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Integrability of Green potentials in fractal domains

Kaj Nyström

Abstract. We prove L^q-inequalities for the gradient of the Green potential (Gf) in bounded, connected NTA-domains in $\mathbf{R}^n$, $n \geq 2$. These domains may have a highly non-rectifiable boundary and in the plane the set of all bounded simply connected NTA-domains coincides with the set of all quasidiscs. We get a restriction on the exponent q for which our inequalities are valid in terms of the validity of a reverse Hölder inequality for the Green function close to the boundary.

1. Introduction

In all of the following, let Ω be an open, connected and bounded subset of $\mathbf{R}^n$, $n \geq 2$. Let $G(x,y)$ denote the Green function of Ω and define for $f \in L^1(\Omega)$, $Gf(x) = \int_\Omega G(x,y) f(y)\, dy$. Then $Gf(x)$ is the Green potential of f. It is a classical fact that if $\partial\Omega$ is sufficiently smooth then the following inequality is valid for $f \in L^p(\Omega)$ with $1/q = 1/p - 1/n$, $n/(n-1) < q < \infty$,

$$\left(\int_\Omega |\nabla Gf|^q \, dx \right)^{1/q} \leq C(\Omega, p) \left(\int_\Omega |f|^p \, dx \right)^{1/p}. \tag{1}$$

The possibility of extending (1) to Lipschitz domains was investigated by Dahlberg [6] and he was able to prove the following.

Theorem. [6, Theorem 1] *Let $\Omega \subset \mathbf{R}^n$ be a Lipschitz domain and put $q_2 = 4$, $q_n = 3$ for $n \geq 3$. Then there exists a number $\varepsilon = \varepsilon(\Omega) > 0$ such that if $n/(n-1) < q < q_n + \varepsilon$ and $1/q = 1/p - 1/n$, then*

$$\left(\int_\Omega |\nabla Gf|^q \, dx \right)^{1/q} \leq C(\Omega, p) \left(\int_\Omega |f|^p \, dx \right)^{1/p}.$$

If $p = 1$ then, $|\{x \in \Omega : |\nabla Gf(x)| > \lambda\}| \leq C(\Omega)(\|f\|_1/\lambda)^{n/(n-1)}$.

Dahlberg also proved that these results are sharp in the following sense. For every $q>q_n$ there exists a Lipschitz domain $\Omega_q \subset \mathbf{R}^n$ such that $\nabla G f \notin L^q(\Omega_q)$ for some $f \in L^\infty(\Omega_q)$.

In this paper we study the possibility of proving similar results on even less smooth domains. Our results are valid for bounded, connected NTA-domains. We recall the definition of NTA-domains (see Section 2). Let in the following $d(\cdot, \partial\Omega)$ denote the Euclidean distance to the boundary, $\partial\Omega$, of Ω.

Definition. A bounded domain $\Omega \subset \mathbf{R}^n$ is called non-tangentially accessible (NTA) when there exist constants M and r_0 such that:

1. Corkscrew condition: For any $Q \in \partial\Omega$, $r<r_0$, there exists $A=A_r(Q)\in\Omega$ such that $M^{-1}r<|A-Q|<r$ and $d(A,\partial\Omega)>M^{-1}r$.
2. $(\complement\Omega)^o$ satisfies the corkscrew condition.
3. Harnack chain condition: If $\varepsilon>0$ and P_1 and P_2 belong to Ω, $d(P_j,\partial\Omega)>\varepsilon$ and $|P_1-P_2|<C\varepsilon$, then there exists a Harnack chain from P_1 to P_2 whose length depends on C and not on ε.

Let $G(x)$ denote the Green function of Ω with fixed pole x_0, $d(x_0,\partial\Omega)\sim \operatorname{diam}(\Omega)$. We make the following definition.

Definition. Let Ω be an open, connected and bounded subset of $\mathbf{R}^n$, $n\geq 2$. Then $\Omega \in \operatorname{Domain}(n, M, r_0, q)$ if the following two conditions are fulfilled.

1. Ω is an NTA-domain with parameters M and r_0,
2. there exists a constant $C=C(\Omega,q)$ independent of Q and r such that the following reverse Hölder inequality is valid for all $Q\in\partial\Omega$, $r<r_0$,

$$J(Q,r,\Omega,x_0,q)\leq CJ(Q,r,\Omega,x_0,1),$$

where

$$J(Q,r,\Omega,x_0,a)=\left(\frac{1}{|B(Q,r)\cap\Omega|}\int_{B(Q,r)\cap\Omega}\left|\frac{G(x)}{d(x,\partial\Omega)}\right|^a dx\right)^{1/a},$$

for $a\in[1,\infty)$. Here $B(Q,r)$ denotes an open ball, centered at Q and of radius r. By $|B(Q,r)\cap\Omega|$ we denote the n-dimensional Lebesgue measure of the set.

It is important to note that if Ω is an NTA-domain with parameters M, r_0, then $\Omega\in\operatorname{Domain}(n,M,r_0,1+1/(1-\beta))$ where $\beta=\beta(M)>0$ is a constant describing the boundary behaviour of the Green function, $G(x)$ (the constant β is the constant appearing in Lemma 3.5 below). That is, for every bounded NTA-domain Ω the reverse Hölder inequality stated above is valid with $q=2$.

We may now state our main theorem.

Main Theorem. *Let* $\Omega\in\mathrm{Domain}(n,M,r_0,q)$ *where* $q>n/(n-1)$. *Then there exists a constant* $C=C(\Omega,q)$ *such that if* $1/q=1/p-1/n$, *then the following inequality is valid for all* $f\in L^p(\Omega)$,

$$\left(\int_\Omega |\nabla Gf|^q\,dx\right)^{1/q}\leq C\left(\int_\Omega |f|^p\,dx\right)^{1/p}.$$

If $p=1$ *then*, $|\{x\in\Omega:|\nabla Gf(x)|>\lambda\}|\leq C(\Omega)(\|f\|_1/\lambda)^{n/(n-1)}$.

There are two questions naturally associated with the statement of this theorem. The first one is the question of estimating q (of course we want q as big as possible) and the second one is the question of finding the sharp exponent. These two related questions are addressed in Section 12. Here we just want to give the reader an idea of the results of Section 12. Let W_Ω denote the Whitney decomposition of $\Omega\subset\mathbf{R}^n$. Let $W_j:=\{Q\in W_\Omega,\ l(Q)=2^{-j}\}$, where $l(Q)$ denotes the sidelength of Q. For $q>0$ we introduce the number,

$$I_q(\Omega):=\sum_{j\geq j_0}2^{j(q-n)}\sum_{Q\in W_j}G(x_Q)^q.$$

Here we choose j_0 to avoid the pole of $G(x)=G(x,x_0)$. The point x_Q is the center of the cube Q. As a special case of Corollary 12.1 we may now formulate the following theorem.

Theorem. *Let* $\Omega\subset\mathbf{R}^2$ *be von Koch's snowflake. Then the following is true. If* $I_q(\Omega)<\infty$ *then there exists a constant* $C=C(\Omega,q)$ *such that if* $1/q=1/p-1/2$, $q>2$, $f\in L^p(\Omega)$, *then*

$$\left(\int_\Omega |\nabla Gf|^q\,dx\right)^{1/q}\leq C\left(\int_\Omega |f|^p\,dx\right)^{1/p}.$$

Furthermore, if $I_q(\Omega)=\infty$, *then there exists* $f\in L^\infty(\Omega)$ *such that* $\nabla Gf\notin L^q(\Omega)$.

The difficult part of this paper is the proof of our Main Theorem. Section 3 to Section 10 all contain essential contributions to the final proof in Section 11. The order in which lemmas and theorems are proved follows Dahlberg [6] and we once and for all acknowledge our debt to his work. Though the basic philosophy is the same as in Dahlberg [6] several other ideas are needed and the sources of these ideas are essentially three: Jerison–Kenig [14], Jones [16] and Nyström [21]. We will briefly describe the method used. The methods Dahlberg used are by now classical tools in harmonic analysis and are presented in Coifman–Fefferman [4], Muckenhoupt–Wheeden [20], Burkholder–Gundy [3], Stein [22], De Guzman [11]. Dahlberg used these powerful techniques together with results on harmonic functions and potential theory on Lipschitz domains developed by himself [5], [7], [8]. Several of Dahlberg's results were generalized in [14] by Jerison–Kenig to NTA-domains. Their paper supplies us with several lemmas on the boundary behaviour of harmonic functions which we make frequent use of. The connection to [16] is that our approach contains extensions of harmonic functions and Green potentials to a neighbourhood of our domain Ω. In this way we get rid of the difficult geometry of $\partial\Omega$ and may then instead work on cubes centered on the boundary. These extensions are defined by using both the Whitney decomposition of $\Omega(W_\Omega)$ and of $(\complement\Omega)^o(W_{(\complement\Omega)^o})$. The definition on $Q\in W_{(\complement\Omega)^o}$ of a quantity is related to the definition on an associated reflected cube $Q^*\in W_\Omega$. This type of quasiconformal reflection technique is of course not unique for [16] but our extensions are in the same spirit. The fourth source of ideas is Nyström [21] and concerns inequalities for subsets of the Sobolev spaces defined on Ω. Their deduction is based on non-linear potential theory and the work of Maz'ya [18], and the approach involves estimates of capacities.

The plan of the paper is as follows. There are altogether 12 sections of which Section 1 is this introduction. As several of these sections are quite technical we usually in the beginning of each such section state the lemmas and theorems that will be used later on, i.e. the reader may very well just read those for a start. Still we have enumerated the theorems and lemmas in each section in the sequence they are proved. That is why we, for instance in Section 5, start off by stating Lemma 5.1 and Lemma 5.4. This means that these two lemmas are the only ones to be used in other sections, but that there are still two lemmas in between, Lemma 5.2 and Lemma 5.3, which are used only in the proof of Lemma 5.4.

In Section 2 we define and explain the geometric notions we are working with and in Section 3 we present those results of Jerison–Kenig that we will make use of later on. Section 4 contains the basic facts about the Whitney decomposition of our domain and the reflection principle which we will make use of is presented. In Section 5 we define our extensions and prove some lemmas. Section 6 contains the proof of some reverse Hölder inequalities on arbitrary cubes. In Section 7 several results on the integrability of the Green function as well as Green potentials are presented. The results of Nyström [21] used, are contained in this section. In Section 8 we present some important lemmas which we need in Section 9 where we prove a good-λ-inequality for an operator T and a maximal operator K associated to our problem. Section 10 contains the last preliminary results before we in Section 11 present the proofs of the final results. At this level the proofs become quite short. In Section 12 we are concerned with the reverse Hölder inequality for the Green function and the question of sharpness.

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Notation. We here just give the basic notation used in the paper. Note specifically the convention on the denotation of constants stated below.

Ω will be an open, connected and bounded subset of $\mathbf{R}^n$ with boundary $\partial\Omega$. By $B(x,r)$ we denote an open ball with center x and radius r. If $Q\in\partial\Omega$ then $\Delta(Q,r):=B(Q,r)\cap\partial\Omega$. $w(x,F,\Omega)$ is the harmonic measure of $F\subset\partial\Omega$ relative to Ω at $x\in\Omega$. $G(x,y)$ will denote the positive Green function of Ω with pole at $y\in\Omega$. If $x\in\mathbf{R}^n$ and B is a closed subset of $\mathbf{R}^n$ then by $d(x,B)$ we denote the Euclidean distance from x to B. If $f\in L^p(\Omega)$, $Gf(x)$ is the Green potential of f. By $W^{m,p}(\Omega)$ we mean the Banach space of those functions in $L^p(\Omega)$, which have distributional derivatives up to order m in $L^p(\Omega)$ normed with the sum of the L^p norms of the derivatives. As usual, $W_0^{m,p}(\Omega)$ denotes the closure of $C_0^\infty(\Omega)$ in the same norm. $\nabla^k u$ is the vector of all (weak) partial derivatives of u of order k. By $m_n(E)$ and $|E|$ we mean the n-dimensional Lebesgue measure. All the cubes used are closed cubes with sides parallel to a fixed system of coördinate axes.

Conventions on constants. Most of our constants will depend on $\Omega\subset\mathbf{R}^n$. Although most constants, appearing in the formulation of lemmas and so on, will depend on the constant C appearing in the formulation of the reverse Hölder inequality for the Green function (see the definition above or Definition 2.1), this will not be explicitly stated. $c(a_1,a_2,\dots,a_n)$ will mean that the constant c only depends on the parameters $a_1,a_2,\dots,a_n$. By an absolute constant we mean a constant that just depends on characteristic data of Ω (i.e. on M and C) and the space dimension in a non-local way. By $A\sim B$ we will mean that the quotient of the parameters A and B is bounded from above and below by absolute constants. By $A\lesssim B$ we mean that A/B is bounded from above by an absolute constant. If we in a sequence of deductions use the same constant c all the time, this means that the original dependence of c is the same after as before the operations were carried out. Otherwise, the constants appearing are usually given with the parameters they depend on or are described at the point they appear.

2. Geometry

Let in the following $G(x)$ denote the Green function of Ω with fixed pole x_0, $d(x_0,\partial\Omega)\sim\operatorname{diam}(\Omega)$. Here $d(\,\cdot\,,\partial\Omega)$ denotes the distance to the boundary. We make the following definition,

Definition 2.1. Let Ω be an open, connected and bounded subset of $\mathbf{R}^n$, $n\geq 2$. Then $\Omega\in\text{Domain}(n,M,r_0,q)$ if the following two conditions are fulfilled.

1. Ω is an NTA-domain with parameters M and r_0,

2. there exists a constant $C=C(\Omega,q)$ independent of Q and r such that the following reverse Hölder inequality is valid for all $Q\in\partial\Omega$, $r<r_0$,

$$J(Q,r,\Omega,x_0,q)\leq CJ(Q,r,\Omega,x_0,1),$$

where

$$J(Q,r,\Omega,x_0,a)=\left(\frac{1}{|B(Q,r)\cap\Omega|}\int_{B(Q,r)\cap\Omega}\left|\frac{G(x)}{d(x,\partial\Omega)}\right|^a dx\right)^{1/a},$$

for $a\in[1,\infty)$.

In this section we will describe at some length the geometric notions of Condition 1 in Definition 2.1. Condition 2 is investigated in Section 12.

Definition 2.2 [16, p. 73]. $\Omega\subset\mathbf{R}^n$ is an (ε,δ)-domain if for any pair of points $x,y\in\Omega$, $|x-y|<\delta$, there exists a rectifiable arc $\gamma\subset\Omega$ joining x and y and satisfying

1. $l(\gamma)\leq|x-y|/\varepsilon$,
2. $d(z,\partial\Omega)\geq\varepsilon(|x-z|\,|y-z|)/|x-y|$ for all $z\in\gamma$.

This condition on Ω has been proved useful in extension theorems for function spaces defined on Ω. See [15], [16]. In [14], Jerison–Kenig defined a class of domains which they named NTA-domains. These are the domains we will be working with. Their connection to (ε,δ)-domains is that an NTA-domain is an (ε,∞)-domain with an additional thickness condition on the complement of Ω. The definition is in itself adapted to the study of harmonic functions on Ω. Definition 2.3 and 2.4 below are verbally taken from [14, p. 93]. In the following M will be a fixed constant depending on $\Omega\subset\mathbf{R}^n$.

Definition 2.3. An M non-tangential ball in a domain Ω is a ball $B(A,r)$ in Ω whose distance from $\partial\Omega$ is comparable to its radius: $Mr>d(B(A,r),\,\partial\Omega)>M^{-1}r$. For $P_1,P_2\in\Omega$, a Harnack chain from P_1 to P_2 in Ω is a sequence of M non-tangential balls such that the first ball contains P_1, the last contains P_2, and such that consecutive balls have non-empty intersection.

Remark 2.1. Suppose u is a positive harmonic function in Ω. Then $C^{-1}u(P_1)\leq u(P_2)\leq Cu(P_1)$, where C depends only on the length of the Harnack chain connecting P_1 and P_2 by the Harnack inequality.

Definition 2.4. A bounded domain $\Omega\subset\mathbf{R}^n$ is called non-tangentially accessible (NTA) when there exist constants M and r_0 such that:

1. Corkscrew condition: For any $Q\in\partial\Omega$, $r<r_0$, there exists $A=A_r(Q)\in\Omega$ such that $M^{-1}r<|A-Q|<r$ and $d(A,\partial\Omega)>M^{-1}r$.

2. $(\complement\Omega)^o$ satisfies the corkscrew condition.

3. Harnack chain condition: If $\varepsilon>0$ and P_1 and P_2 belong to Ω, $d(P_j,\partial\Omega)>\varepsilon$ and $|P_1-P_2|<C\varepsilon$, then there exists a Harnack chain from P_1 to P_2 whose length depends on C and not on ε.

Condition 1 is named the corkscrew condition because the union of non-tangential balls of radius $\frac{1}{2}M^{-1}r$, as r tends to zero, forms a non-tangential approach region tending towards Q, which is a twisting replacement for the usual conical approach region for Lipschitz domains. Condition 2 implies that NTA-domains are regular for the Dirichlet problem. Conditions 1 and 3 may be combined to the following equivalent condition:

4. If $\varepsilon>0$, P_1, P_2 belong to Ω, $d(P_j,\partial\Omega)>\varepsilon$, and $|P_1-P_2|<2^k\varepsilon$, then there exists a Harnack chain from P_1 to P_2 of length Mk. Moreover, for each ball B in the chain, $\text{radius}(B)\geq M^{-1}\min(d(P_1,B),d(P_2,B))$.

If Ω is an NTA-domain then it is also an (ε,δ)-domain for all $\delta<\infty$ as mentioned above. We will sometimes use this description of Ω and we note that the relation between the crucial parameters M and ε is essentially $\varepsilon=1/M$.

We will now state three important results.

Lemma 2.1. [16, Lemma 2.3] *Suppose $\Omega\subset\mathbf{R}^n$, $n\geq2$, is an* NTA-*domain. Then $m_n(\partial\Omega)=0$.*

The second one is more of an observation and tells us that the important parameter M is invariant under dilations of $\mathbf{R}^n$.

Lemma 2.2. *Let $\Omega\subset\mathbf{R}^n$, $n\geq2$, be an* NTA-*domain with parameters M and r_0. Define Ω', $x\in\Omega \Leftrightarrow (x-p)/a\in\Omega'$, for some $a>0$ and some $p\in\mathbf{R}^n$. Then Ω' is an* NTA-*domain with parameters M' and r_0', where $M'=M$ and $r_0'=r_0/a$.*

The third result is a geometric localization theorem.

Theorem 2.1. [17] *Suppose $\Omega\subset\mathbf{R}^n$, $n\geq2$, is a bounded* NTA-*domain. Then there exists $r_0>0$ depending only on Ω such that for all $Q\in\partial\Omega$ and all $r<r_0$ there exists an* NTA-*domain $\Omega_{Q,r}\subset\Omega$ such that*

$$B(Q,M^{-1}r)\cap\Omega\subset\Omega_{Q,r}\subset B(Q,Mr)\cap\Omega.$$

Furthermore, the constant M in the NTA-*definition of $\Omega_{Q,r}$ is independent of Q and r.*

This localization for NTA-domains replaces the local starshapedness explored on Lipschitz domains.

For a discussion of the geometry of NTA-domains in $\mathbf{R}^n$, $n\geq2$, we refer to [14, p. 90–94]. We will briefly just describe the geometry of NTA-domains in the plane.

This short exposition reveals the close connection between NTA-domains and the theory of quasiconformal mappings. By a quasicircle is meant the image of a circle under a quasiconformal mapping. A domain bounded by a quasicircle is called a quasidisc. For the theory of quasiconformal mappings we refer to Gehring [9] and Väisälä [23].

Remark 2.2. Gehring and Väisälä [10] have proved that while the Hausdorff dimension of a quasicircle is always less than 2 it can take any value λ, $1\leq\lambda<2$.

A simple, closed curve in the plane is said to satisfy the Ahlfors' three point condition, if for any points Z_1, Z_2 on the curve and any point Z_3 on the arc between Z_1 and Z_2 of smaller diameter, the quotient between the distance between Z_1 and Z_3 and the distance between Z_1 and Z_2 is bounded by a fixed constant.

One may prove the following theorem.

Theorem 2.2. *Let Ω be a bounded and simply connected subset of the plane. Then the following are equivalent statements.*

(1) *Ω is a quasidisc.*

(2) *$\partial\Omega$ satisfies the Ahlfors' three point condition.*

(3) *Ω is an* NTA-*domain.*

Proof. (1)$\Leftrightarrow$(2) is due to Ahlfors [2]. (1)$\Leftrightarrow$(3) is due to Jones [15].

3. Harmonic functions on NTA-domains

In this section we summarize the results of Jerison–Kenig [14] that we will frequently make use of in the forthcoming sections. $\Omega\subset\mathbf{R}^n$ will all the time denote a bounded, connected NTA-domain with parameters M, r_0. The definition of NTA-domains was given in Section 2. Recall that for all $Q\in\partial\Omega$, $r<r_0$, $A_r(Q)$ denotes a point in Ω fulfilling $M^{-1}r<|A_r(Q)-Q|<r$ and $d(A_r(Q),\partial\Omega)\geq M^{-1}r$. All the constants appearing in the following lemmas only depend on the original value of M. $\Delta(Q,r):=B(Q,r)\cap\partial\Omega$.

Lemma 3.1. [14, Lemma 4.4] *If u is a positive harmonic function in Ω which vanishes continuously on $\Delta(Q,2r)$, then $u(x)\leq Mu(A_r(Q))$ for all $x\in B(Q,r)\cap\Omega$.*

Lemma 3.2. [14, Lemma 4.8] *If $2r<r_0$ and $x\in\Omega\setminus B(Q,2r)$, then*

$$M^{-1}<\frac{w(x,\Delta(Q,r),\Omega)}{r^{n-2}G(x,A_r(Q))}<M.$$

Lemma 3.3. [14, Lemma 4.10] *Let r be such that $Mr<r_0$. Suppose that u and v are positive harmonic functions in Ω vanishing continuously on $\Delta(Q,Mr)$ for some $Q\in\partial\Omega$. Then,*

$$M^{-1}\frac{u(A_r(Q))}{v(A_r(Q))}\le\frac{u(x)}{v(x)}\le M\frac{u(A_r(Q))}{v(A_r(Q))},$$

for all $x\in B(Q,M^{-1}r)\cap\Omega$.

Lemma 3.4. [14, Lemma 4.1] *There exists $\beta=\beta(M)>0$ such that for all $Q\in\partial\Omega$, $r<r_0$, and for every positive harmonic function u in Ω, such that u vanishes continuously on $\Delta(Q,r)$, the following is valid. If $x\in\Omega\cap B(Q,r)$ then $u(x)\le M(|x-Q|/r)^\beta C(u)$, where $C(u):=\sup\{u(y):y\in\partial B(Q,r)\cap\Omega\}$.*

Using these lemmas we may prove the following estimate of the Green function.

Lemma 3.5. *Let $Q_0\in\partial\Omega$. Then there exists a constant $C=C(n)$ such that if $Cr<r_0$ and $x\in\Omega\setminus B(Q_0,Cr)$, then the following estimate is valid with $\beta=\beta(M)>0$ for all $y\in B(Q_0,r)\cap\Omega$,*

$$G(x,y)\le C(M,n)\frac{d(y,\partial\Omega)^\beta}{r^{n-2+\beta}}w(x,\Delta(Q_0,r),\Omega).$$

Proof. From Lemmas 3.4, 3.1 and 3.2 we have,

$$G(x,y)\le C(M,n)\frac{|y-Q_0|^\beta}{r^{n-2+\beta}}w(x,\Delta(Q_0,r),\Omega). \tag{1}$$

Fix $y\in B(Q_0,r)\cap\Omega$. Then there exists a Whitney cube $Q_y=Q$ such that $y\in Q$. Choose $p\in\partial\Omega$ such that $d(p,Q)=d(Q,\partial\Omega)$. Then $|p-Q_0|\le Cr$ where $C=C(n)$. Furthermore, $y\in\Omega\cap B(p,Cr)$ and $B(p,Cr)\subseteq B(Q_0,2Cr)$. If $x\in\Omega\setminus B(Q_0,6Cr)$, we get by applying (1) to the ball $B(p,Cr)$,

$$G(x,y)\lesssim\frac{d(y,\partial\Omega)^\beta}{r^{n-2+\beta}}w(x,\Delta(Q_0,2Cr),\Omega)\lesssim\frac{d(y,\partial\Omega)^\beta}{r^{n-2+\beta}}w(x,\Delta(Q_0,r),\Omega).$$

Theorem 3.1. [14, Theorem 7.9] *Let Ω be an* NTA*-domain and let V be an open set. Let K be a compact subset of V. Then there exists a number $\alpha>0$ such that for all positive harmonic functions u and v in Ω that vanish continuously on $\partial\Omega\cap V$, the function $u(x)/v(x)$ is Hölder continuous of order α in $K\cap\bar{\Omega}$. In particular, $\lim_{x\to Q}u(x)/v(x)$ exists for all $Q\in K\cap\partial\Omega$.*

The next lemma contains two basic estimates of the Green function. The first can be found in Wu [25, Lemma 2] and the second part in Widman [24, Lemma 2.1].

Lemma 3.6. *Let $\Omega\subset\mathbf{R}^n$, $n\geq 2$, be a bounded and simply connected* NTA-*domain. Then the following is valid.*

1. *If $x,y\in\Omega$, $|x-y|<d(y,\partial\Omega)/2$ then*

$$\log\frac{d(y,\partial\Omega)}{|x-y|}\leq G(x,y)\leq 5\log\frac{d(y,\partial\Omega)}{|x-y|},\quad n=2$$
$$\tfrac{1}{2}|x-y|^{-n+2}\leq G(x,y)\leq|x-y|^{-n+2},\qquad n\geq 3.$$

2. *$G(x,y)\leq C(n)|x-y|^{-n+1}$ if $x,y\in B(z,1)$ for some $z\in\Omega$ and $n\geq 2$.*

We will also make frequent use of continuity properties of Riesz potentials. Define for $0<\alpha<n$,

$$I_\alpha f(x):=\frac{1}{\gamma(\alpha)}\int_{\mathbf{R}^n}\frac{f(y)}{|x-y|^{n-\alpha}}\,dy.$$

Theorem 3.2. [22, p. 119] *Let $0<\alpha<n$, $1<p\leq q<\infty$, $1/q=1/p-\alpha/n$. Then*

1. *$\|I_\alpha f\|_q\leq A_{p,q}\|f\|_p$,*
2. *suppose $f\in L^1(\Omega)$ and $1/q=1-\alpha/n$, then $|\{x:|I_\alpha f(x)|>\lambda\}|\leq A_{p,q}(\|f\|/\lambda)^q$.*

4. Whitney cubes and other cubes

In this section we gather the information about the behaviour of the Whitney decomposition of an NTA-domain with parameters M and r_0 needed in the extension procedure carried out in the next section. As pointed out in Section 2, an NTA-domain $\Omega\subset\mathbf{R}^n$ is an (ε,δ)-domain for all $\delta<\infty$ with $\varepsilon=1/M$. We let $\delta=a\ll r_0$ be one fixed such δ. In the following we work with closed cubes with sides parallel to a fixed system of coordinate axes. By x_Q we mean the center of the cube Q and by kQ, where $k>0$, we mean the cube Q dilated with respect to x_Q by a factor k. By $l(Q)$ we mean the sidelength of Q. We start with the Whitney decomposition.

Theorem 4.1. [22, p. 16] *Let Ω be an open subset of $\mathbf{R}^n$. Then*

1. *$\Omega=\bigcup_{k=1}^{\infty}Q_k$,*
2. *$Q_j^o\cap Q_k^o=\emptyset$ if $j\neq k$,*
3. *there exists $c_1,c_2>0$ so that $c_1 l(Q_k)\leq d(Q_k,\complement\Omega)\leq c_2 l(Q_k)$.*

We denote by W_Ω the Whitney decomposition of Ω. Define $W_1:=W_\Omega$, $W_2:=W_{(\complement\Omega)^o}$, $W_3:=\{Q\in W_2:l(Q)\leq a\}$. The constant C in the following lemmas depends at most on n and M.

Lemma 4.1. [16, Lemma 2.4] *If $Q_j\in W_3$ then there exists a cube $S_k\in W_1$ such that*

$$1\leq\frac{l(S_k)}{l(Q_j)}\leq 4,\quad (Q_j,S_k)\leq Cl(Q_j).$$

For each $Q_j \in W_3$ we fix $S_k \in W_1$ according to Lemma 4.1 and define $Q_j^* := S_k$. Q_j^* is called the reflected cube associated to Q_j.

Lemma 4.2. [16, Lemma 2.6] *If $S_k \in W_1$ then there are at most C cubes Q_j, $Q_j \in W_3$, such that $Q_j^* = S_k$.*

Lemma 4.3. [16, Lemma 2.7] *If $Q_j, Q_k \in W_3$ and $Q_j \cap Q_k \neq \emptyset$, then*

$$d(Q_j^*, Q_k^*) \leq Cl(Q_j).$$

We will also make use of the following consequence of the definition of NTA-domains.

Lemma 4.4. *Let Q be a cube such that $Q \cap \partial\Omega \neq \emptyset$, $l(Q) < r_0$. Then there exists a cube $Q_* \subset Q$ such that either $Q_* \subset \Omega$ or $Q_* \subset (\complement\Omega)^o$ and such that $l(Q_*) \geq Cl(Q)$.*

We always denote by Q_* the largest such cube. In the forthcoming sections we will be working with arbitrary cubes Q with sides parallel to the coordinate axes and lying in a band of the boundary. We will split the class of cubes into several subclasses depending on whether $Q \subset \Omega$, $Q \subset (\complement\Omega)^o$ or $Q \cap \partial\Omega \neq \emptyset$. In the first and the second case we will also distinguish between the two cases $l(Q) < \frac{1}{10} d(Q, \partial\Omega)$ and $l(Q) \geq \frac{1}{10} d(Q, \partial\Omega)$. The three types of cubes may be referred to as "small" cubes, "large" cubes and "boundary" cubes. The analysis on "large" and "boundary" cubes will in general be the same, but the analysis on the "small" cubes will often be quite different. Lemma 4.5–4.9 below all contain trivial but very important facts about this splitting and we will refer to these results frequently.

Lemma 4.5. *Suppose $Q \subset \Omega$ or $Q \subset (\complement\Omega)^o$. Then the following is valid.*
1. *If $l(Q) < \frac{1}{10} d(Q, \partial\Omega)$, then $d(x, \partial\Omega) \sim d(Q, \partial\Omega)$ for all $x \in Q$.*
2. *If $l(Q) \geq \frac{1}{10} d(Q, \partial\Omega)$, then $d(x_Q, \partial\Omega) \sim l(Q)$.*

Lemma 4.6. *Let $l(Q) < \frac{1}{10} d(Q, \partial\Omega)$. Define $W_1(Q) := \{Q_j \in W_1 : \frac{17}{16} Q_j \cap Q \neq \emptyset\}$ if $Q \subset \Omega$ and $W_2(Q) := \{Q_j \in W_2 : \frac{17}{16} Q_j \cap Q \neq \emptyset\}$ if $Q \subset (\complement\Omega)^o$. Then $\# W_1(Q) \leq C$ and $\# W_2(Q) \leq C$, with $C = C(n)$. Furthermore, if $Q_j \in W_1(Q)$ or $Q_j \in W_2(Q)$, then $l(Q_j) \sim d(Q, \partial\Omega)$.*

Lemma 4.7. *Let $Q \subset \Omega$ or $Q \subset (\complement\Omega)^o$ be such that $l(Q) \geq \frac{1}{10} d(Q, \partial\Omega)$. Then there exists a cube Q', $Q' \in W_1$ or $Q' \in W_2$, such that $l(Q') \sim l(Q)$ and $m_n(Q' \cap Q) \sim m_n(Q)$.*

Lemma 4.8. *Let Q be a cube such that $Q \subset (\complement\Omega)^o$, $l(Q) \geq \frac{1}{10} d(Q, \partial\Omega)$, or $Q \cap \partial\Omega \neq \emptyset$. Then $Q \cap (\complement\Omega)^o \subset B(p, Cl(Q))$ for some $p \in \partial\Omega$ and $C = C(n)$. Suppose $Q_1 \in W_3$ and $\frac{17}{16} Q_1 \cap B(p, Cl(Q)) \neq \emptyset$. Then $Q_1^* \subset B(p, C^* l(Q)) \cap \Omega$ where $C^* = C^*(M, n)$ and Q_j^* is the reflected cube associated to Q_j.*

Lemma 4.9. *Let $Q\cap\partial\Omega\neq\emptyset$. Fix $A>1$. Then there exists a constant $C_0=C_0(M,n,A)>0$ such that if $Q_j\in W_3$, $\frac{17}{16}Q_j\cap\{\mathbf{R}^n\setminus C_0Q\}\neq\emptyset$, then $Q_j^*\cap AQ=\emptyset$. Here Q_j^* is the reflected cube associated to Q_j.*

Proof. Suppose $Q_j^*\cap AQ\neq\emptyset$. Then by Lemma 4.1 we have $l(Q_j)\lesssim Al(Q)$ and $d(Q_j,Q)\lesssim Al(Q)$. But this contradicts the assumption that $\frac{17}{16}Q_j\cap\{\mathbf{R}^n\setminus C_0Q\}\neq\emptyset$ if we just choose C_0 sufficiently large.

5. Extensions of harmonic functions and potentials

In this section we extend positive harmonic functions $u(x)$ defined on Ω, the Green function of Ω and Green potentials to a band around Ω. Of course our extended function Eu will not be harmonic but the reflection principle, described in Lemmas 4.1–4.3, which is the crucial technique, will ensure that the Harnack inequality is preserved in a strong sense. We remind the reader that $\Omega\subset\mathbf{R}^n$ denotes an NTA-domain with parameters M and r_0. Recall that $W_1:=W_\Omega$, $W_2:=W_{(\complement\Omega)^\circ}$, $W_3:=\{Q\in W_2:l(Q)\leq a\}$. This notation was introduced in Section 4. Let $\{\psi_{Q_j}\}$ denote a partition of unity associated to W_3 such that for all $Q_j\in W_3$ we have $\psi_{Q_j}\in C_0^\infty(\mathbf{R}^n)$, $\operatorname{supp}\psi_{Q_j}\subseteq\frac{17}{16}Q_j$ and $\sum_{Q_j\in W_3}\psi_{Q_j}(x)=1$ for all $x\in\bigcup_{Q\in W_3}Q$. Let in the same way $\{\varphi_{Q_i}\}$ be a partition of unity associated to W_1.

Definition 5.1. Let $u(x)$ be a positive harmonic function defined on Ω. We define an extended function $Eu(x)$ for all $x\in\bar\Omega\cup\bigcup_{Q\in W_3}Q$ in the following way.

$$Eu(x):=\chi_{\bar\Omega}(x)u(x)+\chi_{(\complement\Omega)^\circ}(x)\sum_{Q_j\in W_3}u(x_{Q_j^*})\psi_{Q_j}(x),$$

where Q_j^* is the reflected cube associated to Q_j as described in Lemma 4.1.

We will also define an extended "Green function". To do this in a way that fits our purpose we introduce for all $Q\in W_3$ the following sets

$$B(Q):=\{Q_j\in W_1:Q_j\cap Q^*\neq\emptyset\},\quad A(Q):=W_1\setminus B(Q).$$

The interpretation of the sets $B(Q)$ and $A(Q)$ for a fixed cube $Q\in W_3$ is that $B(Q)$ denotes the Whitney cubes in Ω which are close to the reflected cube Q^*. $A(Q)$ is just the complementary set.

Definition 5.2. Let $y\in\Omega$ and $x\in\bar\Omega\cup\bigcup_{Q\in W_3}Q$. We define the extended "Green function" as follows,

$$EG(x,y):=\chi_{\bar\Omega}(x)G(x,y)+\chi_{(\complement\Omega)^\circ}(x)\sum_{Q_1\in W_3}\psi_{Q_1}(x)A(x_{Q_1},y),$$

where

$$A(x_{Q_1},y):=\sum_{Q_2\in W_1}\varphi_{Q_2}(y)a_{Q_1,Q_2}(y),$$
$$a_{Q_1,Q_2}(y):=G(x_{Q_1^*},y) \qquad \text{if } Q_2\in A(Q_1),$$
$$a_{Q_1,Q_2}(y):=1/d(y,\partial\Omega)^{n-2} \qquad \text{if } Q_2\in B(Q_1).$$

The extended potential is defined $EGf(x):=\int_\Omega EG(x,y)f(y)\,dy$ for all $f\in L^\infty(\Omega)$, $\operatorname{supp} f\subseteq\Omega$ and $x\in\bar{\Omega}\cup\bigcup_{Q\in W_3}Q$.

We note that $EGf(x)$ is continuous on $\bar{\Omega}\cup\bigcup_{Q\in W_3}Q$ if $f\in C_0^\infty(\Omega)$. On Ω this is trivial as $EGf(x)=Gf(x)$ if $x\in\bar{\Omega}$ and $Gf(x)$ is continuous. See Helms [13, Theorem 6.22]. If $x\in(\complement\Omega)^o$ then by definition

$$EGf(x)=\sum_{Q_1\in W_3}\psi_{Q_1}(x)EGf(x_{Q_1}),$$

which of course is a continuous function. What therefore remain in the proof of the continuity of $EGf(x)$ is to prove that if $Q\in\partial\Omega$ and $x_j\in(\complement\Omega)^o$, $x_j\to Q$, then $EGf(x_j)\to 0$. But this follows immediately from the definition.

The definition of the extended "Green function" is such that the properties of the Green function $G(x,y)$, as x is not situated too close to the pole y, are preserved under the extension. In the critical case, that is when $x\in Q\in W_3$ and $y\in Q^*$, we have defined $EG(x,y)$ as a truncation of $G(\cdot\,,y)$ near y at an appropriate level. This implies that $EG(x,y)$ is bounded on $(\complement\Omega)^o$ in the following way. The proof is presented below.

Lemma 5.1. *Let $y\in\Omega$. Then for all $x\in\bigcup_{Q\in W_3}Q$ the following is valid,*

$$EG(x,y)\le\frac{c(n)}{d(y,\partial\Omega)^{n-2}}.$$

The definition also implies the following.

Lemma 5.4. *Let $Q\in W_3$. Let $Q\to Q^*$ by reflection. Then for all $f\in L^1(\Omega)$, $f\ge 0$ and $\operatorname{supp} f\subseteq\Omega$ the following is valid,*

$$\int_Q\left|\frac{EGf(x)}{d(x,\partial\Omega)}\right|^{n/n-1}dx\le C\int_{Q^*}\left|\frac{Gf(x)}{d(x,\partial\Omega)}\right|^{n/n-1}dx,$$

for some constant $C=C(n,M)$.

The rest of this section is devoted to the proof of Lemma 5.1 and Lemma 5.4. Lemma 5.2 and Lemma 5.3 below, are used in the proof of Lemma 5.4.

Proof of Lemma 5.1. For all x, y the number of nonzero terms in the definition of $EG(x,y)$ is bounded by a constant $C=C(n)$ by the underlying properties of the Whitney decomposition. So what we have to prove is that if $y\in Q_2\in W_1$, $Q_1\in W_3$ and $Q_2\cap Q_1^*=\emptyset$, then $G(x_{Q_1^*},y)\le C(n)/d(y,\partial\Omega)^{n-2}$. But this follows from Lemma 3.6 and the maximum principle.

Lemma 5.2. *Let $y\in\Omega$ and define $S_y:=\{Q\in W_1:\varphi_Q(y)\neq 0\}$. Suppose $Q_1,Q_2\in W_3$, $Q_1\cap Q_2\neq\emptyset$. Then*
1. $G(x_{Q_1^*},y)\sim G(x_{Q_2^*},y)$ *if* $S_y\subset A(Q_1)\cap A(Q_2)$,
2. $G(x_{Q_1^*},y)\ge C(M,n)/d(y,\partial\Omega)^{n-2}$ *if* $S_y\cap B(Q_2)\neq\emptyset$ *and* $S_y\cap B(Q_1)=\emptyset$.

Proof. We start with the conclusion in 1. The conclusion is trivial if $Q_1^*\cap Q_2^*\neq\emptyset$. We therefore assume that $Q_1^*\cap Q_2^*=\emptyset$. Let B denote a ball centered at y and of radius $c_1 l(Q_2)$, where $c_1\ll 1$ is a constant to be fixed later on. $G(x,y)$ is then a positive harmonic function on $\Omega\setminus B$ as a function of x. As Ω is an (ε,δ)-domain there exists a curve $\gamma(t)$ connecting $\gamma(0)=x_{Q_1^*}$, $\gamma(1)=x_{Q_2^*}$ such that by Lemma 4.3,

$$l(\gamma)\le M|x_{Q_1^*}-x_{Q_2^*}|\le C(M,n)l(Q_2). \tag{1}$$

Furthermore for all $z\in\gamma$ we have,

$$d(z,\partial\Omega)\ge\frac{1}{M}\frac{|x_{Q_1^*}-z|\,|x_{Q_2^*}-z|}{|x_{Q_1^*}-x_{Q_2^*}|}. \tag{2}$$

Define $t_1:=\inf\{t_0:\gamma(t)\notin Q_1^*,\ \forall t\ge t_0\}$ and $t_2:=\inf\{t_0:\gamma(t)\in Q_2^*,\ \forall t\ge t_0\}$. Then for all $z=\gamma(t)$, with $t\in[t_1,t_2]$, we get from (2), Lemmas 4.1 and 4.3 that

$$d(z,\partial\Omega)\ge C(M,n)l(Q_2). \tag{3}$$

If $\{\gamma(t):t\in[t_1,t_2]\}\cap B=\emptyset$ then we leave $\gamma(t)$ unchanged. If $\{\gamma(t):t\in[t_1,t_2]\}\cap B\neq\emptyset$ then define $t_3:=\inf\{t_0:\gamma(t)\notin B,\ \forall t\le t_0\}$ and $t_4:=\inf\{t_0:\gamma(t)\notin B,\ \forall t\ge t_0\}$. Then $\{\gamma(t):t\in[0,t_3)\cup(t_4,1]\}\cap B=\emptyset$. Replace $\{\gamma(t):t\in[t_3,t_4]\}$ with one of the arcs on ∂B defined by the points $\gamma(t_3)$ and $\gamma(t_4)$ and denote the modified curve by $\gamma^*(t)$. Choose $C_1=C(M,n)/2$ where $C(M,n)$ is the constant in (3). This construction has lead us to a rectifiable arc $\gamma^*(t)$, connecting $x_{Q_1^*}$ and $x_{Q_2^*}$, such that $l(\gamma^*)\le C(M,n)l(Q_2)$, $d(z,\partial\Omega)\ge C(M,n)l(Q_2)$ and $d(z,y)\ge C(M,n)l(Q_2)$ for all $z\in\gamma^*$. Covering γ^* with balls of size $\sim l(Q_2)$ and applying the Harnack inequality completes the proof.

We now prove the conclusion in 2. By assumption there exist cubes $Q_3\in W_1$ and $Q_4\in W_1$ such that $Q_3\cap Q_4\neq\emptyset$, $Q_3\cap Q_2^*\neq\emptyset$, $\varphi_{Q_4}(y)\neq 0$ and $S_y\cap B(Q_1)=\emptyset$. It follows from Lemma 3.6 and the Harnack inequality that

$$\inf_{x\in Q_2^*}G(x,y)\ge\frac{C(n)}{d(y,\partial\Omega)^{n-2}}. \tag{4}$$

Fix $x_0 \in Q_2^*$ such that $|x_0 - y| \geq l(Q_2^*)/4$. By the same argument as in the proof of 1, we get that $G(x_{Q_1^*}, y) \sim G(x_0, y)$. The conclusion then follows from (4).

We now define an auxilary mapping $Q \in W_3 \to Q^* \in W_1$ in the following way. Let $x \in Q$. Then

$$\varphi_{Q,Q^*}(x) = \frac{l(Q^*)}{l(Q)}(x - x_Q) + x_{Q^*}.$$

By this map Q is mapped bijectively onto Q^*.

Lemma 5.3. *Let $x \in Q \in W_3$, $y \in \Omega$. Then*

$$EG(x,y) \leq CG(\varphi_{Q,Q^*}(x), y),$$

for some constant $C = C(M,n)$.

Proof. Define $W(Q) = \{Q_j \in W_3 : Q_j \cap Q \neq \emptyset\}$. Fix $y \in \Omega$ and define $S_y = \{Q_i \in W_1 : \varphi_{Q_i}(y) \neq \emptyset\}$. Then by Definition 5.2 we have

$$EG(x,y) = \sum_{Q_j \in W(Q)} \sum_{Q_i \in S_y} \psi_{Q_j}(x) \varphi_{Q_i}(y) a_{Q_j, Q_i}(y). \tag{1}$$

Suppose

$$S_y \subset \bigcap_{Q_j \in W(Q)} A(Q_j).$$

Then (1) reduces to

$$EG(x,y) = \sum_{Q_j \in W(Q)} \psi_{Q_j}(x) G(x_{Q_j^*}, y). \tag{2}$$

As $\# W(Q) \leq C(n)$ it follows from Lemma 5.2 and (2) that,

$$EG(x,y) \leq C(M,n) G(x_{Q^*}, y). \tag{3}$$

But as $S_y \subset A(Q)$ we get by the Harnack inequality that

$$G(x_{Q^*}, y) \lesssim \min_{z \in Q^*} G(z,y) \lesssim G(\varphi_{Q,Q^*}(x), y). \tag{4}$$

Combining (3) and (4) completes the proof in this case. Suppose

$$S_y \not\subset \bigcap_{Q_j \in W(Q)} A(Q_j).$$

This implies that there exists $Q_{j_0}\in W(Q)$ such that $S_y\cap B(Q_{j_0})\neq\emptyset$. By Lemma 5.2 we always have $EG(x,y)\le C(n)/d(y,\partial\Omega)^{n-2}$ so what we have to prove is that

$$\min_{z\in Q^*} G(z,y)\ge C(M,n)/d(y,\partial\Omega)^{n-2}. \tag{5}$$

If $S_y\cap B(Q)\neq\emptyset$ this is trivial. If $S_y\cap B(Q)=\emptyset$, (5) follows from part 2 of Lemma 5.2. This completes the proof.

Proof of Lemma 5.4. As $l(Q)\sim l(Q^*)$ by Lemma 4.1, all we have to prove is that

$$\int_Q |EGf(x)|^{n/n-1}\,dx\lesssim\int_{Q^*}|Gf(x)|^{n/n-1}\,dx.$$

But $EGf(x)\lesssim Gf(\varphi_{Q,Q^*}(x))$ by Lemma 5.4. A simple change of variables then completes the proof.

6. Reverse Hölder inequalities on cubes

In this section we prove Theorem 6.1 and Lemma 6.1 below.

Theorem 6.1. *Let $\Omega\in\mathrm{Domain}(n,M,r_0,q)$. Let Q_0 be a cube centered at $x_0\in\partial\Omega$ such that $l(Q_0)=\delta$, where $\delta\ll r_0$ is a fixed number. Define $S:=\{Q\subseteq Q_0,\ Q \text{ cube}\}$. Then the following inequalities are valid with $C=C(M,n,q)$ for all $Q\in S$ and all positive functions u, u harmonic in $\Omega\cap B(x_0,l(Q_0)C_0)$ and vanishing continuously on $\Delta(Q_0,C_0l(Q_0))$. Here $C_0=C_0(M,n)\gg 1$.*

1. *If $Q\subset\Omega$ then*

$$C^{-1}\left(\frac{1}{|Q|}\int_Q\left|\frac{u(x)}{d(x,\partial\Omega)}\right|^q dx\right)^{1/q}\le\frac{u(x_Q)}{d(x_Q,\partial\Omega)}\le C\frac{1}{|Q|}\int_Q\frac{u(x)}{d(x,\partial\Omega)}\,dx.$$

2. *If $Q\subset(\complement\Omega)^o$ then*

$$C^{-1}\left(\frac{1}{|Q|}\int_Q\left|\frac{Eu(x)}{d(x,\partial\Omega)}\right|^q dx\right)^{1/q}\le\frac{Eu(x_Q)}{d(x_Q,\partial\Omega)}\le C\frac{1}{|Q|}\int_Q\frac{Eu(x)}{d(x,\partial\Omega)}\,dx.$$

3. *If $Q\cap\partial\Omega\neq\emptyset$ then*

$$C^{-1}\left(\frac{1}{|Q|}\int_Q\left|\frac{Eu(x)}{d(x,\partial\Omega)}\right|^q dx\right)^{1/q}\le\frac{Eu(x_{Q_*})}{d(x_{Q_*},\partial\Omega)}\le C\frac{1}{|Q|}\int_Q\frac{Eu(x)}{d(x,\partial\Omega)}\,dx,$$

where $Q_*\subset Q$ is the cube existing by Lemma 4.4.

Remark 6.1. E is the extension operator defined in Definition 5.1.

Define a measure λ on Q_0 by

$$\lambda(A)=\int_A\left|\frac{Eu(x)}{d(x,\partial\Omega)}\right|^{n/(n-1)}dx\quad\text{for } A\subseteq Q_0.$$

The next lemma shows that λ is a doubling measure on cubes.

Lemma 6.1. *Let $Q\subset Q_0$ be a cube such that $\alpha Q\subseteq Q_0$. Then there exists a constant $C=C(M,n,\alpha)$ such that $\lambda(\alpha Q)\leq C\lambda(Q)$.*

The rest of this section is devoted to the proof of Theorem 6.1. Lemma 6.1 is a simple consequence of Theorem 6.1.

Proof of Theorem 6.1. We start with the case $Q\subset\Omega$ and divide the proof into two subcases depending on whether or not $l(Q)<d(Q,\partial\Omega)/10$.

In case $l(Q)<d(Q,\partial\Omega)/10$ the proof follows immediately from the Harnack inequality. Suppose therefore that $l(Q)\geq d(Q,\partial\Omega)/10$. Take $p\in\partial\Omega$, $d(Q,\partial\Omega)=d(p,\partial\Omega)$. Then for some $C=C(n)$ we have $\operatorname{diam}(Q)+d(Q,\partial\Omega)\leq Cl(Q)$. Put

$$\begin{aligned}B_0&=B(p,Cl(Q)),\\ B_1&=B(p,M^2Cl(Q))\setminus B(p,Cl(Q)),\end{aligned}$$

where M is the constant appearing in Lemma 3.3. By Definition 2.4 there exists $x_1\in B_1\cap\Omega$ such that $d(x_1,\partial\Omega)\sim l(Q)$. By Lemma 3.3 the following is valid for all $x\in B_0\cap\Omega$,

$$\frac{u(x)}{G(x,x_0)}\sim\frac{u(x_1)}{G(x_1,x_0)},\tag{1}$$

where x_0 denotes the fixed pole used in the statement of the reverse Hölder inequality for the Green function (see Definition 2.1). Using (1) and our assumption on Ω,

$$\begin{aligned}\frac{1}{|Q|}\int_Q\left|\frac{u(x)}{d(x,\partial\Omega)}\right|^q dx&\leq\frac{1}{|Q|}\int_{\Omega\cap B_0}\left|\frac{G(x,x_0)}{d(x,\partial\Omega)}\right|^q\left|\frac{u(x)}{G(x,x_0)}\right|^q dx\\ &\lesssim\left|\frac{u(x_1)}{G(x_1,x_0)}\right|^q\frac{1}{|Q|}\int_{\Omega\cap B_0}\left|\frac{G(x,x_0)}{d(x,\partial\Omega)}\right|^q dx\\ &\lesssim\left|\frac{u(x_1)}{G(x_1,x_0)}\right|^q\left(\frac{1}{|Q|}\int_{\Omega\cap B_0}\frac{G(x,x_0)}{d(x,\partial\Omega)}\,dx\right)^q.\end{aligned}\tag{2}$$

We now examine the integral

$$I:=\int_{\Omega\cap B_0}\frac{G(x,x_0)}{d(x,\partial\Omega)}\,dx.$$

Define $A_j:=\{Q_i\in W_\Omega:l(Q_i)=2^{-j},\ Q_i\cap B_0\neq\emptyset\}$. Then by the Harnack inequality

$$I\lesssim\sum_{j=j_0}^{\infty}\sum_{Q_i\in A_j}2^{j(1-n)}|G(x_{Q_i},x_0)|. \tag{3}$$

By Lemma 3.2 we have

$$G(x_{Q_i},x_0)\sim w(x_0,\Delta_i,\Omega)2^{j(n-2)}, \tag{4}$$

where $\Delta_i:=\partial\Omega\cap B(p_i,c2^{-j})$ for some $p_i\in\partial\Omega$, $c=c(n)$, $d(p_i,Q_i)=d(Q_i,\partial\Omega)$. Furthermore, for all $x\in\partial\Omega$,

$$\sum_{Q_i\in A_j}\chi_{\Delta_{Q_i}}(x)\leq c=c(n). \tag{5}$$

Let $\Delta=\Delta(p,Cl(Q))$. Using estimate (4) and (5) in (3),

$$\begin{aligned}I&\lesssim\sum_{j=j_0}^{\infty}\sum_{Q_i\in A_j}2^{j(1-n)}w(x_0,\Delta_i,\Omega)2^{j(n-2)}\\&\lesssim w(x_0,\Delta,\Omega)\sum_{j=j_0}^{\infty}2^{-j}\lesssim w(x_0,\Delta,\Omega)2^{-j_0}.\end{aligned} \tag{6}$$

But $2^{-j_0}\sim l(Q)$. Using this we can conclude from (6) that

$$I\lesssim w(x_0,\Delta,\Omega)l(Q). \tag{7}$$

Using (2), (7), Lemma 3.2, the Harnack chain condition on Ω and the fact that by Lemma 4.5, $l(Q)\sim d(x_Q,\partial\Omega)$, we get the conclusion.

We now prove the inequality in the opposite direction. By Lemma 4.7 there exists a cube $Q'\in W_1$ such that

$$l(Q')\sim l(Q)\quad\text{and}\quad m_n(Q\cap Q')\sim m_n(Q). \tag{8}$$

We get,

$$\begin{aligned}I&:=\frac{1}{|Q|}\int_Q\frac{u(x)}{d(x,\partial\Omega)}\,dx\geq\frac{1}{|Q|}\int_{Q\cap Q'}\frac{u(x)}{d(x,\partial\Omega)}\,dx\\&\gtrsim\frac{1}{|Q|}\frac{m_n(Q\cap Q')}{l(Q)}u(x_{Q'})\gtrsim\frac{u(x_Q)}{d(x_Q,\partial\Omega)},\end{aligned}$$

by (8).

We now deal with the case $Q\subset(\complement\Omega)^o$, i.e. case 2. Suppose first that $l(Q)< d(Q,\partial\Omega)/10$. Let $W(Q):=\{Q_j\in W_3:\operatorname{supp}\psi_{Q_j}\cap Q\neq\emptyset\}$. By Lemma 4.5,

$$\#W(Q)\leq C,\quad l(Q_j)\sim d(Q,\partial\Omega) \text{ for all } Q_j\in W(Q). \tag{9}$$

Fix one j_0 such that $Q_{j_0}\in W(Q)$. By (9), the definition of Eu, Lemma 4.3 and the Harnack inequality it follows that for all $x\in Q$ we may deduce,

$$Eu(x)\sim u(x_{Q_{j_0}}),\quad d(x,\partial\Omega)\sim d(x_Q,\partial\Omega). \tag{10}$$

That is, by (10)

$$\int_Q\left|\frac{Eu(x)}{d(x,\partial\Omega)}\right|^q dx\sim|Q|\left|\frac{u(x_{Q_{j_0}})}{d(x_Q,\partial\Omega)}\right|^q\sim|Q|\left|\frac{Eu(x_Q)}{d(x_Q,\partial\Omega)}\right|^q,$$

which completes the proof in this case.

Now suppose that $l(Q)\geq d(Q,\partial\Omega)/10$. Take $p\in\partial\Omega$, $d(Q,\partial\Omega)=d(Q,p)$. Let as above $W(Q):=\{Q_j\in W_3:\operatorname{supp}\psi_{Q_j}\cap Q\neq\emptyset\}$. It follows from Lemma 4.8 that if $Q_j\in W(Q)$, then $Q_j^*\subset B(p,Cl(Q))\cap\Omega$ for some $C=C(M,n)$. Using Lemma 4.2 and the Harnack inequality we therefore get

$$\frac{1}{|Q|}\int_Q\left|\frac{Eu(x)}{d(x,\partial\Omega)}\right|^q dx\lesssim\frac{1}{|Q|}\int_{\Omega\cap B(p,Cl(Q))}\left|\frac{u(x)}{d(x,\partial\Omega)}\right|^q dx. \tag{11}$$

In the same way as the case $Q\subset\Omega$, $l(Q)\geq d(Q,\partial\Omega)/10$ was analyzed, we may therefore conclude that

$$\frac{1}{|Q|}\int_Q\left|\frac{Eu(x)}{d(x,\partial\Omega)}\right|^q dx\lesssim\left|\frac{u(x_1)}{l(Q)}\right|^q, \tag{12}$$

for some $x_1\in\Omega$, $d(x_1,\partial\Omega)\sim l(Q)$. We have $l(Q)\sim d(x_Q,\partial\Omega)$ by Lemma 4.5. Let $S_Q:=\{Q_j\in W_3:\psi_{Q_j}(x_Q)\neq\emptyset\}$. Fix $Q_{j_0}\in S_Q$. Then $Eu(x_Q)\sim u(x_{Q^*_{j_0}})$. It follows from the reflection principle and Lemma 4.5 that

$$d(x_{Q^*_{j_0}},\partial\Omega)\sim d(x_{Q_{j_0}},\partial\Omega)\sim d(x_Q,\partial\Omega)\sim l(Q) \tag{13}$$

and $|x_{Q^*_{j_0}}-x_1|\lesssim l(Q)$. By the Harnack chain condition we therefore get, $Eu(x_Q)\sim u(x_{Q^*_{j_0}})\sim u(x_1)$ and we are through. The proof of the other inequality is similar to the proof in the case $Q\subset\Omega$, $l(Q)\geq d(Q,\partial\Omega)/10$.

Left is now just case 3, i.e. the case $Q\cap\partial\Omega\neq\emptyset$. By Lemma 4.4 there exists $Q_*\subset Q$ such that either $Q_*\subset\Omega$ or $Q_*\subset(\complement\Omega)^o$. Furthermore, $l(Q)\sim l(Q_*)$. Using this, case 1 and 2 we get

$$\frac{1}{|Q|}\int_Q\left|\frac{Eu(x)}{d(x,\partial\Omega)}\right|dx\geq\frac{|Q_*|}{|Q|}\frac{1}{|Q_*|}\int_{Q_*}\left|\frac{Eu(x)}{d(x,\partial\Omega)}\right|dx\gtrsim\frac{Eu(x_{Q_*})}{d(x_{Q_*},\partial\Omega)}.$$

Left to prove is the other inequality. But in the same way as in the analysis of the case $Q\subset(\complement\Omega)^o$, $l(Q)\geq d(Q,\partial\Omega)/10$ we have

$$\frac{1}{|Q|}\int_Q\left|\frac{Eu(x)}{d(x,\partial\Omega)}\right|^q dx\lesssim\frac{1}{|Q|}\int_{\Omega\cap B(p,Cl(Q))}\left|\frac{u(x)}{d(x,\partial\Omega)}\right|^q dx,$$

where $p\in\partial\Omega$ and $c=c(M,n)$. Redoing the deduction made in the case $Q\subset(\complement\Omega)^o$, $l(Q)\geq d(Q,\partial\Omega)/10$ then completes the proof.

7. Integrability of the Green function

In this section we prove several results on the integrability of the Green function and Green potentials on $\Omega\subset\mathbf{R}^n$, where Ω is a bounded NTA-domain. We start by stating the results of this section that we will make use of in the forthcoming sections.

Theorem 7.1. *Let $\Omega\subset\mathbf{R}^n$, $n\geq 2$, be a bounded, connected* NTA-*domain with parameters M and r_0. Let $y\in\Omega$, $d(y,\partial\Omega)\ll r_0$. Then*

$$\int_\Omega\left|\frac{G(x,y)}{d(x,\partial\Omega)}\right|^{n/(n-1)}dx\leq C,$$

where $C=C(M,r_0,\delta,\operatorname{diam}(\Omega),n)$ (δ is the δ of Theorem 6.1).

Lemma 7.1. *Let Q_0 be a cube centered at $x_0\in\partial\Omega$ with $l(Q_0)=\delta>0$. Put $S:=\{Q\subseteq Q_0:Q\cap\Omega\neq\emptyset,\ Q \text{ cube}\}$. Then the following is valid for all $Q\in S$ if $C_0=C_0(M,n)$ is sufficiently large,*

$$\int_{\Omega\setminus C_0Q}\sup_{y\in Q\cap\Omega}\left|\frac{G(x,y)}{d(x,\partial\Omega)}\right|^{n/(n-1)}dx\leq C(M,r_0,\delta,\operatorname{diam}(\Omega),n).$$

The following lemma is needed in Section 8.

Lemma 7.2. *Let Q_0 and S be defined according to Lemma* 7.1. *Take $Q_j \in S$ and let $b_j \in L^1(Q_j)$, $b_j \geq 0$ and* $\operatorname{supp} b_j \subseteq Q_j \cap \Omega$. *If $C_0 = C_0(M,n)$ is sufficiently large then*

$$\int_{Q_0 \setminus C_0 Q_j} \left| \frac{EGb_j(x)}{d(x,\partial\Omega)} \right|^{n/(n-1)} dx \leq C \|b_j\|_{L^1(Q_j)}^{n/(n-1)},$$

where $EGb_j(x)$ is the extended potential defined in Definition 5.2 *and $C = C(M, r_0, \delta, \operatorname{diam}(\Omega), n)$.*

Theorem 7.3. *Let $\Omega \subset \mathbf{R}^n$, $n \geq 2$, be a bounded* NTA-*domain. Then there exists a constant $C = C(M, r_0, \operatorname{diam}(\Omega), n)$ such that if $f \in L^2(\Omega)$ then*

$$\int_\Omega \left| \frac{Gf(x)}{d(x,\partial\Omega)} \right|^2 dx \leq C \int_\Omega |f|^2 d(x,\partial\Omega)^2 \, dx.$$

The rest of this section is devoted to the proofs of these results.

Proof of Theorem 7.1. Let $y \in Q \in W_\Omega$ and define j_0 by $l(Q) = 2^{-j_0}$. Choose $p \in \partial\Omega$, $d(p,Q) = d(Q,\partial\Omega)$. Let for simplicity $r_0 = 2^{-i_0}$ for some i_0. Let $k(n) > 0$ be the smallest integer such that $Q \subset B(p, 2^k 2^{-j_0})$. Let $a(n)$ be the smallest integer such that $2^a \geq C(n)$, where $C(n)$ is the constant appearing in Lemma 3.5. Assume $i_0 < j_0 - 1 - a - k$. This can always be arranged if we restrict y to be situated closer to $\partial\Omega$. Define $I := \{i_0, i_0+1, \dots, j_0 - 1 - a - k\}$. Define further for each $i \in I$, $B_i := B(p, 2^{-i}) \setminus B(p, 2^{-(i+1)})$. Define also $B_0 := B(p, 2^{-(j_0-a-k)})$, $\Omega_1 = \Omega \setminus B(p, 2^{-i_0})$. Then

$$\Omega = (B_0 \cap \Omega) \left(\bigcup_{i \in I} (B_i \cap \Omega) \right) \cup \Omega_1.$$

Let δ, C_0 be the constants appearing in Theorem 6.1. Put $t_0 = \delta / 4C_0$. Define

$$\begin{aligned} \Omega_0 &:= \{x \in \Omega_1 : d(x,\partial\Omega) \geq t_0 = \delta/4C_0\}, \\ \Omega_i &:= \{x \in B_i \cap \Omega : d(x,\partial\Omega) \geq (\delta/4C_0) 2^{-i}\}, \end{aligned}$$

for all $i \in I$. Define

$$I(E) := \int_E \left| \frac{G(x,y)}{d(x,\partial\Omega)} \right|^{n/(n-1)} dx \quad \text{for } E \subseteq \Omega.$$

We have $I(\Omega_1) = I(\Omega_0) + I(\Omega_1 \setminus \Omega_0)$. We may cover $\Omega_1 \setminus \Omega_0$ by $N = N(C_0, \operatorname{diam}(\Omega), n, \delta) = N(M, n, \operatorname{diam}(\Omega), \delta)$ cubes with sidelength $\sim t_0$ and apply Theorem 6.1 on each of these cubes. This gives us,

$$I(\Omega_1 \setminus \Omega_0) \leq C \sum_{k=1}^N G(x_k, y)^{n/(n-1)} \quad \text{for some } x_k \in \Omega_1.$$

But by Lemma 3.5, $G(x_k,y)\le C/r_0^{n-2+\beta}$, i.e., $I(\Omega_1\setminus\Omega_0)\le C(M,n,\delta,\operatorname{diam}(\Omega),r_0)$. Furthermore, it is trivial that $I(\Omega_0)\le C(M,n,\delta,\operatorname{diam}(\Omega),r_0)$. Left to estimate is $I(\Omega\setminus\Omega_1)$. Dilate Ω in the following way: $x'\in\Omega' \Leftrightarrow 2^{-j_0}x'+p\in\Omega$. Now the Green function on Ω', $G'(x',y')$, is related to the Green function of Ω in the following way,

$$G'(x',y')=2^{-j_0(n-2)}G(2^{-j_0}x'+p,2^{-j_0}y'+p).$$

Using this with $r=2^{-j_0}$ we get,

$$\begin{aligned}I(E):&=\int_E\left|\frac{G(x,y)}{d(x,\partial\Omega)}\right|^{n/(n-1)}dx\\ &=\int_{E'}\left|\frac{G(rx'+p,ry'+p)}{d(x',\partial\Omega')}\right|^{n/(n-1)}r^{n-n/(n-1)}\,dx\\ &=\int_{E'}\left|\frac{G'(x',y')}{d(x',\partial\Omega')}\right|^{n/(n-1)}dx,\end{aligned}\tag{1}$$

i.e., the integral is dilation invariant in the sense described in (1). By Lemma 2.2 we know that Ω' is an NTA-domain with the same parameter M as Ω. This means that Theorem 6.1 is valid on Ω' with a constant independent of the dilation parameter $r=2^{-j_0}$. Furthermore, Lemma 3.5 is valid for $G'(x',y')$ with the same constants as for $G(x,y)$. Put $E_i:=B_i\cap\Omega\setminus\Omega_i$ for $i\in I$. Left to estimate is $I(E_i)$, $I(\Omega_i)$ and $I(B_0\cap\Omega)$. By Lemma 3.5 we have for all $x'\in E_i'\cup\Omega_i'$,

$$G'(x',y')\lesssim\frac{1}{2^{(j_0-i)(n-2+\beta)}}.\tag{2}$$

We may cover E_i' by $N=N(M,n,\delta)$ cubes of sidelength $t_0 2^{j_0-i}$ and apply Theorem 6.1 to each of these cubes as $G'(x',y')$ is a positive harmonic function in a "C_0-neighbourhood" of these cubes. Applying (1), (2) we therefore get, as $d(x',\partial\Omega')\ge t_0 2^{j_0-i}$ for all $x'\in\Omega_i'$ and $|\Omega_i'|\le c2^{(j_0-i)n}$,

$$\begin{aligned}I(E_i\cup\Omega_i)&\le C2^{(j_0-i)n}2^{-(j_0-i)n/(n-1)}2^{-(j_0-i)(n-2+\beta)n/(n-1)}\\ &\le C2^{-(j_0-i)\beta n/(n-1)},\end{aligned}\tag{3}$$

where $C=C(M,n,\delta)$, $\beta=\beta(M)$. Summing the estimates in (3) we get

$$I\Bigl(\bigcup_{i\in I}(B_i\cap\Omega)\Bigr)\le C\sum_{i\in I}2^{-(j_0-i)\beta n/(n-1)}\le C.\tag{4}$$

Left to estimate is $I(B_0\cap\Omega)$. Let $y\to y'$ by the dilation. By (1) and Lemma 3.6 we have

$$I(B(y,d(y,\partial\Omega)/2))\le\int_{B(y',d(y',\partial\Omega')/2)}\frac{1}{|x'-y'|^{n-1}}\,dx'\le C.$$

By a covering argument of $\partial\Omega\cap B_0$ as above, an application of Theorem 6.1, (1) and the fact that by the maximum principle and Lemma 3.6, $G'(x',y')\le C=C(n)$ for all $x'\in\Omega'\setminus B(y',d(y',\partial\Omega')/2)$, we may complete the proof.

Proof of Lemma 7.1. Fix $x\in\Omega\setminus C_0Q$. If $l(Q)<d(Q,\partial\Omega)/10$ then it follows from the Harnack inequality that if $C_0=10$, then $G(x,y)\sim G(x,y_Q)$ for all $y\in Q$. The result then follows from Theorem 7.1. If $l(Q)\ge d(Q,\partial\Omega)/10$ or $Q\cap\partial\Omega\ne\emptyset$ then it follows from Lemma 3.1 that if $C_0=C_0(M,n)$ is sufficiently large, then $G(x,y)\lesssim G(x,y_0)$ for some $y_0\in\Omega$ and for all $y\in Q$. An application of Theorem 7.1 completes the proof.

Proof of Lemma 7.2. Put $U_j:=Q_0\setminus C_0Q_j$. We divide the proof into the cases $Q_j\subset\Omega$ and $Q_j\cap\partial\Omega\ne\emptyset$. As by Lemma 2.1, $m_n(\partial\Omega)=0$, we have

$$\begin{aligned}\int_{U_j}\left|\frac{EGb_j(x)}{d(x,\partial\Omega)}\right|^{n/(n-1)}dx&=\int_{U_j\cap\Omega}\left|\frac{Gb_j(x)}{d(x,\partial\Omega)}\right|^{n/(n-1)}dx\\&\quad+\int_{U_j\cap(\complement\Omega)^\circ}\left|\frac{EGb_j(x)}{d(x,\partial\Omega)}\right|^{n/(n-1)}dx.\end{aligned}$$

But

$$\int_{U_j\cap\Omega}\left|\frac{Gb_j(x)}{d(x,\partial\Omega)}\right|^{n/(n-1)}dx\le C\|b_j\|_{L^1(Q_j)}^{n/(n-1)},$$

by Lemma 7.1. Left to estimate is

$$I:=\int_{U_j\cap(\complement\Omega)^\circ}\left|\frac{EGb_j(x)}{d(x,\partial\Omega)}\right|^{n/(n-1)}dx. \tag{1}$$

Let $Q_j\subset\Omega$. We first suppose that $l(Q_j)<d(Q_j,\partial\Omega)/10$. Define $W(Q_j):=\{Q_k\in W_\Omega:\operatorname{supp}\varphi_{Q_k}\cap Q_j\ne\emptyset\}$. $W(Q_j)$ denotes the Whitney cubes close to or intersecting Q_j. We also define an associated set $AW(Q_j)$ by

$$AW(Q_j):=\{Q_m\in W_\Omega:\exists Q_k\in W(Q_j)\text{ such that }Q_m\cap Q_k\ne\emptyset\}.$$

Then $AW(Q_j)$ consists of $W(Q_j)$ and all the Whitney cubes that have at least one neighbour in the set $W(Q_j)$. We define two more sets of cubes.

$$B(Q_j):=\{Q\in W_3:Q\cap U_j\ne\emptyset,\ Q^*\in AW(Q_j)\}, \tag{2}$$

$$G(Q_j):=\{Q\in W_3:Q\cap U_j\ne\emptyset,\ Q^*\notin AW(Q_j)\}. \tag{3}$$

Then $B(Q_j)$ consists of those Whitney cubes $Q\in W_3$, $Q\cap U_j\neq\emptyset$, such that their associated reflected cubes Q^* are close to Q_j in the sense explained above. $G(Q_j)$ is just the complementary set. Using this notation we get

$$(4)\quad I\leq \sum_{Q\in B(Q_j)}\int_Q\left|\frac{EGb_j(x)}{d(x,\partial\Omega)}\right|^{n/(n-1)}dx+\sum_{Q\in G(Q_j)}\int_Q\left|\frac{EGb_j(x)}{d(x,\partial\Omega)}\right|^{n/(n-1)}dx=I_1+I_2.$$

We first estimate I_2 in (4). By Lemma 5.4 and Lemma 4.2 we have

$$(5)\qquad I_2\lesssim \sum_{Q\in G(Q_j)}\int_{Q^*}\left|\frac{Gb_j(x)}{d(x,\partial\Omega)}\right|^{n/(n-1)}dx\lesssim\int_{\Omega\setminus\Omega(Q_j)}\left|\frac{Gb_j(x)}{d(x,\partial\Omega)}\right|^{n/(n-1)}dx,$$

where $\Omega(Q_j):=\bigcup_{Q_m\in AW(Q_j)}Q_m$. But if $x\in\Omega\setminus\Omega(Q_j)$, then by the Harnack inequality,

$$(6)\qquad \left|\frac{Gb_j(x)}{d(x,\partial\Omega)}\right|^{n/(n-1)}\lesssim\left|\frac{G(x,y_{Q_j})}{d(x,\partial\Omega)}\right|^{n/(n-1)}\|b_j\|^{n/(n-1)}_{L^1(Q_j)}.$$

Combining (5), (6) and Theorem 7.1 we get the required estimate of I_2. Left is to estimate I_1 in (4). As $l(Q_j)<d(Q_j,\partial\Omega)/10$ we know by Lemma 4.5 that $\#AW(Q_j)\leq C(n)$ and that all cubes in $AW(Q_j)$ have size comparable to $d(Q_j,\partial\Omega)$. Using Lemma 4.2 we may conclude that $\#B(Q_j)\leq C(M,n)$ and that if $Q\in B(Q_j)$ then $l(Q)\sim d(Q_j,\partial\Omega)$. Using this we get the estimate

$$(7)\qquad I_1\lesssim\left|\frac{1}{d(Q_j,\partial\Omega)}\right|^{n/(n-1)}\sum_{Q\in B(Q_j)}\int_Q\left|EGb_j(x)\right|^{n/(n-1)}dx.$$

By Lemma 5.1 the following is valid for all $y\in Q_j$,

$$(8)\qquad EG(x,y)\leq\frac{C(n)}{d(y,\partial\Omega)^{n-2}}\sim\frac{C(n)}{d(Q_j,\partial\Omega)^{n-2}}.$$

Combining (7) and (8) we get

$$(9)\qquad I_1\lesssim d(Q_j,\partial\Omega)^{-n/(n-1)-(n-2)n/(n-1)}\left(\sum_{Q\in B(Q_j)}|Q|\right)\|b_j\|^{n/(n-1)}_{L^1(Q_j)}.$$

But

$$(10)\qquad \sum_{Q\in B(Q_j)}|Q|\lesssim d(Q_j,\partial\Omega)^n\#B(Q_j)\lesssim d(Q_j,\partial\Omega)^n.$$

This completes the proof in the case $Q_j\subset\Omega$ and $l(Q_j)<d(Q_j,\partial\Omega)/10$.

We now examine the case $l(Q_j)\geq d(Q_j,\partial\Omega)/10$. Let us fix a constant $A\gg1$ to be determined later on. By Lemma 4.9 there exists a constant $C_0=C_0(A,M,n)$ such that if $Q\in W_3$ and $\operatorname{supp}\psi_Q\cap\{\mathbf{R}^n\setminus C_0Q_j\}\neq\emptyset$, then $Q^*\cap AQ_j=\emptyset$. Using this we get quite painlessly that

$$
(11)\qquad
\begin{aligned}
I:=\int_{U_j\cap(\complement\Omega)^o}\left|\frac{EGb_j(x)}{d(x,\partial\Omega)}\right|^{n/(n-1)}dx&=\int_{\{Q_0\setminus C_0Q_j\}\cap(\complement\Omega)^o}\left|\frac{EGb_j(x)}{d(x,\partial\Omega)}\right|^{n/(n-1)}dx\\
\leq\sum_{\substack{Q\in W_3\\ Q\cap\{Q_0\setminus C_0Q_j\}\neq\emptyset}}\int_Q\left|\frac{EGb_j(x)}{d(x,\partial\Omega)}\right|^{n/(n-1)}dx&\lesssim\int_{\Omega\setminus AQ_j}\left|\frac{Gb_j(x)}{d(x,\partial\Omega)}\right|^{n/(n-1)}dx
\end{aligned}
$$

by Lemma 4.2 and Lemma 5.4. Choosing $A=A(M,n)$ sufficiently large we get by Lemma 3.1 for some $y_0\in\Omega$, $G(x,y)\lesssim G(x,y_{Q_j})$ for all $x\in\Omega\setminus AQ_j$ and $y\in Q_j$. Combining this with Theorem 7.1 completes the proof.

The case $Q_j\cap\partial\Omega\neq\emptyset$ is analyzed similarly to the previous case.

We will close this section with the proof of Theorem 7.3. The material presented here may be found in Nyström [21]. We can prove the following.

Theorem 7.2. *Let $\Omega\subset\mathbf{R}^n$, $n\geq2$, be a bounded* NTA*-domain. Then there exists a constant $C=C(M,r_0,\operatorname{diam}(\Omega),n)$ such that for all $u\in W_0^{1,2}(\Omega)$,*

$$\int_\Omega\left|\frac{u(x)}{d(x,\partial\Omega)}\right|^2dx\leq C\int_\Omega|\nabla u|^2\,dx.$$

Proof. Using [21] all we have to verify is that Ω satisfies a uniform capacity condition in the following sense. Let $Q\in W_\Omega$ and let $d(p_Q,Q)=d(Q,\partial\Omega)$ where $p_Q\in\partial\Omega$. Define C_Q to be the smallest cube centered at p_Q and containing Q with sides parallel to the axes. What we then have to prove is that

$$(1)\qquad \inf_{Q\in W_\Omega}C_{1,p_0}(C_Q\cap(\complement\Omega)^o)\geq Cl(Q)^{n-p_0},$$

for some $p_0\in(2n/(n+2),2)$ and some $C=C(M,r_0,\operatorname{diam}(\Omega),n,p_0)$. Here $C_{1,p_0}(E)$ denotes the Bessel capacity. For its definition, see [1], [19]. As Ω is a bounded and connected NTA-domain we may conclude that for every $Q\in W_\Omega$ there exists a ball $B_Q\subset C_Q\cap(\complement\Omega)^o$ with radius at least

$$(2)\qquad \frac{C(M,r_0,n)}{\operatorname{diam}(\Omega)}l(Q)=Cl(Q).$$

Using this,

$$C_{1,p_0}(C_Q\cap(\complement\Omega)^o)\geq C_{1,p_0}(B_Q)\geq Cl(Q)^{n-p_0}, \tag{3}$$

by the monotonicity of the capacity and the fact that $C_{1,p_0}(B(x,r))\sim r^{n-p_0}$. The conclusion in (1) is therefore established and Theorem 7.2 proved.

Using Green's formula we have the following consequence of Theorem 7.2.

Corollary. *Suppose $u\in W_0^{1,2}(\Omega)$ and $\Delta u\in L^2(\Omega)$. Then*

$$\int_\Omega\left|\frac{u(x)}{d(x,\partial\Omega)}\right|^2 dx\leq C\int_\Omega|\Delta u|^2 d(x,\partial\Omega)^2\,dx,$$

where C is the constant given in Theorem 7.2.

Theorem 7.3 is now a simple consequence of this corollary as if $f\in L^2(\Omega)$, then $\Delta Gf=-f$ in the sense of distributions and $Gf\in W_0^{1,2}(\Omega)$.

8. Comparison of potentials and harmonic functions

In this section we present the essential lemmas needed in Section 9 where we establish a good-λ-inequality which is at the heart of the method we are using. Lemma 8.1 and Lemma 8.2 below could have been formulated in just one lemma, but we believe that the way these very crucial steps are presented will make it a bit easier for the reader to follow the ideas. Lemma 8.2 should be considered as a dilated version of Lemma 8.1. Though the proof of Lemma 8.2 is identical in spirit to the proof of Lemma 8.1 we present it to be able to point out the dilation invariances we are using. Furthermore, Lemma 8.1 and 8.2 are proved just to be able to prove Lemma 8.3. The dilation argument is needed to avoid the constants in Lemma 8.3 to blow up for small cubes. We start by formulating the lemmas and then complete the section with the proofs. All the cubes appearing have sides parallel to a fixed system of coordinate axes.

Lemma 8.1. *Let $\Omega\in\mathrm{Domain}(n,M,r_0,q)$, $q>n/(n-1)$. Let Q_0 be a cube centered at $x_0\in\partial\Omega$ with $l(Q_0)=\delta$. Let $u(x)$ be a positive harmonic function on Ω which vanishes continuously on $\Delta(Q_0,r_0)$ and is such that $u(x_1)=1$ for some $x_1\in\Omega\cap Q_0$ fulfilling $d(x_1,\partial\Omega)\geq l(Q_0)/4$. Define $M_1(Q_0):=\{Q\subseteq Q_0:l(Q)\geq l(Q_0)/100\sqrt{n},\ Q\cap\Omega\neq\emptyset\}$. Pick $Q\in M_1(Q_0)$ and let $f\in L^1(Q)$, $f\geq0$, $\operatorname{supp} f\subseteq Q\cap\Omega$ and $\int_Q f(x)\,dx< m_n(Q)\sim\delta^n$. Then there exist constants $\alpha=\alpha(M,n)$ and $C=C(M,r_0,\delta,n,\mathrm{diam}(\Omega),$ $\mathrm{dimloc}(\partial\Omega))$ such that*

$$\int_E\left|\frac{Eu(x)}{d(x,\partial\Omega)}\right|^{n/(n-1)} dx\leq C\left(\int_Q f(x)\,dx\right)^\alpha,$$

where $E{:=}\{x{\in}Q{:}EGf(x){\geq}Eu(x)\}$.

We now formulate Lemma 8.2 which is a dilated version of Lemma 8.1 and deals with cubes in $M_2(Q_0){:=}\{Q{:}Q{\subseteq}Q_0,\ l(Q){<}l(Q_0)/100\sqrt{n};\ Q{\cap}\partial\Omega{\neq}\emptyset$ or $Q{\subset}\Omega,\ l(Q){\geq}d(Q,\partial\Omega)/10\}$. Choose $Q{\in}M_2(Q_0)$ and associate a cube A_Q centered at $p_Q{\in}\partial\Omega$ and defined as the smallest cube with sides parallel to the coordinate axes and containing Q. Here $d(p_Q,Q){=}d(Q,\partial\Omega)$ in case $l(Q){\geq}d(Q,\partial\Omega)/10$, $Q{\subset}\Omega$, and $p_Q{\in}Q{\cap}\Omega$ if $Q{\cap}\partial\Omega{\neq}\emptyset$. Then $l(A_Q){\sim}l(Q)$. Associate a dilation factor r_Q to Q by $r_Q{=}l(A_Q)/l(Q_0)$. We now dilate Ω with respect to $r{=}r_Q$ and $p{=}p_Q$ to get Ω'. If G' is the Green function of Ω', then $G'(x',y'){=}r^{n-2}G(rx'{+}p,ry'{+}p)$. Define $E'G'(x',y'){=}r^{n-2}EG(rx'{+}p,ry'{+}p)$. That is, in extending $G'(x',y')$ to $\mathbf{R}^n$ for fixed $y'{\in}\Omega'$, we just make use of the fact that we have extended $G(x,y)$ to $EG(x,y)$. We furthermore define, $q(x'){=}Eu(rx'{+}p)/Eu(x_{Q_1})$, where $Q_1{=}Q$ if $Q{\cap}\partial\Omega{=}\emptyset$ and $Q_1{=}Q_*$ if $Q{\cap}\partial\Omega{\neq}\emptyset$. Here Q_* is the cube existing by Lemma 4.4. Put

$$k(x'):=\left|\frac{q(x')}{d(x',\partial\Omega')}\right|^{n/(n-1)}.$$

Let by the dilation, $A_Q{\to}A'_Q$. Define for $E'{\subset}A'_Q$,

$$\lambda'(E'):=\int_{E'}k(x')\,dx'.$$

Lemma 8.2. *Let* Ω, Q_0 *and* $u(x)$ *be as in the statement of Lemma* 8.1. *Let* $Q{\in}M_2(Q_0)$, *where* $M_2(Q_0){:=}\{Q{:}Q{\subseteq}Q_0,l(Q){<}l(Q_0)/100\sqrt{n};\ Q{\cap}\partial\Omega{\neq}\emptyset$ *or both* $Q{\subset}\Omega,\ l(Q){\geq}d(Q,\partial\Omega)/10\}$. *Let* $Q{\to}Q'$ *by the dilation described above. Define* $E'{:=}\{x'{\in}Q'{:}E'G'F(x'){\geq}q(x')\}$ *where* $F{\in}L^1(Q')$, $F{\geq}0$, $\operatorname{supp}F{\subseteq}Q'{\cap}\Omega'$ *and* $\int_{Q'}F\,dx'{<}m_n(Q'){\sim}\delta^n$. *Then*

$$\lambda'(E')\leq C\biggl(\int_{Q'}F(x')\,dx'\biggr)^{\alpha},$$

where C *and* α *are the constants described in Lemma* 8.1.

Lemma 8.1 and Lemma 8.2 are important in the proof of Lemma 8.3.

Lemma 8.3. *Let* Ω, Q_0 *and* $u(x)$ *be as in the statement of Lemma* 8.1. *Then there exist constants* C *and* α *as in Lemma* 8.1 *and* $C_1{=}C_1(M,n,\delta)$ *such that the following is valid. If* $\gamma{\in}(0,C_1)$ *and if* $f{\in}L^1(Q)$, $f{\geq}0$, $\operatorname{supp}f{\subseteq}Q{\cap}\Omega$ *for some* $Q{\subseteq}Q_0$, $Q{\cap}\Omega{\neq}\emptyset$, *fulfills*

$$\int_Q f\,dx\leq\gamma\biggl(\int_Q\left|\frac{Eu(x)}{d(x,\partial\Omega)}\right|^{n/(n-1)}dx\biggr)^{(n-1)/n},$$

then

$$\int_E \left|\frac{Eu(x)}{d(x,\partial\Omega)}\right|^{n/(n-1)} dx \le C\gamma^\alpha \int_Q \left|\frac{Eu(x)}{d(x,\partial\Omega)}\right|^{n/(n-1)} dx,$$

where $E:=\{x\in Q: EGf(x)\ge Eu(x)\}$.

Proof of Lemma 8.1. Put $\gamma=\int_Q f\,dx$. There is no loss of generality to assume that Q is a dyadic cube. Consider Q as our universe and take the Calderón–Zygmund decomposition of f with threshold $\lambda=1$ (see [41, pp. 17–19]). That is, $f(x)=f_1(x)+f_2(x)$ where,

$$f_2(x)=\sum b_j(x), \quad \operatorname{supp} b_j\subseteq Q_j\cap\Omega,\ Q_j\cap\Omega\ne\emptyset,\ Q_j\subseteq Q, \tag{1}$$

$$\sum \|b_j\|_{L^1(Q_j)}\le C\gamma, \quad m_n\Big(\bigcup Q_j\Big)\le C\gamma. \tag{2}$$

Here $C=C(n)$ and $\{Q_j\}$ is a disjoint collection of cubes. As $f_1\le 1$ a.e. on Q we have

$$\int_Q f_1^2\,dx \le \int_Q f_1\,dx=\gamma. \tag{3}$$

Define

$$\lambda(E):=\int_E \left|\frac{Eu(x)}{d(x,\partial\Omega)}\right|^{n/(n-1)} dx, \quad E\subset Q.$$

Put $E_i:=\{x\in Q: EGf_i(x)\ge Eu(x)/2\}$ for $i=1,2$. Then $E\subset E_1\cup E_2$. To prove the lemma we intend to prove that

$$\lambda(E_i)\le C_i\gamma^{\alpha_i}, \quad i=1,2, \tag{4}$$

with C_i and α_i as in the statement of the lemma. We start to estimate $\lambda(E_1)$. We have that

$$\begin{aligned}\lambda(E_1)&\le \int_Q \left|\frac{EGf_1(x)}{d(x,\partial\Omega)}\right|^{n/(n-1)} dx\\ &\le \int_{Q\cap\Omega} \left|\frac{Gf_1(x)}{d(x,\partial\Omega)}\right|^{n/(n-1)} dx+\int_{Q\cap(\complement\Omega)^\circ} \left|\frac{EGf_1(x)}{d(x,\partial\Omega)}\right|^{n/(n-1)} dx.\end{aligned}$$

But by Lemmas 5.4 and 4.2 we have

$$\int_{Q\cap(\complement\Omega)^\circ} \left|\frac{EGf_1(x)}{d(x,\partial\Omega)}\right|^{n/(n-1)} dx \lesssim \int_\Omega \left|\frac{Gf_1(x)}{d(x,\partial\Omega)}\right|^{n/(n-1)} dx.$$

Therefore by the Hölder inequality,

$$\lambda(E_1) \le C(M,n,\operatorname{diam}(\Omega)) \left(\int_\Omega \left| \frac{Gf_1(x)}{d(x,\partial\Omega)} \right|^2 dx \right)^{n/(2n-2)}.$$

As $f_1 \in L^2(\Omega)$ we get from Theorem 7.3 and (3) that

$$\lambda(E_1) \le C\gamma^{n/(2n-2)},$$

which completes the estimate in this case.

We now estimate $\lambda(E_2)$. Let C_0 be the constant appearing in Lemma 7.2. Put $U := Q \setminus \bigcup C_0 Q_j$. Using Lemma 7.2 and the Minkowski inequality we get

$$\begin{aligned} \lambda(E_2 \cap U)^{(n-1)/n} &\le \left(\int_{E_2 \cap U} \left| \frac{EGf_2(x)}{d(x,\partial\Omega)} \right|^{n/(n-1)} dx \right)^{(n-1)/n} \\ &\le \sum_j \left(\int_{E_2 \cap U} \left| \frac{EGb_j(x)}{d(x,\partial\Omega)} \right|^{n/n-1} dx \right)^{(n-1)/n} \\ &\le \sum_j \left(\int_{Q \setminus C_0 Q_j} \left| \frac{EGb_j(x)}{d(x,\partial\Omega)} \right|^{n/(n-1)} dx \right)^{(n-1)/n} \\ &\le C \sum_j \|b_j\|_{L^1(Q_j)} \le C\gamma. \end{aligned} \tag{5}$$

By Lemma 6.1 we get

$$\lambda(E_2 \setminus U) \le \sum_j \lambda(C_0 Q_j) \le C \sum_j \lambda(Q_j) = C\lambda\Big(\bigcup Q_j\Big), \tag{6}$$

as $\{Q_j\}$ is a disjoint collection. There are never any problems connected to the application of Lemma 6.1 as we may if necessary just choose a smaller initial cube Q_0. Let $A \subseteq Q$ and let $q > n/(n-1)$ be a q for which the reverse Hölder inequalities of Theorem 6.1 are valid. We then get,

$$\lambda(A) \le C|A|^{1-n/(qn-q)} |Q|^{n/(qn-q)} \left| \frac{Eu(x_{Q_1})}{d(x_{Q_1},\partial\Omega)} \right|^{n/(n-1)}, \tag{7}$$

where $Q_1 = Q$ or $Q_1 = Q_*$, depending on whether or not $Q \cap \partial\Omega \ne \emptyset$. By assumption $l(Q) \sim \delta$. Therefore using (7),

$$\lambda(A) \le C|A|^{1-n/(qn-q)} l(Q)^{n^2/(qn-q)-n/(n-1)} |Eu(x_{Q_1})|^{n/(n-1)}. \tag{8}$$

But $Eu(x_{Q_1})\sim u(x_1)=1$, so may conclude that there exists a constant $C=C(M,n,\delta)$ such that

$$\lambda(A)\leq C|A|^{1-n/(qn-q)}. \tag{9}$$

Combining (2), (6) and (9) completes the proof.

We now present the proof of Lemma 8.2 to help the reader understand how we use the fact that the results of Section 6 and 7 are valid with uniform constants for arbitrary small cubes. It turns out that the extra weight in the inequality of Theorem 7.3 is crucial.

Proof of Lemma 8.2. Put $\gamma=\int_{Q'}F(x')\,dx'$. Choose just as in the proof of Lemma 8.1 a Calderón–Zygmund decomposition of $F=F_1+F_2$, $F_2=\sum B_j$, $\operatorname{supp} B_j\subseteq Q'_j\cap\Omega'$, $\sum\int_{Q'_j}|B_j|\,dx'\leq c\gamma$, $|\bigcup Q'_j|\leq c\gamma$, $F_1(x')\leq 1$ a.e. on Q'. Again we have $E'\subset E'_1\cup E'_2$, where $E'_i:=\{x'\in Q':E'G'F_i(x')\geq q(x')/2\}$. Again we want to estimate $\lambda'(E'_1)$, $\lambda'(E'_2)$. We first estimate $\lambda'(E'_1)$. Define f_1 by

$$E'G'F_1(x')=EGf_1(rx'+p).$$

Then $f_1(y)=F_1(y-p/r)/r^2$ and we get

$$\begin{aligned}\lambda'(E'_1)&\leq\left(\int_{Q'}\left|\frac{EGf_1(rx'+p)}{d(x',\partial\Omega')}\right|^{n/(n-1)}dx'\right)\\&\leq\left(\int_{Q'}\left|\frac{EGf_1(rx'+p)}{d(x',\partial\Omega')}\right|^{2}dx'\right)^{n/(2n-2)}\\&=r^{(2-n)n/(2n-2)}\left(\int_{Q}\left|\frac{EGf_1(x)}{d(x,\partial\Omega)}\right|^{2}dx\right)^{n/(2n-2)}\\&\leq C(M,n)r^{(2-n)n/(2n-2)}\left(\int_{\Omega}\left|\frac{Gf_1(x)}{d(x,\partial\Omega)}\right|^{2}dx\right)^{n/(2n-2)}\end{aligned}\tag{1}$$

as before. But by Theorem 7.3,

$$\int_{\Omega}\left|\frac{Gf_1(x)}{d(x,\partial\Omega)}\right|^{2}dx\leq C\int_{Q\cap\Omega}|f_1(x)|^2d(x,\partial\Omega)^2\,dx \tag{2}$$

as $\operatorname{supp} f_1\subseteq Q\cap\Omega$. As $Q\in M_2(Q_0)$ we always have $d(x,\partial\Omega)\leq C(n)l(Q)$ for all $x\in Q\cap\Omega$. Combining (1) and (2) we may conclude that

$$\lambda'(E'_1)\leq Cr^{(2-n)n/(2n-2)}l(Q)^{n/(n-1)}\left(\int_{Q}|f_1|^2\,dx\right)^{n/(2n-2)}. \tag{3}$$

But

$$\int_Q |f_1|^2\,dx = r^{n-4}\int_{Q'} |F_1(x')|^2\,dx' \le r^{n-4}\gamma. \tag{4}$$

Combining (3) and (4) we obtain

$$\lambda'(E_1') \le C r^{(2-n)n/(2n-2)} l(Q)^{n/(n-1)} r^{(n-4)n/(2n-2)} \gamma^{n/(2n-2)}. \tag{5}$$

But $r \sim l(Q)/l(Q_0) = l(Q)/\delta$. Therefore, $\lambda'(E_1') \le C\gamma^{n/(2n-2)}$, with C as in the deduction of Lemma 8.1.

We now estimate $\lambda'(E_2')$. Put $U' = Q' \setminus \bigcup C_0 Q_j'$, where C_0 is the constant of Lemma 7.2. Define $b_j \in L^1(Q_j)$ by

$$E'G'B_j(x') = EGb_j(rx'+p).$$

Then $b_j(y) = B_j((y-p)/r)/r^2$ by the scaling law for the Green function described above. We have by Lemma 7.2

$$\begin{aligned}\lambda'(E_2' \cap U')^{(n-1)/n} &\le r^{2-n} \sum_j \left(\int_{Q \setminus C_0 Q_j} \left| \frac{EGb_j(x)}{d(x,\partial\Omega)} \right|^{n/(n-1)} dx \right)^{(n-1)/n} \\ &\le C r^{2-n} \sum_j \|b_j\|_{L^1(Q_j)}.\end{aligned}$$

But $\int_{Q_j} |b_j|\,dx = r^{n-2} \int_{Q_j'} |B_j|\,dx'$ and $\sum_j \int_{Q_j'} |B_j|\,dx' \le c\gamma$ which give us

$$\lambda'(E_2' \cap U') \le C\gamma^{n/(n-1)}. \tag{6}$$

Again

$$\lambda'(E_2' \setminus U') \le C\lambda'\Bigl(\bigcup Q_j'\Bigr). \tag{7}$$

Let $A' \subseteq Q'$. Then

$$\lambda'(A') \le |A'|^{1-n/(qn-q)} \left(\int_{Q'} \left| \frac{q(x')}{d(x',\partial\Omega')} \right|^q dx' \right)^{n/(qn-q)}. \tag{8}$$

But

$$\begin{aligned}&\left(\int_{Q'} \left| \frac{q(x')}{d(x',\partial\Omega')} \right|^q dx' \right)^{n/(qn-q)} \\ &\qquad = \left| \frac{1}{Eu(x_{Q_1})} \right|^{n/(n-1)} \left(\int_{Q'} \left| \frac{Eu(rx'+p)}{d(x',\partial\Omega')} \right|^q dx' \right)^{n/(qn-q)} \\ &\qquad = \left| \frac{1}{Eu(x_{Q_1})} \right|^{n/(n-1)} r^{(q-n)n/(qn-q)} \left(\int_Q \left| \frac{Eu(x)}{d(x,\partial\Omega)} \right|^q dx \right)^{n/(qn-q)}\end{aligned}$$

Using Theorem 6.1 we may continue our estimate,

$$(9)\qquad \left(\int_{Q'}\left|\frac{q(x')}{d(x',\partial\Omega')}\right|^q dx'\right)^{n/(qn-q)} \lesssim r^{(q-n)n/(qn-q)}|Q|^{n/(qn-q)}\frac{1}{l(Q)^{n/(n-1)}}.$$

But $r\sim l(Q)/\delta$. Combining (7), (8) and (9) we get the required estimate with the same constant as in Lemma 8.1. This completes the proof.

We now apply these lemmas to prove Lemma 8.3. It is necessary to carry out a dilation argument to get uniform constants.

Proof of Lemma 8.3. We divide the proof into two cases:
1. $Q\subset\Omega$, $l(Q)<d(Q,\partial\Omega)/10$,
2. $Q\subset\Omega$, $l(Q)\geq d(Q,\partial\Omega)/10$ or $Q\cap\partial\Omega\neq\emptyset$.

In each case we will carry out a dilation argument but with different dilation factors. In case 1 we put $r=d(Q,\partial\Omega)$, $p=x_Q$. In case 2 we put, if $l(Q)<l(Q_0)/100\sqrt{n}$, $r=l(A_Q)/l(Q_0)\sim l(Q)/l(Q_0)$ and $p\in\partial\Omega$ as above. If $l(Q)\geq l(Q_0)/100\sqrt{n}$ we do not carry out any dilations. In case 1 the dilation implies that $Q\rightarrow Q'$, where Q' is situated at unit distance from $\partial\Omega'$. In case 2, Q' will be a cube of the same size as the initial cube Q_0, i.e. $l(Q')\sim\delta$. In case 2 we will only present the proof in the case $l(Q)<l(Q_0)/100\sqrt{n}$. The proof in the case $l(Q)\geq l(Q_0)/100\sqrt{n}$ just differs in the sense that we then use Lemma 8.1 instead of Lemma 8.2. As before $\Omega\rightarrow\Omega'$, $Q\rightarrow Q'$ under the dilation and $q(x')=Eu(rx'+p)/Eu(x_{Q_1})$. Here $Q_1=Q$ if $Q\subset\Omega$ and $Q_1=Q_*$ if $Q\cap\partial\Omega\neq\emptyset$. Let

$$k(x'):=\left|\frac{q(x')}{d(x',\partial\Omega')}\right|^{n/(n-1)}.$$

We get

$$(1)\qquad \int_E\left|\frac{Eu(x)}{d(x,\partial\Omega)}\right|^{n/(n-1)}dx=Eu(x_{Q_1})^{n/(n-1)}r^{n-n/(n-1)}\int_{E'}k(x')\,dx'.$$

That is, what we have to prove is that

$$(2)\qquad \int_{E'}k(x')\,dx'\leq c\gamma^\alpha\int_{Q'}k(x')\,dx',$$

where $E':=\{(x-p)/r:x\in E\}$. We now rephrase the assumption in its dilated form. As before, $EGf(rx'+p)=E'G'F(x')$, where $F(x')=f(rx'+p)r^2$. By the assumption we have

$$
\begin{aligned}
\int_{Q'} F(x')\,dx' = r^{2-n}\int_Q f(x)\,dx &\le \gamma r^{2-n}\left(\int_Q\left|\frac{Eu(x)}{d(x,\partial\Omega)}\right|^{n/(n-1)}dx\right)^{(n-1)/n}\\
&= \gamma r^{2-n} Eu(x_{Q_1}) r^{(n-n/(n-1))(n-1)/n}\left(\int_{Q'} k(x')\,dx'\right)^{(n-1)/n} \qquad (3)\\
&= \gamma Eu(x_{Q_1})\left(\int_{Q'} k(x')\,dx'\right)^{(n-1)/n}
\end{aligned}
$$

by (1). Define $h(x')=F(x')/Eu(x_{Q_1})$. We get $E':=\{(x-p)/r : x\in E\}=\{x'\in Q': E'G'F(x')\ge Eu(x_{Q_1})q(x')\}=\{x'\in Q': E'G'h(x')\ge q(x')\}$. After these changes to the dilated scale we may formulate our problem as follows. Show that,

$$E' = \{x'\in Q' : E'G'h(x') \ge q(x')\}, \tag{4}$$

$$\int_{Q'} h(x')\,dx' \le \gamma\left(\int_{Q'} k(x')\,dx'\right)^{(n-1)/n}, \tag{5}$$

$$\operatorname{supp} h \subseteq Q'\cap\Omega', \tag{6}$$

imply,

$$\int_{E'} k(x')\,dx' \le c\gamma^\alpha \int_{Q'} k(x')\,dx'. \tag{7}$$

We now treat case 1 and case 2 separately. First case 1. In this case $Q\subset\Omega$. Therefore no extension operators are involved. In this case Q' is at unit distance from $\partial\Omega'$. By the Harnack inequality, $q(x')\ge C$ and $k(x')\le C$ for all $x'\in Q'$, with $C=C(n)$. This gives us $E'\subset\{x'\in Q' : G'h(x')\ge C\}$. By Lemma 3.6, $G'(x',y')\le C(n)|x'-y'|^{1-n}$ for all $x',y'\in Q'$. Therefore, $E'\subset\{x'\in Q' : I_1h(x')\ge C\}$. Using this together with Theorem 3.2 and (5) we may deduce

$$
\begin{aligned}
\int_{E'} k(x')\,dx' &\le C|E'| \le |\{x'\in Q' : I_1 h(x')\ge C\}|\\
&\lesssim \left(\int_{Q'} h(x')\,dx'\right)^{n/(n-1)} \le C\gamma^{n/(n-1)}\int_{Q'} k(x')\,dx',
\end{aligned}
$$

which completes the proof in this case.

We now examine case 2. We just supply the proof when $Q\cap\partial\Omega\ne\emptyset$ as this is the only case when the extension is involved. In this case we get from Theorem 6.1

and (1) that

$$(8)\quad \begin{aligned}\int_{Q'} k(x')\,dx' &= \left|\frac{1}{Eu(x_{Q_1})}\right|^{n/(n-1)} r^{-n+n/(n-1)} \int_Q \left|\frac{Eu(x)}{d(x,\partial\Omega)}\right|^{n/(n-1)} dx \\ &\sim \frac{|Q|}{Eu(x_{Q_1})^{n/(n-1)}} \frac{1}{r^{n-n/(n-1)}} \left|\frac{Eu(x_{Q_1})}{d(x_{Q_1},\partial\Omega)}\right|^{n/(n-1)} \sim C(M,n,\delta)\end{aligned}$$

by our choice of r in this case, i.e., $\int_{Q'} k\,dx' \le C(M,n,\delta)$ in this case. Restrict γ to $(0,C_1)$, where $C_1 = m_n(Q')(1/C(M,n,\delta))^{(n-1)/n} \sim \delta^n (1/C(M,n,\delta))^{(n-1)/n}$. Then using (5) we have

$$(9)\qquad \int_{Q'} h(x')\,dx' < m_n(Q') \sim \delta^n.$$

(9) in combination with Lemma 8.2 and (5) give us

$$\begin{aligned}\int_{E'} k(x')\,dx' &\le C\left(\int_{Q'} h(x')\,dx'\right)^{\alpha} \\ &\le C\gamma^{\alpha}\left(\int_{Q'} k(x')\,dx'\right)^{\alpha(n-1)/n} \le C\gamma^{\alpha} \le C\gamma^{\alpha}\int_{Q'} k(x')\,dx',\end{aligned}$$

as by (8), $\int_{Q'} k(x')\,dx' \sim C(M,n,\delta)$. This completes the proof of Lemma 8.3.

9. Proof of a good-λ-inequality

In this section we deduce a good-λ-inequality for an operator $Tf(x)$ and an associated maximal function $Kf(x)$. We then integrate this to get norm inequalities. We ask the reader to refresh his/her knowledge of the extensions made in Definitions 5.1 and 5.2. In the following Q_0 will always denote the initial cube centered at $x_0 \in \partial\Omega$ with $l(Q_0)=\delta$ which we have used before. $u(x)$ will be a positive harmonic function fulfilling the requirements that u vanishes continuously on $\partial\Omega \cap B(x_0,r_0)$ and $u(x_1)=1$ for some $x_1 \in \Omega \cap Q_0$ such that $d(x_1,\partial\Omega) \ge l(Q_0)/4$.

As usual we start by summarizing the results of the section and then supply the proofs at the end.

Definition 9.1. Let $f \in C_0^\infty(\Omega)$. Define

$$Tf(x) := \chi_\Omega(x)\frac{Gf(x)}{u(x)} + \chi_{(\complement\Omega)^\circ}(x) \sum_{Q\in W_3} \frac{EGf(x_Q)}{Eu(x_Q)}\psi_Q(x)$$

if $x \in (\Omega \cup \bigcup_{Q\in W_3} Q)) \cap 2Q_0$ and

$$Tf(x) = \lim_{x_j\to x,\, x_j\in\Omega} Tf(x_j)$$

if $x \in \frac{3}{2}Q_0 \cap \partial\Omega$.

Lemma 9.1. *Let $f \geq 0$, $f \in C_0^\infty(\Omega)$. Then Tf is a continuous function on Q_0.*

This is a crucial lemma, as we will see below, because it implies that the set $Q_0^o \cap \{Tf(x) > \lambda\}$ is an open set for all $\lambda > 0$ to which we may apply the Whitney decomposition. In the proof of our main results, presented in Section 11, we may by an approximation argument assume that $f \in C_0^\infty(\Omega)$, $f \geq 0$. We may therefore all the time assume that f has these smooth properties. The proof of Lemma 9.1 relies on the fact that if $f \in C_0^\infty(\Omega)$, $f \geq 0$, then $Gf(x)$ is a positive harmonic function in a small neighbourhood of $\partial\Omega$.

Let $C_0 = C_0(M, n) \gg 1$ be the constant defined in Lemma 9.3 below. Let Q_1 be a cube with the same center as Q_0 but with sidelength $l(Q_0)/5C_0$.

Definition 9.2. Let $f \geq 0$, $f \in L^1(\mathbf{R}^n)$, $\operatorname{supp} f \subseteq \Omega$. Put $M_x := \{Q : x \in Q,\ Q \subseteq Q_0\}$. Define

$$Kf(x) := \sup_{Q \in M_x} \int_{Q \cap \Omega} f\,dx / \lambda(Q)^{(n-1)/n},$$

where

$$\lambda(Q) := \int_Q \left| \frac{Eu(x)}{d(x, \partial\Omega)} \right|^{n/(n-1)} dx.$$

Lemma 9.2. *Let $f \geq 0$, $f \in C_0^\infty(\Omega)$, $\operatorname{supp} f \subseteq Q_1 \cap \Omega$, where Q_1 is defined above. Then there exist constants $\alpha = \alpha(M, n)$, $\beta = \beta(M, n)$ and $C = C(M, n, \delta, r_0, \operatorname{diam}(\Omega))$ such that if $Q \subseteq 5Q_1$ and if $Tf(x) \leq 1$ for some $x \in Q$, then for all $\gamma \in (0, C_1)$, $C_1 = C_1(M, n, \delta)$, we have the following good-λ-inequality*

$$\lambda\{x \in Q : Tf(x) > \beta,\ Kf(x) \leq \gamma\} \leq C\gamma^\alpha \lambda(Q).$$

Integrating this we will prove the following.

Lemma 9.4. *For all q such that the reverse Hölder inequalities of Theorem 6.1 are valid, there exists a constant $C = C(M, n, \delta, r_0, q, \operatorname{diam}(\Omega))$ such that if $f \geq 0$, $f \in C_0^\infty(\Omega)$ and $\operatorname{supp} f \subset Q_1$ (Q_1 is the cube described above), then*

$$\int_{Q_1} |Tf|^q \, d\mu_q(x) \leq C \int_{Q_1} |Kf|^q \, d\mu_q(x),$$

with $d\mu_q(x) = |Eu(x)/d(x, \partial\Omega)|^q dx$.

The important thing is that if $x \in Q_1 \cap \Omega$ then

$$|Tf(x)|^q d\mu_q(x) = \left| \frac{Gf(x)}{d(x, \partial\Omega)} \right|^q dx.$$

The rest of this section is devoted to the proof of these results.

Proof of Lemma 9.1. As $f\in C_0^\infty(\Omega)$ there exists a $\tau>0$ such that $Gf(x)$ is a positive harmonic function in $B_\tau:=\{x\in 3Q_0\cap\Omega: d(x,\partial\Omega)<\tau\}$. As Ω is regular for the Dirichlet problem we further know that $Gf(x)$ vanishes continuously on $\partial\Omega$. See Helms [13, Lemma 6.24]. It therefore follows from Theorem 3.1 that $Tf(x)$ is well defined on $\partial\Omega\cap\frac{3}{2}Q_0$. By the same theorem it also follows that $Tf(x)$ is Hölder continuous in $B_\tau\cup\partial\Omega$ for some Hölder exponent α. That $Tf(x)$ is continuous at $x\in\Omega\cap(Q_0\setminus B_\tau)$ is trivial as for these points, $u(x)\neq 0$ and $Gf(x)$ is continuous on $\bar{\Omega}$. See Helms [13, Theorem 6.22]. To prove the continuity of $Tf(x)$ on Q_0 all we therefore have to do is to prove that if $p\in Q_0\cap\partial\Omega$ then,

$$\lim_{x_j\to p, x_j\in(\complement\Omega)^o} Tf(x_j)=Tf(p). \tag{1}$$

We start by proving that the limit in (1) actually exists. Put $N_\tau:=\{x\in\Omega: d(x,\partial\Omega)<\tau/10\}$. Let r_j be a sequence of numbers, to be fixed later on, such that $r_j\to 0$ as $j\to\infty$ and $r_{j+1}<r_j$. Define $\Omega_j:=(\complement\Omega)^o\cap B(p,r_j)$ and

$$M_j:=\sup_{x\in\Omega_j} Tf(x),\quad m_j:=\inf_{x\in\Omega_j} Tf(x),$$
$$A_j:=\{Q\in W_3:\ \operatorname{supp}\psi_Q\cap\Omega_j\neq\emptyset\}.$$

Let $A_j\to A_j^*$ by the reflection principle described in Section 4. We prove that $M_j/m_j\to 1$ as $j\to\infty$. By Lemma 4.1 there exists a $C=C(M,n)$ such that,

$$\bigcup_{Q\in A_j^*} Q\subset B(p,Cr_j)\cap\Omega. \tag{2}$$

Choose r_1 so that $M^2Cr_1\ll\tau/10$. By Theorem 2.1 we know that there exists Ω_j^* such that Ω_j^* is an NTA-domain with the same parameter M as Ω and such that

$$B(p,Cr_j)\cap\Omega\subset\Omega_j^*\subset B(p,CM^2r_j)\cap\Omega. \tag{3}$$

Take $x\in\Omega_j$ and put $S_x:=\{Q\in A_j:\psi_Q(x)\neq 0\}$. By the definition we get

$$Tf(x)\le\max_{Q\in S_x}\frac{EGf(x_Q)}{Eu(x_Q)},\quad Tf(x)\ge\min_{Q\in S_x}\frac{EGf(x_Q)}{Eu(x_Q)}. \tag{4}$$

Define

$$M_j^*:=\sup_{x\in\Omega_j^*} Tf(x),\quad m_j^*:=\inf_{x\in\Omega_j^*} Tf(x).$$

We now examine the definitions of $EGf(x)$ and $Eu(x)$ a bit closer. Let $Q\in A_j$. Then $Eu(x_Q)=u(x_{Q^*})$. Furthermore, $EGf(x_Q)=\int_{\Omega\setminus B_\tau} EG(x_Q,y)f(y)\,dy$. But $Q^*\subset N_\tau$. By Definition 5.2, $EG(x_Q,y)=G(x_{Q^*},y)$ for all $y\in\Omega\setminus B_\tau$. Therefore $EGf(x_Q)=Gf(x_{Q^*})$. (4) may therefore be restated as,

$$(5)\qquad Tf(x)\le \max_{Q\in S_x}\frac{Gf(x_{Q^*})}{u(x_{Q^*})},\quad Tf(x)\ge \min_{Q\in S_x}\frac{Gf(x_{Q^*})}{u(x_{Q^*})}.$$

From (2), (3) and (5) we get that $M_j\le M_j^*$, $m_j\ge m_j^*$, i.e., $M_j/m_j-1\le M_j^*/m_j^*-1$. Put $r_{j+1}=M^{-1}r_j$, where M is the NTA-constant of Ω. Using Theorem 3.1 it follows that $M_j^*/m_j^*-1\to 0$ as $Gf(x)$ and $u(x)$ are positive harmonic functions in B_τ. This proves that the limit in (1) exists. That the limit value is $Tf(p)$ is obvious.

Proof of Lemma 9.2. Recall that $Q_1=Q_0/5C_0$, where C_0 is the constant appearing in Lemma 9.3 below, $C_0=C_0(M,n)$. By the assumption, $C_0Q\subseteq Q_0$ for all $Q\subseteq Q_1$. Let Q be a fixed cube such that $Tf(x_0)\le 1$ for some $x_0\in Q$. Put $g(x)=f(x)$ if $x\in C_0Q$ and $g(x)\equiv 0$ otherwise. Put $h(x)=f(x)-g(x)$. By Lemma 9.3 below we get

$$(1)\qquad \sup_{x\in Q} Th(x)\lesssim \inf_{x\in Q} Th(x)\le C(M,n)Tf(x_0)\le C(M,n).$$

Let $\varepsilon>0$ be a constant to be fixed below. By (1) it follows that there exists $\beta=\beta(M,n,\varepsilon)$ such that

$$\{x\in Q:Tf(x)>\beta,\ Kf(x)\le\gamma\}\subseteq\{x\in Q:Tg(x)>\varepsilon,\ Kf(x)\le\gamma\}:=E.$$

Assume $E\neq\emptyset$ and take $x_1\in E$. Then

$$(2)\qquad \int_{C_0Q} g\,dx=\int_{C_0Q} f\,dx\le\lambda(C_0Q)^{(n-1)/n}Kf(x_1)\le\gamma\lambda(C_0Q)^{(n-1)/n}.$$

We note that if $C_0Q\cap\Omega=\emptyset$ and if $\beta=\beta(M,n)$ is sufficiently large, then $E=\emptyset$ by (1). We may therefore assume that $C_0Q\cap\Omega\neq\emptyset$. By Lemma 8.3 we get

$$(3)\qquad \lambda(\{x\in C_0Q:EGg(x)\ge Eu(x)\})\le C\gamma^\alpha\lambda(C_0Q).$$

$\lambda(C_0Q)\le C\lambda(Q)$ by Lemma 6.1. What remains to prove is that $E\subseteq\{x\in Q:EGg(x)\ge Eu(x)\}$. That is, what we have to prove is that there exists $\varepsilon=\varepsilon(M,n)$ such that if $x\in Q$ and $Tg(x)>\varepsilon$, then $EGg(x)\ge Eu(x)$. We note that it is enough that this is true for all $x \in Q \setminus \partial\Omega$ as $|\partial\Omega|=0$. If $x\in\Omega$ then this is trivial of course. Let $x\in(\complement\Omega)^o$. Then

$$\begin{aligned} Tg(x) &= \sum_{Q\in W_3} \psi_Q(x)\frac{EGg(x_Q)}{Eu(x_Q)} \\ &\sim |Eu(x)|^{-1} \sum_{Q\in W_3} \psi_Q(x) EGg(x_Q) = |Eu(x)|^{-1} EGg(x), \end{aligned}$$

which completes the proof.

We note that the lemma implies that if $\eta>0$ and $Tf(x)\le\eta$ for some $x\in Q$, then

$$\lambda\{x\in Q: Tf(x)>\beta\eta,\ Kf(x)\le\gamma\eta\}\le C\gamma^\alpha\lambda(Q).$$

Lemma 9.3. *Let Q_0 be our original initial cube centered at $x_0\in\partial\Omega$. Let $Q\subseteq Q_0$. Then there exists a constant $C_0=C_0(M,n)$ such that if $h\in C_0^\infty(\Omega)$ and* $\operatorname{supp} h\cap C_0Q=\emptyset$ *then,*

$$\sup_{x\in Q} Th(x)\le C(M,n)\inf_{x\in Q} Th(x).$$

Proof. We divide the proof into three cases:
1. $Q\subset\Omega$;
2. $Q\subset(\complement\Omega)^o$;
3. $Q\cap\partial\Omega\neq\emptyset$.

In the first case we have $Th(x)=Gh(x)/u(x)$ if $x\in Q$. The result therefore follows easily from the fact that if $y\in\Omega\setminus C_0Q$, $C_0=C_0(M,n)$ sufficiently large, then

$$\sup_{x\in Q}\frac{G(x,y)}{u(x)}\le C(M,n)\inf_{x\in Q}\frac{G(x,y)}{u(x)}. \tag{1}$$

If $l(Q)<d(Q,\partial\Omega)/10$ then (1) follows from the Harnack inequality with $C_0=10$. If $l(Q)\ge d(Q,\partial\Omega)/10$ then (1) is a consequence of Lemma 3.3.

In the second case

$$Th(x)=\sum_{Q_j\in W_3}\frac{EGh(x_{Q_j})}{Eu(x_{Q_j})}\psi_{Q_j}(x).$$

Put $W(Q):=\{Q_j\in W_3:\operatorname{Supp}\psi_{Q_j}\cap Q\neq\emptyset\}$. Suppose $l(Q)<d(Q,\partial\Omega)/10$. Then by Lemma 4.5, $\#W(Q)\le C$ and $l(Q_j)\sim d(Q,\partial\Omega)$ if $Q_j\in W(Q)$. Fix j_0 such that $Q_{j_0}\in W(Q)$. Then by the reflection principle and the Harnack inequality we have for all $x\in Q$,

$$Th(x)\sim\frac{1}{Eu(x_{Q_{j_0}})}\sum_{Q_j\in W(Q)}\psi_{Q_j}(x)EGh(x_{Q_j}). \tag{2}$$

So what we have to prove is that,

$$\max_{Q_j\in W(Q)} EGh(x_{Q_j}) \lesssim \min_{Q_j\in W(Q)} EGh(x_{Q_j}). \tag{3}$$

This follows if we can prove that if $y\in\Omega$ then,

$$\max_{Q_j\in W(Q)} EG(x_{Q_j},y) \lesssim \min_{Q_j\in W(Q)} EG(x_{Q_j},y).$$

But this is proved in the same way as Lemmas 5.2 and 5.3 were proved.

We now suppose that $l(Q)\ge d(Q,\partial\Omega)/10$. Choose $p\in\partial\Omega$, $d(p,Q)=d(Q,\partial\Omega)$. As before there exists $C=C(M,n)$ such that if $Q_j\in W(Q)$ then $Q_j\subset B(p,Cl(Q))$ and $Q_j^*\subset B(p,Cl(Q))$. Therefore, by the definition of $EGh(x)$, there exists $C_0=C_0(M,n)$ such that if $\operatorname{supp} h\cap C_0Q=\emptyset$ and $x\in Q$ then

$$\inf_{Q_j\in W(Q)} \frac{Gh(x_{Q_j^*})}{u(x_{Q_j^*})} \le Th(x) \le \sup_{Q_j\in W(Q)} \frac{Gh(x_{Q_j^*})}{u(x_{Q_j^*})}. \tag{4}$$

The conclusion

$$\inf_{Q_j\in W(Q)} \frac{Gh(x_{Q_j^*})}{u(x_{Q_j^*})} \sim \sup_{Q_j\in W(Q)} \frac{Gh(x_{Q_j^*})}{u(x_{Q_j^*})} \tag{5}$$

follows in the same way as (1) was proved if we just choose $C_0=C_0(M,n)$ sufficiently large and $\operatorname{supp} h\cap C_0Q=\emptyset$.

Left is the proof in the third case, that is when $Q\cap\partial\Omega\neq\emptyset$. Let $p\in Q\cap\partial\Omega$. By the same argument as above we get if $C_0=C_0(M,n)$ is sufficiently large,

$$\sup_{x\in Q} Th(x) \lesssim \sup_{x\in B(p,Cl(Q))} \frac{Gh(x)}{u(x)} \sim \inf_{x\in B(p,Cl(Q))} \frac{Gh(x)}{u(x)} \lesssim \inf_{x\in Q} Th(x). \tag{6}$$

The deduction in (6) completes the proof.

We end this section by proving Lemma 9.4. It is well known that Lemma 9.4 is a consequence of Lemma 9.2 but we supply the proof for completion (see e.g. [4], [20]).

Proof of Lemma 9.4. As we may assume Q_1 to be a dyadic cube we may adjust a dyadic net $\{\pi_j\}$ on $\mathbf{R}^n$ to Q_1 so that $Q_1\in\pi_j$ for some $j=j(Q_1)$. Let $B(x_{Q_1},10l(Q_1))$ be an open ball. By Lemma 9.1, the set

$$B_\eta := \{x : Tf(x)>\eta\}\cap B(x_{Q_1},10l(Q_1)) \tag{1}$$

is an open set. Let W_η be a Whitney decomposition of B_η such that $d(Q_j,\complement B_\eta)/4\leq l(Q_j)$ for all $Q_j\in W_\eta$. Put

$$E_\eta := Q_1\cap\{x: Tf(x)>\eta\},$$
$$S_\eta := Q_1\cap\{x: Tf(x)>\beta\eta\},$$
$$T_\eta := Q_1\cap\{x: Tf(x)>\beta\eta,\ Kf(x)\leq\gamma\eta\}.$$

We have

$$\begin{aligned}\int_{Q_1}|Tf|^q\,d\mu_q(x) &= \beta^q q\int_0^\infty \mu_q(S_\eta)\eta^{q-1}\,d\eta \\ &\leq \beta^q q\int_0^\infty \mu_q(T_\eta)\eta^{q-1}\,d\eta + C(\gamma,\beta,q)\int_{Q_1}|Kf|^q\,d\mu_q(x).\end{aligned} \tag{2}$$

We will use Lemma 9.2 to estimate the first integral in (2). We first note the following. Suppose $T_{\eta_0}\neq\emptyset$ for some $\eta_0\in(0,\infty)$. Choose $x_0\in\Omega\cap\partial(3Q_1)$ such that $d(x_0,\partial\Omega)\sim l(Q_1)$. Then

$$Tf(x_0)=\frac{Gf(x_0)}{u(x_0)}\leq M\frac{G(x_0,A)}{u(x_0)}\int_{Q_1}f(y)\,dy \tag{3}$$

by Lemma 3.1, where $A\in Q_1$. As $T_{\eta_0}\neq\emptyset$ we get for some $p\in T_{\eta_0}$,

$$Tf(x_0)\leq M\frac{G(x_0,A)}{u(x_0)}\lambda(Q_1)^{(n-1)/n}Kf(p)\leq C(M,n)\gamma\frac{G(x_0,A)}{u(x_0)}\lambda(Q_1)^{(n-1)/n}\eta_0.$$

But $G(x_0,A)\lesssim 1/l(Q_1)^{n-2}$ by Lemma 3.6 and

$$\lambda(Q_1)^{(n-1)/n}\lesssim|Q_1|^{(n-1)/n}u(x_0)/l(Q_1)$$

by Theorem 6.1 and the Harnack inequality, i.e., $Tf(x_0)\leq C(M,n)\gamma\eta_0$. If we restrict γ to $(0,1/C)$ we therefore get

$$Tf(x_0)\leq\eta_0. \tag{4}$$

(4) implies that if $Q_j\cap E_\eta\neq\emptyset$ and $Q_j\in W_\eta$, then $Q_j\subseteq Q_1$, and by properties of the Whitney cubes, there exists $p_j\in 5Q_j$ such that $Tf(p_j)\leq\eta$. This is valid for all $\eta\in(0,\infty)$. As $S_\eta\subseteq E_\eta$ we get a covering

$$S_\eta\subseteq\bigcup_{Q_j\in W_\eta,Q_j\subseteq Q_1}5Q_j.$$

As furthermore, $T_\eta \subseteq S_\eta$, we may deduce that

$$\mu_q(T_\eta) \le \sum_{Q_j \in W_\eta, Q_j \subseteq Q_1} \mu_q(5Q_j \cap T_\eta). \tag{5}$$

Theorem 6.1 implies that $d\mu_q \in A_\infty(d\lambda)$. See [4, Lemma 5]. By Lemma 9.2 and Lemma 6.1 we therefore get the following for some $\varepsilon = \varepsilon(q) > 0$,

$$\begin{aligned} \mu_q(T_\eta) &\le C\gamma^{\alpha\varepsilon} \sum_{Q_j \in W_\eta, Q_j \subseteq Q_1} \mu_q(5Q_j) \\ &\le C\gamma^{\alpha\varepsilon} \sum_{Q_j \in W_\eta, Q_j \subseteq Q_1} \mu_q(Q_j) = C\gamma^{\alpha\varepsilon} \mu_q(E_\eta). \end{aligned}$$

Using (2),

$$\int_{Q_1} |Tf|^q \, d\mu_q(x) \le C\beta^q q \gamma^{\alpha\varepsilon} \int_{Q_1} |Tf|^q \, d\mu_q(x) + C(\gamma) \int_{Q_1} |Kf|^q \, d\mu_q(x).$$

Choosing $\gamma < \gamma_0$, where $C\beta^q q \gamma_0^{\alpha\varepsilon} = \frac{1}{2}$, and taking the restrictions on γ into account, we may complete the proof.

10. An inequality for the maximal function

In this section we present the last two lemmas before we in the next section prove the final results.

Lemma 10.1. *Let the cubes Q_1 and Q_0 be as in Lemma 9.4. Let $q > n/(n-1)$ and suppose that for all cubes $Q \subseteq Q_0$ we have*

$$\left(\frac{1}{|Q|} \int_Q \left| \frac{Eu(x)}{d(x, \partial\Omega)} \right|^q dx \right)^{1/q} \le N \frac{1}{|Q|} \int_Q \frac{Eu(x)}{d(x, \partial\Omega)} \, dx,$$

with $N = N(M, n, q)$ as in Theorem 6.1. Then for all $f \in C_0^\infty(\Omega)$, $\operatorname{supp} f \subset Q_1$, $f \ge 0$ we have

$$\int_{Q_1 \cap \Omega} \left| \frac{Gf(x)}{d(x, \partial\Omega)} \right|^q dx \le C \|f\|_p^q,$$

where $1/q = 1/p - 1/n$ and $C = C(M, n, q, r_0, \delta, \operatorname{diam}(\Omega))$.

The next lemma is a weak type estimate for $p = 1$.

Lemma 10.2. *Let f and Q_1 be as in Lemma* 10.1. *Let $t>0$. Then there exists $C=C(M,n,q,r_0,\delta,\operatorname{diam}(\Omega))$ such that*

$$|\{x\in Q_1\cap\Omega : Gf(x)>td(x,\partial\Omega)\}|\le C(\|f\|_1/t)^{n/(n-1)}.$$

Proof of Lemma 10.1. By Lemma 9.4 we have

$$\int_{Q_1\cap\Omega}\left|\frac{Gf(x)}{d(x,\partial\Omega)}\right|^q dx\le\int_{Q_1}|Tf|^q\,d\mu_q(x)\le C\int_{Q_1}|Kf|^q\,d\mu_q(x).$$

To prove the lemma all we have to prove is that

$$\left(\int_{Q_1}|Kf|^q\,d\mu_q(x)\right)^{1/q}\le C\left(\int_{Q_1}|f|^p\,dx\right)^{1/p}. \tag{1}$$

The way (1) is proved is a kind of standard deduction for maximal functions involving a Besicovitch type covering and Marcinkiewicz interpolation theorem. The details are carried out in Dahlberg [6, Lemma 11].

Proof of Lemma 10.2. This proof follows the same lines as the proof in [6, Lemma 12]. Take the Calderón–Zygmund decomposition of f with threshold $z>\|f\|_1/m_n(Q_1)$. Then $f=f_1+f_2$, $f_2=\sum b_j$, $\operatorname{supp} b_j\subseteq Q_j\subseteq Q_1$, $\operatorname{supp} b_j\subseteq Q_j\cap\Omega$, $f_1\le z$ a.e. on Q_1, $\sum_j\int_{Q_j}b_j(x)\,dx\le C\|f\|_1$ and $|\bigcup Q_j|\le c\|f\|_1/z$. Put $E_i:=\{x\in Q_1\cap\Omega: Gf_i(x)>td(x,\partial\Omega)/2\}$. Using Lemma 10.1 with $q>n/(n-1)$,

$$|E_1|\le\frac{C}{t^q}\int_{Q_1\cap\Omega}\left|\frac{Gf_1(x)}{d(x,\partial\Omega)}\right|^q dx\le C\|f_1\|_p^q\frac{1}{t^q}\le C\frac{z^{q(1-1/p)}}{t^q}\|f\|_1^{q/p}. \tag{2}$$

Let $U=Q_1\setminus\bigcup C_0Q_j$, where C_0 is the constant appearing in Lemma 7.2. Then

$$\begin{aligned}\left(\int_{U\cap\Omega}\left|\frac{Gf_2(x)}{d(x,\partial\Omega)}\right|^{n/(n-1)}dx\right)^{(n-1)/n}&\le\sum_j\left(\int_{U\cap\Omega}\left|\frac{Gb_j(x)}{d(x,\partial\Omega)}\right|^{n/(n-1)}dx\right)^{(n-1)/n}\\&\le C\sum_j\|b_j\|_{L^1(Q_j)}\le C\|f\|_1.\end{aligned} \tag{3}$$

(3) implies that

$$|E_2\cap U|\le C\left(\frac{\|f\|_1}{t}\right)^{n/(n-1)}. \tag{4}$$

We further have

$$|E_2\cap(Q_1\setminus U)|\le C\left|\bigcup Q_j\right|\le C\|f\|_1/z. \tag{5}$$

From (2), (4) and (5) we may therefore conclude that,

$$|E|\le|E_1|+|E_2|\le C\left(\|f\|_1^{q/p}\frac{z^{q(1-1/p)}}{t^q}+\left(\frac{\|f\|_1}{t}\right)^{n/(n-1)}+\frac{\|f\|_1}{z}\right).$$

Choosing $z=t^{n/(n-1)}\|f\|_1^{-1/(n-1)}$ completes the proof.

11. Proof of the main theorem

As $|\nabla Gf(x)|\le C(\Omega)(I_1 f(x)+Gf(x)/d(x,\partial\Omega))$ if $f\ge 0$, it follows that our Main Theorem is a consequence of Theorem 11.1 below and Theorem 3.2.

Theorem 11.1. *Let $\Omega\subset\mathbf{R}^n$, $n\ge 2$, fulfill the requirements of the Main Theorem. Then there exist a constant $C=C(\Omega,q)$ such that if, $1/q=1/p-1/n$, then the following inequality is valid for all $f\in L^p(\Omega)$,*

$$\left(\int_\Omega\left|\frac{Gf(x)}{d(x,\partial\Omega)}\right|^q dx\right)^{1/q}\le C\left(\int_\Omega |f|^p\,dx\right)^{1/p}.$$

If $p=1$ then $|\{x\in\Omega:|Gf(x)|>td(x,\partial\Omega)\}|\le C(\|f\|_1/t)^{n/(n-1)}$.

Proof of Theorem 11.1. We note that by an approximation argument we just have to prove the theorem for $f\ge 0$ and $f\in C_0^\infty(\Omega)$. Let as before Q_0 be our initial cube and let Q_1 be the cube associated to Q_0 as described at the beginning of Section 9. Let Q_2 be a cube with the same center as Q_1 and sidelength $l(Q_2)=l(Q_1)/C_0$, where $C_0=C_0(M,n)$ is such that if $v(x)$ is a harmonic function on $Q_1\cap\Omega$ vanishing continuously on $\partial\Omega\cap Q_1$, then

$$\int_{Q_2\cap\Omega}\left|\frac{v(x)}{d(x,\partial\Omega)}\right|^q dx\le C|Q_2|\left|\frac{v(x_2)}{d(x_2,\partial\Omega)}\right|^q \tag{1}$$

for some $x_2\in Q_2\cap\Omega$, $d(x_2,\partial\Omega)\ge Cl(Q_2)$. That this always can be arranged follows from Theorem 6.1. Let Q_k be a covering of $\partial\Omega$ with cubes centered at $x_k\in\partial\Omega$ and such that each Q_k is related to a cube Q_k^* in the same way as Q_2 is related to Q_1. Let $\Omega_0:=\Omega\setminus\bigcup Q_k$. We may assume that $d(\Omega_0,\partial\Omega)>Cl(Q_2)$. Then

$$\int_\Omega\left|\frac{Gf(x)}{d(x,\partial\Omega)}\right|^q dx\le\int_{\Omega_0}\left|\frac{Gf(x)}{d(x,\partial\Omega)}\right|^q dx+\sum_k\int_{\Omega\cap Q_k}\left|\frac{Gf(x)}{d(x,\partial\Omega)}\right|^q dx. \tag{2}$$

The first integral in (2) is estimated by Lemma 3.6 and Theorem 3.2. Fix k and let $\varphi\in C_0^\infty(Q_k^*)$ such that $\varphi(x)=1$ on $(1-\varepsilon)Q_k^*$ for some small ε. Put $g_k(x)=(1-\varphi(x))f(x)$ and $h_k(x)=\varphi(x)f(x)$. Then

$$\int_{\Omega\cap Q_k}\left|\frac{Gf(x)}{d(x,\partial\Omega)}\right|^q dx\le\int_{\Omega\cap Q_k}\left|\frac{Gh_k(x)}{d(x,\partial\Omega)}\right|^q dx+\int_{\Omega\cap Q_k}\left|\frac{Gg_k(x)}{d(x,\partial\Omega)}\right|^q dx. \tag{3}$$

Call the integrals I_1 and I_2. Using Lemma 10.1 we get,

$$I_1\le C\|h_k\|_{L^p}^q\le C\|f\|_{L^p}^q. \tag{4}$$

Using (1),

$$I_2\le C|Gg_k(x_k)|^q\le C\|g_k\|_{L^1}^q\le C\|f\|_{L^1}^q. \tag{5}$$

This completes the proof of the first part. The second part follows from Lemma 10.2 and the weak-type estimate for Riesz potentials stated in Theorem 3.2.

12. Sharpness and geometry

In this section we try to disentangle the condition imposed on Ω in the sense that we try to rephrase it in a condition which is more easy to handle. We also address the question of sharp conditions for the validity of our main theorem. It is important to note, as mentioned before, that by using Lemma 3.5 and a simple calculation, it follows that if Ω is an NTA-domain with parameters M, r_0, then $\Omega\in\mathrm{Domain}(n, M, r_0, 1+1/(1-\beta))$ where $\beta=\beta(M)>0$ is the constant appearing in Lemma 3.5. That is for any bounded NTA-domain there exist $q>2$, such that the requirements of our main theorem are fulfilled.

In this section we assume for simplicity that $\mathrm{diam}(\Omega)=1$. If Ω is a bounded NTA-domain with Green function $G(x)=G(x,x_0)$, $d(x_0,\partial\Omega)\sim\mathrm{diam}(\Omega)$ we define for all $Q\in\partial\Omega$, $r<r_0$,

$$A(Q,r,\Omega,x_0,q)=\frac{J(Q,r,\Omega,x_0,q)}{J(Q,r,\Omega,x_0,1)},$$

where as before,

$$J(Q,r,\Omega,x_0,a)=\left(\frac{1}{|B(Q,r)\cap\Omega|}\int_{B(Q,r)\cap\Omega}\left|\frac{G(x)}{d(x,\partial\Omega)}\right|^a dx\right)^{1/a},$$

for $a\in[1,\infty)$. Let W_Ω denote the Whitney decomposition of Ω. Let $W_j:=\{Q\in W_\Omega : l(Q)=2^{-j}\}$ where $l(Q)$ denotes the sidelength of Q. For $q>0$ we introduce the number,

$$I_q(\Omega):=\sum_{j\geq 4}2^{j(q-n)}\sum_{Q\in W_j}G(x)^q.$$

Take $Q\in\partial\Omega$, $Mr<r_0$, where M, r_0 are the NTA parameters of Ω. According to Theorem 2.1 there exists an NTA-domain $\Omega_{Q,r}$ with the same NTA-constant as Ω and such that $B(Q,r)\cap\Omega\subset\Omega_{Q,r}$. Furthermore, there exists $x_{Q,r}\in\Omega_{Q,r}$ such that $d(x_{Q,r},\partial\Omega)\sim d(x_{Q,r},\partial\Omega_{Q,r})\sim r$ and $x_{Q,r}\cap B(Q,Mr)=\emptyset$. Define $\Omega^*_{Q,r}$ in the following way,

$$x^*\in\Omega^*_{Q,r}\quad\text{if and only if}\quad rx^*+x_{Q,r}\in\Omega_{Q,r}.$$

Let $G_{Q,r}(x)=G_{Q,r}(x,x_{Q,r})$ be the Green function of $\Omega_{Q,r}$ with pole at $x_{Q,r}$. Let furthermore $G^*_{Q,r}(x)=G^*_{Q,r}(x,x^*_{Q,r})$ be the Green function of $\Omega^*_{Q,r}$ with pole at $x^*_{Q,r}$. Then with the notation introduced above we get using Lemma 3.3 and the dilation invariance of the quantity A,

$$A(Q,r,\Omega,x_0,q)\sim A(Q,r,\Omega_{Q,r},x_{Q,r},q)\sim A(Q^*,1,\Omega^*_{Q,r},x^*_{Q,r},q).$$

$\Omega^*_{Q,r}$ is now an NTA-domain with diameter ~ 1, having the same NTA-parameter M as Ω. In particular we have,

$$A(Q^*,1,\Omega^*_{Q,r},x^*_{Q,r},q)\sim\left(\int_{B(Q^*,1)\cap\Omega^*_{Q,r}}\left|\frac{G^*_{Q,r}(x,x^*_{Q,r})}{d(x,\partial\Omega^*_{Q,r})}\right|^q dx\right)^{1/q}\lesssim I_q(\Omega^*_{Q,r})^{1/q}.$$

We now make the following definition.

Definition 12.1. Ω is said to be selfsimilar of order q if Ω is a bounded NTA-domain and if there exists a constant $C=C(\Omega,q)$ such that for all $Q\in\partial\Omega$, $Mr<r_0$, the following is valid,

$$I_q(\Omega^*_{Q,r})\le CI_q(\Omega),$$

where $\Omega^*_{Q,r}$ is defined above. We denote this class of domains by $\mathrm{SF}(q)$.

To say that $\Omega\in\mathrm{SF}(q)$ is a way of quantifying that the geometry on the small scale looks like the geometry on the global scale. To be more precise, the condition is actually a condition on the localization property in the following sense. If we may choose the localized domains, $\Omega_{Q,r}$, as perfect copies of Ω, then the condition in Definition 12.1 is trivially fulfilled as we then essentially have $\Omega^*_{Q,r}=\Omega$. Through the deductions made above we have proved the following theorem.

Theorem 12.1. *Let $\Omega\in\mathbf{R}^n$ and $\Omega\in\mathrm{SF}(q)$. Then the following is true. If $I_q(\Omega)<\infty$, $q>n/(n-1)$, then the conclusion of the Main Theorem is valid.*

In other words, if $\Omega\in\mathrm{SF}(q)$ we are able to rephrase the reverse Hölder inequality condition in terms of a sum over Whitney cubes. To determine, for different values of q, the finiteness of that sum you probably need to use a computer, something we have not spent any time doing.

In the following we will need the following lemma, which may be deduced from the results in Nyström [21] in the same way as Theorem 7.2 was deduced.

Lemma 12.1. *Let $\Omega\in\mathbf{R}^n$ be a bounded* NTA-*domain. Then there exists a constant $C=C(\Omega,q)$ such that if $u\in W^{1,q}_0(\Omega)$ then*

$$\left(\int_\Omega\left|\frac{u(x)}{d(x,\partial\Omega)}\right|^q dx\right)^{1/q}\le C\left(\int_\Omega|\nabla u(x)|^q\,dx\right)^{1/q}.$$

Theorem 12.2. *Let $\Omega\in\mathbf{R}^n$ be a bounded* NTA-*domain. Let $I_q(\Omega)=\infty$. Then there exists $f\in L^\infty(\Omega)$ such that $\nabla Gf\notin L^q(\Omega)$.*

Proof. Let f be the characteristic function of $Q_0\subset\Omega$, $d(Q_0,\partial\Omega)\sim\mathrm{diam}(\Omega)$. Suppose that $I_q(\Omega)=\infty$ and that $Gf\in W^{1,q}(\Omega)$. As Ω is regular for the Dirichlet problem and as Ω is an (ε,δ)-domain (i.e. an extension domain for Sobolev spaces [16]) we get by using the Spectral synthesis for Sobolev spaces (see Hedberg [12]) that $Gf\in W_0^{1,q}(\Omega)$. Using Lemma 12.1 we have

$$\infty > \int_\Omega |\nabla u(x)|^q\,dx \geq C\int_\Omega \left|\frac{Gf(x)}{d(x,\partial\Omega)}\right|^q dx \geq CI_q = \infty.$$

We have reached a contradiction and the theorem is proved.

We may formulate the following corollary,

Corollary 12.1. *Let $\Omega\in\mathbf{R}^n$, $\Omega\in\mathrm{SF}(q)$. Then the conclusion of the Main Theorem is valid if and only if $I_q(\Omega)<\infty$.*

It is obvious that a domain like the snowflake domain fulfills the requirement of Corollary 12.1. We may therefore conclude that the theorem for the snowflake stated in the introduction is true.

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Kaj Nyström
Department of Mathematics
University of Umeå
S-90187 Umeå
Sweden
Current address:
Department of Mathematics
The University of Chicago
5734 University Avenue
Chicago, IL 60637
U.S.A.
email: kaj@math.uchicago.edu