THE ANALYSIS OF ADDITIVE SET FUNCTIONS IN EUCLIDEAN SPACE

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1. Introduction

Suppose that I_0 is a closed rectangle in Euclidean n-space and that $\mathcal{B}$ is the field of Borel subsets of I_0. We confine our attention to those completely additive set functions F, having a finite value $F(E)$ for each E of $\mathcal{B}$, and left undefined for sets E not in $\mathcal{B}$; we reserve the name "set function" for such special set functions. Each set function corresponds to a point function $\Phi(x)$, and an additive interval function $\Phi(I)$ such that $F(E) = \Phi^*(E)$ for each $E \in \mathcal{B}$, where Φ^* denotes the Lebesgue–Stieltjes measure generated by Φ. Thus corresponding to each of our decomposition theorems about set functions, there is a decomposition theorem for point functions of bounded variation, and for additive interval functions of bounded variation. (See Saks [5], Chap. 3, for the exact correspondence.)

Using the classical Lebesgue theory, set functions have been classified as follows. (i) A set function F is *absolutely continuous*, if $F(E) = 0$ for every $E \in \mathcal{B}$ with zero Lebesgue measure; it is *singular*, if there is a set $E_0 \in \mathcal{B}$ of zero Lebesgue measure such that $F(E) = F(E \cap E_0)$ for each $E \in \mathcal{B}$. (ii) A set function F is said to be *diffuse*, if $F(\{x\}) = 0$ for each point $x \in I_0$; it is said to be *atomic*, if there is a finite or enumerable set $E_0 = x_1, x_2, \ldots$ such that $F(E) = F(E \cap E_0)$ for each $E \in \mathcal{B}$. Now it is well known that, given any set function F, there are 3 component set functions F_1, F_2, F_3, such that

$$F(E) = F_1(E) + F_2(E) + F_3(E)$$

for all E of $\mathcal{B}$, and

F_1 is absolutely continuous,

F_2 is singular and diffuse,

F_3 is atomic.

Further, this decomposition is unique. In the present paper our object is to analyse the singular diffuse set functions, endeavouring to break them down into components which are uniform in some respect.

Recently Dr. H. Kober proved that any singular diffuse set function can be uniquely expressed as the sum of a continuous set function and a step function. A set function F is continuous at a point $x \in I_0$, if $F(R) \to 0$ as the Lebesgue measure of R tends to zero, R denoting any rectangle with $x \in R \subset I_0$ whose sides are parallel to the sides of I_0. The set function is continuous if it is continuous at all $x \in I_0$. The set function F is called a step function, if there is a set $E_0 \subset I_0$ consisting of a finite or enumerable number of hyperplanes each parallel to a face of I_0, such that $F(E) = F(E \cap E_0)$ for all E of $\mathfrak{B}$. Although Dr. Kober's decomposition is very interesting, it seems to us to suffer from the following disadvantages. (1) Only "concentrations" on hyperplanes parallel to a face of the rectangle I_0 are picked out; thus a set function concentrated on a straight line not parallel to an edge of I_0 would be said to be continuous, provided it was diffuse. (2) Even if the definition of continuous were altered to allow "skew" rectangles, one still would not pick out concentrations on hypersurfaces which are "curved" at every point. (3) The step function is not sub-divided into a part "spread over" the $(n-1)$-dimensional hyperplanes and a part concentrated on sets of lower dimension. Our work is a natural development of Dr. Kober's and uses similar methods, it is largely motivated by the desire to overcome these difficulties. We are most grateful to Dr. Kober for telling us of his work before its publication and for interesting us in the problem.

Before we describe our results it is convenient to consider set functions "concentrated" on certain sets. Suppose, for example, that S is an r-dimensional surface $(1 \leqslant r \leqslant n-1)$ with an r-dimensional volume element dS, and $f(x)$ is a function defined on S, but not taking the value zero there; then provided the integral is defined and completely additive, there will be a corresponding set function F defined by

$$F(E) = \int_{E \cap S} f(x)\, dS.$$

This set function is concentrated on S in the sense that $F(E) = F(E \cap S)$ for each $E \in \mathfrak{B}$; it is spread over S in the sense that, if T is any measurable subset of S with

$$\int_T dS > 0,$$

there will be a set $T' \subset T$ such that

$$F(T') = \int_{T'} f(x)\, dS \neq 0;$$

and it is continuous relative to S in the sense that, if T is a subset of S with

$$\int_T dS = 0,$$

then $F(T') = 0$. If we use the concept of an integral with respect to a general Hausdorff measure (for the appropriate definitions see § 2), it is easy to give a more general example of a set function which is in appropriate senses concentrated on, spread over and continuous relative to a sub-set of I_0. Let $h(t)$ be a Hausdorff measure function. Let S be a Borel sub-set of I_0 with positive σ-finite h-measure. Write $\mu(E) = h\text{-}m(E \cap S)$ for all $E \in \mathcal{B}$. Let $f(x)$ be a point function, which does not take the value zero for any point of S, and which is (absolutely) μ-integrable. Then, in the obvious sense, the set function defined by

$$F(E) = \int_{E \cap S} f \, d\mu, \tag{1}$$

for all $E \in \mathcal{B}$, is concentrated, on S. Further, using the h-measure to measure the size of Borel sub-sets of S, the set function is spread over and continuous relative to S. We shall show later (§ 3) that in a certain sense this example is the most general possible.

Naturally more complicated set functions F can be formed by adding a countable number of set functions of the type (1) corresponding to different sets S (not necessarily disjoint) associated with different Hausdorff measure functions $h(t)$. This suggests that we should aim at a general decomposition theorem for arbitrary set functions, which when applied to such a sum would separate out the different component set functions used to construct it. We obtain a general decomposition theorem (Theorem 6) for arbitrary set functions, which will effect this re-decomposition of a sum of such components, provided the corresponding h-measures are all mutually comparable and provided the maximal system of h-measure functions, used to effect the decomposition, contains the particular h-measures corresponding to the original terms of the sum.

Before we discuss the general decomposition theorem we consider a much simpler decomposition involving only a single h-measure. In § 3 we show (Theorem 1) that, if F is a set function and $h(t)$ is a Hausdorff measure function, then there is a unique decomposition

$$F = F_1 + F_2 + F_3,$$

where F_1 is strongly continuous with respect to the h-measure in the sense that $F_1(E) = 0$ for every E with a σ-finite h-measure, where F_2 is both absolutely continuous with respect to the h-measure and concentrated on a set of σ-finite h-measure, and where F_3 is concentrated on a set of zero h-measure. By using the Radon–Nikodym theorem, we show that, provided it does not vanish, the set function F_2 will in fact have a representation in the form (1) and that it will thus be concentrated on, spread over and continuous relative to a set S of positive σ-finite h-measure.

Decomposition theorems of this type are the main tools for obtaining more complete decompositions. In § 4 we confine our attention to the Hausdorff measure functions $h(t)$ of the form t^α, with $0 < \alpha \leqslant n$, corresponding to the fractional dimension measures. We prove (Theorem 3) that given any finite singular diffuse set function F, there is a finite or enumerable sequence $\{\alpha_i\}$, with $0 \leqslant \alpha_i \leqslant n$, of distinct dimensions and a unique decomposition

$$F = F^{(d)} + F^{(1)} + F^{(2)} + \ldots,$$

such that $F^{(i)}$ is concentrated on a set of dimension α_i, but $F^{(i)}(E) = 0$ for each E of $\mathcal{B}$ with dimension less than α_i, and $F^{(d)}$ has no component concentrated on a set of definite dimension. The component $F^{(i)}$ corresponding to the dimension α_i can be decomposed by using the corresponding h-function t^{α_i} into three components

$$F_1^{(i)},\ F_2^{(i)},\ F_3^{(i)},$$

which will in general be all non-trivial. But, if the component $F^{(d)}$ with a "diffuse real dimension spectrum" is decomposed, using any Hausdorff measure function h which is comparable with the class of functions t^α, the middle component $F_2^{(d)}$ will automatically be zero. We describe the structure of this component with a diffuse real dimension spectrum in § 4; in § 5 we give an example of such a set function.

In § 6 we introduce the concept of a maximal system of mutually comparable Hausdorff measure functions. This enables us to prove (Theorem 6) that a set function F can be decomposed in the form

$$F = F^{(d)} + F^{(1)} + F^{(2)} + \ldots,$$

where now each component $F^{(i)}$, $i = 1, 2, \ldots$, corresponds to a definite section of the maximal completely ordered system of measure functions used, but the component $F^{(d)}$ has no component corresponding to a definite section. In terms of the prescribed maximal system of measure functions, this decomposition is also unique. Again the component with a diffuse dimension spectrum has a definite structure, and the discrete components $F^{(i)}$ which correspond to sections defined by Hausdorff measure functions of the system can be decomposed as before into components $F_1^{(i)}$, $F_2^{(i)}$, $F_3^{(i)}$.

The exact relationships between the different components obtained by using the two decomposition theorems is in general rather complicated. We discuss these relationships in detail in § 6 and we show that, if a maximal system of mutually comparable Hausdorff measure functions is used, and if this system includes all the functions t^α (with $0 < \alpha \leqslant n$), the decomposition obtained is a refinement of the decomposition using only the special α-dimensional measures.

Our analysis raises a number of problems that we have not been able to solve (see § 6). Until some of these problems have been solved it is difficult to judge whether our decomposition theorem is really satisfactory or whether some other approach to the subject would be preferable.

While all our results are expressed in terms of finite set functions defined for Borel subsets of I_0, they extend immediately (i) to set functions which are σ-finite, and (ii) to set functions defined for all Borel sets in Euclidean n-space.

2. Definitions and preliminary results

A function $h(t)$ defined for $t > 0$ will be called a Hausdorff measure function, if $h(t)$ is continuous, monotonic increasing and $\lim_{t \to 0+} h(t) = 0$. For any set E in Euclidean n-space, the outer h-measure of E, denoted by $h\text{-}m_*(E)$, is defined by

$$h\text{-}m_*(E) = \lim_{\delta \to 0} \left[\inf_{E \subset \bigcup_{i=1}^{\infty} C_i} \sum_{i=1}^{\infty} h\{d(C_i)\} \right], \quad d(C_i) < \delta$$

where $\{C_i\}$ is a sequence of convex sets covering E, and $d(C_i)$ denotes the diameter of the set C_i (the value $+\infty$ is allowed for the function $h\text{-}m_*(E)$). Then $h\text{-}m_*$ is an outer Caratheodory measure and defines a class of h-measurable sets which, for any $h(t)$, includes all Borel sets. When E is h-measurable we write $h\text{-}m(E) = h\text{-}m_*(E)$ and call the set function $h\text{-}m(E)$ the h-measure of E.

A special notation is used for the measure functions $h(t) = t^\alpha$, $\alpha > 0$. In this case $h\text{-}m(E)$ is written $\Lambda^\alpha(E)$. The following properties follow easily from the definition:

$$\begin{aligned} &\text{if} \quad \Lambda^\alpha(E) > 0, &&\text{then} \quad \Lambda^\beta(E) = +\infty &&\text{for } \beta < \alpha, \\ &\text{if} \quad \Lambda^\alpha(E) < +\infty, &&\text{then} \quad \Lambda^\beta(E) = 0 &&\text{for } \beta > \alpha. \end{aligned}$$

For any set E we can therefore define $\dim_B(E)$ as the infinum of numbers α for which $\Lambda^\alpha_*(E) = 0$: it is known as the (Besicovitch) dimension of E. We also put $\Lambda^0(E) = +\infty$, if E contains an infinite number of points, and $\Lambda^0(E) = q$, if E contains q points ($q = 0, 1, 2, \ldots$).

The class $\mathcal{J}$ of our set functions forms a linear space over the field of reals, if we define a set function $\lambda_1 F_1 + \lambda_2 F_2$ by

$$(\lambda_1 F_1 + \lambda_2 F_2)(E) = \lambda_1 F_1(E) + \lambda_2 F_2(E)$$

for all sets $E \in \mathfrak{B}$. If $F_1, F_2 \in \mathfrak{F}$ are such that $F_1(E) \geqslant F_2(E)$ for every $E \in \mathfrak{B}$, we say that $F_1 \geqslant F_2$. This clearly defines a partial ordering in the class $\mathfrak{F}$. We say that $F_1 > F_2$ if $F_1 \geqslant F_2$ and in addition $F_1(I_0) > F_2(I_0)$.

A lattice structure (we use the notation of Birkoff [1]) exists in the space $\mathfrak{F}$ if we define $F = F_1 \cup F_2$ as the smallest set function in $\mathfrak{F}$ which satisfies $F \geqslant F_1$ and $F \geqslant F_2$; and $F_1 \cap F_2$ as the largest element in $\mathfrak{F}$ satisfying both $F \leqslant F_1$ and $F \leqslant F_2$. Thus

$$F_1 \cup F_2 \geqslant F_1 \geqslant F_1 \cap F_2,$$
$$F_1 \cup F_2 \geqslant F_2 \geqslant F_1 \cap F_2;$$

but if $F(E) < (F_1 \cup F_2)(E)$ for some $E \in \mathfrak{B}$, then there will be some set $E' \in \mathfrak{B}$ for which

$$F(E') < \max[F_1(E'), F_2(E')];$$

and if $F(G) > (F_1 \cap F_2)(G)$ for some $G \in \mathfrak{B}$, then there will be some set $G' \in \mathfrak{B}$ for which

$$F(G') > \min[F_1(G'), F_2(G')].$$

The proof of the existence of the set functions $F_1 \cup F_2$ and $F_1 \cap F_2$ is essentially due to Riesz [4]. It can be shown that $F_1 + F_2 = F_1 \cup F_2 + F_1 \cap F_2$. The structure of $\mathfrak{F}$, which we have already discussed, shows that $\mathfrak{F}$ is a Riesz space.

Now the Jordan decomposition theorem shows that any $F \in \mathfrak{F}$ can be decomposed uniquely into $F = F^+ - F^-$ where F^+ and F^- are non-negative set functions of $\mathfrak{F}$, and the interval I_0 can be decomposed (though not uniquely) into Borel sets P and N such that $I_0 = P \cup N$ and $F^+(N) = 0$, $F^-(P) = 0$. We denote the sum of the positive and negative components by $|F| = F^+ + F^-$. It is clear that $|F| = F \cup (-F)$.

A subset $\mathfrak{S} \subset \mathfrak{F}$ is said to be bounded above, if there is an element $F_0 \in \mathfrak{F}$ such that $F_0 \geqslant F$ for every $F \in \mathfrak{S}$. The space $\mathfrak{F}$ considered as a lattice is conditionally complete in the terminology of Birkoff [1] (for a proof see Riesz [4]). That is, every subset $\mathfrak{S} \subset \mathfrak{F}$, which is bounded above, has a least upper bound F_0, such that $F_0 \geqslant F$ for every $F \in \mathfrak{S}$, but, if $F_1(E) < F_0(E)$ for at least one set $E \in \mathfrak{B}$, then $F_1 \ngeqslant F$ for at least one $F \in \mathfrak{S}$.

Following Bourbaki, [2], we call a subset $\mathfrak{S}$ of $\mathfrak{F}$, a *band*, if:

(i) it is a vector subspace of $\mathfrak{F}$;

(ii) if $G \in \mathfrak{S}$, then $F \in \mathfrak{S}$ for every $F \in \mathfrak{F}$ such that $|F| \leqslant |G|$;

(iii) $\mathfrak{S}$ is a sublattice of $\mathfrak{F}$ which is conditionally complete; i.e. every subset $\mathfrak{T} \subset \mathfrak{S}$ which is bounded above has its least upper bound in $\mathfrak{F}$ lying in $\mathfrak{S}$.

Two set functions $F_1, F_2 \in \mathfrak{F}$ are said to be *complementary*, if $|F_1| \cap |F_2| = \emptyset$, the null function. For example, the positive and negative components F^+ and F^- of a set function $F \in \mathfrak{F}$ are complementary. We need

THEOREM A. *If $\mathfrak{T} \subset \mathfrak{F}$ and $\mathcal{S}$ is the class of set functions $F \in \mathfrak{F}$ which are complementary to every element of $\mathfrak{T}$, then $\mathcal{S}$ forms a band in $\mathfrak{F}$. If $\mathfrak{T}$ is also a band, then $\mathfrak{T}$ consists of all the set functions $F \in \mathfrak{F}$ which are complementary to every element of $\mathcal{S}$.*

This theorem follows almost immediately from the definitions; for a proof see N. Bourbaki [2]. If $\mathcal{S}$ and $\mathfrak{T}$ are bands in $\mathfrak{F}$ such that $\mathfrak{T}$ consists of all the set functions complementary to every element of $\mathcal{S}$ we say that $\mathcal{S}$ and $\mathfrak{T}$ are *complementary bands* in $\mathfrak{F}$.

THEOREM B. *Given two complementary bands $\mathcal{S}_1$, $\mathcal{S}_2$ in $\mathfrak{F}$ each $F \in \mathfrak{F}$ can be expressed uniquely as*

$$F = F_1 + F_2,$$

where $F_1 \in \mathcal{S}_1$, $F_2 \in \mathcal{S}_2$, and $|F| = |F_1| + |F_2|$.

This is essentially due to Riesz [4].

THEOREM C. *Suppose μ is a measure defined for some Borel field $\mathcal{Q} \supset \mathcal{B}$ of subsets of I_0 and I_0 has σ-finite μ-measure, then $F \in \mathfrak{F}$ is absolutely continuous with respect to μ, if, and only if, there is a μ-measurable point function $f(x)$ such that*

$$F(E) = \int_E f(x)\, d\mu$$

for every $E \in \mathcal{B}$.

This is the theorem of Radon-Nikodym expressed in our terminology.

3. Decomposition with respect to a Hausdorff measure

Suppose that $h(t)$ is a fixed Hausdorff measure function. We call a set function F of $\mathfrak{F}$ h-continuous, or absolutely continuous with respect to the corresponding h-measure, if $F(E) = 0$ for every E of $\mathcal{B}$ with $h\text{-}m(E) = 0$; and we use $\mathcal{C}(h)$ to denote the set of all such set functions. Similarly we call a set function F of $\mathfrak{F}$ strongly h-continuous, if $F(E) = 0$ for every set E of $\mathcal{B}$ with a σ-finite h-measure (1); and we use $\mathcal{C}^*(h)$ to denote the set of all such set functions. Note that clearly $\mathcal{C}^*(h) \subset \mathcal{C}(h)$.

We say that a set function F of $\mathfrak{F}$ is h-singular, or singular with respect to the h-measure, if there is a set E_0 of $\mathcal{B}$ with $h\text{-}m(E_0) = 0$ such that

$$F(E) = F(E \cap E_0) \tag{2}$$

for all E of $\mathcal{B}$; we use $\mathcal{S}(h)$ to denote the set of all h-singular set functions. The equation

(1) Note, however, that this condition is automatically satisfied, if $F(E) = 0$, for every set E of $\mathcal{B}$ with a finite h-measure.

(2) shows that the h-singular set function F is concentrated on the set E_0 of zero h-measure. Further we say that a set function F of $\mathfrak{F}$ is almost h-singular, if there is a set E_0 in $\mathcal{B}$ of σ-finite h-measure, such that

$$F(E) = F(E \cap E_0),$$

for each E in $\mathcal{B}$; and we use $\mathcal{S}^*(h)$ to denote this set of almost singular set functions. Such almost singular set functions are concentrated on sets of σ-finite h-measure. Note that clearly $\mathcal{S}^*(h) \supset \mathcal{S}(h)$.

Our aim now is to prove that $\mathcal{C}(h)$, $\mathcal{S}(h)$ and $\mathcal{C}^*(h)$, $\mathcal{S}^*(h)$ form pairs of complementary bands in $\mathfrak{F}$. We find it convenient to prove a more general result about bands generated in this sort of way and obtain the particular bands as a special case.

LEMMA 1. *Let $\mathcal{A}$ be a system of sets of $\mathcal{B}$ with the following properties:*

(i) *if $E_1 \in \mathcal{A}$ and $E_2 \in \mathcal{B}$ then $E_1 \cap E_2 \in \mathcal{A}$;*

(ii) *if $E_i \in \mathcal{A}$, $i = 1, 2, 3, \ldots$ then $\bigcup_{i=1}^{\infty} E_i \in \mathcal{A}$.*

Then the set $\mathcal{S}_{\mathcal{A}}$ of all functions F of $\mathfrak{F}$, such that for some $E_0 \in \mathcal{A}$ we have

$$F(E) = F(E \cap E_0)$$

for all E in $\mathcal{B}$, is a band.

Proof. (i) If the set functions F_1 and F_2 are concentrated on sets E_1 and E_2 of $\mathcal{A}$, in the sense that

$$F_i(E) = F_i(E \cap E_i), \quad i = 1, 2,$$

for all E in $\mathcal{B}$, it is clear that for all real numbers λ_1, λ_2, the set function $\lambda_1 F_1 + \lambda_2 F_2$ is concentrated on the set $E_1 \cup E_2$ of $\mathcal{A}$. Hence $\mathcal{S}_{\mathcal{A}}$ is a vector subspace of $\mathfrak{F}$.

(ii) Let F be a set function of $\mathcal{S}_{\mathcal{A}}$ concentrated on a set E_0 of $\mathcal{A}$, and let G be a set function of $\mathfrak{F}$ satisfying $|G| \leqslant |F|$. Then, using the decomposition

$$F = F_+ - F_-, \qquad I_0 = P \cup N,$$

where F_+ and F_- are non-negative set functions of $\mathfrak{F}$ and P, N are Borel sets such that $F_+(N) = 0$, $F_-(P) = 0$, we see that, for each E in $\mathcal{B}$,

$$\begin{aligned} |F|(E) &= F_+(E) + F_-(E) \\ &= F(E \cap P) - F(E \cap N) \\ &= F(E \cap P \cap E_0) - F(E \cap N \cap E_0) \\ &= |F|(E \cap E_0). \end{aligned}$$

So $|F|$ is concentrated on E_0. Further, as $|G| \leqslant |F|$, for each E in $\mathcal{B}$ we have

$$|G(E) - G(E \cap E_0)| = |G(E \cap (I_0 - E_0))| \leqslant |F(E \cap (I_0 - E_0))| = 0.$$

Thus G is also concentrated on E_0 and so lies in S_A. This shows that S_A satisfies the second condition for a band.

(iii) Let $\mathcal{T}$ be a subset of S_A which has an upper bound in $\mathcal{F}$. Then there is set function, G say, of $\mathcal{F}$ such that for each F in $\mathcal{T}$ we have $F \leqslant G$. Then, if $T_1, T_2, \ldots, T_k$ is any finite set of elements of $\mathcal{T}$ we have

$$T_1 \cup T_2 \cup \ldots \cup T_k \leqslant G.$$

So we can define a real number h_0 by the formula

$$h_0 = \sup (T_1 \cup T_2 \cup \ldots \cup T_k)(I_0), \tag{3}$$

the supremum being taken over all finite sets of elements of $\mathcal{T}$. We can now choose a strictly increasing sequence of positive integers $n_1, n_2, \ldots$, and a sequence of set functions $T_1, T_2, \ldots$ of $\mathcal{T}$ such that the set function

$$H_k = T_1 \cup T_2 \cup \ldots \cup T_{n_k} \tag{4}$$

satisfies

$$H_k(I_0) > h_0 - 2^{-k}, \quad k = 1, 2, \ldots. \tag{5}$$

Now, as

$$|H_k| \leqslant |T_1| + |T_2| + \cdots + |T_{n_k}|,$$

it follows from (i) and (ii) above that $H_k \in S_A$. But, also, as $T \leqslant G$ for each T in $\mathcal{T}$, we have

$$H_k = T_1 \cup T_2 \cup \ldots \cup T_{n_k} \leqslant G.$$

So, for each E of $\mathcal{B}$, we have

$$H_1(E) \leqslant H_2(E) \leqslant H_3(E) \leqslant \ldots \leqslant G(E).$$

Hence we can define a set function H by the formula

$$H(E) = \lim_{k \to \infty} H_k(E). \tag{6}$$

Since, for each k, the function H_k is less than or equal to the least upper bound of $\mathcal{T}$, this ensures that H is also less than or equal to this least upper bound. It follows without difficulty from the theory of dominated monotonic convergence that H is a completely additive set function having a finite value for each set E of $\mathcal{B}$. Thus $H \in \mathcal{F}$. Further, there will be sets $E_1, E_2, \ldots$ of $\mathcal{A}$ such that H_k is concentrated on E_k, for $k = 1, 2, \ldots$. Then the set

$$E_0 = \bigcup_{k=1}^{\infty} E_k$$

also belongs to $\mathcal{A}$. But, for each E of $\mathcal{B}$ we have, by (6),

$$\begin{aligned} H(E) &= \lim_{k\to\infty} H_k(E) \\ &= \lim_{k\to\infty} H_k(E \cap E_k) \\ &= \lim_{k\to\infty} H_k(E \cap E_k \cap E_0) \\ &= \lim_{k\to\infty} H_k(E \cap E_0) = H(E \cap E_0). \end{aligned}$$

Hence $H \in \mathcal{S}_A$. Since we have already noted that H is less than or equal to the least upper bound of the subset $\mathcal{T}$, in order to prove that H is the least upper bound of $\mathcal{T}$, it remains to show that $H \geqslant T$ for every $T \in \mathcal{T}$.

Now, for each E in $\mathcal{B}$, we have

$$\begin{aligned} H(E) - H_k(E) &= \lim_{h\to\infty} \{H_h(E) - H_k(E)\} \\ &= \lim_{h\to\infty} \{H_h(I_0) - H_k(I_0)\} - \lim_{h\to\infty} \{H_h(CE) - H_k(CE)\} \\ &\leqslant H(I_0) - H_k(I_0), \end{aligned}$$

since, by (4), $H_k \leqslant H_h$ when $h > k$. Using (5), it follows that

$$H(E) - H_k(E) \leqslant 2^{-k} \quad (k = 1, 2, \ldots). \tag{7}$$

Suppose that there is a set function T_0 of $\mathcal{T}$, which does not satisfy $T_0 \leqslant H$. Then there is a set R_0 of $\mathcal{B}$ and a positive integer k such that

$$T_0(R_0) - H(R_0) > 2^{-k}.$$

Hence
$$(T_0 \cup H_k)(R_0) \geqslant T_0(R_0) > H(R_0) + 2^{-k}. \tag{8}$$

But, by (7),
$$H_k(CR_0) - H(CR_0) \geqslant -2^{-k},$$

so that
$$(T_0 \cup H_k)(CR_0) \geqslant H_k(CR_0) \geqslant H(CR_0) - 2^{-k}. \tag{9}$$

Adding (8) and (9) we obtain

$$(T_0 \cup H_k)(I_0) > H(I_0) = h_0,$$

or
$$(T_0 \cup T_1 \cup \ldots \cup T_{n_k})(I_0) > h_0.$$

This contradicts (3). Consequently there can be no such function T_0 in $\mathcal{T}$ and so H is the least upper bound for $\mathcal{T}$ and lies in $\mathcal{S}_A$. Hence $\mathcal{S}_A$ is a band.

LEMMA 2. *Let* $\mathcal{A}$ *be a system of sets of* $\mathcal{B}$ *with the properties:*

(i) *if* $E_1 \in \mathcal{A}$ *and* $E_2 \in \mathcal{B}$ *then* $E_1 \cap E_2 \in \mathcal{A}$;

(ii) *if* E_1, $E_2 \in \mathcal{A}$ *then* $E_1 \cup E_2 \in \mathcal{A}$.

The set $\mathcal{C}_A$ *of all* F *of* $\mathfrak{J}$ *satisfying* $F(E) = 0$ *for each* E *of* $\mathcal{A}$, *is a band.*

LEMMA(1) 3. *If* $\mathcal{A}$ *is a system of sets of* $\mathcal{B}$ *satisfying the conditions of Lemma* 1 (*and*, a fortiori, *of Lemma* 2) *then* $\mathcal{C}_A$ *and* $\mathcal{S}_A$ *are complementary bands in* $\mathfrak{J}$.

Lemma 2 can be proved directly by a simpler version of the proof of Lemma 1 (conditional completeness is easier to prove). However, it is easier to deduce Lemma 2 from Lemma 3.

Proof of Lemma 3. First suppose that $F \in \mathcal{C}_A$ and $T \in \mathcal{S}_A$. Then for some $E_0 \in \mathcal{A}$ we have

$$|T|(E) = |T|(E \cap E_0),$$

for all E in $\mathcal{B}$. Hence for each E in $\mathcal{B}$,

$$\begin{aligned}(|F| \cap |T|)(E) &= (|F| \cap |T|)(E \cap E_0) + (|F| \cap |T|)(E \cap (I_0 - E_0)) \\ &\leqslant |F|(E \cap E_0) + |T|(E \cap (I_0 - E_0)) = 0.\end{aligned}$$

Thus $F \in \mathcal{C}_A$ is complementary to every element T in the band $\mathcal{S}_A$.

Suppose now that there is some F in $\mathfrak{J}$, not in $\mathcal{C}_A$, which is nevertheless complementary to every element of $\mathcal{S}_A$. Then there is $E_0 \in \mathcal{A}$ such that $|F|(E_0) > 0$. Define a set function T by the condition

$$T(E) = |F|(E \cap E_0)$$

for all E in $\mathcal{B}$. Clearly $T \in \mathcal{S}_A$ as $E_0 \in \mathcal{A}$. But, as

$$|F| \geqslant T,$$

we have

$$|F| \cap |T| = |F| \cap T = T$$

and so

$$|F| \cap |T|(I_0) = T(I_0) = |F|(E_0) > 0.$$

This is contrary to the supposition that F is complementary to each element of $\mathcal{S}_A$. Hence each element complementary to each element of $\mathcal{S}_A$ belongs to $\mathcal{C}_A$. By theorem A it follows that $\mathcal{C}_A$ is the band complementary to $\mathcal{S}_A$.

(1) *Note added in Proof.* Results which are essentially equivalent to our Lemma 3 have been recently obtained by somewhat different methods by E. T. Mickle and T. Rado (*Rendiconti del Circolo Mat. di Palermo*, Serie 2, 7 (1958), 5—33). They use their results for a different purpose.

COROLLARY 1. *The sets $\mathcal{C}(h)$ and $\mathcal{S}(h)$ are complementary bands of $\mathcal{F}$.*

Proof. The result follows from the lemma on taking $\mathcal{A}$ to be the system of all sets E of $\mathcal{B}$ with $h\text{-}m(E) = 0$.

COROLLARY 2. *The sets $\mathcal{C}^*(h)$ and $\mathcal{S}^*(h)$ are complementary bands of $\mathcal{F}$.*

Proof. The result follows from the lemma on taking $\mathcal{A}$ to be the system of all sets E of $\mathcal{B}$ with σ-finite h-measure.

Proof of Lemma 2. Let $\mathcal{A}$ be a system of sets of $\mathcal{B}$ with the properties (i) and (ii). Let $\mathcal{A}^*$ be the system of all sets which can be expressed as countable unions of sets of the system $\mathcal{A}$. Then it is easy to verify that the system $\mathcal{A}^*$ satisfies the conditions (i) and (ii) of Lemma 1. So, by Lemma 3, the set functions of $\mathcal{C}_{\mathcal{A}^*}$ form a band. But a set function F of $\mathcal{F}$ satisfies the condition $F(E) = 0$ for each E of $\mathcal{A}$, if, and only if, it satisfies the condition $F(E) = 0$ for each E of $\mathcal{A}^*$. Hence $\mathcal{C}_{\mathcal{A}}$ coincides with $\mathcal{C}_{\mathcal{A}^*}$ and so is a band.

We can now decompose any F of $\mathcal{F}$ with respect to a given Hausdorff measure.

THEOREM 1. *Given any set function $F \in \mathcal{F}$ and a Hausdorff measure function $h(t)$, there is a unique decomposition*

$$F = F_1 + F_2 + F_3,$$

where F_1 is strongly h-continuous, where F_2 is h-continuous and almost h-singular, and where F_3 is h-singular.

If the component F_2 does not vanish, it has a representation in the form

$$F_2(E) = \int_{E \cap S} f(x)\, d\, h\text{-}m(x),$$

where S is a Borel set of positive σ-finite h-measure, and f is a h-measurable point function not taking the value zero on S.

Proof. By Theorem B, and Corollary 2 to Lemma 3, the set function F has a unique decomposition of the form $F = F_0 + F_1$, where $F_0 \in \mathcal{S}^*(h)$ and $F_1 \in \mathcal{C}^*(h)$.

Applying Theorem B and corollary 1 of Lemma 3, the set function F_0 has a unique decomposition of the form $F_0 = F_2 + F_3$, where $F_2 \in \mathcal{C}(h)$, $F_3 \in \mathcal{S}(h)$. Since $|F_2| \leqslant |F_0|$, F_2 must belong to $\mathcal{S}^*(h) \cap \mathcal{C}(h)$. Consequently F has a unique decomposition of the required form.

Since $F_2 \in \mathcal{S}^*(h)$ there is a set T of σ-finite h-measure such that

$$F_2(E) = F_2(E \cap T)$$

for all E of $\mathcal{B}$. We define a measure ν by the equation

$$\nu(E) = h\text{-}m(E \cap T)$$

for all E of $\mathcal{B}$. Then ν is a measure over I_0 which is σ-finite, since T has σ-finite h-measure. Further the class of ν-measurable sets contains $\mathcal{B}$. Now, if E is any set of $\mathcal{B}$ with $\nu(E) = 0$, then

$$F_2(E) = F_2(E \cap T) = 0,$$

since $h\text{-}m(E \cap T) = \nu(E) = 0$, while $F_2 \in \mathcal{C}(h)$. Thus F_2 is absolutely continuous with respect to ν. Hence by the Radon–Nikodym theorem (Theorem C) there is a function $f = f(x)$ defined for x in I_0, which is ν-measurable and ν-integrable over I_0, and which satisfies

$$F_2(E) = \int_E f(x)\, d\nu,$$

for all E of $\mathcal{B}$. Since $F_2(E) = F_2(E \cap T)$ for all E of $\mathcal{B}$, and ν corresponds to h-measure for Borel subsets of T, we can write this in the form

$$F_2(E) = \int_{E \cap T} f(x)\, d\,h\text{-}m(x).$$

Let R be the set of points of T for which $f(x) = 0$, and let $Q \supset R$ be such that $h\text{-}m(Q) = h\text{-}m(R)$ and $Q \in \mathcal{B}$. Now put $S = T - Q$. Then $f(x) \neq 0$ for x in S, $S \in \mathcal{B}$, and S has σ-finite h-meaure. Further $\int_{E \cap Q} f(x)\,d\,h\text{-}m(x) = 0$ and so

$$F_2(E) = \int_{E \cap S} f(x)\, d\,h\text{-}m(x)$$

as required.

It is interesting to note that in the case when $h(t) = t^n$, every set E contained in I_0 has finite h-measure, so that the decomposition reduces to the Lebesgue decomposition into the absolutely continuous set function F_2 and the singular function F_3. Further, in the limiting case when the h-measure of a set is taken to be the number of points it contains, the set function F_3, being concentrated on a set with no point, is zero and the decomposition reduces to the decomposition into the diffuse set function F_1 and the atomic set function F_2.

Theorem 1 is stated as a decomposition of a set function F with respect to a Hausdorff measure. This is what is relevant in the context of the present paper for we want to analyse the set function into "uniform" components; and Hausdorff measures are invariant under translations and rotations in Euclidean space. However, the theorem is true for a decomposition with respect to any measure in Euclidean space which is regular. Thus it forms a generalisation of (i) the Lebesgue decomposition theorem, and (ii) the Radon-Nikodym theorem to the case of measures which are non σ-finite.

THEOREM 1A. *Let μ be any measure defined for Borel sets in Euclidean n-space, and suppose F is a completely additive set function also defined for Borel sets. Then there is a unique decomposition*

$$F = F_1 + F_2 + F_3,$$

where F_1 vanishes for any set of σ-finite μ-measure, F_2 is absolutely continuous with respect to μ-measure and concentrated on a Borel set of σ-finite μ-measure, and F_3 is singular with respect to μ-measure. If the component F_2 does not vanish, it has a representation in the form

$$F_2(E) = \int_{E \cap S} f(x)\, d\mu,$$

where S is a Borel set of positive σ-finite μ-measure and f is a μ-measurable point function not taking the value zero on S.

This theorem can be proved by the same method as was used in the proof of Theorem 1.

4. The decomposition spectrum with respect to measures of fractional dimension

In this section we work only with the fractional dimension measure Λ^α, where $0 \leqslant \alpha \leqslant n$; but instead of confining our attention to a single h-measure at a time we work with the whole system of these fractional dimension measures, and we need to introduce some further concepts.

We call a set function F α-dimension-continuous, if $F(E) = 0$ for every E of $\mathcal{B}$ which has dimension less than α; and we use $\mathcal{C}_\alpha$ to denote the set of all such set functions. Similarly we call a set function F of $\mathcal{J}$ strongly α-dimension-continuous, if $F(E) = 0$ for every set E of $\mathcal{B}$ of dimension less than or equal to α; and we use $\mathcal{C}_\alpha^*$ to denote the set of all such functions. Note that $\mathcal{C}_\alpha^* \subset \mathcal{C}_\alpha$.

We say that a set function F of $\mathcal{J}$ is α-dimension-singular, if there is a set E_0 of $\mathcal{B}$, which can be expressed as the union of a countable number of sets E_i of $\mathcal{B}$, each of dimension less than α, and which is such that

$$F(E) = F(E \cap E_0), \tag{10}$$

for all E of $\mathcal{B}$. We use $\mathcal{S}_\alpha$ to denote the set of all α-dimension-singular set functions. Further, we say that a set function F of $\mathcal{J}$ is almost α-dimension-singular, if there is a set E_0 in $\mathcal{B}$ of dimension less than or equal to α, such that (10) holds for each E in $\mathcal{B}$; and we use $\mathcal{S}_\alpha^*$ to denote the set of almost α-dimension-singular set functions. Note that we have $\mathcal{S}_\alpha^* \supset \mathcal{S}_\alpha$.

These concepts are clearly closely related to the corresponding concepts introduced in the last section. The situation is conveniently summarized by the inclusion relationships:

$$\mathcal{C}_\alpha^* \subset \mathcal{C}^*(t^\alpha) \subset \mathcal{C}(t^\alpha) \subset \mathcal{C}_\alpha;$$

$$\mathcal{S}_\alpha^* \supset \mathcal{S}^*(t^\alpha) \supset \mathcal{S}(t^\alpha) \supset \mathcal{S}_\alpha.$$

LEMMA 4. *If $0 \leqslant \alpha \leqslant n$ the pairs $\mathcal{C}_\alpha$, $\mathcal{S}_\alpha$ and $\mathcal{C}_\alpha^*$, $\mathcal{S}_\alpha^*$ are pairs of complementary bands in $\mathfrak{J}$.*

Proof. First take $\mathcal{A}$ to be the system of all sets E of $\mathcal{B}$ which can be expressed as a countable union of sets E_i of $\mathcal{B}$ of dimension less than α. It follows without difficulty from Lemma 3 that $\mathcal{C}_\alpha$ and $\mathcal{S}_\alpha$ are complementary bands in $\mathfrak{J}$.

Secondly take $\mathcal{A}$ to be the system of all sets E of $\mathcal{B}$ which are of dimension less than or equal to α. It follows without difficulty from Lemma 3 that $\mathcal{C}_\alpha^*$ and $\mathcal{S}_\alpha^*$ are complementary bands in $\mathfrak{J}$.

Remark. When $\alpha = 0$ we have $\mathcal{C}_0 = \mathfrak{J}$ and $\mathcal{S}_0 = \mathfrak{Z}$, where $\mathfrak{Z}$ is the system of set functions containing only the null function. When $\alpha = n$ we have $\mathcal{C}_n^* = \mathfrak{Z}$ and $\mathcal{S}_n^* = \mathfrak{J}$.

We also need to introduce the idea of a set function having an exact dimension. We say that a function F of $\mathfrak{J}$ has the exact dimension α, if F is α-dimension continuous and almost α-dimension singular. Such a function F is concentrated on a set of dimension α and $F(E) = 0$ for each set E of $\mathcal{B}$ of dimension less than α.

Notice that if we apply Theorem 1 to $F \in \mathfrak{J}$ with $h(t) = t^\alpha$, $(0 < \alpha < n)$ the component F_2 obtained will have the exact dimension α. However, the class of set functions of exact dimension α is wider than the class of components F_2 arising in this way, because $\mathcal{C}_\alpha^*$ is a proper subset of $\mathcal{C}^*(t^\alpha)$ and $\mathcal{C}(t^\alpha)$ is a proper subset of $\mathcal{C}_\alpha$.

THEOREM 2. *Given any set function $F \in \mathfrak{J}$, for each α with $0 \leqslant \alpha \leqslant n$, there is a unique decomposition*

$$F = F_1^{(\alpha)} + F_2^{(\alpha)} + F_3^{(\alpha)},$$

where $F_1^{(\alpha)}$ is strongly α-dimension-continuous, where $F_2^{(\alpha)}$ is of exact dimension α, and where $F_3^{(\alpha)}$ is α-dimension-singular.

Further for each set E of $\mathcal{B}$ the function $|F_1^{(\alpha)}|(E)$, is monotonic decreasing for $0 \leqslant \alpha \leqslant n$, while the function $|F_3^{(\alpha)}|(E)$, is monotonic increasing for $0 \leqslant \alpha \leqslant n$.

Proof. The first part of the theorem follows from Theorem B and Lemma 4, just as the first part of Theorem 1 follows from Theorem B and the corollaries to Lemma 3. Note that the corresponding decomposition for the set function $|F|$ is given by the identity:

$$|F| = |F_1^{(\alpha)}| + |F_2^{(\alpha)}| + |F_3^{(\alpha)}|.$$

Now suppose that $0 \leqslant \alpha < \beta \leqslant n$. Then $|F_1^{(\beta)}|$ is strongly α-dimension-continuous.

Further $|F_2^{(\beta)}|+|F_3^{(\beta)}|$ has a decomposition

$$|F_2^{(\beta)}|+|F_3^{(\beta)}|=G_1^{(\alpha)}+G_2^{(\alpha)}+G_3^{(\alpha)},$$

where $G_1^{(\alpha)}$ is non-negative and strongly α-dimension-continuous, where $G_2^{(\alpha)}$ is non-negative, α-dimension-continuous and almost α-dimension-singular, and where $G_3^{(\alpha)}$ is non-negative and α-dimension-singular. Hence by the uniqueness of the decomposition we have

$$|F_1^{(\alpha)}|=|F_1^{(\beta)}|+G_1^{(\alpha)},$$
$$|F_2^{(\alpha)}|=G_2^{(\alpha)},$$
$$|F_3^{(\alpha)}|=G_3^{(\alpha)}.$$

Since $G_1^{(\alpha)}$ is a non-negative set function, this shows that

$$|F_1^{(\alpha)}|(E)\geqslant|F_1^{(\beta)}|(E),$$

for each E of $\mathfrak{B}$. Hence $|F_1^{(\alpha)}|(E)$ is a monotonic decreasing function of α for $0\leqslant\alpha\leqslant n$. A similar proof shows that $|F_3^{(\alpha)}|(E)$ is monotonic increasing.

The following lemma gives some further information about the decomposition of this theorem.

LEMMA 5. (i) *If $F\in\mathfrak{F}$ is decomposed by Theorem 2, then $|F_1^{(\alpha)}|(I_0)$ is continuous on the right as a function of α for $0\leqslant\alpha<n$, and $|F_3^{(\alpha)}|(I_0)$ is continuous on the left as a function of α for $0<\alpha\leqslant n$.*

(ii) *Discontinuities of the two functions $|F_1^{(\alpha)}|(I_0)$, $|F_3^{(\alpha)}|(I_0)$ occur for the same values of α in the range $0<\alpha<n$ and $|F_2^{(\gamma)}|(I_0)$ is the size of the discontinuity of either function at $\alpha=\gamma$.*

Proof. (i) $|F_3^{(\alpha)}|$ is α-dimension-singular. Hence, if $\alpha>0$, there is a set $E_0\in\mathfrak{B}$ such that $E_0=\bigcup_{i=1}^{\infty}E_i$, $\dim_B(E_i)<\alpha$ $(i=1,2,3,\ldots)$, and $|F_3^{(\alpha)}|(E)=|F_3^{(\alpha)}|(E\cap E_0)$ for every $E\in\mathfrak{B}$. We may assume that the sets $E_i\in\mathfrak{B}$ are disjoint, so that

$$|F_3^{(\alpha)}|(I_0)=\sum_{i=1}^{\infty}|F_3^{(\alpha)}|(E_i). \tag{11}$$

Given $\varepsilon>0$, we can choose $\delta>0$ such that

$$\sum_{\alpha-\delta\leqslant\dim_B(E_i)<\alpha}|F_3^{(\alpha)}|(E_i)<\varepsilon. \tag{12}$$

Then, if $\alpha>\beta>\alpha-\delta$,

$$|F_3^{(\beta)}|(I_0)\geqslant|F_3^{(\alpha-\delta)}|(I_0)\geqslant\sum_{\dim_B(E_i)<\alpha-\delta}|F_3^{(\alpha)}|(E_i)$$

and therefore, by (11) and (12),

$$|F_3^{(\alpha)}|(I_0) \geq |F_3^{(\beta)}|(I_0) \geq |F_3^{(\alpha)}|(I_0) - \varepsilon.$$

This proves that $|F_3^{(\alpha)}|(I_0)$ is continuous on the left for $0 < \alpha \leq n$. A similar proof shows that $|F_1^{(\alpha)}|(I_0)$ is continuous on the right for $0 \leq \alpha < n$.

(ii) Now suppose $0 < \gamma \leq n$ and write

$$d(\gamma) = \lim_{h \to 0+} \{|F_1^{(\gamma-h)}|(I_0) - |F_1^{(\gamma)}|(I_0)\}. \tag{13}$$

Now
$$\{|F_1^{(\alpha)}|(I_0) + |F_2^{(\alpha)}|(I_0)\} + |F_3^{(\alpha)}|(I_0) = |F|(I_0)$$

is independent of α for $0 \leq \alpha \leq n$, so that $|F_1^{(\alpha)}|(I_0) + |F_2^{(\alpha)}|(I_0)$ is monotonic decreasing and continuous on the left as a function of α. Hence

$$|F_2^{(\gamma)}|(I_0) \geq d(\gamma).$$

But
$$|F_1^{(\gamma-h)}|(I_0) \geq |F_1^{(\gamma)}|(I_0) + |F_2^{(\gamma)}|(I_0)$$

so that
$$|F_2^{(\gamma)}|(I_0) \leq d(\gamma).$$

This proves that
$$|F_2^{(\gamma)}|(I_0) = d(\gamma) \text{ for } 0 < \gamma \leq n. \tag{14}$$

Similary it can be proved that, for $0 \leq \gamma < n$,

$$|F_2^{(\gamma)}|(I_0) = \lim_{h \to 0+} \{|F_3^{(\gamma+h)}|(I_0) - |F_3^{(\gamma)}|(I_0)\}. \tag{15}$$

(13), (14) and (15) together show that, for $0 < \alpha < n$, the discontinuities of $|F_1^{(\alpha)}|(I_0)$, $|F_3^{(\alpha)}|(I_0)$ coincide in position and size.

COROLLARY. *For any $E \in \mathfrak{B}$, the real function $F_1^{(\alpha)}(E)$ is continuous on the right, and $F_3^{(\alpha)}(E)$ is continuous on the left. The discontinuities of these two functions coincide and have the same magnitude.*

Proof. The corollary follows from the Lemma applied to the positive and negative parts of the function $G \in \mathfrak{J}$ given by $G(Q) = F(E \cap Q)$ for any $Q \in \mathfrak{B}$.

By means of Theorem 2 it is possible to pick out some components of a set function $F \in \mathfrak{J}$ with an exact real dimension. These are the set functions $|F_2^{(\alpha)}|$ which do not vanish. To describe what is left when these discrete components have been removed we need the following.

DEFINITION. *A set function $F \in \mathfrak{J}$ is said to have a diffuse real dimension spectrum, if the component $F_2^{(\alpha)}$ of Theorem 2 vanishes identically for each α with $0 \leq \alpha \leq n$.*

The following lemma gives insight into the structure of these set functions.

LEMMA 6. *For a function $F \in \mathfrak{J}$ to have a diffuse real dimension spectrum, either of the following conditions is both necessary and sufficient.*

1. *If $F_1^{(\alpha)}$, $F_3^{(\alpha)}$ are the components defined by Theorem 2, then $|F_1^{(\alpha)}|(I_0)$, $|F_3^{(\alpha)}|(I_0)$, are each continuous functions of α for $0 \leqslant \alpha \leqslant n$.*

2. *There is no function $G \in \mathfrak{J}$ with an exact real dimension such that $0 < G \leqslant |F|$.*

Proof. 1. The necessity and sufficiency of the continuity of $|F_1^{(\alpha)}|(I_0)$, $|F_3^{(\alpha)}|(I_0)$ for $0 < \alpha < n$ follows from Lemma 5 (ii). The special case of the end points follows from (13) and (15).

2. If this condition is satisfied then $|F_2^{(\alpha)}|$ must be null for every α, $0 \leqslant \alpha \leqslant n$. Conversely suppose there is a non-null function $G \in \mathfrak{J}$ with exact real dimension γ and $0 < G \leqslant |F|$. Then $0 < G \leqslant |F_2^{(\gamma)}|$ and the function $F \in \mathfrak{J}$ has not got a diffuse real dimension spectrum.

We can now state the main theorem of this section which gives a decomposition of $F \in \mathfrak{J}$ into a "discrete" spectrum and a diffuse spectrum.

THEOREM 3. *Given any set function F of $\mathfrak{J}$ there is a finite or enumerable sequence $\alpha_1, \alpha_2, \ldots$ with $0 \leqslant \alpha_i \leqslant n$ and $\alpha_i \neq \alpha_j$ when $i \neq j$ and a decomposition*

$$F = F^{(d)} + F^{(\alpha_1)} + F^{(\alpha_2)} + \cdots,$$

where $F^{(d)}$ has a diffuse real dimension spectrum, and $F^{(\alpha_i)}$ is a set function of exact dimension α_i for $i = 1, 2, \ldots$. The set of numbers α_i, and the decomposition (apart from the order of its terms) are uniquely determined by F.

Proof. Let S be the set of real numbers α such that in the decomposition of Theorem 2, $|F_2^{(\alpha)}|(I_0) \neq 0$. Then S is finite or enumerable, for

$$\sum_{\alpha \in S} |F_2^{(\alpha)}|(I_0) \leqslant |F|(I_0) < +\infty.$$

Let $\alpha_1, \alpha_2, \ldots$ be any enumeration of the elements of S such that $\alpha_i \neq \alpha_j$ when $i \neq j$. Then for each $E \in \mathfrak{B}$,

$$G(E) = \Sigma F_2^{(\alpha_i)}(E) \tag{16}$$

is defined, and $G \in \mathfrak{J}$, since the series in (16) is absolutely and uniformly convergent. Let $F^{(d)} = F - G$. Then $F^{(d)} \in \mathfrak{J}$. Now, if G is decomposed by Theorem 2,

$$G_2^{(\alpha)} = F_2^{(\alpha_i)}, \quad \text{when } \alpha = \alpha_i,$$

$$G_2^{(\alpha)} = 0, \quad \text{when } \alpha \neq \alpha_i, \text{ any } i.$$

Since the decomposition of Theorem 2 is unique, it follows that:

(i) if $\alpha = \alpha_i$, $i = 1, 2, \ldots$

$$F_2^{(d)(\alpha)} = F_2^{(\alpha_i)} - F_2^{(\alpha_i)} = 0;$$

(ii) if $\alpha \neq \alpha_i$, any i,

$$F_2^{(d)(\alpha)} = F_2^{(\alpha)} - G_2^{(\alpha)} = 0 - 0 = 0.$$

Thus the set function $F^{(d)}$ has a diffuse real dimension spectrum. This proves that the decomposition of Theorem 3 is possible.

To prove that the decomposition is unique, let $F = G^{(d)} + G^{(\beta_1)} + G^{(\beta_2)} + \cdots$, where $G^{(d)} \in \mathfrak{J}$ and has a diffuse real dimension spectrum, and $G^{(\beta_i)} \in \mathfrak{J}$ and has the exact real dimension β_i $(i = 1, 2, \ldots)$. Let $\varphi(F, \alpha) = |F_3^{(\alpha)}|(I_0)$, where $0 \leqslant \alpha \leqslant n$, $F \in \mathfrak{J}$ and $F_3^{(\alpha)}$ is defined by Theorem 2. Then

$$\varphi(F, \alpha) = \varphi(F^{(d)}, \alpha) + \sum_i \varphi(F^{(\alpha_i)}, \alpha) \tag{17}$$

$$= \varphi(G^{(d)}, \alpha) + \sum_i \varphi(G^{(\beta_i)}, \alpha). \tag{18}$$

The functions $\varphi(F^{(\alpha_i)}, \alpha)$ are monotonic functions which are constant except for a single step at α_i, so that (17) represents a decomposition of the monotonic function $\varphi(F, \alpha)$ into a continuous monotonic function $\varphi(F^{(d)}, \alpha)$ and a step function $\sum_i \varphi(F^{(\alpha_i)}, \alpha)$. (18) is another such decomposition. Hence

$$\varphi(F^{(d)}, \alpha) = \varphi(G^{(d)}, \alpha) + k,$$

where k is a constant. But when $\alpha = 0$, $\varphi(F^{(d)}, \alpha) = 0 = \varphi(G^{(d)}, \alpha)$ so that $k = 0$. It follows that the functions $\varphi(F^{(\alpha_i)}, \alpha)$, are the same as $\varphi(F^{(\beta_i)}, \alpha)$, though the order may be different. Rename the $G^{(\beta_i)}$ functions $G^{(\alpha_i)}$.

Thus it is sufficient to prove that for each i,

$$G^{(\alpha_i)} = F^{(\alpha_i)}.$$

Then we know that $F^{(\alpha_i)} = F_2^{(\alpha_i)}$. But $F_2^{(\alpha_i)}$ can be obtained by applying Theorem 2 to each of the functions $G^{(d)}, G^{(\alpha_j)}$ $(j = 1, 2, \ldots)$ and adding. The only one to make a non-null contribution is $G^{(\alpha_i)}$. Hence $F_2^{(\alpha_i)} = G^{(\alpha_i)}$, and the theorem is proved.

Remark. Each of the components $F^{(\alpha_i)}$ of exact dimension α_i of Theorem 3 may be decomposed by Theorem 1 with respect to the measure function t^{α_i}; and in some cases the three components will be non-null.

If $F \in \mathfrak{J}$ has an absolutely continuous component, then $F^{(n)}$ will be not null and when $F^{(n)}$ is decomposed by Theorem 1 using $h(t) = t^n$ the first component will be absolutely continuous in the Lebesgue sense. Similarly if $F \in \mathfrak{J}$ has an atomic component, $F^{(0)}$ will be not null and, when decomposed by Theorem 1A with respect to Λ^0-measure, will give a middle component which is the atomic component of F.

5. A set function which has a diffuse real dimension spectrum.

In Theorem 3 we hawe obtained a decomposition of any set function $F \in \mathfrak{F}$ into a function $F^{(d)}$ with a diffuse real dimension spectrum and a discrete spectrum of at most enumerably many set functions of exact real dimension. It is easy to see that the discrete spectrum can exist. For example, given different real numbers $\alpha_1, \alpha_2, \ldots, \alpha_n, \ldots, 0 < \alpha_i < n$, let E_i be a subset of I_0 such that $\Lambda^{\alpha_i}(E_i) = 2^{-i}$ $(i = 1, 2, \ldots)$, and define $F^{(\alpha_i)}(E) = \Lambda^{\alpha_i}(E \cap E_i)$ for each $E \in B$. Then if $F = \sum_{i=1}^{\infty} F^{(\alpha_i)}$, we have $F(I_0) = 1$, and Theorem 3 must decompose F into the components $F^{(\alpha_i)}$.

It is not trivial to show that the component $F^{(d)}$ need not be void. The object of the present section is to exhibit an example of a non zero set function $F^{(d)}$. Our example is in Euclidean 1-space, but obvious modifications would give an example in n-space. The method is to carry out suitable modifications to the Cantor ternary function, and define a Stieltjes measure F with respect to this function, defined for Borel subsets of $[0, 1]$. Clearly $F \in \mathfrak{F}$.

To obtain a Cantor type set of dimension α, $0 < \alpha < 1$, one proceeds as follows. Let P_0 be the closed interval $[0, 1]$. Obtain P_1 from P_0 by removing an open interval of length λ_α from the middle of $[0, 1]$ where λ_α satisfies

$$[\tfrac{1}{2}(1 - \lambda_\alpha)]^\alpha = \tfrac{1}{2};$$

that is,
$$\lambda_\alpha = 1 - 2^{1-\alpha^{-1}}. \tag{19}$$

If P_n has been defined and contains 2^n intervals of length $l_n = (\frac{1}{2})^{n/\alpha}$, P_{n+1} is obtained by removing from the middle of each interval of P_n an open interval of length $\lambda_\alpha l_n$. Finally put $P = \bigcap_{r=0}^{\infty} P_r$. Then P is a perfect set and $\Lambda^\alpha(P) = 1$. (This was first proved by Hausdorff [3].)

A function $\varphi_\alpha(x)$ is defined for x in $[0, 1] = I_0$ as follows. First put $\varphi_\alpha(x) = \frac{1}{2}$ on $(I_0 - P_1)$. Now $(P_1 - P_2)$ consists of 2 open intervals; define $\varphi_\alpha(x) = \frac{1}{4}$, $\varphi_\alpha(x) = \frac{3}{4}$ respectively on the left and right of these. Similarly $(P_{n-1} - P_n)$ consist of 2^{n-1} equal open intervals; order these from left to right and put $\varphi_\alpha(x) = 2^{-n}(2r - 1)$ in the rth one. This process will define $\varphi_\alpha(x)$ for all x in $(I_0 - P)$. Define it at points of P so that it is continuous for $0 \leqslant x \leqslant 1$. For any set $A \in \mathfrak{B}$, let $C_A(x)$ be the characteristic function of A and define

$$H_\alpha(A) = \int_0^1 C_A(x)\, d\varphi_\alpha(x),$$

where the integral on the right is a Lebesgue Stieltjes integral with respect to the monotonic function $\varphi_\alpha(x)$. It can be shown that H_α is a positive set function in $\mathfrak{J}$ of exact real dimension α and that

$$H_\alpha(A) = \Lambda^\alpha(P \cap A).$$

We do not need quite as much but we prove:

Lemma 7. *For $0<\alpha<1$, the function $\varphi_\alpha(x)$ satisfies $0 \leqslant \varphi_\alpha(x_2) - \varphi_\alpha(x_1) \leqslant K_\alpha(x_2 - x_1)^\alpha$, for all x_1, x_2 such that $0 \leqslant x_1 \leqslant x_2 \leqslant 1$, where K_α is a suitable finite constant.*

Proof. Since $\varphi_\alpha(x)$ is constant on each interval of $(I_0 - P)$ it is sufficient to prove the lemma when x_1, x_2 are two distinct points of P. Let n be the unique integer such that x_1, x_2 are in the same interval of P_n but in different intervals of P_{n+1}. Now if $[a, b]$ is one of the intervals of P_n we have $\varphi_\alpha(b) - \varphi_\alpha(a) = (\frac{1}{2})^n$ and it follows that

$$\varphi_\alpha(x_2) - \varphi_\alpha(x_1) \leqslant (\tfrac{1}{2})^n.$$

Also
$$x_2 - x_1 \geqslant \lambda_\alpha l_n = \lambda_\alpha (\tfrac{1}{2})^{n/\alpha}.$$

Thus
$$\varphi_\alpha(x_2) - \varphi_\alpha(x_1) \leqslant (\tfrac{1}{2})^n \leqslant \lambda_\alpha^{-\alpha}(x_2 - x_1)^\alpha,$$

and the lemma is proved with $K_\alpha = \lambda_\alpha^{-\alpha}$.

Remark. By taking considerably more care, the above lemma can be proved with $K_\alpha = 1$, and this result is clearly best possible.

Lemma 8. *If A is a Borel subset of I_0 such that $\Lambda^\alpha(A) = 0$, then*

$$H_\alpha(A) = \int_0^1 C_A(x)\, d\varphi_\alpha(x) = 0.$$

Note. This proves that $H_\alpha \in \mathsf{C}_\alpha$. Since P is a set of finite Λ^α-measure for which $H_\alpha(E) = H_\alpha(E \cap P)$ it follows that H_α is a set function of exact real dimension α. In fact more is true: if we apply Theorem 1 to the set function H_α we will obtain:

$$H_\alpha(E) = \int_{E \cap P} 1 \cdot d\Lambda^\alpha,$$

so that H_α is in a certain sense "spread very uniformly" over the set P.

Proof of lemma. Let A be any set $\subset I_0$ such that $\Lambda^\alpha(A)=0$. Then there exists an open set $G \supset A$ consisting of intervals (a_i, b_i) $i=1, 2, \ldots$ such that $\sum_i (b_i-a_i)^\alpha < \varepsilon$. Now

$$H_\alpha(A) \leqslant H_\alpha(G) = \sum_i \{\varphi_\alpha(b_i)-\varphi_\alpha(a_i)\} \leqslant K_\alpha \sum_i (b_i-a_i)^\alpha < K_\alpha \varepsilon.$$

Since ε is arbitrary, $H_\alpha(A)=0$, and the lemma is proved.

Given any set E in Euclidean n-space, we can define the dimension of E at the point x, denoted by $\dim(x, E)$, by the relation

$$\dim(x, E) = \lim_{r\to 0+} \dim_B(E \cap S(x, r)), \tag{20}$$

where $S(x, r)$ is the open sphere centre x and radius r.

The Cantor set P defined above has the same dimension α at each of its points. Our object now is to modify the definition of P so that it has a different dimension at each of its points. Note that, by (19), λ_α is a monotonic decreasing function of α for $0<\alpha<1$ and $\lim_{\alpha\to 1-} \lambda_\alpha = 0$, $\lim_{\alpha\to 0+} \lambda_\alpha = 1$. The value $\lambda_\alpha = \frac{1}{3}$ corresponds to the Cantor ternary set which has dimension $\alpha = \log 2/\log 3$.

Let $Q_0 = [0, 1]$. Remove from the middle of Q_0 an open interval of length $\lambda_{\frac{1}{2}}$. For x in this open interval put $\varphi(x)=\frac{1}{2}$. The remaining set Q_1 consists of closed intervals $I_1^{(1)}$, $I_2^{(1)}$ of lengths $l_1^{(1)}$, $l_2^{(1)}$ ordered from left to right. Remove from the middle of the interval $I_{(1)}^{(1)}$ an open interval of length $\lambda_{\frac{1}{4}} l_1^{(1)}$, and from the middle of $I_2^{(1)}$ an open interval of length $\lambda_{\frac{3}{4}} l_2^{(1)}$. These open intervals of x correspond to $\varphi(x)=\frac{1}{4}$, $\varphi(x)=\frac{3}{4}$ respectively.

Suppose now that Q_n has been defined and consists of the closed intervals $I_1^{(n)}, I_2^{(n)}, \ldots I_{2^n}^{(n)}$, where the lower suffices agree with the order of the intervals from left to right; and $\varphi(x)$ is already defined on $Q_0 - Q_n$. From the middle of $I_r^{(n(}$ $(r=1, 2, \ldots, 2^n)$ of length $l_r^{(n)}$ remove an open interval of length $\lambda_\gamma l_r^{(n)}$, where

$$\gamma = \frac{r}{2^n} - \frac{1}{2^{n+1}},$$

and, for x in this open interval, put

$$\varphi(x) = \gamma = \frac{r}{2^n} - \frac{1}{2^{n+1}}.$$

The 2^{n+1} closed intervals formed in this way are the components of the set Q_{n+1}. Finally put $Q = \bigcap_{n=1}^{\infty} Q_n$, and complete the definition of the function $\varphi(x)$ at points of

Q so that it is continuous for $0 \leqslant x \leqslant 1$. This definition ensures that $\varphi(x)$ is monotonic and non-decreasing.

If A is any Borel subset of I_0, define

$$H(A) = \int_0^1 C_A(x)\, d\varphi(x).$$

Then $H(A)$ is a positive set function $\in \mathfrak{J}$ with $H(Q) = H(I_0) = 1$. Thus H is a singular set function in the Lebesgue sense, and it is also diffuse since $\varphi(x)$ is continuous. We now prove that $Q \in \mathfrak{B}$ and $H \in \mathfrak{J}$ satisfy the conditions of

THEOREM 4. (i) *There exists a perfect set $Q \subset I_0 = [0, 1]$ such that the dimension of Q at any two of its points is different unless the two points are end-points of the same interval of $I_0 - Q$.*

(ii) *There is a positive set function $H \in \mathfrak{J}$ concentrated on the set Q such that H has a diffuse real dimension spectrum.*

Proof. Let Q be the modified Cantor set defined above and $\varphi(x) = y$ the associated monotonic continuous function. The inverse function $x = \varphi^{-1}(y)$ is uniquely defined for $0 \leqslant y \leqslant 1$ except for the enumerable set of points $y = p \cdot 2^{-r}$ (p, r integers). Define it uniquely for all y with $0 \leqslant y \leqslant 1$ by making it continuous on the left. Thus for any y, $0 \leqslant y \leqslant 1$, $\varphi^{-1}(y)$ will be a point of the set Q.

For $0 < \alpha < 1$, let $x_\alpha = \varphi^{-1}(\alpha)$. If $I_r^{(n)}$ is a closed interval of Q_n such that $I_r^{(n)} \subset (0, x_\alpha)$ and m is the right hand end point of $I_r^{(n)}$, then we may write $\varphi(m) = \gamma < \beta < \alpha$. Now in defining $I_r^{(n)} \cap Q_{n+k}$ for $k = 1, 2, \ldots$, we removed from $I_r^{(n)}$ successively open intervals of length λ times the closed interval where always we have $\lambda \geqslant \lambda_\gamma$. Thus $I_r^{(n)} \cap Q_{n+k}$ will consist of 2^k closed intervals each of length less than

$$[\tfrac{1}{2}(1 - \lambda_\gamma)]^k = (\tfrac{1}{2})^{k/\gamma}.$$

Hence
$$\Lambda_*^\beta (I_r^{(n)} \cap Q) \leqslant \lim_{k \to \infty} 2^k (\tfrac{1}{2})^{k\beta/\gamma} = 0.$$

Now
$$Q \cap (0, x_\alpha) = \bigcup_{I_r^{(n)} \subset (0, x_\alpha)} Q \cap I_r^{(n)},$$

and we can write
$$Q \cap (0, x_\alpha) = \bigcup_{i=1}^{\infty} E_i, \tag{21}$$

where each E_i is a Borel set of dimension less than α.

Similarly, if $I_r^{(n)}$ is an interval of Q_n such that $I_r^{(n)} \subset (x_\alpha, 1)$, and l is the left hand end-point of $I_r^{(n)}$, we have $\varphi(l) = \gamma \geqslant \alpha$. In defining $I_r^{(n)} \cap Q_{n+k}$ we remove from $I_r^{(n)}$

successively open intervals of length λ times the closed interval where always we have $\lambda \leqslant \lambda_\gamma$. Thus $I_r^{(n)} \cap Q_{n+k}$ will consist of 2^k closed intervals each of length at least $(\frac{1}{2})^{k/\alpha} l_r^{(n)}$, where $l_r^{(n)}$ is the length of $I_r^{(n)}$.

Then if x_1, x_2 are in $I_r^{(n)}$ and

$$x_\alpha \leqslant x_1 < x_2 \leqslant 1,$$

the method used in the proof of Lemma 7 shows that

$$0 \leqslant \varphi(x_2) - \varphi(x_1) \leqslant K_\alpha (x_2 - x_1)^\alpha.$$

It follows, as in the proof of Lemma 8 that if A is any set of zero Λ^α-measure, then

$$H(A \cap (x_\alpha, 1)) = 0. \tag{22}$$

Now, since H is diffuse, for any $E \in \mathfrak{B}$

$$H(E) = H(E \cap (0, x_\alpha)) + H(E \cap (x_\alpha, 1)). \tag{23}$$

But the set function

$$H(E \cap (0, x_\alpha)) = H(E \cap Q \cap (0, x_\alpha))$$

clearly belongs to $\mathfrak{J}$, and so by (21) belongs to $\mathcal{S}_\alpha$. Further, using (22), we see that the set function $H(E \cap (x_\alpha, 1))$ belongs to $\mathcal{C}_\alpha$. Since the decomposition

$$H = H_1 + H_2 + H_3$$

of H provided by Theorem 2 is unique, we can make the identification

$$H_1(E) = H(E \cap (x_\alpha, 1)), \quad H_2(E) = 0, \quad H_3(E) = H(E \cap (0, x_\alpha)).$$

Thus H has a diffuse real dimension spectrum as required. This proves (ii).

Now for any $x \in Q$ we want to find the limit of the dimension of $(l, m) \cap Q$ for $l < x < m$ as $l \to x$ from below and $m \to x$ from above. Let

$$d(l, m) = \dim_B \{(l, m) \cap Q\}.$$

Then (i) $d(l, m) \leqslant \varphi(m)$

by (21) and also, since the set function

$$T(E) = H(E \cap (l, 1))$$

is in $\mathcal{C}_{\varphi(l)}$, while

$$T(Q \cap (l, m)) = H((l, m)) = \varphi(m) - \varphi(l) > 0,$$

it follows that (ii) $d(l, m) \geqslant \varphi(l)$.

Since $\varphi(x)$ is continuous, (i) and (ii) show that

$$\limsup_{\substack{l\to x-0\\ m\to x+0}} d(l, m) \leqslant \lim_{m\to x+0} \varphi(m) = \varphi(x) = \lim_{l\to x-0} \varphi(l) = \liminf_{\substack{l\to x-0\\ m\to x+0}} d(l, m),$$

and therefore, by (20), dim $(x,Q) = \varphi(x)$. This completes the proof of (i) since $\varphi(x_1) < \varphi(x_2)$ if x_1, x_2 are points of Q, not end-points of the same interval of $I_0 - Q$ and $x_1 < x_2$.

6. Decompositions relative to a maximal system of comparable Hausdorff measures

Given two Hausdorff measure functions $h_1(t)$, $h_2(t)$ we say

$$h_1(t) \succ h_2(t) \quad \text{if} \quad \lim_{t\to 0+} \frac{h_1(t)}{h_2(t)} = 0,$$

$$h_1(t) \prec h_2(t) \quad \text{if} \quad \lim_{t\to 0+} \frac{h_1(t)}{h_2(t)} = +\infty.$$

In either case two such measure functions are said to be *comparable*. It is clear that not all pairs of measure functions are comparable, but the class $\mathcal{H}$ of all Hausdorff measure functions is given a partial ordering by the relation $\prec$.

If $h_1(t) \succ h_2(t)$ then h_1-$m(E) = 0$ for any set E of finite h_2 outer measure; and h_2-$m_*(E) = +\infty$ for any set E of positive h_1 outer measure.

The class $\mathcal{R}$ of measure functions t^α, for real positive α, forms a completely ordered subset of $\mathcal{H}$. Using the axiom of choice (or Zorn's lemma) we can find a maximal completely ordered set $\mathcal{C}$ such that $\mathcal{R} \subset \mathcal{C} \subset \mathcal{H}$, and $\mathcal{C}$ has the following properties:

(i) any two elements h_1, $h_2 \in \mathcal{C}$ are comparable;

(ii) there is no element $h \in \mathcal{H} - \mathcal{C}$ which is comparable with every element in $\mathcal{C}$.

The class $\mathcal{H}$ of all Hausdorff measure functions and the set $\mathcal{R} \subset \mathcal{H}$ are of power continuum so the maximal system $\mathcal{C}$ must also have power continuum.

Two subsets $L, R \subset \mathcal{C}$ will be said to form a *section* in $\mathcal{C}$ if $L \cap R = \phi$, $L \cup R = \mathcal{C}$; and $h_1 \in L$, $h_2 \in R$ implies that $h_1 \prec h_2$, $h_1 \in L$ implies that $h \in L$ for every $h \in \mathcal{C}$ satisfying $h \prec h_1$, while $h_2 \in R$ implies that $h \in R$ for every $h \in \mathcal{C}$ satisfying $h \succ h_2$. The sections can be ordered using the inclusion relationship, so we can form a completely ordered set $\mathcal{J}$ of elements s, where each s is a section L_s, R_s of $\mathcal{C}$, and we say $s_1 < s_2$, if and only if L_{s_1} is a proper subset of L_{s_2}. For any element $\varphi \in \mathcal{C}$ one can form sections L, R by putting in L all those $h \in \mathcal{C}$ for which $h \prec \varphi$, putting in R all those $h \in \mathcal{C}$ for which $h \succ \varphi$, and finally adding φ either to L or to R. Not all sections of $\mathcal{C}$ are necessarily formed in this way, however. That is, there may be a section s such that L_s has no greatest member in $\mathcal{C}$ and R_s has no least member in

$\mathfrak{C}$. The set $\mathfrak{J}$ of sections is a well-ordered set and clearly has power at least that of the continuum.[1]

We now develop decomposition theorems for the class $\mathfrak{C}$ similar to those developed in § 4 for the class $\mathfrak{R}$. The only changes in method will be those necessitated by the existence of sections s which do not correspond to any element $\varphi \in \mathfrak{C}$. From now on all definitions and results will be in terms of the particular class $\mathfrak{C}$.

Given $s \in \mathfrak{J}$, let L_s, R_s denote the corresponding decomposition of $\mathfrak{C}$ and let L_s', R_s' denote the sets obtained by removing from L_s the greatest element (if it exists) and from R_s the least element (if it exists). Thus the section s corresponding to $\varphi \in \mathfrak{C}$ leads to the subsets L_s' consisting of $h \in \mathfrak{C}$ with $h \prec \varphi$ and R_s' consisting of $h \in \mathfrak{C}$ with $h \succ \varphi$. Now for any $s \in \mathfrak{J}$ define the following classes.

1. C_s is the class of set functions F of $\mathfrak{F}$ which are s-continuous, that is those set functions such that $F(E) = 0$ for any $E \in \mathfrak{B}$ for which there is at least one $h \in L_s'$ with h-$m(E) = 0$.

2. C_s^* is the class of set functions F of $\mathfrak{F}$ which are strongly s-continuous, that is those set functions such that h-$m(E) = 0$ for every $h \in R_s'$ implies that $F(E) = 0$.

3. S_s is the class of set functions F of $\mathfrak{F}$ which are s-singular; that is those set functions for which there is some $E_0 \in \mathfrak{B}$ such that

$$F(E) = F(E \cap E_0) \tag{24}$$

and $E_0 = \bigcup_{i=1}^{\infty} E_i$, where for each E_i there is some $h_i \in L_s'$ for which h_i-$m(E_i) = 0$.

4. S_s^* is the class of set functions F of $\mathfrak{F}$ which are almost s-singular; that is those set functions for which there is some $E_0 \in \mathfrak{B}$ such that (24) is satisfied and h-$m(E_0) = 0$ for every $h \in R_s'$.

Clearly, for any s, we have

$$\mathsf{C}_s^* \subset \mathsf{C}_s \quad \text{and} \quad S_s^* \supset S_s.$$

The measure function t^α is in the set $\mathfrak{C}$, and defines a section $s(\alpha)$. It is interesting to notice the relationships between the classes defined by this section, and the classes previously obtained in §§ 3, 4. These can be summarized by the relationships

$$\left.\begin{array}{l} \mathsf{C}_\alpha^* \subset \mathsf{C}_{s(\alpha)}^* \subset \mathsf{C}^*(t^\alpha) \subset \mathsf{C}(t^\alpha) \subset \mathsf{C}_{s(\alpha)} \subset \mathsf{C}_\alpha, \\ S_\alpha^* \supset S_{s(\alpha)}^* \supset S^*(t^\alpha) \supset S(t^\alpha) \supset S_{s(\alpha)} \supset S_\alpha. \end{array}\right\} \tag{25}$$

LEMMA 9. *If $s \in \mathfrak{J}$ is a section of the maximal system $\mathfrak{C}$ of measure functions, then the pairs C_s, S_s and C_s^*, S_s^* are pairs of complementary bands in $\mathfrak{F}$.*

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(1) Dr. R. O. Davies has proved that $\mathfrak{J}$ has in fact power greater than that of the continuum.

Proof. (i) Let $\mathcal{A}$ be the system of sets $E \in \mathcal{B}$ which can be expressed as a countable union of sets E_i of $\mathcal{B}$ such that for each E_i there exists an $h_i \in L'_s$ with $h_i\text{-}m(E_i) = 0$. Then $\mathcal{A}$ satisfies the conditions of Lemmas 1, 2, 3. It is immediate that $S_A = S_s$. To see that $C_A = C_s$ it is sufficient to notice that if $E = \cup E_i$ is in $\mathcal{A}$, and $F \in C_s$, then

$$|F|(E) \leqslant \sum_{i=1}^{\infty} |F|(E_i) = 0.$$

Hence, by Lemma 3, C_s, S_s are complementary bands.

(ii) Now let $\mathcal{A}'$ be the system of sets $E \in \mathcal{B}$ such that $h\text{-}m(E) = 0$ for every $h \in R'_s$. This system also satisfies the conditions of Lemma 3, so $C_{A'}$, $S_{A'}$ are complementary bands in $\mathfrak{F}$. Since $C^*_s = C_{A'}$, and $S^*_s = S_{A'}$, this completes the proof of the lemma.

DEFINITION. *A set function $F \in \mathfrak{F}$ will be said to have the exact dimension s relative to $\mathfrak{C}$, if F is in C_s and S^*_s.*

Such a function is concentrated on a set E which has zero measure with respect to every $h \in R'_s$, and $F(E) = 0$ for every E for which there is some $h \in L'_s$ with $h\text{-}m(E) = 0$.

Remark. Since $\mathfrak{C}$ is by definition comparable with $h(t) = t^\alpha$ for $0 \leqslant \alpha \leqslant n$, any section $s \in \mathfrak{J}$ corresponds to a unique real dimension α. Thus any set function of exact dimension s relative to $\mathfrak{C}$ will also be of exact real dimension α for the relevant α. However, there will in general be a large number of sections s corresponding to the same real dimension α.

THEOREM 5. *Given any set function $F \in \mathfrak{F}$, for each $s \in \mathfrak{J}$, the class of sections of the maximal system $\mathfrak{C}$ of measure functions, there is a unique decomposition*

$$F = F_1^{(s)} + F_2^{(s)} + F_3^{(s)},$$

where $F_1^{(s)}$ is strongly s-continuous, $F_2^{(s)}$ has the exact dimension s relative to $\mathfrak{C}$, and $F_3^{(s)}$ is s-singular.

Further if s_1, s_2 are two sections of $\mathfrak{C}$ with $s_1 < s_2$, then for every $E \in \mathcal{B}$

$$|F_1^{(s_1)}|(E) \geqslant |F_1^{(s_2)}|(E),$$

and

$$|F_3^{(s_1)}|(E) \leqslant |F_3^{(s_2)}|(E).$$

This theorem follows from Lemma 9 in exactly the same way that Theorem 2 followed from Lemma 4.

DEFINITION. *A set function $F \in \mathfrak{F}$ is said to have a diffuse $\mathfrak{C}$-dimension spectrum if the component $F_2^{(s)}$ of Theorem 5 vanishes identically for every section $s \in \mathfrak{J}$.*

If for every $s \in \mathfrak{J}$ there is defined a set function $F^{(s)} \in \mathfrak{F}$, then we say that $F^{(s)}$ is a continuous function of s at s_0 if, given $\varepsilon > 0$ there exist sections s_1, $s_2 \in \mathfrak{J}$ such that $s_1 < s_0 < s_2$, and for any $E \in \mathfrak{B}$, $s_1 < s < s_2$ we have}

$$|F^{(s)} - F^{(s_0)}| < \varepsilon.$$

Continuity on the left and right can clearly be defined and the development of § 4 leads to

LEMMA 10. *For a set function $F \in \mathfrak{F}$ to have a diffuse $\mathfrak{C}$-dimension spectrum, either of the following conditions is both necessary and sufficient.*

1. *If $F_1^{(s)}$, $F_3^{(s)}$ are the components defined by Theorem 5, then $|F_1^{(s)}|$, $|F_3^{(s)}|$, are each continuous functions of s.*

2. *For no $s \in \mathfrak{J}$ is there a function $G \in \mathfrak{F}$ with exact dimension s relative to $\mathfrak{C}$, and $0 < |G| \leqslant |F|$.*

Remark. If a set function $F \in \mathfrak{F}$ has a diffuse real dimension spectrum, then clearly it has a diffuse $\mathfrak{C}$-dimension spectrum for any maximal system $\mathfrak{C}$.

This immediately gives the decomposition of any set function $F \in \mathfrak{F}$ relative to $\mathfrak{C}$ into a set function with a diffuse $\mathfrak{C}$-dimension spectrum and a discrete sum of set functions each having an exact dimension relative to $\mathfrak{C}$.

THEOREM 6. *Given a maximal system $\mathfrak{C}$ of measure functions and any set function $F \in \mathfrak{F}$, there is a finite or enumerable sequence s_1, s_2, ... of distinct sections of $\mathfrak{C}$ and a decomposition*

$$F = F^{(d)} + F^{(s_1)} + F^{(s_2)} + \cdots,$$

where $F^{(d)}$ has a diffuse $\mathfrak{C}$-dimension spectrum, and $F^{(s_i)}$ is a set function of exact dimension s_i for $i = 1, 2, \ldots$. The set of the sections s_i, and the decomposition (apart from the order of its terms) are uniquely determined by F.

The proof is omitted since only changes of wording are needed in the proof of Theorem 3.

Remark 1. It is natural to compare the decomposition of $F \in \mathfrak{F}$ given by Theorem 6 with that given by Theorem 3. Suppose then, that

$$F = G^{(d)} + G^{(\alpha_1)} + G^{(\alpha_2)} + \cdots,$$

where $G^{(d)}$ has a continuous real dimension spectrum, and $G^{(\alpha_i)}$ is of exact real dimension α_i, $i = 1, 2, \ldots$. $G^{(d)}$ also has a diffuse $\mathfrak{C}$-dimension spectrum, so the component $G^{(d)}$ will form part of the component $F^{(d)}$ of Theorem 6. Now consider the function $G^{(\alpha_i)}$, and suppose it is decomposed by Theorem 1, using $h(t) = t^{\alpha_i}$, into

$$G^{(\alpha_i)} = G_1^{(\alpha_i)} + G_2^{(\alpha_i)} + G_3^{(\alpha_i)}.$$

The set function $G^{(\alpha_i)}$ will reappear in one of the functions $F^{(s_j)}$ of Theorem 6 where s_j is the section of $\mathcal{C}$ defined by the measure function t^{α_i}. On the other hand, the components $G_1^{(\alpha_i)}$, $G_3^{(\alpha_i)}$ will each be decomposed by $\mathcal{C}$ and may contribute to the component $F^{(d)}$ and also give a finite or enumerable number of components $F^{(s_k)}$ of exact dimension s_k where each of the sections s_k will correspond to the real dimension α_i. Thus two different components $G^{(\alpha_i)}$, $G^{(\alpha_j)}$ will not contribute to the same $F^{(s_k)}$ of Theorem 6.

On the other hand, if we regard the decomposition of Theorem 6 as given, it is easy to reconstruct that of Theorem 3. Each of the components $F^{(s_i)}$ will form a contribution to the component $G^{(\alpha_k)}$, where α_k is the real dimension corresponding to the section s_i. The total component $G^{(\alpha_k)}$ will be made up as a countable sum of such contributions $F^{(s_i)}$ together, perhaps, with a contribution from the component $F^{(d)}$. The contribution to $G^{(\alpha)}$ from $F^{(d)}$ will be of the form

$$F_1^{(d)(t)} - F_1^{(d)(s)},$$

where $F_1^{(d)(s)}$, $F_1^{(d)(t)}$ denote the components in the decomposition of $F^{(d)}$ using the sections s and t of $\mathcal{J}$ defined as follows. The left class L_s denotes all the functions $\varphi \in \mathcal{C}$ satisfying $\varphi \prec t^{\beta}$ for some $\beta < \alpha$, and R_s consist of all the other $\varphi \in \mathcal{C}$. The left class L_s denotes all the functions $\varphi \in \mathcal{C}$ satisfying $\varphi \prec t^{\beta}$ for each β with $\beta > \alpha$.

Remark 2. It is slightly unsatisfactory that the decomposition spectrum obtained in Theorem 6 may depend on the particular maximal class $\mathcal{C} \subset \mathcal{H}$. However, there are set functions with an exact dimension corresponding to a measure function φ, which will not be decomposed further by any maximal class $\mathcal{C}$. For example the middle component in Theorem 1 is of this kind. If

$$F(E) = \int_{E \cap T} f(x)\, d\,\varphi\text{-}m(x)$$

where $T \in \mathcal{B}$ has positive σ-finite φ-measure, then for any $\mathcal{C}$ which contains φ, the decomposition of Theorem 6 will yield the single function F of exact dimension s defined by φ. H_α defined in detail in § 5 is an example of such a set function. Whatever maximal class $\mathcal{C}$ is used, the single component H_α will remain.

Remark 3. If a set function $F \in \mathcal{J}$ is formed by adding together an enumerable number of components of type (1), we might hope that the decomposition of Theorem 6 would resolve F into these distinct components. This will only be so if (i) the components F_i of type (1) correspond to Hausdorff measure functions φ_i which are distinct and mutually comparable, and (ii) the maximal system $\mathcal{C}$ used in Theorem 6 includes sections corresponding to each of the φ_i. If there are two components F_i, F_j corresponding to φ_i, φ_j which are not comparable, there is no hope of Theorem 6 separating them out.

Remark 4. In the decomposition of Theorem 6 we have used sections of $\mathfrak{C}$. We have not been able to decide whether or not this is necessary. In other words we have not been able to construct an example of set function $F \in \mathfrak{F}$ which has exact $\mathfrak{C}$-dimension$\cdot s$ where $s \in \mathfrak{I}$, but does not correspond to a measure function $\varphi \in \mathfrak{C}$.

Remark 5. The decomposition of Theorem 6 is in a certain obvious sense (see Remark 1 above) finer than that of Theorem 3. It would be desirable to produce for any $F \in \mathfrak{F}$ an ultimate decomposition in the sense that it described completely the detailed structure of F. We are quite a long way from deciding under what circumstances this is possible. The best one could hope for would be a decomposition theorem (like Theorems 3, 6) into a set function with a diffuse spectrum and an enumerable sum of set functions of exact dimension in some sense each of which could not be decomposed any further into more uniform components. Thus one would like the components of exact dimension $\varphi \in \mathcal{H}$ to be such that Theorem 1 would give an integral representation of the whole component.

By using the continuum hypothesis we have been able to make some progress in this direction. We hope to publish the results in a later paper.

References

[1]. G. Birkoff, *Lattice Theory*. Amer. Math. Soc. Coll. Publ., Vol. 25.
[2]. N. Bourbaki, *Les structures fondamentales de l'analyse*. Book VI, *Integration*. Paris, 1952 and 1956.
[3]. F. Hausdorff, Dimension und äusseres Mass. *Math. Ann.* 79 (1919), 157–179.
[4]. F. Riesz, Sur quelques notions fondamentales dans la théorie générale des opérations linéaires. *Ann. of Math.* (2) 41 (1940), 174–206.
[5]. S. Saks, *Theory of the Integral*. Warsaw, 1937.