Ark. Mat., **32** (1994), 323–371

Resonances for perturbations of a semiclassical periodic Schrödinger operator

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Abstract. In the semi-classical regime we study the resonances of the operator $P_t = -h^2\Delta + V + t\cdot\delta V$ in some small neighborhood of the first spectral band of P_0. Here V is a periodic potential, δV a compactly supported potential and t a small coupling constant. We construct a meromorphic multivalued continuation of the resolvent of P_t, and define the resonances to be the poles of this continuation. We compute these resonances and study the way they turn into eigenvalues when t crosses a certain threshold.

0. Introduction

In a previous paper [Kl] we studied the semi-classical eigenvalue problem for the following operator acting on $L^2(\mathbf{R}^n)$:

$$P_t = -h^2\Delta + V + t\cdot\delta V, \tag{0.1}$$

where V is a periodic potential, δV a compactly supported potential and t a real coupling constant.

For h small enough we proved that there exists a threshold $T_{\delta V}$ ($T_{\delta V}=0$ if $n=1$ or 2, and $T_{\delta V}>0$ otherwise), such that, for $t>T_{\delta V}$, P_t admits a simple eigenvalue $\lambda(t)$ in a neighborhood of the first band of P_0.

In this paper we are dealing with the resonance problem for P_t, and consequently, as we will see later on, with what happens to $\lambda(t)$ when t gets smaller than $T_{\delta V}$.

Resonances have been studied quite extensively for various operators in the last 20 years, and there exist various definitions of them, all of which lead to many investigations (see, for example, the works of P. D. Lax and R. Phillips [LxPh], J. Aguilar and J. M. Combes [AgCo], E. Balslev and J. M. Combes [BaCo], M. Reed and B. Simon [ReSi], W. Hunziker [Hu1], B. Simon [Si1], [Si2], B. Helffer and

J. Sjöstrand [HeSj], B. Helffer and A. Martinez [HeMa], P. Hislop and I. M. Sigal [HiSig], E. Balslev and E. Skibsted [BaSk], A. Orth [Or], etc.).

However, in most of these papers the authors studied resonances for perturbations of the Laplace operator. In our case, we are interested in resonances for perturbations of a periodic Schrödinger operator. The problem of the multivalued meromorphic continuation of the resolvent of an operator of the form (0.1) has already been studied in a quite general setting.

In the one-dimensional case ($n=1$), in [Fi1], N. E. Firsova constructs the meromorphic continuation of the resolvent of Hill operators and perturbed Hill operators on the Riemann surface of quasi-momenta. In [Fi2], she studies resonances for a Hill operator perturbed by an exponentially decreasing potential. She shows that, in each gap (of the Hill operator) of sufficiently high energy, there exists one or an odd number of resonances for the perturbed Hill operator.

In a more general case (no restriction on the dimension), in [Gé2] (see also [Gé1]), C. Gérard constructs a multivalued meromorphic continuation of the resolvent of a periodic Schrödinger operator. He gives a geometric interpretation of the branch points of this continuation, and shows that the branch points contained in a simple band are the critical values of the band function (i.e. the Floquet eigenvalues). At last, he also gives some properties of the possible resonances for a periodic Schrödinger operator perturbed by an exponentially decreasing potential. Nevertheless, under such general assumptions, he is not able to show the existence of resonances.

As we already pointed out, in this paper we will study the resonances for P_t only in some small neighborhood of B, the first band of the spectrum of P_0. Under generic assumptions on V this band will be simple for h small enough. Using the reduction done in [Kl], as a first theorem, we show that $R_t(z)$ (resp. $R_0(z)$) can be continued as a multivalued meromorphic (resp. analytic) operator-valued function in some small complex neighborhood of B. The branch points for both of these functions are the critical values of the band function. And these branch points are of logarithmic type if n is even, and of square root type if n is odd. To prove these results we rewrite P_t via some adapted Fourier transform and we push the relevant momentum space (in our case, it is a torus due to the periodicity of the background potential V) into $\mathbf{C}^n$ to move the essential spectrum away from the real axis (see [Gé2]). This is some kind of analytic dilation method adapted to periodic problems (see [AgCo], [Hu1] and [Cy]).

Then we define the resonances as the possible poles of the continuation of the resolvent. We show that, in dimension 1 or in dimension larger than 3, some neighborhood of the interior of B is free from resonances. So we may only find resonances near the edges of the band B (at least if $n\neq 2$).

If $n \neq 2$ and $n \leq 4$ (resp. $n \geq 5$), we prove the existence of one or more resonances when $t \leq T_{\delta V}$ (resp. $t < T_{\delta V}$) and t is close enough to $T_{\delta V}$. We locate these resonances on the different sheets of the Riemann surface where we could continue $R_t(z)$. We compute the asymptotics of these resonances when t tends to $T_{\delta V}$. So we can follow the way in which these resonances turn into a unique eigenvalue when t increases and crosses $T_{\delta V}$. In a way these resonances are perturbed bound states (see [Hu2]).

Using the previously cited asymptotics, we compute the imaginary part of the resonances we found. One must notice that, in odd dimension, there is always one resonance that is located on the real axis of the second sheet of the Riemann surface associated to the problem. If $n \leq 3$, it is the only existing resonance (at least in the domain we study).

Finally, using these results on resonances, we prove that there are no eigenvalues for P_t embedded in the interior of the band B, at least when t is close enough to $T_{\delta V}$.

The paper is organized along the following lines. After this short introduction, in Section 1 we describe our precise framework and state the main results. In Section 2 we construct the analytic continuation of $R_t(z)$, the technical part being described in the appendix, Section 4. In Section 3 we compute the resonances and prove the absence of embedded eigenvalues.

Acknowledgements. The author would like to thank J. Sjöstrand for suggesting him this subject, as well as B. Helffer, C. Gérard and L. Robbiano for many helpful discussions, and A. Bardot for efficient typing assistance.

I. Definitions and results

1. Definitions

Let L be a lattice: $L = \bigoplus_{j=1}^{n} \mathbf{Z} u_j$ where $(u_j)_{1 \leq j \leq n}$ is a basis of $\mathbf{R}^n$. Let L^* be the dual lattice of L (i.e $L^* = \{\gamma' \in (\mathbf{R}^n)^*; \forall \gamma \in L, \gamma' \cdot \gamma \in 2\pi \mathbf{Z}\}$) and $\mathbf{T} = (\mathbf{R}^n)^*/L^*$, the dual torus. We consider the following periodic Schrödinger hamiltonian acting on $L^2(\mathbf{R}^n)$:

$$P = -h^2 \Delta + V \tag{1.1}$$

where

$$V \in C^\infty(\mathbf{R}^n, \mathbf{R}) \quad \text{and} \quad V \text{ is } L\text{-periodic,} \tag{H.1}$$

that is $\forall x \in \mathbf{R}^n$, $\forall \gamma \in L$, $V(x+\gamma) = V(x)$.

Under assumption (H.1) it is well known that the spectrum of P consists of bands; these bands consist of purely absolutely continuous spectrum. It is in a neighborhood of such a band that we are going to define and study the resonances for certain perturbations of P in the semi-classical limit (i.e $h \to 0$).

Let $\sigma(P)$ be the spectrum of P. First, to isolate one of the bands of $\sigma(P)$ from the rest of the spectrum, we will need two more assumptions on V. Here we will only give an approximative statement of these assumptions, the rigorous statement being found in [Kl]. Suppose

(H.2) for $\delta > 0$ small enough, $\{x \in \mathbf{R}^n; V(x) \leq \delta\}$ has only one non-empty connected component in each cell of the lattice L; this component is compact and its diameter in the Agmon metric is 0. (The Agmon metric is the metric induced by the measure $\sup(V(x) - \delta, 0)dx$.)

We call the connected components of $\{x \in \mathbf{R}^n; V(x) \leq 0\}$, the wells of V; by assumption (H.2), these can be indexed by the points of L. For $\gamma \in L$ we define P_γ to be the operator P where all the wells, except the well corresponding to γ, have been filled (see [Kl] for a precise statement).

Let us suppose that there exist $\mu(h)$, a simple eigenvalue of P_0 and $a(h)$, a positive function of h, such that

(H.3)

(i) $\mu(h) \to 0$, $a(h) \to 0$ and $h \log a(h) \to 0$ when $h \to 0$,

(ii) for h small enough,

$$\sigma(P_0) \cap [\mu(h) - 2a(h), \mu(h) + 2a(h)] = \{\mu(h)\}.$$

One knows that under assumptions (H.1)–(H.3), for h small enough, there exists an analytic function ω_h defined in a neighborhood of $\mathbf{T}$ in $\mathbf{C}^n$, such that, for $\theta \in \mathbf{T}$, $\omega_h(\theta)$ is a simple Floquet eigenvalue for P (i.e. a simple eigenvalue for the operator defined by P on $L^2_\theta = \{u \in L^2_{\text{loc}};\ \forall \gamma \in L,\ u(x+\gamma) = e^{i\gamma\theta} u(x)\}$ (see [Sj])) and that there is a neighborhood of $\mu(h)$ of size $a(h)$ in which the spectrum of P consists of the band $\omega_h(\mathbf{T})$ (see [Or], [Kl]).

Let us consider the operator P_t

$$P_t = P + t\delta V = -h^2\Delta + V + t\delta V, \tag{1.2}$$

where V satisfies (H.1)–(H.3), t is a real parameter and δV satisfies

(H.4) δV is a C^∞ function, compactly supported in a sufficiently small neighborhood of the well 0, non-negative and strictly positive in the well 0 (see [Kl] for a precise statement).

We now recall the reduction theorem stated in [Kl]. Let $F_t (\subset L^2(\mathbf{R}^n))$ be the spectral space associated to P_t and the interval $[\mu(h) - a(h), \mu(h) + a(h)]$ and Π_t the orthogonal projection on F_t. Then

Theorem 1.1. *Assume* (H.1)–(H.4). *There exists some* $h_0>0$ *so that* $\forall h\in(0,h_0)$ *and* $\forall t\in[-a(h)/4, a(h)/4]$,

(a)

$$\sigma(P_t)\cap\left[\mu(h)-\tfrac{3}{2}a(h),\mu(h)+\tfrac{3}{2}a(h)\right]\subset[\mu(h)-a(h),\mu(h)+a(h)],$$

(b) $P_t\Pi_t$ *is unitarily equivalent to* $\Omega_t\colon L^2(\mathbf{T})\to L^2(\mathbf{T})$ *defined for* $f\in L^2(\mathbf{T})$ *by*

$$\Omega_t f=\omega_h\cdot f+b(t)(\Pi_0+K(t))f,$$

where:

(i) ω_h *is the Floquet eigenvalue for* P *defined above,*

(ii) Π_0 *is the orthogonal projection on the vector* 1 *in* $L^2(\mathbf{T})$, *that is, for* $f\in L^2(\mathbf{T})$

$$\Pi_0 f=\frac{1}{\mathrm{Vol}\,(\mathbf{T})}\int_{\mathbf{T}} f(\theta)\,d\theta,$$

(iii) $K(t)\colon L^2(\mathbf{T})\to L^2(\mathbf{T})$ *is an operator whose kernel* $k(t,\theta,\theta')$ *is analytic in* $D(0,a(h)/4)\times W_h\times W_h$, *where* W_h *is a neighborhood of* $\mathbf{T}$ *in* $\mathbf{C}^n$ *and* $D(0,a(h)/4)=\{z\in\mathbf{C};|z|<a(h)/4\}$. *Moreover, there exists* $c>0$ *such that*

$$\sup_{W_h\times W_h}|k(0,\theta,\theta')|\le e^{-c/h},$$

$$\sup_{D(0,a(h)/4)\times W_h\times W_h}|\partial_t k(t,\theta,\theta')|\le e^{-c/h},$$

(iv) $b(t)$ *is a bi-analytic bijection between two neighborhoods of* 0 *in* $\mathbf{C}$.

Remark.

- One knows that

$$b(t)=\varrho\cdot t\cdot(1+tq(t))$$

where q is analytic on $D(0,a(h)/2)$ (see [Kl] Section 3).

- In the sequel, for $t\in[-a(h)/4,a(h)/4]$, we will denote by $\mathcal{F}_t\colon F_t\to L^2(\mathbf{T})$ the unitary equivalence realizing

$$\Omega_t=\mathcal{F}_tP_t\Pi_t\mathcal{F}_t^*.$$

In fact, $\mathcal{F}_t$ is defined on $L^2(\mathbf{R}^n)$ and $\operatorname{Ker}\mathcal{F}_t=(F_t)^\perp$. The construction of $\mathcal{F}_t$ ([Kl] Section 4) shows that, for $t\in(-a(h)/4,a(h)/4)$, $\mathcal{F}_t$ and $\mathcal{F}_t^*$ can be defined as bounded operators depending analytically on t from $L^2(\mathbf{R}^n)$ to $L^2(\mathbf{T})$ and from $L^2(\mathbf{T})$ to $L^2(\mathbf{R}^n)$ respectively.

2. Analytic continuation of the resolvent

The operator P_t being self-adjoint, its resolvent $(z-P_t)^{-1}$ is well defined as a bounded operator-valued analytical function of z for $z\in\mathbf{C}\setminus\mathbf{R}$. We want to continue analytically $(z-P_t)^{-1}$ when z crosses the real axis, z staying in a neighborhood of $\omega(\mathbf{T})$.

Let $t\in[-a(h)/4, a(h)/4]$. Then, by definition

$$P_t=\Pi_t P_t \Pi_t+(1-\Pi_t)P_t(1-\Pi_t). \tag{1.3}$$

So for $z\in\mathbf{C}\setminus\mathbf{R}$, one has,

$$(z-P_t)^{-1}=\Pi_t(z-P_t\Pi_t)^{-1}\Pi_t+(1-\Pi_t)(z-P_t(1-\Pi_t))^{-1}(1-\Pi_t). \tag{1.4}$$

For $z\in D_h=\{z\in\mathbf{C}; d(z,\omega_h(\mathbf{T}))\le a(h)/4\}$ and $t\in[-a(h)/4, a(h)/4]$, we know by Theorem 1.1 that

$$d(z,\sigma(P_t(1-\Pi_t)))>a(h)/4. \tag{1.5}$$

So $(1-\Pi_t)(z-P_t(1-\Pi_t))^{-1}(1-\Pi_t)$ is a bounded operator-valued analytical function of z in D_h.

Now, we want to continue analytically $R(z,t)=\Pi_t(z-P_t\Pi_t)^{-1}\Pi_t$ for z in a neighborhood of the band $\omega_h(\mathbf{T})$ when z crosses $\omega_h(\mathbf{T})$. To do this, we will need some assumptions on the Floquet eigenvalue ω_h. First we give some notations; we call $s_h=\sup_{\theta\in\mathbf{T}}\omega_h(\theta)$, $i_h=\inf_{\theta\in\mathbf{T}}\omega_h(\theta)$ and $f(h)=s_h-i_h$ the supremum, the infimum and the length respectively of the band $\omega_h(\mathbf{T})$. We also renormalize the band defining for $\theta\in W$

$$\widetilde{\omega}_h(\theta)=\frac{\omega_h(\theta)-i_h}{f(h)}.$$

Let us suppose that there exists $h_0>0$ such that the following holds:

(H.5)

(i) one has

$$h\cdot\log f(h)=-S_0+o(1), \quad \text{when } h\to 0,$$

where S_0 is the shortest Agmon distance between 2 distinct wells.

(ii) there exists W a compact complex neighborhood of $\mathbf{T}$ in $\mathbf{C}^n$, such that, for $h\in(0,h_0)$, ω_h is analytic in W, the only critical points of ω_h in W are the points of $\left(\frac{1}{2}L^*\right)/L^*$ and these critical points are non-degenerate.

(iii) there exists $C>0$ so that

$$\sup_{h\in(0,h_0)}\left(\sup_{|\alpha|\le 3}\left(\sup_{\theta\in W}|\partial^\alpha\widetilde{\omega}_h(\theta)|\right)\right)\le C,$$

$$\inf_{h\in(0,h_0)}\Big(\inf_{\theta\in(\frac{1}{2}L^*)/L^*}\big|\det(\mathrm{Hess}(\widetilde{\omega}_h(\theta)))\big|\Big)\geq\frac{1}{C},$$

where $\det(\mathrm{Hess}(f(x)))$ is the determinant of the Hessian matrix of f at the point x.

(iv) for $\varepsilon>0$, there exists $\delta(\varepsilon)>0$ so that

$$\forall h\in(0,h_0), |\nabla\widetilde{\omega}_h(\theta)|>\delta(\varepsilon)\quad\text{if }\theta\in W\text{ and }d\big(\theta,\big(\tfrac{1}{2}L^*\big)/L^*\big)>\varepsilon.$$

(v) there exists $\widetilde{\omega}_0$, an analytic function on W, so that uniformly on W, $\widetilde{\omega}_h\to\widetilde{\omega}_0$ when $h\to 0$.

Remark. Following the appendix of [Kl], it can be proven that, under suitable symmetry assumptions on L and V, (H.5) holds.

We define

$$\Lambda_0=\{\text{the critical values of }\widetilde{\omega}_0\}=\{\tilde{\lambda}_0^j; 1\leq j\leq p\},$$

and

$$\begin{aligned}\Theta_j&=\{\text{the critical points of }\widetilde{\omega}_0\text{ associated to }\tilde{\lambda}_0^j\}\\&=\{\theta_j^k; 1\leq k\leq k_j\}\subset\big(\tfrac{1}{2}L^*\big)/L^*.\end{aligned}$$

In the sequel, these critical values will be ordered increasingly with the index j; so $\tilde{\lambda}_0^1=0$ and $\tilde{\lambda}_0^p=1$. We define the internal critical points to be $\{\theta_j^k; 2\leq j\leq p-1,\ 1\leq k\leq k_j\}$ and the internal critical values to be the critical values associated to these points.

Notice that $\widetilde{\omega}_h$ and $\widetilde{\omega}_0$ have the same critical points. Moreover the critical values of $\widetilde{\omega}_h$ associated to the points of Θ_j are tending to $\tilde{\lambda}_0^j$ when h goes to 0.

We define

$$R_0(z,t)=\mathcal{F}_t^*(z-\omega_h)^{-1}\mathcal{F}_t,\quad\text{and}\tag{1.6}$$

$$\Gamma(z,t)=\mathcal{F}_t^*(\Pi_0+K(t))(z-\omega_h)^{-1}\mathcal{F}_t.\tag{1.7}$$

Thus

$$R(z,t)=R_0(z,t)(1-b(t)\Gamma(z,t))^{-1}.\tag{1.8}$$

One shows

Proposition 1.2. *Assume* (H.1)–(H.4). *Then, for h small enough, $t\in[-a(h)/4,a(h)/4]$ and $z\in\mathbf{C}\backslash\omega_h(\mathbf{T})$,*

(a) $R_0(z,t)$ *is a bounded automorphism of* $L^2(\mathbf{R}^n)$ *satisfying*

$$\|R_0(z,t)\|_{\mathcal{L}(F_t)}\le\|R_0(z,t)\|_{\mathcal{L}(L^2(\mathbf{T}))}\le\frac{1}{d(z,\omega_h(\mathbf{T}))},$$

(b) $\Gamma(z,t)$ *is a compact operator from* $L^2(\mathbf{T})$ *to* $L^2(\mathbf{T})$ *satisfying*

$$\|\Gamma(z,t)\|_{\mathcal{L}(L^2(\mathbf{T}))}\le 2\left|\int\frac{1}{(z-\omega_h(\theta))^2}\,d\theta\right|.$$

Following [Gé1] and [Gé2], we define our set of analytic vectors (i.e. dense subsets of $L^2(\mathbf{R}^n)$ on which we will be able to continue $R(z,t)$) to be, for $a\in\mathbf{R}$,

$$L^2_a(\mathbf{R}^n)=\{u\in\mathcal{D}'(\mathbf{R}^n);\,e^{a|x|}\cdot u\in L^2(\mathbf{R}^n)\}$$

being provided with its natural norm $\|\varphi\|_{L^2_a}=\|e^{a|\cdot|}\cdot\varphi\|_{L^2}$. In fact, $L^2_a(\mathbf{R}^n)$ will be a set of analytic vectors only for $a>0$ small enough.

Remark. One shows that there exists $C>0$ such that, for h small enough, for $t\in[-a(h)/4,a(h)/4]$ and for $0<a<1/Ch$, one has $\Pi_t(L^2_a(\mathbf{R}^n))\subset L^2_a(\mathbf{R}^n)$ (see Section 2).

For $(x,r)\in\mathbf{C}\times\mathbf{R}^+$, define $\square(x,r)$ to be a square box in $\mathbf{C}$ with center x and side-length 2r and $\square_h(x,r)=i_h+f(h)\square(x,r)$. For $E\subset\mathbf{C}$ we define

$$E^\pm=E\cap\{z\in\mathbf{C};\,(\mathrm{Im}(z)\gtrless 0)\text{ or }(\mathrm{Im}(z)=0\text{ and }\mathrm{Re}(z)\notin\omega_h(\mathbf{T}))\}.$$

Let us define

$${}^c\square_h(r_0)=(\omega_h(\mathbf{T})+f(h)\square(0,r_0))\backslash\bigcup_{1\le j\le p}\square_h(\tilde\lambda_0^j,r_0),$$

and ${}^c\square_h^\pm(r_0)=({}^c\square_h(r_0))^\pm$.

In Figure 1, (1) denotes points of $\omega_h(\Theta_j)$, (2) is λ_0^j, (3) the length $2r_0\cdot f(h)$ of the side of the square $\square_h(\lambda_0^j,r_0)$ denoted by (4). The shaded zones marked by (5) are ${}^c\square_h^\pm(r_0)$.

One has

Theorem 1.3. *Assume* (H.1)–(H.5). *Then there exist $h_0>0$, $r_0>0$ and $c>0$ such that for $h\in(0,h_0)$, for $1\le j\le p$, $t\in[-a(h)/4,a(h)/4]$ and for $z\in\square_h^\pm(\tilde\lambda_0^j,r_0)$, the following expansions hold*:

$$f(h)R_0(z,t)=\sum_{k=1}^{k_j}S(\tilde z-\tilde\omega_h(\theta_j^k))\cdot H_{k,0}^\pm(\tilde z,t)+G_0^\pm(\tilde z,t),$$

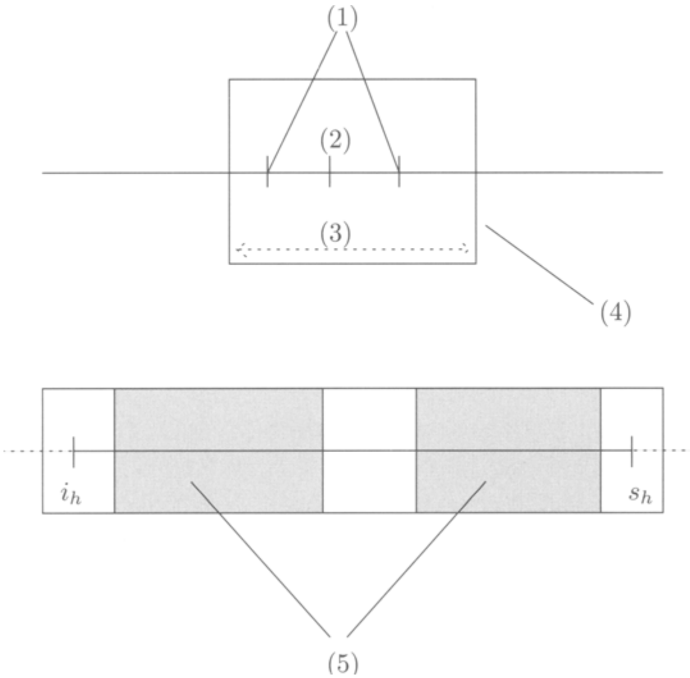

Figure 1.

$$f(h)\Gamma(z,t)=\sum_{k=1}^{k_j} S(\tilde{z}-\widetilde{\omega}_h(\theta_j^k))\cdot H_{k,K}^{\pm}(\tilde{z},t)+G_K^{\pm}(\tilde{z},t),$$

where:

(a) *the* $(H_{k,0}^{\pm})_{1\le k\le k_j}$ *and* $G_{k,0}^{\pm}$ *are* $\mathcal{L}(L_c^2(\mathbf{R}^n),L_{-c}^2(\mathbf{R}^n))$*-valued functions, analytic in* $(\tilde{z},t)$ *for* $(\tilde{z},t)\in\square(\tilde{\lambda}_0^j,r_0)\times D(0,a(h)/4)$,

(b) *the* $(H_{k,K}^{\pm})_{1\le k\le k_j}$ *and* $G_K^{\pm}$ *are* $\mathcal{C}(L_c^2(\mathbf{R}^n),L_c^2(\mathbf{R}^n))$*-valued functions, analytic in* $(\tilde{z},t)$ *for* $(\tilde{z},t)\in\square(\tilde{\lambda}_0^j,r_0)\times D(0,a(h)/4)$,

(c) *if* n *is odd,*

$$S(z)=\frac{\pi}{2}\cdot(-1)^{(n-1)/2}z^{(n-2)/2},$$

if n *is even,*

$$S(z)=\frac{1}{2}\cdot(-1)^{n/2}z^{(n-2)/2}\cdot\log z.$$

Moreover, $R_0(z,t)$ (resp. $\Gamma(z,t)$) can be analytically continued from ${}^c\square_h^{\pm}(r_0)$ to ${}^c\square_h(r_0)$ as an $\mathcal{L}(L^2_c, L^2_{-c})$ (resp. $\mathcal{C}(L^2_c, L^2_c)$)-valued analytic function in z and t.

Remark. $\mathcal{L}(E,F)$ is the set of bounded operators from E to F, $\mathcal{C}(E,F)$ the set of compact ones. Here $z^{1/2}$ and $\log z$ are the principal determinations of the square root and the logarithm.

For F a simply connected domain in $\mathbf{C}$ and $E\subset F$ a domain, $\mathcal{UC}(E,F)$ denotes the universal covering of E in F. For $r_0>0$ and $1\le j\le p$, we define

$$\mathcal{UC}(r_0,j)=\mathcal{UC}\left(\square_h(\tilde\lambda_0^j,r_0)\setminus\bigcup_{1\le k\le k_j}\{\omega_h(\theta_j^k)\},\square_h(\tilde\lambda_0^j,r_0)\right).$$

Let $R_0^{\pm}$ and $\Gamma^{\pm}$ be the analytic continuations of R_0 and Γ defined from

$$\square_h^{\pm}(\tilde\lambda_0^j,r_0)\times D(0,a(h)/4)\quad\text{to}\quad \mathcal{UC}(r_0,j)\times D(0,a(h)/4),$$

and from

$${}^c\square_h^{\pm}(r_0)\times D(0,a(h)/4)\quad\text{to}\quad {}^c\square_h(r_0)\times D(0,a(h)/4).$$

Using (1.8) one immediately gets the following

Corollary 1.4. *Assume* (H.1)–(H.5). *Then there exists $h_0>0$, $r_0>0$ and $c>0$ such that, for $h\in(0,h_0)$ and $1\le j\le p$,*

- *R can be meromorphically continued from*

$$\square_h^{\pm}(\tilde\lambda_0^j,r_0)\times D(0,a(h)/4)\quad\textit{to}\quad \mathcal{UC}(r_0,j)\times D(0,a(h)/4)$$

as $R^{\pm}$, an $\mathcal{L}(L^2_c,L^2_{-c})$-valued meromorphic function in z and t.

- *R can be meromorphically continued from*

$${}^c\square_h^{\pm}(r_0)\times D(0,a(h)/4)\quad\textit{to}\quad {}^c\square_h(r_0)\times D(0,a(h)/4)$$

as $R^{\pm}$, an $\mathcal{L}(L^2_c,L^2_{-c})$-valued meromorphic function in z and t.

Notice that, when one is dealing with the continuations at the edges of the band (i.e. $j=1$ or p), continuation from above the band or below is the same.

3. Resonances

We define

$$\mathrm{Res}^{\pm}(r_0,t)=\{z\in{}^c\square_h(r_0);\, z \text{ is a pole of } R^{\pm}(z,t)\},$$

and, for $\mathcal{O}\subset\mathcal{UC}(r_0,j)$,

$$\mathrm{Res}^{\pm}(\mathcal{O},t)=\{z\in\mathcal{O};\, z \text{ is a pole of } R^{\pm}(z,t)\}.$$

Definition. *z is said to be a resonance for P_t if there exists $r_0>0$ such that*

$$\begin{cases} z\in \mathrm{Res}^{\pm}(r_0,t), & \textit{or} \\ \exists\, 1\le j\le p,\ \exists\, \mathcal{O}\subset \mathcal{UC}(r_0,j) \textit{ such that } z\in \mathrm{Res}^{\pm}(\mathcal{O},t). \end{cases}$$

Remark. In this definition, no difference is made between actual eigenvalues and resonances.

$\mathcal{UC}(r_0,j)$ is naturally provided with a riemannian metric (induced by the euclidian metric on $\mathbf{C}$). For $0<r\le r_0$, let $\mathcal{O}(r,j)$ be the ball of radius r around λ_0^j in $\mathcal{UC}(r_0,j)$.

Our first theorem states that there are no resonances in a neighborhood of the interior of the band $\omega_h(\mathbf{T})$.

Theorem 1.5.

(a) *For any dimension n there exist $h_0>0$, $r_0>0$ such that, for $h\in(0,h_0)$ and $t\in(-a(h)/4,a(h)/4)$ one has*

$$\mathrm{Res}^{\pm}(r_0,t)=\phi.$$

(b) *For $n\ge 3$ and $2\le j\le p-1$, there exists $h_0>0$, $r_0>0$ such that, for $h\in(0,h_0)$ and $t\in(-a(h)/4,a(h)/4)$ one has*

$$\mathrm{Res}^{\pm}(\mathcal{O}(r_0,j),t)=\phi.$$

Remark. In dimension $n=2$ we get no results near the internal critical values. In dimension $n=1$ such critical values do not exist.

So, if P_t admits resonances, these are located near the edges of the band $\omega(\mathbf{T})$. We will only study what happens near the upper edge of the band; obviously, a symmetric study may be done near the lower edge. To compute these resonances we will need one more assumption on the band function ω_h (an assumption that is satisfied under suitable symmetry conditions on L and V when h is small enough (see [Kl] Appendix)); we assume that

(H.6) there exists only one critical point $\theta_s\in\left(\frac{1}{2}L^*\right)/L^*$ such that $\omega_h(\theta_s)=s=\lambda_0^p$ is the maximum of ω_h on $\mathbf{T}$.

We define

$$D_s=|\det(\mathrm{Hess}(\widetilde{\omega}_h(\theta_s)))|^{-1/2}.$$

We know that near the edges of the band, $R^+=R^-$ so, for any t and $\mathcal{O}$, a sufficiently small neighborhood of the edges of the band, one has

$$\mathrm{Res}^+(\mathcal{O},t)=\mathrm{Res}^-(\mathcal{O},t)=\ Res(\mathcal{O},t).$$

Under assumption (H.6), we define the following realisation of $\mathcal{UC}(r_0, s_h) = \mathcal{UC}(r_0, p)$. Define

$$\square_p(r_0, s_h) = s_h + e^{ip\Pi} \cdot (\square_h(r_0, 0) \setminus [-r_0 \cdot f(h), 0]),$$

$$\mathcal{UC}(r_0, s_h) = \{s_h\} \cup \bigcup_{p \in \mathbf{Z}} \square_p(r_0, s_h).$$

We define for $q \in \mathbf{N}$

$$\mathcal{UC}(r_0, s_h, q) = \{s_h\} \cup \bigcup_{-q \leq p \leq q} \square_p(r_0, s_h).$$

We are now able to state the results about resonances near the upper end of the band $\omega_h(\mathbf{T})$. In dimension 1 we get

Theorem 1.6. *Let $n=1$ and assume* (H.1)–(H.6). *Then there exists $h_0 > 0$, $r_0 > 0$, $\tilde{t}_0 > 0$ such that, for $h \in (0, h_0)$, there exist:*

(i) *a function $\lambda: (0, \tilde{t}_0 \cdot f(h)) \to (s, s + r_0 \cdot f(h))$ the values of which are simple eigenvalues of P_t and that admits the following convergent expansion:*

$$\lambda(t) - s_h = f(h) \cdot (\tilde{t})^2 \cdot \left(\sum_{l \in \mathbf{N}} \alpha_l(h) \cdot (\tilde{t})^l \right),$$

where:

- $\tilde{t} = t / f(h)$
- *for any $l \in \mathbf{N}$, $\alpha_l(h) \in \mathbf{R}$*
- $$\alpha_0(h) = \left(\frac{2^{1/2} \cdot \pi \varrho \cdot D_s}{\mathrm{Vol}(\mathbf{T})} \right)^2 \cdot (1 + O(e^{-c/h})) \quad \textit{for some } c > 0,$$

(ii) *a function $\lambda: (-\tilde{t}_0 \cdot f(h), 0) \to i + e^{2i\pi} \cdot (0, r_0 \cdot f(h))$ the values of which are simple resonances of P_t and that admits the following convergent expansion:*

$$\lambda(t) - s_h = e^{2i\pi} \cdot f(h) \cdot (\tilde{t})^2 \cdot \left(\sum_{l \in \mathbf{N}} \alpha_l(h) \cdot (\tilde{t})^l \right).$$

Moreover, for $t \in (-\tilde{t}_0 \cdot f(h), 0) \cup (0, \tilde{t}_0 \cdot f(h))$, one has

$$\mathrm{Res}(\mathcal{UC}(r_0, s_h), t) = \{\lambda(t)\}.$$

Remark. For $t > 0$, $\lambda(t)$ is the eigenvalue already found in [Kl]. One can notice that for any t, $\mathrm{Im}(\lambda(t)) = 0$; so the resonances we get for $t < 0$ are 0-energy resonances.

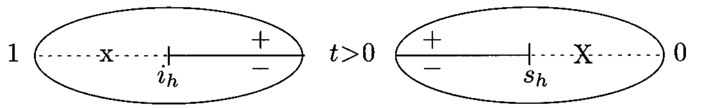

Figure 2a.

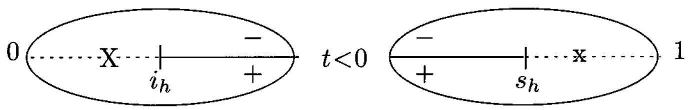

Figure 2b.

Figure 2 shows the resonance picture near both edges of the band. The sheet numbered 0 is the physical sheet and number 1 the non-physical sheet. Figure 2a shows the picture of the resonances when $t>0$; we omitted to draw the (resp. non-) physical sheet near the upper (resp. lower) edge of the band as it contains neither eigenvalue nor resonance. Figure 2b shows the picture of the resonances when $t<0$.

In Figure 2, X denotes an eigenvalue and x a resonance. 0 is the physical sheet and 1 the non-physical one. For $t<0$ or $t>0$, we only drew the sheet where there is either an eigenvalue or a resonance. The + or − signs indicate how the sheets are connected.

In dimension 2 we get

Theorem 1.7. *Let* $n=2$ *and assume* (H.1)–(H.6). *Let* $q\in\mathbf{N}$. *Then there exist* $h_q>0$, $\tilde{t}_q>0$, $r_q>0$ *such that there exists a function*

$$\lambda\colon (0,\tilde{t}_q\cdot f(h))\rightarrow(s_h,s_h+r_q\cdot f(h))$$

the values of which are simple eigenvalues for P_t *and that admits the following convergent expansion:*

$$\lambda(t)-s=f(h)\cdot\exp\Bigg(-\frac{\alpha_0(h)}{\tilde{t}}+\sum_{l\geq 0,m\geq 0}\alpha_{l,m}(h)\cdot(\tilde{t})^{l-m}e^{-m\cdot a_0(h)/\tilde{t}}\Bigg),$$

where:

- $\tilde{t}=t/f(h)$
- *for any* (l,m), $\alpha_{l,m}(h)\in\mathbf{R}$
- $$\alpha_0(h)=\frac{\mathrm{Vol}(\mathbf{T})}{2\varrho\pi\cdot D_s}\cdot(1+O(e^{-c/h}))\quad \textit{for some } c>0.$$

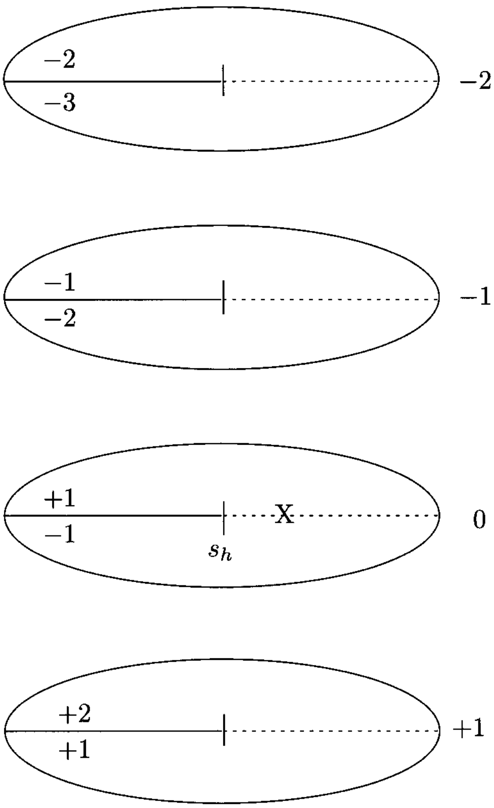

Figure 3.

Moreover, for $t \in (0, \tilde{t}_q \cdot f(h))$, *one has*

$$\operatorname{Res}(\mathcal{U}C(r_q, s_h, q), t) = \{\lambda(t)\},$$

and, for $t \in (-\tilde{t}_q \cdot f(h), 0)$,

$$\operatorname{Res}(\mathcal{U}C(r_q, s_h, q), t) = \phi.$$

Remark. λ is the eigenvalue already found in [Kl]. So, in dimension 2, one does not get any actual resonance near the edges of the band but only eigenvalues (see Figure 3). One may draw a symmetric picture for $t < 0$.

In Figure 3, X is an eigenvalue for $t > 0$. The numbers at the left of the sheets indicate how these are connected.

Let $n\geq 3$. Define

$$I=\frac{1}{\mathrm{Vol}(\mathbf{T})}\cdot\int_{\mathbf{T}}\frac{1}{s_h-\omega_h(\theta)}\,d\theta.$$

Let $T_{\delta V}$ be the threshold for the existence of an eigenvalue outside the band $\omega_h(\mathbf{T})$ (see [Kl]). Then one has

Theorem 1.8. *Let $n=3$ and assume* (H.1)–(H.6). *Then there exist $h_0>0$, $r_0>0$, $\tilde{t}_0>0$ such that, for $h\in(0,h_0)$, there exist*:

(i) *a function $\lambda:(T_{\delta V},T_{\delta V}+\tilde{t}_0\cdot f(h))\to s_h+(0,r_0\cdot f(h))$ that is a simple eigenvalue of P_t and that admits the following convergent expansion*:

$$\lambda(t)-s=f(h)\cdot(\tilde{t})^2\cdot\left(\sum_{l\in\mathbf{N}}\alpha_l(h)\cdot(\tilde{t})^l\right),$$

where:

- $\tilde{t}=(t-T_{\delta V})/f(h)$
- *for any* $l\in\mathbf{N}$, $\alpha_l(h)\in\mathbf{R}$
- $$\alpha_0(h)=\left(\frac{\mathrm{Vol}(\mathbf{T})\cdot\varrho\cdot(f(h)\cdot I)^2}{2^{5/2}\pi^2\cdot D_s}\right)^2\cdot(1+O(e^{-c/h})),\quad \textit{for some } c>0.$$

(ii) *a function λ: $(T_{\delta V}-\tilde{t}_0\cdot f(h),T_{\delta V}]\to s_h+e^{2i\pi}\cdot(0,r_0\cdot f(h))$ that is a simple resonance of P_t and that admits the following convergent expansion*:

$$\lambda(t)-s=e^{2i\pi}\cdot f(h)\cdot(\tilde{t})^2\cdot\left(\sum_{l\in\mathbf{N}}\alpha_l(h)\cdot(\tilde{t})^l\right).$$

Moreover, for $t\in(T_{\delta V}-\tilde{t}_0\cdot f(h),T_{\delta V}+\tilde{t}_0\cdot f(h))$, one has

$$\mathrm{Res}(\mathcal{U}C(r_0,s_h),t)=\{\lambda(t)\}.$$

Remark. Here one sees that, for decreasing t, the eigenvalue turns into a resonance when t crosses the threshold $T_{\delta V}$. The resonance satisfies $\mathrm{Im}(\lambda(t))=0$; it is located on the real axis of the second sheet of the Riemann surface where the resolvent $(z-P_t)^{-1}$ is defined (see Figure 4).

In Figure 4, X is an eigenvalue for $t>T_{\delta V}$ and x a resonance for $t\leq T_{\delta V}$.

To describe the resonances in dimension 4 we will need

Lemma 1.9. *Let $q\in\mathbf{N}$. There exist r_q, r'_q and $r''_q>0$, α a function analytic in a small neighborhood of $(0,0)$ in $\mathbf{C}^2$ that is real if both of its arguments are real and such that $\alpha(z,z')=o(|z|+|z'|)$, and, for each $-q\leq j\leq q$, an open set D_j satisfying*

$$e^{2ij\pi}\cdot(-r''_q,r''_q)\subset\mathcal{D}_j\subset e^{2ij\pi}\cdot(D(0,r'_q)\setminus i\cdot(-r'_q,0]),$$

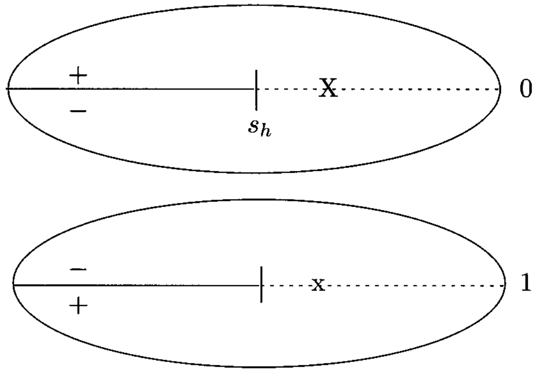

Figure 4.

and such that, if for $z \in \mathcal{D}_j$ *we define*

$$y_j(z) = \frac{-z}{\log z} \cdot \left(1 + \alpha\left(\frac{\log(-\log z)}{\log z}, \frac{1}{\log z}\right)\right),$$

then $y_j : \mathcal{D}_j \to e^{2ij\pi} \cdot (D(0, r_q) \setminus i \cdot (-r_q, 0])$ *is bijective and*

$$-y_j(z) \cdot \log(y_j(z)) = z.$$

We then get

Theorem 1.10. *Let* $q \in \mathbf{N}$ *and assume* (H.1)–(H.6). *Then there exist* $h_q > 0$, $r_q > 0$, $\tilde{t}_q > 0$ *such that, for* $h \in (0, h_q)$, *there exists an analytic function* g *admitting the following convergent expansion near* $(0,0)$:

$$g(v, w) = \sum_{l \geq 0, m \geq 0} \alpha_{m,l}(h) \cdot v^l \cdot w^m,$$

where

- *for any* (l, m), $\alpha_{m,l}(h) \in \mathbf{R}$
- $$\alpha_{0,0}(h) = \frac{\mathrm{Vol}(\mathbf{T}) \cdot \varrho \cdot (f(h) \cdot I)^2}{4\pi^2 \cdot D_s} \cdot (1 + O(e^{-c/h})), \quad \textit{for some } c > 0,$$

such that if we define

$$\tilde{t} = \frac{t - T_{\delta V}}{f(h)} \quad \textit{and} \quad \lambda_p(t) - s = f(h) \cdot y_p(e^{2iq\pi}\tilde{t}) \cdot g(\tilde{t}, y_p(e^{2iq\pi}\tilde{t})/\tilde{t}),$$

for $-q \leq p \leq q$ *and* $t \in ((T_{\delta V} - \tilde{t}_q \cdot f(h), T_{\delta V} + \tilde{t}_q \cdot f(h))$, *then*

- *for* $t > T_{\delta V}$, $\lambda_0(t)$ *is a simple eigenvalue of* P_t,
- *for* $t \leq T_{\delta V}$, $\lambda_0(t)$ *is a simple resonance of* P_t,
- *for* $p \neq 0$ *and for any* t, $\lambda_p(t)$ *is a simple resonance of* P_t.

Moreover, for $t\in(T_{\delta V}-\tilde{t}_q\cdot f(h), T_{\delta V}+\tilde{t}_q\cdot f(h))$*, one has*

$$\mathrm{Res}(\mathcal{U}C(r_q, s_h, q), t) = \bigcup_{-q\le p\le q} \{\lambda_p(t)\}.$$

Remark. We compute the imaginary part of these resonances and get, for $-q\le p\le q$,

$$\mathrm{Im}(\lambda_p(t)) = \alpha_{0,0}(h)\cdot f(h)\cdot\left(\frac{p\cdot\pi\cdot\tilde{t}}{(\log|\tilde{t}|)^2}\right)\cdot(1+o(1)).$$

Figure 5b shows a picture of these resonances.

Let $n\ge 5$ and define

$$\partial I = -\frac{1}{\mathrm{Vol}(\mathbf{T})}\cdot\int_{\mathbf{T}}\frac{1}{(s_h-\omega_h(\theta))^2}\,d\theta,$$

and for $z\in D(0,r_q)\setminus i\cdot(-r_q,0)$

$$g_n(z) = \begin{cases} \frac{1}{2}\pi\cdot(-1)^{(n-1)/2} z^{(n-4)/2} & \text{if } n \text{ is odd},\\ \frac{1}{2}\cdot(-1)^{n/2}\cdot z^{(n-4)/2}\cdot\log z & \text{if } n \text{ is even}.\end{cases}$$

Then one has

Theorem 1.11. *Let* $n\ge 5$, $q\in\mathbf{N}$ *and assume* (H.1)–(H.6). *Then there exist* $h_q>0$, $r_q>0$, $\tilde{t}_q>0$ *such that, for* $h\in(0,h_q)$*, there exists an analytic function* α_n *admitting the following convergent expansion near* $(0,0)$:

$$\alpha_n(v,w) = \sum_{l\ge 0, m\ge 0}\alpha^n_{l,m}(h)\cdot v^l\cdot w^m,$$

where

- *for any* (l,m), $\alpha^n_{l,m}(h)\in\mathbf{R}$
- $\alpha^n_{0,0}(h) = -(\varrho\cdot I^2)/\partial I\cdot(1+O(e^{-c/h}))$,

$$\alpha^n_{0,1}(h) = \frac{2^n\pi^{n/2}}{\Gamma(n/2)\cdot\mathrm{Vol}(\mathbf{T})\cdot f(h)^2}\cdot\frac{(\varrho I^2)^{(n-2)/2}}{(-\partial I)^{n/2}}\cdot D_s\cdot(1+O(e^{-c/h})),$$

for some $c>0$ *such that if we define*

$$\tilde{t} = (t-T_{\delta V})/f(h) \text{ and } \lambda_p(t)-s = e^{2ip\pi}\cdot(t-T_{\delta V})\cdot\alpha_n(\tilde{t}, g_n(e^{2ip\pi}\tilde{t})),$$

for $-q\le p\le q$ *and* $t\in(T_{\delta V}-\tilde{t}_q\cdot f(h), T_{\delta V}+\tilde{t}_q\cdot f(h))$*, then*

- *for* $t\ge T_{\delta V}$, $\lambda_0(t)$ *is a simple eigenvalue of* P_t,
- *for* $t<T_{\delta V}$, $\lambda_0(t)$ *is a simple resonance of* P_t,
- *for* $p\ne 0$ *and for any* t, $\lambda_p(t)$ *is a simple resonance of* P_t.

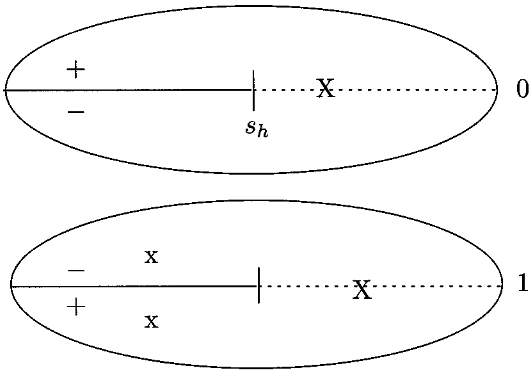

Figure 5a. $n \geq 4$, n odd.

Moreover, for $t \in (T_{\delta V} - \tilde{t}_q \cdot f(h), T_{\delta V} + \tilde{t}_q \cdot f(h))$, *one has*

$$\mathrm{Res}(\mathcal{U}C(r_q, s_h, q), t) = \bigcup_{-q \leq p \leq q} \{\lambda_p(t)\}.$$

Remark. One computes the imaginary part of these resonances to obtain:

- if n is even, for $-q \leq p \leq q$,

$$\mathrm{Im}(\lambda_p(t)) = \pi \cdot p \cdot f(h) \cdot \alpha_{0,1}^n(h) \cdot (-1)^{n/2} \cdot (\tilde{t})^{(n-2)/2},$$

- if n is odd, for p even,

$$\mathrm{Im}(\lambda_p(t)) = 0$$

  and, for p odd,

$$\mathrm{Im}(\lambda_p(t)) = f(h) \cdot \alpha_{0,1}^n(h) \cdot (-1)^{(p+1)/2} \cdot |\tilde{t}|^{(n-2)/2}.$$

Figure 5a, b, c, show pictures of these resonances depending on n.

In all these pictures, x is a resonance for $t < T_{\delta V}$ (or $t \leq T_{\delta V}$ if $n=4$) and X is an eigenvalue or a resonance for $t \geq T_{\delta V}$ (or $t > T_{\delta V}$ if $n=4$) depending on in which sheet it is located.

4. Embedded eigenvalues

A corollary of the preceding study is

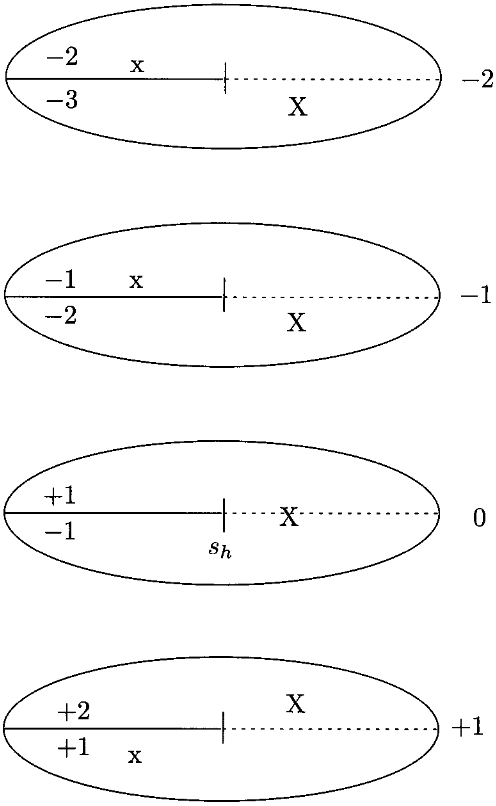

Figure 5b. $n \geq 4$, $n \equiv 0(4)$.

Theorem 1.12. *Let $n \in \mathbf{N}$ and assume* (H.1)–(H.6). *Then there exist $h_0 > 0$ and $t_0 > 0$ such that, for $h \in (0, h_0)$, one has*

(a) *if $n=1$: for $t \in [-t_0 \cdot f(h), t_0 \cdot f(h)]$, there is no eigenvalue of P_t embedded in $[i_h, s_h]$, the band of the essential spectrum, that is*

$$\sigma(P_t) \cap [i_h, s_h] = \sigma_{\text{cont}}(P_t) \cap [i_h, s_h].$$

(b) *if $n \geq 3$: for $t \in [T_{\delta V} - t_0 \cdot f(h), T_{\delta V} + t_0 \cdot f(h)]$, there is no eigenvalue of P_t embedded in (i_h, s_h), the band of the essential spectrum, that is*

$$\sigma(P_t) \cap (i_h, s_h) = \sigma_{\text{cont}}(P_t) \cap (i_h, s_h).$$

Remark. In dimension 2 one can state a similar result outside some neighborhoods of the inner critical points of ω_h.

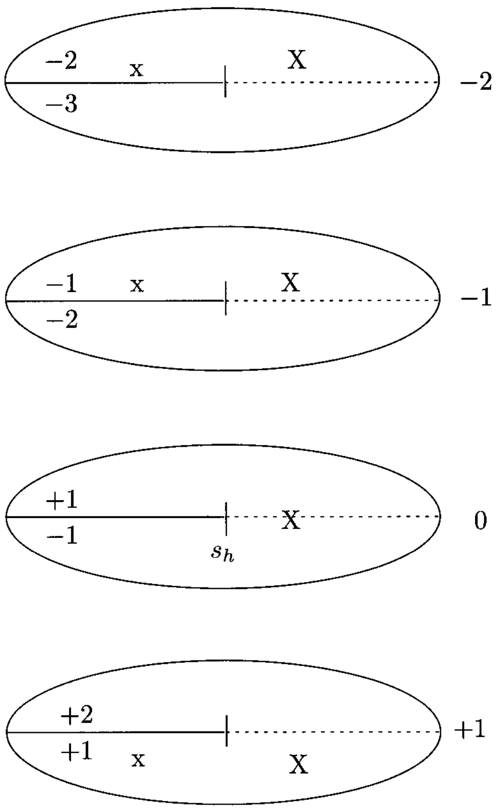

Figure 5c. $n \geq 4$, $n \equiv 2(4)$.

II. Analytic continuation of R_0 and Γ

1. The unitary equivalence $\mathcal{F}_t$

We will first recall some facts from [Kl]. Under assumptions (H.1)–(H.4), let $(\varphi_{t,\gamma})_{\gamma\in L}$ be the Hilbert basis spanning F_t constructed in [Kl]. We know that there exist $h_0>0$ and $C>0$ such that, for $h\in(0,h_0)$, $t\in(-a(h)/4, a(h)/4)$ and for any $\gamma\in L$,

$$\|\varphi_{t,\gamma}(\cdot)e^{|\cdot-\gamma|/Ch}\|_{L^2(\mathbf{R}^n)} \leq C. \tag{2.1}$$

One defines the projector Π_t on F_t, for $\varphi\in L^2(\mathbf{R}^n)$,

$$\Pi_t\varphi = \sum_{\gamma\in L}(\varphi \mid \varphi_{t,\gamma})\varphi_{t,\gamma}, \tag{2.2}$$

where $(\cdot|\cdot)$ denotes the scalar product in $L^2(\mathbf{R}^n)$.

We also define $\mathcal{F}_t: L^2(\mathbf{R}^n)\to L^2(\mathbf{T})$ and $\mathcal{F}_t^*: L^2(\mathbf{T})\to L^2(\mathbf{R}^n)$ for $\varphi\in L^2(\mathbf{R}^n)$, $u\in L^2(\mathbf{T})$ and $\theta\in\mathbf{T}$ by

$$(\mathcal{F}_t\varphi)(\theta)=\sum_{\gamma\in L}(\varphi\,|\,\varphi_{t,\gamma})e^{i\gamma\cdot\theta}, \tag{2.3}$$

and

$$\mathcal{F}_t^*u=\sum_{\gamma\in L}\left(\frac{1}{\mathrm{Vol}(\mathbf{T})}\cdot\int_{\mathbf{T}}e^{-i\gamma\cdot\theta}u(\theta)d\theta\right)\varphi_{t,\gamma}. \tag{2.4}$$

$\mathcal{F}_t$ realises a unitary equivalence from F_t to $L^2(\mathbf{T})$ with inverse $\mathcal{F}_t^*$.

For $a>0$, let $W_a=\mathbf{T}+iB_{\mathbf{C}^n}(0,a)$ and $\mathcal{O}(W_a)$ be the set of bounded analytic functions in W_a provided with the L^∞ norm (here $B_{\mathbf{C}^n}(0,a)$ denotes the ball of center 0 and radius a in $\mathbf{C}^n$). One has

Lemma 2.1. *There exists $C_0>0$ and $h_0>0$ such that for $h\in(0,h_0)$ and $0<a<a'<1/C_0h$*

(a) Π_t *is continuous from* $L^2_a(\mathbf{R}^n)$ *to* $L^2_a(\mathbf{R}^n)$,

(b) $\mathcal{F}_t$ *is compact from* $L^2_a(\mathbf{R}^n)$ *to* $\mathcal{O}(W_{a'})$,

(c) $\mathcal{F}_t^*$ *is continuous from* $\mathcal{O}(W_a)$ *to* $L^2_{a'}(\mathbf{R}^n)$.

These results are uniform in t and h small enough.

Proof. Let $\varphi\in L^2_a(\mathbf{R}^n)$. Then for $\gamma\in L$

$$(\varphi\,|\,\varphi_{t,\gamma})=\int_{\mathbf{R}^n}(e^{-(a|x|+|x-\gamma|/Ch)})\cdot(e^{a|x|}\varphi(x))\cdot(e^{|x-\gamma|/Ch}\varphi_{t,\gamma}(x))\,dx \tag{2.5}$$

so by (2.1)

$$\begin{aligned}|(\varphi\,|\,\varphi_{t,\gamma})|&\le e^{-\inf(a,1/Ch)\cdot|\gamma|}\cdot\|\varphi\|_{L^2_a}\cdot\|\varphi_{t,\gamma}e^{|\cdot-\gamma|/Ch}\|_{L^2}\\&=Ce^{-a\cdot|\gamma|}\cdot\|\varphi\|_{L^2_a}.\end{aligned} \tag{2.6}$$

Then, using (2.2) and taking C_0 large one gets

$$\|\Pi_t\varphi\|_{L^2_a}\le\sum_{\gamma\in L}|(\varphi\,|\,\varphi_{t,\gamma})|\cdot\|e^{a|\cdot|}\varphi_{t,\gamma}\|_{L^2}\le C'\|\varphi\|_{L^2_a}.$$

which proves (a).

Using estimate (2.6), (b) is immediate. To prove (c) one uses Stokes' formula and the analyticity of u to write, for $\gamma\neq 0$

$$\begin{aligned}\left|\int_{\mathbf{T}} e^{-i\gamma\cdot\theta}u(\theta)\,d\theta\right| &= \left|\int_{\mathbf{T}-i(a'\gamma)/|\gamma|} e^{-i\gamma\cdot\theta}u(\theta)\,d\theta\right| \\ &= e^{-a'|\gamma|}\cdot\left|\int_{\mathbf{T}} e^{-i\gamma\cdot\theta}u\left(\theta-i\frac{a'\gamma}{|\gamma|}\right)d\theta\right| \\ &\leq \mathrm{Vol}(\mathbf{T})\cdot e^{-a'|\gamma|}\|u\|_{\infty,W_a}.\end{aligned}$$

Then, using (2.4), one gets (c).

2. The analytic continuation of R_0 and Γ

By (1.6)–(1.7) and (2.3)–(2.4), for $\varphi\in L^2(\mathbf{R}^n)$ one has

$$R_0(z,t)(\varphi)=\sum_{\gamma\in L}\left(\frac{1}{\mathrm{Vol}(\mathbf{T})}\cdot\int_{\mathbf{T}}\frac{e^{-i\gamma\cdot\theta}\mathcal{F}_t(\varphi)(\theta)}{z-\omega_h(\theta)}\,d\theta\right)\varphi_{t,\gamma} \tag{2.7}$$

and

$$\Gamma(z,t)(\varphi)=\mathcal{F}_t^*\left(\int_{\mathbf{T}}\frac{(1+k(t,\cdot,\theta))\cdot\mathcal{F}_t(\varphi)(\theta)}{z-\omega_h(\theta)}\,d\theta\right). \tag{2.8}$$

Let $0<a'<a<1/Ch$. For $\gamma\in L$ we define $\Pi_\gamma(z)\colon\mathcal{O}(W_a)\to\mathbf{C}$ by

$$u\mapsto\frac{1}{\mathrm{Vol}(\mathbf{T})}\cdot\int_{\mathbf{T}}\frac{e^{-i\gamma\cdot\theta}\cdot u(\theta)}{z-\omega_h(\theta)}\,d\theta$$

and $\Gamma_k(z,t)\colon\mathcal{O}(W_a)\to\mathcal{O}(W_a)$ by

$$u\mapsto\frac{1}{\mathrm{Vol}(\mathbf{T})}\cdot\int_{\mathbf{T}}\frac{(1+k(t,\cdot,\theta))\cdot u(\theta)}{z-\omega_h(\theta)}\,d\theta. \tag{2.9}$$

Both of these operators are compact.

After renormalizing the band by introducing $\tilde z=(z-i_h)/f(h)$ we prove, using Proposition A.1, that the operators $f(h)\Pi_\gamma(z)$ and $f(h)\Gamma_k(z,t)$ can be analytically continued from above or below the real axis to a neighborhood of $\omega_h(\mathbf{T})\setminus\{\omega_h(\theta_j^k);1\leq j\leq p,1\leq k\leq k_j\}$ as compact operators from $\mathcal{O}(W_a)$ to $\mathbf{C}$ or $\mathcal{O}(W_a)$ with the following upper bounds for the norm:

$$f(h)\|\Pi_\gamma(z)\|_{\mathcal{O}(W_a)\to\mathbf{C}}\leq Ce^{|\gamma|\cdot|\mathrm{Im}(\tilde z)|},$$

and

$$f(h)\|\Gamma_k(z,t)\|_{\mathcal{O}(W_a)\to\mathcal{O}(W_a)}\leq C$$

for some $C>0$.

Moreover, Proposition A.2 gives us precise expansions for these continuations near the branch points which are the critical points of ω_h. Using this, one easily gets Theorem 1.3 by (2.7) and (2.8).

Estimate (b) of the Proposition A.2 applied to $\partial_t\Gamma_k(z,t)$ combined with the estimates known for the kernel k implies that, for $(\tilde{z},t)\in\square(\tilde{\lambda}_0^j,r_0)\times D(0,a(h)/4)$, one has

$$\sum_{1\leq l\leq k_j}\|\partial_t H^{\pm}_{l,k}(\tilde{z},t)\|_{\mathcal{O}(W_a)\to\mathcal{O}(W_a)}+\|\partial_t G^{\pm}_k(\tilde{z},t)\|_{\mathcal{O}(W_a)\to\mathcal{O}(W_a)}\leq e^{-c/h}. \tag{2.10}$$

III. The spectrum of Γ

By (2.8) and (2.9) Γ and Γ_k are unitarily equivalent. So we will now study the spectrum of Γ_k. Taking z and t as usual, let us define $\Gamma_0(z)\colon\mathcal{O}(W_a)\to\mathcal{O}(W_a)$ by

$$u\mapsto\frac{1}{\operatorname{Vol}(\mathbf{T})}\cdot\int_{\mathbf{T}}\frac{u(\theta)}{z-\omega_h(\theta)}\,d\theta,$$

where the expression to the right is viewed as a constant function. Obviously, the spectrum of Γ_0 consists of 2 eigenvalues, 0 of infinite multiplicity and

$$I(z)=\frac{1}{\operatorname{Vol}(\mathbf{T})}\cdot\int_{\mathbf{T}}\frac{1}{z-\omega_h(\theta)}\,d\theta$$

of multiplicity 1 with eigenvector the constant function 1. For $x\notin\{I(z),0\}$ one computes

$$(x-\Gamma_0(z))^{-1}=\frac{1}{x}\left(1+\frac{1}{x-I(z)}\Gamma_0(z)\right). \tag{2.11}$$

By Proposition A.2 one has an expansion

$$f(h)\Gamma_0(z)=\sum_{l=1}^{k_j}S(\tilde{z}-\tilde{\omega}_h(\theta_l^k))\cdot H^{\pm}_{l,0}(\tilde{z})+G^{\pm}_0(\tilde{z}).$$

Then using (2.11) one gets the estimate

$$\|(x-\Gamma_0(z))^{-1}\|\leq\frac{1}{|x|}+\frac{C|I(z)|}{|x|\cdot|x-I(z)|}. \tag{2.12}$$

Using estimate (b) of Proposition A.2 applied to $\Gamma_k-\Gamma_0$, for $(\tilde{z},t)\in\square(\tilde{\lambda}_0^j,r_0)\times D(0,a(h)/4)$, one gets

$$\sum_{1\leq l\leq k_j}\|H_{l,k}^{\pm}(\tilde{z},t)-H_{l,0}^{\pm}(\tilde{z})\|_{\mathcal{O}(W_a)\to\mathcal{O}(W_a)} + \|G_k^{\pm}(\tilde{z},t)-G_0^{\pm}(\tilde{z})\|_{\mathcal{O}(W_a)\to\mathcal{O}(W_a)}\leq e^{-c/h}. \tag{2.13}$$

1. Investigations near the regular values of ω_h

One gets

Lemma 2.2. *For any $r_0>0$ small enough there exist $h_0>0$ and $C>0$ such that, for $h\in(0,h_0)$ and for $z\in\mathcal{UC}({}^c\square(r_0),{}^c\square^{\pm}(r_0))$,*

$$\sigma(\Gamma_k(z,t))\subset\{0,I(z)\}+D_{\mathbf{C}}\big(0,C\cdot|I(z)|\cdot\|k\|_{\infty,W_a}\big).$$

Proof. By Propositions A.1 and A.3 we know that, for r_0 small enough, $z\in\mathcal{UC}({}^c\square(r_0),{}^c\square^{\pm}(r_0))$ and for some $C>0$ (depending only on r_0),

$$\frac{1}{C\cdot f(h)}\leq|I(z)|\leq\frac{C}{f(h)}. \tag{2.14}$$

By Proposition A.1 we know that, for $z\in\mathcal{UC}({}^c\square(r_0),{}^c\square^{\pm}(r_0))$ and some $C>0$,

$$\begin{aligned}\|\Gamma_k(z,t)-\Gamma_0(z)\|_{\mathcal{O}(W_a)}&\leq\frac{1}{\mathrm{Vol}(\mathbf{T})}\cdot\left\|\int_{\mathbf{T}}\frac{k(t,\cdot,\theta')}{z-\omega_h(\theta')}\,d\theta'\right\|_{\mathcal{O}(W_a)}\\&\leq\frac{C\cdot\|k\|_{\infty,W_a}}{f(h)}.\end{aligned}$$

By (2.12) and (2.14) we get that, if $x\notin\{0,I(z)\}+D_{\mathbf{C}}(0,C\cdot|I(z)|\cdot\|k\|_{\infty,W_a})$ for some $C>0$ and h small enough, then

$$\|(x-\Gamma_0(z))^{-1}\|_{\mathcal{O}(W_a)}\leq\frac{f(h)}{C\cdot\|k\|_{\infty,W_a}}.$$

So using the following resolvent formula

$$(x-\Gamma_k(z,t))^{-1}=(x-\Gamma_0(z))^{-1}\cdot\big(1-(\Gamma_k(z,t)-\Gamma_0(z))\cdot(x-\Gamma_0(z,t))^{-1}\big)^{-1}, \tag{2.15}$$

we get the announced lemma.

2. Investigations near the critical values

Let $1 \le j \le p$ and let $\lambda_0^j = i_h + f(h) \cdot \tilde{\lambda}_0^j$ be a critical value of $\widetilde{\omega}_0$ rescaled to the size of the band. We recall that, for $0 \le r \le r_0$, $\mathcal{O}(r,j)$ is a ball of radius $r \cdot f(h)$ around λ_j^0 in $\mathcal{UC}(r_0, j)$. Then the following holds:

Lemma 2.3. *Assume $n \ge 3$. There exist $h_r > 0$, $C_r > 0$ when $0 < r$ is small such that, for $h \in (0, h_r)$ and $z \in \mathcal{O}(r,j)$, one has*

$$\sigma(\Gamma_k(z,t)) \subset \{0, I(z)\} + D_{\mathbf{C}}(0, C_q \cdot |I(z)| \cdot \|k\|_{\infty, W_a}).$$

Proof. If $n \ge 3$, the proof is exactly the same as for the case of the regular values except that one has (2.14) only for z in the usual complex plane (the first sheet of the universal covering). So the upper bound on $I(z)$ still holds true for z equal to a branch point (as S is continuous at $z=0$). Then by connectedness of the universal covering and continuity of $I(z)$, one gets this bound in a neighborhood of the branch points in the universal covering. The same is true for the estimate of $\|\Gamma_k(z,t) - \Gamma_0(z)\|$. This gives the lemma.

In dimension $n=1$ or 2 we will assume (H.6). We will only study the spectrum of Γ_k near the edges of the band. For $q \in \mathbf{N}$ and $z = s_h$ or i_h define

$$\mathcal{OC}(q,r,z) = \bigcup_{-q \le p \le q} (z + e^{ip\pi} \cdot \square_h(r,0)) \subset \mathcal{OC}(r,z).$$

One has

Lemma 2.4. *Assume $n=1$ or 2. For any $q \in \mathbf{N}$, there exist $h_q > 0$, $r_q > 0$ and $C_q > 0$ such that, for $h \in (0, h_q)$ and $z \in \mathcal{O}(q, r_q, s_h) \cup \mathcal{O}(q, r_q, i_h)$ one has*

$$\sigma(\Gamma_k(z,t)) \subset D_{\mathbf{C}}\left(0, \frac{C_q}{f(h)}\right) \cup D_{\mathbf{C}}(I(z), C_q \cdot |I(z)| \cdot \|k\|_{\infty, W_a}).$$

Remark. Notice that when $\tilde{z}$ tend to $\tilde{s}_h$, $f(h) \cdot I(z)$ tends to ∞, so for $\tilde{z} - \tilde{s}_h$ chosen small enough,

$$D_{\mathbf{C}}\left(0, \frac{C_q}{f(h)}\right) \cap D_{\mathbf{C}}(I(z), C_q \cdot |I(z)| \cdot \|k\|_{\infty, W_a}) = \phi.$$

This holds uniformly in h.

Proof. We will only prove Lemma 2.4 for z close to the maximum; the case of the minimum goes along the same lines. Using the fact that there is only one critical point corresponding to the maximum, say θ_s, by the expansion given in Theorem 1.3 and the computations done in Section 4 one has, for z close to s_h,

$$f(h)\cdot\Gamma_k(z,t)=S(\tilde{z}-\tilde{s}_h)A_k+G(\tilde{z},t).$$

Here $G(z,t)$ is a compact operator such that for some $C_q>0$ and $r_q>0$, for z in $\mathcal{O}C(q,r_q,s_h)$ and t as usual

$$\|G(\tilde{z},t)\|_{\mathcal{O}(W_a)\to\mathcal{O}(W_a)}\le C_q.$$

Moreover, A_k is the rank one operator defined for $u\in\mathcal{O}(W_a)$ by

$$A_k(u)(\theta)=\beta\cdot(1+k(t,\theta,\theta_s))\cdot u(\theta_s),$$

where

$$\frac{1}{C}\le\beta=\frac{2^{n/2}\cdot\mathrm{Vol}(\partial B(0,1))\cdot|\det(\mathrm{Hess}(\widetilde{\omega}_h(\theta_s)))|^{-1/2}}{\mathrm{Vol}(\mathbf{T})}\le C$$

for some $C>0$ independent of h small enough.

By perturbation theory we know that

$$\sigma(f(h)\cdot\Gamma_k(z,t))\subset D_{\mathbf{C}}(0,C_q)\cup D_{\mathbf{C}}\big(\beta\cdot S(\tilde{z}-\tilde{s}_h),C_q\cdot|\beta\cdot S(\tilde{z}-\tilde{s}_h)|\cdot\|k\|_{\infty,W_a}\big). \tag{2.16}$$

By Proposition A.2 we know that there exists $C_q>0$ such that for $z\in\mathcal{O}(q,r_q,s_h)$

$$\frac{1}{C_q}\le\left|\frac{I(z)\cdot f(h)}{S(\tilde{z}-\tilde{s}_h)}\right|\le C_q.$$

So using (2.16) one gets the announced result.

Remark. 1. In odd dimension, because the branch points are of square root type (so the Riemann surface associated to the analytic continuation is only finitely "sheeted"), one may choose r_q, h_q, and C_q independent of q.

2. Here we did not treat the case of the inner critical points in dimension $n=2$ (such points do not exist in dimension $n=1$). In dimension 2, one sees that, when h goes to 0, for any $0\le k\le k_j$, $\widetilde{\omega}_h(\theta_j^k)$ tends to λ_j^0. Hence, for $\tilde{z}$ tending to one of the $\widetilde{\omega}_h(\theta_j^k)$, it is not possible to control the behaviour of the expansions given in Proposition A.2 without further assumptions on the behaviour of the critical points.

As will be seen in the next section, these lemmas will suffice to conclude that resonances can only exist near the edges of the band. So we are going now to study more precisely the spectrum of Γ_k near s_h (the other side of the band can be treated in the same way).

Proposition 2.5. *Assume* (H.6) *and let* $q\in\mathbf{N}$. *Then there exist* $h_q>0$, $r_q>0$ *and* $C_q>0$ *such that, for* $h\in(0,h_q)$, *there exists a function* $vp(z,k)$ *defined on* $\mathcal{OC}(q,r_q,s_h)$ *verifying*:

(a) $vp(z,k)$ *is a simple eigenvalue of* Γ_k *and*

$$\sigma(\Gamma_k(z,t))\cap D_{\mathbf{C}}\left(I(z),\frac{|I(z)|}{4}\right)=\{vp(z,k)\}.$$

(b) *For* $z\in\mathcal{OC}(q,r_q,s_h)$ *we define the coefficients* $a^0_{l,m}\in\mathbf{R}$ *by*

$$f(h)\cdot I(z)=S(\tilde z-\tilde s_h)\cdot\left(\sum_{l\in\mathbf{N}}a^0_{l,0}(\tilde z-\tilde s_h)^l\right)+\sum_{l\in\mathbf{N}}a^0_{l,1}(\tilde z-\tilde s_h)^l,$$

and $a^0_{l,m}=0$ *if* $m\geq 2$.

Then, for $z\in\mathcal{OC}(q,r_q,s_h)$, *the function* $vp(z,k)$ *admits the following uniformly convergent expansion*:

If $n=1$ *or* 2 *then*

$$f(h)\cdot vp(z,k)=S(\tilde z-\tilde s_h)\cdot\sum_{l\in\mathbf{N},m\in\mathbf{N}}a^k_{l,m}(t)(\tilde z-\tilde s_h)^l\cdot(S(\tilde z-\tilde s_h))^{-m},$$

where the coefficients $(a^k_{l,m}(t))_{l\in\mathbf{N},m\in\mathbf{N}}$ *are analytic functions in* $t\in D_{\mathbf{C}}(0,a(h)/4)$, *real valued for t real and*

$$|a^k_{l,m}(t)-a^0_{l,m}|\leq C_q\cdot r_q^{-l}\cdot S(r_q)^m\cdot\|k\|_{\infty,W_a},$$
$$|\partial_t a^k_{l,m}(t)|\leq C_q\cdot r_q^{-l}\cdot S(r_q)^m\cdot\|k\|_{\infty,W_a}.$$

If $n\geq 3$ *then*

$$f(h)\cdot vp(z,k)=\sum_{l\in\mathbf{N},m\in\mathbf{N}}a^k_{l,m}(t)(\tilde z-\tilde s_h)^l\cdot(S(\tilde z-\tilde s_h))^m,$$

where the coefficients $(a^k_{l,m}(t))_{l\in\mathbf{N},m\in\mathbf{N}}$ *are analytic functions in* $t\in D_{\mathbf{C}}(0,a(h)/4)$, *real valued for t real and*

$$|a^k_{l,1}(t)-a^0_{l,0}|\leq C_q\cdot r_q^{-l}\cdot S(r_q)^{-m}\cdot\|k\|_{\infty,W_a},$$
$$|a^k_{l,0}(t)-a^0_{l,1}|\leq C_q\cdot r_q^{-l}\cdot S(r_q)^{-m}\cdot\|k\|_{\infty,W_a},$$
$$|a^k_{l,m}(t)|\leq C_q\cdot r_q^{-l}\cdot S(r_q)^{-m}\cdot\|k\|_{\infty,W_a}\quad \text{for } m\geq 2,$$
$$|\partial_t a^k_{l,m}(t)|\leq C_q\cdot r_q^{-l}\cdot S(r_q)^{-m}\cdot\|k\|_{\infty,W_a}\quad \text{for any } m.$$

Remark.

- The remark following Lemma 2.4 still holds for this proposition.
- Using the notations of appendix A one has

$$a^0_{0,0} = \frac{1}{\mathrm{Vol}(\mathbf{T})}\cdot A^0(1) = \frac{(4\pi)^{n/2}}{\Gamma(n/2)\cdot\mathrm{Vol}(\mathbf{T})}\cdot|\det(\mathrm{Hess}(\widetilde{\omega}_h(\theta_s)))|^{-1/2}, \quad \text{if } 1\leq n\leq 2,$$

and

$$a^0_{0,1} = \frac{1}{\mathrm{Vol}(\mathbf{T})}\cdot A^0(1) = \frac{(4\pi)^{n/2}}{\Gamma(n/2)\cdot\mathrm{Vol}(\mathbf{T})}\cdot|\det(\mathrm{Hess}(\widetilde{\omega}_h(\theta_s)))|^{-1/2}, \quad \text{if } n\geq 3.$$

Moreover, for $n\geq 3$ and $0\leq l\leq (n-3)/2$, $a^0_{l,0}$ is given by the expansion (A.17) for $J(\tilde{z},1)$, that is

$$a^0_{l,0} = \int_{\mathbf{T}} \frac{(-1)^l}{(\tilde{s}_h-\widetilde{\omega}_h(\theta))^{l+1}}\,d\theta.$$

Proof. For h small enough consider the following family of projectors

$$\Pi_{k,\alpha} = \int_{\mathcal{C}} (x-\Gamma_{\alpha\cdot k}(z,t))^{-1}dx, \quad \alpha\in[0,1],$$

where $\mathcal{C}$ is the complex contour $\{x\in\mathbf{C};\, |x-I(z)|=|I(z)|/4\}$ and $\Gamma_{\alpha\cdot k}$ is the operator Γ_k where one has replaced the kernel k by $\alpha\cdot k$. This family is analytic in α in the norm sense. By (2.12), (2.15), and by chosing h small enough to get $\|k\|_{\infty,W_a}\leq\frac{1}{4}$, we see that $\Pi_{k,0}$ is of rank 1. So $\Pi_{k,1}$ is of rank 1 which gives point (a).

Assume $n=1$ or 2. For $y\in\mathbf{C}$ and $\tilde{z}\in D_{\mathbf{C}}(\tilde{s}_h,r_q)$, consider the operator

$$O_k(y,\tilde{z}) = H_k(\tilde{z},t)+y\cdot G_k(\tilde{z},t),$$

where $H_k(\tilde{z},t)$ and $G_k(\tilde{z},t)$ are the operators given in Theorem 1.3 (in this case, because one looks at the edge of the band, these operators do not depend on whether one continues analytically from below or above the band). Then Theorem 1.3 says, that for $z\in\mathcal{U}C(q,r_q,s_h)$

$$f(h)\cdot\Gamma_k(z,t) = S(\tilde{z}-\tilde{s}_h)\cdot O_k\left(\frac{1}{S(\tilde{z}-\tilde{s}_h)},\tilde{z}\right). \tag{2.17}$$

Noticing that

$$\lim_{r\to 0}\left(\sup_{z\in\mathcal{O}(q,r,s_h)}\left|\frac{1}{S(\tilde{z}-\tilde{s}_h)}\right|\right) = 0,$$

and letting $\tilde{z}\to\tilde{s}_h$ in (2.17), one gets

$$O_k(0,\tilde{s}_h)=\lim_{\tilde{z}\to\tilde{s}_h}\left(\frac{1}{I(z)}\cdot\Gamma_k(z,t)\right).$$

By part (a) of this proposition, $O_k(0,\tilde{s}_h)$ admits a simple eigenvalue in $D_{\mathbf{C}}\left(1,\frac{1}{4}\right)$ isolated from the rest of its spectrum.

As $O_k(y,\tilde{z})$ is analytic in y, $\tilde{z}$ and t, there exists $b>0$ such that, for $(y,\tilde{z},t)\in D_{\mathbf{C}}(0,b)\times D_{\mathbf{C}}(\tilde{s}_h,b)\times D_{\mathbf{C}}(0,a(h)/4)$, there exists $v(y,\tilde{z},k(t))$, a simple eigenvalue of $O_k(y,\tilde{z})$ isolated from the rest of the spectrum. This eigenvalue is simple and therefore analytic in its parameters, so it admits the following convergent expansion

$$v(y,\tilde{z},k(t))=\sum_{l\in\mathbf{N},m\in\mathbf{N}} a^k_{l,m}(t)(\tilde{z}-\tilde{s}_h)^l\cdot y^m,$$

where the coefficients $a^k_{l,m}(t)$ are analytic functions in t that are real when t is real. The estimates on $|a^k_{l,m}-a^0_{l,m}|$ and $|\partial_t a^k_{l,m}|$ are immediate consequences of the Cauchy estimates applied to this expansion and the estimates (2.10) and (2.13).

By (2.17), if $1/S(\tilde{z}-\tilde{s}_h)\in D_{\mathbf{C}}(0,b)$, then

$$S(\tilde{z}-\tilde{s}_h)\cdot v\left(\frac{1}{S(\tilde{z}-\tilde{s}_h)},\tilde{z},k(t)\right)=f(h)\cdot vp(z,k(t)).$$

This gives the convergent expansion for $vp(z,k(t))$ in the case $n=1$ or 2.

Assume $n\geq 3$ and let $H_k(\tilde{z},t)$ and $G(\tilde{z},t)$ denote the same operators as above. For $y\in\mathbf{C}$ and $\tilde{z}\in D_{\mathbf{C}}(\tilde{s}_h,r_q)$ consider the operator

$$O_k(y,\tilde{z})=y\cdot H_k(\tilde{z},t)+G(\tilde{z},t).$$

For the same reasons as above there exists $b>0$ such that for $(y,\tilde{z},t)\in D_{\mathbf{C}}(0,b)\times D_{\mathbf{C}}(\tilde{s}_h,b)\times D_{\mathbf{C}}(0,a(h)/4)$ there exists $v(y,\tilde{z},k(t))$, a simple eigenvalue of $O_k(y,\tilde{z})$ isolated from the rest of the spectrum. Moreover,

$$v(S(\tilde{z}-\tilde{s}_h),\tilde{z},k(t))=f(h)\cdot vp(z,k(t)).$$

Now the conclusion follows along the same lines as in the case $n=1$ or 2.

III. Computation of the resonances

By (1.8) and Corollary 1.4, to say that z is a pole of $R^{\pm}(z,t)$ is equivalent to say that 1 is an eigenvalue of $b(t)\Gamma^{\pm}(z,t)$.

1. Investigations away from the edges of the band

(a) *Far away from the internal critical values.* Let r_0 be small enough and h_0 be chosen as in Lemma 2.2. Then there exists a constant $C(r_0)>0$ such that, for $z\in\mathcal{UC}({}^c\square(r_0),\square^{\pm}(r_0))$, one has

$$|I(z)|\leq\frac{C(r_0)}{f(h)}.$$

By Proposition A.3 we know that there exists $c(r_0)>0$ such that

$$z\in\mathcal{UC}({}^c\square(r_0),\square^{\pm}(r_0))\cap\mathbf{R}$$

when $|\operatorname{Im}(I(z))|\geq c(r_0)/f(h)$. So, by Lemma 2.2 and by the estimate on $\|k\|_{\infty,W_a}$ given in Theorem 1.1 one has for $z\in\mathcal{UC}({}^c\square(r_0),\square^{\pm}(r_0))\cap\mathbf{R}$ and h small enough

$$\sigma(\Gamma_k(z,t))\subset D\big(0,e^{-1/Ch}\big)\cup\left\{z\in\mathbf{C};|\operatorname{Im}(z)|\geq\frac{c(r_0)}{f(h)}\right\},\tag{3.1}$$

for a certain $C>0$. But, for $t\in[-a(h)/4,a(h)/4]$, we know that $b(t)\in\mathbf{R}$ and $|b(t)|<2a(h)$. Then, by (3.1), for h small enough, 1 can not be an eigenvalue for $b(t)\cdot\Gamma_k(z,t)$, so z can not be a resonance of P_t.

(b) *Close to the internal critical values for $n\geq3$.* Except for the fact that one uses Lemma 2.3 instead of Lemma 2.2 the proof is the same as above.

Remark. In dimension 1, the only critical values are the extrema. In dimension 2, the problem near the internal critical values (i.e. that are no extrema) comes from the fact that if there are 2 critical values of ω_h that are asymptotically equal when h goes to 0, then in the expansions given for $I(z)$ near these values there may occur compensations for $\operatorname{Im}(I(z))$ (i.e. this imaginary part may become 0). Consequently, the largest eigenvalue of $b(t)\cdot\Gamma_k(z,t)$ may be real and equal to 1 (see the remark following Lemma 2.4).

2. Computation of the resonances near the edges of the band

We will only study what happens in a neighborhood of s_h, the maximum of ω_h.

(a) *Proof of Theorem* 1.6. Let $n=1$. Using Lemma 2.4 and the expansion given in Theorem 1.1 for $b(t)$, we see that for $t\in[-r_0\cdot f(h),r_0\cdot f(h)]$ (for a certain $r_0>0$ given by Lemma 2.4), the only eigenvalues of $b(t)\cdot\Gamma_k(z,t)$ that may be equal to 1 are the ones contained in $D_{\mathbf{C}}(b(t)\cdot I(z), |b(t)\cdot I(z)|/4)$. So by Proposition 2.5 we just have to solve

$$b(t)\cdot vp(z,t)=1. \tag{3.2}$$

Let us first make a change of variables. Let $\tilde{t}=t/f(h)$. Then

$$\frac{b(t)}{f(h)}=\varrho\cdot\tilde{t}\cdot(1+f(h)\cdot(\tilde{t}\cdot q(\tilde{t}f(h))))=\varrho\cdot\tilde{t}\cdot(1+\tilde{q}(\tilde{t})),$$

where $\tilde{q}$ is analytic in $D_{\mathbf{C}}(0,a(h)/f(h))$ and satisfies $|\tilde{q}|\le C\cdot f(h)$.

By Proposition 2.5 and the remark following it, we know that there exists $r_0>0$ such that for z in $\mathcal{UC}(r_0,s_h)$

$$f(h)\cdot vp(z,k)=S(\tilde{z}-\tilde{s}_h)\cdot\sum_{l\in\mathbf{N},m\in\mathbf{N}} a^k_{l,m}(t)(\tilde{z}-\tilde{s}_h)^l\cdot(S(\tilde{z}-\tilde{s}_h))^{-m}.$$

Therefore (3.2) becomes

$$\varrho\cdot\tilde{t}\cdot(1+\tilde{q}(\tilde{t}))\cdot S(\tilde{z}-\tilde{s}_h)\cdot\sum_{l\in\mathbf{N},m\in\mathbf{N}} a^k_{l,m}(f(h)\cdot\tilde{t})(\tilde{z}-\tilde{s}_h)^l\cdot(S(\tilde{z}-\tilde{s}_h))^{-m}=1. \tag{3.3}$$

For z in $\mathcal{UC}(r_0,s_h)$ set

$$\tilde{z}-\tilde{s}_h=\tilde{u}^2, \tag{3.4}$$

where $\tilde{u}\in D_{\mathbf{C}}(0,r_0)$.

Doing this, we uniformize the function S on $\mathcal{UC}(r_0,s_h)$. Plug (3.4) into (3.3) and use the definition of S to get

$$\frac{\pi\cdot\varrho}{2}\cdot\left(\frac{\tilde{t}}{\tilde{u}}\right)\cdot(1+\tilde{q}(\tilde{t}))\cdot\sum_{l,m} a^k_{l,m}(\tilde{t}\cdot f(h))\cdot\left(\frac{\pi}{2}\right)^m\cdot\tilde{u}^{2l+m}=1. \tag{3.5}$$

For $\tilde{t}\neq 0$ and $\tilde{u}\neq 0$, (3.5) becomes

$$g(\tilde{t},\tilde{u})=0, \tag{3.6}$$

where

$$g(\tilde{t},\tilde{u})=\frac{\pi\cdot\varrho}{2}\cdot\tilde{t}\cdot(1+\tilde{q}(\tilde{t}))\cdot\sum_{l,m} a^k_{l,m}(\tilde{t}\cdot f(h))\cdot\left(\frac{\pi}{2}\right)^m\cdot\tilde{u}^{2l+m}-\tilde{u}. \tag{3.7}$$

It is clear that g is analytic in a neighborhood of $(0,0)$, and that

$$g(0,0)=0 \quad \text{and} \quad \partial_{\tilde{u}} g(0,0)=1. \tag{3.8}$$

We can apply the implicit function theorem to equation (3.6) in a neighborhood of $(0,0)$ in $\mathbf{C}^2$. So there exist $\tilde{t}_0>0$, $r_0>0$ and an analytic function $\tilde{u}: D_{\mathbf{C}}(0,\tilde{t}_0)\to D_{\mathbf{C}}(0,r_0)$ such that, for $\tilde{t}\in D_{\mathbf{C}}(0,\tilde{t}_0)$,

$$g(\tilde{t},\tilde{u}(\tilde{t}))=0.$$

Moreover, we compute

$$\tilde{u}(\tilde{t})=\frac{\pi\cdot\varrho\cdot a_{0,0}^k(0)}{2}\cdot\tilde{t}\cdot(1+\tilde{v}(\tilde{t})), \tag{3.9}$$

where $\tilde{v}$ is function analytic in $D_{\mathbf{C}}(0,\tilde{t}_0)$.

As the coefficients $a_{l,m}^k(t)$ are real for real t, $g(\tilde{t},\tilde{u})$ is real when $\tilde{t}$ and $\tilde{u}$ are real. Hence the coefficients of the power series expansions of $\tilde{u}$ and $\tilde{v}$ are real. The estimate of the leading coefficient of $\tilde{u}(\tilde{t})$ comes from (3.6), (A.16) and Proposition 2.5.

Plugging (3.9) into (3.4) when $\tilde{t}\neq 0$, we get the announced result. Of course, one can do a symmetric study in $\mathcal{UC}(r_0,i_h)$. $\square$

Proof of Theorem 1.7. Let $n=2$. By the same arguments as in the proof of Theorem 1.6, it is clear that we only have to solve equation (3.2) for $vp(z,t)$, the eigenvalue of $\Gamma_k(z,t)$ given by Proposition 2.5.

Let $q\in\mathbf{N}$. Using the expansion of $vp(z,t)$ given by Proposition 2.5, for $z\in\mathcal{UC}(q,r_q,s_h)$, we have to solve equation (3.3). Again we will uniformize S on $\mathcal{UC}(r_0,s_h)$, the change of variable now being

$$\tilde{z}-\tilde{s}_h=\exp(\tilde{u}), \tag{3.10}$$

where $\tilde{u}\in(-\infty,-R_0)+i\mathbf{R}$ for some $R_0>0$. For some $R_q>0$, exp is an analytic embedding from $(-\infty,-R_q)+i(-q\cdot\pi+\pi/2,q\cdot\pi+\pi/2)$ into $\mathcal{UC}(q,r_q,s_h)$.

Plug (3.10) into (3.3) and use the definition of S to get

$$-\frac{\varrho}{2}\cdot\tilde{t}\cdot\tilde{u}\cdot(1+\tilde{q}(\tilde{t}))\cdot\sum_{l,m} a_{l,m}^k(\tilde{t}\cdot f(h))\cdot\left(-\frac{1}{2}\right)^m\cdot\tilde{u}^{-m}\cdot\exp(l\cdot\tilde{u})=1. \tag{3.11}$$

Let $\tilde{v}=1/\tilde{u}$. Notice that, for $\tilde{u}\in(-\infty,-R_q)+i(-q\cdot\pi+\pi/2,q\cdot\pi+\pi/2)$, one has $\operatorname{Re}\tilde{v}<0$. For $\tilde{t}\neq 0$ and $\tilde{v}\neq 0$, (3.11) becomes

$$g(\tilde{t},\tilde{v})=0, \tag{3.12}$$

where

$$g(\tilde{t},\tilde{v}) = -\frac{\varrho}{2}\cdot\tilde{t}\cdot(1+\tilde{q}(\tilde{t}))\cdot\sum_{l,m} a^k_{l,m}(\tilde{t}\cdot f(h))\cdot\left(-\frac{1}{2}\right)^m\cdot\tilde{v}^m\cdot\exp\left(\frac{l}{\tilde{v}}\right)-\tilde{v}. \tag{3.13}$$

Obviously g can be defined as a function with continuous derivatives in some neighborhood of $(0,0)$ in $\mathbf{C}\times\{z\in\mathbf{C};\operatorname{Re}(z)<0\}$ and then it satisfies (3.8). So we can apply the implicit function theorem to get a unique solution to equation (3.12) which, moreover, is of the form

$$\tilde{v}(\tilde{t}) = -\frac{\varrho\cdot a^k_{0,0}(0)}{2}\cdot\tilde{t}\cdot(1+\tilde{w}(\tilde{t})). \tag{3.14}$$

The condition $\operatorname{Re}(\tilde{v}(\tilde{t}))<0$ tells us that there is no solution of (3.12) when $\tilde{t}<0$. To get some precision on $\tilde{w}$, we look for solutions to (3.12) of the form

$$\tilde{v}(\tilde{t}) = -\frac{\varrho\cdot a^k_{0,0}(0)}{2}\cdot\tilde{t}\cdot(1+\tilde{t}\cdot\widetilde{w}(\tilde{t}))^{-1},$$

where $\widetilde{w}$ is a function such that $\widetilde{w}(0)=C$, a constant to be chosen later on. Plug this ansatz into (3.13) to get

$$\begin{aligned} &g\left(\tilde{t},-\frac{\varrho\cdot a^k_{0,0}(0)}{2}\cdot\tilde{t}\cdot(1+\tilde{t}\cdot\widetilde{w})^{-1}\right)\\ &\quad= -\frac{\varrho}{2}\cdot\tilde{t}\cdot(1+\tilde{q}(\tilde{t}))\cdot\sum_{l,m} a^k_{l,m}(\tilde{t}\cdot f(h))\cdot\left(\frac{\varrho\cdot a^k_{0,0}(0)\cdot\tilde{t}}{4\cdot(1+\tilde{t}\cdot\widetilde{w})}\right)^m\\ &\quad\quad\times\exp\left(\frac{-2\cdot l\cdot(1+\tilde{t}\cdot\widetilde{w})}{\varrho\cdot a^k_{0,0}(0)\cdot\tilde{t}}\right)+\frac{\varrho\cdot a^k_{0,0}(0)\cdot\tilde{t}}{2\cdot(1+\tilde{t}\cdot\widetilde{w})}. \end{aligned} \tag{3.15}$$

So, for $\tilde{t}\neq 0$, (3.12) becomes

$$h\left(\tilde{t},\frac{1}{\tilde{t}}\cdot\exp\left(-\frac{2}{\varrho\cdot a^k_{0,0}(0)\cdot\tilde{t}}\right),\widetilde{w}\right)=0, \tag{3.16}$$

where

$$\begin{aligned} h(\tilde{t},\tilde{x},\widetilde{w}) = \tilde{t}\cdot f(\tilde{t},\tilde{x},\widetilde{w}) &= -\frac{a^k_{0,0}(0)}{1+\tilde{t}\cdot\widetilde{w}}+(1+\tilde{q}(\tilde{t}))\cdot\sum_{l,m} a^k_{l,m}(\tilde{t}\cdot f(h))\\ &\quad\times\left(\frac{\varrho\cdot a^k_{0,0}(0)\cdot\tilde{t}}{4\cdot(1+\tilde{t}\cdot\ w)}\right)^m\cdot\tilde{t}^l\cdot\tilde{x}^l\cdot\exp\left(\frac{-2\cdot l\cdot\widetilde{w}}{\varrho\cdot a^k_{0,0}(0)}\right). \end{aligned} \tag{3.17}$$

The function f is analytic in some neighborhood of $(0,0,0)$ in $\mathbf{C}^3$ and satisfies

$$
\begin{aligned}
f(0,0,0) &= \partial_{\tilde{t}} h(0,0,0) \\
&= C\cdot a^k_{0,0}(0) - \frac{\varrho\cdot a^k_{0,1}(0)\cdot a^k_{0,0}(0)}{4} + a^k_{0,0}(0)\cdot\partial_{\tilde{t}}\tilde{q}(0) + f(h)\cdot\partial_t a^k_{0,0}(0)
\end{aligned}
\tag{3.18}
$$

and

$$\partial_{\widetilde{w}} f(0,0,0) = -a^k_{0,0}(0)\cdot\left(1-\frac{\varrho\cdot a^k_{0,1}(0)}{4}\right).$$

Choose C such that $f(0,0,0)=0$. Using (A.16) and the estimates on $a^k_{l,m}$ given in Proposition 2.5, we can apply the implicit function theorem uniformly for h small enough to construct $N\subset\mathbf{C}^2$, a neighborhood of $(0,0)$, and an analytic function $\widetilde{w}: N\to\mathbf{C}$ such that $f(\tilde{t},\tilde{x},\widetilde{w}(\tilde{t},\tilde{x}))=0$. Then by (3.16), we get that $\tilde{v}$, the solution of (3.12), satisfies

$$\tilde{v}(\tilde{t}) = -\frac{\varrho\cdot a^k_{0,0}(0)}{2}\cdot\tilde{t}\cdot\left(1+\widetilde{w}\left(\tilde{t},\frac{1}{\tilde{t}}\cdot\exp\left(-\frac{2}{\varrho\cdot a^k_{0,0}(0)\cdot\tilde{t}}\right)\right)\right).$$

Using the properties of the coefficient $a^k_{l,m}(0)$ one completes the proof of Theorem 1.7. □

Proof of Theorem 1.8. Let $n=3$. Let $\tilde{t}_0>0$ be fixed and arbitrarily large. We know that $I(z)/f(h)$ is bounded in $\mathcal{UC}(r_0,s_h)$ for a certain $r_0>0$. By Lemma 2.3 and using the fact that $\|k\|_{\infty,W_a}\le e^{-c/h}$ (for a certain $c>0$), it is clear that, for h small enough (depending on $\tilde{t}_0$) and $t\in[-\tilde{t}_0\cdot f(h),\tilde{t}_0\cdot f(h)]$, the only eigenvalues of $b(t)\cdot\Gamma_k(z,t)$ that may be equal to 1 are the ones contained in $D_{\mathbf{C}}(b(t)\cdot I(z),|b(t)\cdot I(z)|/4)$.

We only have to solve equation (3.2) for $vp(z,t)$, the eigenvalue of $\Gamma_k(z,t)$ given by Proposition 2.5. We recall that by Proposition 2.5 there exists $r_0>0$ such that, for z in $\mathcal{UC}(r_0,s_h)$,

$$f(h)\cdot vp(z,k) = \sum_{l\in\mathbf{N},m\in\mathbf{N}} a^k_{l,m}(t)(\tilde{z}-\tilde{s}_h)^l\cdot(S(\tilde{z}-\tilde{s}_h))^m,$$

where the properties of the coefficients $a^k_{l,m}$ are given in Proposition 2.5.

Letting $\tilde{z}\to\tilde{s}_h$, we see that we need to take $t>0$ for (3.2) to admit a solution in $\mathcal{UC}(r_0,s_h)$ for $r_0>0$ small enough. (3.2) can be rewritten

$$\varrho\cdot\tilde{t}\cdot(1+\tilde{q}(\tilde{t}))\cdot\sum_{l\in\mathbf{N},m\in\mathbf{N}} a^k_{l,m}(t)(\tilde{z}-\tilde{s}_h)^l\cdot\left(-\frac{\pi}{2}\cdot(\tilde{z}-\tilde{s}_h)^{1/2}\right)^m = 1. \tag{3.19}$$

For $t\in[-a(h)/4, a(h)/4]$ let $g(\tilde{t})=\tilde{t}\cdot(1+q(\tilde{t}))\cdot a_{0,0}^k(\tilde{t})$. Then $|\partial_{\tilde{t}}g(\tilde{t})|>c$ for a certain $c>0$ and h small enough. So the equation $g(\tilde{t})=1$ admits a unique solution $\widetilde{T}_{\delta V}$ in $[-a(h)/4, a(h)/4]$. One has

$$\widetilde{T}_{\delta V}=\frac{\varrho}{\tilde{I}(\tilde{s}_h)}\cdot(1+\mathcal{O}(e^{-c/h})).$$

(see [Kl] for more details). We define $T_{\delta V}=f(h)\cdot\widetilde{T}_{\delta V}$.

We will solve (3.19) for $\tilde{t}$ close to $\widetilde{T}_{\delta V}$ and z in $\mathcal{UC}(r_0, s_h)$. To uniformize S we make the following ansatz for $z\in\mathcal{UC}(r_0, s_h)$

$$\tilde{z}-\tilde{s}_h=\tilde{u}^2.$$

Plugging this into (3.19) we are to solve the equation

$$g(\tilde{t}, u(\tilde{t}))=1, \tag{3.20}$$

where

$$g(\tilde{t}, u)=\varrho\cdot\tilde{t}\cdot(1+\tilde{q}(\tilde{t}))\cdot\sum_{l\in\mathbf{N}, m\in\mathbf{N}} a_{l,m}^k(f(h)\cdot\tilde{t})\cdot\left(-\frac{\pi}{2}\right)^m\cdot\tilde{u}^{m+2l}. \tag{3.21}$$

We notice that g is analytic in $(\tilde{t}, u)$ in a neighborhood of $(\widetilde{T}_{\delta V}, 0)$ and satisfies

$$\begin{aligned} g(\widetilde{T}_{\delta V}, 0)&=1,\\ \partial_{\tilde{u}}g(\widetilde{T}_{\delta V}, 0)&=-\frac{\pi}{2}\cdot(1+q(\widetilde{T}_{\delta V}))\cdot\varrho\cdot\widetilde{T}_{\delta V}\cdot a_{0,1}^k(T_{\delta V}).\end{aligned}$$

So by the estimates on the coefficients $a_{l,m}^k$ given in Proposition 2.5, we can use the implicit function theorem, uniformly for h small enough, to get $\tilde{t}_0>0$ and $\tilde{u}$, a function analytic in $D_{\mathbf{C}}(\widetilde{T}_{\delta V}, \tilde{t}_0)$, such that $g(\tilde{t}, \tilde{u}(\tilde{t}))=1$ for $\tilde{t}\in D_{\mathbf{C}}(\widetilde{T}_{\delta V}, \tilde{t}_0)$. Moreover, one gets

$$\tilde{u}(\tilde{t})=\frac{2\varrho\cdot(1+q(\widetilde{T}_{\delta V}))\cdot a_{0,0}^k(T_{\delta V})}{\pi\cdot a_{0,1}^k(T_{\delta V})}\cdot\tilde{t}\cdot(1+\tilde{v}(\tilde{t})),$$

where $\tilde{v}$ is analytic in $D_{\mathbf{C}}(\widetilde{T}_{\delta V}, \tilde{t}_0)$ and $\tilde{v}(\widetilde{T}_{\delta V})=0$.

One ends the proof of this theorem in the same way as the proof of Theorem 1.6. □

Proof of Lemma 1.9 *and Theorem* 1.10. Let $n=4$. In this case $S(z)=\frac{1}{2}\cdot z\cdot\log z$. Fix $q\in\mathbf{N}$ and $-q\le j\le q$. It is immediate to see that, for $r_q>0$ small enough, $-z\cdot\log z$

maps $e^{2ij\pi}\cdot(D(0,r_q)\backslash i\cdot(-r_q,0])$ to an open set $\mathcal{D}_j$ such that the following holds for some r'_q and $r''_q>0$ (depending on r_q):

$$e^{2ij\pi}\cdot(-r''_q,r''_q)\subset\mathcal{D}_j\subset e^{2ij\pi}\cdot\big(D(0,r'_q)\backslash i\cdot(-r'_q,0]\big).$$

We now make the following ansatz

$$z(u)=-\frac{u}{\log u}\cdot(1+g(u)), \tag{3.22}$$

where g will be a function defined in some neighborhood of 0 such that $g(0)=0$. We try to find g such that for $u\in\mathcal{D}_j$

$$-z(u)\cdot\log(z(u))=u. \tag{3.23}$$

Plugging (3.22) into (3.23), we get

$$u\cdot(1+g(u))\cdot\left(1-\frac{\log(-\log(u))}{\log u}+\frac{1}{\log u}\cdot\log(1+g(u))\right)=u.$$

Hence, for $u\neq0$,

$$f(g(u),v(u),w(u))=(1+g(u))\cdot(1-w(u)+v(u)\cdot\log(1+g(u)))=1, \tag{3.24}$$

where $f(\alpha,v,w)=(1+\alpha)\cdot(1-w+v\cdot\log(1+\alpha))$, $w(u)=\log(-\log(u))/\log u$, and $v(u)=1/\log u$.

The function f is analytic in some neighborhood of $(0,0,0)$ in $\mathbf{C}^3$. Moreover, one computes $f(0,0,0)=1$ and $\partial_\alpha f(0,0,0)=1$. Hence we can apply the implicit function theorem to find N, a neighborhood of $(0,0)$ in $\mathbf{C}^2$ and $\alpha(w,v)\colon N\to\mathbf{C}$, an analytic function such that $f(\alpha(w,v),w,v)=1$ for $(w,v)\in N$.

Now, noticing that

$$\frac{\log(-\log(u))}{\log u}\to0\quad\text{and}\quad\frac{1}{\log u}\to0,$$

when $u\to0$ in $\mathcal{D}_j$ we obtain that, for some $r_q>0$ small enough, for $z\in\mathcal{D}_j$, then (3.20) is satisfied if

$$h(z)=-\frac{z}{\log z}\cdot\left(1+\alpha\left(\frac{\log(\log(z))}{\log z},\frac{1}{\log z}\right)\right). \tag{3.25}$$

This finishes the proof of Lemma 1.9.

To prove Theorem 1.10, one just uses Lemma 1.9 and the technique used in the proof of Theorem 1.8, restricting the study to $\square_p(r_q, s_h)$. $\square$

Proof of Theorem 1.11. Let $n\geq 5$. For the same reasons as in the case $n=3$ or 4, we only have to solve equation (3.2) for $vp(z,t)$, the eigenvalue of $\Gamma_k(z,t)$ given by Proposition 2.5.

For $n\geq 5$ the situation is different from the ones previously discussed because the leading order term in the expansion of $vp(z,t)$ is not S any more, it is $z-s_h$. For $q\in\mathbf{N}$, we are going to solve equation (3.2) in $\mathcal{O}C(r_q, s_h, q)$ (if $q>2$ and n is odd, since the singularity of S is of square root type, this is equivalent to solving (3.2) on $\mathcal{U}C(r_q, s_h)$).

As in the proof of Theorem 1.8 one computes the threshold $T_{\delta V}=\varrho/I(s_h)\cdot(1+\mathcal{O}(e^{-c/h}))$.

Fix $q\in\mathbf{N}$ and $-q\leq j\leq q$. For $z\in\square_j(r_q, s_h)$, we make the ansatz

$$\tilde{z}-\tilde{s}_h=e^{2ij\pi}\cdot C\cdot\tilde{t}\cdot(1+u(\tilde{t})), \tag{3.26}$$

with $u(0)=0$ and C a constant to be chosen later on.
Using the definition of S, we get that:
(3.27)

$$\begin{aligned} S(\tilde{z}-\tilde{s}_h) &= \begin{cases} \frac{1}{2}\pi\cdot(-1)^{(n-1)/2}\cdot(C\tilde{t}\cdot(1+u(\tilde{t})))^{(n-2)/2} & \text{if } n \text{ is odd,}\\ \frac{1}{2}\cdot(-1)^{n/2}\cdot(C\tilde{t})^{(n-2)/2}\cdot\big(\log\tilde{t}+\log(C(1+u(\tilde{t})))\big) & \text{if } n \text{ is even,}\end{cases}\\ &= C^{(n-2)/2}\cdot\tilde{t}\cdot f_n(\tilde{t}, g_n(\tilde{t}), u(\tilde{t})),\end{aligned}$$

where

$$f_n(t,g,u)=\begin{cases} g\cdot(1+u)^{(n-2)/2} \\ \quad \text{if } n \text{ is odd,}\\ g\cdot(1+u)^{(n-2)/2}+\frac{1}{2}\cdot(-1)^{n/2}\cdot(C\cdot t)^{(n-2)/2}\cdot\log(C(1+u)) \\ \quad \text{if } n \text{ is even,}\end{cases}$$

and g_n is defined in Section 1. Notice that, because $n\geq 5$, $g_n(z)\to 0$ when $z\to 0$ and $f_n(0,0,0)=0$.

Plugging this ansatz into equation (3.2) we get
(3.28)

$$\begin{aligned}&\varrho\cdot(\tilde{t}+\widetilde{T}_{\delta V})\cdot(1+\tilde{q}(\tilde{t}+\widetilde{T}_{\delta V}))\\ &\quad\times\sum_{l\in\mathbf{N}, m\in\mathbf{N}} a^n_{l,m}(t)(\tilde{t})^{l+m}\cdot C^{l+(n-2)/2m}\cdot(1+u(\tilde{t}))^l\cdot f_n(\tilde{t}, g_n(\tilde{t}), u(\tilde{t}))^m=1.\end{aligned}$$

By the definition of $T_{\delta V}$ we get for $\tilde{t}\neq 0$

$$(\tilde{t}+\widetilde{T}_{\delta V})\cdot\sum_{l+m\geq 1} a^n_{l,m}(t)(\tilde{t})^{l+m-1}\cdot C^{l+m(n-2)/2}\cdot(1+u(\tilde{t}))^l\cdot\big(f_n(\tilde{t},g_n(\tilde{t}),u(\tilde{t}))\big)^m + a^n_{0,0}(t)+T_{\delta V}\cdot\frac{a^n_{0,0}(t)-a^n_{0,0}(T_{\delta V})}{t-T_{\delta V}}=0. \tag{3.29}$$

So to be able to solve this equation for $\tilde{t}=0$, we must choose the constant C such that

$$\widetilde{T}_{\delta V}\cdot a^n_{1,0}(T_{\delta V})\cdot C+a^n_{0,0}(T_{\delta V})+T_{\delta V}\cdot\partial_t a^n_{0,0}(T_{\delta V})=0,$$

that is

$$C=-\frac{\varrho\cdot I^2}{\partial I}\cdot(1+O(e^{-c/h}))>0.$$

Using the implicit function theorem we find a function $u(\tilde{t},g)$, analytic in the neighborhood of $(0,0)$, such that in this neighborhood

$$\begin{aligned} F(\tilde{t},g,u(\tilde{t},g)) &= a^n_{0,0}(t)+T_{\delta V}\cdot\frac{a^n_{0,0}(t)-a^n_{0,0}(T_{\delta V})}{t-T_{\delta V}}+(\tilde{t}+\widetilde{T}_{\delta V})\\ &\quad\times\sum_{l+m\geq 1} a^n_{l,m}(t)(\tilde{t})^{l+m-1}\cdot C^{l+m(n-2)/2}\\ &\quad\times(1+u(\tilde{t},g))^l\cdot(f_n(\tilde{t},g,u(\tilde{t},g)))^m=0\end{aligned} \tag{3.30}$$

and $u(0,0)=0$ (one checks that $\partial_u F(0,0,0)=\widetilde{T}_{\delta V}\cdot C^{(n-2)/2}\cdot a^n_{1,0}(T_{\delta V})$).

Then the solution we are looking for is

$$\tilde{z}-\tilde{s}_h=C\cdot\tilde{t}\cdot(1+u(\tilde{t},g_n(\tilde{t}))).$$

Using equation (3.30) one computes

$$\begin{aligned}\partial_g u(0,0) &= -\frac{\partial_g F(0,0,0)}{\partial_u F(0,0,0)}=-C^{(n-4)/2}\cdot\frac{a^n_{0,1}(T_{\delta V})}{a^n_{1,0}(T_{\delta V})}\\ &=\frac{2^n\pi^{n/2}}{\Gamma(n/2)\cdot\mathrm{Vol}(\mathbf{T})\cdot f(h)^2}\cdot\frac{(\varrho I^2)^{(n-2)/2}}{(-\partial I)^{n/2}}\cdot D_s\cdot(1+O(e^{-c/h})).\end{aligned}$$

This completes the proof of Theorem 1.10. □

3. Embedded eigenvalues: Proof of Theorem 1.12

Let $n\in\mathbf{N}$ and $n\neq 2$. Pick t as in Theorem 1.12. Suppose that λ is an eigenvalue of P_t in (i_h,s_h). Then, by Theorem 1.1, there exists $u\in L^2(\mathbf{T})$ such that, for $\theta\in\mathbf{T}$,

$$(\lambda-\omega_h(\theta))\cdot u(\theta)=b(t)\cdot((\Pi_0+K(t))u)(\theta)=v(\theta). \tag{3.31}$$

Moreover we know that v and ω_h are analytic in W, some complex neighborhood of $\mathbf{T}$. So equation (3.31) shows that one can define u as an analytic function in $W\backslash\omega_h^{-1}(\lambda)$. If we show that u can be defined as an analytic function in some complex neighborhood of $\mathbf{T}$, we will be done. Indeed, if this is true, then for $\mathrm{Im}(z)\neq 0$

$$b(t)\cdot\Gamma(z,t)(v_\lambda)=(z-\lambda)\cdot b(t)\cdot\Gamma(z,t)(u)+v_\lambda, \tag{3.32}$$

where $v_\lambda=(\lambda-\omega_h)\cdot u$. When $z\to\lambda\pm i0$, using the asymptotic behaviour of $\Gamma(z,t)$ given in Theorem 1.3, we obtain

$$b(t)\cdot\Gamma^{\pm}(\lambda,t)(v_\lambda)=v_\lambda.$$

So λ will be a resonance of P_t (according to our definition), and we know that this is not possible.

Let $\theta^0\in\mathbf{T}$ be such that $\omega_h(\theta^0)=\lambda$ and θ^0 is not a critical point of ω_h. Then there exist N_0, a neighborhood of 0 in $\mathbf{C}^n$, N_θ, a neighborhood of θ^0 in $\mathbf{C}^n$, and $D\colon N_0\to N_\theta$, an analytic bijection such that:

(a) $D(N_0\cap\mathbf{R})=N_\theta\cap\mathbf{R}$,

(b) $\forall\theta=(\theta_1,\dots,\theta_n)\in N_0$, $\omega_h(D(\theta))=\lambda-\theta_1$.

Then, by (3.31), for $\theta\in N_0\cap\mathbf{R}$ one gets

$$\theta_1\cdot u\circ D(\theta)=v\circ D(\theta). \tag{3.33}$$

As $v\circ D$ is analytic and $u\circ D$ is in L^2_{loc}, we see that, if $\theta\in N_\theta\cap\mathbf{R}$ and $\theta_1=0$, then $v\circ D(\theta)=0$. So, in some complex neighborhood of 0 one can factorize

$$v\circ D(\theta)=\theta_1\cdot\tilde{v}(\theta),$$

where $\tilde{v}$ is an analytic function.

Using (3.32) one can continue u analytically in some complex neighborhood of θ_0 by $u(\theta)=\tilde{v}\circ D^{-1}(\theta)$. So, if λ is an eigenvalue of P_t embedded in (i_h,s_h) and λ is not a critical value of ω_h, then any eigenfunction associated to λ can be continued analytically in some neighborhood of $\mathbf{T}$. Hence λ will be a resonance which contradicts the results already proven.

Let λ be a critical value of ω_h, non extremal if $n\geq 3$, and $\theta^0\in\mathbf{T}$ be one of the isolated critical points of ω_h associated to λ. Then there exist $p\notin\{1,n\}$, N_0, a neighborhood of 0 in $\mathbf{C}^n$, N_θ, a neighborhood of θ^0 in $\mathbf{C}^n$, and $D\colon N_0\to N_\theta$, an analytic bijection such that:

(a) $D(N_0\cap\mathbf{R})=N_\theta\cap\mathbf{R}$,

(b) $\forall\theta=(\theta_1,\dots,\theta_n)\in N_0$, $\omega_h(D(\theta))=\lambda-(\sum_{1\leq l\leq p}\theta_l^2-\sum_{p+1\leq l\leq n}\theta_l^2)$.

Then (3.31) becomes

$$\left(\sum_{1\le l\le p}\theta_l^2-\sum_{p+1\le l\le n}\theta_l^2\right)\cdot(u\circ D(\theta))=v\circ D(\theta). \tag{3.34}$$

Write, for $\theta=(\theta_1,\theta')$,

$$\sum_{1\le l\le p}\theta_l^2-\sum_{p+1\le l\le n}\theta_l^2=\theta_1^2+b(\theta')=w(\theta). \tag{3.35}$$

We know that, if θ is a real regular point of w, then $w(\theta)=0$ implies $v(\theta)=0$. Since there exists a sequence of real regular points of w converging to 0, we know that, for $\theta\in N_0\cap\mathbf{R}^n$, $w(\theta)=0$ implies $v(\theta)=0$.

By Weierstrass' preparation theorem we can write

$$v\circ D(\theta)=w(\theta)\cdot g(\theta)+\theta_1\cdot a_1(\theta')+a_0(\theta'), \tag{3.36}$$

where a_0 and a_1 are analytic functions in some neighborhood of 0 in $\mathbf{C}^{n-1}$. So, for $\theta=(\theta_1,\theta')\in N_0\cap\mathbf{R}^n$ such that $w(\theta)=0$,

$$\theta_1\cdot a_1(\theta')+a_0(\theta')=0. \tag{3.37}$$

We notice that $w(\theta_1,\theta')=0$ implies $w(-\theta_1,\theta')=0$. Hence, by (3.35), $a_1(\theta')=a_0(\theta')=0$ if $\theta_1\neq 0$.

Pick $\tilde\theta\in N_0\cap\mathbf{R}^n$ such that $\tilde\theta_1\neq 0$ and $w(\tilde\theta)=0$. Then $b(\tilde\theta')<0$. So for θ' real close to $\tilde\theta'$ there exists $0\neq\theta_1$ real such that $w(\theta_1,\theta')=w(-\theta_1,\theta')=0$. So $a_1(\theta')=a_0(\theta')=0$ for θ' in some real neighborhood of $\tilde\theta'$. As a_0 and a_1 are analytic, they are equal to 0.

One then concludes, by (3.34), that

$$v\circ D(\theta)=w(\theta)\cdot g(\theta),$$

which, in turn, says that u can be defined in some complex neighborhood of θ_0 by the following analytic function

$$u(\theta)=g\circ D^{-1}(\theta).$$

So, for λ an eigenvalue of P_t embedded in (i_h,s_h) which is the same as a critical value of ω_h, we get the same contradiction as in the case when λ is not a critical value of ω_h.

If $n=1$ and λ is a critical point of ω_h (e.g. the minimum), then equation (3.34) becomes $\theta^2\cdot(u\circ D(\theta))=v\circ D(\theta)$ for θ in some real neighborhood of 0. As $u\in L^2(\mathbf{T})$, we know that, for some function g analytic in a neighborhood of 0 in $\mathbf{C}$,

$$v\circ D(\theta)=\theta^2\cdot g(\theta),$$

so u can be continued as an analytic function of θ in a complex neighborhood of the critical points of ω_h.

This completes the proof of Theorem 1.12. $\square$

IV. Appendix

1. Analytic continuation of some integrals

Analogues of the integrals we will study in this chapter have already been studied by several specialists in algebraic geometry (see [La], [FFLP], [P]). Here we construct hand-made proofs to get h-uniform results.

Let $c>0$ and $u(x,\theta)$ be a function analytic in θ in $W_c=\mathbf{T}+iB(0,c)$ uniformly for $x\in X$. We define, for $0<c'<c$,

$$\|u(x)\|_{\infty,c'}=\sup_{\theta\in W_{c'}}|u(x,\theta)|.$$

We study $\tilde{I}(z,u)=\int_{\mathbf{T}}u(x,\theta)/(\tilde{z}-\widetilde{\omega}_h(\theta))\,d\theta$ in the neighborhood of the band $\widetilde{\omega}_h(\mathbf{T})$. Let $\varepsilon\in\mathbf{R}$ and consider the transformation $D_\varepsilon:W_{1/Ch}\to\mathbf{C}^n$ given by

$$\theta\mapsto\theta+i\varepsilon\cdot\overline{\nabla\widetilde{\omega}_h(\theta)}. \tag{A.1}$$

Then, for $\theta\in W_{1/Ch}$, one has

$$\nabla D_\varepsilon(\theta)=\mathrm{Id}+i\varepsilon(\overline{\mathrm{Hess}(\widetilde{\omega}_h(\theta))}). \tag{A.2}$$

So by assumption (H.5), for $C>0$, h small enough, for $|\varepsilon|$ small enough (depending on C), D_ε is an analytic embedding.

Let $\mathrm{Im}(\tilde{z})>0$ and $\varepsilon<0$. Using Stokes' formula one gets

$$\begin{aligned}\tilde{I}(z,u)&=\int_{D_\varepsilon(\mathbf{T})}\frac{u(x,\theta)}{\tilde{z}-\widetilde{\omega}_h(\theta)}\,d\theta=\int_{\mathbf{T}}\frac{u(x,D_\varepsilon(\theta))\cdot\mathrm{Jac}(D_\varepsilon(\theta))}{\tilde{z}-\widetilde{\omega}_h(D_\varepsilon(\theta))}\,d\theta\\&=\int_{\mathbf{T}}\frac{u(x,\theta+i\varepsilon(\nabla\widetilde{\omega}_h(\theta)))\cdot\mathrm{Jac}(D_\varepsilon(\theta))}{\tilde{z}-\widetilde{\omega}_h(\theta)-i\varepsilon|\nabla\widetilde{\omega}_h(\theta)|^2+O(\varepsilon^2)}\,d\theta,\end{aligned} \tag{A.3}$$

(where by (H.5), $O(\varepsilon^2)$ is uniform in θ and h for h small enough).

Let us recall some notations from Section 1,

$$\Lambda_0=\{\text{the critical values of }\widetilde{\omega}_0\}=\{\tilde{\lambda}_0^j;1\le j\le p\}.$$

For $r_0>0$, $\square(\tilde{\lambda}_0^j,r_0)$ denotes a complex square box centered at $\tilde{\lambda}_0^j$ with side r_0, and

$${}^c\square(r_0)=(\widetilde{\omega}_h(\mathbf{T})+\square(0,r_0))\setminus\bigg(\bigcup_{1\le j\le p}\square(\tilde{\lambda}_0^j,r_0)\bigg).$$

Using (A.3) one immediately gets

Proposition A.1. *Let $\widetilde{\omega}_h$ satisfy assumptions* (H.5)*. Then there exist $h_0>0$ and $r_0>0$ such that, for $h\in(0,h_0)$, $\tilde{I}(z,u)$ can be analytically continued from the upper (or lower) half plane to ${}^c\square(r_0)$. Moreover, for these continuations there exists $C>0$ such that, for $h\in(0,h_0)$ and $z\in{}^c\square(r_0)$*

$$|\tilde{I}_\pm(z,u)|<C\|u(x)\|_{\infty,c'}.$$

We will now continue $\tilde{I}(z,u)$ in the neighborhood of the critical values of $\widetilde{\omega}_h$. For $1\le j\le p$ the following holds true.

Proposition A.2. *There exist $a>0$ and $h_0>0$ such that, for $\tilde{z}\in\square^\pm(\tilde{\lambda}_0^j,a)$, one has*

$$\tilde{I}(z,u)=\sum_{k=0}^{k_j}S(\tilde{z}-\widetilde{\omega}_h(\theta_j^k))\cdot H_{j,k}^\pm(\tilde{z},u)+G_j^\pm(\tilde{z},u),$$

where:

(a) *$H_{j,k}^\pm$ and $G_j^\pm$ are holomorphic for $\tilde{z}$ in $\square(\tilde{\lambda}_0^j,a)$.*

(b) *There exists $C>0$ such that*

$$\sup_{\tilde{z}\in\square(\tilde{\lambda}_0^j,a)}(|H_{j,k}^\pm(\tilde{z},u)|+|G_j^\pm(\tilde{z},u)|)<C\|u(x)\|_{\infty,c'}.$$

(c) *One has*

$$H_{j,k}^\pm(\widetilde{\omega}_h(\theta_j^k),u)=2^{n/2}\cdot(\pm i)^{p_k}\cdot\mathrm{Vol}(\partial B(0,1))\cdot D_s(\theta_j^k)\cdot u(x,\theta_j^k)$$

where $D_s(\theta_j^k)=|\det(\mathrm{Hess}(\widetilde{\omega}_h(\theta_j^k)))|^{-1/2}$.

(d) *If n is even, $S(z)=\frac{1}{2}\cdot(-1)^{n/2}z^{(n-2)/2}\cdot\log z$,*
if n is odd, $S(z)=\frac{1}{2}\pi\cdot(-1)^{(n-1)/2}z^{(n-2)/2}$.

(*Here $\log z$ and $z^{1/2}$ are the principal determinations of these functions*).

Proof. We will only study the analytic continuation of $\tilde{I}(z,u)$ from above the real axis, the procedure being identical when coming from the other side. Let $\mathrm{Im}(\tilde{z})>0$ and $\varepsilon<0$. For $\varrho>0$ let us define $\mathbf{T}_j(\varrho)=\mathbf{T}\setminus\bigcup_{1\le k\le k_j}B(\theta_j^k,\varrho)$; it contains none of the critical points of $\widetilde{\omega}_0$ associated to $\tilde{\lambda}_0^j$.

One has, using Stokes' theorem,

$$\begin{aligned}\tilde{I}(z,u)&=\sum_{1\le k\le k_j}\int_{B(\theta_j^k,\varrho_0)}\frac{u(x,\theta)}{\tilde{z}-\widetilde{\omega}_h(\theta)}\,d\theta+\int_{\mathbf{T}_j(\varrho)}\frac{u(x,\theta)}{\tilde{z}-\widetilde{\omega}_h(\theta)}\,d\theta\\&=\sum_{1\le k\le k_j}\int_{B(\theta_j^k,\varrho_0)}\frac{u(x,\theta)}{\tilde{z}-\widetilde{\omega}_h(\theta)}\,d\theta+\int_{D_\varepsilon(\mathbf{T}_j(\varrho))}\frac{u(x,\theta)}{\tilde{z}-\widetilde{\omega}_h(\theta)}\,d\theta\\&\quad+\int_{\partial(\bigcup_{\zeta\in[0,\varepsilon]}D_\zeta(\mathbf{T}_j(\varrho)))}\frac{u(x,\theta)}{\tilde{z}-\widetilde{\omega}_h(\theta)}\,d\theta.\end{aligned}$$

The function $\int_{D_\varepsilon(\mathbf{T}_j(\varrho))} u(x,\theta)/(\tilde{z}-\widetilde{\omega}_h(\theta))d\theta$ is analytic for $\tilde{z}$ in a small neighborhood of $\tilde{\lambda}_0^j$ in $\mathbf{C}$ (depending only on ε and ϱ but not on h small enough).

For ϱ,ε and h small enough

$$\bigcup_{\zeta\in[0,\varepsilon]} D_\zeta(\mathbf{T}_j(\varrho)) = \bigcup_{1\leq k\leq k_j}\Bigg(\bigcup_{\zeta\in[0,\varepsilon]} D_\zeta(S(\theta_j^k,\varrho))\Bigg)$$

contains no critical point of $\widetilde{\omega}_h$. On this compact set, using the Taylor formula one sees that $\operatorname{Im}\widetilde{\omega}_h(\theta)<0$. Hence, using Stokes' theorem and regular deformations, one can continue analytically as a function of $\tilde{z}$ in some small neighborhood of $\tilde{\lambda}_j^0$, the following integral

$$\int_{\partial(\bigcup_{\zeta\in[0,\varepsilon]} D_\zeta(\mathbf{T}_j(\varrho)))} \frac{u(x,\theta)}{\tilde{z}-\widetilde{\omega}_h(\theta)}\,d\theta.$$

We just have to continue analytically an integral of the form

$$J(\tilde{z},u)=\int_{B(\theta_j^k,\varrho)} \frac{u(x,\theta)}{\tilde{z}-\widetilde{\omega}_h(\theta)}\,d\theta.$$

Assumption (H.5) ensures that one can prove an h-uniform Morse lemma (for h small enough), that is, there exist $\varrho_0,\varrho_1>0$ such that, for $1\leq k\leq k_j$ and h small enough, there exist $p_k\in\mathbf{N}$ and a local analytic diffeomorphism D_k defined from a complex neighborhood of θ_j^k to a complex neighborhood of 0 in $\mathbf{C}^n$ such that

- $B_{\mathbf{R}^n}(\theta_k,\varrho_1)\subset(D_k)^{-1}(B_{\mathbf{R}^n}(0,2\varrho_0))$ and $B_{\mathbf{C}^n}(\theta_k,\varrho_1)\subset(D_k)^{-1}(B_{\mathbf{C}^n}(0,2\varrho_0))$,
- for $\theta\in B_{\mathbf{C}^n}(0,2\varrho_0)$,

$$\widetilde{\omega}_h(D_k^{-1}(\theta))=\widetilde{\omega}_h(\theta_j^k)+\sum_{1\leq l\leq p_k}\theta_l^2-\sum_{p_k+1\leq l\leq n}\theta_l^2,$$

- $\det(\operatorname{Jac}(D_k^{-1})(0))=2^{n/2}|\det(\operatorname{Hess}(\widetilde{\omega}_h(\theta_k^j)))|^{-1/2}$,
- $(p_k,n-p_k)$ is the signature of $\operatorname{Hess}(\widetilde{\omega}_h(\theta_k^j))$.

Using this and regular deformations as before one gets

$$\begin{aligned} J(\tilde{z},u)=&\int_{B(0,\varrho_0)} \frac{u(x,D_k^{-1}(\theta))\cdot\operatorname{Jac}(D_k^{-1}(\theta))}{\tilde{z}-\widetilde{\omega}_h(D_k^{-1}(\theta))}\,d\theta \\ &+\int_{B(\theta_j^k,\varrho_1)\setminus D_k^{-1}(B(0,\varrho_0))} \frac{u(\theta,x)}{\tilde{\lambda}-\widetilde{\omega}_h(\theta)}\,d\theta. \end{aligned} \tag{A.4}$$

The second integral of the right hand side of (A.4) can be analytically continued for $\tilde{z}$ close enough to $\tilde{\lambda}_0^i$ (as we integrate over a domain free of critical points). Let

$$J(\tilde{z},u)=\int_{B(0,\varrho_0)} \frac{u(x,D_k^{-1}(\theta))\cdot\operatorname{Jac}(D_k^{-1}(\theta))}{\tilde{z}-\widetilde{\omega}_h(\theta_j^k)-\sum_{1\leq l\leq p_k}\theta_l^2+\sum_{p_k+1\leq l\leq n}\theta_l^2}\,d\theta.$$

Consider the following group of deformations. Define $R_\alpha\colon B(0,\varrho_0)\to\mathbf{C}^n$ for $\alpha\in[0,\pi/2]$ by

$$(\theta_j)_{1\le j\le n}\mapsto((e^{i\alpha}\theta_j)_{1\le j\le p_k},(\theta_j)_{p_k+1\le j\le n}).$$

The R_α are embeddings so $\bigcup_{\alpha\in[0,\pi/2]}R_\alpha(B(0,\varrho_0))$ is an (n+1)-dimensional submanifold of $\mathbf{C}^n$. Then by Stokes' formula

$$\begin{aligned}J(\tilde z,u)=\int_{B(0,\varrho_0)}&\frac{u(x,D_k^{-1}\circ R_{\pi/2}(\theta))\cdot\mathrm{Jac}(D_k^{-1}\circ R_{\pi/2}(\theta))}{\tilde z-\widetilde\omega_h(\theta_j^k)+\sum_{1\le l\le p_k}\theta_l^2+\sum_{p_k+1\le l\le n}\theta_l^2}\,d\theta\\&+\int_{\bigcup_{\alpha\in[0,\pi/2]}(R_\alpha(\partial B(0,\varrho_0)))}\frac{u(x,D_k^{-1}(\theta))\cdot\mathrm{Jac}(D_k^{-1}(\theta))}{\tilde z-\widetilde\omega_h(D_k^{-1}(\theta))}\,d\theta.\end{aligned}$$

The points of $\bigcup_{\alpha\in[0,\pi/2]}R_\alpha(\partial B(0,\varrho_0))$ are regular for $\widetilde\omega_h\circ D_k^{-1}$ and on these points one has $\mathrm{Im}\,\widetilde\omega_h(\theta)\le 0$. So, using regular deformations like D_ε, one sees that the second integral defines a function analytic in a neighborhood of $\tilde\lambda_0^j$.

We are now only left with studying

$$J(\tilde z,u)=\int_0^{\varrho_0}\frac{f(\varrho,u)}{\tilde z-\widetilde\omega_h(\theta_j^k)+\varrho^2}\cdot\varrho^{n-1}\,d\varrho,\tag{A.5}$$

where

$$f(\varrho,u)=\int_{\partial B(0,1)}u(x,D_k^{-1}\circ R_{\pi/2}(\varrho\sigma))\cdot\mathrm{Jac}(D_k^{-1}\circ R_{\pi/2})(\varrho\sigma)\,d\sigma.\tag{A.6}$$

Obviously, $u(x,D_k^{-1}\circ R_{\pi/2}(\varrho\sigma))\cdot\mathrm{Jac}(D_k^{-1}\circ R_{\pi/2})(\varrho\sigma)$ is analytic in ϱ in $D_{\mathbf{C}}(0,2\varrho_0)$. Moreover, expanding this function in a power series

$$u(x,D_k^{-1}\circ R_{\pi/2}(\varrho\sigma))\cdot\mathrm{Jac}(D_k^{-1}\circ R_{\pi/2})(\varrho\sigma)=\sum_{p=0}^{+\infty}\Bigg(\sum_{\alpha\in\mathbf{N}^n;|\alpha|=p}a_\alpha\sigma^\alpha\Bigg)\cdot r^p,$$

one gets the following Cauchy estimate,

$$|a_\alpha|\le C\cdot\left(\tfrac{3}{2}\varrho_0\right)^{-|\alpha|}\cdot\|u\|_{\infty,c}.\tag{A.7}$$

Here, for $\alpha=(\alpha_1,...,\alpha_n)\in\mathbf{N}^n$ and $\sigma=(\sigma_1,...,\sigma_n)\in\partial B(0,1)\subset\mathbf{R}^n$, we define $|\alpha|=\sum_{i=1}^n\alpha_i$ and $\sigma^\alpha=\prod_{i=1}^n\sigma_i^{\alpha_i}$, and $C>0$ is a constant independent of h small enough.

Now, using (A.7) and carrying out the integration in (A.6), one gets

$$f(\varrho,u)=\sum_{p=0}^{+\infty}A_p(u)\varrho^p,$$

where

$$|A_p(u)| = \left| \sum_{\alpha\in\mathbf{N}^n;|\alpha|=p} a_\alpha \cdot \int_{\partial B(0,1)} \sigma^\alpha d\sigma \right| \le C\cdot\Big(\sum_{\alpha\in\mathbf{N}^n;|\alpha|=p} |a_\alpha| \Big) \le C'\cdot\left(\frac{4}{3}\varrho_0\right)^{-p}\cdot\|u\|_{\infty,c}. \tag{A.8}$$

Obviously $A_{2p+1}(u)=0$. So $f(\varrho,u)$ is analytic in $D_{\mathbf{C}}(0,\varrho_0)$ admitting a power series expansion that is well controlled in u.

By (A.5) one gets the following

$$J(\tilde z,u) = \sum_{p\in\mathbf{N}} A_{2p}(u)\int_0^{\varrho_0} \frac{\varrho^{2p+n-1}}{\tilde z - \widetilde\omega_h(\theta_j^k)+\varrho^2}\,d\varrho. \tag{A.9}$$

Now, we just have to continue analytically

$$I_p(\tilde z) = \int_0^{\varrho_0} \frac{\varrho^{2p+n-1}}{\tilde z+\varrho^2}\,d\varrho$$

for $\tilde z$ in a neighborhood of 0 in $\mathbf{C}$.

Let us write $n=2k+\nu$ where $\nu=1$ or 2 and $k\ge 0$. Then

$$\begin{aligned} I_p(\tilde z) &= \int_0^{\varrho_0} \frac{\varrho^{2(p+k)}\cdot\varrho^{\nu-1}}{\tilde z+\varrho^2}\,d\varrho \\ &= \int_0^{\varrho_0}\left(\sum_{l=0}^{p+k}\binom{p+k}{l}(-1)^{p+k-l}\tilde z^{p+k-l}(\tilde z+\varrho^2)^l\right)\frac{\varrho^{\nu-1}}{\tilde z+\varrho^2}\,d\varrho \\ &= (-1)^{p+k}\tilde z^{p+k}\int_0^{\varrho_0}\frac{\varrho^{\nu-1}}{\tilde z+\varrho^2}\,d\varrho + R_p(\tilde z), \end{aligned} \tag{A.10}$$

where

$$R_p(\tilde z) = \sum_{l=0}^{p+k-1}\binom{p+k}{l+1}(-\tilde z)^{p+k-l+1}\int_0^{\varrho_0}\varrho^{\nu-1}\cdot(\tilde z+\varrho^2)^l\,d\varrho$$

is obviously analytic in $\tilde z$ and satisfies, for $\tilde z\in D_{\mathbf{C}}(0,\varrho_0^2/4)$ and a certain $C>0$,

$$|R_p(\tilde z)| \le C\left(\tfrac{5}{4}\varrho_0^2\right)^{p+k}. \tag{A.11}$$

Easy computations show that

$$\int_0^{\varrho_0}\frac{1}{\tilde z+\varrho^2}d\varrho = \frac{\pi}{2}\cdot\tilde z^{-1/2} - \int_{\varrho_0}^{+\infty}\frac{1}{\tilde z+\varrho^2}\,d\varrho \tag{A.12}$$

and

$$\int_0^{\varrho_0} \frac{\varrho}{\tilde{z}+\varrho^2} d\varrho = -\frac{1}{2}\cdot\log(\tilde{z})+\frac{1}{2}\log(\varrho_0^2+\tilde{z}). \tag{A.13}$$

Remark. Here we use the principal determinations of the square root and the logarithm.

So one gets

$$I_p(\tilde{z}) = (-1)^p \tilde{z}^p \cdot S(\tilde{z}) + R_p(\tilde{z}),$$

where S is defined in the statement of Proposition A.2, and $R_p(\tilde{z})$ is analytic for $\tilde{z}\in D_{\mathbf{C}}(0,\varrho_0^2/4)$ and satisfies (A.11). Then, using (A.9), the sums being absolutely convergent by (A.8) and (A.11), one gets

$$J(\tilde{z},u) = S(\tilde{z}-\widetilde{\omega}_h(\theta_j^k))\cdot H_{j,k}^+(\tilde{z},u) + G_{j,k}^+(\tilde{z},u),$$

where

$$H_{j,k}^+(\tilde{z},u) = \sum_{p\in\mathbf{N}} (-1)^p A_{2p}(u)(\tilde{z}-\widetilde{\omega}_h(\theta_j^k))^p, \tag{A.14}$$

and

$$G_{j,k}^+(\tilde{z},u) = \sum_{p\in\mathbf{N}} A_{2p}(u) R_p(\tilde{z}-\widetilde{\omega}_h(\theta_j^k)), \tag{A.15}$$

(both of these sums being uniformly convergent for $\tilde{z}\in D_{\mathbf{C}}(0,\varrho_0^2/4)$.) One computes

$$A_0(u) = 2^{n/2}\cdot i^{p_k}\cdot \mathrm{Vol}(\partial B(0,1))\cdot |\det(\mathrm{Hess}(\widetilde{\omega}_h(\theta_j^k)))|^{-1/2}\cdot u(\theta_j^k,x). \tag{A.16}$$

Of course, the same study can be done for analytic continuation from above the band. This ends the proof of Proposition A.2. $\square$

Remark. Let us assume (H.6) and that $n\geq 3$. Let k_n denote the largest integer smaller or equal to $(n-3)/2$. Using the Taylor formula one gets for $\tilde{z}\notin[\tilde{i}_h,\tilde{s}_h]$

$$\begin{aligned} J(\tilde{z},u) = {} & \sum_{k=0}^{k_n}\left(\int_{\mathbf{T}} \frac{(-1)^k u(x,\theta)}{(\tilde{s}_h-\widetilde{\omega}_h(\theta))^{k+1}} d\theta\right)\cdot(\tilde{z}-\tilde{s}_h)^k \\ & +(\tilde{z}-\tilde{s}_h)^{k_n+1}\cdot\int_0^1\left(\int_{\mathbf{T}} \frac{(-1)^{k_n+1}\cdot u(x,\theta)}{(t\cdot(\tilde{\lambda}-\tilde{s}_h)+\tilde{s}_h-\widetilde{\omega}_h(\theta))^{k_n+2}} d\theta\right)\cdot t^{k_n+1}\, dt. \end{aligned} \tag{A.17}$$

One may use the same technique as before to continue analytically the last integral in formula (A.17) for $\tilde{z}$ close to $\tilde{s}_h$.

2. Another point of view on $I(z,u)$

Let u be a function analytic in some complex neighborhood of $\mathbf{T}$. Let φ_u be the distribution on $\mathbf{R}$ defined for $g\in C^\infty(\mathbf{R})$ by

$$\langle\varphi_u, g\rangle = \int_{\mathbf{T}} g(\widetilde{\omega}_h(\theta))\cdot u(\theta)\,d\theta.$$

Then φ_u is of order 0 and compactly supported. More precisely, $\operatorname{supp}(\varphi_u)\subset\widetilde{\omega}_h(\mathbf{T})$. If $\tilde{x}\in\widetilde{\omega}_h(\mathbf{T})$ and $\tilde{x}$ is a regular value of $\widetilde{\omega}_h$, then

$$\varphi_u(\tilde{x}) = \int_{\{\theta\in\mathbf{T};\widetilde{\omega}_h(\theta)=\tilde{x}\}} u(\theta)d\sigma(\theta), \tag{A.18}$$

where $d\sigma$ is the measure induced on $\{\theta\in\mathbf{T};\widetilde{\omega}_h(\theta)=\tilde{x}\}$, a smooth compact submanifold of $\mathbf{T}$, by the Lebesgue measure on $\mathbf{T}$.

(A.18) shows that φ_u is analytic in a complex neighborhood of the regular values of $\widetilde{\omega}_h$.

For $\tilde{z}$ such that $\operatorname{Im}\tilde{z}\neq 0$, by definition

$$\tilde{I}(\tilde{z},u) = \left\langle \varphi_u, \frac{1}{\tilde{z}-\cdot}\right\rangle. \tag{A.19}$$

So, if $\tilde{x}\in\widetilde{\omega}_h(\mathbf{T})$ and $\tilde{x}$ is a regular value of $\widetilde{\omega}_h$, then

$$\lim_{y\to 0^\pm} \operatorname{Im}\tilde{I}(\tilde{x}+iy,u) = \pm\pi\cdot\varphi_u(\tilde{x}). \tag{A.20}$$

This gives us

Proposition A.3.

(a) *For any $n\in\mathbf{N}$, there exists $r_0>0$ such that $\forall r\in(0,r_0]$, there exist $h_r>0$ and $c_r>0$ such that, $\forall h\in(0,h_r)$ and $\forall z\in\mathcal{UC}({}^c\square(r),\square^\pm(r))$*

$$|\operatorname{Im} I(z)| > \frac{c_r}{f(h)}. \tag{$*$}$$

For $n\geq 3$ and $j\notin\{1,p\}$, there exists $r_j>0$ such that $\forall r\in(0,r_j]$, there exists $h_r>0$ and $c_r>0$ such that $\forall h\in(0,h_r)$ and $\forall z\in\mathcal{O}(r,j)$ the inequality $(*)$ holds.

Proof. Notice that, by (A.18), for $\tilde{x}\in\widetilde{\omega}_h(\mathbf{T})$ and $\tilde{x}$ a regular value of $\widetilde{\omega}_h$, we know that $\varphi_1(\tilde{x})>0$. So, using (A.20), we get Proposition A.3.

For $n\geq 3$, one just uses the expansion given by Proposition A.2.

Instead of the study we did in the first part of this section, we could also have studied the singularities of φ_u at the critical values of $\widetilde{\omega}_h$, and then have used the Cauchy formula (A.18) to get the information on $\tilde{I}(\tilde{z},u)$.

References

[AgCo] AGUILAR, J. and COMBES, J. M., A class of analytic perturbations for one-body Schrödinger hamiltonians, *Comm. Math. Phys.* **22** (1971), 269–279.

[BaCo] BALSLEV, E. and COMBES, J. M., Spectral properties of many-body Schrödinger operators with dilation analytic interactions, *Comm. Math. Phys.* **22** (1971), 280–294.

[BaSk] BALSLEV, E. and SKIBSTED, E., Resonance theory of two-body Schrödinger operators, *Ann. Inst. H. Poincaré Phys. Théor.* **51** (1989), 129–154.

[Cy] CYCON, H. L., Resonances defined by modified dilations, *Helv. Phys. Acta* **58** (1985), 969–981.

[Fi1] FIRSOVA, N. E., Riemann surfaces of quasi-momentum and Scattering theory for the perturbed Hill operator, *Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov (LOMI)* **51** (1975), 183–196 (Russian). English Transl.: *J. Soviet Math.* **11** (1979), 487–497.

[Fi2] FIRSOVA, N. E., Resonances of a Hill operator perturbed by an exponentially decreasing additive potential, *Mat. Zametki* **36** (1984), 711–724 (Russian). English transl.: *Math. Notes* **36** (1984), 854–861.

[FFLP] FOTIADI, D., FROISSART, M., LASCOUX, J. and PHAM, F., Applications of an isotopy theorem, *Topology* **4** (1965), 159–191.

[Gé1] GÉRARD, C., Resonance theory in atom surface scattering, *Comm. Math. Phys.* **126** (1989), 263–290.

[Gé2] GÉRARD, C., Resonance theory for periodic Schrödinger operators, *Preprint École Polytechnique*, 1988.

[HeMa] HELFFER, B. and MARTINEZ, A., Comparaison entre les diverses notions de résonances, *Helv. Phys. Acta* **60** (1987), 992–1003.

[HeSj] HELFFER, B. and SJÖSTRAND, J., Résonances en limite semi-classique, *Mém. Soc. Math. France* **24–25** (1986).

[HiSig] HISLOP, P. and SIGAL, I., *Semi-classical Theory of Shape Resonances in Quantum Mechanics, Mem. Amer. Math. Soc.* **399**, Amer. Math. Soc., Providence, R. I., 1989.

[Hu1] HUNZIKER, W., Distortion Analyticity and Molecular Resonances Curves, *Ann. Inst. H. Poincaré Phys. Théor.* **45** (1986), 339–358.

[Hu2] HUNZIKER, W., Resonances, Metastable States and Exponential Decay Laws in Perturbation Theory, *Comm. Math. Phys.* **132** (1990), 177–188.

[Kl] KLOPP, F., Étude semi-classique d'une perturbation d'un opérateur de Schrödinger périodique, *Ann. Inst. H. Poincaré Phys. Théor.* **55** (1991), 459–509.

[La] LAMOTKE, K., The topology of complex projective varieties after S. Leftschetz, *Topology* **20** (1981), 15–51.

[LxPh] LAX, P. D. and PHILLIPS, R., Scattering theory for transport phenomena, *Functional Analysis, Proc. Conf. Irvine, Calif., 1966*, pp. 119–130, Academic Press, London and Thompson Book, Co., Washington, D. C., 1967.

[Or] ORTH, A., Quantum mechanical resonance and limiting absorption: the many-body case, *Comm. Math. Phys.* **126** (1990), 559–573.

[P] PHAM, F., *Introduction à l'étude topologique des singularités de Landau, Mémorial des Sciences Mathématiques* **164**, Gauthiers-Villars, Paris, 1967.

[ReSi] REED, M. and SIMON, B., *Methods of Modern Mathematical Physics, Vol IV: Analysis of Operators*, Academic Press, New York, 1978.

[Si1] SIMON, B., The theory of resonances in n-body quantum systems for dilatation analytic potentials and the foundations of time-dependent scattering theory, *Ann. of Math.* **97** (1973), 247–274.

[Si2] SIMON, B., Resonances and complex scaling: a rigorous overview, *Internat. J. Quantum Chem.* **14** (1978), 529–542.

[Sj] SJÖSTRAND, J., Microlocal analysis for periodic magnetic Schrödinger equation and related questions, in *Microlocal analysis and applications* (Cattabriga, L. and Rodino, L., eds.), *Lecture Notes in Math.* **1495**, pp. 237–332, Springer-Verlag, Berlin–Heidelberg, 1991.

Received April 29, 1993

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