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Uniform bound for Hecke L-functions

by

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1. Introduction

Our principal aim in the present article is to establish a uniform hybrid bound for individual values on the critical line of Hecke L-functions associated with cusp forms over the full modular group. This is rendered in the statement that for $t \geqslant 0$,

$$H_j\left(\tfrac{1}{2}+it\right) \ll (\varkappa_j+t)^{1/3+\varepsilon}, \tag{1.1}$$

$$H_{j,k}\left(\tfrac{1}{2}+it\right) \ll (k+t)^{1/3+\varepsilon}, \tag{1.2}$$

with the common notation to be made precise in the course of discussion.

Most of the arithmetically significant Dirichlet series, such as the Riemann zeta-function $\zeta(s)$, Dirichlet L-functions, and Hecke L-functions associated with various cusp forms, satisfy Riemannian functional equations connecting values at $s=\sigma+it$ and $1-s$ of the respective functions. Essentially best possible estimates for these functions near the lines $\sigma=1$ and $\sigma=0$ can usually be deduced from the definition of the respective functions and their functional equations. From this, bounds in the critical strip $0<\sigma<1$, in particular on the critical line $\sigma=\frac{1}{2}$, follow readily via the Phragmén–Lindelöf convexity principle; thus they are called *convexity bounds*. In general, there is a quantity $B(g,t)$ characterising the size of a function $g\left(\frac{1}{2}+it\right)$ of the above kind in a given t-range in such a way that the convexity bound is stated as

$$g\left(\tfrac{1}{2}+it\right) \ll B(g,t)^{1/2+\varepsilon}, \quad t>0, \tag{1.3}$$

with the usual usage of the symbol ε (see Convention 1 at the end of this section). For instance, $B(\zeta,t)=t^{1/2}$, or perhaps more naturally $B(\zeta^2,t)=t$. In view of the generalised

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Lindelöf hypothesis asserting that the exponent on the right of (1.3) be ε, any improvement upon (1.3), i.e., *subconvexity bounds*, are of considerable interest. One may call $\varpi<\frac{1}{2}$ a *Lindelöf constant*, provided that (1.3) holds with $\varpi+\varepsilon$ in place of $\frac{1}{2}+\varepsilon$. The classical Lindelöf constant for ζ^2 is $\varpi=\frac{1}{3}$, which has been successively improved, though not very drastically. A natural task would then be to achieve at least the same for wide classes of Dirichlet series. We shall consider this fundamental problem dealing mainly with Hecke L-functions associated with real-analytic cusp forms.

To this end, we shall first make our objects precise; for details we refer to the monograph [23]. Thus, let Γ be the full modular group $\mathrm{PSL}_2(\mathbf{Z})$; throughout the sequel we shall work with Γ, although our argument appears to be effective in a considerably general setting. Let $L^2(\Gamma\backslash\mathbf{H})$ be the Hilbert space composed of all Γ-automorphic functions on the hyperbolic upper half-plane $\mathbf{H}=\{x+iy:x\in\mathbf{R},\, y>0\}$ which are square integrable over the quotient $\Gamma\backslash\mathbf{H}$ with respect to the hyperbolic measure. If a function in $L^2(\Gamma\backslash\mathbf{H})$ is an eigenfunction of the hyperbolic Laplacian $\mathcal{L}=-y^2(\partial_x^2+\partial_y^2)$, then it is called a *real-analytic cusp form*. The subspace spanned by all such functions has a maximal orthonormal system $\{\psi_j:j=1,2,\ldots\}$, where $\mathcal{L}\psi_j=\left(\frac{1}{4}+\varkappa_j^2\right)\psi_j$ with $0<\varkappa_1\leqslant\varkappa_2\leqslant\ldots$, and

$$\psi_j(x+iy)=\sqrt{y}\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\varrho_j(n)K_{i\varkappa_j}(2\pi|n|y)\exp(2\pi inx),\quad x+iy\in\mathbf{H}, \tag{1.4}$$

with K_ν being the K-Bessel function of order ν. The $\varrho_j(n)$ are called the *Fourier coefficients of* ψ_j. In addition, we may suppose that ψ_j are simultaneous eigenfunctions of all Hecke operators with corresponding eigenvalues $\tau_j(n)\in\mathbf{R}$; that is, for each positive integer n,

$$\frac{1}{\sqrt{n}}\sum_{ad=n}\sum_{b \bmod d}\psi_j\left(\frac{az+b}{d}\right)=\tau_j(n)\psi_j(z),\quad z\in\mathbf{H}. \tag{1.5}$$

We have, for any $m,n>0$,

$$\tau_j(m)\tau_j(n)=\sum_{d|(m,n)}\tau_j\left(\frac{mn}{d^2}\right). \tag{1.6}$$

We may assume further that

$$\psi_j(-\bar{z})=\epsilon_j\psi_j(z),\quad \epsilon_j=\pm1. \tag{1.7}$$

Then the Hecke L-function associated with ψ_j is defined by

$$H_j(s)=\sum_{n=1}^{\infty}\tau_j(n)n^{-s},\quad \mathrm{Re}\, s>1. \tag{1.8}$$

This continues to an entire function, satisfying the functional equation

$$H_j(s)=\chi_j(s)H_j(1-s), \tag{1.9}$$

with

$$\chi_j(s)=\epsilon_j\pi^{2s-1}\frac{\Gamma\big(\frac12\big(1-s+i\varkappa_j+\frac12(1-\epsilon_j)\big)\big)\,\Gamma\big(\frac12\big(1-s-i\varkappa_j+\frac12(1-\epsilon_j)\big)\big)}{\Gamma\big(\frac12\big(s+i\varkappa_j+\frac12(1-\epsilon_j)\big)\big)\,\Gamma\big(\frac12\big(s-i\varkappa_j+\frac12(1-\epsilon_j)\big)\big)} \tag{1.10}$$

$$=2^{2s-1}\pi^{2(s-1)}\Gamma(1-s+i\varkappa_j)\Gamma(1-s-i\varkappa_j)[\epsilon_j\cosh\pi\varkappa_j-\cos\pi s]. \tag{1.11}$$

Since we have $\tau_j(n)\ll n^{1/4+\varepsilon}$ uniformly in ψ_j (see [23, (3.1.18)]), the equation (1.9) implies that $H_j(s)$ is of polynomial growth with respect to both s and $\varkappa_j$ in any fixed vertical strip of the s-plane.

We shall also need holomorphic cusp forms over Γ, and corresponding Hecke L-functions. Thus, if ψ is holomorphic over $\mathbf{H}$, vanishing at $i\infty$, and $\psi(z)(dz)^k$ with a positive integer k is Γ-invariant, then we call it a *holomorphic cusp form of weight* $2k$. The space composed of all such functions is a finite-dimensional Hilbert space. We denote the dimension by $\vartheta(k)$, and let $\{\psi_{j,k}:1\leqslant j\leqslant\vartheta(k)\}$ be a corresponding orthonormal basis. Note that $\vartheta(k)=0$ for $k\leqslant 5$. The Fourier coefficient $\varrho_{j,k}(n)$ of $\psi_{j,k}$ is defined by the expansion

$$\psi_{j,k}(z)=\sum_{n=1}^{\infty}n^{k-1/2}\varrho_{j,k}(n)\exp(2\pi inz),\quad z\in\mathbf{H}. \tag{1.12}$$

We may assume that $\psi_{j,k}$ are simultaneous eigenfunctions of all Hecke operators, so that there exist real numbers $\tau_{j,k}(n)$ such that

$$\frac{1}{\sqrt{n}}\sum_{ad=n}\left(\frac{a}{d}\right)^k\sum_{b \bmod d}\psi_{j,k}\left(\frac{az+b}{d}\right)=\tau_{j,k}(n)\psi_{j,k}(z),\quad z\in\mathbf{H}. \tag{1.13}$$

Then the Hecke L-function associated with $\psi_{j,k}$ is defined by

$$H_{j,k}(s)=\sum_{n=1}^{\infty}\tau_{j,k}(n)n^{-s},\quad \operatorname{Re}s>1. \tag{1.14}$$

This continues to an entire function; and it satisfies the functional equation

$$H_{j,k}(s)=-2^{2s-1}\pi^{2(s-1)}\Gamma\big(\tfrac12-s+k\big)\Gamma\big(\tfrac32-s-k\big)\cos(\pi s)H_{j,k}(1-s). \tag{1.15}$$

Now, returning to our original subject, let H be a particular function among the H_j and $H_{j,k}$. Comparing (1.11) and (1.15) with the functional equation

$$\zeta^2(s)=2^{2s-1}\pi^{2(s-1)}\Gamma^2(1-s)(1-\cos\pi s)\zeta^2(1-s), \tag{1.16}$$

and invoking what is stated above about the size of $\zeta^2(\frac{1}{2}+it)$, we might put $B(H,t)=t$; and an expected subconvexity bound would be

$$H\left(\tfrac{1}{2}+it\right) \ll t^{1/3+\varepsilon}, \quad t \geqslant 1. \tag{1.17}$$

In the case of holomorphic cusp forms, this was proved by A. Good [4] as a corollary of an asymptotic formula for the mean square of $H(\frac{1}{2}+it)$, which he achieved by an appeal to the spectral theory of real-analytic automorphic functions (see also [22]). An alternative and conceptually simpler proof, based solely on functional properties of H and its twists with additive characters (see (8.8) below), was devised by the first author [8]. His argument turned out to be applicable also to the real-analytic case, as shown by T. Meurman [19], yielding a proof of (1.17). Good's mean value result itself was later extended to this case by the first author [10], which implies (1.17) in yet another way.

In the light of these developments, it should be desirable to have bounds uniform in ψ_j. More precisely, (1.9)–(1.11) suggest that we may choose $B(H_j,t)=\varkappa_j+t$ for $t\geqslant 0$, and hence a hypothetical uniform subconvexity bound would be (1.1), although (1.1) is not subconvex under a particular localization of parameters, to be made precise following (3.50) below. As a support, the first author [13] showed recently that

$$H_j\left(\tfrac{1}{2}+it\right) \ll t^{1/3+\varepsilon}, \quad t \gg \varkappa_j^{3/2-\varepsilon}, \tag{1.18}$$

which supersedes Meurman's estimate, with respect to uniformity. This is in fact a consequence of the following result on the spectral mean square (loc. cit.):

$$\sum_{K\leqslant \varkappa_j\leqslant K+G} \alpha_j \left|H_j\left(\tfrac{1}{2}+it\right)\right|^2 \ll (GK+t^{2/3})^{1+\varepsilon}, \quad t\geqslant 0,\ 1\leqslant G\leqslant K, \tag{1.19}$$

where $\alpha_j=|\varrho_j(1)|^2/\cosh\pi\varkappa_j$. Hence, when t is relatively large, the bound (1.1) holds indeed, in view of the lower bound $\alpha_j\gg\varkappa_j^{-\varepsilon}$ due to H. Iwaniec [7]. The assertion (1.19) has an essential relevance to our discussion in §8, where a brief description of its proof is given.

The real interest is, however, in the range

$$0\leqslant t\leqslant \varkappa_j^{3/2}, \tag{1.20}$$

since here the discrete quantity $\varkappa_j$ seems to overwhelm the influence of the continuous parameter t. In this circumstance, what A. Ivić [5] had achieved prior to (1.18) was a breakthrough. He succeeded in proving (1.1) for $t=0$ by a method quite different from those previously applied; see [14] and [25] for the developments preceding [5]. His starting point was an identity due to the second author [23, Lemma 3.8] for the spectral average

$$\sum_{j=1}^{\infty} \alpha_j \tau_j(f) H_j^2\left(\tfrac{1}{2}\right) h(\varkappa_j), \tag{1.21}$$

where h is a weight function satisfying certain regularity and decay condition. As is precisely presented in Lemma 3 below, this identity transforms the sum (1.21) into a purely arithmetic expression involving, in particular, the divisor function $d(n)$, which Ivić could exploit effectively. His bound for $H_j\left(\frac{1}{2}\right)$ is a corollary of the following result thus obtained:

$$\sum_{K\leqslant\varkappa_j\leqslant K+G} \alpha_j H_j^3\left(\tfrac{1}{2}\right) \ll GK^{1+\varepsilon}, \quad 1\leqslant G\leqslant K, \tag{1.22}$$

and the assertion $H_j\left(\frac{1}{2}\right)\geqslant 0$ due to S. Katok and P. Sarnak [15]. It should be remarked that a spectral sum of cubic powers of $H_j\left(\frac{1}{2}\right)$ appeared for the first time in an explicit spectral expansion of the weighted fourth moment of $\zeta\left(\frac{1}{2}+it\right)$ due to the second author [20] (see also [23, Chapter 4]). Motivated by this advance with the cubic moment, the first author [12] turned to the fourth moment, establishing

$$\sum_{K\leqslant\varkappa_j\leqslant K+G} \alpha_j H_j^4\left(\tfrac{1}{2}\right) \ll GK^{1+\varepsilon}, \quad K^{1/3}\leqslant G\leqslant K. \tag{1.23}$$

The same identity for the sum (1.21) played again a crucial role in his proof. Also, as a new basic ingredient, use was made of an explicit spectral decomposition of the binary additive divisor sum

$$D(f;W)=\sum_{n=1}^{\infty} d(n)d(n+f)W\left(\frac{n}{f}\right), \quad f\geqslant 1, \tag{1.24}$$

due to the second author [21] (see Lemma 5 below). It should be stressed that (1.23) proves Ivić's bound for $H_j\left(\frac{1}{2}\right)$ without the non-negativity assertion quoted after (1.22).

Having stated this, it is now natural to investigate the spectral fourth moment

$$\mathcal{S}(G,K)=\sum_{K\leqslant\varkappa_j\leqslant K+G} \alpha_j \left|H_j\left(\tfrac{1}{2}+it\right)\right|^4, \quad t\geqslant 0,\ 1\leqslant G\leqslant K, \tag{1.25}$$

trying to retain the same bound as (1.23) with uniformity in the parameter t. Indeed, it gives rise to a proof of (1.1):

THEOREM 1. *Let K be sufficiently large, and*

$$G=(K+t)^{4/3}K^{-1+\varepsilon}, \quad 0\leqslant t\ll K^{3/2-\varepsilon}. \tag{1.26}$$

Then we have

$$\mathcal{S}(G,K)\ll GK^{1+\varepsilon}. \tag{1.27}$$

In particular, the bound (1.1) holds uniformly for any $t\geqslant 0$ and for any real-analytic cusp form ψ_j.

This embodies the main result of the present article. The second assertion follows immediately from (1.19) and (1.27). For orientation, it should be remarked that the estimate

$$\mathcal{S}(G,K) \ll (K+t)^{2+\varepsilon}, \tag{1.28}$$

with the same specification as in (1.25), follows immediately from (3.50) below.

The proof of (1.27) that we shall develop below is in principle an elaboration of the argument in [12]. However, the prerequisite that the whole of our procedure be uniform in the parameter t necessitates major changes of argument as well. First of all, we are unable to exploit a peculiar property of the central values of Hecke series, on which both [5] and [12] rely via the explicit formula for (1.21). Thus instead we appeal to the sum formula of R. W. Bruggeman [1] and N. V. Kuznetsov [16], and in tandem to the sum formula of Voronoĭ. This is made at an earlier phase, i.e., §4, of the reduction process, and causes already a considerable complication; nevertheless, it leads us to an instance of the additive divisor sum $D(f;W)$. The subsequent procedure is far more involved than the corresponding steps in [12], as will be seen in §§5 and 6. Moreover, only when t is relatively small, i.e., $t \leqslant K^{2/3}$, the end result thus reached is appropriate for an application of the spectral large sieve (see Lemma 7 below) to produce what we desire. The analogy with [12] ceases here. For larger t in the range (1.26), the same combination yields only an assertion short of (1.27). Thence, we enter into the second phase of our discussion. That is about a spectral hybrid mean value of Hecke series, an implement to extract (1.27) out of the aforementioned end result. This part might raise a particular interest, because a significant contribution of holomorphic cusp forms takes place. It is thus suggested that what we deal with in the present article is of quite a different nature from any problem in analytic number theory to which the spectral theory of automorphic forms was applied, e.g., the fourth moment of the Riemann zeta-function, where the role of holomorphic cusp forms was in fact negligible.

More precisely, the spectral hybrid mean value is concerned with the expression

$$\mathcal{T}(K,t) = \sum_{K \leqslant \varkappa_j \leqslant 2K} \alpha_j H_j^2\left(\tfrac{1}{2}\right) \left| H_j\left(\tfrac{1}{2}+it\right)\right|^2. \tag{1.29}$$

THEOREM 2. *We have, for any $K,t \geqslant 0$,*

$$\mathcal{T}(K,t) \ll \left(K^2 + t^{4/3}\right)^{1+\varepsilon}. \tag{1.30}$$

This is in fact an auxiliary result; thus it should be noted that no attempt is made to prove the best result obtainable by present-day methods. The proof developed in

§7 starts with the explicit formula for (1.21) and follows to some extent the arguments of [12] and [23, §3.4]. We encounter an additive divisor sum of the type

$$D(f;\alpha,\beta;W)=\sum_{n=1}^{\infty}\sigma_\alpha(n)\sigma_\beta(n+f)W\left(\frac{n}{f}\right),\quad f\geqslant 1, \tag{1.31}$$

where $\sigma_\alpha(n)=\sum_{d|n}d^\alpha$, and in our situation α and β are complex. We can appeal to an explicit formula for this sum due to the second author [21]; however, the subsequent discussion is quite subtle. We shall have two instances of $D(f;\alpha,\beta;W)$; and to deal with the first, we require a counterpart of (1.18)–(1.19) for holomorphic cusp forms. This is precisely the peculiarity of our problem mentioned above. Thus, uniformly for any $\psi_{j,k}$,

$$H_{j,k}\left(\tfrac{1}{2}+it\right)\ll t^{1/3+\varepsilon},\quad t\gg k^{3/2-\varepsilon}. \tag{1.32}$$

Also, under the same specification as in (1.19),

$$\sum_{K\leqslant k\leqslant K+G}\ \sum_{j=1}^{\vartheta(k)}\alpha_{j,k}\left|H_{j,k}\left(\tfrac{1}{2}+it\right)\right|^2\ll(GK+t^{2/3})^{1+\varepsilon}, \tag{1.33}$$

where $\alpha_{j,k}=8(4\pi)^{-2k-1}(2k-1)!\,|\varrho_{j,k}(1)|^2$. The former is of course a consequence of the latter together with an obvious analogue of the lower bound for α_j. On the other hand, with another instance of $D(f;\alpha,\beta;W)$, we require instead (1.19) in an analogous configuration. Therefore, the holomorphic and the real-analytic cusp forms stand at parity in our discussion of $\mathcal{T}(K,t)$.

A proof of (1.33) is given in the final section. It depends on an observation about a crucial role played by the divisor function in our discussion so far laid out. We are then led not only to (1.33) but also to the following counterpart of Theorem 1:

THEOREM 3. *We have, under* (1.26),

$$\sum_{K\leqslant k\leqslant K+G}\ \sum_{j=1}^{\vartheta(k)}\alpha_{j,k}\left|H_{j,k}\left(\tfrac{1}{2}+it\right)\right|^4\ll GK^{1+\varepsilon}. \tag{1.34}$$

In particular, the bound (1.2) *holds uniformly for any* $t\geqslant 0$ *and for any holomorphic cusp form* $\psi_{j,k}$.

With this, we look into the structure of our argument, in order to envisage further extensions of our main result (1.1); and we come to a circle of problems on the size of Rankin–Selberg L-functions. We shall indicate that our method is capable of yielding new results in such a generality as well.

In passing, it should be added that the bounds (1.1)–(1.2) could be stated more uniformly, if we refer to basic terms from the theory of the Γ-automorphic representations of the Lie group $\mathrm{PSL}_2(\mathbf{R})$, which can be found in [3], for instance. Thus, we have

$$H_V\left(\tfrac{1}{2}+it\right) \ll (|\nu_V|+t)^{1/3+\varepsilon}, \tag{1.35}$$

uniformly for $t\geqslant 0$ and for any Hecke invariant irreducible cuspidal representation V with the spectral data ν_V, occurring in $L^2(\Gamma\backslash\mathrm{PSL}_2(\mathbf{R}))$. In the final section we shall make a digression relevant to this aspect of our work.

Throughout our discussion, the common symbol ε plays a basic role. Here we make precise our usage of it, in terms of a convention. This is to avoid any confusion that might arise otherwise:

Convention 1. The symbol ε denotes a sufficiently small positive parameter, which in general takes different values at each occurrence. An $\varepsilon_0>0$ could actually be fixed initially so that a local value of ε is an integral multiple of ε_0, and each inequality holds with an implied constant which depends solely on our choice of ε_0. Thus, except being stated together with extra dependencies, the notation $X\ll Y$, with $Y>0$, implies that $|X|/Y$ is bounded by a constant depending on ε_0 at most, and $X\asymp Y$ means that $1\ll|X/Y|\ll 1$. It is implicit in our argument how to choose multiples of ε_0 to have a particular inequality and a specific reasoning valid.

Notation and conventions, including those above, are introduced where they are needed for the first time, and will continue to be effective thereafter.

2. Basic identities

Our proof of Theorem 1 is comprised of a series of various transformations and approximations applied to spectral and arithmetic objects. Here we collect identities which will give rise to fundamental transformations of $\mathcal{S}(G,K)$ in later sections.

LEMMA 1. *Let $h(r)$ be even and regular in the strip $|\mathrm{Im}\, r|<\frac{1}{4}+\varepsilon$, and there $|h(r)|\ll (1+|r|)^{-2-\varepsilon}$. Put*

$$\hat{h}(x)=\frac{2i}{\pi}\int_{-\infty}^{\infty}\frac{rh(r)}{\cosh\pi r}J_{2ir}(x)\,dr=\frac{2i}{\pi}\int_{0}^{\infty}\frac{rh(r)}{\cosh\pi r}(J_{2ir}(x)-J_{-2ir}(x))\,dr, \tag{2.1}$$

with J_ν being the J-Bessel function of order ν. Then we have

$$\begin{aligned}\sum_{j=1}^{\infty}\alpha_j\tau_j(m)\tau_j(n)h(\varkappa_j)=&-\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\sigma_{2ir}(m)\sigma_{2ir}(n)}{(mn)^{ir}|\zeta(1+2ir)|^2}h(r)\,dr\\&+\frac{\delta_{m,n}}{\pi^2}\int_{-\infty}^{\infty}r\tanh(\pi r)h(r)\,dr\\&+\sum_{l=1}^{\infty}\frac{1}{l}S(m,n;l)\,\hat{h}\left(\frac{4\pi\sqrt{mn}}{l}\right),\end{aligned}\tag{2.2}$$

where $\delta_{m,n}$ is the Kronecker delta, and

$$S(m,n;l)=\sum_{\substack{q=1\\(q,l)=1}}^{l}\exp\left(\frac{2\pi i(mq+n\tilde{q})}{l}\right),\quad q\tilde{q}\equiv 1 \bmod l,\tag{2.3}$$

is a Kloosterman sum.

This is a refined version of the Kloosterman-spectral sum formula of Bruggeman and Kuznetsov. See [23, §2.6] for a proof.

LEMMA 2. *We have, for any integers $k,m,n\geqslant 1$,*

$$\begin{aligned}\sum_{j=1}^{\vartheta(k)}\alpha_{j,k}\tau_{j,k}(m)\tau_{j,k}(n)=&\frac{1}{2\pi^2}\delta_{m,n}(2k-1)\\&+\frac{(-1)^k}{\pi}(2k-1)\sum_{l=1}^{\infty}\frac{1}{l}S(m,n;l)J_{2k-1}\left(\frac{4\pi\sqrt{mn}}{l}\right).\end{aligned}\tag{2.4}$$

This is the sum formula of H. Petersson. A proof is given in [23, §2.2].

LEMMA 3. *Let $h(r)$ be an even entire function satisfying*

$$h\left(\pm\tfrac{1}{2}i\right)=0,\tag{2.5}$$

and

$$h(r)\ll\exp(-\varepsilon|r|^2),\tag{2.6}$$

in any fixed horizontal strip. Put

$$\Psi^+(x;h)=2\pi\int_0^1\left(y(1-y)\left(1+\frac{y}{x}\right)\right)^{-1/2}\int_{-\infty}^{\infty}rh(r)\tanh(\pi r)\left(\frac{y(1-y)}{x+y}\right)^{ir}dr\,dy\tag{2.7}$$

and

$$\Psi^-(x;h)=\int_0^\infty\left(\int_{(a)}x^s(y(y+1))^{s-1}\frac{\Gamma^2(\frac12-s)}{\Gamma(1-2s)\cos\pi s}\,ds\right)\times\left(\int_{-\infty}^\infty rh(r)\left(\frac{y}{y+1}\right)^{ir}dr\right)dy, \tag{2.8}$$

with $-\frac32<a<\frac12$, $a\neq-\frac12$, *where* (a) *is the vertical line* $\mathrm{Re}\,s=a$. *Then we have, for any* $f\geqslant1$,

$$\sum_{j=1}^\infty\alpha_j\tau_j(f)H_j^2(\tfrac12)h(\varkappa_j)=\sum_{\nu=1}^7\mathcal{H}_\nu(f;h), \tag{2.9}$$

where

$$\mathcal{H}_1(f;h)=-i\frac{2}{\pi^3}\frac{d(f)}{\sqrt f}\times\int_{-\infty}^\infty\left(2(\gamma_E-\log2\pi\sqrt f)\frac{\Gamma'}{\Gamma}\left(\frac12+ir\right)+\left(\frac{\Gamma'}{\Gamma}\left(\frac12+ir\right)\right)^2\right)rh(r)\,dr, \tag{2.10}$$

with the Euler constant γ_E, *and*

$$\mathcal{H}_2(f;h)=\frac{1}{\pi^3}\sum_{m=1}^\infty m^{-1/2}d(m)d(m+f)\Psi^+\left(\frac mf;h\right), \tag{2.11}$$

$$\mathcal{H}_3(f;h)=\frac{1}{\pi^3}\sum_{m=1}^\infty (m+f)^{-1/2}d(m)d(m+f)\Psi^-\left(1+\frac mf;h\right), \tag{2.12}$$

$$\mathcal{H}_4(f;h)=\frac{1}{\pi^3}\sum_{m=1}^{f-1} m^{-1/2}d(m)d(f-m)\Psi^-\left(\frac mf;h\right), \tag{2.13}$$

$$\mathcal{H}_5(f;h)=-\frac{1}{2\pi^3}\frac{d(f)}{\sqrt f}\Psi^-(1;h), \tag{2.14}$$

$$\mathcal{H}_6(f;h)=-\frac{12}{\pi^2}f^{1/2}\sigma_{-1}(f)h'(-\tfrac12 i), \tag{2.15}$$

$$\mathcal{H}_7(f;h)=-\frac1\pi\int_{-\infty}^\infty f^{-ir}\sigma_{2ir}(f)\frac{|\zeta(\frac12+ir)|^4}{|\zeta(1+2ir)|^2}h(r)\,dr. \tag{2.16}$$

This is [23, Lemma 3.8]. Note that we have invoked the formulas [23, (3.3.41) and (3.3.45)]. The decay condition on h could be far less stringent than (2.6).

LEMMA 4. *Let* $D(f;\alpha,\beta;W)$ *be defined by* (1.31), *where* W *is a smooth function compactly supported in the positive reals, and* $|\mathrm{Re}\,\alpha|,|\mathrm{Re}\,\beta|<\varepsilon$. *Then we have*

$$D(f;\alpha,\beta;W)=[D_r+D_d+D_h+D_c](f;\alpha,\beta;W), \tag{2.17}$$

where

$$D_r(f;\alpha,\beta;W)=\int_0^\infty W(x)Y_f(x;\alpha,\beta)\,dx, \tag{2.18}$$

$$D_d(f;\alpha,\beta;W)=2(2\pi)^{\beta-1}f^{(\alpha+\beta+1)/2}\sum_{j=1}^\infty \alpha_j\tau_j(f) \times H_j\big(\tfrac12(1-\alpha-\beta)\big)\,H_j\big(\tfrac12(1+\alpha-\beta)\big)\,(\Psi_+ +\epsilon_j\Psi_-)(i\varkappa_j;\alpha,\beta;W), \tag{2.19}$$

$$D_h(f;\alpha,\beta;W)=2(2\pi)^{\beta-1}f^{(\alpha+\beta+1)/2}\sum_{k=1}^\infty\sum_{j=1}^{\vartheta(k)}(-1)^k\alpha_{j,k}\tau_{j,k}(f) \times H_{j,k}\big(\tfrac12(1-\alpha-\beta)\big)\,H_{j,k}\big(\tfrac12(1+\alpha-\beta)\big)\,\Psi_+\big(k-\tfrac12;\alpha,\beta;W\big), \tag{2.20}$$

$$D_c(f;\alpha,\beta;W)=4(2\pi)^{\beta-2}f^{(\alpha+\beta+1)/2} \times\int_{-\infty}^\infty f^{-i\varkappa}\sigma_{2i\varkappa}(f)\frac{Z(i\varkappa;\alpha,\beta)}{\zeta(1+2i\varkappa)\zeta(1-2i\varkappa)}(\Psi_+ +\Psi_-)(i\varkappa;\alpha,\beta;W)\,d\varkappa. \tag{2.21}$$

Here

$$\begin{aligned} Y_f(x;\alpha,\beta)&=\sigma_{1+\alpha+\beta}(f)\frac{\zeta(1+\alpha)\zeta(1+\beta)}{\zeta(2+\alpha+\beta)}x^\alpha(x+1)^\beta\\ &\quad+f^\alpha\sigma_{1-\alpha+\beta}(f)\frac{\zeta(1-\alpha)\zeta(1+\beta)}{\zeta(2-\alpha+\beta)}(x+1)^\beta\\ &\quad+f^\beta\sigma_{1+\alpha-\beta}(f)\frac{\zeta(1+\alpha)\zeta(1-\beta)}{\zeta(2+\alpha-\beta)}x^\alpha\\ &\quad+f^{\alpha+\beta}\sigma_{1-\alpha-\beta}(f)\frac{\zeta(1-\alpha)\zeta(1-\beta)}{\zeta(2-\alpha-\beta)}, \end{aligned} \tag{2.22}$$

$$\begin{aligned} Z(\xi;\alpha,\beta)&=\zeta\big(\tfrac12(1-\alpha-\beta)+\xi\big)\,\zeta\big(\tfrac12(1+\alpha-\beta)+\xi\big)\\ &\quad\times\zeta\big(\tfrac12(1-\alpha-\beta)-\xi\big)\,\zeta\big(\tfrac12(1+\alpha-\beta)-\xi\big), \end{aligned} \tag{2.23}$$

and

$$\Psi_+(\xi;\alpha,\beta;W)=\frac{1}{4\pi i}\cos\big(\tfrac12\pi\alpha\big)\int_{-i\infty}^{i\infty}\cos(\pi s)\Gamma(s+\xi)\Gamma(s-\xi) \times\Gamma\big(\tfrac12(1-\alpha-\beta)-s\big)\,\Gamma\big(\tfrac12(1+\alpha-\beta)-s\big)\,W^*\big(s+\tfrac12(\alpha+\beta+1)\big)\,ds, \tag{2.24}$$

$$\Psi_-(\xi;\alpha,\beta;W)=\frac{1}{4\pi i}\cos(\pi\xi)\int_{-i\infty}^{i\infty}\sin\big(\pi\big(s+\tfrac12\beta\big)\big)\Gamma(s+\xi)\Gamma(s-\xi) \times\Gamma\big(\tfrac12(1-\alpha-\beta)-s\big)\,\Gamma\big(\tfrac12(1+\alpha-\beta)-s\big)\,W^*\big(s+\tfrac12(\alpha+\beta+1)\big)\,ds, \tag{2.25}$$

where W^* is the Mellin transform of W, and the last integrals are such that the path separates the poles of the first three factors in the integrand and those of the remaining factors to the left and the right of the path, respectively.

Proof. This is asserted in [21, (3.45)–(3.47)], save for a minor modification applied to the D_h-term. Also the formulas [21, (3.42) and (3.49)] are invoked. The above condition on the real parts of α and β is imposed only for the sake of convenience, and thus by no means essential. In fact, the explicit formula (2.17) holds for all complex α and β in the context of analytic continuation. In [21] it is implicitly assumed that W is real-valued, but in fact the argument there allows us to drop it; hence in the above, W can be complex-valued.

LEMMA 5. *Let $D(f;W)$ be defined by (1.24), with W being as in the previous lemma, and let*

$$\begin{aligned}Y_f(u) = \log(u)\log(u+1) + \left(c - \log f + 2\frac{\sigma_1'}{\sigma_1}(f)\right)\log(u(u+1))\\ + (c-\log f)^2 - 4\left(\frac{\zeta'}{\zeta}\right)'(2) + 4\frac{\sigma_1'}{\sigma_1}(f)(c-\log f) + \frac{\sigma_1''}{\sigma_1}(f),\end{aligned} \tag{2.26}$$

where $\sigma_\xi^{(\nu)} = (d/d\xi)^\nu \sigma_\xi$ and $c = 2\gamma_E - 2(\zeta'/\zeta)(2)$. Then we have

$$D(f;W) = [D_r + D_d + D_h + D_c](f;W), \tag{2.27}$$

where

$$D_r(f;W) = \frac{6}{\pi^2}\sigma_1(f)\int_0^\infty Y_f(u)W(u)\,du, \tag{2.28}$$

$$D_d(f;W) = f^{1/2}\sum_{j=1}^\infty \alpha_j \tau_j(f) H_j^2\left(\tfrac12\right)\Psi(i\varkappa_j;W), \tag{2.29}$$

$$D_h(f;W) = f^{1/2}\sum_{\substack{k=6\\ 2|k}}^\infty \sum_{j=1}^{\vartheta(k)} \alpha_{j,k}\tau_{j,k}(f) H_{j,k}^2\left(\tfrac12\right)\Psi\left(k-\tfrac12;W\right), \tag{2.30}$$

$$D_c(f;W) = \frac{f^{1/2}}{\pi}\int_{-\infty}^\infty f^{-i\varkappa}\sigma_{2i\varkappa}(f)\frac{\left|\zeta\left(\frac12+i\varkappa\right)\right|^4}{|\zeta(1+2i\varkappa)|^2}\Psi(i\varkappa;W)\,d\varkappa. \tag{2.31}$$

Here

$$\begin{aligned}\Psi(\xi;W) = \frac12\int_0^\infty \mathrm{Re}\Bigg[\left(1-\frac{1}{\sin\pi\xi}\right)\frac{\Gamma^2\left(\frac12+\xi\right)}{\Gamma(1+2\xi)}\\ \times {}_2F_1\left(\frac12+\xi, \frac12+\xi; 1+2\xi; -\frac1u\right)u^{-1/2-\xi}\Bigg]W(u)\,du,\end{aligned} \tag{2.32}$$

with the Gaussian hypergeometric function ${}_2F_1$.

Proof. This is a corollary of the last lemma. See [21] for details. Note that here it is invoked that $H_j(\frac{1}{2})=0$ if $\epsilon_j=-1$, and $H_{j,k}(\frac{1}{2})=0$ if k is odd, as the functional equations (1.9) and (1.15) imply, respectively.

Lemmas 3–5 should be compared with the corresponding assertions claimed by Kuznetsov [17]. We add also that in the light of [3] the explicit formula (2.17) could be derived directly from the spectral structure of $L^2(\Gamma\backslash\mathrm{PSL}_2(\mathbf{R}))$, that is, without appealing to the spectral theory of sums of Kloosterman sums on which [17] and [21] rely.

3. Basic inequalities

In the present section we shall prepare those implements which are crucial in our approximation procedures pertaining to estimations of our key objects. Asymptotics in this context will be supplied mostly by the saddle point method. The proof of Lemma 7 below furnishes typical instances which could be referred to at later applications of the method. Note that Convention 1 is always in force hereafter.

To facilitate the relevant reasoning and in fact the whole of our discussion, the following formulation of the treatment of off-saddle integrals will turn out to be highly instrumental:

LEMMA 6. *Let A be a smooth function compactly supported in a finite interval $[a,b]$; and assume that there exist two quantities A_0 and A_1 such that for each integer $\nu\geqslant 0$ and for any x in the interval,*

$$A^{(\nu)}(x)\ll A_0A_1^{-\nu}. \tag{3.1}$$

Also, let B be a function which is real-valued on $[a,b]$, and regular throughout the complex domain composed of all points within the distance ϱ from the interval; and assume that there exists a quantity B_1 such that

$$0<B_1\ll|B'(x)| \tag{3.2}$$

for any point x in the domain. Then we have, for each fixed integer $P\geqslant 0$,

$$\int_{-\infty}^{\infty}A(x)\exp(iB(x))\,dx\ll A_0(A_1B_1)^{-P}\left(1+\frac{A_1}{\varrho}\right)^P(b-a). \tag{3.3}$$

Proof. With a multiple application of integration by parts, we see that the integral is equal to

$$i^P\int_a^b[(\mathcal{D}_B)^PA](x)\exp(iB(x))\,dx, \tag{3.4}$$

where $\mathcal{D}_B$ is the operator $g \mapsto (g/B')'$. We have

$$(\mathcal{D}_B)^P A(x) = \sum_{\nu_1+\ldots+\nu_P \leqslant P} a(\nu_1,\ldots,\nu_P) A^{(P-\nu_1-\ldots-\nu_P)}(x) \left(\frac{1}{B'(x)}\right)^{(\nu_1)} \cdots \left(\frac{1}{B'(x)}\right)^{(\nu_P)}, \quad (3.5)$$

with certain constants $a(\nu_1,\ldots,\nu_P)$. The assumption (3.2) gives $(1/B'(x))^{(\nu)} \ll B_1^{-1} \varrho^{-\nu}$ on $[a,b]$ via Cauchy's integral formula for derivatives. Thus, (3.1) implies that in (3.4),

$$(\mathcal{D}_B)^P A(x) \ll A_0 (A_1 B_1)^{-P} \sum_{\nu_1+\ldots+\nu_P \leqslant P} \left(\frac{A_1}{\varrho}\right)^{\nu_1+\ldots+\nu_P}, \quad (3.6)$$

from which (3.3) follows.

LEMMA 7. *Let* $1 \leqslant G \leqslant K$ *and* $N \geqslant 1$. *Then we have, for any complex vector* $\{a(n)\}$,

$$\sum_{K \leqslant \varkappa_j \leqslant K+G} \alpha_j \left| \sum_{N \leqslant n \leqslant 2N} \tau_j(n) a(n) \right|^2 \ll (GK+N)(KN)^{\varepsilon} \sum_{N \leqslant n \leqslant 2N} |a(n)|^2. \quad (3.7)$$

Proof. This version of the spectral large sieve of Iwaniec [6] is due to the first author [11] (see [23, Theorem 3.3] for a refinement). Here we shall show a new approach to (3.7). A truncation procedure in our argument, i.e., (3.12) below, will turn out to be fundamental for our discussion of $\mathcal{S}(G,K)$ that starts in the next section. It should be noted that smooth and compactly supported weights attached to integers could be avoided in the present proof proper; their use is made rather for the sake of later purpose.

Obviously we may assume that $K^{\varepsilon} \ll G \ll K^{1-\varepsilon}$, with the basic parameter K which is larger than a constant depending solely on ε_0. The case $N \gg K^{1/\varepsilon}$ can be settled by an application of a duality principle and the theory of Rankin zeta-functions (see [23, pp. 137–138]). Thus, we may assume also that $N \ll K^{1/\varepsilon}$. With this, let

$$h(r) = K^{-2}(r^2 + \tfrac{1}{4}) \left[\exp\left(-\left(\frac{r-K}{G}\right)^2\right) + \exp\left(-\left(\frac{r+K}{G}\right)^2\right) \right]. \quad (3.8)$$

It suffices to prove that

$$\sum_{j=1}^{\infty} \alpha_j \left| \sum_{N \leqslant n \leqslant 2N} \phi(n) \tau_j(n) a(n) \right|^2 h(\varkappa_j) \quad (3.9)$$

is bounded by the right-hand side of (3.7), where ϕ is an arbitrary real-valued smooth function which is supported in $[N, 2N]$ and $\phi^{(\nu)}(y) \ll N^{-\nu}$ for each $\nu \geqslant 0$. Expand the square, take the spectral sum inside, and apply (2.2). The contribution to (3.9) of the first term on the right of (2.2) is negative, and can be discarded. That of the second term is obviously absorbed into the right-hand side of (3.7). Then, let $\mathcal{A}$ be the part of (3.9) corresponding to the sum of Kloosterman sums on the right of (2.2). In $\mathcal{A}$, the sum over l can be truncated to $1\leqslant l\ll K^{1/\varepsilon}$, under Convention 1. This can be seen by shifting the contour of the first integral in (2.1) to $\operatorname{Im} r=-1$. In fact, we have

$$\hat{h}(x) \ll \int_{-\infty}^{\infty} \frac{(|r|+1)h(r)}{\cosh \pi r} |J_{2+2ir}(x)|\,dr \ll \frac{Gx^2}{K} \tag{3.10}$$

via Poisson's formula

$$J_\nu(x) = \frac{\left(\frac{1}{2}x\right)^\nu}{\sqrt{\pi}\,\Gamma\left(\nu+\frac{1}{2}\right)} \int_{-1}^{1} (1-y^2)^{\nu-1/2} \cos(xy)\,dy, \tag{3.11}$$

which is valid for $x>0$, $\operatorname{Re}\nu>-\frac{1}{2}$.

We shall show that the remaining part of $\mathcal{A}$ could be truncated to

$$1\leqslant l \ll N(GK)^{-1}\log K. \tag{3.12}$$

To this end, we invoke the representation

$$J_{2ir}(x)-J_{-2ir}(x) = \frac{2}{\pi i}\sinh(\pi r)\operatorname{Re}\int_{-\infty}^{\infty} \exp(ix\cosh u - 2iru)\,du \tag{3.13}$$

(see the formula (12) in [27, p. 180]), which we use with $x=4\pi\sqrt{mn}/l$. When $|u|>\log^2 K$, we perform integration by parts with respect to the factor $\exp(ix\cosh u)$, getting, for $r\geqslant 0$,

$$\begin{aligned} J_{2ir}(x)-J_{-2ir}(x) = \frac{2}{\pi i}\sinh(\pi r)\operatorname{Re}\int_{-\log^2 K}^{\log^2 K} \exp(ix\cosh u - 2iru)\,du \\ +O\big((r+1)\exp\big(\pi r-\tfrac{1}{2}\log^2 K\big)\big). \end{aligned} \tag{3.14}$$

Thus, via the second expression in (2.1), we obtain, after a rearrangement,

$$\begin{aligned} \hat{h}(x) = K^{-2}\operatorname{Re}\int_{-\infty}^{\infty} R(u)\eta(u)\exp(-(Gu)^2)\exp(ix\cosh u - 2iKu)\,du \\ +O\big(\exp\big(-\tfrac{1}{3}\log^2 K\big)\big), \end{aligned} \tag{3.15}$$

where R is a certain polynomial on u, G and K, and $\eta(u)$ is a smooth weight such that $\eta(u)=1$ for $|u|\leqslant(\log K)/G$, $\eta(u)=0$ for $|u|\geqslant 2(\log K)/G$, as well as $\eta^{(\nu)}(u)\ll(G/\log K)^\nu$ for each $\nu\geqslant 0$. In fact, the expression follows first with the range $|u|\leqslant\log^2 K$, but without the weight; then the truncation to $|u|\leqslant(\log K)/G$ can be imposed; and the result is modified as (3.15). With this, we assume temporarily that $x\ll GK/\log K$. Then Lemma 6 is applicable to the last integral, with $A_0=GK^3$, $A_1=G^{-1}\log K$, $B_1=K$ and $\varrho\approx G^{-1}$. We find that $\hat{h}(x)$ is negligibly small. Hence, we may restrict ourselves to $x \gg GK/\log K$, which is the same as (3.12). In passing, we note that

$$R(u)\eta(u) = 4\pi^{-3/2}GK^3(1+O(K^{-\varepsilon}))\eta(u). \tag{3.16}$$

We may thus equip $\mathcal{A}$ with weights $\phi_0(l)$, where ϕ_0 is a smooth function such that $\phi_0(y)=0$ for $y\leqslant 0$, $\phi_0(y)=1$ for $1\leqslant y\leqslant L_0$, and $\phi_0(y)=0$ for $y\geqslant 2L_0$; here L_0 is a dyadic number such that $L_0 \ll (N\log K)/(GK)$, and the error thus caused is negligible. We replace $\phi_0(y)$ by a sum of smooth $\phi_1(y;L)$ such that it is supported in $\left[\frac{1}{2}L, 2L\right]$ with a dyadic $L\leqslant L_0$, and $\phi_1^{(\nu)}(y) \ll L^{-\nu}$ for each $\nu\geqslant 0$. Hence, it suffices to deal with

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\phi(m)\phi(n)a(m)\overline{a(n)}\sum_{l=1}^{\infty}\frac{\phi_1(l)}{l}S(m,n;l)\,\hat{h}\left(\frac{4\pi\sqrt{mn}}{l}\right), \tag{3.17}$$

where $\phi_1(y)=\phi_1(y;L)$. Obviously we require that

$$GKL \ll N\log K. \tag{3.18}$$

Then we shall show that provided (3.18) we have

$$\hat{h}(x) = \frac{4}{\pi}\sqrt{\frac{2}{x}}\,GKR_1(x)\exp(-(Gu_0)^2)\cos\left(x\cosh u_0 - 2Ku_0 + \tfrac{1}{4}\pi\right) + O(K^{-1/\varepsilon}), \tag{3.19}$$

where $\sinh u_0 = 2K/x$ or $u_0 = \log\left(2K/x + \sqrt{1+4(K/x)^2}\,\right)$, and

$$R_1(x) = \sum_{\nu} b_\nu(G,K)x^\nu, \tag{3.20}$$

with a finite number of terms satisfying $b_\nu(G,K)x^\nu \ll K^{-\varepsilon}$ for $\nu\neq 0$ and $b_0(G,K)=1$. Note that any regular function of u_0 is a power series of K/x, and could be approximated by a polynomial in K/x with arbitrary accuracy; this is relevant to our reasoning below.

Before entering into the proof of (3.19), let us make a useful observation: What matters in estimating (3.17) is to fix the leading term in (3.20); that is, in the asymptotic evaluation of $\hat{h}(x)$, which we are about to develop using the saddle point method, we may restrict our attention to the main term, provided it is clear that the argument yields, in fact, an expansion of the type (3.19) with (3.20). This is due to the fact that the contribution to (3.17) of the term $b_\nu(G,K)x^\nu$ ($\nu\neq 0$) of (3.20) could be dealt with in just the same way as that of the term with $\nu=0$, because it corresponds to a change of weight:

$$\phi(m)\phi(n)\phi_1(l) \longmapsto b_\nu(G,K)\left(\frac{N}{L}\right)^{\nu}\left[\phi(m)\left(\frac{m}{N}\right)^{\nu/2}\right]\left[\phi(n)\left(\frac{n}{N}\right)^{\nu/2}\right]\left[\phi_1(l)\left(\frac{l}{L}\right)^{-\nu}\right]. \tag{3.21}$$

The new weight thus obtained is of the same type as the original, but only smaller, as the factor $b_\nu(G,K)(N/L)^\nu$ is $\ll K^{-\varepsilon}$ for $\nu\neq 0$.

With this, we consider (3.15). Then the above observation allows us to treat instead the simpler

$$J(x)=4\pi^{-3/2}GK\int_{-\infty}^{\infty}\exp(-(Gu)^2)\exp(ix\cosh u-2iKu)\,du, \tag{3.22}$$

since it will be clear from our discussion that the factor $R(u)$ gives rise to a factor of the type $R_1(x)$, and since the weight $\eta(u)$ can obviously be removed. We now apply the saddle point method to (3.22), which is routine but better to be performed with some details because of our later purpose. Thus, u_0 is the saddle point, and (3.12) or (3.18) gives $u_0\ll K/x\ll G^{-1}\log K$. We put $u=v+\xi\exp\big(\frac14\pi i\big)$. We move the last contour to $C_-+C_0+C_+$, where $C_-=\{u:v<u_0,\xi=-\xi_0\}$, $C_0=\{u:v=u_0,-\xi_0\leqslant\xi\leqslant\xi_0\}$ and $C_+=\{u:u_0<v,\xi=\xi_0\}$, with an obvious orientation and $\xi_0=x^{-1/(2+\varepsilon)}$. Accordingly, the dissection $J(x)=(J^{(-)}+J^{(0)}+J^{(+)})(x)$ follows. Note that we have $\xi_0\ll(GK^\varepsilon)^{-1}$, because of (3.18), $G\ll K^{1-\varepsilon}$ and Convention 1. In particular, $\exp(-(Gu)^2)\ll\exp\big(-\frac12(Gv)^2\big)$ on the new contour. We have also

$$\begin{aligned}x\cosh u-2Ku&=x\cosh v-2Kv+\xi\exp\big(\tfrac14\pi i\big)(x\sinh v-2K)\\&\quad+\tfrac12 ix\xi^2\cosh v+x\sum_{j=3}^{\infty}\frac{1}{j!}\big(\xi\exp\big(-\tfrac14\pi i\big)\big)^j\cosh\big(v+\tfrac12 j\pi i\big).\end{aligned}\tag{3.23}$$

This implies that $\mathrm{Im}(x\cosh u-2Ku)>\frac13 x\xi_0^2\cosh v$ on $C_\pm$, since $\pm(x\sinh v-2K)>0$ throughout $C_\pm$. Hence we have $(J^{(-)}+J^{(+)})(x)\ll K\exp\big(-\frac13 x\xi_0^2\big)$, which is negligible. On the other hand, we have

$$\begin{aligned}J^{(0)}(x)&=4\pi^{-3/2}GK\exp\big(ix\cosh u_0-2iKu_0+\tfrac14\pi i\big)\\&\quad\times\int_{-\xi_0}^{\xi_0}\exp(-(Gu)^2)\exp\big(-\tfrac12 x\xi^2\cosh u_0+i\Sigma\big)\,d\xi,\end{aligned}\tag{3.24}$$

where Σ is the last term of (3.23) with $v=u_0$. Since $\Sigma\ll x\xi_0^3$, the factor $\exp(i\Sigma)$ can be replaced by a polynomial on Σ with a negligible error; and the power series in Σ is truncated with the same effect. Also, $\exp(-(Gu)^2)$ is replaced by $\exp(-(Gu_0)^2)$ times a polynomial in a similar fashion. This and applications of integration by parts over $[-\xi_0,\xi_0]$ to $\xi^\nu\exp(-\frac12 x\xi^2\cosh u_0)$ $(\nu=1,2,\ldots)$ give that $J^{(0)}(x)$ is equal to a multiple by a factor of the type $R_1(x)$ of

$$\begin{aligned}&4\pi^{-3/2}GK\exp(-(Gu_0)^2)\exp\big(ix\cosh u_0-2iKu_0+\tfrac14\pi i\big)\\&\qquad\times\int_{-\xi_0}^{\xi_0}\exp\big(-\tfrac12 x\xi^2\cosh u_0\big)\,d\xi+O(K^{-1/\varepsilon}),\end{aligned}\tag{3.25}$$

which leads us to the assertion (3.19).

Now we return to the estimation of (3.17). As observed above, it suffices to consider the contribution of (3.19) with $R_1(x)$ being replaced by 1. Then we put

$$\theta_*(s)=\int_0^\infty \theta(x)\exp(-(Gu_0)^2)\exp\big(ix\cosh u_0-2iKu_0+\tfrac{1}{4}\pi i\big)\,x^{s-1}\,dx, \tag{3.26}$$

where $\theta(x)$ is a smooth function which is equal to 1 over $[2\pi N/L, 16\pi N/L]$, supported in $[\pi N/L, 20\pi N/L]$, and $\theta^{(\nu)}(x)\ll x^{-\nu}$ for each $\nu\geqslant 0$. We find that the estimation of the part of $\mathcal{A}$ under consideration is, via the Mellin inversion of (3.26) and the definition (2.3), reduced to that of

$$GKL^{-1/2}\sum_{L\leqslant l\leqslant 2L}\sum_{\substack{q=1\\(q,l)=1}}^{l}\int_{-\infty}^{\infty}\bigg|\sum_{N\leqslant n\leqslant 2N}\frac{\phi(n)a(n)}{n^{(1/2+iv)/2}}\exp\Big(\frac{2\pi iqn}{l}\Big)\bigg|^2|\theta_*(iv)|\,dv. \tag{3.27}$$

This is, by the hybrid large sieve inequality (see [23, Lemma 3.11]),

$$\ll GK(LN)^{-1/2}\sum_{V\geqslant 1}(N+L^2(V+1))\sup_{V\leqslant|v|\leqslant 2V}|\theta_*(iv)|\sum_{N\leqslant n\leqslant 2N}|a(n)|^2, \tag{3.28}$$

where V runs over dyadic numbers.

To bound $\theta_*(iv)$, we note that

$$\frac{d}{dx}(v\log x+x\cosh u_0-2Ku_0)=\frac{v}{x}+\sqrt{1+4\Big(\frac{K}{x}\Big)^2}. \tag{3.29}$$

The saddle point x_0 of the integral (3.26) with $s{=}iv$ is close to $-v$, and has to be inside the support of θ, since otherwise $\theta_*(iv)$ could be seen to be negligibly small by Lemma 6, with $A_0{=}L/N$, $A_1{=}N/L$, $B_1{=}1{+}VL/N$ and $\varrho\approx N/L$. In particular, we may assume that

$$V\approx\frac{N}{L}. \tag{3.30}$$

With this, we shall further estimate $\theta_*(iv)$ by the saddle point method. Thus, let $\varrho_0=(N/L)^{2/5}$, and let θ_1 be a smooth function such that $\theta_1(x){=}\theta(x)$ for $|x-x_0|\leqslant x_0/\varrho_0$, and $\theta_1(x){=}0$ for $|x-x_0|\geqslant 2x_0/\varrho_0$, as well as $\theta_1^{(\nu)}(x)\ll(x_0/\varrho_0)^{-\nu}$ for each $\nu\geqslant 0$. Also, let $\theta_2{=}\theta-\theta_1$. Then Lemma 6 implies that $(\theta_2)_*(iv)$ is negligibly small; in fact, this results in the specification $A_0{=}L/N$, $A_1{=}(N/L)^{3/5}$, $B_1{=}(N/L)^{-2/5}$ and $\varrho\approx(N/L)^{3/5}$. Thus, via the Taylor expansion of the integrand of (3.26) at $x{=}x_0$, we have that

$$\begin{aligned}\theta_*(iv)&=\exp(-(Gu_0)^2)\exp\big(ix_0\cosh u_0-2iKu_0+\tfrac{1}{4}\pi i\big)\,x_0^{iv-1}\\&\quad\times\int_{-\infty}^{\infty}R_2(x)\theta_1(x)\exp\big(\tfrac{1}{2}iX(x-x_0)^2\big)\,dx+O(K^{-1/\varepsilon}).\end{aligned} \tag{3.31}$$

Here u_0 is specialised with $x=x_0$, $X\approx L/N$ is the derivative of (3.29) at $x=x_0$, and $R_2(x)$ is a polynomial on $x-x_0$ whose constant term is equal to 1 and other terms are all $\ll K^{-\varepsilon}$, provided $\theta_1(x)\neq 0$. The last integral is divided into two parts according to whether $|x-x_0|\leqslant X^{-1/2}$ or not. The first part is bounded trivially, and the second after an application of integration by parts. We obtain

$$\theta_*(iv)\ll\left(\frac{N}{L}\right)^{-1/2}. \tag{3.32}$$

Being inserted into the part of (3.28) corresponding to (3.30), this gives rise to the assertion (3.7).

LEMMA 8. *With the same specifications as in the previous lemma, we have*

$$\sum_{K\leqslant k\leqslant K+G}\sum_{j=1}^{\vartheta(k)}\alpha_{j,k}\left|\sum_{N\leqslant n\leqslant 2N}\tau_{j,k}(n)a(n)\right|^2\ll(GK+N)(KN)^{\varepsilon}\sum_{N\leqslant n\leqslant 2N}|a(n)|^2. \tag{3.33}$$

Proof. To show this counterpart of (3.7), we put $h_1(r)=\exp(-((K-r)/G)^2)$, multiply the inner sum on the left of (3.33) by the factor $h_1(k)$, and sum over all integers $k\geqslant 1$. Then using (2.4) leads us to a sum of Kloosterman sums as in the previous proof, but with $\hat{h}(x)$ being replaced by

$$\sum_{k=1}^{\infty}(-1)^k(2k-1)h_1(k)J_{2k-1}(x). \tag{3.34}$$

By the Poisson integral

$$J_{2k-1}(x)=\frac{(-1)^k}{\pi}\int_{-\pi/2}^{\pi/2}\sin((2k-1)u-x\cos u)\,du \tag{3.35}$$

and the Poisson sum formula, one may see that (3.34) can be replaced by

$$\frac{2}{\sqrt{\pi}}GK\int_{-(\log K)/G}^{(\log K)/G}\sin(2Ku-x\cos u)\exp(-(Gu)^2)\,du. \tag{3.36}$$

With this, the rest of the proof is analogous to the above.

LEMMA 9. *Let K be a large positive parameter. Let*

$$K^{\varepsilon}\ll G\ll K^{1-\varepsilon},\quad 0\leqslant t\ll K^{1/\varepsilon}, \tag{3.37}$$

and put

$$T=\frac{1}{4\pi^2}(K+t)(|K-t|+G). \tag{3.38}$$

Then, uniformly for all cusp forms ψ_j *with*

$$|K-\varkappa_j|\ll G, \tag{3.39}$$

we have

$$\begin{aligned} H_j^2\left(\tfrac12+it\right)\ll(\log^2K)\sum_{M\leqslant 2T}\sum_{q=1}^{\infty}\frac1q \\ \times\int_{\gamma^{-1}-i\gamma^2}^{\gamma^{-1}+i\gamma^2}\left|\sum_{m=1}^{\infty}\phi(q^2m;M)d(m)\tau_j(m)m^{-\xi-1/2-it}\right||d\xi|, \end{aligned}\tag{3.40}$$

with $\gamma=\log^2K$ *and dyadic numbers* M. *Here the smooth function* $\phi(y;M)$ *depends solely on the interval* $\left[\frac12M,2M\right]$, *in which it is supported and* $\phi^{(\nu)}(y;M)\ll M^{-\nu}$, *with the implied constant depending only on* ν.

Proof. We consider the integral

$$\mathcal{R}=\frac{1}{2\pi i\gamma}\int_{(3)}H_j^2\left(\xi+\tfrac12+it\right)T^{\xi}\Gamma\left(\frac{\xi}{\gamma}\right)d\xi. \tag{3.41}$$

Since (1.6) gives

$$H_j^2(s)=\zeta(2s)\sum_{n=1}^{\infty}d(n)\tau_j(n)n^{-s},\quad \operatorname{Re}s>1, \tag{3.42}$$

we have

$$\mathcal{R}=\sum_{n\leqslant\alpha T}\check\tau_j(n)n^{-1/2-it}\exp\left(-\left(\frac{n}{T}\right)^{\gamma}\right)+O(e^{-K}), \tag{3.43}$$

where $2\leqslant\alpha\leqslant4$ is arbitrary and

$$\check\tau_j(n)=\sum_{q^2|n}d\left(\frac{n}{q^2}\right)\tau_j\left(\frac{n}{q^2}\right). \tag{3.44}$$

Shifting the path of (3.41) to $\left(-\frac12\gamma\right)$ and recalling the functional equation (1.10), we get

$$\mathcal{R}=H_j^2\left(\tfrac12+it\right)+\sum_{n=1}^{\infty}\check\tau_j(n)n^{-1/2+it}\mathcal{R}_j(n), \tag{3.45}$$

where

$$\begin{aligned} \mathcal{R}_j(n)=\frac{\pi^{4it}}{2\pi i\gamma}\int_{(-\gamma/2)}\left(\frac{\Gamma\left(\frac12\left(\frac12-\xi-it+i\varkappa_j+\frac12(1-\epsilon_j)\right)\right)}{\Gamma\left(\frac12\left(\frac12+\xi+it-i\varkappa_j+\frac12(1-\epsilon_j)\right)\right)}\right)^2 \\ \times\left(\frac{\Gamma\left(\frac12\left(\frac12-\xi-it-i\varkappa_j+\frac12(1-\epsilon_j)\right)\right)}{\Gamma\left(\frac12\left(\frac12+\xi+it+i\varkappa_j+\frac12(1-\epsilon_j)\right)\right)}\right)^2(\pi^4nT)^{\xi}\Gamma\left(\frac{\xi}{\gamma}\right)d\xi. \end{aligned}\tag{3.46}$$

By Stirling's formula coupled with the functional equation $\Gamma(s)\Gamma(1-s)=\pi/\sin\pi s$, this integrand is

$$\ll (16\pi^4 nT)^{-\gamma/2}(|\xi+i(t-\varkappa_j)|\,|\xi-i(\varkappa_j+t)|)^\gamma \exp\Big(-\frac{\pi|\xi|}{2\gamma}\Big); \tag{3.47}$$

and thus

$$\mathcal{R}_j(n) \ll \Big(\frac{n}{T}\Big)^{-\gamma/2}. \tag{3.48}$$

In fact, when $|\xi|<\gamma^2$ we see that the factor $(|\xi+i(\varkappa_j-t)|\,|\xi-i(\varkappa_j+t)|)^\gamma$ is $\ll(4\pi^2T)^\gamma$ in view of (3.39), and when $|\xi|\geqslant\gamma^2$ the integrand itself is negligible due to the factor $\exp(-\pi|\xi|/2\gamma)$. The estimate (3.48) allows us to truncate the sum in (3.45) to $n\leqslant\alpha T$. In this way, we have, uniformly for all ψ_j satisfying (3.39),

$$\begin{aligned} H_j^2\big(\tfrac12+it\big) = \sum_{n\leqslant\alpha T} \check\tau_j(n) n^{-1/2-it}\exp\Big(-\Big(\frac{n}{T}\Big)^\gamma\Big) \\ - \sum_{n\leqslant\alpha T} \check\tau_j(n) n^{-1/2+it}\mathcal{R}_j(n)+O(K^{-1}). \end{aligned} \tag{3.49}$$

We equip both sums with smooth and compactly supported weights in much the same way as performed preceding (3.17); here the parameter α plays a role. Then the first sum in (3.49) is readily seen to be bounded by the right-hand side of (3.40). As to the second sum, we modify $\mathcal{R}_j$ by shifting the path in (3.46) to $(-\gamma^{-1})$, and take the sum over n inside the integral. Considering the absolute value of the resulting integrand, we may eliminate the Γ-factors of (3.46) except for $\Gamma(\xi/\gamma)$. This gives rise to (3.40).

A combination of Lemmas 7 and 9 yields readily the following result:

Corollary. *We have, under* (3.37),

$$\mathcal{S}(G,K) \ll (K+t)(|K-t|+G)K^\varepsilon. \tag{3.50}$$

Note that this contains the convexity bound $H_j\big(\frac12+it\big)\ll(\varkappa_j+t)^{1/4+\varepsilon}(|\varkappa_j-t|+1)^{1/4}$, which is better than (1.1) only when $|\varkappa_j-t|\ll\varkappa_j^{1/3}$.

With these preparations we shall start our discussion of $\mathcal{S}(G,K)$ in the next section. Technically it is a layered application of those approximation–estimation procedures employed in the proof of Lemma 7. To avoid excessive repetitions of details, we introduce the following in the form of a convention:

Convention 2. All subsequent approximations are to hold with the basic parameter K that is assumed to be larger than a quantity depending solely on ε_0. With this, let $\mathcal{X}$ be a particular object that we need to bound. Suppose that an expression $\mathcal{Y}$ comes up in a relevant discussion, and we have an approximation $\mathcal{Y}=\mathcal{Y}_0+\mathcal{Y}_1+O(\mathcal{Z})$, in which $\mathcal{Y}_0$ is dominant, $\mathcal{Y}_1$ oscillates in the same mode as $\mathcal{Y}_0$, while $\mathcal{Z}$ contributes negligibly to $\mathcal{X}$. Then the notation $\mathcal{Y}\sim\mathcal{Y}_0$ indicates an actuation of a procedure in which the treatment of $\mathcal{Y}_1$ is a repetition of that of $\mathcal{Y}_0$, and the replacement of $\mathcal{Y}$ by $\mathcal{Y}_0$ causes no differences in bounding $\mathcal{X}$.

For instance, in the proof of Lemma 7, the polynomial factor $R_1(x)$ of (3.19) is essentially irrelevant for the estimation of (3.17); and this could be denoted as $R_1(x)\sim 1$. More drastically, as we shall do in the sequel, this economy of reasoning could have been applied to (3.17) from the very beginning of the proof, as (3.21) endorses. We shall employ devices analogous to (3.21), without mentioning them persistently.

4. Reduction

We begin our discussion of $\mathcal{S}(G,K)$. We assume that K is as in Convention 2, and that (1.26) holds. Note that $G\ll K^{1-\varepsilon}$ under Convention 1.

Let h be defined by (3.8), but with the present specification of the parameters. Then, by Lemma 9, it suffices to treat

$$\sum_{j=1}^{\infty}\alpha_j\left|\sum_{m=1}^{\infty}\phi_0(m)d(m)\tau_j(m)m^{-1/2-it}\right|^2 h(\varkappa_j), \tag{4.1}$$

where $\phi_0(x)=\phi(q^2x;M)x^{-\xi}$ with ξ and $\phi(q^2x;M)$ as in (3.40), while T is defined by (3.38) with the present G. Thus, $\phi_0(x)$ is smooth, compactly supported accordingly, and $\phi_0^{(\nu)}(x)\ll((\log^4K)/x)^\nu$.

We proceed just in the same way as in the proof of Lemma 7. What is essential for our purpose is to bound the Kloosterman-sum part of (4.1) thus obtained. In view of (3.12), we may assume that the corresponding truncation has already been performed to the present sum over the moduli of Kloosterman sums. Thus, more specifically, we shall consider

$$\begin{aligned}\mathcal{S}_1=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\phi(m)\phi(n)d(m)d(n)(mn)^{-1/2}\left(\frac{m}{n}\right)^{it}\\ \times\sum_{l=1}^{\infty}\frac{\phi_1(l)}{l}S(m,n;l)\hat{h}\left(\frac{4\pi\sqrt{mn}}{l}\right),\end{aligned} \tag{4.2}$$

which corresponds to (3.17). Here ϕ and ϕ_1 are real-valued smooth functions, which are compactly supported in $\left[\frac{1}{2}M,2M\right]$ and $\left[\frac{1}{2}L,2L\right]$, respectively, with

$$\frac{GKL}{\log K}\ll M\ll T. \tag{4.3}$$

Also, we have $\phi^{(\nu)}(x)\ll((\log^4 K)/M)^\nu$ and $\phi_1^{(\nu)}(x)\ll L^{-\nu}$ for each $\nu\geqslant 0$. Note that the present M stands for M/q^2 in (4.1). The symbols M, L, ϕ and ϕ_1 will retain the current specifications till the end of §6. We have

$$\mathcal{S}(G,K)\ll GK^{1+\varepsilon}+|\mathcal{S}_1|K^\varepsilon. \tag{4.4}$$

In the sequel, we shall modify or transform the sum $\mathcal{S}_1$ in several steps. The most significant contribution will be denoted by $\mathcal{S}_\nu$, $\nu=2,3,4$; accordingly, the estimation of $\mathcal{S}_1$ is reduced to the same as of $\mathcal{S}_4$.

We return to the second expression in (2.1). We note that the integration can be restricted to

$$|r-K|\ll G\log K, \tag{4.5}$$

because of the uniform bound $|J_{2ir}(x)|\leqslant(\cosh 2\pi r)^{1/2}$ which follows from (3.11). We then evaluate the integral (3.13) asymptotically; we require $x=4\pi\sqrt{mn}/l$ to appear in (4.2), i.e., $\phi(m)\phi(n)\phi_1(l)\neq 0$. Obviously, we may proceed in much the same way as in (3.22)–(3.25), and get

$$\begin{aligned} J_{2ir}(x)-J_{-2ir}(x)&\sim\frac{\sqrt{2}}{i\sqrt{\pi x\cosh u_1}}e^{\pi r}\cos\left(x\cosh u_1-2ru_1+\tfrac{1}{4}\pi\right)\\ &\sim\frac{1}{i}\sqrt{\frac{2}{\pi x}}\,e^{\pi r}\cos\left(\omega(r,x)+\tfrac{1}{4}\pi\right), \end{aligned}\tag{4.6}$$

where $x\sinh u_1=2r$ and

$$\omega(r,x)=x\left(1-2\left(\frac{r}{x}\right)^2\right). \tag{4.7}$$

That is, the left-hand side of (4.6) is asymptotically equal, within a negligible error, to the right-hand side multiplied by a factor similar to $R_1(x)$ defined at (3.20). Here we have used the facts that $x\cosh u_1=x+2r^2/x+O(r^4/x^3)$, $ru_1=2r^2/x+O(r^4/x^3)$ and $r^4/x^3\ll K^4/(GK/\log K)^3\ll K^{-\varepsilon}$ because of (1.26) and (4.3).

Hence, by Convention 2, it suffices to consider the expression

$$\begin{aligned} \frac{2^{3/2}}{\pi^2}\int_{K-G\log K}^{K+G\log K} rh(r)\Bigg[&\sum_{m=1}^\infty\sum_{n=1}^\infty\phi(m)\phi(n)d(m)d(n)(mn)^{-3/4}\left(\frac{m}{n}\right)^{it}\\ &\times\sum_{l=1}^\infty\frac{\phi_1(l)}{\sqrt{l}}S(m,n;l)\cos\left(\omega\left(r,\frac{4\pi\sqrt{mn}}{l}\right)+\frac{1}{4}\pi\right)\Bigg]\,dr. \end{aligned}\tag{4.8}$$

This reduces the estimation of $\mathcal{S}_1$ to that of

$$\begin{aligned} \mathcal{S}_2=\sum_{m=1}^\infty\frac{\phi(m)d(m)}{m^{3/4-it}}&\sum_{l=1}^\infty\frac{\phi_1(l)}{\sqrt{l}}\sum_{\substack{q=1\\(q,l)=1}}^{l}\exp\left(\frac{2\pi iqm}{l}\right)\\ &\times\sum_{n=1}^\infty\frac{\phi(n)d(n)}{n^{3/4+it}}\exp\left(\frac{2\pi i\tilde{q}n}{l}\right)\exp\left(\delta_1 i\omega\left(r,\frac{4\pi\sqrt{mn}}{l}\right)\right), \end{aligned}\tag{4.9}$$

where $\delta_1=\pm1$ and $q\tilde{q}\equiv1 \bmod l$. Note that $\mathcal{S}_2$ is a function of r. We have, via (4.4),

$$\mathcal{S}(G,K)\ll GK^{1+\varepsilon}\big(1+\sup_r|\mathcal{S}_2|\big), \tag{4.10}$$

where r is in the range (4.5).

To transform $\mathcal{S}_2$, we apply the sum formula of Voronoĭ (see, e.g., [8, Theorem 1.7]) to the inner-most sum of (4.9): Thus, it is equal to

$$\begin{aligned}\frac{1}{l}\int_0^\infty&(\log y+2\gamma_E-2\log l)p(y)\,dy\\&+\frac{1}{l}\sum_{n=1}^\infty d(n)\int_0^\infty\bigg[4\exp\Big(\frac{2\pi inq}{l}\Big)K_0\Big(\frac{4\pi\sqrt{ny}}{l}\Big)\\&\qquad-2\pi\exp\Big(-\frac{2\pi inq}{l}\Big)Y_0\Big(\frac{4\pi\sqrt{ny}}{l}\Big)\bigg]p(y)\,dy,\end{aligned} \tag{4.11}$$

where K_0 and Y_0 are Bessel functions in the notation of [27], and

$$p(y)=\phi(y)y^{-3/4-it}\exp\Big(\delta_1 i\omega\Big(r,\frac{4\pi\sqrt{my}}{l}\Big)\Big). \tag{4.12}$$

For the sake of a later purpose, we stress that (4.11) is a simple consequence of the functional equation for the Hecke–Estermann zeta-function (see, e.g., [23, Lemma 3.7]): Let

$$D\Big(s,\xi;\frac{q}{l}\Big)=\sum_{n=1}^\infty\sigma_\xi(n)\exp\Big(\frac{2\pi inq}{l}\Big)n^{-s},\quad (q,l)=1. \tag{4.13}$$

Then

$$\begin{aligned}D\Big(s,\xi;\frac{q}{l}\Big)&=2(2\pi)^{2s-\xi-2}l^{\xi-2s+1}\Gamma(1-s)\Gamma(1+\xi-s)\\&\times\Big[\cos\big(\tfrac12\pi\xi\big)D\Big(1-s,-\xi;\frac{\tilde{q}}{l}\Big)-\cos\big(\pi(s-\tfrac12\xi)\big)D\Big(1-s,-\xi;-\frac{\tilde{q}}{l}\Big)\Big],\end{aligned} \tag{4.14}$$

which is actually equivalent to the automorphy of the real-analytic Eisenstein series of weight 0 over Γ.

The leading term of (4.11) is negligible by Lemma 6. In fact, we may set $\varrho\approx M$; and in the relevant domain of y,

$$\frac{d}{dy}\Big(-t\log y+\delta_1\omega\Big(r,\frac{4\pi\sqrt{my}}{l}\Big)\Big)=\frac{1}{y}\Big(-t+2\pi\frac{\delta_1}{l}\sqrt{my}+\delta_1\frac{r^2l}{4\pi\sqrt{my}}\Big). \tag{4.15}$$

Here we have $r^2l/\sqrt{my}\ll K^2L/M\ll G^{-1}K\log K$ by (4.3) and (4.5); and $\sqrt{m|y|}/l\gg GK/\log K\gg tK^\varepsilon$ by (1.26). Thus, we see that Lemma 6 works with $A_0=(\log K)/M^{3/4}$,

$A_1 = M/\log^4 K$, $B_1 = GK/(M\log K)$ and $\varrho \approx M$; note that we have used the bound $\phi^{(\nu)}(y) \ll ((\log^4 K)/M)^\nu$. This confirms our claim. Also, the part of (4.11) which contains the Bessel function K_0 is negligible, because of the exponential decay of the function and the fact that $\sqrt{y}/l \gg GK/(\sqrt{M}\log K) \gg (K+t)^{4/3}/\sqrt{T} \gg (K+t)^{1/3}$. As to the Y_0-part, we use the fact that $Y_0(x) \sim (2/\pi x)^{1/2}\sin(x - \frac{1}{4}\pi)$ according to [27, formula (4) on p. 199]. Thus, the main part of (4.11) is

$$-\sqrt{\frac{2}{l}}\sum_{n=1}^{\infty}\frac{d(n)}{n^{1/4}}\exp\left(-\frac{2\pi inq}{l}\right)\int_0^\infty y^{-1/4}p(y)\sin\left(\frac{4\pi\sqrt{ny}}{l} - \frac{1}{4}\pi\right)dy. \tag{4.16}$$

Inserting this into (4.9), we see that instead of $\mathcal{S}_2$ we may deal with

$$\mathcal{S}_3 = \sum_{m=1}^{\infty}\frac{\phi(m)d(m)}{m^{3/4-it}}\sum_{l=1}^{\infty}\frac{\phi_1(l)}{l}\sum_{n=1}^{\infty}\frac{d(n)}{n^{1/4}}c_l(m-n)I(l,m,n;\delta_1,\delta_2), \tag{4.17}$$

where c_l is the Ramanujan sum $\bmod\, l$ and

$$I(l,m,n;\delta_1,\delta_2) = \int_0^\infty \phi(y)y^{-1-it}\exp\left(\delta_1 i\omega\left(r, \frac{4\pi\sqrt{my}}{l}\right) + \frac{4\pi\delta_2 i\sqrt{ny}}{l}\right)dy, \tag{4.18}$$

with $\delta_2 = \pm 1$. By Convention 2, we have, in place of (4.10),

$$\mathcal{S}(G,K) \ll GK^{1+\varepsilon}\big(1 + \sup_r |\mathcal{S}_3|\big). \tag{4.19}$$

We apply Lemma 6 to the last integral. If $\delta_1 = \delta_2$, then the integral is similar to the leading term of (4.11), and can be discarded. Thus, hereafter we shall have $\delta_2 = -\delta_1$. We may set $\varrho \approx M$ again, and in the relevant domain of y we have

$$\begin{aligned}\frac{d}{dy}&\left(-t\log y + \delta_1\omega\left(r, \frac{4\pi\sqrt{my}}{l}\right) - \frac{4\pi\delta_1\sqrt{ny}}{l}\right)\\ &= \frac{1}{y}\left(-t + 2\pi\delta_1\frac{\sqrt{y}}{l}(\sqrt{m} - \sqrt{n}) + \delta_1\frac{r^2 l}{4\pi\sqrt{my}}\right).\end{aligned} \tag{4.20}$$

Let us assume that $|m-n| \gg L(t + K^2L/M)K^\varepsilon$. Then throughout the domain we have $\big(\sqrt{|y|}/l\big)\big|\sqrt{m} - \sqrt{n}\big| \gg \big(t + r^2 l/\sqrt{m|y|}\big)K^\varepsilon$. Hence Lemma 6 works with $A_0 = 1/M$, $A_1 = M/\log^4 K$, $B_1 = (t + K^2L/M)K^\varepsilon/M$ and $\varrho \approx M$. Note that $A_1B_1 \gg K^\varepsilon$; in fact, if $t \geqslant 1$ then this is obvious, and otherwise (3.38) and (4.3) yield the same. Thus, (4.18) is negligibly small, provided the above lower bound for $|m-n|$. In other words, we may proceed with the truncation

$$m - n \ll L\left(t + \frac{K^2L}{M}\right)K^\varepsilon \ll \frac{M}{GK}\left(t + \frac{K}{G}\right)K^\varepsilon. \tag{4.21}$$

Let us settle the case $m=n$; that is, we are dealing with the diagonal part of $\mathcal{S}_3$:

$$\sum_{n=1}^{\infty}\frac{\phi(n)d^2(n)}{n^{1-it}}\sum_{l=1}^{\infty}\frac{\phi_1(l)}{l}\varphi(l)I(l,n,n;\delta_1,-\delta_1), \tag{4.22}$$

where φ is Euler's totient function. We have

$$I(l,n,n;\delta_1,-\delta_1)=\int_1^{\infty}\phi(y)y^{-1-it}\exp\biggl(-\delta_1 i\frac{r^2 l}{2\pi\sqrt{ny}}\biggr)\,dy, \tag{4.23}$$

since $\phi(y)=0$ for $y\leqslant 1$. We can assume that $L\gg K^{\varepsilon}$, for otherwise (4.22) could obviously be ignored. Consider then the situation $t\ll K^{\varepsilon}$; in particular, $T\approx K^2$, and $M\ll K^2$ by (4.3). We may apply Lemma 6 to (4.23), with $A_0=1/M$, $A_1=M/\log^4 K$, $B_1=K^2L/M^2$ and $\varrho\approx M$, since $K^2L/M\gg L\gg tK^{\varepsilon}$ under Convention 1. That is, this case can be ignored. Let us move to the situation $t\gg K^{\varepsilon}$. We shall employ an argument based on Mellin inversion; one may use the saddle point method as well. With the Mellin transform ϕ^* of ϕ, (4.23) is equal to

$$\begin{aligned}\frac{1}{2\pi i}&\int_{(\varepsilon)}\phi^*(s)\int_1^{\infty}y^{-1-it-s}\exp\biggl(-\delta_1 i\frac{r^2 l}{2\pi\sqrt{ny}}\biggr)\,dy\,ds\\ &=\frac{1}{2\pi i}\int_{(\varepsilon)}\phi^*(s)\biggl(\int_0^{\infty}-\int_0^1\biggr)y^{-1-it-s}\exp\biggl(-\delta_1 i\frac{r^2 l}{2\pi\sqrt{ny}}\biggr)\,dy\,ds.\end{aligned} \tag{4.24}$$

Note that $\phi^*(s)$ is of fast decay with respect to s in any fixed vertical strip. The double integral arising from the last finite integral vanishes, as can be seen by performing integration by parts in the y-integral and exchanging the order of integration. We have

$$I(l,n,n;\delta_1,-\delta_1)=\frac{1}{\pi i}\int_{(\varepsilon)}\phi^*(s)\Gamma(2(s+it))e^{-\delta_1\pi i(s+it)}\biggl(\frac{r^2 l}{2\pi\sqrt{n}}\biggr)^{-2(s+it)}ds, \tag{4.25}$$

which converges absolutely. Thus, the inner sum of (4.22) can be written as

$$-\frac{1}{2\pi^2}\int_{(2)}\phi_1^*(s_1)\int_{(\varepsilon)}\biggl(\frac{r^4}{4\pi^2 n}\biggr)^{-s-it}\frac{\zeta(s_1+2(s+it))}{\zeta(s_1+2(s+it)+1)}\phi^*(s)\Gamma(2(s+it))e^{-\delta_1\pi i(s+it)}\,ds\,ds_1, \tag{4.26}$$

where ϕ_1^* is the Mellin transform of ϕ_1. This double integral can be truncated to $|s|$, $|s_1|\ll K^{\varepsilon}$. Moving the s_1-contour to the vertical line (ε), we do not encounter any poles under Convention 1, and find that (4.26) is $\ll K^{\varepsilon}$, which settles the present case. Hence the diagonal part of $\mathcal{S}_3$ can be ignored.

We turn to the non-diagonal part of $\mathcal{S}_3$,

$$\begin{aligned}&\sum_{f\leqslant f_0}\sum_{l=1}^{\infty}\frac{\phi_1(l)c_l(f)}{l}\sum_{n=1}^{\infty}\frac{\phi(n)d(n)d(n+f)}{n^{3/4-it}(n+f)^{1/4}}I(l,n,n+f;\delta_1,-\delta_1)\\ &\quad+\sum_{f\leqslant f_0}\sum_{l=1}^{\infty}\frac{\phi_1(l)c_l(f)}{l}\sum_{n=1}^{\infty}\frac{\phi(n+f)d(n)d(n+f)}{(n+f)^{3/4-it}n^{1/4}}I(l,n+f,n;\delta_1,-\delta_1)=\mathcal{S}_3^-+\mathcal{S}_3^+,\end{aligned} \tag{4.27}$$

say, where by (4.21)

$$f_0 \ll L\left(t+\frac{K^2L}{M}\right)K^{\varepsilon}. \tag{4.28}$$

We have got instances of the additive divisor sum. Let us put

$$\begin{aligned} W_-(u) &= \phi(fu)u^{-3/4+it}(u+1)^{-1/4}I_-(f,l,u),\\ W_+(u) &= \phi(f(u+1))(u+1)^{-3/4+it}u^{-1/4}I_+(f,l,u), \end{aligned} \tag{4.29}$$

where

$$I_\pm(f,l,u) = \int_0^\infty \frac{\phi(y)}{y^{1+it}} \exp\left(\pm\delta_1 i \frac{4\pi\sqrt{fy}}{l(\sqrt{u}+\sqrt{u+1})} - \delta_1 i \frac{r^2 l}{2\pi\sqrt{fy(u+a_\pm)}}\right) dy, \tag{4.30}$$

with $a_\pm = \frac{1}{2}(1\pm 1)$. Then (4.27) can be written as

$$\mathcal{S}_3^\pm = \sum_{f\leqslant f_0} \frac{1}{f^{1-it}} \sum_{l=1}^{\infty} \frac{\phi_1(l)c_l(f)}{l} \sum_{m=1}^{\infty} d(m)d(m+f)W_\pm\left(\frac{m}{f}\right). \tag{4.31}$$

Let us consider $\mathcal{S}_3^-$. With the change of variable $v=4\pi l^{-1}\left(u+\frac{1}{2}\right)^{-1/2}\sqrt{fy}$, we rewrite it as

$$2(4\pi)^{2it} \sum_{f\leqslant f_0} \frac{1}{f^{1-2it}} \sum_{l=1}^{\infty} \frac{\phi_1(l)c_l(f)}{l^{1+2it}} \sum_{m=1}^{\infty} d(m)d(m+f)W_-^{(1)}\left(\frac{m}{f}\right), \tag{4.32}$$

where

$$W_-^{(1)}(u) = \phi(fu)u^{-3/4}(u+1)^{-1/4}\left(1+\frac{1}{2u}\right)^{-it} I_-^{(1)}(f,l,u), \tag{4.33}$$

with

$$\begin{aligned} I_-^{(1)}(f,l,u) = \int_0^\infty &\phi\left(\frac{(lv)^2}{16\pi^2 f}\left(u+\frac{1}{2}\right)\right)\\ &\times \exp\left(-\delta_1 i \frac{v\sqrt{u+1/2}}{\sqrt{u}+\sqrt{u+1}} - 2\delta_1 i \frac{r^2}{v\sqrt{u(u+1/2)}}\right) v^{-1-2it}\, dv. \end{aligned} \tag{4.34}$$

Here we could introduce the truncation

$$u \approx \frac{M}{f} \quad \text{and} \quad v \approx \frac{f}{L}, \tag{4.35}$$

in which the former is obvious, and the latter is due to the presence of the ϕ-factor in (4.34).

We are going to simplify $W_-^{(1)}$ under Conventions 1 and 2. To this end we note first that

$$u^{-3/4}(u+1)^{-1/4}\left(1+\frac{1}{2u}\right)^{-it} \sim \frac{1}{u}\exp\left(-\frac{it}{2u}+\frac{it}{8u^2}\right), \tag{4.36}$$

since $t/u^3 \ll t(f/M)^3 \ll t(GK)^{-3}(t+K/G)^3 K^\varepsilon \ll (GK)^{-3}(t+K)^4 K^\varepsilon$, which is $\ll K^{-\varepsilon}$ because of (1.26). Also

$$\frac{v\sqrt{u+1/2}}{\sqrt{u}+\sqrt{u+1}} \sim \frac{v}{2}\left(1+\frac{1}{32u^2}\right), \tag{4.37}$$

since $v/u^3 \ll L^3 M^{-3}(t+K/G)^4 K^\varepsilon \ll (GK)^{-3}(t+K/G)^4 K^\varepsilon \ll K^{-\varepsilon}$. Further,

$$\frac{r^2}{v\sqrt{u(u+1/2)}} \sim \frac{r^2}{uv}\left(1-\frac{1}{4u}\right), \tag{4.38}$$

since $r^2/u^3 v \ll K^2(L/M)^3(t+K/G)^2 K^\varepsilon \ll (GK)^{-3}(t+K)^4 K^\varepsilon \ll K^{-\varepsilon}$.

This leads us to

$$\mathcal{S}_4 = \sum_{f=1}^{\infty} \frac{\phi_2(f)}{f^{1-2it}} \sum_{l=1}^{\infty} \frac{\phi_1(l) c_l(f)}{l^{1+2it}} \sum_{m=1}^{\infty} d(m) d(m+f) X\!\left(\frac{m}{f}\right). \tag{4.39}$$

Here ϕ_2 is a smooth function compactly supported in $\left[\frac{1}{2}F, 2F\right]$, with

$$F \ll L\left(t+\frac{K}{G}\right)K^\varepsilon, \tag{4.40}$$

as follows from (4.28); and

$$X(u) = \frac{1}{u}\int_0^\infty \xi(f,l,u,v)\exp(iY)\frac{dv}{v^{1+2it}}, \tag{4.41}$$

with

$$\xi(f,l,u,v) = \phi(fu)\phi\!\left(\frac{(lv)^2}{16\pi^2 f}u\right) \tag{4.42}$$

and

$$Y = -\frac{t}{2u}\left(1-\frac{1}{4u}\right) - \frac{1}{2}\delta_1 v\left(1+\frac{1}{32u^2}\right) - 2\delta_1\frac{r^2}{uv}\left(1-\frac{1}{4u}\right). \tag{4.43}$$

The equality (4.42) depends on the fact that

$$\phi\!\left(\frac{(lv)^2}{16\pi^2 f}\left(u+\frac{1}{2}\right)\right) \sim \phi\!\left(\frac{(lv)^2}{16\pi^2 f}u\right).$$

We note also that $\phi_2^{(\nu)}(x) \ll F^{-\nu}$ for each $\nu \geqslant 0$, as usual.

The transformation of $\mathcal{S}_3^+$ is analogous. In fact, we end up with the same expression as $\mathcal{S}_4$ except for the change of the definition (4.43) into

$$\frac{t}{2u}\left(1-\frac{3}{4u}\right) + \frac{1}{2}\delta_1 v\left(1+\frac{1}{32u^2}\right) - 2\delta_1\frac{r^2}{uv}\left(1-\frac{3}{4u}\right). \tag{4.44}$$

This should imply that the discussion of $\mathcal{S}_3^+$ can be done with unessential alterations to that of $\mathcal{S}_4$. Hence it suffices to treat $\mathcal{S}_4$; that is, we have, in place of (4.19),

$$\mathcal{S}(G,K)\ll GK^{1+\varepsilon}\big(1+\sup_r|\mathcal{S}_4|\big), \tag{4.45}$$

with a minor abuse of reasoning. For later convenience we note that (4.35) can be stated as

$$u\approx\frac{M}{F}\quad\text{and}\quad v\approx\frac{F}{L}, \tag{4.46}$$

with (1.26), (3.38), (4.3), (4.4), (4.5) and (4.40) being provided. The assertion (4.45) is naturally dependent on a reasoning similar to (3.21).

5. Lower range

We have reduced the estimation of $\mathcal{S}(G,K)$, a spectral object, to that of $\mathcal{S}_4$, an arithmetic object. With this, we now return to the spectra. That is to say, we apply Lemma 5 to $\mathcal{S}_4$:

$$\mathcal{S}_4=S_r+S_d+S_h+S_c, \tag{5.1}$$

in an obvious correspondence to the terms on the right of (2.27). In the present and the subsequent sections we shall deal with S_r and S_d in two ranges of the parameter t. The parts S_h and S_c will be briefly treated; they are analogous to S_d and turn out to be negligible.

As vaguely indicated in the introduction, the range of t is divided into three sections according to the size of the spectral data under consideration. This is rendered in the division

$$0\leqslant t\leqslant K^{2/3},\quad K^{2/3}\leqslant t\leqslant K^{3/2},\quad K^{3/2}\leqslant t. \tag{5.2}$$

We call these intervals the *lower*, the *intermediate* and the *upper ranges*, respectively. The bound (1.1) for the upper range follows from the spectral mean square (1.19); the difference caused by the factors $\varkappa_j^{\pm\varepsilon}$ and $K^{\pm\varepsilon}$ is immaterial for our current discussion. Thus we consider the remaining two ranges. In the present section we shall deal with the lower range, or more precisely, we shall consider the situation

$$0\leqslant t\ll\frac{K^{1+\varepsilon}}{G},\quad G\approx K^{1/3+\varepsilon}. \tag{5.3}$$

Note that consequentially we have $T\approx K^2$ and $M\ll K^2$. As a matter of fact, this case has already been settled in the announcement article [14], and thus could be skipped. Nevertheless, there is a certain need to fill in some details missing in [14], and above all what we are about to develop here should motivate effectively the reasoning in the next section, where we shall treat the intermediate range. Also, we shall depart from [14] in a few technical aspects in order to show a variety of available methods.

It should, however, be noted that our division of the range of t is not imperative; a refinement of the argument of the next section should make the present section redundant, at the cost of accessibility.

Thus, we assume (5.3), and consider the spectral expansion (5.1). Then we observe that

$$Y \sim Q = -\frac{1}{2}\delta_1 v - 2\delta_1 \frac{r^2}{uv}. \tag{5.4}$$

In fact, in (4.43) we have $t/u \ll tF/M \ll t(K/G)(L/M)K^{\varepsilon} \ll (t/G^2)K^{\varepsilon}$, as (4.46), (4.40) and (4.3) successively imply. Also, $v/u^2 \ll F^3/LM^2 \ll (K/G)^3(L/M)^2K^{\varepsilon} \ll K^{1+\varepsilon}/G^5$ and $r^2/u^2v \ll K^2FL/M^2 \ll K^{2+\varepsilon}(K/G)(L/M)^2 \ll K^{1+\varepsilon}/G^3$. Thus terms of Y, save for those two on the right of (5.4), are all negligible under Conventions 1 and 2.

We shall consider $\mathcal{S}_4$ with Y being replaced by Q. We begin with S_r. We may naturally take into account only the leading term on the right of (2.26), since the other terms are treated similarly. Noting that $\log(u)\log(u+1) \sim \log^2 u$, we need to bound the expression

$$\sum_{f=1}^{\infty} \frac{\phi_2(f)\sigma_1(f)}{f^{1-2it}} \sum_{l=1}^{\infty} \frac{\phi_1(l)c_l(f)}{l^{1+2it}} \iint \xi(f,l,u,v)(\log^2 u)\exp(iQ)\frac{du\,dv}{uv^{1+2it}}, \tag{5.5}$$

where the range of integration is indicated by (4.46).

Performing the change of variable $u \mapsto w/v$, we consider instead

$$\sum_{f=1}^{\infty} \frac{\phi_2(f)\sigma_1(f)}{f^{1-2it}} \sum_{l=1}^{\infty} \frac{\phi_1(l)c_l(f)}{l^{1+2it}} \iint \xi\Big(f,l,\frac{w}{v},v\Big)\Big(\log^2\frac{w}{v}\Big)\exp(iQ_1)\frac{dv\,dw}{v^{1+2it}w}, \tag{5.6}$$

where

$$v \approx \frac{F}{L}, \quad w \approx \frac{M}{L} \tag{5.7}$$

and

$$Q_1 = -\frac{1}{2}\delta_1 v - 2\delta_1 \frac{r^2}{w}. \tag{5.8}$$

We integrate with respect to w first. Lemma 6 is applicable with $A_0 = (\log^2 K)L/M$, $A_1 = M/(L\log^4 K)$, $B_1 = (KL/M)^2$ and $\varrho \approx M/L$. Thus we have that the w-integral of (5.6) is $\ll (K^2L/(M\log^4 K))^{-P}\log^2 K$; and (5.6) can be discarded if $L \gg K^{\varepsilon}$, since we have $M \ll K^2$. That is, we may assume that $L \ll K^{\varepsilon}$. Then obviously the case $F \ll K^{\varepsilon}$ can be ignored. In particular, we may assume that $F/L \gg K^{\varepsilon}$ as well, under Convention 1. With this, we integrate, in (5.6), with respect to v first. We have

$$\frac{\partial}{\partial v}(Q_1 - 2t\log v) = -\frac{1}{2}\delta_1 - 2\frac{t}{v}. \tag{5.9}$$

In the case $\delta_1=1$, Lemma 6 can be applied with $A_0=(\log^2 K)L/F$, $A_1=F/(L\log^4 K)$, $B_1=1$ and $\varrho\approx F/L$. We see readily that this case can be ignored. Likewise the case $\delta_1=-1$ with $t\ll K^\varepsilon$ can be dropped from consideration under Convention 1, because we now have $F/L\gg K^\varepsilon$ and Lemma 6 works as in the previous case.

On the other hand, if $\delta_1=-1$ and $K^\varepsilon\ll t$, then we may compute the v-integral asymptotically with the saddle point method. The saddle point is at $v=4t$. We divide the integral into two parts according to whether $|v-4t|<4t/\varrho_1$ or not, with $\varrho_1=t^{2/5}$. Then we proceed in a fashion much similar to the proof of (3.31). Thus, under the current situation, the integral of (5.6) is seen to be

$$\sim\sqrt{\frac{\pi}{t}}\exp\bigl(-2it\log 4t+\tfrac14\pi i\bigr)\int\xi\Bigl(f,l,\frac{w}{4t},4t\Bigr)\log^2\Bigl(\frac{w}{4t}\Bigr)\exp(iQ_1(4t))\,\frac{dw}{w},\tag{5.10}$$

where $Q_1(4t)=Q_1|_{v=4t}$. That is, the estimation of (5.6) has been reduced to that of

$$\frac{1}{\sqrt{t}}\sum_{l\leqslant K^\varepsilon}\frac1l\int\Biggl|\sum_{f=1}^{\infty}\phi_2(f)\,\frac{\sigma_1(f)\,c_l(f)}{f^{1-2it}}\,\xi\Bigl(f,l,\frac{w}{4t},4t\Bigr)\Biggr|\,\frac{dw}{w}.\tag{5.11}$$

By invoking (4.42), this sum over f is equal to

$$\begin{aligned}\frac{i}{8\pi^3}\int_{(0)}\int_{(0)}\int_{(2)}\frac{(4\pi)^{2s_2}(4t)^{s_1-s_2}}{w^{s_1+s_2}l^{2s_2}}\,\phi^*(s_1)\,\phi^*(s_2)\,\phi_2^*(s_3)\,C_l(s_1-s_2+s_3+1-2it)\\ \times\zeta(s_1-s_2+s_3-2it)\,\zeta(s_1-s_2+s_3+1-2it)\,ds_1\,ds_2\,ds_3,\end{aligned}\tag{5.12}$$

where ϕ^* and ϕ_2^* are Mellin transforms of the respective functions, and

$$C_l(s)=\sum_{d_1d_2d_3d_4=l}\frac{\mu(d_3)\mu(d_4)}{(d_1d_2d_3^2)^s}\,d_1d_2^2d_3^2,\tag{5.13}$$

with the Möbius function μ. In fact, we have, in the region of absolute convergence,

$$\begin{aligned}\zeta(s)\zeta(s-1)\sum_{l=1}^{\infty}\frac{C_l(s)}{l^\lambda}&=\zeta(s)\zeta(s-1)\,\frac{\zeta(s+\lambda-1)\zeta(s+\lambda-2)}{\zeta(\lambda)\zeta(2s+\lambda-2)}\\ &=\sum_{n=1}^{\infty}\frac{\sigma_1(n)}{n^s}\,\frac{\sigma_{1-\lambda}(n)}{\zeta(\lambda)}=\sum_{l=1}^{\infty}\frac{1}{l^\lambda}\sum_{n=1}^{\infty}\frac{\sigma_1(n)\,c_l(n)}{n^s},\end{aligned}\tag{5.14}$$

where we have used two well-known formulas of Ramanujan [26, (1.3.3) and (1.5.4)]. Then, after the truncation to $|s_j|\leqslant K^\varepsilon$ for all j in (5.12), we move the s_1-contour to the imaginary axis. Under Convention 1 and $t\gg K^\varepsilon$, we encounter no singularities. In this way we find that

$$S_r\ll K^\varepsilon.\tag{5.15}$$

Next, let us consider S_d, the contribution of the discrete spectrum; note (2.29) and (2.32). We need first to approximate $\Psi(i\varkappa;X)$, where $\varkappa\in\mathbf{R}$, and X is defined by (4.41) with $Y=Q$. To this end, we invoke the identity

$$\begin{aligned}&{}_2F_1\left(\frac{1}{2}+i\varkappa,\frac{1}{2}+i\varkappa;1+2i\varkappa;-\frac{1}{u}\right)\\&\quad=\left(\frac{1}{2}\left(1+\sqrt{1+\frac{1}{u}}\right)\right)^{-1-2i\varkappa}{}_2F_1\left(\frac{1}{2},\frac{1}{2}+i\varkappa;1+i\varkappa;\left(\frac{1-\sqrt{1+1/u}}{1+\sqrt{1+1/u}}\right)^2\right),\end{aligned}\tag{5.16}$$

which is an instance of the quadratic transformations of the Gaussian hypergeometric function (see, e.g., [18, (9.6.12)]). This implies that uniformly in $\varkappa$,

$${}_2F_1\left(\frac{1}{2}+i\varkappa,\frac{1}{2}+i\varkappa;1+2i\varkappa;-\frac{1}{u}\right)\sim\left(\frac{1}{2}\left(1+\sqrt{1+\frac{1}{u}}\right)\right)^{-1-2i\varkappa},\tag{5.17}$$

with u as in (4.46); note that we have currently $u\gg G^2K^{-\varepsilon}$. Thus, the estimation of S_d is reduced to that of

$$\begin{aligned}&\sum_{f=1}^{\infty}\frac{\phi_2(f)}{f^{1/2-2it}}\sum_{l=1}^{\infty}\frac{\phi_1(l)c_l(f)}{l^{1+2it}}\sum_{j=1}^{\infty}2^{2\delta_3 i\varkappa_j}\alpha_j\tau_j(f)H_j^2(\tfrac{1}{2})\\&\qquad\times\left(1+\frac{\delta_3 i}{\sinh\pi\varkappa_j}\right)\frac{\Gamma^2(\frac{1}{2}+\delta_3 i\varkappa_j)}{\Gamma(1+2\delta_3 i\varkappa_j)}\Xi(f,l,\varkappa_j,\delta_1,\delta_3),\end{aligned}\tag{5.18}$$

where

$$\Xi(f,l,\varkappa,\delta_1,\delta_3)=\iint\xi(f,l,u,v)\exp(iQ)\left(\sqrt{u}+\sqrt{u+1}\right)^{-1-2\delta_3 i\varkappa}\frac{du\,dv}{uv^{1+2it}}\tag{5.19}$$

with $\delta_3=\pm1$.

We have

$$\frac{\partial}{\partial u}\left(Q-2\delta_3\varkappa\log\left(\sqrt{u}+\sqrt{u+1}\right)\right)=2\delta_1\frac{r^2}{u^2v}-\frac{\delta_3\varkappa}{\sqrt{u(u+1)}}.\tag{5.20}$$

On the right-hand side, provided $\varkappa\gg(K/G)\log K$, the second term is dominant, and Lemma 6 becomes relevant with $A_0=(M/F)^{-3/2}$, $A_1=M/(F\log^4 K)$, $B_1=\varkappa F/M$ and $\varrho\approx M/F$. In fact, we have $r^2/u^2v\ll FK^2L/M^2\ll((K/G)\log K)(F/M)$ in the relevant domain, by (4.3) and (4.46). Thus, this case can be ignored. That is, we may truncate the inner-most sum of (5.18) to $\varkappa_j\ll(K/G)\log K$. We have then $\left(\sqrt{u}+\sqrt{u+1}\right)^{i\varkappa_j}\sim\left(2\sqrt{u}\right)^{i\varkappa_j}$, as $\varkappa_j/u\ll FK(\log K)/GM\ll K^{1+\varepsilon}/G^3\ll K^{-\varepsilon}$ because of (4.3), (4.40), (5.3) and Convention 1.

Hence the estimation of S_d is further reduced to that of

$$\sum_{\varkappa_j \ll (K/G)\log K} \frac{\alpha_j}{\sqrt{\varkappa_j}} H_j^2(\tfrac{1}{2}) \left| \sum_{f=1}^{\infty} \frac{\phi_2(f)\tau_j(f)}{f^{1/2-2it}} \sum_{l=1}^{\infty} \frac{\phi_1(l)c_l(f)}{l^{1+2it}} \Xi_1(f,l,\varkappa_j,\delta_1,\delta_3) \right|, \tag{5.21}$$

with

$$\Xi_1(f,l,\varkappa,\delta_1,\delta_3) = \iint \xi(f,l,u,v)\exp(iQ)\, \frac{du\,dv}{u^{3/2+\delta_3 i\varkappa} v^{1+2it}}. \tag{5.22}$$

As before, we perform the change of variable $u \mapsto w/v$. Then an application of the Mellin inversion gives

$$\Xi_1(f,l,\varkappa,\delta_1,\delta_3) = -\frac{1}{4\pi^2} \int_{(0)} \int_{(0)} \phi^*(s_1)\phi^*(s_2) f^{s_2-s_1} \left(\frac{4\pi}{l}\right)^{2s_2} \tag{5.23}$$

$$\times \left(\int_0^{\infty} \frac{\exp\left(-\frac{1}{2}\delta_1 iv\right)}{v^{1/2-\delta_3 i\varkappa+2it-s_1+s_2}}\, dv \right) \left(\int_0^{\infty} \frac{\exp(-2\delta_1 ir^2/w)}{w^{3/2+\delta_3 i\varkappa+s_1+s_2}}\, dw \right) ds_1\, ds_2,$$

which can be verified as (4.25). Thus,

$$\Xi_1(f,l,\varkappa,\delta_1,\delta_3) = \frac{\delta_1 i}{\pi^2 4^{1+it} r^{1+2\delta_3 i\varkappa}} \int_{(0)} \int_{(0)} \phi^*(s_1)\phi^*(s_2) f^{s_2-s_1} \left(\frac{2\pi}{l}\right)^{2s_2}$$

$$\times r^{-2s_1-2s_2} \exp(\delta_1 \pi i(-\delta_3 i\varkappa + it - s_1)) \tag{5.24}$$

$$\times \Gamma\left(\tfrac{1}{2}+\delta_3 i\varkappa - 2it + s_1 - s_2\right) \Gamma\left(\tfrac{1}{2}+\delta_3 i\varkappa + s_1 + s_2\right) ds_1\, ds_2.$$

Inserting this assertion into (5.21), we encounter

$$\sum_{l=1}^{\infty} \frac{\phi_1(l)c_l(f)}{l^{1+2it+2s_2}} = \frac{1}{2\pi i} \int_{(0)} \phi_1^*(s_3) \frac{\sigma_{-2it-2s_2-s_3}(f)}{\zeta(1+2it+2s_2+s_3)}\, ds_3, \tag{5.25}$$

where the Ramanujan identity [26, (1.5.4)] has been used.

Hence

$$S_d \ll K^{-1+\varepsilon} \int_{(0)} \int_{(0)} \int_{(0)} |\phi^*(s_1)\phi^*(s_2)\phi_1^*(s_3)|$$

$$\times \sum_{\varkappa_j \ll (K/G)\log K} \frac{\alpha_j}{\sqrt{\varkappa_j}} H_j^2(\tfrac{1}{2}) \left| \sum_{f=1}^{\infty} \phi_2(f)\tau_j(f) \frac{\sigma_{-2it-2s_2-s_3}(f)}{f^{1/2-2it+s_1-s_2}} \right| |ds_1|\,|ds_2|\,|ds_3|$$

$$\ll K^{-1+\varepsilon} \sup_U \sup_{t_1,t_2} \frac{1}{\sqrt{U}} \sum_{U \leqslant \varkappa_j \leqslant 2U} \alpha_j H_j^2(\tfrac{1}{2}) \left| \sum_{f=1}^{\infty} \phi_2(f)\tau_j(f) \frac{\sigma_{-it_1}(f)}{f^{1/2-it_2}} \right|, \tag{5.26}$$

where $U \ll (K/G)\log K$ and $|t_\nu - 2t| \ll K^\varepsilon$ $(\nu=1,2)$, after an obvious truncation of the triple integral. This and Lemma 7 imply that

$$S_d \ll K^{-1+\varepsilon} \left(\frac{K}{G}\right)^{1/2} \left(\frac{K}{G} + F^{1/2}\right) \ll \left(\frac{K}{G^3}\right)^{1/2} K^\varepsilon \ll K^\varepsilon, \tag{5.27}$$

because $F\ll(K/G)^2K^\varepsilon$. Here we have used a well-known bound for the spectral fourth moment of $H_j\left(\frac{1}{2}\right)$ that follows from, e.g., (1.28).

The discussion of S_c is analogous to the above, up to (5.26). In fact, the change is only that (5.26) is to be replaced by the expression

$$K^{-1+\varepsilon}\sup_U\sup_{t_1,t_2}\int_{-U}^{U}\frac{\left|\zeta\left(\frac{1}{2}+i\varkappa\right)\right|^4}{|\zeta(1+2i\varkappa)|^2}\left|\sum_{f=1}^{\infty}\phi_2(f)\frac{\sigma_{2i\varkappa}(f)\sigma_{-it_1}(f)}{f^{1/2+i\varkappa-it_2}}\right|\frac{d\varkappa}{\sqrt{|\varkappa|+1}}. \tag{5.28}$$

To this we could apply a continuous analogue of Lemma 7, but we take a different way to motivate a later purpose. Thus, we note first that the part corresponding to $|\varkappa|\ll K^\varepsilon$ contributes $K^{-1+\varepsilon}F^{1/2}\ll K^\varepsilon/G$. To treat the part with $|\varkappa|\gg K^\varepsilon$, we use Mellin inversion. The last sum is equal to

$$\begin{aligned}\frac{1}{2\pi i}\int_{(2)}\phi_2^*(s)\zeta\left(s+\tfrac{1}{2}+i\varkappa-it_2\right)\zeta\left(s+\tfrac{1}{2}-i\varkappa-it_2\right)\zeta\left(s+\tfrac{1}{2}+i\varkappa+i(t_1-t_2)\right)&\\ \times\zeta\left(s+\tfrac{1}{2}-i\varkappa+i(t_1-t_2)\right)\left(\zeta(2s+1+it_1-2it_2)\right)^{-1}ds,&\end{aligned} \tag{5.29}$$

again by the formula of Ramanujan [26, (1.3.3)]. After truncating to $|s|\leqslant\frac{1}{2}|\varkappa|$, we shift the contour to (0). We see that the integral is

$$\begin{aligned}\ll\left((1+|\varkappa|)^2(1+|\varkappa-2t|)(1+|\varkappa+2t|)\right)^{1/6}K^\varepsilon&\\ +F^{1/2}\left((1+|\varkappa-2t|)^{-1/\varepsilon}+(1+|\varkappa+2t|)^{-1/\varepsilon}\right)K^\varepsilon.&\end{aligned} \tag{5.30}$$

The second term comes from the possible simple poles at $\frac{1}{2}\pm i\varkappa+it_2$, since one may assume that $|\varkappa|\geqslant4|t_1-t_2|$, under Convention 1. Inserting this into (5.28) and invoking Ingham's well-known bound [26, §7.6] for the fourth moment of $\zeta\left(\frac{1}{2}+it\right)$, we find that the relevant part of the integral in (5.28) is $\ll(K/G)^{7/6+\varepsilon}$. Hence

$$S_c\ll\frac{K^\varepsilon}{G}+K^{-1+\varepsilon}\left(\frac{K}{G}\right)^{7/6}, \tag{5.31}$$

which is negligible.

It remains to consider S_h. From (2.30), it is

$$\ll\sum_{f=1}^{\infty}\frac{\phi_2(f)}{f^{1/2}}\sum_{l=1}^{\infty}\frac{\phi_1(l)}{l}|c_l(f)|\sum_{\substack{k=6\\2|k}}^{\infty}\sum_{j=1}^{\vartheta(k)}\alpha_{j,k}|\tau_{j,k}(f)|H_{j,k}^2\left(\tfrac{1}{2}\right)\left|\Psi\left(k-\tfrac{1}{2};X\right)\right|. \tag{5.32}$$

The Ψ-factor is, by (2.32) and (4.41),

$$\ll\frac{\Gamma(k)^2}{\Gamma(2k)}\int u^{-k-1}{}_2F_1\left(k,k;2k;-\frac{1}{u}\right)du\ll\left(\frac{F}{M}\right)^{k-\varepsilon}. \tag{5.33}$$

Here the range of integration is given by (4.46), and the bound is uniform in k; the latter can be seen by using Gauss' integral representation of ${}_2F_1$ (see, e.g., [18, (9.1.4)]). Also, we invoke Deligne's bound

$$|\tau_{j,k}(f)| \leqslant d(f) \tag{5.34}$$

and its elementary consequence

$$H_{j,k}\left(\tfrac{1}{2}\right) \ll k^{1/2+\varepsilon}. \tag{5.35}$$

Further,

$$\sum_{j=1}^{\vartheta(k)} \alpha_{j,k} \ll k \tag{5.36}$$

(see [23, Lemma 3.3]). One could replace (5.34) by the bound given in [23, (3.1.22)], for instance, and (5.35) by an easier bound. At any event, the above combination gives that

$$S_h \ll F^{1/2} L\left(\frac{F}{M}\right)^6 K^{\varepsilon}, \tag{5.37}$$

which is negligible.

Collecting (4.45), (5.1), (5.15), (5.27), (5.31) and (5.37), we end the proof of (1.27) with the condition (5.3). In particular, we have proved (1.23) and consequently Ivić's bound for $H_j\left(\frac{1}{2}\right)$ as well, in a wider context.

6. Intermediate range

Now, we enter into the intermediate range; or more precisely we shall consider $\mathcal{S}_4$, with (5.1), under the conditions

$$\frac{K^{1+\varepsilon}}{G} \ll t \ll K^{3/2-\varepsilon} \quad \text{and} \quad |K-t| \gg GK^{\varepsilon}. \tag{6.1}$$

Here the quantity $K^{1+\varepsilon}/G$ should be equal to the same in (5.3) for an obvious reason. The second condition is by no means a restriction, because otherwise (3.50) already gives what we desire. The conditions (1.26), (4.3), (4.4) with (3.38) and (4.5) are of course retained, but (4.40) now becomes

$$F \ll tLK^{\varepsilon}. \tag{6.2}$$

We shall be brief occasionally, since the reasoning is analogous, though not quite, to that developed in the preceding section.

We begin with S_r. We consider, instead of (5.6), the expression

$$\sum_{f=1}^{\infty}\frac{\phi_2(f)\sigma_1(f)}{f^{1-2it}}\sum_{l=1}^{\infty}\frac{\phi_1(l)c_l(f)}{l^{1+2it}}\iint\xi\Big(f,l,\frac{w}{v},v\Big)\frac{\log^2(w/v)}{v^{1+2it}w}\exp(Y_1 i)\,dv\,dw, \tag{6.3}$$

where the integration range is (5.7) and

$$Y_1=-\frac{tv}{2w}\left(1-\frac{v}{4w}\right)-\frac{1}{2}\delta_1 v\left(1+\frac{v^2}{32w^2}\right)-2\delta_1\frac{r^2}{w}\left(1-\frac{v}{4w}\right). \tag{6.4}$$

We integrate first with respect to v. We have

$$\frac{\partial}{\partial v}(Y_1-2t\log v)=-2\frac{t}{v}\left(1+\frac{v}{4w}-\frac{v^2}{8w^2}\right)-\frac{1}{2}\delta_1\left(1+\frac{3}{32}\frac{v^2}{w^2}-\frac{r^2}{w^2}\right), \tag{6.5}$$

which is close to $-2t/v-\frac{1}{2}\delta_1$; in fact, we have $v/w\ll F/M\ll tK^\varepsilon/(GK)\ll t^{-1/3}$ and $r^2/w^2\ll(G/\log K)^{-2}$, by (1.26), (4.3) and (6.2). Hence, if $\delta_1=1$, then (6.3) is negligible, as can be confirmed with the procedure following (5.9) but with $B_1=Lt/F$. Thus, let us assume that $\delta_1=-1$. Then we apply the saddle point method. The saddle point is at $v_0\sim 4t$, or more precisely, it satisfies the recursive equation

$$v_0=4t\left(1+\frac{v_0}{4w}-\frac{v_0^2}{8w^2}\right)\left(1+\frac{3}{32}\frac{v_0^2}{w^2}-\frac{r^2}{w^2}\right)^{-1}. \tag{6.6}$$

When $|v-v_0|<v_0/\varrho_1$, with $\varrho_1=t^{2/5}$, we have

$$\begin{aligned}Y_1-2t\log v=Y_1(v_0)-2t\log v_0+2t\sum_{j=2}^{\infty}\frac{1}{j}\left(\frac{v_0-v}{v_0}\right)^j\\ +\frac{1}{2}Y_1^{(2)}(v_0)(v-v_0)^2+\frac{1}{6}Y_1^{(3)}(v_0)(v-v_0)^3,\end{aligned} \tag{6.7}$$

where $Y_1^{(\nu)}(v_0)=((\partial/\partial v)^\nu)_{v=v_0}Y_1$. We note that

$$Y_1^{(2)}(v_0)\ll\frac{t}{w^2}\ll\frac{tK^\varepsilon}{(GK)^2}\quad\text{and}\quad Y_1^{(3)}(v_0)\ll\frac{K^\varepsilon}{(GK)^2}.$$

Thus,

$$Y_1-2t\log v\sim Y_1(v_0)-2t\log v_0+t\left(\frac{v_0-v}{v_0}\right)^2, \tag{6.8}$$

and the integral of (6.3) is

$$\sim e^{\pi i/4}\sqrt{\frac{\pi}{t}}\int\xi\Big(f,l,\frac{w}{v_0},v_0\Big)\log^2\Big(\frac{w}{v_0}\Big)\exp(iY_1(v_0)-2it\log v_0)\,\frac{dw}{w}. \tag{6.9}$$

This corresponds to (5.10).

We shall show that (6.9) is negligibly small, if $l \gg K^\varepsilon$. To this end, we note that

$$\frac{d}{dw}(Y_1(v_0)-2t\log v_0)=\frac{tv_0}{2w^2}\Big(1-\frac{v_0}{2w}\Big)-\frac{v_0^3}{32w^3}-2\frac{r^2}{w^2}\Big(1-\frac{v_0}{2w}\Big), \tag{6.10}$$

since the left-hand side is equal to $(Y_1)_w(v_0)$ by the definition of v_0. It is expedient to note here that $w \gg GK/\log K \gg (K+t)^{4/3}$ because of (1.26) and (5.7). With this, inserting the approximation $v_0=4t(1+t/w)+O(t(t+K)^2/w^2)$, which follows readily from (6.6), we get

$$\frac{d}{dw}(Y_1(v_0)-2t\log v_0)=2\frac{t^2-r^2}{w^2}\Big(1-2\frac{t}{w}\Big)+O\Big(\frac{(t(t+K))^2}{w^4}\Big). \tag{6.11}$$

Thus, with T as in (3.38),

$$\frac{d}{dw}(Y_1(v_0)-2t\log v_0)\approx\frac{T}{w^2}. \tag{6.12}$$

In fact, $t^2-r^2\approx T$ and

$$\frac{(t(t+K))^2}{w^2T}\ll\frac{t^2(t+K)}{(GK)^2|K-t|}\ll\frac{t^2}{(t+K)^{5/3}|K-t|}K^{-\varepsilon}\ll\frac{t^{1/3}}{|K-t|}K^{-\varepsilon}, \tag{6.13}$$

which is negligibly small; here we have used the second condition in (6.1). Then Lemma 6 is applied to (6.9), with $A_0=(GK)^{-1}\log^2K$, $A_1=M/(L\log^4K)$, $B_1=T(L/M)^2$ and $\varrho\approx M/L$. The integral is $\ll(LT/M)^{-P}K^\varepsilon$. Hence, (6.9) is indeed negligibly small if $L\gg K^\varepsilon$, because we have (4.3).

That is, in dealing with (6.3), we may assume that we have $L\ll K^\varepsilon$ together with (6.9). We are, thus, in a situation much analogous to that with S_r in the lower range. In this way we are led again to

$$S_r\ll K^\varepsilon. \tag{6.14}$$

We turn to S_d. The reduction to (5.18)-(5.19) does not need to be altered, except for the replacement of Q by Y. With this, we shall consider Ξ. We have

$$\begin{aligned}&\frac{\partial}{\partial u}\big(Y-2\delta_3\varkappa\log(\sqrt{u}+\sqrt{u+1})\big)\\&\qquad=\frac{t}{2u^2}\Big(1-\frac{1}{2u}\Big)+\frac{1}{32}\delta_1\frac{v}{u^3}+2\delta_1\frac{r^2}{u^2v}\Big(1-\frac{1}{2u}\Big)-\frac{\delta_3\varkappa}{\sqrt{u(u+1)}}.\end{aligned} \tag{6.15}$$

This shows in particular that the part with $\varkappa_j\gg K^{1/\varepsilon}$ of S_d can be discarded, as Lemma 6 implies. Thus we have the initial truncation $\varkappa_j\ll K^{1/\varepsilon}$.

We then integrate with respect to v. We have

$$\frac{\partial}{\partial v}(Y-2t\log v)=-\frac{1}{2}\delta_1\left(1+\frac{1}{32u^2}\right)+2\delta_1\frac{r^2}{uv^2}\left(1-\frac{1}{4u}\right)-\frac{2t}{v}. \tag{6.16}$$

Note that $r^2/(uv)\ll K^2L/M\ll(K/G)\log K\ll tK^{-\varepsilon}$ because of (4.3) and the first condition in (6.1). Hence we may adopt the argument following (5.9), and see that the case $\delta_1=1$ can be discarded. Hereafter we shall assume that $\delta_1=-1$. The v-integral has a saddle point at v_1, which satisfies the recursive equation

$$v_1=4t\left(1+\frac{r^2}{uv_1t}\left(1-\frac{1}{4u}\right)\right)\left(1+\frac{1}{32u^2}\right)^{-1}; \tag{6.17}$$

and $v_1=4t(1+O(K^{-\varepsilon}))$, since $u\gg(K+t)^{4/3}/t$, which follows from (1.26), (4.3), (4.46) and (6.2); note that this lower bound is needed in what follows as well. In particular, (5.7) gives

$$F\approx tL, \tag{6.18}$$

which replaces (6.2). The saddle point method yields that Ξ defined by (5.19) with $Q=Y$ is

$$\sim e^{\pi i/4}\sqrt{\frac{\pi}{t}}\int\xi(f,l,u,v_1)\exp(iY(v_1)-2it\log v_1)\left(\sqrt{u}+\sqrt{u+1}\right)^{-1-2\delta_3 i\varkappa}\frac{du}{u}, \tag{6.19}$$

where $Y(v_1)=Y|_{v=v_1}$. In this, we have, by the definition of v_1,

$$\frac{\partial}{\partial u}\left(Y(v_1)-2t\log v_1-2\delta_3\varkappa\log\left(\sqrt{u}+\sqrt{u+1}\right)\right)=(Y)_u(v_1)-\frac{\delta_3\varkappa}{\sqrt{u(u+1)}}. \tag{6.20}$$

We shall show that

$$(Y)_u(v_1)=\frac{t}{2u^2}\left(1-\left(\frac{r}{t}\right)^2\right)(1+O(K^{-\varepsilon}))\approx\frac{T}{u^2t}. \tag{6.21}$$

It suffices to prove the asymptotics; and to this end we may assume that $t\geqslant K^{1-\varepsilon}$, since otherwise the assertion follows immediately from the first line of (6.22) below. Then we note that (6.17) gives $v_1/(16ut)=1/(4u)+O(K^{\varepsilon}/u^2)$ and

$$\frac{4r^2}{v_1t}=\left(\frac{r}{t}\right)^2\frac{1-(r/t)^2}{4u}+O\left(\frac{K^{\varepsilon}}{u^2}\right).$$

Thus,

$$\begin{aligned}(Y)_u(v_1)&=\frac{t}{2u^2}\left(1-\frac{1}{2u}-\frac{v_1}{16ut}-4\frac{r^2}{v_1t}\left(1-\frac{1}{2u}\right)\right)\\&=\frac{t}{2u^2}\left(1-\left(\frac{r}{t}\right)^2\right)\left(1-\frac{3}{4u}-\frac{1}{4u}\left(\frac{r}{t}\right)^2\right)+O\left(\frac{tK^{\varepsilon}}{u^4}\right).\end{aligned} \tag{6.22}$$

Also, by (1.26) and Convention 1,

$$\frac{1}{u^2}\left(1-\left(\frac{r}{t}\right)^2\right)^{-1} \ll \frac{t^{1/3}}{|K-t|}K^{-\varepsilon}, \tag{6.23}$$

which proves (6.21), because of the second condition in (6.1). In passing, we stress that both assumptions in (6.1) are indeed required above.

With (6.20)-(6.21), Lemma 6 allows us to impose the truncation

$$\varkappa_j \ll \varkappa_0 = \frac{TK^\varepsilon}{GK}. \tag{6.24}$$

In fact, we may set $\varrho \approx M/F$, and in the relevant domain of u we have $T/ut \ll TK^\varepsilon/(GK)$. Thus, provided $\varkappa \gg \varkappa_0$, a specification is given by $A_0=(M/F)^{-3/2}$, $A_1=M/(F\log^4 K)$ and $B_1=\varkappa F/M$. Note that $\varkappa_0 \gg K^\varepsilon$ under Convention 1.

Hence the estimation of S_d has been reduced to that of

$$\frac{1}{\sqrt{tU}}\sum_{U\leqslant\varkappa_j\leqslant 2U}\alpha_j H_j^2\left(\tfrac12\right)\left|\sum_{f=1}^{\infty}\frac{\phi_2(f)\tau_j(f)}{f^{1/2-2it}}\sum_{l=1}^{\infty}\frac{\phi_1(l)c_l(f)}{l^{1+2it}}\Xi_2(f,l,\varkappa_j,\delta_3)\right|, \tag{6.25}$$

where $U \ll \varkappa_0$ and

$$\begin{aligned}\Xi_2(f,l,\varkappa,\delta_3) = \int_0^\infty &\kappa(u)\xi(f,l,u,v_1)\\ &\times\exp(iY(v_1)-2it\log v_1)\left(\sqrt{u}+\sqrt{u+1}\right)^{-1-2i\delta_3\varkappa}\frac{du}{u}.\end{aligned} \tag{6.26}$$

Here κ is a smooth weight whose role is analogous to that of θ in (3.26).

Appealing to Mellin inversion, we find that (6.25) is

$$\begin{aligned}\ll \frac{K^\varepsilon}{\sqrt{tU}}\int_{(0)}\int_{(0)}\int_{(0)}&|\phi^*(s_1)\phi^*(s_2)\phi_1^*(s_3)|\sum_{U\leqslant\varkappa_j\leqslant 2U}\alpha_j H_j^2\left(\tfrac12\right)\\ &\times|\Lambda(\varkappa_j,\delta_3;s_1,s_2)|\left|\sum_{f=1}^{\infty}\phi_2(f)\tau_j(f)\frac{\sigma_{-2it-2s_2-s_3}(f)}{f^{1/2-2it+s_1-s_2}}\right||ds_1|\,|ds_2|\,|ds_3|,\end{aligned} \tag{6.27}$$

where

$$\Lambda(\varkappa,\delta_3;s_1,s_2)=\int_0^\infty \kappa(u)\exp(iY(v_1)-2it\log v_1)\left(\sqrt{u}+\sqrt{u+1}\right)^{-1-2i\delta_3\varkappa}\frac{du}{u^{1+s_1+s_2}}. \tag{6.28}$$

This should be compared with (5.26).

We shall bound Λ. Naturally we could truncate (6.27) to $|s_1|, |s_2|, |s_3| \ll K^\varepsilon$, as we assume now. We have, by (6.20)–(6.21),

$$\begin{aligned}\frac{\partial}{\partial u}\big(Y(v_1)-2t\log v_1+i(s_1+s_2)\log u-2\delta_3\varkappa\log(\sqrt{u}+\sqrt{u+1})\big)\\ =2\pi^2\operatorname{sgn}(t-r)\frac{T}{tu^2}(1+O(K^{-\varepsilon}))+i\frac{s_1+s_2}{u}-\frac{\delta_3\varkappa}{\sqrt{u(u+1)}}.\end{aligned} \tag{6.29}$$

When $\delta_3=\operatorname{sgn}(r-t)$ and $L+\varkappa\gg K^\varepsilon$, we may appeal to Lemma 6 with $A_0=(M/F)^{-3/2}$, $A_1=M/F$, $B_1=(TL/M+\varkappa)F/M$ and $\varrho\approx M/F$. In fact, this assertion on B_1 follows from the Taylor expansion of the left-hand side of (6.29) around any real point in the relevant domain, coupled with the fact that $ut\approx uv_1\approx M/L$ by (4.46), and $T\gg M$ as well as $|s_1+s_2|\ll K^\varepsilon$, thus under Convention 1. We find that Λ is negligibly small with the present supposition. That is, provided $\delta_3=\operatorname{sgn}(r-t)$, we may impose the truncation $L+\varkappa_j\ll K^\varepsilon$ in (6.27); in particular, $U\ll K^\varepsilon$, and $F\ll tK^\varepsilon$ by (6.18). Then applying Lemma 7 to the sum over $\varkappa_j$ of (6.27), we immediately find that the case $\delta_3=\operatorname{sgn}(r-t)$ can be dropped.

Hence we assume now that $\delta_3=\operatorname{sgn}(t-r)$. With this, let u_1 be the saddle point of the integral in (6.28). By the second line of (6.29) we have $1/u_1\ll(t/T)(\varkappa+K^\varepsilon)$. This implies that if $\varkappa_j\ll K^\varepsilon$, then $L\ll K^\varepsilon$, because (6.18) gives $tL/M\approx F/M\approx 1/u_1$. That is, we can ignore this situation as well; and hence we may assume that $\delta_3=\operatorname{sgn}(t-r)$ and $\varkappa\gg K^\varepsilon$. Then the saddle point method yields, in a fashion similar to the argument leading up to (3.32), that

$$\Lambda(\varkappa,\operatorname{sgn}(t-r);s_1,s_2)\ll\frac{1}{\sqrt{u_1^3|\Lambda_0|}}, \tag{6.30}$$

where

$$\Lambda_0=\left(\frac{\partial}{\partial u}\right)^2_{u=u_1}\big(Y(v_1)-2t\log v_1+i(s_1+s_2)\log u-2\delta_3\varkappa\log(\sqrt{u}+\sqrt{u+1})\big), \tag{6.31}$$

with $\delta_1=-1$ in the definition of Y. We have

$$\Lambda_0\approx\frac{T}{tu_1^3}. \tag{6.32}$$

Inserting this into (6.27) via (6.30) together with the aforementioned truncation of the triple integral, we find that

$$S_d\ll K^\varepsilon\sup_U\sup_{t_1,t_2}\frac{1}{\sqrt{TU}}\sum_{U\leqslant\varkappa_j\leqslant 2U}\alpha_jH_j^2(\tfrac12)\left|\sum_{f=1}^\infty\phi_2(f)\tau_j(f)\frac{\sigma_{-it_1}(f)}{f^{1/2-it_2}}\right|, \tag{6.33}$$

with

$$U \ll \frac{TK^\varepsilon}{GK}, \quad |t_\nu - 2t| \ll K^\varepsilon \ (\nu = 1, 2) \quad \text{and} \quad F \ll Ut. \tag{6.34}$$

The bound for F follows from the observation that $M/F \approx u_1 \approx T/(t\varkappa)$. The assertion (6.33) is obviously an extension of (5.26), since $T \approx K^2$ in the lower range.

The discussions of S_c and S_h are analogous to that in the previous section; and it can readily be seen that their contributions are again negligible.

Collecting (6.14), (6.33) and the last assertion, we conclude that our problem has been reduced to the estimation of the spectral sum in (6.33). However, unlike the case of the lower range, the sole application of Lemma 7 to (6.33) does not settle our problem. In fact, we end up with

$$\mathcal{S}(G, K) \ll \left(GK + \sqrt{Tt}\right)^{1+\varepsilon}, \tag{6.35}$$

which yields the inferior exponent $\frac{3}{8}$ in place of $\frac{1}{3}$ in (1.1).

To resolve this difficulty, we have to devise yet another spectral mean value result, whose discussion is to be developed in the next section. To make our next aim clearer, we perform a transformation analogous to (5.29) to the sum over f in (6.33). It is expressed as

$$\frac{1}{2\pi i} \int_{(0)} \phi_2^*(s) \frac{H_j\left(s + \frac{1}{2} - it_2\right) H_j\left(s + \frac{1}{2} + i(t_1 - t_2)\right)}{\zeta(2s + 1 + it_1 - 2it_2)} \, ds, \tag{6.36}$$

which can be truncated to $|s| \ll K^\varepsilon$. Hence, in the intermediate range we have

$$\mathcal{S}(G, K) \ll GK^{1+\varepsilon} \Big(1 + \sup_U \sup_{t_3} \sqrt{\mathcal{T}(U, t_3)(U/T)}\Big), \tag{6.37}$$

where $t_3 \approx t$, $U \ll TK^\varepsilon/(GK)$, and $\mathcal{T}$ is defined by (1.29). Appealing to Theorem 2, we would be able to end the proof of (1.27) immediately.

7. Hybrid moment

Now, we begin our discussion of the mean value $\mathcal{T}(K, t)$. Our aim is to prove (1.30). In the course of discussion we shall encounter two instances of applications of Lemma 4, as indicated in the introduction. Accordingly, the present section is divided into two parts, with the second starting at (7.27). It should be understood that the basic parameters are independent of those utilised above. On the other hand, smooth weights attached to sums over integers are as before, and bounds for their derivatives will be applied without mentioning. Also, we shall not give details about applications of Lemma 6 and the saddle point method, since they are quite similar to those that we have encountered above.

First of all, we observe that we may restrict ourselves to the situation

$$K^{1+\varepsilon} \ll t \ll K^{2-\varepsilon}, \tag{7.1}$$

with a sufficiently large K. In fact, the case $0\leqslant t\ll K^{1+\varepsilon}$ is contained in (1.28). On the other hand, the case $K^{2-\varepsilon}\ll t$ is readily settled with a combination of Ivić's bound for $H_j(\frac{1}{2})$ and (1.19). Thus (7.1) is assumed throughout the present section.

Then we put

$$h(r)=K^{-2P_0}\prod_{p=0}^{P_0-1}\left(r^2+(p+\tfrac{1}{2})^2\right)\left(\exp\left(-\left(\frac{r-K}{G}\right)^2\right)+\exp\left(-\left(\frac{r+K}{G}\right)^2\right)\right), \tag{7.2}$$

with

$$G=K^{1-\varepsilon}, \tag{7.3}$$

and consider

$$\sum_{j=1}^{\infty}\alpha_j H_j^2(\tfrac{1}{2})\left|H_j(\tfrac{1}{2}+it)\right|^2 h(\varkappa_j). \tag{7.4}$$

Obviously it suffices to prove that this is bounded by the right-hand side of (1.30). The integer $P_0\geqslant 1$ is to be fixed later, at (7.30), where the effect of the polynomial factor of high order will become apparent.

In much the same way as (3.40) one may show that

$$H_j(\tfrac{1}{2}+it)\ll(\log^2 K)\sum_{M\ll t}\int_{\gamma^{-1}-i\gamma^2}^{\gamma^{-1}+i\gamma^2}\left|\sum_{m=1}^{\infty}\phi(m;M)\tau_j(m)m^{-1/2-it-\xi}\right||d\xi|, \tag{7.5}$$

with $\gamma=\log^2 K$ and dyadic numbers M, uniformly in ψ_j under consideration. Thus, in place of the factor $\left|H_j(\frac{1}{2}+it)\right|^2$ of (7.4), we may put

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{\phi_0(m)\overline{\phi_0(n)}}{\sqrt{mn}}\left(\frac{m}{n}\right)^{it}\tau_j(m)\tau_j(n), \tag{7.6}$$

where ϕ_0 is as in (4.1) with $q=1$. We apply Mellin inversion to ϕ_0 and subsequent truncation of the integration range, and also invoke (1.6). In this way, our problem (7.1) is reduced to the estimation of

$$\sum_{n=1}^{\infty}\phi(n)\frac{\sigma_{2it_1}(n)}{n^{1/2+it_2}}\sum_{j=1}^{\infty}\alpha_j\tau_j(n)H_j^2(\tfrac{1}{2})h(\varkappa_j), \tag{7.7}$$

where $|t_\nu-t|\ll K^\varepsilon$ ($\nu=1,2$), and ϕ is as in (4.2) but with $M\ll t^2$.

We apply Lemma 3 or (2.9) to the last inner sum. We have

$$\begin{aligned}&\mathcal{H}_1(n;h)\ll d(n)n^{-1/2}GK\log^2 K,\quad \mathcal{H}_3(n;h)\ll\exp(-\log^2 K),\\ &\mathcal{H}_5(n;h)\ll d(n)n^{-1/2}\exp(-K),\quad \mathcal{H}_6(n;h)\ll\sigma_{-1}(n)n^{1/2}\exp\left(-\left(\frac{K}{G}\right)^2\right),\end{aligned} \tag{7.8}$$

which can be verified following the discussion in [23, p. 128]. As to the contribution of $\mathcal{H}_7$, it is expressed as

$$\begin{aligned}\frac{i}{2\pi^2}\int_{-\infty}^{\infty} h(r)\frac{\left|\zeta\left(\frac{1}{2}+ir\right)\right|^4}{|\zeta(1+2ir)|^2}\int_{(1)}&\phi^*(s)\zeta\left(s+\tfrac{1}{2}-ir+it_2\right)\\ &\times\zeta\left(s+\tfrac{1}{2}+ir+it_2\right)\zeta\left(s+\tfrac{1}{2}-ir+it_2-2it_1\right)\\ &\times\zeta\left(s+\tfrac{1}{2}+ir+it_2-2it_1\right)\zeta(2s+1+2it_2-2it_1)^{-1}\,ds\,dr,\end{aligned}\tag{7.9}$$

with the Mellin transform ϕ^* of ϕ. We truncate the inner integral to $s\ll K^\varepsilon$, and shift the contour to the imaginary axis. Because of the lower bound for t in (7.1), we do not encounter any singularity, and (7.9) is seen to be $\ll Kt^{2/3}$. Hence, the contribution to (7.7) of the terms in (2.9) except for $\mathcal{H}_2$ and $\mathcal{H}_4$ is $\ll GK\log^5 K+Kt^{2/3}$.

The discussion of the contributions of $\mathcal{H}_2$ and $\mathcal{H}_4$ remains. We shall treat $\mathcal{H}_2$ first. The argument is an adaptation of [12, Chapters 2–3] and [23, §3.4], except for the treatment of a contribution of holomorphic cusp forms, i.e., (7.22) below.

Thus, we need to estimate the sum

$$\sum_{f=1}^{\infty}\frac{d(f)}{f^{1+it_2}}\sum_{n=1}^{\infty}\sigma_{2it_1}(n)\,d(n+f)\,W_f^+\left(\frac{n}{f}\right),\tag{7.10}$$

where

$$W_f^+(x)=\phi(fx)\,x^{-1/2-it_2}\,\Psi^+\left(\frac{1}{x};h\right),\tag{7.11}$$

with Ψ^+ defined by (2.7). We may truncate the outer sum to $f\ll K^{1/\varepsilon}$, as can be confirmed by moving the r-contour of (2.7) to $\operatorname{Im} r=-\frac{3}{4}$. We have, with $x=n/f$,

$$\begin{aligned}\Psi^+\left(\frac{1}{x};h\right)\sim 4\pi^{3/2}GK\operatorname{Re}\int_0^1&(y(1-y)(1+xy))^{-1/2}\\ &\times\left(\frac{y(1-y)}{1/x+y}\right)^{iK}\exp\left(-\left(\frac{G}{2}\log\frac{y(1-y)}{1/x+y}\right)^2\right)dy.\end{aligned}\tag{7.12}$$

The maximum value of $y(1-y)/(1/x+y)$ is less than $(1+x^{-1/2})^{-2}$. Thus, $\Psi^+(1/x;h)$ is negligibly small when $x\ll G^2K^{-\varepsilon}$. Therefore we may further impose the truncation $f\ll K^\varepsilon M/G^2$. With this, we compute the last integral asymptotically by the saddle point method. We get, after some simplification,

$$\Psi^+\left(\frac{1}{x};h\right)\sim 2^{3/2}\pi^2GK^{1/2}x^{-1/4}\operatorname{Re}\exp\left(\frac{2iK}{\sqrt{x}}-\frac{G^2}{x}\right).\tag{7.13}$$

Then, via (7.11), we may consider, instead of (7.10), the estimation of

$$GK^{1/2}\sum_{f=1}^{\infty}\frac{\phi_2(f)\,d(f)}{f^{1+it_2}}\sum_{n=1}^{\infty}\sigma_{2it_1}(n)\,d(n+f)\,V_f^+\left(\frac{n}{f};\delta\right),\tag{7.14}$$

where

$$V_f^+(x;\delta)=\phi(fx)x^{-3/4-it_2}\exp\left(\frac{2i\delta K}{\sqrt{x}}-\frac{G^2}{x}\right),\quad \delta=\pm1, \tag{7.15}$$

and ϕ_2 is as in (4.39) but with

$$F\ll\frac{K^\varepsilon M}{G^2}. \tag{7.16}$$

To the inner sum of (7.14) we apply the spectral decomposition (2.17), with $\alpha=2it_1$ and $\beta=0$. The leading term D_r has the factor $Y_f(x;2it_1,0)$ in the integrand, and it should be understood as a limit of the right-hand side of (2.22). The definition (7.15) implies that Lemma 6 applies here and that the contribution of D_r is negligible.

We turn to the contribution of the part D_d defined by (2.19). Thus, let us study the $\Psi_\pm$-factors. By (2.24)–(2.25),

$$\Psi_+(i\varkappa;2it_1,0;V_f^+(\,\cdot\,;\delta))=\frac{1}{4\pi i}\cosh(\pi t_1)\int_{(\varepsilon)}\cos(\pi s)\Gamma(s+i\varkappa)\Gamma(s-i\varkappa) \tag{7.17}$$
$$\times\Gamma\left(\tfrac12-it_1-s\right)\Gamma\left(\tfrac12+it_1-s\right)\left(\int_0^\infty x^{s+it_1-1/2}V_f^+(\,\cdot\,;\delta)\,dx\right)ds$$

and

$$\Psi_-(i\varkappa;2it_1,0;V_f^+(\,\cdot\,;\delta))=\frac{1}{4\pi i}\cosh(\pi\varkappa)\int_{(\varepsilon)}\sin(\pi s)\Gamma(s+i\varkappa)\Gamma(s-i\varkappa) \tag{7.18}$$
$$\times\Gamma\left(\tfrac12-it_1-s\right)\Gamma\left(\tfrac12+it_1-s\right)\left(\int_0^\infty x^{s+it_1-1/2}V_f^+(\,\cdot\,;\delta)\,dx\right)ds,$$

with $\varkappa$ being real. We shift the s-contour to $\left(-P_1+\frac14\right)$ with P_1 a non-negative integer. On the new contour the integrands are

$$\ll\exp(\pi|\lambda|-\pi\max\{|\lambda|,|\varkappa|\}-\pi\max\{|\lambda|,t_1\})$$
$$\times(1+|\lambda|K^{-\varepsilon})^{-P_2}\left(\frac{F}{M}\,\frac{(1+|\lambda+t_1|)(1+|\lambda-t_1|)}{(1+|\lambda+\varkappa|)(1+|\lambda-\varkappa|)}\right)^{P_1}, \tag{7.19}$$

with $\operatorname{Im}s=\lambda$ and any integer $P_2\geqslant0$, where the implied constant depends only on ε, P_1 and P_2; the residues are bounded analogously, with $|\lambda|=|\varkappa|$ and $j<P_1$ in place of the exponent P_1. In fact, the Γ-factors are easy to bound, and the x-integral is $\ll(1+|\lambda|K^{-\varepsilon})^{-P_2}(F/M)^{P_1}$, which can be confirmed by Lemma 6, while noting (7.3), $|t_1-t_2|\ll K^\varepsilon$ and $x\approx M/F$ with (7.16). Thus, taking P_1 sufficiently large, we see that the truncation $\varkappa_j\ll K^{1/\varepsilon}$ can be introduced. With this, we set $P_1=0$, getting the truncation $\lambda\ll K^\varepsilon$ and subsequently $\varkappa_j\ll K^\varepsilon$, under Convention 1. Then we shift the truncated s-contour to (P_3), with an integer $P_3\geqslant0$. No singularities are encountered. On the new

contour, the s-integrands are bounded by (7.19), but with the exponent P_1 being replaced by $-P_3$. We get

$$\Psi_\pm(i\varkappa; 2it_1, 0; V_f^+) \ll \left(\frac{M}{F}\frac{K^\varepsilon}{t^2}\right)^{P_3}, \quad |\varkappa| \ll K^\varepsilon. \tag{7.20}$$

Taking P_3 sufficiently large, we see that we may impose the truncation $F \ll K^\varepsilon$ as well. Then we insert (7.15) into (7.17)–(7.18), and move the f-sum to the innermost part. The contribution of D_d to (7.14) is now seen to be

$$\ll GK^{1/2+\varepsilon} {\sum_{\varkappa_j \ll K^\varepsilon}}' \alpha_j \left|H_j\left(\tfrac{1}{2}+it_1\right)\right|^2 \int_{t^2/K^\varepsilon}^\infty \left| \sum_{f \ll K^\varepsilon} \phi(fx)\phi_2(f)\frac{d(f)\tau_j(f)}{f^{1/2+i(t_2-t_1)}}\right| \frac{dx}{x^{5/4}}, \tag{7.21}$$

where the lower bound for x is due to (7.20). Then we appeal to either Meurman's bound or (1.18) for Hecke L-functions as well as to a uniform bound for $\tau_j(f)$ (see [23, (3.1.18)]), and find that (7.21) is $\ll K^{3/2}t^{1/6}$, which implies that the D_d-part of (7.14) is negligible.

The discussion of the contribution of D_c defined by (2.21) is analogous to the above, and can be skipped. It should be noted that the assertion (7.20), i.e., $F \ll K^\varepsilon$, is actually not necessary when we deal with the part involving D_d. The truncation $\varkappa_j \ll K^\varepsilon$ suffices. However, the part involving D_c requires (7.20), for the function $f^{-i\varkappa}\sigma_{2i\varkappa}(f)$ with a small $\varkappa$ is non-oscillating, unlike $\tau_j(f)$.

As to the D_h-part of (7.14), we need to bound

$$\begin{aligned} GK^{1/2} \sum_{f=1}^\infty \frac{\phi_2(f)d(f)}{f^{1/2+i(t_2-t_1)}} \sum_{k=6}^\infty \sum_{j=1}^{\vartheta(k)} (-1)^k \alpha_{j,k}\tau_{j,k}(f)\left|H_{j,k}\left(\tfrac{1}{2}+it_1\right)\right|^2 \\ \times \Psi_+\left(k-\tfrac{1}{2}; 2it_1, 0; V_f^+(\,\cdot\,;\delta)\right). \end{aligned} \tag{7.22}$$

Apart from a constant multiplier, the factor Ψ_+ is equal to

$$\begin{aligned} \cosh(\pi t_1) \int_{(0)} \frac{\Gamma\left(k+s-\frac{1}{2}\right)}{\Gamma\left(k+\frac{1}{2}-s\right)} \Gamma\left(\frac{1}{2}+it_1-s\right)\Gamma\left(\frac{1}{2}-it_1-s\right) \\ \times \int_0^\infty \phi(fx)x^{s-5/4+i(t_1-t_2)} \exp\left(\frac{2i\delta K}{\sqrt{x}} - \frac{G^2}{x}\right) dx\, ds. \end{aligned} \tag{7.23}$$

We suppose first that $k \gg K^{1/\varepsilon}$, and shift the s-contour far to the left, without passing over any pole. We see readily that the integral is negligibly small. Thus, we get the truncation of (7.22) to $k \ll K^{1/\varepsilon}$. This implies that we may introduce the truncation $|s| \ll K^\varepsilon$ as before. Then, shifting the truncated s-contour far to the right, we see that $t(F/M)^{1/2}K^{-\varepsilon} \ll k$ can be assumed. In particular, if $k \ll K^\varepsilon$, then $F \ll K^\varepsilon M/t^2 \ll K^\varepsilon$, under Convention 1. We are led to a situation analogous to (7.21), and invoking the bound (1.32) for $H_{j,k}\left(\frac{1}{2}+it\right)$, which is yet to be proved, as well as (5.34) or any uniform bound for $\tau_{j,k}(n)$, we can settle this case. Note that Good's bound mentioned immediately after (1.17) should not be utilised here, because it appears not to be uniform in the relevant cusp forms, unlike Meurman's bound applied to (7.21). At any event, we may assume that $k \gg K^\varepsilon$. Then, shifting the contour far to the left again, we are led to the truncation

$$\max\left\{K^\varepsilon, t\left(\frac{F}{M}\right)^{1/2} K^{-\varepsilon}\right\} \ll k \ll t\left(\frac{F}{M}\right)^{1/2} K^{-\varepsilon}. \tag{7.24}$$

With this, we insert (7.23) into (7.22), and move the f-sum to the innermost part. We see that (7.22) is

$$\begin{aligned} \ll GK^{1/2+\varepsilon} \sup_{U,F} \frac{1}{U} \int_{G^2/K^\varepsilon}^\infty \sum_{U \leqslant k \leqslant 2U} \sum_{j=1}^{\vartheta(k)} \alpha_{j,k} \left|H_{j,k}\left(\tfrac{1}{2}+it_1\right)\right|^2 \\ \times \left| \sum_{f=1}^\infty \phi_2(f)\phi(fx) \frac{d(f)\tau_{j,k}(f)}{f^{1/2+i(t_2-t_1)}} \right| \frac{dx}{x^{5/4}}, \end{aligned} \tag{7.25}$$

where the lower bound for x comes from (7.16). To this we apply (1.32)–(1.33) and Lemma 8. More precisely, to one factor of $\left|H_{j,k}\left(\frac{1}{2}+it_1\right)\right|^2$ we apply (1.32), which is possible, because $t \gg U^{3/2}$ with U in the range (7.24); and to the remaining part of (7.25) we apply (1.33) and Lemma 8. We find that (7.25) is

$$\ll K^{1+\varepsilon} t^{1/3} \sup_{U,F} \frac{1}{U}((U^2+t^{2/3})(U^2+F))^{1/2} \ll (Kt^{2/3}+t^{4/3})^{1+\varepsilon}. \tag{7.26}$$

This finishes the treatment of the contribution of $\mathcal{H}_2$ to (7.7), up to the proof of (1.32)–(1.33).

Now, we move to the contribution to (7.7) of $\mathcal{H}_4$ defined by (2.13). This is equal to

$$\sum_{n=1}^\infty \frac{\phi(n)\sigma_{2it_1}(n)}{n^{1/2+it_2}} \sum_{m=1}^{n-1} m^{-1/2} d(m)d(n-m)\Psi^-\left(\frac{m}{n};h\right). \tag{7.27}$$

We should remark first that the present choice (7.2) of the function h allows us to move the vertical line of (2.8) to the left freely as far as $a > -P_0 - \frac{1}{2}$ and $a \neq -\frac{1}{2}, \ldots, -P_0 + \frac{1}{2}$. This can be seen by a simple extension of the argument in [23, pp. 113 and 121]. Also we may replace $\Psi^-(x;h)$ by

$$\begin{aligned} GK \int_0^\infty \left(\int_{(a)} x^s (y(y+1))^{s-1} \frac{\Gamma^2\left(\frac{1}{2}-s\right)}{\Gamma(1-2s)\cos \pi s} \, ds \right) \\ \times \left(\frac{y}{y+1}\right)^{i\delta K} \exp\left(-\left(\frac{G}{2}\log\frac{y}{y+1}\right)^2\right) dy, \end{aligned} \tag{7.28}$$

with $\delta=\pm 1$, and further by

$$GK\int_{G/\log K}^{\infty}\left(\int_{a-i\log^2 K}^{a+i\log^2 K}\frac{(xy^2)^s\Gamma^2\left(\frac{1}{2}-s\right)}{\Gamma(1-2s)\cos\pi s}\,ds\right)\exp\left(-\frac{\delta iK}{y}-\frac{1}{4}\left(\frac{G}{y}\right)^2\right)\frac{dy}{y^2}. \tag{7.29}$$

In particular,

$$\Psi^-(x;h)\ll K(xG^2)^{-P_0}. \tag{7.30}$$

Hence, taking P_0 sufficiently large, we may consider, instead of (7.27),

$$\sum_{n=1}^{\infty}\frac{\phi(n)\sigma_{2it_1}(n)}{n^{1/2+it_2}}\sum_{m\ll K^\varepsilon M/G^2} m^{-1/2}d(m)\,d(n-m)\,\Psi^-\left(\frac{m}{n};h\right). \tag{7.31}$$

We may further replace this by

$$\sum_{f=1}^{\infty}\frac{\phi_2(f)\,d(f)}{f^{1+it_2}}\sum_{n=1}^{\infty}d(n)\,\sigma_{2it_1}(n+f)\,V_f^-\left(\frac{n}{f}\right), \tag{7.32}$$

where ϕ_2 is as in (7.14), with (7.16), and

$$V_f^-(x)=\phi(f(x+1))(x+1)^{-1/2-it_2}\Psi^-((x+1)^{-1};h). \tag{7.33}$$

Now, we shall proceed with

$$a=0. \tag{7.34}$$

We apply (2.17) to (7.32), with $\alpha=0$ and $\beta=2it_1$. We consider first the contribution of D_r. We need to take a limit on the right-hand side (2.22), but obviously this procedure can be ignored. Then, for instance, the leading term of (2.22) yields the expression

$$\begin{aligned} &GK\int_{-i\log^2 K}^{i\log^2 K}\frac{\Gamma^2\left(\frac{1}{2}-s\right)}{\Gamma(1-2s)\cos\pi s}R_\delta(s;G,K)\\ &\qquad\times\sum_{f=1}^{\infty}\phi_2(f)\frac{d(f)\sigma_{1+2it_1}(f)}{f^{1+it_2}}\int_{G^2/K^\varepsilon}^{\infty}\frac{\phi(f(x+1))}{(x+1)^{s+1/2-2it_1+it_2}}\,dx\,ds, \end{aligned} \tag{7.35}$$

where

$$R_\delta(s;G,K)=\int_{G/\log K}^{\infty}y^{2(s-1)}\exp\left(-\frac{\delta iK}{y}-\frac{1}{4}\left(\frac{G}{y}\right)^2\right)dy. \tag{7.36}$$

Lemma 6 implies that the innermost integral of (7.35) is negligibly small, and the same assertion holds for the contribution of D_r to (7.32).

Let us consider the contribution of D_d. We need to study the $\Psi_{\pm}$-functions defined by (2.24)–(2.25), with our current specifications. In view of (7.29) with $a=0$ and (7.33), we may deal instead with the expressions

$$GK\int_{-i\log^2 K}^{i\log^2 K}\frac{\Gamma^2(\frac{1}{2}-s)}{\Gamma(1-2s)\cos\pi s}R_\delta(s;G,K)\Delta_\pm(\varkappa,f,s)\,ds, \tag{7.37}$$

respectively, where

$$\Delta_+(\varkappa,f,s)=\frac{1}{4\pi i}\int_{(\varepsilon)}\cos(\pi s_1)\Gamma(s_1+i\varkappa)\Gamma(s_1-i\varkappa)\Gamma^2(\tfrac{1}{2}-it_1-s_1) \tag{7.38}$$
$$\times\int_{G^2/K^\varepsilon}^\infty \phi(f(x+1))\,x^{s_1+it_1-1/2}(x+1)^{-s-1/2-it_2}\,dx\,ds_1,$$

$$\Delta_-(\varkappa,f,s)=\frac{1}{4\pi i}\cosh(\pi\varkappa)\int_{(\varepsilon)}\sin(\pi(s_1+it_1))\Gamma(s_1+i\varkappa)\Gamma(s_1-i\varkappa) \tag{7.39}$$
$$\times\Gamma^2(\tfrac{1}{2}-it_1-s_1)\int_{G^2/K^\varepsilon}^\infty \phi(f(x+1))\,x^{s_1+it_1-1/2}(x+1)^{-s-1/2-it_2}\,dx\,ds_1.$$

We observe that we may suppose, in both expressions, that $|\varkappa|\ll K^{1/\varepsilon}$. To see this, it suffices to adopt the reasoning following (7.19). Note that we have $s\ll\log^2 K$.

Let us then consider Δ_+. The Γ-factor is

$$\ll\exp(-\pi\max\{|\lambda_1|,|\varkappa|\}-\pi|\lambda_1+t_1|),\quad \operatorname{Im}s_1=\lambda_1. \tag{7.40}$$

Thus, the case $|\lambda_1+t_1|\geqslant\frac{1}{2}t$ can readily be ignored. Otherwise, the inner integral is obviously negligibly small by Lemma 6. Hence, Δ_+ can be discarded.

We turn to Δ_-. We note that the factor $\cosh(\pi\varkappa)\sin\pi(s_1+it_1)$ is cancelled by the Γ-factor. Before shifting the s_1-contour, we shall show that we may truncate it to $\lambda_1\ll K^\varepsilon$. In fact, concerning the last x-integral, we have

$$\frac{d}{dx}((\lambda_1+t_1)\log x-(\lambda+t_2)\log(x+1))=\frac{1}{x}\left(\lambda_1+\frac{t}{x+1}+O(K^\varepsilon)\right), \tag{7.41}$$

with $\operatorname{Im}s=\lambda$, since $\lambda\ll\log^2K$ and $|t_\nu-t|\ll K^\varepsilon$ ($\nu=1,2$). Provided $|\lambda_1|\gg K^\varepsilon$ and under Convention 1, the absolute value of the right-hand side of (7.41) is $\gg|\lambda_1|/x$ because of the bound $t/(x+1)\ll K^\varepsilon$, which is implied by (7.1) and (7.3). Thus Lemma 6 works, and the part of Δ_- with $|\lambda_1|\gg K^\varepsilon$ can be discarded as claimed. With this, we shift the s_1-contour far to the right, without encountering any pole, and obtain the truncation $t(F/M)^{1/2}K^{-\varepsilon}\ll|\varkappa|$. Thus, if $\varkappa\ll K^\varepsilon$, then $F\ll K^\varepsilon$. That is, this case is settled by Meurman's bound as before. On the other hand, if $|\varkappa|\gg K^\varepsilon$, then we may shift the

s_1-contour far to the left under Convention 1, again without encountering any pole, and come to the following analogue of (7.24):

$$\max\left\{K^\varepsilon, t\left(\frac{F}{M}\right)^{1/2}K^{-\varepsilon}\right\} \ll |\varkappa| \ll t\left(\frac{F}{M}\right)^{1/2}K^\varepsilon. \tag{7.42}$$

Hence, the contribution of D_d to (7.32) is

$$\begin{aligned}\ll K^{1+\varepsilon}\sup_{U,F}\frac{1}{U}\int_{G^2/K^\varepsilon}^\infty \sum_{U\leqslant\varkappa_j\leqslant 2U} \alpha_j\left|H_j\left(\tfrac12+it_1\right)\right|^2 \\ \times\left|\sum_{f=1}^\infty \phi_2(f)\phi(f(x+1))\frac{d(f)\tau_j(f)}{f^{1/2+i(t_2-t_1)}}\right|\frac{dx}{x},\end{aligned} \tag{7.43}$$

with U in the range (7.42). The rest is similar to the discussion of (7.25). This time we appeal instead to (1.18)–(1.19) and Lemma 7. We find that (7.43) is $\ll(Kt^{2/3}+t^{4/3})^{1+\varepsilon}$.

As to the D_c-part, we follow the above reasoning, and get, instead of (7.43), the expression

$$\begin{aligned}K^{1+\varepsilon}\sup_{U,F}\frac{1}{U}\int_{G^2/K^\varepsilon}^\infty\int_U^{2U}\frac{\left|\zeta\left(\frac12+i(t_1+\varkappa)\right)\right|^2\left|\zeta\left(\frac12+i(t_1-\varkappa)\right)\right|^2}{|\zeta(1+2i\varkappa)|^2} \\ \times\left|\sum_{f=1}^\infty \phi_2(f)\phi(f(x+1))\frac{d(f)\sigma_{2i\varkappa}(f)}{f^{1/2+i\varkappa+i(t_2-t_1)}}\right|d\varkappa\,\frac{dx}{x}.\end{aligned} \tag{7.44}$$

We have $\zeta\left(\frac12+i(t_1\pm\varkappa)\right)\ll t^{1/6}$ because of the upper bound in (7.42). We apply the Mellin inversion of ϕ and ϕ_2, with an appropriate truncation of the resulting new double integral. Then the sum over f is expressed in terms of the zeta-function. We may shift the two contours to the imaginary axis without encountering any pole, because of the lower bound in (7.42). The inner integral of (7.44) is $\ll t^{2/3}UK^\varepsilon$. Hence (7.44) itself is $\ll K^{1+\varepsilon}t^{2/3}$.

The treatment of the D_h-part is analogous to that pertaining to Δ_+, and the contribution is negligibly small.

We conclude that the $\mathcal{H}_2$- and $\mathcal{H}_4$-parts of (7.7) are both $\ll(Kt^{2/3}+t^{4/3})^{1+\varepsilon}$. Combined with the assertion adjacent to (7.9), this ends the proof of Theorem 2 and thus of Theorem 1, leaving one essential step yet to be confirmed. What remains is to prove (1.33). That is to be done in the next section.

8. Discussion

Here we shall first develop a brief proof of (1.33), and a proof of Theorem 3. Then we shall observe a structure that makes such extensions possible. With this, the feasibility of further extensions will be discussed.

To prove (1.33), we follow closely the argument of [13]. We first look into the case $t \gg K^{3+\varepsilon}$ with a large K. The main difference from the corresponding part of [13] is that instead of a Voronoĭ summation formula involving the coefficients $\tau_j(n)$ we work with its counterpart for $\tau_{j,k}(n)$. This means replacing the Bessel function $J_{2ir}(x)$ with a real $r \approx K$ by $J_{2k-1}(x)$ with an integer $k \approx K$. The argument in [13] relies on the fact that if $x \gg K^{2+\varepsilon}$, then

$$\frac{J_{2ir}(x)-J_{-2ir}(x)}{2\sinh \pi r} \sim i\sqrt{\frac{2}{\pi x}}\sin\left(x-\frac{1}{4}\pi\right) \tag{8.1}$$

(see (4.6)). With the same assumption, we have

$$J_{2k-1}(x) \sim (-1)^{k-1}\sqrt{\frac{2}{\pi x}}\sin\left(x-\frac{1}{4}\pi\right), \tag{8.2}$$

if k is a natural number (see (8.5) below). The analogy is perfect as far as the Bessel functions are concerned. Also, to resulting sums involving the coefficients $\tau_{j,k}(n)$ we apply Lemma 8 in place of Lemma 7 that is used in [13]. Hence, the argument of [13] can be repeated word by word if $t \gg K^{3+\varepsilon}$.

Thus we assume that $t \ll K^{3+\varepsilon}$ as well as $K^{\varepsilon} \ll G \ll K^{1-\varepsilon}$. Then we have, analogously to (7.5), that

$$H_{j,k}\left(\tfrac{1}{2}+it\right) \ll (\log^2 K) \sum_{M \ll K+t} \int_{\gamma^{-1}-i\gamma^2}^{\gamma^{-1}+i\gamma^2} \left|\sum_{m=1}^{\infty} \phi(m;M)\tau_{j,k}(m)m^{-1/2-it-\xi}\right| |d\xi|, \tag{8.3}$$

where M runs over dyadic numbers. With this, the case $t \ll K^{1+\varepsilon}$ is readily settled by Lemma 8. Hence it remains to consider the intermediate range $K^{1+\varepsilon} \ll t \ll K^{3+\varepsilon}$. Here the proof of Lemma 8 is relevant. Thus, we are to deal with

$$\sum_{l} \frac{1}{l} \sum_{m,n} \frac{\phi_0(m)\overline{\phi_0(n)}}{\sqrt{mn}} \left(\frac{m}{n}\right)^{it} S(m,n;l)(h_1)^{\circ}\left(\frac{4\pi\sqrt{mn}}{l}\right), \tag{8.4}$$

where ϕ_0 is as in (7.6), $(h_1)^{\circ}$ is the expression (3.34), and the truncation (3.12) has already been applied, but with N being replaced by $M \ll t$. The Kloosterman sums are expanded according to their definition, and the assertion (3.36) is invoked. Then we end up with a double exponential sum over m and n, essentially the same as the corresponding sum in [13]. This ends the proof of (1.33). Consequently, we have finished the proof of Theorem 1.

As to the proof of Theorem 3, it depends solely on the observation that the procedure developed in §4 is as a matter of fact a reduction of the original problem to additive divisor sums. Applied to the left-hand side of (1.34), this argument leads us to exactly the same additive divisor sums, albeit there exist differences coming from the use of Lemma 2 in place of Lemma 1, and from the fact that $T \approx (K+t)^2$. There is virtually no difference in terms of asymptotics. This is endorsed by the truncation (3.12), which is applicable to the present situation as mentioned above, and by the formula

$$J_{2k-1}(x) \sim (-1)^{k-1}\sqrt{\frac{2}{\pi x}}\sin\left(\omega(ik,x)-\frac{1}{4}\pi\right). \tag{8.5}$$

The former fact corresponds to (4.3), and the latter to (4.6). The rest of the proof is the same as that of Theorem 1. In fact, it is slightly simpler, because the second condition in (6.1) is unnecessary, due to the fact that we have ik in (8.5) in place of r in (4.6).

We shall expand our observation about the role of additive divisor sums. To this end, we return to (3.40). The L-series that yields the Dirichlet series on the right is associated with the Rankin–Selberg convolution of the Eisenstein series and the relevant cusp form. The divisor function there is a Fourier coefficient of an automorphic function. The structure of our subsequent reasoning, which is admittedly involved, could be summarised as follows:

(1) Appearance at (4.2) of Kloosterman sums via Lemma 1;
(2) Basic truncation (4.3) of moduli of Kloosterman sums;
(3) Application of the Voronoĭ sum formula at (4.11);
(4) Another basic truncation at (4.21);
(5) Appearance at (4.27) of additive divisor sums;
(6) Application at (5.1) of the spectral decomposition (2.27);
(7) Truncation of the spectral range at (5.21)/(6.24);
(8) Appearance at (6.33) of a simpler spectral sum;
(9) Reduction at (6.37) to a hybrid moment.

Note that step (3) is performed upon the divisor function, which is not our main concern at (3.40). The subsequent analysis is, however, wholly relevant to these Fourier coefficients of the Eisenstein series. It is true that Kloosterman sums replace Fourier coefficients of original cusp forms, and thus the latter objects are actually playing a role in the background. Nevertheless, the operations following (3) are made possible because of the presence of the divisor function. Moreover, the decisive step (8) is due solely to (5). In other words, the divisor function is indeed the protagonist of our scenario, despite its obscure entrance at (3.40). Or perhaps more correctly, an orchestration of automorphic waves conducted by the sum formulas due to Bruggeman, Kuznetsov and Petersson makes it possible for the divisor function to conjure the uniform bounds (1.1) and (1.2).

Now, if (3.40) can be regarded as a statement concerning a Rankin–Selberg convolution, then what has been developed above could be a typical instance of a general mechanism arising from automorphy; by no means a serendipity. We shall indicate, with a plausible inference, that this should be the case.

Thus, let ψ be a Hecke invariant cusp form, either holomorphic or real-analytic. Let $\tau_\psi(n)$ be its Hecke eigenvalue. We are interested in bounding the Rankin–Selberg L-function

$$L(s,\psi\otimes\psi_j)=\zeta(2s)\sum_{n=1}^{\infty}\tau_\psi(n)\tau_j(n)n^{-s} \tag{8.6}$$

on the critical line. Note that the function $(s-1)L(s,\psi\otimes\psi_j)$ is entire, and also that one may naturally replace ψ_j by $\psi_{j,k}$, and proceed analogously.

We need to treat the expression

$$\sum_{K\leqslant\varkappa_j\leqslant K+G}\alpha_j\left|\sum_{n=1}^{\infty}\phi(n)\tau_\psi(n)\tau_j(n)n^{-1/2-it}\right|^2, \tag{8.7}$$

where (1.26) is effective, and ϕ is as in (4.2) with $M\ll T_\psi$, where T_ψ is defined analogously to (3.38). We may apply steps (1) and (2) without any change. The third step is equivalent to an appeal to the functional equation for the Hecke–Estermann zeta-function

$$\sum_{n=1}^{\infty}\tau_\psi(n)\exp\left(\frac{2\pi iqn}{l}\right)n^{-s},\quad (q,l)=1, \tag{8.8}$$

which is an extension of (4.14) and a consequence of the automorphy of ψ. Essentially the same as (4.17) comes out, with d being replaced by τ_ψ. Here might, however, arise a problem relevant to the change in the function I, which should be taken into account if the uniformity in ψ is to be maintained. The same can be said about the extension of step (4). Step (5) is now with the sum

$$\sum_{n=1}^{\infty}\tau_\psi(n)\tau_\psi(n+f)W\left(\frac{n}{f}\right). \tag{8.9}$$

When ψ is holomorphic, there exists a complete analogue of Lemma 5 which is due to the second author (implicit in [22]). Hence this case should not cause any extra difficulty as far as step (8). With a real-analytic ψ, there might arise a new issue, because we lack any complete extension of Lemma 5 to this case. There exists, however, a relevant result, an asymptotic extension due to the first author [9]. That might serve well for our purpose. Despite this, we should better try to achieve a complete extension of Lemma 5 to the real-analytic case, mainly for the sake of a fuller understanding of this fascinating mechanism. In fact, such a programme is being undertaken by the second author (see [24]); the key seems to be the harmonic analysis on the Lie group $\mathrm{PSL}_2(\mathbf{R})$. Thus, we may envisage with a good reason that we could go through steps (5)–(7) in the new context as well; that is, an analogue of (6.33) should hold with ψ in general. There the factor $H_j^2\left(\frac{1}{2}\right)$ is to be replaced by the inner product $\langle|\psi|^2, \psi_j\rangle$ or a quantity closely related, with an appropriate normalisation of the metric. We need an analogue for $\langle|\psi|^2, \psi_j\rangle$ of the spectral fourth moment of $H_j\left(\frac{1}{2}\right)$. Such a result, in fact the spectral mean square of the inner product, is proved by Good [4]. Also, its extension to the real-analytic case is obtained by the first author [9]. Other parts of (6.33) do not need to be changed substantially. Therefore, it is highly probable that a counterpart of (1.27), and consequently a subconvexity bound

$$L\left(\tfrac{1}{2}+it, \psi\otimes\psi_j\right) \ll \varkappa_j^{2/3+\varepsilon}, \tag{8.10}$$

be within our reach, at least when t is relatively small compared with $\varkappa_j$. Indeed, we have proved already that this is the case when ψ is holomorphic, with a meaningful uniformity in ψ and t. The situation with a real-analytic ψ should be analogous, though we have not worked out the details yet.

It remains to ponder about a fuller analogue of Theorem 1. Here we are, however, to realise that we were in a fortuitous situation with $\mathcal{S}(G, K)$. A reason why the hybrid mean value worked fine with (6.33) is that the latter has the factor $H_j^2\left(\frac{1}{2}\right)$, as this fact was exploited to reach (6.37). Such a splitting of the corresponding factor $\langle|\psi|^2, \psi_j\rangle$ does not appear to be possible in general. Thus, we should better stop our plausible inference here. It should, however, be added that there are other directions of the extension. For instance, we may replace the group Γ by $\Gamma_0(q)$; and the twist of $H_j(s)$ with a Dirichlet character can be treated with the same strategy as above, taking into account the uniformity in the modulus of the character. Another possibility is to include the Bianchi groups. All basic machineries needed for this purpose are laid out in [2].

Finally, we stress that there exists a possibility that one might come to step (8) directly from the original spectral sum. That is, the use of Kloosterman sums and the Voronoĭ sum formula could be avoided altogether. This is suggested by the recent work [3], where the spectral decomposition of the fourth power moment of the Riemann zeta-function is grasped as a special instance of that of a Poincaré series on the group $\mathrm{PSL}_2(\mathbf{R})$, yielding a new approach to the subject closely related to ours. In the perspective thus opened, the functional equations and the Bessel transforms which are in the core of our analysis developed above are understood to be realisations of the action of the Weyl element of the group under various circumstances. To this and the above observation on Rankin–Selberg L-functions we shall return elsewhere.

Concluding remark. After finishing the present work, we found that Sarnak had developed in [25] an approach to the subconvexity bound of Rankin–Selberg L-functions. He worked mainly with holomorphic cusp forms; nevertheless, the initial stage of his approach is analogous to ours in the sense that the corresponding steps up to (5) are observable there, though with a different configuration. It is indeed hard to conceive any other way to take. However, from the stage corresponding to (6) and onwards, Sarnak's strategy differs considerably from ours, and the bound that his method gives rise to is tangibly weaker than our assertion pertaining to (8.10), as far as the full modular group is concerned. A talk on this subject, and in fact a summary of the present article, were delivered by the authors at the Tagung 'Theory of the Riemann Zeta and Allied Functions' (Mathematisches Forschungsinstitut Oberwolfach, September 20, 2004).

Added on July 26, 2005. The referee kindly informed us that Z. Peng (Ph.D. thesis, Princeton University, 2001) had proved an analogue of (1.22) and thus of (1.2) with $t=0$ for holomorphic cusp-forms; however, we have not seen his work yet. Also, we should mention that the case with real-analytic ψ in (8.10) is successfully treated in our forthcoming work.

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