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Renewal theory and the almost sure convergence of branching processes

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Let $Z(t)$, $t \geqslant 0$, $Z(0)=1$, be an age-dependent branching process, defined in the usual way on a probability space of the family tree type. Let the probability law be determined by the right continuous life-length distribution G, $G(0)=0$, and the reproduction law $\{p_k\}_{k=0}^{\infty}$, $p_0+p_1<1$, $h(s)=\sum_{k=0}^{\infty} p_k s^k$. Thus, in an applied language, we count the number of individuals at time t, $Z(t)$, in a population where each member has a random life-length, distributed according to G, and where any individual at its death is substituted by a random number ν of new individuals, $P\{\nu=k\}=p_k$. Different individuals are supposed to act independently of one another and independence is also assumed between the life-length and the reproduction of any specific individual.

The distribution of $Z(t)$ is determined by its generating function, $F(s,t)= E[s^{Z(t)}]$, E for expectation, F being in its turn determined by the integral equations

$$F(s,t)=s[1-G(t)]+\int_0^t h\{F(s,t-u)\}dG(u), \quad |s|\leqslant 1,$$

as is well known. Differentiating this for $s<1$ and passing to the limit $s\uparrow 1$ with a certain care yields a renewal equation for the mean M, $M(t)=E[Z(t)]$,

$$M(t)=1-G(t)+m\int_0^t M(t-u)\,dG(u),$$

$m=h'(1)$, supposed finite [2, p. 140]. From an analogous relation for the generating function of the vector $(Z(t),Z(t+\tau))$, $\tau\geqslant 0$, equations for the second moment k_τ, $k_\tau(t)=E[Z(t)Z(t+\tau)]$, $\tau\geqslant 0$, may be obtained in a similar manner,

$$k_\tau(t)=1-G(t+\tau)+m\int_t^{t+\tau} M(t+\tau-u)\,dG(u)+h''(1)\int_0^t M(t-u)M(t+\tau-u)\,dG(u)$$
$$+m\int_0^t k_\tau(t-u)\,dG(u),$$

$h''(1)$ assumed finite [2, p. 144]. With the help of renewal theory [cf. 1], these equations may be used to investigate the asymptotic behaviour of the process, as time passes. If G is not a lattice distribution, i.e. its points of increase are not integer multiples of some positive number, $m > 1$ and α satisfies

$$m \int_0^\infty e^{-\alpha t}\, dG(t) = 1,$$

then
$$M(t) \sim a e^{\alpha t}, \quad t \to \infty, \qquad a = \frac{m-1}{\alpha m^2 \int_0^\infty t e^{-\alpha t}\, dG(t)}.$$

If further $h''(1) < \infty$, then

$$k_\tau(t) \sim c e^{\alpha\tau + 2\alpha t}, \quad t \to \infty,\ \tau \geq 0, \qquad c = \frac{h''(1) a^2 \int_0^\infty e^{-2\alpha u}\, dG(u)}{1 - m \int_0^\infty e^{-2\alpha u}\, dG(u)}$$

and from this it follows that

$$W(t) = \frac{Z(t)}{a e^{\alpha t}}$$

converges in mean square, as $t \to \infty$, to some random variable W with expectation 1 but a strictly positive variance [2, p. 146]. In 1960 Harris proved that if $m > 1$, $h''(1) < \infty$, G is non-lattice and, furthermore,

$$\int_0^\infty E[(W(t) - W)^2]\, dt < \infty,$$

then $W(t)$ converges also with probability one to W [2, p. 147]. If G has a density g a more manageable criterion is known. Suppose that $m > 1$, $h''(1) < \infty$ and that

$$\int_0^\infty [g(t)]^p\, dt < \infty$$

for some $p > 1$, then $W(t) \to W$ almost surely, [2, p. 147]. Here we shall prove that the last condition in these theorems may simply be discarded:

Theorem. *If $m > 1$, $h''(1) < \infty$, and G is not a lattice distribution, then $W(t) \to W$ almost surely, as $t \to \infty$.*

The idea is to show that Harris' condition

$$\int_0^\infty E[(W(t) - W)^2]\, dt < \infty$$

is actually satisfied under the assumptions of the theorem. For the sake of clarity the proof will be given as a chain of simpler propositions.

We shall use traditional convolution notation:

$$f * g(t) = \int_0^t f(t-u)\,dg(u),$$

$$f^{*0}(t) = \begin{cases} 0, & \text{if} \quad t < 0, \\ 1, & \text{if} \quad t \geqslant 0, \end{cases}$$

$$f^{*n} = f * f^{*(n-1)}, \quad n \geqslant 1,$$

f and g supposed to be functions such that the definitions make sense. Integrals $\int_a^b$ should be interpreted as ranging over the half-closed interval $(a, b]$, except when $a = 0$. Then zero is included in the integration. The statement "$\int_a^\infty f(t)\,dg(t)$ converges" simply means that $\lim_{r\to\infty} \int_a^r f(t)\,dg(t)$ exists.

Our main tool of proof will be renewal theory, where some results of independent interest will be derived, though Tauberian arguments and Laplace transforms might seem nearest to hand. Let us therefore begin by stating the so-called key renewal theorem in a form suitable for our purposes.

Proposition 1. *If μ is a non-lattice probability on $(0, \infty)$ with finite second moment, f is of bounded variation on finite intervals and converges to zero as its argument tends to infinity and*

$$\int_0^\infty f(t)\,dt$$

converges, then the solution x of the renewal equation $x = f + x * \mu$ satisfies

$$\lim_{t\to\infty} x(t) = \frac{\int_0^\infty f(t)\,dt}{w},$$

where $w = \int_0^\infty t\,d\mu(t)$.

This theorem is well known, though seldom deduced in detail. A proof can be given by applying the fact that, v^2 denoting $\int_0^\infty t^2 d\mu(t)$,

$$0 \leqslant \sum_{n=0}^{\infty} \mu^{*n}(t) - \frac{t}{w} \to \frac{v^2}{2w^2},$$

as $t \to \infty$ [1, p. 357], to the formal solution of the renewal equation:

$$x = f * \sum_{n=0}^{\infty} \mu^{*n}.$$

Elementary approximations will yield Proposition 1.

We shall also need a much simpler fact, the renewal theorem for defective measures [1, p. 361]:

Proposition 2. *Assume that μ is a non-decreasing function on $[0, \infty)$ with $\mu(0)=0$* and $\mu(\infty)<1$, *and that $\lim_{t\to\infty} f(t)=f(\infty)$ exists. Then, if $x=f+x*\mu$,*

$$\lim_{t\to\infty} x(t)=\frac{f(\infty)}{1-\mu(\infty)}.$$

This statement is a direct consequence of the following well-known theorem.

Proposition 3. *If f is bounded, $\lim_{t\to\infty} f(t)=A$, $g(t)$ increases to B, as $t\to\infty$, and $f*g$ is well defined, then $\lim_{t\to\infty} f*g(t)=AB$.*

Proposition 4. *Suppose that μ is a probability on $(0, \infty)$, satisfying $v^2=\int_0^\infty t^2 d\mu(t)<\infty$, that $\int_0^\infty f(t)dt=0$ and that $\lim_{t\to\infty}\int_0^t (t-u)f(u)du=r$ exists finitely. Then, if $x=f+x*\mu$,*

$$\int_0^\infty x(t)\,dt=\frac{r}{w},$$

and is, thus, convergent.

Proof. Introducing

$$X(t)=\int_0^t x(u)\,du$$

and integrating the equation for x, we obtain

$$X(t)=\int_0^t f(u)\,du+\int_0^t X(t-u)\,d\mu(u),$$

after a change of the order of integration in the double integral. This is again a renewal equation and since

$$\int_0^t du\int_0^u f(y)\,dy=\int_0^t (t-u)f(u)\,du,$$

the assertion made follows from Proposition 1.

An analogous assertion for defective measures can be deduced by the same trick from Proposition 2:

Proposition 5. *If $\mu(0)=0$, $\mu(t)\uparrow\mu(\infty)<1$, as $t\to\infty$, and $\int_0^\infty f(t)dt$ converges, then the solution of the equation $x=f+x*\mu$ has a convergent integral on $[0, \infty)$*

$$\int_0^\infty x(t)\,dt=\frac{\int_0^\infty f(t)\,dt}{1-\mu(\infty)}.$$

Proposition 6. *If the conditions of Proposition 4 are satisfied, f is of bounded variation on every finite interval, $f(t)=o(t^{-1})$, as $t\to\infty$, and $\int_0^\infty tf(t)\,dt$ converges, then $\int_0^\infty t\,dx(t)$ is convergent, too.*

Proof. Introducing $\bar{x}$ by

$$\bar{x}(t)=\int_0^t u\,dx(u)$$

(evidently this integral is well defined), we conclude that

$$\bar{x}(t)=\int_0^t u\,df(u)+\int_0^t x(t-u)\,u\,d\mu(u)+\int_0^t \bar{x}(t-u)\,d\mu(u)$$

in the following way: Integrate u from 0 to t with respect to x and apply the relation $x=f+x*\mu$ to obtain

$$\bar{x}(t)=\int_0^t u\,df(u)+\int_0^t u\,d(x*\mu)(u).$$

Here, integrate the last term by parts, apply Fubini's theorem and perform another integration by parts. But the relation thus obtained is again a renewal equation, for $\bar{x}$. Since

$$\int_0^t u\,df(u)=tf(t)-\int_0^t f(u)\,du,$$

it follows that

$$\lim_{t\to\infty}\int_0^t u\,df(u)=0 \quad \text{and that} \quad \int_0^\infty dt\int_0^t u\,df(u)$$

converges. But because

$$\lim_{t\to\infty} x(t)=\frac{\int_0^\infty f(t)\,dt}{w}=0,$$

it is also true that

$$\int_0^t x(t-u)\,u\,d\mu(u)\to 0, \quad \text{as} \quad t\to\infty,$$

by Proposition 3. This one combined with Proposition 4 also shows that

$$\int_0^v dt\int_0^t x(t-u)\,u\,d\mu(u)=\int_0^t u\,d\mu(u)\int_0^{v-u} x(t)\,dt\to w\frac{r}{w}=r,$$

as $v\to\infty$. Thus, the key renewal theorem, Proposition 1, may be applied, guaranteeing the existence of

$$\lim_{t\to\infty}\bar{x}(t)=w^{-1}\left[\int_0^\infty tf(t)\,dt-\int_0^\infty dt\int_0^t f(u)\,du+r\right]=\frac{\int_0^\infty tf(t)\,dt}{w}.$$

Proposition 7. *Under the assumptions of Proposition 6, it holds that* $x(t)=O(t^{-1})$, *as* $t\to\infty$. *Furthermore, under these assumptions* $x(t)=o(t^{-1})$ *if and only if* $\int_0^t f(u)\,du = o(t^{-1})$, *as* $t\to\infty$.

The proof is immediate: Integration by parts shows that

$$tx(t)=\int_0^t u\,dx(u)+\int_0^t x(u)\,du.$$

As $t\to\infty$ the right-hand side has the limit

$$w^{-1}\int_0^\infty tf(t)\,dt+\int_0^\infty x(t)\,dt=w^{-1}\left[\int_0^\infty tf(t)\,dt+\lim_{t\to\infty}\int_0^t (t-u)f(u)\,du\right]$$

$$=\lim_{t\to\infty}\frac{t\int_0^t f(u)\,du}{w}.$$

We now apply our results to branching processes with a nonlattice life-length distribution and $1<m<\infty$.

Proposition 8. $M(t)\sim ae^{\alpha t}$, *as* $t\to\infty$ *and*

$$\int_0^\infty [e^{-\alpha t}M(t)-a]\,dt$$

converges

Proof. M is known to satisfy $M=1-G+mM*G$. Multiplication by $e^{-\alpha t}$ yields a renewal equation, since $\bar{G}$,

$$\bar{G}(t)=m\int_0^t e^{-\alpha u}\,dG(u),$$

is a probability distribution on $(0,\infty)$. But $\bar{G}$ has finite moments of all orders and, thus, the key renewal theorem is applicable showing that $e^{-\alpha t}M(t)\to a$, a well-known fact mentioned also earlier in this paper.

As to the second assertion of the proposition we note that

$$e^{-\alpha t}M(t)-a=e^{-\alpha t}[1-G(t)]-a[1-\bar{G}(t)]+\int_0^t [e^{-\alpha(t-u)}M(t-u)-a]\,d\bar{G}(u).$$

The fact that

$$\int_0^\infty \{e^{-\alpha t}[1-G(t)]-a[1-\bar{G}(t)]\}\,dt=0$$

is easily checked and in order to apply Proposition 4 it remains to show that the function φ,

$$\varphi(t)=\int_0^t \{e^{-\alpha u}[1-G(u)]-a[1-\bar{G}(u)]\}\,du,$$

has a convergent integral on $[0, \infty)$. But

$$\varphi(t)=a\int_t^\infty [1-\bar{G}(u)]\,du-\int_t^\infty e^{-\alpha u}[1-G(u)]\,du.$$

However, by Fubini's theorem

$$\int_0^\infty dt\int_t^\infty e^{-\alpha u}[1-G(u)]\,du=\int_0^\infty e^{-\alpha u}[1-G(u)]\,du\int_0^u dt=\int_0^\infty ue^{-\alpha u}[1-G(u)]\,du<\infty.$$

And, similarly,

$$\int_0^\infty dt\int_t^\infty [1-\bar{G}(u)]\,du=\int_0^\infty u[1-\bar{G}(u)]\,du<\infty.$$

This completes the proof.

Proposition 9. *As* $t\to\infty$, $e^{-\alpha t}M(t)-a=o(t^{-1})$.

Proof. By Proposition 7 we have only to prove that

$$\lim_{t\to\infty} t\{e^{-\alpha t}[1-G(t)]-a[1-\bar{G}(t)]\}=0,$$

that

$$\varphi(t)=o(t^{-1}),$$

and that

$$\int_0^\infty t\{e^{-\alpha t}[1-G(t)]-a[1-\bar{G}(t)]\}\,dt$$

converges. But these facts are immediate.

Consider now again the integral equation for k_τ, $k_\tau(t)=E[Z(t)Z(t+\tau)]$,

$$k_\tau(t)=1-G(t+\tau)+m\int_t^{t+\tau} M(t+\tau-u)\,dG(u)+h''(1)\int_0^t M(t-u)\,M(t+\tau-u)\,dG(u)$$
$$+m\int_0^t k_\tau(t-u)\,dG(u),$$

$h''(1)$ assumed finite. Multiplying this by $e^{-\alpha\tau-2\alpha t}$, we verify the above-mentioned fact that

$$\lim_{t\to\infty} e^{-\alpha\tau-2\alpha t}k_\tau(t)=c,$$

uniformly in $\tau\geqslant 0$, using Proposition 2, since $m\int_0^\infty e^{-2\alpha t}dG(u)<1$. Subtracting c, we get the equation

$$e^{-\alpha\tau-2\alpha t}k_\tau(t)-c=e^{-\alpha\tau-2\alpha t}[1-G(t+\tau)]+e^{-\alpha t}m\int_t^{t+\tau}e^{-\alpha(t+\tau-u)}M(t+\tau-u)e^{-\alpha u}dG(u)$$

$$+h''(1)\int_0^t e^{-\alpha(t-u)}M(t-u)e^{-\alpha(t+\tau-u)}M(t+\tau-u)e^{-2\alpha u}dG(u)$$

$$-c\left[1-m\int_0^t e^{-2\alpha u}dG(u)\right]+m\int_0^t[e^{-\alpha\tau-2\alpha(t-u)}k_\tau(t-u)-c]e^{-2\alpha u}dG(u).$$

Let us check whether this equation is such that Proposition 5 is applicable. The first and second terms at the right-hand side are evidently integrable $[0,\infty)$, since $e^{-\alpha t}M(t)$, $t\geqslant 0$, is bounded. The two subsequent ones may be rewritten as follows,

$$h''(1)\int_0^t e^{-\alpha(t-u)}M(t-u)e^{-\alpha(t+\tau-u)}M(t+\tau-u)e^{-2\alpha u}dG(u)-c\left[1-m\int_0^t e^{-2\alpha u}dG(u)\right]$$

$$=h''(1)a^2\int_0^\infty e^{-2\alpha u}dG(u)-c\left[1-m\int_0^t e^{-2\alpha u}dG(u)\right]-h''(1)a^2\int_t^\infty e^{-2\alpha u}dG(u)$$

$$+h''(1)\int_0^t[e^{-\alpha(t-u)}M(t-u)e^{-\alpha(t+\tau-u)}M(t+\tau-u)-a^2]e^{-2\alpha u}dG(u).$$

But

$$h''(1)a^2\int_0^\infty e^{-2\alpha u}dG(u)-c\left[1-m\int_0^t e^{-2\alpha u}dG(u)\right]=-c\int_t^\infty e^{-2\alpha u}dG(u).$$

However,
$$\int_0^\infty dt\int_t^\infty e^{-2\alpha u}dG(u)=\int_0^\infty ue^{-2\alpha u}dG(u)<\infty.$$

Hence, it remains to prove that

$$\lim_{r\to\infty}\int_0^r dt\int_0^t[e^{-\alpha(t-u)}M(t-u)e^{-\alpha(t+\tau-u)}M(t+\tau-u)-a^2]e^{-2\alpha u}dG(u)$$

exists. Since the double integral equals

$$\int_0^r e^{-2\alpha u}dG(u)\int_0^{r-u}[e^{-\alpha t}M(t)e^{-\alpha(t+\tau)}M(t+\tau)-a^2]dt$$

it is enough to prove that

$$\int_0^\infty[e^{-\alpha t}M(t)e^{-\alpha(t+\tau)}M(t+\tau)-a^2]dt$$

converges and then apply Proposition 3. But by an elementary algebraic identity

$$e^{-\alpha t}M(t)e^{-\alpha(t+\tau)}M(t+\tau)-a^2=[e^{-\alpha(t+\tau)}M(t+\tau)-a][e^{-\alpha t}M(t)-a]$$
$$+a[e^{-\alpha t}M(t)-a]+a[e^{-\alpha(t+\tau)}M(t+\tau)-a].$$

Proposition 9, however, shows that

$$[e^{-\alpha(t+\tau)}M(t+\tau)-a][e^{-\alpha t}M(t)-a]=o(t^{-2}),$$

as $t\to\infty$, whereas the integral

$$\int_0^\infty [e^{-\alpha(t+\tau)}M(t+\tau)-a]\,dt$$

converges for any $\tau\geqslant 0$ by Proposition 8. This completes the proof of

Proposition 10. *If* $h''(1)<\infty$ *and* $\tau\geqslant 0$, *then*

$$\int_0^\infty [e^{-\alpha\tau-2\alpha t}k_\tau(t)-c]\,dt$$

converges.

Consider now for $\tau\geqslant 0$

$$E[(W(t+\tau)-W(t))^2]=\frac{k_0(t+\tau)}{a^2e^{2\alpha(t+\tau)}}+\frac{k_0(t)}{a^2e^{2\alpha t}}-2\frac{k_\tau(t)}{a^2e^{\alpha\tau+2\alpha t}}.$$

Since $W(t)\to W$ in mean square, as $t\to\infty$, a fact following from this relation, it is also true that

$$\lim_{\tau\to\infty}E[(W(t+\tau)-W(t))^2]=E[(W(t)-W)^2]$$

and that
$$\frac{k_\tau(t)}{ae^{\alpha(t+\tau)}}=E[W(t+\tau)Z(t)]$$

tends to some limit $k(t)$, as $\tau\to\infty$. Dividing the integral equation for $k_\tau(t)$ by $ae^{\alpha(t+\tau)}$ and then passing to the limit $\tau\to\infty$, we get, by dominated convergence,

$$k(t)=m\int_t^\infty e^{-\alpha u}dG(u)+h''(1)\int_0^t M(t-u)e^{-\alpha u}dG(u)+m\int_0^t k(t-u)e^{-\alpha u}dG(u).$$

Multiplication of this by $ae^{-\alpha t}$ and an application of Proposition 2 show that

$$\lim_{t\to\infty}ae^{-\alpha t}k(t)=c.$$

We have thus proved the first part of

Proposition 11. *If* $h''(1) < \infty$, *then*

$$ae^{-\alpha t}k(t) \to c, \quad as \quad t \to \infty, \quad and$$

$$\int_0^\infty [ae^{-\alpha t}k(t) - c]dt$$

converges

The rest of the deduction follows the pattern of earlier proofs, apply Proposition 5 to the integral equation for $ae^{-\alpha t}k(t) - c$.

Since $\lim_{\tau\to\infty} e^{-2\alpha(t+\tau)}k_0(t+\tau) = c$, it follows that

$$a^2E[(W(t) - W)^2] = c + e^{-2\alpha t}k_0(t) - 2ae^{-\alpha t}k(t) = [e^{-2\alpha t}k_0(t) - c] - 2[ae^{-\alpha t}k(t) - c],$$

and, hence the integral

$$\int_0^\infty E[(W(t) - W)^2]dt$$

converges by propositions 10 and 11. This completes the proof that $W(t) \to W$ almost surely, as $t \to \infty$.

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