Ark. Mat., **32** (1994), 157–193

Sampling and interpolation of entire functions and exponential systems in convex domains

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1. Introduction

The possibility of representing functions which are analytic in a bounded convex domain $G\subset\mathbf{C}$ by means of exponential series, was first investigated by Leont′ev [5] (see also his monograph [6] and the paper [13] for the history of such representations); Leont′ev was concerned with series which are convergent in a "weak" compactwise topology and in other naturally related topologies.

In the light of Leont′ev's work, it seems interesting to look for a counterpart, for Hilbert spaces of analytic functions in convex domains, of the highly developed theory of nonharmonic Fourier series in the classical space $L^2(-\pi,\pi)$. Via duality arguments, this theme, in its turn, is closely connected to sampling and interpolation problems in special spaces of entire functions.

In the present paper, we obtain solutions to the latter type of problems, and we "translate" these results into results concerning exponential systems in convex domains. Our results and techniques are inspired by work of Beurling [1] in the classical situation. It should be noted, however, that in our setting, Beurling's approach yields complete results, while this is *not* the case in the classical Hilbert space setting (where the concern is about exponential systems in $L^2(-\pi,\pi)$ and sampling and interpolation in the Paley–Wiener space of entire functions). We shall see that what is crucial for such completeness to be obtainable, is the structure (to be made precise below) of the boundary of the convex domain in question.

Throughout this paper, G denotes a bounded convex set in $\mathbf{C}$. We consider exponential expansions in the *Smirnov space* $E^2(G)$, which is the closure of the set

(1) Research supported by the Norwegian Research Council for Science and Humanities (NAVF).

of all polynomials in z with respect to the norm

$$\|f\|^2_{E^2(G)} = \int_{\partial G} |f(z)|^2\,|dz|.$$

The functions in $E^2(G)$ are analytic in G and have nontangential limits at almost every point of ∂G. $E^2(G)$ is a Hilbert space with inner product

$$\langle f, g\rangle_{E^2(G)} = \int_{\partial G} f(z)\overline{g(z)}\,|dz|.$$

It is clear that $E^2(G)$ is an extension of the Hardy space H^2 in the unit disk. For basic facts about Smirnov spaces, we refer to [16].

Let $\Lambda = \{\lambda_n\}$ be a sequence of distinct complex numbers. To such a sequence we associate a system of exponentials in $E^2(G)$,

$$\mathcal{E}(\Lambda) = \{e_n(z)\},$$

where $e_n(z) = c_n e^{\overline{\lambda_n} z}$ and $c_n > 0$ is chosen so that $\|e_n\|_{E^2(G)} = 1$. It was proved in [9] that if G is a convex polygon, one may construct a set Λ in such a way that the system $\mathcal{E}(\Lambda)$ constitutes a Riesz basis in $E^2(G)$. Later, it turned out (see [13]) that for more general domains G even very regular sets Λ do not generate Riesz bases in $E^2(G)$.

Instead of Riesz bases, it seems natural then to study *frames*. The notion of a frame was introduced by Duffin and Schaeffer in the context of nonharmonic Fourier series [3]; we say that $\mathcal{E}(\Lambda)$ is a frame in $E^2(G)$ if there are positive constants A and B such that

$$A\|f\|^2_{E^2(G)} \le \sum_n |\langle f, e_n\rangle_{E^2(G)}|^2 \le B\|f\|^2_{E^2(G)}$$

for all $f \in E^2(G)$. By a standard duality argument (see, e.g., [3]) we find that $\mathcal{E}(\Lambda)$ is a frame if and only if every function $f \in E^2(G)$ can be expressed as an exponential series

$$f(z) = \sum_n a_n e_n(z),$$

convergent with respect to the norm $\|\cdot\|_{E^2(G)}$, and

$$B^{-1}\|f\|^2_{E^2(G)} \le \sum_n |a_n|^2 \le A^{-1}\|f\|^2_{E^2(G)}.$$

Assuming (roughly speaking) that ∂G is smooth and all points of ∂G have positive curvature, we obtain a condition on Λ which is necessary and sufficient for

$\mathcal{E}(\Lambda)$ to constitute a frame in $E^2(G)$. This condition is given as a relation between angular densities of Λ and the curvature of ∂G. In particular, it follows that $\mathcal{E}(\Lambda)$ can never constitute a Riesz basis in $E^2(G)$; this remains true if we assume that only some arbitrary part of ∂G is curved, a result which, we should add, has also been obtained independently by Lutsenko and Yulmuhametov [10].

Our results concerning properties of the moment sequences $\{\langle f, e_n\rangle_{E^2(G)}\}$, $f\in E^2(G)$, rely on the fact that there exists a Paley–Wiener theorem for convex sets [12] (see also [11], where a more general result is proved), i.e., one may establish a natural correspondence between the dual space of $E^2(G)$ and a certain space of entire functions of exponential type. We prove sampling and interpolation theorems for spaces of entire functions and then use the Paley–Wiener theorem to interpret these results as statements about the moment sequences $\{\langle f, e_n\rangle_{E^2(G)}\}$.

We have found it more convenient to study the sampling and interpolation problems in appropriate spaces of entire functions of the second order rather than in spaces of functions of exponential type (as in the classical setting). In fact, the study of these problems forms the body of the paper.

In order to explain our independent interest in these problems and to provide some background, we recall the main results of [17, 18], on which our analysis is partly based. For $\alpha>0$, let $d\mu_\alpha(z)=e^{-2\alpha|z|^2}dm(z)$, where m denotes area measure in $\mathbf{C}$, and define the Bargmann–Fock space F^2_α to be the collection of entire functions $f(z)$ with the norm

$$\|f\|_{\alpha,2}=\iint_{\mathbf{C}}|f(z)|^2d\mu_\alpha(z)<\infty.$$

A discrete set $\Gamma=\{\gamma_n\}$ of complex numbers is a *set of sampling* for F^2_α if there exist positive numbers A and B such that

$$A\|f\|^2_{\alpha,2}\le\sum_{\gamma_n\in\Gamma}e^{-2\alpha|\gamma_j|^2}|f(\gamma_j)|^2\le B\|f\|^2_{\alpha,2} \tag{1}$$

for all $f\in F^2_\alpha$. If to every l^2-sequence $\{a_j\}$ of complex numbers there exists an $f\in F^2_\alpha$ such that $e^{-\alpha|\gamma_j|^2}f(\gamma_j)=a_j$ for all j, the set $\Gamma=\{\gamma_j\}$ is said to be a *set of interpolation* for F^2_α. Sets of sampling and interpolation are described in terms of Landau's generalization of Beurling's notion of uniform densities [4]. We consider then *uniformly discrete sets*, i.e., discrete sets $\Gamma=\{\gamma_j\}$ for which $q=q(\Gamma)=\inf_{j\ne k}|\gamma_j-\gamma_k|>0$. We fix a compact set I of measure 1 in the complex plane, whose boundary has measure 0. $n^-(r)$ and $n^+(r)$ denote respectively the smallest and largest numbers of points from Γ to be found in a translate of rI, and we define the lower and upper uniform densities of Γ to be

$$D^-(\Gamma)=\liminf_{r\to\infty}\frac{n^-(r)}{r^2}\quad\text{and}\quad D^+(\Gamma)=\limsup_{r\to\infty}\frac{n^+(r)}{r^2},$$

respectively. Landau proved that these limits are independent of I.

The main theorems of [17, 18] are the following.

Theorem 1.1. *Γ is a set of sampling for F^2_α if and only if Γ can be expressed as a finite union of uniformly discrete sets and contains a uniformly discrete subset Γ' for which $D^-(\Gamma')>2\alpha/\pi$.*

Theorem 1.2. *Γ is a set of interpolation for F^2_α if and only if Γ is uniformly discrete and $D^+(\Gamma)<2\alpha/\pi$.*

A basic tool for proving these theorems is the *shift invariance* of F^2_α, namely, the translations

$$(T_\zeta f)(z)=(T^\alpha_\zeta f)(z)=e^{-\alpha\bar{\zeta}z-(\alpha/2)|\zeta|^2}f(z+\zeta)$$

act isometrically in F^2_α. In fact, F^2_α enjoys a group theoretical interpretation; it is a space of projective representation $\zeta\mapsto T_\zeta$ of the group of all complex numbers. We note that the classical spaces of bandlimited functions (the Paley–Wiener spaces) are connected in a similar way to the group of real translations; for that reason, the technique of using shift invariance in [17, 18] could essentially be copied from Beurling's work on related problems for bandlimited functions [1].

In our situation, we encounter spaces of entire functions which do not admit such a group theoretical interpretation, although they are closely connected to the classical Bargmann–Fock spaces. It is quite natural to expect that the reason for a set to possess the sampling or interpolating property is of an *analytic* nature rather than due to the group structure of the subjects at hand. In the present paper, we try to separate these two aspects in order to understand what analytic properties and what "remainings" of the group structure are really essential.

The main definitions and results on sampling and interpolation in spaces of entire functions of the second order are formulated in Section 2. The main theorems are proved in Sections 3–5. In Section 6, we transform them to apply to entire functions of exponential type. In Section 7, we obtain a solution to the frame problem in the Smirnov spaces. Section 8 is an appendix containing some details of a paper of Sodin and the first author [15] concerning the construction of entire functions of a certain prescribed growth. This work has previously been published only as a preprint in Russian. It is essential for our proofs, and for the convenience of the reader, we have elaborated some of its details. We are grateful to M. L. Sodin for his kind permission to include this material.

For the terminology and basic properties of entire functions used in this paper, we refer to Chapter I of [7].

2. Sampling and interpolation of entire functions: Main results

In this section we state our main theorems on sampling and interpolation of entire functions.

Let h be a 2π-periodic 2-trigonometrically convex function, put

$$d\mu_h(z) = e^{-2h(\arg z)|z|^2}\, dm(z),$$

and define the following generalized Bargmann–Fock spaces,

$$F_h^2 = \{ f : f \text{ entire},\ \|f\|_{h,2} = \iint_{\mathbf{C}} |f(z)|^2\, d\mu_h(z) < \infty \},$$
$$F_h^\infty = \{ f : f \text{ entire},\ \|f\|_{h,\infty} = \sup |f(z)| e^{-h(\arg z)|z|^2} < \infty \}.$$

Throughout the paper, we shall assume that

$$h \in C^2[0,2\pi] \quad \text{and} \quad \delta(\theta) = h(\theta) + \tfrac{1}{4}h''(\theta) > 0,\ \theta \in [0,2\pi]. \tag{2}$$

We see that $h(\theta) \equiv \alpha > 0$ corresponds to the classical situation treated in [17, 18].

A sequence Γ of distinct complex numbers is said to be a *set of sampling* for F_h^2 provided there are constants $0 < A \le B < \infty$ such that

$$A\|f\|_{h,2}^2 \le \sum_{\gamma \in \Gamma} |f(\gamma)|^2 e^{-2h(\arg \gamma)|\gamma|^2} \le B\|f\|_{h,2}^2$$

for all $f \in F_h^2$; Γ is a *set of sampling* for F_h^∞ if there is a constant $K > 0$ such that

$$K\|f\|_{h,\infty} \le \sup_{\gamma \in \Gamma} |f(\gamma)| e^{-h(\arg \gamma)|\gamma|^2}$$

for all $f \in F_h^\infty$. The sequence Γ is a *set of interpolation* for F_h^p ($p=2$ or ∞) if

$$f \mapsto \{ f(\gamma) e^{-|\gamma|^2 h(\arg \gamma)} \}_{\gamma \in \Gamma}$$

maps F_h^p into and onto l^p.

We shall generalize Beurling and Landau's notion of uniform densities in order to describe sets of sampling and interpolation for F_h^p. To this end, assume that $\Gamma = \{\gamma_k\}$ is a uniformly discrete set. Let $\#$ denote cardinality of a finite set, put $D(a,r) = \{ z : |z-a| < r \}$, and define

$$n_\varepsilon^-(r,\theta) = \liminf_{\varrho \to \infty} \Big\{ \inf_{|\phi - \theta| < \varepsilon} \#(\Gamma \cap D(\varrho e^{i\phi}, r)) \Big\},$$

$$D_\varepsilon^-(\Gamma,\theta)=\liminf_{r\to\infty}\frac{n_\varepsilon^-(r,\theta)}{\pi r^2},$$

and then

$$D^-(\Gamma,\theta)=\lim_{\varepsilon\to 0} D_\varepsilon^-(\Gamma,\theta).$$

Likewise, define

$$n_\varepsilon^+(r,\theta)=\limsup_{\varrho\to\infty}\Big\{\sup_{|\phi-\theta|<\varepsilon}\#(\Gamma\cap D(\varrho e^{i\phi},r))\Big\},$$

$$D_\varepsilon^+(\Gamma,\theta)=\limsup_{r\to\infty}\frac{n_\varepsilon^+(r,\theta)}{\pi r^2},$$

and then finally

$$D^+(\Gamma,\theta)=\lim_{\varepsilon\to 0} D_\varepsilon^+(\Gamma,\theta).$$

$D^-(\Gamma,\theta)$ and $D^+(\Gamma,\theta)$ will be referred to as *lower* and *upper angular densities.*

Note that we may repeat Landau's analysis in order to see that $D(a,r)$ could be replaced by $a+rI$, I of measure $\sqrt{\pi}$, ∂I of measure 0, without altering the above definitions.

The main theorems are given below. In all these statements, we assume that h and δ satisfy (2).

Theorem 2.1. *Γ is a set of sampling for F_h^2 if and only if it can be expressed as a finite union of uniformly discrete sets and contains a uniformly discrete subset Γ' for which*

$$\inf_{0\le\theta\le 2\pi}(D^-(\Gamma',\theta)-2\delta(\theta)/\pi)>0.$$

Theorem 2.2. *Γ is a set of sampling for F_h^∞ if and only if it contains a uniformly discrete subset Γ' for which*

$$\inf_{0\le\theta\le 2\pi}(D^-(\Gamma',\theta)-2\delta(\theta)/\pi)>0.$$

Theorem 2.3. *Γ is a set of interpolation for F_h^p (p=2 or ∞) if and only if*

$$\sup_{0\le\theta\le 2\pi}(D^+(\Gamma,\theta)-2\delta(\theta)/\pi)<0.$$

Note that Theorems 2.1–2.3 reduce to the main theorems of [17, 18] when $h(\theta)\equiv\alpha$; see also [14], where another approach is suggested to describe sets of sampling for F_α^2.

Theorems 2.1–2.3 will be proved in Sections 3–5, while in Section 6 we will discuss some slight extensions of these theorems relevant for the application to exponential systems in Smirnov spaces.

3. Preliminaries and some auxiliary results

In this section we describe some notational conventions and introduce some tools to be used in the proofs.

We let

$$\|f \mid \Gamma\|_{h,2} = \left(\sum_{\gamma \in \Gamma} e^{-2h(\arg \gamma)\alpha|\gamma|^2} |f(\gamma)|^2 \right)^{1/2},$$

and similarly,

$$\|f \mid \Gamma\|_{h,\infty} = \sup_{\gamma \in \Gamma} e^{-h(\arg \gamma)|\gamma|^2} |f(\gamma)|.$$

Sometimes we will have $h(\theta) \equiv \alpha$, in which case we replace h by α in these expressions. This somewhat sloppy notation should not cause any confusion, since it will always be clear from the context what the index refers to.

A sequence Q_j of closed sets converges strongly to Q, denoted $Q_j \to Q$, if $[Q, Q_j] \to 0$; here $[Q, R]$ denotes the Fréchet distance between two closed sets Q and R, i.e., $[Q, R]$ is the smallest number t such that $Q \subset \{ z : d(z, R) \le t \}$ and $R \subset \{ z : d(z, Q) \le t \}$, where $d(\cdot,\cdot)$ denotes Euclidean distance in $\mathbf{C}$. Q_j converges weakly to Q, denoted $Q_j \rightharpoonup Q$, if for every compact set D $(Q_j \cap D) \cup \partial D \to (Q \cap D) \cup \partial D$. $\Gamma' = \lim_{j\to\infty}(\Gamma - a_j)$ will mean that $\Gamma - a_j \rightharpoonup \Gamma'$. Following Beurling, for a closed set Γ, we let $W(\Gamma)$ denote the collection of weak limits of translates $\Gamma + z$. We note that $W(\Gamma)$ is compact in the sense that every sequence of elements $\Gamma_j \in W(\Gamma)$ has a subsequence converging weakly to some element in $W(\Gamma)$.

$W(\Gamma)$ will, as in [17], be an important concept in our analysis. However, now we need to take into account *how* a set $\Gamma' \in W(\Gamma)$ is obtained, i.e., in what direction Γ is translated when approaching the limit Γ'. In order to make precise statements, we introduce the following concepts. Let $A = \{a_j\} \subset \mathbf{C}$, $a_j \to \infty$, $\arg(a_j) \in [0, 2\pi)$. Put

$$\operatorname{dir}(A) = \bigcap_n \overline{\{\arg(a_j)\}_{j>n}}$$

(the bar denotes closure of the set); when taking the closure, we assume $0 = 2\pi$. Then define

$$\Delta_-(A) = \min\{ \delta(\alpha) : \alpha \in \operatorname{dir}(A) \} \quad \text{and} \quad \Delta_+(A) = \max\{ \delta(\alpha) : \alpha \in \operatorname{dir}(A) \}.$$

The spaces of interest when approaching the limit $\Gamma' = \lim(\Gamma - a_j)$, will be $F^p_{\Delta_-(A)}$ and $F^p_{\Delta_+(A)}$, respectively, i.e., classical Bargmann–Fock spaces. Then "in the limit", we may use the techniques developed in [17, 18].

We introduce now an operation of directional translation in F^∞_h. Let

$$v(z, a) = e^{-|a|^2 h(\arg a) - z\bar{a}[2h(\arg a) - ih'(\arg a)] + (\bar{a}/a) z^2 [(1/2) i h(\arg a) + (1/4) h''(\arg a)]},$$

and define

$$T_a^h f(z) = v(z,a) f(a+z);$$

note that for $h(\theta) \equiv \alpha$, we are back to the translation operator for the classical spaces.

T_a^h is in general *not* isometric. On the contrary, it may very well take us out of the space. Locally, however, it behaves much like the classical translation operator, as shown by the following lemma.

Lemma 3.1. *$T_a^h f$ satisfies*

$$|f(a+z)|e^{-|a+z|^2 h(\arg(a+z))} = e^{o(1)} |T_a^h f(z)| e^{-\delta(\arg a)}, \tag{3}$$

as $|a| \to \infty$, z belonging to a compact subset of $\mathbf{C}$, *and*

$$|f(a+z)|e^{-|a+z|^2 h(\arg(a+z))} = (1+O(|z|)) |T_a^h f(z)| e^{-\delta(\arg a)|z|^2} \tag{4}$$

as $|z| \to 0$, independently of $|a|$.

Proof. We write

$$\begin{aligned}
|f(a+z)|e^{-|a+z|^2 h(\arg(a+z))} &= |f(a+z)|\, |e^{-|a|^2 h(\arg a) - |z|^2 h(\arg a) - 2z\bar{a} h(\arg a)}| \\
&\quad \times |e^{\{-|a|^2 [h(\arg(a+z)) - h(\arg a)]\}} \times e^{\{-2z\bar{a}[h(\arg(a+z)) - h(\arg a)]\}} \\
&\quad \times e^{\{-|z|^2 [h(\arg(a+z)) - h(\arg a)]\}}| \\
&= |f(a+z)|\, |e^{-|a|^2 h(\arg a) - |z|^2 h(\arg a) - 2z\bar{a} h(\arg a)}| \times |e^{\{III\}} e^{\{II\}} e^{\{I\}}|.
\end{aligned}$$

Clearly $\{I\} \to 0$ as $|a| \to \infty$. Using Taylor's formula, we obtain

$$\begin{aligned}
\operatorname{Re}\{II\} &= \operatorname{Re}\Big\{ i\frac{\bar{a}}{a} z^2 h'(\arg a) \Big\} + o(1), \quad |a| \to \infty; \\
\operatorname{Re}\{III\} &= \operatorname{Re}\Big\{ i\bar{a}z h'(\arg a) - \frac{i}{2}\frac{\bar{a}}{a} z^2 h'(\arg a) \\
&\quad + \frac{1}{4}\frac{\bar{a}}{a} z^2 h''(\arg a) - \frac{1}{4}|z|^2 h''(\arg a) \Big\} + o(1), \quad |a| \to \infty,
\end{aligned}$$

where both $o(1)$'s are uniform with respect to z, z belonging to a compact subset of $\mathbf{C}$. Collecting our estimates, we see that (3) is proved.

(4) follows in a similar fashion for large a; for small a the estimate is obvious. $\square$

As a first application of this lemma, we prove a Bernstein-type estimate. Here, and in the sequel, we let $S(f,h)(z) = e^{-h(\arg z)|z|^2} f(z)$.

Lemma 3.2. *For every* $f \in F_h^\infty$ *we have*

$$\big|\,|S(f,h)(a)|-|S(f,h)(a+z)|\,\big| \le O(|z|)\|f\|_{h,\infty},$$

where the bound in $O(|z|)$ *depends only on* h.

Proof. In view of (4), we may write

$$\begin{aligned}\big|\,|S(f,h)(a)|-|S(f,h)(a+z)|\,\big| \le |S(T_a^h f,\delta(\arg a))(0)-S(T_a^h f,\delta(\arg a))(z)| \\ +O(|z|)|S(T_a^h f,\delta(\arg a))(z)|.\end{aligned}$$

Now the result follows from the Bernstein-type estimate in the classical situation (see Lemma 3.1 of [17]) and (3) of Lemma 3.1. □

The next lemma is an immediate consequence of Lemma 3.1 and the corresponding estimate in the classical situation, which in turn is a consequence of the mean value inequality for subharmonic functions; see (3) of [17].

Lemma 3.3. *For every* $f \in F_h^\infty$ *and* $z \in \mathbf{C}$ *we have*

$$|S(f,h)(z)| \le C(h,r) \iint_{D(z,r)} |f(\zeta)|^2 \, d\mu_h(\zeta). \tag{5}$$

The next auxiliary result reveals the existence of certain functions in F_h^∞ which grow in a very regular manner. These functions are analogues of the sine-type functions introduced by Levin [7] (see also [13]).

Below, if f is an entire function, $Z(f)$ denotes the zero set (counting multiplicities) of f; the symbol $\asymp$ between two positive quantities means that their ratio is bounded from below and above by positive constants.

Lemma 3.4. *For every* h *of the prescribed type there exists a function* $g(z)$ *with simple zeros such that*

$$|g(z)| \asymp d(Z(g),z)e^{h(\arg z)|z|^2}. \tag{6}$$

This lemma was proved in [15]. In Section 8, we have included an outline of the construction of the function g. It follows from this construction that

$$D^-(Z(g),\theta)=D^+(Z(g),\theta)=\delta(\theta)/\pi$$

for each $\theta \in [0,2\pi)$ (a more precise statement about the distribution of the zeros can be made, as follows from the explicit construction of g). Note that when $h(z)\equiv\alpha$, g

can be chosen to be the Weierstrass σ-function corresponding to a suitable square lattice.

We shall now see an important consequence of the above lemma. In some sense, the next result is the converse of Lemma 3.1: Lemma 3.1 is the vehicle to be used when we go from F_h^p to a space $F^p_{\Delta_-(A)}$, while the next lemma is used when we "return" to F_h^p from $F^p_{\Delta_-(A)}$.

Let

$$p_{a,R}(z)=\prod_{\lambda\in Z(g)\cap D(a,R)}\left(1-\frac{z-a}{\lambda-a}\right),$$

where g is the function of Lemma 3.4. We define

$$w(z,a)=g(z)/p_{R,a}(z),$$

and prove the following estimates.

Lemma 3.5. *Let $a_j\to\infty$ and $\arg a_j\to\theta$. Then, as $j\to\infty$,*

$$|w(z,a_j)|\asymp e^{h(\arg z)|z|^2}e^{-\delta(\theta)|z-a_j|^2} \tag{7}$$

for $|z-a|<R$, and

$$|w(z,a_j)|\le e^{h(\arg z)|z|^2}e^{-\delta(\theta)R^2(1+\log(|z-a_j|^2/R))+O(1)} \tag{8}$$

for $|z-a|\ge R$. These estimates hold for any fixed $R>0$ with bounds which are independent of R.

Proof. To estimate $p_{R,a}$, we introduce the function

$$\begin{aligned} u_{R,a}(z)&=\frac{\delta(\theta)}{\pi}\iint_{|a-\lambda|\le R}\log\left|1-\frac{z-a}{\lambda-a}\right|dm(\lambda)\\ &=\frac{\delta(\theta)}{\pi}\iint_{|\lambda|\le R}\log\left|1-\frac{z-a}{\lambda}\right|dm(\lambda). \end{aligned} \tag{9}$$

A direct calculation shows that

$$u_{R,0}(z)=\begin{cases}\delta(\theta)|z|^2, & |z|<R\\ \delta(\theta)R^2\log\dfrac{|z|}{R}+\delta(\theta)R^2, & |z|\ge R.\end{cases} \tag{10}$$

A modification of the computations in Steps 2, 6, and 7 in Section 8 shows that

$$|p_{R,a_j}(z)|=e^{u_{R,a_j}+O(1)} \tag{11}$$

as $j\to\infty$, when, say, $d(z,Z(g)\cap D(a_j,R))\geq\varepsilon>0$ and with $O(1)$ depending on ε but not on R.

Finally, if $\lambda\in Z(g)\cap D(a_j,R)$, we have

$$\text{(12)}\qquad \frac{p_{R,a_j}(\lambda+z)}{z}e^{-\delta(\theta)|\lambda-a_j+z|^2}=\left|\frac{T^{\delta(\theta)}_{\lambda-a_j}p_{R,a_j}(z)}{z}\right|e^{-\delta(\theta)}\asymp 1$$

for $|z|\leq\varepsilon<\frac{1}{2}q(Z(g))$ by the maximum and minimum principle and what has been shown above. The lemma now follows if we combine (10), (11), and (12) with Lemma 3.4. □

4. Proofs of Theorems 2.1 and 2.2

We start by noting that we may restrict our attention to uniformly discrete sets.

Lemma 4.1. *If Γ is a set of sampling for F_h^∞, then Γ contains a uniformly discrete subset that is also a set of sampling for F_h^∞.*

Proof. This follows from Lemma 7.1 (compare the proof of Theorem 2 in [1, p. 344]). □

Lemma 4.2. *There exists a positive constant B such that*

$$\|f\mid\Gamma\|_{h,2}^2\leq B\|f\|_{h,2}^2$$

for all $f\in F_h^2$ if and only if Γ can be expressed as a finite union of uniformly discrete sets.

Proof. As the proof of Lemma 7.1 of [17]; note that (5) is needed. □

Lemma 4.3. *If Γ is a set of sampling for F_h^2, then Γ contains a uniformly discrete subset that is also a set of sampling for F_h^2.*

Proof. For $\varepsilon>0$ we construct (as we may) a uniformly discrete subset Γ' of Γ such that $d(\zeta,\Gamma')<\varepsilon$ for each $\zeta\in\Gamma$. We have then $\Gamma=\bigcup_{\zeta'\in\Gamma'}(\Gamma\cap D(\zeta',\varepsilon))$. By the preceding lemma there is a uniform bound, say P, on the number of points in $\Gamma\cap D(\zeta',\varepsilon)$, $\varepsilon\leq\frac{1}{2}$.

Proceeding as in the proof of Lemma 3.2, we easily deduce the estimate

$$\left||f(\zeta)|e^{-h(\arg\zeta)|\zeta|^2}-|f(\zeta')|e^{-h(\arg\zeta')|\zeta'|^2}\right|\leq C|\zeta-\zeta'|\left(\iint_{D(\zeta,1)}|f(z)|^2\,d\mu_h(z)\right)^{1/2},$$

which holds for, say, $|\zeta-\zeta'|\le\frac{1}{2}$, C depending only on h. We square this inequality and sum over Γ (for each $\zeta\in\Gamma$ we pick some point $\zeta'\in\Gamma'\cap D(\zeta,\varepsilon)$):

$$\begin{aligned}\sum_{\zeta\in\Gamma}\left|\,|f(\zeta)|e^{-h(\arg\zeta)|\zeta|^2}-|f(\zeta')|e^{-h(\arg\zeta')|\zeta'|^2}\,\right|^2 &\le C\varepsilon^2\sum_{\zeta\in\Gamma}\iint_{D(\zeta,1)}|f(z)|^2\,d\mu_h(z)\\ &\le PC\varepsilon^2\|f\|_{h,2}^2.\end{aligned}$$

This gives us

$$\left(\sum_{\zeta\in\Gamma}|f(\zeta)|^2e^{-2h(\arg\zeta)|\zeta|^2}\right)^{1/2}-\left(\sum_{\zeta\in\Gamma}|f(\zeta')|^2e^{-2h(\arg\zeta')|\zeta'|^2}\right)^{1/2}\le\sqrt{PC}\varepsilon\|f\|_{h,2}^2,$$

and hence

$$\|f\,|\,\Gamma\|_{h,2}\le P\|f\,|\,\Gamma'\|_{h,2}+C\varepsilon\|f\|_{h,2}.$$

The proof is finished since ε is arbitrary. □

Therefore, we assume for the rest of this section that all sets Γ are uniformly discrete.

The basic ingredient in the proof of the necessity parts of Theorems 2.1 and 2.2 is the following lemma.

Lemma 4.4. *If Γ is a set of sampling for F_h^p ($p=2$ or ∞) and $\Gamma'=\lim(\Gamma-a_j)$, $a_j\to\infty$, then Γ' is a set of sampling for $F^p_{\Delta_-(A)}$.*

Proof. Consider first the case $p=2$, and suppose the lemma is false. We put $\alpha=\Delta_-(A)$ and see that the assumption implies that for any $\varepsilon>0$, we can find an $f\in F_\alpha^2$ such that $\|f\|_{\alpha,2}=1$ and $\|f\,|\Gamma'\|_{\alpha,2}\le\varepsilon$. Let $f^{\langle N\rangle}$ denote the Nth partial sum of the Maclaurin series of f. Choose N so large that we have $\|f^{\langle N\rangle}\|_{\alpha,2}\ge\frac{1}{2}$ and $\|f^{\langle N\rangle}|\Gamma'\|_{\alpha,2}\le 2\varepsilon$. This is clearly possible since $f^{\langle N\rangle}$ approximates f uniformly on compact sets and approaches f in the norm of F_α^2.

We define

$$f_j(z)=w(z,a_j)f^{\langle N\rangle}(z-a_j),$$

where $w(z,a_j)$ is the function of Lemma 3.5. We choose R so large that

$$\iint_{|z|<R}|f^{\langle N\rangle}(z)|^2\,d\mu_\alpha(z)\ge\frac{1}{4},$$

and at the same time

$$\iint_{|z|>R-1}|f^{\langle N\rangle}(z)|^2e^{-\alpha R^2\log(|z|/R)-(\alpha(0)/2)R^2}\,dm(z)\le\varepsilon, \tag{13}$$

which clearly is possible since $f^{\langle N\rangle}$ is a polynomial. By Lemma 3.5, (10), and (11), we find that

$$\|f_j\|_\alpha^2 \geq C_1$$

and

$$\sum_{\zeta\in\Gamma'+a_j\cap D(a_j,R)} |f_j(\zeta)|^2 e^{-2\alpha|\zeta|^2} \leq C_2\varepsilon.$$

These estimates hold for all sufficiently large j, where C_1 and C_2 are independent of R and j. We finally choose j large enough to obtain

$$\sum_{\gamma\in\Gamma\cap D(a_j,R)} |f_j(\gamma)|^2 e^{-2h(\arg\gamma)|\gamma|^2} \leq 2C_2\varepsilon.$$

Then by (13) and (5), we have

$$\sum_{\gamma\in\Gamma} |f_j(\gamma)|^2 e^{-2h(\arg\gamma)|\gamma|^2} \leq C_2'\varepsilon.$$

This finishes the proof when $p=2$.

We argue in the same way when $p=\infty$. Suppose we can find an $f\in F_\alpha^\infty$ such that $\|f\|_{\alpha,\infty}=1$ and $\|f|\Gamma'\|_{\alpha,\infty}\leq\varepsilon$. We may also assume that $|f(0)|>\frac{1}{2}$. We shall again pass from f to a polynomial, but the passage is now a little harder. It may be noted that the trick to be used here, is an adaption of that used in the proof of Lemma 4.2 of [17].

We write

$$f(z)=\sum_{k=0}^{\infty} c_k z^k$$

and introduce the function

$$f_\delta^{\langle N\rangle}(z)=\sum_{k=0}^{N} c_k(1-\delta)^k z^k,$$

where N and $0<\delta<1$ are to be chosen so that we may replace f by $f_\delta^{\langle N\rangle}$.

In order to compare f and $f_\delta^{\langle N\rangle}$, we need some estimates. Cauchy's formula gives

$$|c_k| \leq \|f\|_{\alpha,\infty}\left(\frac{\alpha e}{k}\right)^{k/2},$$

and by Stirling's formula, we obtain

$$\sum_{k}^{\infty} |c_k|\,|z|^k \leq C\|f\|_{\alpha,\infty}|z|^2 e^{\alpha|z|^2}, \tag{14}$$

C an absolute positive constant. Thus,

$$|f_\delta^{\langle N\rangle}(z)| \le C\|f\|_{\alpha,\infty}|z|^2 e^{\alpha(1-\delta)^2|z|^2}.$$

Let $\varrho=\varepsilon^{-1}$, and choose N so that

$$\sup_{|z|<\varrho} |f(z)-f^{\langle N\rangle}(z)|e^{-\alpha|z|^2} \le \varepsilon.$$

We choose $\delta=\varepsilon^{3/2}$ so that $|z-(1-\delta)z|\le\varepsilon^{1/2}$ when $|z|<R$. By our choice of N and Lemma 3.2, we obtain

$$|f(\gamma')|e^{-\alpha|\gamma'|^2} \le C\varepsilon^{1/2} \tag{15}$$

for $\gamma'\in D(0,\varrho)\cap\Gamma'$. On the other hand, when $|z|\ge\varrho$, we have

$$(1-\delta)^2|z|^2-|z|^2 \le -(2\varepsilon^{3/2}-\varepsilon^3)|z|^2 \le -|z|^{1/2},$$

assuming $\varepsilon<1$. Hence, in view of (14), we have

$$|f_\delta^{\langle N\rangle}(z)|e^{-\alpha|z|^2} \le C\varepsilon$$

for $|z|>\varrho$, so (15) holds for *all* $\gamma'\in\Gamma'$.

We now define

$$f_j(z)=w(z,a_j)f_\delta^{\langle N\rangle}(z-a_j),$$

and see that we may finish the proof as we did in the case when $p=2$. $\square$

Proof of the necessity parts of Theorems 2.1 *and* 2.2. By the construction of our lower angular density, we have

$$\inf D^-(\Gamma_\theta) = D^-(\Gamma,\theta)$$

where the infimum is taken over all $\Gamma_\theta=\lim(\Gamma-a_j)$ for which $\operatorname{dir}(A)=\{\theta\}$. By Theorems 2.1 and 2.3 of [17] (which are the same as Theorems 2.1 and 2.2 above when $h(\theta)\equiv\alpha$) and Lemma 4.4, it follows that

$$D^-(\Gamma,\theta)-2\delta(\theta)/\pi>0.$$

This relation yields the results since the compactness of $W(\Gamma)$ then ensures that

$$\inf_\theta(D^-(\Gamma,\theta)-2\delta(\theta)/\pi)>0. \quad \square$$

In the proof of the sufficiency part of Theorem 2.2, we shall need the following lemma.

Lemma 4.5. *If* $\inf(D^-(\Gamma,\theta)-\delta(\theta)/\pi)>0$, *then* Γ *is a set of uniqueness for* F_h^∞.

Proof. Suppose the assertion is false. Then there is some nonzero function $f\in F_h^\infty$ vanishing on Γ. We may assume that $0\notin\Gamma$ and that $f(0)=1$. We apply Carleman's formula to the upper half-plane, thus

$$\begin{aligned}\sum_{r_k<R,0<\theta_k<\pi}\left(\frac{1}{r_k}-\frac{r_k}{R^2}\right)\sin\theta_k &= \frac{1}{\pi R^2}\int_0^\pi \log|f(Re^{i\theta})|\sin\theta\,d\theta \\ &\quad+\frac{1}{2\pi}\int_1^R\left(\frac{1}{x^2}-\frac{1}{R^2}\right)\log|f(x)f(-x)|\,dx+C,\end{aligned}\tag{16}$$

where $\Gamma=\{r_ke^{i\theta_k}\}$ and C is a constant which does not depend on R. We have

$$\text{r.h.s. of (16)}\le C+I_1(R)+I_2(R),\tag{17}$$

where

$$I_1(R)=\frac{R}{\pi}\int_0^\pi h(\theta)\sin\theta\,d\theta,$$
$$I_2(R)=\frac{R}{3\pi}(h(\pi)+h(0)).$$

The left-hand side can be approximated in the following way. We divide the half-plane into disjoint cells $Q_{n,j}$, $j\le n$, such that

$$\int_{Q_{n,j}}(\delta(\theta)+\varepsilon)r\,dr\,d\theta=1,\tag{18}$$

where

$$\varepsilon=\tfrac{1}{2}\inf(D^-(\Gamma,\theta)-2\delta(\theta)/\pi).$$

This is done by letting

$$t_n=\left(\int_0^\pi(2\delta(\theta)/\pi-\varepsilon)\,d\theta\right)^{-1}\sqrt{n(n+1)}$$

and then dividing the half-plane into n sectors, corresponding to numbers $0=\theta_{n,0}<\theta_{n,1}<\ldots<\theta_{n,n}=\pi$, such that

$$Q_{n,j}=\{z:t_{n-1}\le|z|<t_n\},\quad \theta_{n,j-1}\le\arg z\le\theta_{n,j},$$

satisfy (18) for all $j=1,2,\ldots,n$.

For each pair n, j, let $\zeta_{n,j}=\varrho_{n,j}e^{i\phi_{n,j}}\in Q_{n,j}$ be such that

$$\left(\frac{1}{\varrho_{n,j}}-\frac{\varrho_{n,j}}{R^2}\right)\sin(\phi_{n,j})=\int_{Q_{n,j}}\left(\frac{1}{r}-\frac{r}{R^2}\right)\sin\theta\,(\delta(\theta)+\varepsilon)r\,dr\,d\theta.$$

By the definition of our angular densities, it is clear that we can find a subsequence $\{z_{n,j}\}_{j\le n}\subset\Gamma$ such that

$$\sup_{n,j}|\zeta_{n,j}-z_{n,j}|<\infty.$$

A straightforward estimate then shows that

$$\text{l.h.s. of (16)}=\int_0^R\int_0^\pi\left(\frac{1}{r}-\frac{r}{R^2}\right)\sin\theta\left(\frac{1}{\pi}\big(h(\theta)+\tfrac{1}{4}h''(\theta)\big)+\varepsilon\right)r\,dr\,d\theta+O(\log R).$$

We remove the second-derivative by integrating twice by parts and obtain

$$\text{l.h.s. of (16)}=I_1(R)+I_2(R)+\tfrac{4}{3}\varepsilon R+O(\log R).$$

This contradicts (17). $\square$

Proof of the sufficiency part of Theorem 2.2. Suppose Γ is not a set of sampling. Then we may choose a sequence $A=\{a_j\}\subset\mathbf{C}$ and $\{f_j\}\subset F_h^\infty$ in such a way that

$$f_j(a_j)e^{-h(\arg a_j)|a_j|^2}=1;\quad \|\,f_j\,|_\Gamma\|_{h,\infty}\le 1/j,$$
$$\sup_j\|\,f_j\,\|_{h,\infty}=K<\infty.$$

We may assume $\arg(a_j)\to 0$, and by the above lemma that $|a_j|\to\infty$, since otherwise we have already obtained a contradiction. So we have

$$|f_j(z+a_j)|e^{-|z+a_j|^2h(\arg(z+a_j))}\le K.$$

We now form a new sequence of functions,

$$\phi_j(z)=v(z,a_j)f_j(z+a_j),$$

where v is as in Lemma 3.1. We have then $\phi_j(0)=1$, and in view of (3), we obtain a nonzero limit function $\phi\in F_{\delta(0)}$ vanishing on Γ'. This implies $D^-(\Gamma')\le 2\delta(0)/\pi$, and we have thus proved the sufficiency part of Theorem 2.2. $\square$

Proof of the sufficiency part of Theorem 2.1. In view of Lemma 4.2, we need to show that there exists a positive constant A such that

$$\|f\,|\,\Gamma\|_{h,2}^2\ge A\|f\|_{h,2}$$

for all $f \in F_{h,2}$. We shall apply an argument which was used in [2] in order to obtain an interpolation formula which will solve our problem.

Let

$$F_h^{\infty,0} = \{ g \in F_h^\infty : e^{-h(\arg z)|z|^2} |g(z)| \to 0 \text{ as } |z| \to \infty \};$$

$F_h^{\infty,0}$ is a closed subspace of F_h^∞. Suppose Γ is uniformly discrete and

$$2\varepsilon/\pi = \inf(D^-(\Gamma,\theta) - 2\delta/\pi) > 0.$$

Then, by Theorem 2.2, Γ is a set of sampling for $F_{h+\varepsilon}^\infty$, and therefore also for $F_{h+\varepsilon}^{\infty,0}$. By standard arguments (see [2, Section 5]) this implies, for all $\zeta \in \mathbf{C}$, the existence of a sequence of numbers $\{g_j(\zeta)\}$ such that

$$e^{-(h(\arg\zeta)+\varepsilon)|\zeta|^2} f(\zeta) = \sum_{\gamma_j \in \Gamma} g_j(\zeta) f(\gamma_j) e^{-(h(\arg\gamma_j)+\varepsilon)|\gamma_j|^2} \tag{19}$$

with

$$\sum_{\gamma_j \in \Gamma} |g_j(\zeta)| \le M,$$

M independent of ζ.

Now let $f \in F_h^2$. We apply (19) to the function

$$f_{\zeta,\varepsilon}(z) = e^{2\varepsilon(\bar\zeta z - |\zeta|^2)} f(z),$$

which is seen to belong to $F_{h+\varepsilon}^{\infty,0}$. A straightforward computation (as in [2, Section 5]) leads to the desired conclusion. □

5. Proof of Theorem 2.3

In the sequel, p may be either 2 or ∞.

Note first that if Γ is a set of interpolation for F_h^p, there exists a constant K_p, depending on h and Γ, so that the interpolation problem can be solved in such a way that

$$\|f\|_{h,p} \le K_p \|f \mid \Gamma\|_{h,p}. \tag{20}$$

This follows by a standard argument involving the Closed Graph Theorem. This observation, in conjunction with Lemmas 3.2 and 3.3, yields the following result (compare Lemmas 5.1 and 6.1 of [17]).

Lemma 5.1. *Every set of interpolation for F_h^p is uniformly discrete.*

The next lemma is the main auxiliary result needed for proving the necessity part of Theorem 2.3.

Lemma 5.2. *If Γ is a set of interpolation for F_h^p, and $\Gamma'=\lim_{j\to\infty}(\Gamma-a_j)$ for some $a=\{a_j\}$, $a_j\to\infty$, then Γ' is a set of interpolation for $F^p_{\Delta_+(a)}$.*

Proof. We may assume $\arg a_j\to 0$. First note that Γ' is also uniformly discrete with separating distance not exceeding that of Γ. Let $\Gamma'=\{z_k\}_1^\infty$, and let $|z_k|\le |z_{k+1}|$, $k=1,2,\dots$. We have to solve the interpolation problem

$$f(z_k)=w_k e^{\delta(0)|z_k|^2},\quad f\in F^p_{\delta(0)}$$

for every sequence $w=\{w_k\}\in l^p$. Let us fix such a sequence and also some $R>0$, and let $N(R)$ denote the number of points from Γ' contained in the disk $D(0,R)$. We index $\Gamma-a_j=\{z_k^{(j)}\}$ so that $z_k^{(j)}\to z_k$ as $j\to\infty$.

Using the assumption on Γ, we may construct a function $f_j(z)\in F_h^p$ such that

$$f_j(z_k^{(j)}+a_j)=\begin{cases} v(z_k^{(j)},a_j)^{-1}w_k e^{\delta(0)|z_{k,j}|^2}, & k\le N(R)\\ 0, & \text{otherwise}\end{cases}$$

with v as in Lemma 3.1. By (3) of that lemma, we have for sufficiently large j,

$$\| f_j\mid\Gamma\|_{h,p}\le C,$$

C independent of R. We define

$$\phi_j(z)=v(z,a_j)f_j(z+a_j)$$

so that

$$\phi_j(z_k^{(j)})=w_k e^{\delta(0)|z_k^{(j)}|^2},\quad k=1,2,\dots,N(R).$$

Using (3) of Lemma 3.1 and (20), we have for sufficiently large j,

$$|\phi_j(z)|\le K_2 e^{\delta(0)|z|^2},\quad |z|<R$$

when $p=\infty$, and

$$\iint_{D(0,R)}|\phi_j(z)|^2\,d\mu_{\delta(0)}(z)\le K_3$$

when $p=2$, where K_2 and K_3 are absolute positive constants.

We let $j\to\infty$, choose an appropriate subsequence $\{j_l\}$, and obtain a limit function $\phi(z)$ in the disk $D(0,R)$ such that

$$\phi(z_k)=w_k e^{\delta(0)|z_k|^2},\quad k=1,2,\dots,N(R);$$

if $p=\infty$, we have

$$|\phi(z)|\le K_2 e^{\delta(0)|z|^2},\quad |z|<R,$$

K_2 independent of R, and if $p=2$,

$$\iint_{D(0,R)} |\phi(z)|^2\, d\mu_{\delta_0}(z)\le K_3,$$

K_3 independent of R.

We finally let $R\to\infty$ and obtain the desired limit function f. $\square$

Proof of the necessity part of Theorem 2.3. We have

$$\sup D^+(\Gamma_\theta)=D^+(\Gamma,\theta),$$

where the supremum is taken over all sequences $\Gamma_\theta=\lim(\Gamma-a_j)$ for which $\mathrm{dir}(a)=\{\theta\}$. By Theorems 2.2 and 2.4 of [17] and the two lemmas above, it follows that

$$D^+(\Gamma,\theta)-2\delta(\theta)/\pi<0,$$

which yields the desired conclusion in view of the compactness of $W(\Gamma)$. $\square$

Proof of the sufficiency part of Theorem 2.3. We shall construct an explicit interpolation formula. For that purpose, we need the function g of Lemma 3.4 constructed for $h(\theta)-\varepsilon$, with $2\varepsilon/\pi<\sup(D^+(\Gamma,\theta)-2\delta(\theta)/\pi)$. Arguing as in [1, p. 356], we can find a uniformly discrete sequence $\Gamma'=\{\gamma'_j\}$ which contains Γ and is uniformly close to the zero set $Z(g)=\{z_j\}$ of g, i.e., we have

$$\sup_j |z_j-\gamma'_j|<\infty.$$

Since both Γ and $Z(g)$ are uniformly discrete, we may also assume that

$$\inf_{k\neq j} |z_k-\gamma'_j|>0.$$

We fix j and form the infinite products

$$k_j(z)=\prod_{k\neq j}\left(1-\frac{z}{\gamma'_k-\gamma'_j}\right)\exp\left(\frac{z}{\gamma'_k-\gamma'_j}+\frac{1}{2}\frac{z^2}{(z_k-\gamma'_j)^2}\right)$$

and

$$g_j(z)=(z-z_j)\prod_{k\neq j}\left(1-\frac{z}{z_k-\gamma_j'}\right)\exp\left(\frac{z}{z_k-\gamma_j'}+\frac{1}{2}\frac{z^2}{(z_k-\gamma_j')^2}\right),$$

which clearly are well-defined entire functions.

Repeating the technique of estimation leading to Lemma 2.2 of [18], we obtain that

$$|k_j(z)|\le C\frac{|g_j(z)|}{d(z,Z_g-z_j)}e^{c|z|\log|z|}$$

with C and c constants depending on Γ' and $Z(g)$, but not on j. Now clearly

$$g(z)=g_j(z-z_j)w_j(z),$$

w_j some zero-free entire function satisfying

$$|w_j(\gamma_j')|=C_je^{(h(\arg\gamma_j')-\varepsilon)|\gamma_j'|^2},$$

$C_j\asymp 1$ independently of j. We define

$$\phi_j(z)=k_j(z-\gamma_j')\frac{w_j(z)}{w_j(\gamma_j')}$$

for which we then have

$$|\phi_j(z)|\le Ce^{(h(\arg\gamma_j')-\varepsilon)|\gamma_j'|^2}e^{(h(\arg z)-\varepsilon)|z|^2}e^{c|z-\gamma_j'|\log|z-\gamma_j'|}.$$

Now it is easy to see that that the formula

$$f(z)=\sum_j a_je^{2\varepsilon(\overline{\gamma_j'}z-|\gamma_j'|^2)}\phi_j(z) \tag{21}$$

solves our interpolation problem both for $p=2$ and $p=\infty$ (in fact for all $0<p\le\infty$). □

6. Two modifications of the main theorems

When applying our theory of sampling and interpolation to exponential systems in Smirnov spaces, we shall need some results which differ slightly from those in Section 3. It is the purpose of this section to point out how to make the appropriate extensions.

Let

$$d\mu_{h,s}(z) = (1+|z|)^{2s} d\mu_h(z),$$

s a real number, and define correspondingly

$$F^2_{h,s} = \left\{ f \text{ entire} : \iint_{\mathbf{C}} |f(z)|^2 \, d\mu_{h,s}(z) < \infty \right\}.$$

Similarly,

$$F^\infty_{h,s} = \left\{ f \text{ entire} : \sup_{\mathbf{C}} |f(z)|(1+|z|)^s e^{-h(\arg z)|z|^2} < \infty \right\}.$$

The definitions of sets of sampling and interpolation are modified in the obvious way. To sum up, the following holds.

Theorem 6.1. *The theorems of Section* 2 *remain valid when replacing* F^p_h *by* $F^p_{h,s}$ *in all statements, where* s *is an arbitrary real number.*

Proof. We introduce the function

$$q(z) = s\log(1+|z|),$$

and note that

$$q(a+z) = q(a) + o(1) \tag{22}$$

as $a \to \infty$ for z belonging to a compact set and that

$$|q(a+z) - q(a)| \le s\log(1+|z|), \tag{23}$$

independently of a. Using these properties of q, all proofs in the previous sections can be modified in a straightforward manner.

We indicate some of the changes. In Lemma 2.1, introduce a factor $\exp(q(a))$ in the definition of $v(z,a)$, and the lemma remains true. In Lemma 2.2, observe that we need both properties (22) and (23). In the proof of Lemma 4.4, we replace the function f_j ($p=2$ or ∞) by the function $f_j(z)e^{-q(a)}$ and use (23) to verify that the same method of proof applies. □

The second extension concerns what happens when we have a certain symmetry. More precisely, suppose now $h(\theta)$ is π-periodic and that $\Gamma = -\Gamma$. Denote by $F^p_{h,s,e}$ the closed subspace of even functions in $F^p_{h,s}$. The definition of sets of sampling remains the same, while the definition of sets of interpolation for $F^p_{h,s,e}$ needs an obvious modification: we require $f(\gamma) = f(-\gamma)$. Then the following statement holds.

Theorem 6.2. *If h is π-periodic, $\Gamma=-\Gamma$, then all the theorems of Section 2 remain valid when replacing F_h^p by $F_{h,s;e}^p$.*

Proof. The sufficiency parts of the theorems are obvious. As to the necessity parts, note that Lemmas 4.1, 5.3, and 6.2 can be restated with $F_{h,s,e}^p$ in place of F_h^p. To see this, we argue as follows. In the proofs of Lemmas 4.1 and 5.3, replace $f_j(z)$ by $f_j(z)+f_j(-z)$, and in the proof of Lemma 6.2, construct even functions f_j, and proceed as before. The rest of the proofs are unchanged. □

The point of this theorem is that it allows us to pass from functions of order 2 to functions of exponential type by means of the transformation $\zeta=z^2$. As an example, we formulate a counterpart of the classical Cartwright theorem (see [7, p. 206]; a similar theorem holds in the L^2 case, which, notably, is *not* the case in the classical setting).

Let k be a 2π-periodic trigonometrically convex function satisfying

$$k\in C^2[0,\pi] \quad \text{and} \quad \varrho(\phi)=k(\phi)+k''(\phi)>0,\ \phi\in[0,2\pi). \tag{24}$$

We say that Λ is *normalizing for* k if there exists a positive number $\varepsilon(k,\Lambda)$ such that for any positive $\varepsilon<\varepsilon(k,\Lambda)$ the conditions

$$\sup_{\lambda\in\Lambda}|f(\lambda)|e^{-k(\arg\lambda)|\lambda|}\le 1$$

and

$$\sup_{\zeta}|f(\zeta)|e^{-(k(\arg\zeta)+\varepsilon)|\zeta|}<\infty,$$

f an entire function, imply

$$\sup_{\zeta}|f(\zeta)|e^{-k(\arg\zeta)|\zeta|}\le K_\varepsilon,$$

K_ε independent of f.

In order to reformulate the results proved for functions of order 2, we let $\Lambda^{1/2}$ denote the set of all complex square roots of the elements in Λ. Thus for $\lambda\in\Lambda$ *both* values of $\sqrt{\lambda}$ are contained in $\Lambda^{1/2}$.

Using Theorems 6.2 and 2.2, and the interpolation formula used to prove the sufficiency part of Theorem 2.1, we obtain the following statement.

Theorem 6.3. *Λ is normalizing for k if and only if $\Lambda^{1/2}$ contains a uniformly discrete subset Γ for which*

$$\inf_{0\le\phi\le 2\pi}(D^-(\Gamma,\phi/2)-2\varrho(\phi)/\pi)>0.$$

7. Exponential systems in Smirnov spaces

Let G be a convex domain with support function k satisfying (24). We note that the function $\varrho=k+k''$ has a geometric interpretation: $\varrho(\phi_0)$ is the radius of curvature of the supporting point of the ray $\phi=\phi_0$.

We may now state the solution to the frame problem described in the introduction.

Theorem 7.1. *The system of exponentials $\mathcal{E}(\Lambda)$ is a frame in $E^2(G)$ if and only if $\Lambda^{1/2}$ is a finite union of uniformly discrete sets and contains a uniformly discrete subset Γ for which*

$$\inf_{\phi}(D^-(\Gamma,\phi/2)-2\varrho(\phi)/\pi)>0.$$

The following theorem is also of interest; here the moment space of $\mathcal{E}(\Lambda)$ in $E^2(G)$ is the set of all the moment sequences, $\{\langle f,e_n\rangle_{E^2(G)}\}$, $f\in E^2(G)$.

Theorem 7.2. *The moment space of the system of exponentials $\mathcal{E}(\Lambda)$ in $E^2(G)$ is l^2 if and only if $\Lambda^{1/2}$ is uniformly discrete set and*

$$\sup_{\phi}(D^+(\Lambda^{1/2},\phi/2)-2\varrho(\phi)/\pi)<0.$$

We shall see that these two theorems follow from Theorem 6.2 by an application of the following Paley–Wiener theorem. For convenience, we use here the norm

$$\|f\|^2_{F^2_{h,s}}=4\iint_{\mathbf{C}}|f(z)|^2e^{-2h(\arg z)|z|^2}|z|\,dm(z),$$

which clearly is equivalent to the norm that we used in the previous section.

Lemma 7.3. *The transformation $\varphi\mapsto\varphi_f$, given by*

$$f_\varphi(z)=\int_{\partial G}\overline{\varphi(z)}e^{\zeta^2 z}\,|dz|,$$

is a bijection of $E^2(G)$ onto $F^2_{h,1/2,e}$, where

$$h(\theta)=k(-\theta/2),$$

and

$$\|f_\varphi\|_{F^2_{h,s}}\asymp\|\varphi\|_{E^2(G)}.$$

Proof. We have that $f \mapsto \psi_f(\zeta) = f(\sqrt{\zeta}\,)$, is an isomorphism between the Hilbert spaces $F^2_{h,1/2,e}$ and

$$P^2_{k,-1/2} = \left\{ \psi \text{ entire}: \int_{\mathbf{C}} |\psi(\zeta)|^2 e^{-k(\arg\zeta)|\zeta|} |\zeta|^{-1/2}\, dm(\zeta) < \infty \right\},$$

and so we need to show that $\varphi \mapsto f_\varphi(\sqrt{z}\,)$ is bijective from $E^2(G)$ to $P^2_{k,-1/2}$ and norm-preserving up to equivalence in norms. But this follows from [12] and [13]: We combine the Paley–Wiener theorem for convex sets [12] with Lemma 2.9 of [13]. $\square$

Using this lemma and the fact that

$$\|e^{\bar{\lambda} z}\|^2_{E^2(G)} = (1+|\lambda|)^{-1/2} e^{k(-\arg\lambda)|\lambda|},$$

(see Lemma 2.10 of [13]), we obtain the two theorems above from Theorem 6.2.

One may show that these theorems are valid for the whole scale of spaces E^2_α considered in [13].

It is interesting to observe that in particular, no system of exponentials $\mathcal{E}(\Lambda)$ is a Riesz basis in $E^2(G)$. This statement remains true if we require only

$$k \in C^2[\phi_0, \phi_1] \quad \text{and} \quad \varrho(\phi) = k(\phi) + k''(\phi) > 0, \;\; \phi \in [\phi_0, \phi_1], \tag{25}$$

as shown by the two theorems below. Note, however, that when the conditions (24) hold, the zeros of the function g of Lemma 3.4 generate a system $\mathcal{E}(\Lambda)$ which is complete and minimal and forms a basis of summation in $E^2(G)$ [13].

Theorem 7.4. *Suppose the condition* (25) *holds with* $0 \le \theta_0 < \theta_1 \le 2\pi$. *Then if the system of exponentials* $\mathcal{E}(\Lambda)$ *is a frame in* $E^2(G)$, *there exists a uniformly discrete subset* Γ *of* $\Lambda^{1/2}$ *for which*

$$\inf_{\phi \in I} (D^-(\Gamma, \phi/2) - 2\varrho(\phi)/\pi) > 0$$

for every closed interval $I \subset (\theta_0, \theta_1)$.

Theorem 7.5. *Suppose the condition* (25) *holds with* $0 \le \theta_0 < \theta_1 \le 2\pi$. *Then if the moment space of the system of exponentials* $\mathcal{E}(\Lambda)$ *in* $E^2(G)$ *is* l^2, *we have*

$$\sup_{\phi \in I} (D^+(\Lambda^{1/2}, \phi/2) - 2\varrho(\phi)/\pi) < 0$$

for every closed interval $I \subset (\theta_0, \theta_1)$.

Theorem 7.4 follows directly by repeating the arguments in Section 5. As to Theorem 7.5, let g in Lemma 3.4 be constructed with respect to a function $\tilde{h}$ that satisfy (24) for all θ, coincide with h on some closed interval contained in (θ_0, θ_1), and enjoy the estimate $\tilde{h}(\theta) \le h(\theta)$ for all θ; it is easy to see that such a function $\tilde{h}$ exists. This function g can be used to obtain an analogue of Lemma 4.4, which implies Theorem 7.3.

It is reasonable to ask if convex polygons with a finite number of sides are the only domains for which it is possible to construct Riesz bases.

8. Appendix: Analogues of sine type functions

What we refer to in the title of this section, is that the function g of Lemma 3.4 has growth properties analogous to those of the sine type functions, which were introduced by Levin in his study of exponential bases in $L^2([-\pi, \pi])$ [8]. We present here a proof of Lemma 3.4 which is a variation of the construction in [15].

We put $H(z) = |z|^2 h(\arg(z))$, which is a subharmonic function. The density ϱ of the Riesz measure ν of H with respect to Lebesgue measure is

$$\varrho(z) = \frac{d\nu}{dm} = \frac{2}{2\pi}\delta(\theta), \quad z = re^{i\theta}. \tag{26}$$

We shall first construct a uniformly discrete sequence $\{\zeta_\xi\}$ which is the zero set of an entire function f of order 2 such that

$$\log|f(z)| - H(z) - q\log|z| = \log \mathrm{d}(z, Z(f) + O(1)), \tag{27}$$

where

$$q = \frac{1}{12}\left(2 - \pi\int_0^{2\pi} \frac{d\phi}{\delta(\phi)}\right).$$

This construction contains the main step of the proof of Lemma 3.1.

We first construct $\{\zeta_\xi\}$. It is enough to obtain the sequence with the desired properties for $H(sz)$, $s>0$, and then pick the sequence $\{\zeta_\xi/s\}$. We let s be such that

$$\frac{1}{\pi}\int_0^{2\pi} \delta(\phi)\, d\phi = 2.$$

We next divide $\mathbf{C}$ into a union of disjoint annuli,

$$R_n = \{\zeta : n-1 \le |\zeta| < n\}, \quad n = 1, 2, \dots .$$

We have

$$\nu(R_n) = \frac{1}{\pi}\int_{n-1}^{n}\int_0^{2\pi} \delta(\phi)\, d\phi\, t\, dt = n^2 - (n-1)^2 = 2n - 1.$$

Each R_n may be divided into $2n-1$ cells $\{Q_{n,j}\}$ such that

$$\nu(Q_{n,j})=1, \quad j=1,2,...,2n-1. \tag{28}$$

Indeed, choose numbers $\{\psi_{n,j}\}_{j=0}^{2n-1}$ satisfying $0\le\psi_{n,0}<...<\psi_{n,2n-1}<2\pi$, such that

$$\frac{1}{\pi}\int_{\psi_{n,j-1}}^{\psi_{n,j}} \delta(\phi)=\frac{1}{n-\frac{1}{2}}. \tag{29}$$

Then the cells $Q_{n,j}=\{\,\zeta\in R_n:\psi_{n,j-1}\le\arg(\zeta)<\psi_{n,j}\,\}$ all satisfy (18).

Let

$$\zeta_{n,j}=\iint_{Q_{n,j}} \zeta\, d\nu(\zeta), \tag{30}$$

which is the center of mass of $Q_{n,j}$ with respect to $d\nu$. By the assumptions on ν, there exists an $\varepsilon>0$ such that $D(\zeta_{m,i},\varepsilon)\cap D(\zeta_{n,j},\varepsilon)=\emptyset$ when $(m,i)\ne(n,j)$. It is also clear that the diameters of the sets $Q_{n,j}$ are uniformly bounded.

Let $\xi=(n,j)$, and consider the discrete integer valued measure

$$\eta=\sum_{\xi}\delta_{\zeta_\xi},$$

where δ_{ζ_ξ} is the Dirac measure located at ζ_ξ. Define correspondingly the following δ-subharmonic function

$$w(z)=\iint_{\mathbf{C}}\log\left|1-\frac{z}{\zeta}\right| d(\nu(\zeta)-\eta(\zeta)). \tag{31}$$

We shall mainly be concerned with a detailed estimation of this function, and in particular, we shall obtain the estimate

$$|w(z)-q\log|z||\le C_\varepsilon \tag{32}$$

for $z\notin E_\varepsilon=\bigcup_\xi D(\zeta_\xi,\varepsilon)$, ε some arbitrary positive number.

Having this estimate, we argue as follows. The function $V-w$ is subharmonic with a discrete integer-valued Riesz measure ν and so

$$H(z)-w(z)=\log|f(z)|,$$

where f is an entire function whose zero set is $\{\zeta_\xi\}$. By (32),

$$\log|f(z)|=H(z)+q\log|z|+O(1)$$

for $z \notin E_\varepsilon$. (27) will then follow from an application of Lemma 3.1.

We prepare for the proof of (32). Let

$$\Pi_N = \{\, \zeta : 2^{N-1} \le |\zeta| < 2^N \,\}, \quad N = 1, 2, 3, \dots$$

and $\Pi_0 = D(0,1)$. Let $z \in \mathbf{C} \setminus E_\varepsilon$ be fixed. Choose $N = N(z)$ such that

$$2^{N-(1/2)} \le |z| < 2^{N+(1/2)}. \tag{33}$$

Then

$$N \log 2 = \log |z| + O(1) \tag{34}$$

as $z \to \infty$. Let $\Pi_{(z)} = \Pi_N \cup \Pi_{N+1}$, and rewrite (31) in the form

$$w(z) = w_1(z) + w_2(z) + w_3(z),$$

where

$$w_1(z) = \sum_{n=0}^{N-1} \sum_{Q_\xi \subset \Pi_n} \iint_{Q_\xi} \left\{ \log\left|1 - \frac{z}{\zeta}\right| - \log\left|1 - \frac{z}{\zeta_\xi}\right| \right\} d\nu(\zeta),$$

$$w_2(z) = \sum_{Q_\xi \subset \Pi_{(z)}} \iint_{Q_\xi} \left\{ \log\left|1 - \frac{z}{\zeta}\right| - \log\left|1 - \frac{z}{\zeta_\xi}\right| \right\} d\nu(\zeta),$$

$$w_3(z) = \sum_{n=N+2}^{\infty} \sum_{Q_\xi \subset \Pi_n} \iint_{Q_\xi} \left\{ \log\left|1 - \frac{z}{\zeta}\right| - \log\left|1 - \frac{z}{\zeta_\xi}\right| \right\} d\nu(\zeta).$$

We shall obtain the estimates

$$w_1(z) = q \log |z| + O(1), \tag{35}$$

$$w_2(z) = O(1), \tag{36}$$

$$w_3(z) = O(1), \tag{37}$$

from which (32) will follow.

These estimates are obtained through steps 1–7 below.

Step 1: *Estimates of the summands in* (31). Let

$$B_\xi(z,\zeta) = \log\left|1 - \frac{z}{\zeta}\right| - \log\left|1 - \frac{z}{\zeta_\xi}\right| - \operatorname{Re}\left\{ \left(\frac{1}{\zeta_\xi - z} - \frac{1}{\zeta_\xi} \right)(\zeta - \zeta_\xi) \right\}.$$

By (30), we have

$$\iint_{Q_\xi} (\zeta-\zeta_\xi)\,d\nu(z)=0,$$

and hence,

$$\iint_{Q_\xi} \left\{\log\left|1-\frac{z}{\zeta}\right|-\log\left|1-\frac{z}{\zeta_\xi}\right|\right\} d\nu(\zeta)=\iint_{Q_\xi} B_\xi(z,\zeta)\,d\nu(\zeta). \tag{38}$$

The needed estimates for $B_\xi(z,\zeta)$ are the following. (a) For $Q_\xi\subset\Pi_n$, $n\geq N+2$, $\zeta\in Q_\xi$,

$$|B_\xi(z,\zeta)|\leq\frac{K|z|}{|\zeta_\xi|^3}, \tag{39}$$

(b) If $\zeta\in Q_\xi\subset\Pi_{(z)}$ and Q_ξ has no common boundary points with the cell Q_λ containing z, then

$$\left|B_\xi(z,\zeta)+\frac{1}{2}\operatorname{Re}\frac{(\zeta-\zeta_\xi)^2}{(\zeta_\xi-z)^2}\right|\leq\frac{K_1}{|z-\zeta_\xi|^3}+\frac{K_2}{|\zeta_\xi|^2}, \tag{40}$$

(c) For $\zeta\in Q_\xi\subset\Pi_n$, $n\leq N-1$,

$$\left|B_\xi(z,\zeta)-\frac{1}{2}\operatorname{Re}\frac{(\zeta-\zeta_\xi)^2}{\zeta_\xi^2}\right|\leq\frac{K_3}{|z|^2}+\frac{K_4}{|\zeta_\xi|^3}. \tag{41}$$

In order to prove these estimates, note first that the function

$$L(\zeta)=\log\left(1-\frac{z}{\zeta}\right)$$

is analytic in Q_ξ since $\zeta\notin Q_\xi$. We see that

$$B_\xi(z,\zeta)=\operatorname{Re}\{L(\zeta)-L(\zeta_\xi)-L'(\zeta_\xi)(\zeta-\zeta)\}.$$

When proving (a), we start from the identity

$$L(\zeta)-L(\zeta_\xi)-L'(\zeta_\xi)(\zeta-\zeta_\xi)=\int_{\zeta_\xi}^{\zeta} L''(s)(\zeta-s)\,ds;$$

when proving (b), our starting point is the identity

$$L(\zeta)-L(\zeta_\xi)-L'(\zeta_\xi)(\zeta-\zeta_\xi)+\frac{1}{2}\left(\frac{\zeta-\zeta_\xi}{\zeta-z}\right)=\frac{1}{2}\left(\frac{\zeta-\zeta_\xi}{\zeta_\xi}\right)+\frac{1}{2}\int_{\zeta_\xi}^{\zeta} L'''(s)(\zeta-s)^s\,ds;$$

and when proving (c), we begin with the formula

$$L(\zeta)-L(\zeta_\xi)-L'(\zeta_\xi)(\zeta-\zeta_\xi)-\frac{1}{2}\left(\frac{\zeta-\zeta_\xi}{\zeta_\xi}\right)=-\frac{1}{2}\left(\frac{\zeta-\zeta_\xi}{\zeta-z}\right)+\frac{1}{2}\int_{\zeta_\xi}^{\zeta}L'''(s)(\zeta-s)^s\,ds.$$

Straightforward estimates (which we omit) lead from these formulas to the desired conclusions.

Step 2: *Estimate of* w_3. We first calculate the number α_k of cells Q_ξ which are contained in Π_k:

$$\alpha_k=\sum_{Q_\xi\subset\Pi_k}1=\sum_{m=2^{k-1}+1}^{2^k}(2m-1)=3\cdot 2^{k-2}. \tag{42}$$

Using (38) and (39), we obtain from this

$$\begin{aligned}|w_3(z)|&=\left|\sum_{n=N+2}^{\infty}\sum_{Q_\xi\subset\Pi_n}\iint_{Q_\xi}B_\xi(z,\zeta)\,d\nu(\zeta)\right|\\ &\le\sum_{n=N+2}^{\infty}\sum_{Q_\xi\subset\Pi_n}\iint_{Q_\xi}|B_\xi(z,\zeta)|\,d\nu(\zeta)\\ &\le\sum_{n=N+2}^{\infty}\sum_{Q_\xi\subset\Pi_n}\frac{K|z|}{|\zeta_\xi|^3}\le K2^N\sum_{n=N+2}^{\infty}\frac{\alpha_n}{2^{3n}}\le K.\end{aligned}$$

So estimate (37) is proved. Note that, in particular, we have then proved the convergence of the integral which defines w.

Step 3: *Partial estimate of* $w_2(z)$. First note that, for $z\notin E_\varepsilon$, the contribution to $w_2(z)$ from the cell Q_ξ containing z, as well as from its nearest neighbor cells, is bounded. We let $Q(z)$ denote the union of the cell Q_ξ containing z and its neighbor cells, i.e., those cells which have common boundary points with Q_ξ. We write

$$w_2(z)=-\frac{1}{2}\operatorname{Re}\left\{\sum_{Q_\xi\subset\Pi_{(z)}}\frac{1}{(\zeta_\xi-z)^2}\iint_{Q_\xi}(\zeta-\zeta_\xi)^2\,d\nu(\zeta)\right\}+S, \tag{43}$$

where, here and in the sequel, the prime denotes omission of those terms corresponding to $Q_\xi\subset Q(z)$. Using (38) and (40), we obtain the estimate

$$|S|\le K_1\sum_{Q_\xi\subset\Pi_{(z)}}\frac{1}{|z-\zeta|^3}+K_2\sum_{Q_\xi\subset\Pi_{(z)}}\frac{1}{|\zeta_\xi|^2}. \tag{44}$$

We find that

$$\sum_{Q_\xi\subset\Pi_{(z)}}\frac{1}{|\zeta_\xi|^2}\le\frac{K\alpha_N}{2^{2N}}\le 2K.$$

It is likewise clear that the first summand is bounded; this is a consequence of the regular distribution of the points ζ_ξ and the fact that $(1+x^2+y^2)^{-3/2}$ is integrable with respect to Lebesgue measure over $\mathbf{R}^2$.

We are thus left with the following expression for w_2,

$$w_2(z)=-\frac{1}{2}\operatorname{Re}\sum_{Q_\xi\subset\Pi_{(z)}}\frac{1}{(\zeta_\xi-z)^2}\iint_{Q_\xi}(\zeta-\zeta_\xi)^2\,d\nu(\zeta)+O(1). \tag{45}$$

This sum will be estimated in Step 6 below.

Step 4: *Partial estimate of* w_1. By (3.1) and (4.4), we have

$$w_1(z)=-\frac{1}{2}\operatorname{Re}\sum_{n=0}^{N-1}\sum_{Q_\xi\subset\Pi_n}\frac{1}{\zeta_\xi^2}\iint_{Q_\xi}(\zeta-\zeta_\xi)^2\,d\nu(\zeta)+S,$$

where

$$|S|\le\sum_{n=0}^{N-1}\sum_{Q_\xi\subset\Pi_n}\left(\frac{K_1}{|z|^2}+\frac{K_2}{|\zeta_\xi|^3}\right).$$

It is easy to deduce from (42) that the right-hand side of this expression is bounded. Hence,

$$w_1(z)=-\frac{1}{2}\operatorname{Re}\sum_{Q_\xi\subset D(0,t_{2N-1})}\frac{1}{\zeta_\xi^2}\iint_{Q_\xi}(\zeta-\zeta_\xi)^2\,d\nu(\zeta)+O(1). \tag{46}$$

From Steps 3 and 4 we see that it is crucial to calculate the integral

$$I_\xi=\iint_{Q_\xi}(\zeta-\zeta_\xi)^2\,d\nu(\zeta).$$

This is done in the next step.

Step 5: *Estimate of* I_ξ. We claim that we have

$$I_\xi=\frac{1}{12}e^{2i\arg(\zeta_\xi)}\left(1-\frac{\pi}{\delta(\arg\zeta_\xi)}\right)+O(n^{-1}). \tag{47}$$

In order to prove the claim, we show first that

$$I_\xi=\iint_{Q_\xi}(\zeta-\zeta_\xi^*)^2\,\frac{dm(\zeta)}{m(Q_\xi)}+O(n^{-1}), \tag{48}$$

as $n\to\infty$, where

$$\zeta_\xi^*=\iint_{Q_\xi}\zeta\,\frac{dm(\zeta)}{m(Q_\xi)}$$

is the center of mass of Q_ξ with respect to Lebesgue measure, and then we perform an explicit evaluation of the integral.

Let

$$\phi_{n,j}=\tfrac{1}{2}(\psi_{n,j-1}+\psi_{n,j}). \tag{49}$$

An elementary estimate (which we omit) shows that

$$\zeta_{n,j}=\left(n-\tfrac{1}{2}\right)e^{i\phi_{n,j}}(1+O(n^{-2})). \tag{50}$$

In order to prove (48), we write

$$I_\xi=\iint_{Q_\xi}\left(\zeta-\iint_{Q_\xi}\lambda\,d\nu(\lambda)\right)^2 d\nu(\zeta)=\iint_{Q_\xi}\left\{\iint_{Q_\xi}(\zeta-\lambda\,d\nu(\lambda))\right\}^2 d\nu(\zeta). \tag{51}$$

Continuity of δ and $\nu(Q_\xi)=1$ imply

$$d\nu(\zeta)=(1+O(n^{-1}))\frac{dm(\zeta)}{m(Q_\xi)}, \tag{52}$$

Q_ξ being the cell containing ζ. We insert (52) into (51) and obtain (48).

We now evaluate the integral in (48). Let $\xi=(n,j)$ and $\Delta_{n,j}=\frac{1}{2}(\psi_{n,j}-\psi_{n,j-1})$ so that we may write

$$m(Q_{n,j})=2n\Delta_{n,j}.$$

Direct calculations give

$$\zeta_{n,j}^*=\left(n-\frac{1}{24n}+O(n^{-3})\right)e^{i\phi_{n,j}}\,\frac{\sin\Delta_{n,j}}{\Delta_{n,j}} \tag{54}$$

and

$$\int_{Q_{n,j}}\zeta^2\,\frac{dm(\zeta)}{m(Q_{n,j})}=n^2e^{i2\phi_{n,j}}\,\frac{\sin 2\Delta_{n,j}}{2\Delta_{n,j}}. \tag{55}$$

Inserting (54) and (55) into (48), we obtain

$$I_\xi = n^2 e^{i2\phi_{n,j}} \left(\frac{\sin 2\Delta_{n,j}}{2\Delta_{n,j}} - \frac{\sin^2 \Delta_{n,j}}{\Delta_{n,j}^2} \right) + \tfrac{1}{12} e^{i2\phi_{n,j}} \frac{\sin \Delta_{n,j}}{\Delta_{n,j}} + O(n^{-2}). \tag{56}$$

We use (50) and the fact that

$$2\Delta_{n,j} = \frac{\pi}{2n\delta(\phi_{n,j})} + O(n^{-2})$$

to estimate (56), and arrive at (47) after some elementary computation.

For future convenience, we rewrite (47) as

$$I_\xi = \Psi(\zeta_\xi) + O(n^{-1}), \tag{57}$$

where we have defined

$$\Psi(\zeta) = \frac{1}{12} e^{2i\arg(\zeta)} \left(1 - \frac{\pi}{\delta(\arg \zeta)} \right);$$

note that (57) holds with ζ_ξ replaced by any point $\zeta \in Q_\xi$.

Step 6: *Final estimate of* $w_1(z)$. In view of (46), we need to estimate the sum

$$\Sigma_1 = \sum_{Q_\xi \subset D(0,2^{N-1})} \frac{I_\xi}{\zeta_\xi^2}.$$

By (57), we may write

$$\Sigma_1 = \sum_{Q_\xi \subset D(0,2^{N-1})} \frac{\Psi(\zeta_\xi)}{\zeta_\xi^2} + O(1).$$

We have

$$I_1(z) = \iint_{1 \le |\zeta| \le |z|} \frac{\Psi(\zeta)}{\zeta^2}\, d\nu(\zeta) = q \log|z|,$$

while on the other hand,

$$\Sigma_1 = I(z) + O(1).$$

This proves (35).

Step 7: *Final estimate of* $w_2(z)$. Now we need to estimate the sum

$$\Sigma_2 = \sum_{Q_\xi \subset \Pi_{(z)}} \frac{I_\xi}{(\zeta - \zeta_\xi)^2}.$$

We shall compare the sum with the integral

$$I_2(w)=\text{p.v.}\int_{1/2<|s|<2}\Psi(s)\,\frac{d\nu(s)}{(s-w)^2},$$

where $w=2^{-N}z$. Since $1/\sqrt{2}\leq|w|<\sqrt{2}$, it is clear that

$$I_2(w)=O(1).$$

We define

$$J_N(z)=\iint_{\Pi_{(z)}\setminus Q(z)}\Psi(\zeta)\,\frac{d\nu(\zeta)}{(\zeta-z)^2},$$

and find that

$$I_2(2^{-N}z)=J_N(z)+O(1).$$

Simple estimates (which we omit) show that

$$\Sigma_2=J_N(z)+O(1),$$

and this relation completes the proof of (36).

In order to obtain the estimate (27), we have to investigate what happens close to some point $\zeta\in Z(f)$. By Lemma 3.1, we have

$$\left|\frac{f(\zeta+z)}{z}\right|e^{-h(\arg(\zeta+z))|\zeta+z|^2}=(1+O(|z|))\left|\frac{T^h_\zeta f(z)}{z}\right|e^{-\delta(\arg\zeta)|z|^2}. \tag{58}$$

We know that (27) holds in E_ε, and thus by the maximum and minimum principle applied to $T^h_\zeta f(z)/z$ in $D(0,\varepsilon)$, we see that (27) holds in the whole plane.

The construction is not complete, since we need to get rid of the term $q\log|z|$ in (27). We choose $\varepsilon>0$ such that

$$\varepsilon<\inf_\theta\delta(\theta). \tag{59}$$

Then the function

$$h_1(\theta)=h(\theta)-\varepsilon$$

is of the prescribed type. We construct an entire function f_1 with respect to h_1 as described above, and another function f_2 with respect to $h_2(\theta)\equiv\varepsilon$ in a slightly different way. We choose f_2 in such a way that by moving a part of its zero set, we obtain the desired growth.

To ensure that the zero set of the product $f_1(z)f_2(z)$ is again uniformly discrete, we proceed in the following way. Let

$$s_1=\sqrt{\frac{\left(2-\frac{2}{k^2}\right)\pi}{\int_0^{2\pi}\delta(\theta)\,d\theta}},$$

and choose

$$\varepsilon=\frac{1}{\pi s_1^2k^2}.$$

We see that (59) is satisfied if we let k be a sufficiently large integer. We define

$$\delta_1(\theta)=s_1^2(\delta(\theta)-\varepsilon)$$

and

$$\delta_2(\theta)=s_1^2k^2\varepsilon.$$

Then

$$\frac{1}{\pi}\int_0^{2\pi}\delta_1(\theta)\,d\theta=\frac{1}{\pi}\int_0^{2\pi}\delta_2(\theta)\,d\theta=2.$$

We construct a function with respect to δ_1 as above; it means that the zeros of f_1 will be concentrated along the circles $|z|=s_1(n+\frac{1}{2})$. We then construct a function with respect to δ_2 whose zeros are concentrated along the circles $|z|=n$, so that the zeros of f_2 are concentrated along the circles $|z|=s_1kn$.

We turn to the construction of f_2. Let now

$$R_n=\{\,w:t_{n-1}\le|w|<t_n\,\},\quad n=1,2,\ldots,$$

where $t_n=\sqrt{n(n+1)}$. We put $d\nu_2(z)=\delta_2(\arg z)dm(z)$, and note that

$$\nu(R_n)=\int_{n-1}^{n}\int_0^{2\pi}\delta(\phi)\,d\phi\,t\,dt=t_n^2-t_{n-1}^2=2n.$$

We divide each R_n into $2n$ cells $\{Q_{n,j}\}$ as above, but take care to choose $Q_{n,1}$ so that

$$\iint_{Q_{n,1}}w\,d\nu_2(w)>0.$$

We repeat the above arguments and obtain a product which satisfies

$$f_1(z)f_2(z)=e^{H(z)+(q_1+q_2)\log|z|+O(1)}.$$

In order to get rid of the term involving $\log|z|$, we modify f_2 in the following way. Let $\{\lambda_n\}_{n=1}^{\infty}$ denote the positive zeros of f_2 ordered so that $\lambda_n<\lambda_{n+1}$ for $n=1,2,3,\dots$. Since $\lambda_n=s_1kn+O(n^{-1})$ as $n\to\infty$, we have that

$$\phi(z)=z\prod_{n=1}^{\infty}\left(1-\frac{z^2}{\lambda_n^2}\right)$$

satisfies

$$\log|\phi(z)|=\pi(s_1k)^{-1}|\operatorname{Im} z|+O(1)$$

when $d(z,\{\lambda_n\})>\varepsilon$. Consider the function

$$\phi_\gamma(z)=z\prod_{n=1}^{\infty}\left(1-\frac{z^2}{(s_1k)^2(n+\gamma)^2}\right);$$

using the Stirling formula in a standard way, we obtain

$$\log|\phi_\gamma(z)|=\pi(s_1k)^{-1}|\operatorname{Im} z|-4\gamma\log|z|+O(1)$$

when $d(z,Z(\phi_\gamma))>\varepsilon$. We finally put $\gamma=(q_1+q_2)/4$ and define

$$g(z)=\frac{f_1(z)f_2(z)\phi_\gamma(z)}{\phi(z)}.$$

We claim that this function is of the desired type.

In order to prove the claim, two things must be checked. Firstly, we may have double zeros. However, since the zeros of f_1, say $\{\zeta_{n,j}\}$, satisfy

$$|\zeta_{n,j}|=s_1\left(n-\tfrac{1}{2}\right)+O(n^{-1}),$$

and the zeros of f_2, say $\{w_{n,j}\}$, satisfy

$$|w_{n,j}|=s_1kn+O(n^{-1}),$$

these can occur either in some finite disk or along the real axis. Double zeros along the real axis can be avoided by choosing the zeros of f_1 to have some positive distance to the axis (it follows from the construction that this is possible), and double zeros in a finite disk can be removed by splitting them into two simple zeros without changing the asymptotic behavior of the function (since only finitely many points are involved).

Secondly, we must check that the asymptotic behavior is of the right type also in small disks around the zeros of ϕ_γ. To see that this is the case, we use Lemma 3.1 and obtain an estimate which is similar to (59).

A final remark is due: If h is known to be π-periodic, we may construct an even function g in the same way as f_2 was constructed above (in other words, we do not need f_1). This construction is the original one, which is described in detail in [15].

Acknowledgements. This research was started in June 1991 when the first author visited Trondheim, and most of its details were finished during the second author's stay in Kharkov in July/August 1992. The authors gratefully acknowledge the hospitality offered to them by the respective host institutions.

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Received December 7, 1992

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