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Compact groups and Dirichlet series

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1. In a previous paper [4] I have tried to show that the function-theoretic properties of Dirichlet series are associated with subalgebras of $L^1(-\infty, \infty)$, rather than with almost-periodic functions and compactifications of the line. Now I want to weaken that point by proving a convergence theorem for Dirichlet series, considered as Fourier series on a compact group, that does not apply to Fourier series of analytic type in general.

This paper carries on ideas introduced in [3], but does not refer to the theorems about cocycles proved there.

We start with a subgroup Γ of R_d, the discrete real line. The dual of Γ is a compact group K with normalized Haar measure σ. (The case where K is a circle is uninteresting, and is excluded.) A summable function f on K has Fourier series

$$f(x) \sim \sum_{\lambda} a(\lambda)\, \chi_\lambda(x) \quad (\lambda \in \Gamma), \tag{1}$$

where χ_λ is the character on K defined by $\chi_\lambda(x) = x(\lambda)$. For $p \geqslant 1$, $H^p(K)$ is the subspace of $L^p(K)$ consisting of those functions f in whose Fourier series $a(\lambda)=0$ for all $\lambda < 0$.

A *Dirichlet sequence* in Γ is a sequence of real numbers λ_n in Γ such that

$$0 \leqslant \lambda_1 < \lambda_2 < \ldots; \qquad \lambda_n \to \infty.$$

Let $H^p(K, \Lambda)$ be the space of all functions f in $H^p(K)$ in whose Fourier series $a(\lambda)=0$ except for λ in the Dirichlet sequence Λ. A Dirichlet sequence satisfies the *condition of Bohr* if there are constants c, k such that

$$(\lambda_{n+1} - \lambda_n)^{-1} \leqslant k \exp(c\lambda_n) \quad (n = 1, 2, \ldots).$$

This condition prevents λ_n from increasing too slowly, or from bunching up too densely. It is satisfied in the case of ordinary Dirichlet series: $\lambda_n = \log n$.

2. Let e_t be the element of K defined by $e_t(\lambda) = e^{it\lambda} (\lambda \in \Gamma)$, for each real number t. Then $K_0 = \{e_t\}$ is a dense subgroup of K of measure 0. To a measurable function f on K we associate functions $f_x(t) = f(x + e_t)$ defined and measurable, for almost every x, on the line. If f is in $L^2(K)$, then for almost every x the restriction f_x is square-summable over the line for the measure $d\mu(t) = (1+t^2)^{-1}\,dt$. Finally, for f in $H^2(K)$, $f_x(t)$ is (for almost every x) the boundary function of $f_x(z)$ analytic in the upper half-plane, and $f_x(t+iu)$ belongs uniformly to $L^2(\mu)$ for $u>0$. These results are given in [3].

To keep track of the analytic extensions we may write formally

$$f_x(t+iu) \sim \sum_{\lambda} a(\lambda)\, e^{-\lambda u}\, e^{i\lambda t}\, \chi_\lambda(x).$$

The relation between f and the analytic extensions $f_x(z)$ is general but abstract. Our result is that for certain Dirichlet series this relation is simple and direct.

Theorem. *If f is in $H^2(K, \Lambda)$, where Λ is a Dirichlet sequence in Γ satisfying the condition of Bohr, then the series*

$$f_x(t+iu) \sim \sum_{n=1}^{\infty} a(\lambda_n)\, e^{-\lambda_n u}\, e^{i\lambda_n t}\, \chi_{\lambda_n}(x) \tag{2}$$

converges in the half-plane $u>0$, except in a null-set of x.

For fixed x, (2) is a Dirichlet series in the ordinary sense, whose half-plane of convergence is determined by the coefficients in a manner described by a classical formula. These coefficients are $a(\lambda_n)\chi_{\lambda_n}(x)$, with x regarded as fixed. To establish convergence for each $u>0$ we have to show for almost every x that

$$F_x(\lambda) = \sum_{\lambda_n \leqslant \lambda} a(\lambda_n)\, \chi_{\lambda_n}(x)$$

is $0(e^{\lambda u})$ as $\lambda \to \infty$, for each $u>0$. Our hypothesis is simply

$$\sum_{n=1}^{\infty} |a(\lambda_n)|^2 < \infty.$$

In simple cases, for example if f is a trigonometric polynomial, it is easy to verify that the Fourier transform of $e^{-\lambda u}F_x(\lambda)$, with u fixed, is $f_x(t+iu)/(t+iu)$. The former function is therefore the inverse transform of the latter one. Now approximate f by the partial sums of its Fourier series (1). For almost every x and for each u, $f_x(t+iu)$ is the limit of the partial sums of (2) in the norm of $L^2(\mu)$. Taking inverse Fourier transforms gives the result we want: $e^{-\lambda u}F_x(\lambda)$ is, for almost every x, the inverse Fourier transform of $f_x(t+iu)/(t+iu)$. A consequence is that $e^{-\lambda u}F_x(\lambda)$ is square-summable over the line for almost all x. We have to go further and show this function is bounded.

Since f is in $L^2(K)$, it belongs also to $L^p(K)$ for $p<2$. Denote by P_u the singular measure of mass 1 obtained by placing the Poisson kernel

$$\frac{1}{\pi}\,\frac{u}{u^2+t^2}$$

on K_0. Then $P_u * f$ is in $L^p(K)$ for $p<2$, so that

$$\int |f_x(t+iu)|^p d\sigma(x) < \infty$$

for fixed t, u. By the Fubini theorem, for $1 < p < 2$ we have

$$\int_{-\infty}^{\infty} \left| \frac{f_x(t+iu)}{t+iu} \right|^p dt < \infty \qquad \text{(almost all } x).$$

The Young-Hausdorff theorem implies now that the inverse Fourier transform $e^{-\lambda u} F_x(\lambda)$ belongs to $L^q(-\infty, \infty)$ for $2 < q < \infty$:

$$\int_{-\infty}^{\infty} e^{-\lambda u q} |F_x(\lambda)|^q d\lambda < \infty. \tag{3}$$

Set $b_n = |F_x(\lambda_n)|$. Then (3) can be written

$$\sum_{n=1}^{\infty} e^{-\lambda_n u q} b_n^q (\lambda_{n+1} - \lambda_n) < \infty.$$

The condition of Bohr (which has not been used to this point) means that

$$\lambda_{n+1} - \lambda_n \geqslant k^{-1} \exp(-c\lambda_n)$$

for some constants k, c. Hence

$$\sum_{n=1}^{\infty} e^{-\lambda_n (uq+c)} b_n^q < \infty.$$

The terms of this series must tend to 0. On taking qth roots we find

$$b_n = o(e^{\lambda_n(u+\varepsilon)}), \quad \varepsilon = c/q.$$

Since q can be taken as large as we please, and since u is an arbitrary positive number, we have

$$F_x(\lambda_n) = o(e^{\lambda_n u})$$

for every $u > 0$, which is enough to prove the theorem.

3. Let us assume that Λ is a semigroup. Then $H^\infty(K, \Lambda)$ is an algebra, and each $H^p(K, \Lambda)$ admits multiplication by functions of $H^\infty(K, \Lambda)$. Say that a function f is *outer in* $H^p(K, \Lambda)$ if f belongs to $H^p(K, \Lambda)$, and the set of all products fg, g in $H^\infty(K, \Lambda)$, is dense in $H^p(K, \Lambda)$.

The definition is analogous to the definition of an outer function in $H^p(K)$: f should belong to $H^p(K)$, and the set of products fg, g in $H^\infty(K)$, should be dense in $H^p(K)$. The following result is proved by standard methods involving conjugate functions: *if f is outer in $H^2(K)$, and if f belongs to $H^q(K)$ for some $q > 2$, then f is* *outer in* $H^q(K)$. Unfortunately the proof does not work in $H^q(K, \Lambda)$, and no corresponding result is known.

It is obvious that each f outer in $H^p(K, \Lambda)$ is outer in $H^p(K)$, and therefore satisfies Beurling's condition

$$\int \log |f(x)|\, d\sigma = \log |a(0)| > -\infty.$$

Rudin has shown on the torus that the converse implication is false [5].

These remarks indicate that invariant subspaces for Dirichlet sequences are not yet well understood. Rudin has studied double power series, a special case which may illuminate the general problem. From one point of view it is more natural to consider power series in infinitely many variables, corresponding to Fourier series on an infinite-dimensional torus, and to ordinary Dirichlet series. Bohr called attention to the relation between power series in infinitely many variables and ordinary Dirichlet series [1].

4. Take for Λ the semigroup of ordinary Dirichlet series: $\lambda_n = \log n$. Define functions Z_u, $u > \frac{1}{2}$, by

$$Z_u(x) \sim \sum_{n=1}^{\infty} n^{-u} \chi_{\log n}(x).$$

Here Γ is the group of all $\log r$, r a positive rational, and K is an infinite-dimensional torus.

The restriction of Z_u to K_0 is formally the Riemann zeta-function. Our theorem gives some properties of Z_u on K.

Theorem. *For each* $u > \frac{1}{2}$, Z_u *and* Z_u^{-1} *are outer functions in* $H^q(K, \Lambda)$ *for every finite* q.

Z_u obviously belongs to $H^2(K, \Lambda)$. Its reciprocal has Fourier series

$$Z_u^{-1}(x) \sim \sum_{n=1}^{\infty} \mu(n)\, n^{-u}\, \chi_{\log u}(x);$$

μ is the Möbius function taking only the values $1, 0, -1$. Therefore Z_u^{-1} is in $H^2(K, \Lambda)$.

Z_u^2 is in $H^1(K, \Lambda)$ with Fourier series

$$Z_u^2(x) \sim \sum_{n=1}^{\infty} d(n)\, n^{-u}\, \chi_{\log n}(x),$$

where $d(n)$ is the number of divisors of n. Now $d(n) = 0(n^{\varepsilon})$ for every positive ε [2, p. 260]; therefore Z_u^2 is in $L^2(K)$, or Z_u is in $H^4(K, \Lambda)$. The coefficients of Z_u^{-2} are dominated by those of Z_u^2, so that Z_u^{-1} is also in $H^4(K, \Lambda)$.

Now we proceed by induction. The coefficient of Z_u^3 of order n is

$$n^{-u}\sum_{r|n} d(r).$$

The sum contains $d(n)$ terms, each smaller than n^{ε} (if n is large enough). It follows that Z_u^3 is in $H^2(K, \Lambda)$, or Z_u in $H^6(K, \Lambda)$, and the process continues. The coefficients of Z_u^{-n} are dominated by those of Z_u^n. Thus finally Z_u, Z_u^{-1} belong to $H^q(K, \Lambda)$ for every finite q.

If Z_u is not outer in $H^q(K, \Lambda)$ there is some h in $L^p(K)$, $p^{-1}+q^{-1}=1$, perpendicular to $Z_u g$ for each g in $H^\infty(K, \Lambda)$ but not to the constant functions. (For if $Z_u g$ can approximate constant functions, it can approximate any function in $H^q(K, \Lambda)$.) The orthogonality relations mean that the integral of $\bar{h}Z_u$ multiplied by any trigonometric polynomial with frequencies in Λ is 0. Now $\bar{h}Z_u$ belongs to $L^r(K)$ for each $r<p$. In the dual space $L^s(K)$, $r^{-1}+s^{-1}=1$, Z_u^{-1} is the limit of such trigonometric polynomials. Hence the integral of $\bar{h}=\bar{h}Z_u Z_u^{-1}$ is 0, a contradiction. So Z_u must have been outer in $H^q(K, \Lambda)$. A similar argument shows that Z_u^{-1} is outer.

5. If f is outer merely in $H^2(K)$, the analytic function $f_x(z)$ is an outer function in the upper half-plane in the ordinary sense, for almost every x [3]. So $f_x(z)$ never vanishes on such half-planes. It seems possible that more can be said if f is outer in $H^2(K, \Lambda)$ and $f_x(z)$ is the sum of a convergent Dirichlet series. A plausible conjecture would be: *if Λ is a semigroup and satisfies the condition of Bohr, so the series (2) converges in the upper half-plane for almost every x when f is in $H^2(K, \Lambda)$, then $f_x(z)$ never vanishes on half-planes of convergence when f is outer in $H^2(K, \Lambda)$.*

Some doubt is thrown on the conjecture, or at least on the ease of proving the conjecture, by this consequence of it: *if the conjecture is true, then the Riemann hypothesis is false*. For the Riemann hypothesis implies the convergence of

$$\sum_{n=1}^{\infty} \mu(n)\, n^{-u}$$

for $u>\frac{1}{2}$ [6, p. 315], thus establishing the half-plane over K_0 as a half-plane of convergence for the outer function Z_u^{-1}. Nevertheless Z_u^{-1} has a zero in this half-plane, corresponding to the pole of zeta at 1, contradicting the conjecture.

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