smaller order of magnitude (for large $|x|$) then the last term in the partial sum,

$$G'_{\alpha\beta}(x) = -\frac{1}{\pi}\sum_{k=1}^{n}\frac{(-1)^k}{k!}\frac{\Gamma(\alpha k+1)}{x|x|^{\alpha k}}\sin k\left(\frac{\alpha\pi}{2}+\beta-\alpha\arg x\right) + 0[|x|^{-\alpha(n+1)-1}] \tag{5}$$

($\arg x = \pi$ for $x < 0$), $|x| \to \infty$.

At the same time we give the convergent series

$$G'_{\alpha\beta}(x) = \frac{1}{\pi}\sum_{k=0}^{\infty}(-1)^k\frac{\Gamma\left(\frac{k+1}{\alpha}\right)}{k!\,\alpha}x^k\cos\left[k\left(\frac{\pi}{2}+\frac{\beta}{\alpha}\right)+\frac{\beta}{\alpha}\right] \tag{6}$$

for $\alpha > 1$ and the asymptotic expansion

$$G'_{\alpha\beta}(x) = \frac{1}{\pi}\sum_{k=0}^{n}(-1)^k\frac{\Gamma\left(\frac{k+1}{\alpha}\right)}{k!\,\alpha}x^k\cos\left[k\left(\frac{\pi}{2}+\frac{\beta}{\alpha}\right)+\frac{\beta}{\alpha}\right] + 0(|x|^{n+1}) \tag{7}$$

for $0 < \alpha < 2$, $|x| \to 0$.

For the proof we shall use the Fouriertransform.

The asymptotic expansions are of importance in that case when $G_{\alpha\beta}(t)$ is the limiting distribution of a sequence of distributions. We shall return to this question in an other connection.

2. **Proof**

As $|\gamma_{\alpha\beta}(t)|$ is integrable in $(-\infty, \infty)$ for $\alpha > 0$, $\cos\beta > 0$, the inversion

$$G'_{\alpha\beta}(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-itx}\gamma_{\alpha\beta}(t)\,dt \tag{8}$$

is permitted. We consider real x with $|x| > 0$. Then we can put

$$G'_{\alpha\beta}(x) = \frac{1}{2\pi}[u(x) + \overline{u(x)}] \tag{9}$$

with

$$u(x) = \int_0^\infty e^{-itx} \gamma_{\alpha\beta}(t)\, dt. \tag{10}$$

It is now possible to choose φ_0 in the interval $\frac{-\pi}{2} \le \varphi_0 \le \frac{\pi}{2}$ such that

$$\beta_1 = \pi - \beta + \alpha\varphi_0,$$

and

$$\beta_2 = \frac{3\pi}{2} - \arg x + \varphi_0$$

belong to the interval $\left(\frac{\pi}{2} + \delta, \frac{3\pi}{2} - \delta\right)$ with some δ, $0 < \delta < \frac{\pi}{2}$. (We choose $\varphi_0 < 0$ for $x > 0$ and $\arg x = \pi$, $\varphi_0 > 0$ for $x < 0$).

We can then transform the integral in the right side of (10) into

$$u(x) = e^{i\varphi_0} \int_0^\infty e^{\tau|x|e^{i\beta_2} + \tau^\alpha e^{i\beta_1}} d\tau. \tag{11}$$

For that reason we consider

$$\int e^{-itx - t^\alpha e^{-i\beta}} dt$$

taken along a contour C in the $re^{i\varphi}$-plane where C is defined by the following relations:

$\varphi = 0$, $\varphi = \varphi_0$, r between r_0 and r_1, $r = r_0$, $r = r_1$, φ between 0 and φ_0.

This integral is zero and the part of the integral taken along the curved portions of C vanishes when $r_1 \to \infty$, $r_0 \to 0$. Thus (11) holds. Consider the expansion

$$e^z = \sum_{k=0}^{n} \frac{z^k}{k!} + z^{n+1} M_{n+1}. \tag{12}$$

Here M_{n+1} is smaller than a constant only depending on n, if $R(z) \le 0$.[1]

Putting $z = \tau^\alpha e^{i\beta_1}$ and combining (11) and (12), we obtain

$$u(x) = \sum_{k=0}^{n} \frac{1}{k!} e^{ik\beta_1 + i\varphi_0} \int_0^\infty e^{\tau|x|e^{i\beta_2}} \tau^{\alpha k}\, d\tau + e^{i\varphi_0 + i(n+1)\beta_1} \cdot \int_0^\infty M_{n+1}\, e^{\tau|x|e^{i\beta_2}} \tau^{\alpha(n+1)}\, d\tau. \tag{13}$$

[1] We derive the expansion from Cauchy's integral formula and then we can obtain the remainder term in the form $z^{n+1} M_{n+1}$,

$$M_{n+1} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{a+iy}}{(a+iy)^{n+1}(a+iy-z)} dy$$

with any $a > 0$.

The remainder term is here smaller than

$$\operatorname*{Max}_{z} |M_{n+1}| \cdot \frac{\Gamma[\alpha(n+1)+1]}{|x \cos \beta_2|^{\alpha(n+1)+1}} = 0\left[|x|^{-\alpha(n+1)-1}\right].$$

By a transformation analogous to that one used for (9) we get

$$\int_0^\infty e^{\tau |x| e^{i\beta_2}} \tau^{\alpha k}\, d\tau = e^{i(\pi-\beta_2)(\alpha k+1)} \int_0^\infty e^{-\varrho |x|} \varrho^{\alpha k}\, d\varrho = \tag{14}$$

$$= e^{i(\pi-\beta_2)(\alpha k+1)} \frac{\Gamma(\alpha k+1)}{|x|^{\alpha k+1}}.$$

From (9), (13) and (14) we get (5).

In order to obtain (4) in the case $0 < \alpha < 1$ we change e^z in (10) against the infinite Taylor-expansion and observe that term by term integration is permitted.[1]

Considering the expansion (10) and the corresponding Taylor-expansion with $z = \tau |x| e^{i\beta_2}$ we prove (7) and (6) in the same way as we have proved (5) and (4). Then we have to observe the relation[2]

$$\int_0^\infty \varrho^k e^{-\varrho^\alpha} d\varrho = \frac{1}{\alpha} \Gamma\left(\frac{k+1}{\alpha}\right).$$

We consider the following special cases.

a) $$\alpha < 1, \quad \beta = \frac{\alpha\pi}{2}.$$

Then every member in the right side of (4) is equal to 0 for $x < 0$, i.e. we have $G'_{\alpha\beta}(x) = 0$ for $x < 0$.

b) $$\alpha < 1, \quad \beta = -\frac{\alpha\pi}{2}.$$

Then every member in the right side of (4) is equal to 0 for $x > 0$, i.e. we have $G'_{\alpha\beta}(x) = 0$ for $x > 0$

c) $$1 < \alpha < 2, \quad \beta = -\pi + \frac{\alpha\pi}{2}.$$

Then every member except the remainder term in the right side of (5) is equal to 0 for $x < 0$, i.e. we have $G'_{\alpha\beta}(x) = 0(|x|^{-m})$ with arbitrarily large m for $x < 0$.

d) $$1 < \alpha < 2, \quad \beta = \pi - \frac{\alpha\pi}{2}.$$

Then every member except the remainder term is equal to 0 for $x > 0$, i.e. we have $G'_{\alpha\beta}(x) = 0(x^{-m})$ with arbitrarily large m for $x > 0$.

[1] We can apply a test in E. W. HOBSON, The theory of functions of a real variable II, Cambridge (1950) p. 306.

[2] Compare W. GRÖBNER und N. HOFREITER, Integraltafel II, Springer-Verlag (1950) p. 67.

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