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Regularity of the free boundary in parabolic phase-transition problems

by

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Introduction

In this paper we start the study of the regularity properties of the free boundary, for parabolic two-phase free boundary problems.

May be the best known example of a parabolic two-phase free boundary problem is the Stefan problem, a simplified model describing the melting (or solidification) of a material with a solid-liquid interphase.

The concept of solution can be stated in several ways (classical solution, weak solution on divergence form, or viscosity solution) and as usual, one would like to prove that the (weak) solutions that may be constructed, are in fact as smooth and classical as possible.

Locally, a classical solution of the Stefan problem may be described as following:

On the unit cylinder $Q_1 = B_1 \times (-1,1)$ we have two complementary domains, Ω and $Q_1 \setminus \Omega$, separated by a smooth surface $S = (\partial\Omega) \cap Q_1$.

In Ω and $Q_1 \setminus \Omega$ we have two smooth solutions, u_1 and u_2, of the heat equations

$$\Delta u_1 - a_1 D_t u_1 = 0 = \Delta u_2 - a_2 D_t u_2$$

with $u_2 \leqslant 0 \leqslant u_1$.

The functions u_i are C^1 up to S, and along S, both $u_i \equiv 0$ and the interphase energy balance conditions

$$\frac{(u_i)_t}{(u_i)_\nu} = [u_\nu] = (u_{1_\nu} - u_{2_\nu})$$

are satisfied.

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What one is able to construct for all times are weak solutions, i.e., continuous (see [CE]) solutions u to the equation

$$\Delta u \in \beta(u)_t$$

with $\beta(u)=a_1u^+-a_2u^-+\operatorname{sign} u$. Then heuristically, $u_1=u^+$ in $\Omega=\{u>0\}$, $u_2=-u^-$ in $\complement\Omega=\{u\leqslant 0\}$, and $\partial\{u>0\}$ becomes the (weak) free boundary.

In this paper we start developing a regularity theory for such free boundaries.

May be a parallel with minimal surface theory is in order. One has the classical definition of smooth solutions to the minimal surface problem and the weak solutions obtained for instance from the calculus of variations.

Then, the regularity theory has several approaches.

(a) Lipschitz minimal surfaces are smooth.

(b) "Flat" minimal surfaces (in some "Lebesgue" differentiability sense) are smooth.

(c) Generalized minimal surfaces are smooth except on some small set.

Here we develop (for two-phase parabolic problems) part (a) of the theory, that is, free boundaries, Lipschitz in space and time, are regular, and with it, a good part of the techniques necessary to do (b) and (c), which we hope to treat in a forthcoming work.

Part (a) has nevertheless interest on its own since many interesting geometries and initial data provide solutions satisfying (a) (melting in insulated, infinite cylindrical domains, concentric starlike domains, Lipschitz shapes, etc.).

Finally, a last obstacle encountered in the parabolic theory is the fact (see the counter example in §10) that free boundaries may not regularize instantaneously (i.e., Lipschitz free boundaries may remain only Lipschitz for an interval of time, even for strictly two-phase problems).

This fact is not that surprising anymore, given the similar phenomena occuring in degenerate parabolic equations (see for instance [ACV]), where again the "hyperbolicity" of the "free boundary conditions" gives rise to these abnormalities.

In conclusion, let us mention that the case of one space dimension has been treated extensively, among others, by J. R. Cannon, C. D. Hill, A. Fasano and M. Primicerio.

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1. Basic definitions and main result

In this section we introduce the class of free boundary problems we are going to deal with and the notion of viscosity solution. Let $Q_1=B_1\times(-1,1)$, where $B_1=B_1(0)$ is the unit ball in $\mathbf{R}^n$, centered at 0.

We start with the definition of classical solution.

Definition 1. Let v be a continuous function in Q_1. Then v is called a classical subsolution (supersolution) to a free boundary problem if, for $a_j>0$, $j=1,2$,

(i) $\Delta v-a_1v_t\geqslant 0$ ($\leqslant 0$) in $\Omega^+:=Q_1\cap\{v>0\}$,

(ii) $\Delta v-a_2v_t\geqslant 0$ ($\leqslant 0$) in $\Omega^-:=Q_1\cap\{v<0\}$,

(iii) $v\in C^1(\bar\Omega^+)\cap C^1(\bar\Omega^-)$,

(iv) for any $(x,t)\in\partial\Omega^+\cap Q_1$, $\nabla_x v^+(x,t)\neq 0$ and

$$V_\nu\geqslant -G(\nu,v_\nu^+,v_\nu^-)\quad(\leqslant)$$

where V_ν is the speed of the surface $F_t:=\partial\Omega^+\cap\{t\}$ in the direction $\nu:=\nabla_x v^+/|\nabla_x v^+|$ and G is increasing in v_ν^+, decreasing in v_ν^-, continuous in all its arguments, and $G\to+\infty$ when $v_\nu^+-v_\nu^-\to+\infty$.

We say that v is a classical solution to a free boundary problem if it is both a subsolution and a supersolution. The set $F:=\partial\Omega^+\cap Q_1$ is called the free boundary.

Remark. In the above definition condition (iv) can be replaced by

$$\frac{v_t^+}{v_\nu^+}\leqslant G(\nu,v_\nu^+,v_\nu^-)\quad(\geqslant).$$

Definition 2. Let u be a continuous function in Q_1. u is called a viscosity subsolution (supersolution) to a free boundary problem if, for any subcylinder Q of Q_1 and for every classical supersolution (subsolution) v in Q, $u\leqslant v$ ($u\geqslant v$) on ∂_pQ implies that $u\leqslant v$ ($u\geqslant v$) in Q. The function u is called a viscosity solution if it is both a viscosity supersolution and a viscosity subsolution.

A weak solution of the two-phase Stefan problem is a viscosity solution, according to the above definition.

In this paper we consider viscosity solutions whose free boundary is given (locally) by a Lipschitz graph. In this case they enjoy further regularity and other properties that, for the reader's convenience, we list in the theorems below. For the proofs see [ACS].

THEOREM A. *Suppose u is a viscosity solution to a free boundary problem in $Q_2=B_2\times(-2,2)$, whose free boundary, F, contains the origin and is given by the graph $\{x_n=f(x',t):(x',t)\in\mathbf{R}^{n-1}\times\mathbf{R}\}$ of the Lipschitz function f with Lipschitz constant L. Moreover, suppose $u\left(e_n,-\frac{3}{2}\right)=1$, where e_n is the unit vector in the x_n direction, and $\sup_{Q_2}u=M$. Then, in Q_1:*

(1) *there exists an $(n+1)$-dimensional cone $\Gamma(\theta,e_n)$ (cone of monotonicity) with opening $\theta=\theta(n,L,M,a_1,a_2)$ and axis e_n, such that, along every direction $\tau\in\Gamma(\theta,e_n)$, u is monotone increasing,*

(2) *there exists* $c = c(n, L, M, a_1, a_2)$ *such that*

$$c^{-1}\frac{u(x,t)}{d_{x,t}} \leqslant |\nabla u(x,t)| \leqslant c\frac{u(x,t)}{d_{x,t}}$$

($\nabla = (\nabla_x, D_t)$, $d_{x,t}$ *denotes the distance between* F *and* (x,t)),

(3) u *is Lipschitz continuous.*

Clearly the Lipschitz continuity of u is an optimal result. We can be more precise on the behavior of u near the free boundary, specifying in which sense u satisfies the free boundary relation. This is the content of the next theorem. The symbol $\langle \cdot, \cdot \rangle$ will denote the inner product in $\mathbf{R}^n$ or $\mathbf{R}^{n+1}$.

THEOREM B. *Let* u *be as in Theorem* A. *Then:*

(1) (*Asymptotic development near regular points from the right or the left.*)

Let $(x_0, t_0) \in F$ *and assume there exists an* $(n+1)$*-dimensional ball* $B^{(n+1)} \subset \Omega^+$ *such that* $\bar{B}^{(n+1)} \cap F = \{(x_0, t_0)\}$, *i.e.,* (x_0, t_0) *is a regular point from the right. Let* ν *be the inward spatial normal at* (x_0, t_0) *of* $B^{(n+1)} \cap \{t = t_0\}$ *and* $d(x,t)$ *be the distance between* (x,t) *and* (x_0, t_0). *Then there exist numbers* $\alpha_+, \alpha_-, \beta_+, \beta_-$ *such that near* (x_0, t_0),

$$u(x,t) \geqslant (\alpha_+\langle \nu, x - x_0\rangle + \beta_+(t - t_0))^+ - (\alpha_-\langle \nu, x - x_0\rangle + \beta_-(t - t_0))^- + o(d(x,t)), \qquad \text{(a)}$$

with $\alpha_+ > 0$, $\alpha_- \geqslant 0$ *and equality holding on the hyperplane* $t = t_0$,

$$\beta_+ \geqslant \alpha_+ G(\nu, \alpha_+, \alpha_-), \quad \beta_- \geqslant \alpha_- G(\nu, \alpha_+, \alpha_-). \qquad \text{(b)}$$

If $B^{(n+1)} \subset \Omega^-$ (*i.e.,* (x_0, t_0) *is a regular point from the left*) *the inequalities in* (a) *and* (b) *are reversed,* $\alpha_+ \geqslant 0$, $\alpha_- > 0$ *and* ν *is the outward spatial normal.*

(2) (*Asymptotic development at "good points".*)

Near almost all points (x_0, t_0) *of differentiability of* F (*with respect to surface or caloric measures*) u *has the asymptotic behavior in* (a) *with equality sign in both* (a) *and* (b). *In this case* $\alpha_+ \geqslant 0$, $\alpha_- \geqslant 0$ *and* ν *is the normal to the tangent plane to* F_{t_0} *at* (x_0, t_0).

Our main concern will be to study the regularity of the free boundary. As the counter example in §10 in the two-phase Stefan problem shows, we cannot expect, in general, smoothing of F. Therefore we concentrate on a class of problems, that we call non-degenerate for which the regularity of F can be pushed to C^1. The non-degeneracy condition, which we refer to, states, roughly speaking, that the two heat fluxes are not vanishing simultaneously on F. In some cases the validity of this condition can be inferred by global considerations (see for instance [N]). On the other hand we expect this situation to be generic in a sense that will be explained in a forthcoming paper.

The main result of this paper can be stated in the following way.

MAIN THEOREM. *Let u be a viscosity solution of a free boundary problem in Q_2, whose free boundary, F, is given by the graph of a Lipschitz function $x_n=f(x',t)$ with Lipschitz constant L. Assume that $M=\sup_{Q_2} u$, $u(e_n,-\frac{3}{2})=1$, $(0,0)\in F$, and that*

(i) *$G=G(\nu,a,b)\colon \partial B_1\times\mathbf{R}^2\to\mathbf{R}$ is a Lipschitz function in all of its arguments, with Lipschitz constant L_G, and for some positive number c^*,*

$$D_a G\geqslant c^* \quad \text{and} \quad D_b G\leqslant -c^*,$$

(ii) *(non degeneracy condition) there exists $m>0$ such that, if $(x_0,t_0)\in F$ is a regular point from the right or from the left, then, for any small r,*

$$\fint_{B_r(x_0)} |u|\geqslant mr.$$

Then, the following conclusions hold:

(1) *In Q_1 the free boundary is a C^1 graph in space and time. Moreover, for any η, $0<\eta$, there exists a positive constant $C_1=C_1(n,L,M,L_G,c^*,m,a,a_2,\eta)$ such that, for every $(x',x_n,t),(y',y_n,s)\in F$,*

$$|\nabla_{x'} f(x',t)-\nabla_{x'} f(y',t)|\leqslant C_1(-\log|x'-y'|)^{-3/2+\eta}, \tag{a}$$

$$|D_t f(x',t)-D_t f(x',s)|\leqslant C_1(-\log|t-s|)^{-1/2+\eta}. \tag{b}$$

(2) *$u\in C^1(\bar\Omega^+)\cup C^1(\bar\Omega^-)$ and on $F\cap Q_1$,*

$$u_\nu^+\geqslant C_2>0$$

with $C_2=C_2(n,L,M,L_G,c^*,m,a_1,a_2,\eta)$.

Therefore u is a classical solution.

Strategy of the proof. To describe the strategy of the proof, a few simple observations.

(a) One way to prove that the level surfaces of a function u are hyperplanes perpendicular to the unit vector e, is to prove that $D_\nu u\geqslant 0$ for any vector ν with angle$(\nu,e)=:$ $\alpha(\nu,e)\leqslant\frac{1}{2}\pi$.

(b) If we can only prove that $D_\nu u\geqslant 0$ for any unit vector ν lying in the cone

$$\Gamma(e,\theta)=\{\nu:\alpha(\nu,e)\leqslant\theta<\tfrac{1}{2}\pi\},$$

the level surfaces of u are Lipschitz graphs (in the e direction) with Lipschitz norm less than $\tan\left(\frac{1}{2}\pi-\theta\right)$.

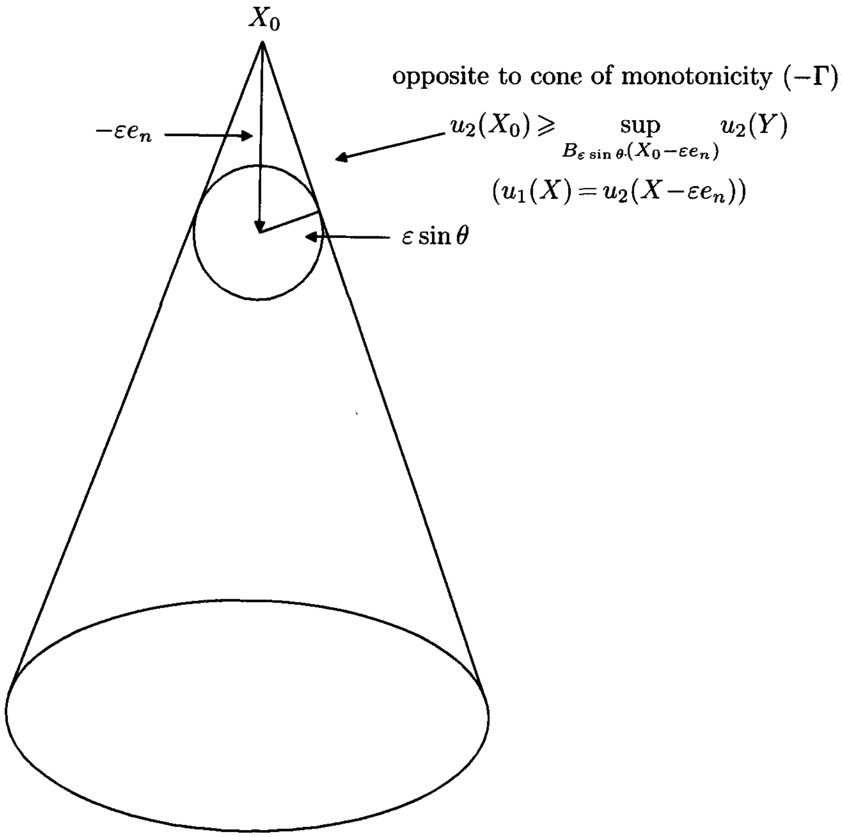

Fig. 1

(c) To say that u is monotone for any direction ν in the cone $\Gamma(e, \theta)$ is equivalent to saying that for any small ε (see Figure 1),

$$u(x) \geqslant \sup_{y \in B_{\varepsilon \sin \theta}(x)} u(y - \varepsilon e).$$

(In particular, if

$$u(x) \geqslant \sup_{y \in B_{\varepsilon}(x)} u(y - \varepsilon e)$$

the level surfaces of u are hyperplanes.)

(d) Heuristically, for any ε, δ,

$$\sup_{y \in B_{\delta}(x)} u(y - \varepsilon e)$$

is a subsolution of the same free boundary problem as u.

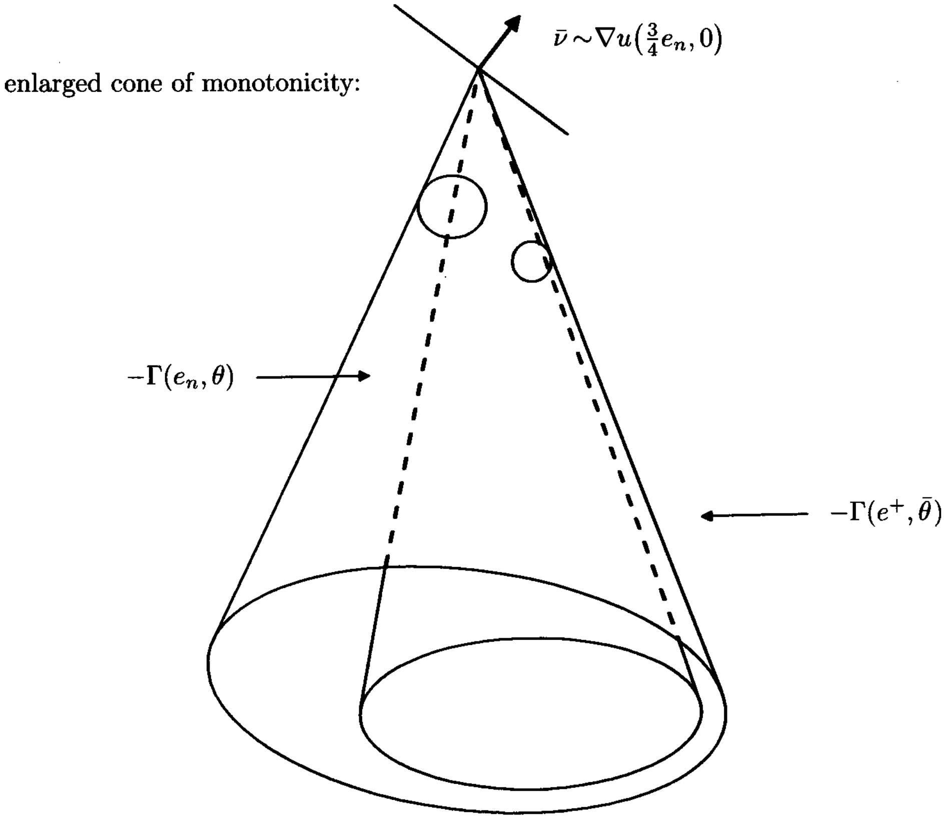

Fig. 2

With these observations at hand, the proof of the theorem consists in showing that on a sequence of dyadically contracting cylinders around a free boundary point, u becomes monotone on a sequence of cones with opening θ, closer and closer to $\frac{1}{2}\pi$.

This is achieved by comparing in these consecutive domains the subsolutions

$$u_{\varepsilon,\theta} = \sup_{y \in B_{\varepsilon \sin \theta}(x)} u(y - \varepsilon e)$$

with our function u, and showing that in smaller and smaller domains, $u_{\varepsilon,\theta}$ remains below u for values of θ closer and closer to $\frac{1}{2}\pi$ (we refer to this as enlargening the cone of monotonicity).

Since the heat equation scales parabolically (i.e. by dilations of the form λ in space, λ^2 in time) and the free boundary condition scales hyperbolically (λ in space, λ in time), there is a delicate balance in the arguments between both homogeneities and thus the unusual moduli of differentiability, and the choice of the sequence of contracting domains neither hyperbolic nor parabolic.

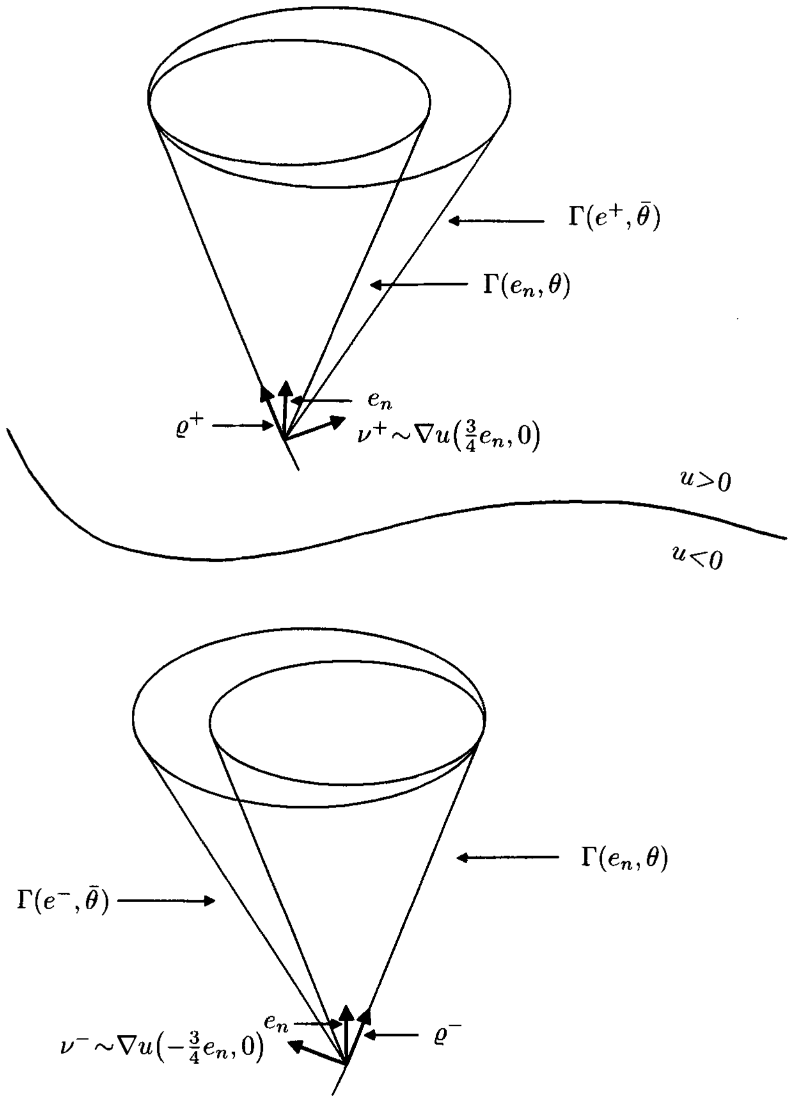

Fig. 3

In §2, we show that away from the free boundary, in parabolic homogeneity the enlargement of the cone of monotonicity (with a suitable choice of new axis) is a simple consequence of Harnack's inequality (Figure 2).

The improvement, though, could be in opposite directions in $\{u>0\}$ and $\{u<0\}$, and since we do not know which part is the commanding one (Figure 3) we must rule out this possibility (§3).

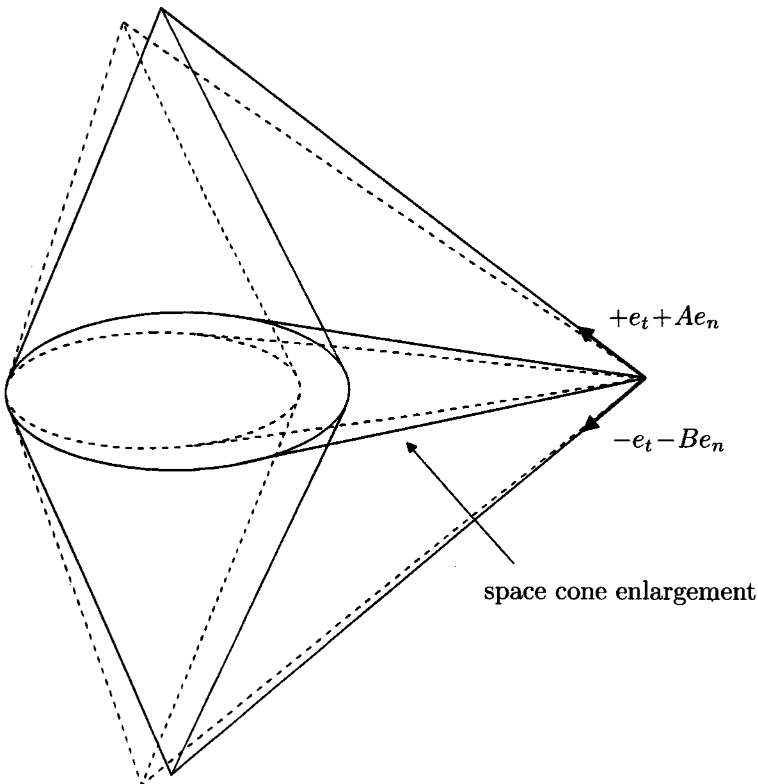

Fig. 4. Space-time enlargement of the monotonicity cone

In §§ 4 and 5, we refine our estimates to show that (always away from the free boundary) a gain in monotonicity can be achieved in a hyperbolically scaling region. In §6 we construct continuous families of subsolutions by varying the radius of the ball in which we take supremum. This is a preparatory step in improving the cone of monotonicity of u across the free boundary (§7).

In §§ 8 and 9 we carefully account for the gain in space and in space-time that our estimates produce. Note that the geometry of the space-time "balls" have to be carefully selected (in $l^1(l^2(\mathbf{R}^n)\times\mathbf{R})$, see Figure 4).

Although the balance is delicate, the fact that at the end we should have regularity is determined by the underlying reason that estimates deteriorate only as the opening of the cones of monotonicity approaches $\frac{1}{2}\pi$.

The various constants c that will appear in the sequel, may vary from formula to formula and, unless explicitly stated, will depend only on some of the relevant parameters $n, L, L_G, M, c^*, m, a_1, a_2$.

2. A Harnack principle

Throughout this section as well as in the subsequent ones we assume that u is a viscosity solution of our free boundary problem in $B_1(0)\times(-1,1)$ $((0,0)\in F)$ which is monotone in a cone of directions. We, also, assume that the coefficients of u_t in the equation are small enough and that $B_{1/4}(\frac{3}{4}e_n)\times(-1,1)\subset\{u>0\}$, $B_{1/4}(-\frac{3}{4}e_n)\times(-1,1)\subset\{u<0\}$. This is no loss of generality since we can start from a very small neighborhood of a free boundary point and perform a scaling of the type $u(\lambda x,\lambda t)/\lambda$ followed, if necessary, by $u(x,\lambda t)$.

The purpose of this section is to prove that a caloric function u, monotone in a cone of directions Γ (i.e., $D_\nu u\geqslant 0$ for $\nu\in\Gamma$) in a parabolic domain, is monotone in a larger cone in any smaller domain.

We start with a geometric lemma which is going to be used in this work. Its proof can be found in [C1]. We denote by $\alpha(e_1,e_2)$ the angle between the vectors e_1 and e_2, and, as usual, by $\Gamma(e,\theta)$ a circular cone with axis in the e-direction and opening θ ($\nu\in\Gamma$ if $\alpha(\nu,e)\leqslant\theta$).

LEMMA 1. *Let* $0<\theta^*<\theta<\frac{1}{2}\pi$, $\delta:=\frac{1}{2}\pi-\theta$, *and*

$$\Gamma(e,\theta)\subset\Gamma\big(\nu,\tfrac{1}{2}\pi\big).$$

For $\tau\in\Gamma\big(e,\frac{1}{2}\theta\big)$, *let*

$$E(\tau)=\tfrac{1}{2}\pi-\big(\alpha(\tau,\nu)+\tfrac{1}{2}\theta\big),$$

and for h small define

$$\varrho(\tau):=|\tau|\sin\big(\tfrac{1}{2}\theta+hE(\tau)\big).$$

Finally, let

$$S_h:=\bigcup_{\tau\in\Gamma(e,\theta/2)}B(\tau)_{\varrho(\tau)}.$$

Then there exist $\bar\theta$ *and* $\bar e$ *such that*

$$\Gamma(e,\theta)\subset\Gamma(\bar e,\bar\theta)\subset S_\mu$$

and

$$\frac{\bar\delta}{\delta}\leqslant b(\theta^*,h)<1$$

where $\bar\delta=\frac{1}{2}\pi-\bar\theta$. □

In the next comparison principle, u_1 may be thought of as a translation of u_2. Thus the assumption $\nu_\varepsilon\leqslant u_2$ means that u_2 is monotone in some cone of directions (see Figure 1).

LEMMA 2 (Harnack principle). *Let $u_1 \leqslant u_2$ be two viscosity solutions to a free boundary problem in Q_1 and monotone increasing in every $\tau \in \Gamma(\nu, \theta)$, $\nu \in \mathbf{R}^{n+1}$.*

Suppose that for $\varepsilon > 0$ small and $a = \min\{a_1, a_2\}$,

(a) $v_\varepsilon(x,t) := \sup_{(y,s)\in B_\varepsilon^{(n+1)}(x,t)} u_1(y,s) \leqslant u_2(x,t)$,

(b) $u_2(p) - v_\varepsilon(p) \geqslant \sigma\varepsilon u_2(p)$ *for* $p = \left(\frac{3}{4}e_n, -\frac{1}{4}a^2\right)$

and

(c) $B_{1/4}\left(\frac{3}{4}e_n\right) \times (-1,1) \subset \Omega_1^+ := \{u_1 > 0\} \cap Q_1$.

Then there exist constants C and h such that in $B_{1/8}\left(\frac{3}{4}e_n\right) \times \left(0, \frac{1}{4}a^2\right)$ we have

$$u_2(x,t) - v_{(1+h\sigma)\varepsilon}(x,t) \geqslant C\sigma\varepsilon u_2(p).$$

Proof. For any unit vector $\bar{\nu}$ we write

$$u_2(x,t) - u_1((x,t) + \varepsilon\bar{\nu}(1+\sigma h)) = w(x,t) + z(x,t)$$

where

$$w(x,t) := u_2(x,t) - u_1((x,t) + \varepsilon\bar{\nu}), \quad z(x,t) := u_1((x,t) + \varepsilon\bar{\nu}) - u_1((x,t) + (1+\sigma h)\varepsilon\bar{\nu}).$$

Now, $w(x,t)$ is nonnegative and caloric in $B_{1/4-\varepsilon}\left(\frac{3}{4}e_n\right) \times (-1+\varepsilon, 1-\varepsilon)$. Then by Harnack's inequality for $(x,t) \in B_{1/8}\left(\frac{3}{4}e_n\right) \times \left(0, \frac{1}{4}a^2\right)$ we have

$$w(x,t) \geqslant Cw(p) \geqslant C\sigma\varepsilon u_2(p).$$

By Theorem A we have, for $(x,t) \in B_{1/8}\left(\frac{3}{4}e_n\right) \times \left(0, \frac{1}{4}a^2\right)$,

$$|\nabla u(x,t)| \leqslant Cu(x,t) \leqslant Cu(p).$$

Hence

$$w(x,t) + z(x,t) \geqslant C\sigma\varepsilon u_2(p) + Ch\sigma\varepsilon u_2(p) \geqslant \overline{C}\sigma\varepsilon u_2(p)$$

for h small enough. □

It is not hard to see that the above Harnack principle remains valid if supremum is taken only over n-dimensional balls, more precisely:

COROLLARY 1. *Let u_1 and u_2 be as in Lemma 2. Suppose that for $\varepsilon > 0$ small,*

(a) $\sup_{y\in B_\varepsilon(x)} u_1(y,t) \leqslant u_2(x,t)$ *in* Q_1,

(b) $u_2\left(\frac{3}{4}e_n, -\frac{1}{4}a^2\right) - \sup_{y\in B_\varepsilon(3e_n/4)} u_1\left(y, -\frac{1}{4}a^2\right) \geqslant \sigma\varepsilon u_2\left(\frac{3}{4}e_n, -\frac{1}{4}a^2\right)$

and

(c) $B_{1/4}\left(\frac{3}{4}e_n\right) \times (-1,1) \subset \Omega_1^+ := \{u_1 > 0\} \cap Q_1$.

Then there exist constants $C>0$, $h>0$ *such that in* $B_{1/8}\left(\frac{3}{4}e_n\right)\times\left(0,\frac{1}{4}a^2\right)$ *we have*

$$u_2(x,t)-\sup_{B_{(1+h\sigma)\varepsilon}(x)} u_1(y,t)\geqslant C\sigma\varepsilon u_2\left(\tfrac{3}{4}e_n,-\tfrac{1}{4}a^2\right).$$

Now, let us apply the above Corollary 1 to the positive part of our viscosity solution u, in the case where $\nu=e_n$. Denote by $\Gamma_x(e_n,\theta)$ the section in space of $\Gamma(e_n,\theta)$. Let $\tau\in\Gamma_x\left(e_n,\frac{1}{2}\theta\right)$ be small and $\varepsilon=|\tau|\sin\frac{1}{2}\theta$. Setting $u_1(x,t):=u(x-\tau,t)$, for $y\in B_\varepsilon\left(\frac{3}{4}e_n\right)$ we have

$$u_1\left(y,-\tfrac{1}{4}a^2\right)=u\left(y-\tau,-\tfrac{1}{4}a^2\right)=u\left(\tfrac{3}{4}e_n-\bar{\tau},-\tfrac{1}{4}a^2\right)=u\left(\tfrac{3}{4}e_n,-\tfrac{1}{4}a^2\right)-D_{\bar{\tau}}u\left(x^*,-\tfrac{1}{4}a^2\right)|\bar{\tau}|$$

where $\bar{\tau}=\frac{3}{4}e_n-y+\tau$. Since $D_{\bar{\tau}}u$ is a nonnegative caloric function in Ω^+ $\left(\alpha(\tau,\bar{\tau})<\frac{1}{2}\theta\right)$, by Harnack's inequality,

$$\inf_{y\in B_{|\tau|+\varepsilon}(3e_n/4)} D_{\bar{\tau}}u\left(y,-\tfrac{1}{4}a^2\right)\geqslant CD_{\bar{\tau}}u\left(\tfrac{3}{4}e_n,-\tfrac{1}{2}a^2\right)=C\left|\nabla_x u\left(\tfrac{3}{4}e_n,-\tfrac{1}{2}a^2\right)\right|\cdot|\bar{\tau}|\cos\alpha(\bar{\nu},\tau)$$

where $\bar{\nu}=\nabla_x u\left(\frac{3}{4}e_n,-\frac{1}{2}a^2\right)$. Since $|\bar{\tau}|>\frac{1}{2}|\tau|$ and $\alpha(\bar{\nu},\bar{\tau})\leqslant\alpha(\bar{\nu},\tau)+\frac{1}{2}\theta$, by Theorem A we have for $y\in B_\varepsilon\left(\frac{3}{4}e_n\right)$,

$$u_1\left(y,-\tfrac{1}{4}a^2\right)\leqslant(1-\sigma\varepsilon)u\left(\tfrac{3}{4}e_n,-\tfrac{1}{4}a^2\right)$$

where $\sigma:=C\left(\frac{1}{2}\pi-\left(\alpha(\bar{\nu},\tau)+\frac{1}{2}\theta\right)\right)$. Thus applying Corollary 1, we have proved:

LEMMA 3. *There exist constants* $C=C(n,\theta)$ *and* $h=h(n,\theta)$ *such that*

$$\sup_{B_{(1+h\sigma)\varepsilon}(x)} u_1(y,t)\leqslant u(x,t)-C\sigma\varepsilon u\left(\tfrac{3}{4}e_n,-\tfrac{1}{4}a^2\right)$$

for every $(x,t)\in B_{1/8}\left(\frac{3}{4}e_n\right)\times\left(0,\frac{1}{4}a^2\right)$.

The above Lemma 3, with the aid of the geometric Lemma 1 (with $\nu=\bar{\nu}$ and $E(\tau)=\sigma$), gives us the monotonicity of u in a strictly larger cone $\Gamma_x(e^+,\theta_+)\supset\Gamma_x(e_n,\theta)$ with

$$\frac{\frac{1}{2}\pi-\bar{\theta}}{\frac{1}{2}\pi-\theta}=\bar{b}_0(h,n,\theta)<1$$

(see Figure 2). Note that the region where this gain holds is of parabolic size. This is insufficient for our purposes since, as it will become apparent later on, we need the region to be invariant under hyperbolic scaling. This will be done in §4.

Analogous results of Lemmas 2, 3 and Corollary 1 hold if we are on the negative side of our viscosity solution.

In the next section we shall make sure that there is a common enlarged cone on both sides of the free boundary.

3. Common gain

In the previous section we showed how to enlarge the spatial section $\Gamma_x(e_n,\theta)$ of the cone $\Gamma(\theta,e_n)$ in the interior of $\{u>0\}$ and $\{u<0\}$, to new cones $\Gamma_x^+=\Gamma_x(e^+,\theta_+)$ and $\Gamma_x^-=\Gamma_x(e^-,\theta_-)$, respectively.

Now, if $\nu^\pm=\nabla u\left(\pm\frac{3}{4}e_n,-\frac{1}{2}a^2\right)$ and

$$\alpha(\nu^+,e_n) \text{ or } \alpha(\nu^-,e_n)<\delta:=\tfrac{1}{2}\pi-\theta,$$

it is easily seen that we can use respectively ν^+ or ν^- as our special direction ν in Lemma 1 and obtain a gain on both sides with the same corrected cone $\Gamma_x(\theta^*,e^*)$. The problem arises when both the angles are close to δ, say

$$\delta-\eta=\alpha(\nu^+,e_n)\leqslant\alpha(\nu^-,e_n)\leqslant\delta$$

with $\eta\ll\delta$, because the new cone axes e^+, necessary for the gain on the positive side of u, may be far from that on the other side (see Figure 3).

For instance, it could be that

$$\alpha(\nu^+,e_n)=\alpha(\nu^-,e_n)=\delta,$$

and, if $\nu^+\neq\nu^-$, the corresponding cones Γ_x^+, and Γ_x^- will be tangent to $\Gamma_x(\theta,e_n)$ at different points, making it impossible to have a common gain.

We will rule out this situation by estimating the distance between e^+ and e^- in terms of δ, showing that we can always choose a common cone on both sides. We start with the following lemma. The vector ν we find here is the candidate for the axis of the common cone $\Gamma(\nu,\bar\theta)$.

LEMMA 4. *There exists a universal constant K and a normal unit vector ν at some point of Lebesgue differentiability of the free boundary such that simultaneously*

$$\langle\nu,\varrho^+\rangle\leqslant K[\delta-\alpha(\nu^+,e_n)]:=K\mu^+,$$
$$\langle\nu,\varrho^-\rangle\leqslant K[\delta-\alpha(\nu^-,e_n)]:=K\mu^-,$$

where $\varrho^\pm$ denote the unit vectors on the boundary of the cone $\Gamma_x(\theta,e_n)$, opposite to $\nu^\pm$ ($\langle\varrho^\pm,\nu^\pm\rangle$ minimum among $\varrho^\pm\in\Gamma_x(\theta,e_n)$).

Proof. Normalize u^+ so that

$$\langle\nu^+,e_n\rangle=D_{e_n}u\left(\tfrac{3}{4}e_n,-\tfrac{1}{2}a^2\right)=1.$$

In particular:

(1) $D_{\varrho^+}u\geqslant 0$,

(2) $D_{\varrho^+}u\left(\frac{3}{4}e_n,-\frac{1}{2}a^2\right)\leqslant c\mu^+$.

Denote by ω the caloric measure in $\Omega^+ \cap \{t \geqslant -20a_1^2\}$, evaluated at $\left(\frac{3}{4}e_n, -\frac{1}{2}a^2\right)$, and set $E = \{(x,t) \in F : -10a_1^2 < t < -a_1^2\}$. Then, by (1) and (2),

$$0 \leqslant \int_E D_{\varrho^+} u^+ \, d\omega \leqslant C\mu^+.$$

Caloric and surface measure being equicontinuous in our case, the set

$$E \cap \{D_{\varrho^+} u^+ \geqslant K\mu^+\}$$

has surface measure as small as we want, depending only on K.

Also since $D_{e_n} u^+$ is equivalent to caloric measure, the set

$$E \cap \{D_{e_n} u^+ < 1/K\}$$

also has Lebesgue measure as small as we want. Similarly for u^-, ν^-, ϱ^-.

Thus for K large there is a common point of Lebesgue differentiability along the free boundary where

$$D_{e_n} u^+ \geqslant 1/K, \quad D_{e_n} u^- \geqslant 1/K$$

and

$$D_{\varrho^+} u^+ \leqslant K\mu^+, \quad D_{\varrho^-} u^- \leqslant K\mu^-.$$

Thus, if ν is the normal unit vector at that point,

$$\frac{\langle \nu, \varrho^+ \rangle}{\langle \nu, e_n \rangle} = \frac{D_{\varrho^+} u^+}{D_{e_n} u^+} \leqslant K^2 \mu^+.$$

This completes the proof. □

We want to show that we can associate to ν, by the gain process indicated in Lemma 1, a cone $\bar{\Gamma}_x$ that, for a small enough gain, will be contained in both the cones Γ_x^- and Γ_x^+ associated to ν^+ and ν^-.

The cones Γ_x^+ and Γ_x^- are constructed on the basis that, after normalization, any directional derivative $D_\tau u$, with τ on $\partial\Gamma_x$, has in a neighborhood of $\left(\pm\frac{3}{4}e_n, 0\right)$ a gain of

$$D_\tau u \sim \langle \tau, \nu^\pm \rangle.$$

Therefore, to obtain a common enlarged cone on both sides, we need to control $\langle \tau, \nu^\pm \rangle$ by below by $\langle \tau, \nu \rangle$. This is achieved in the following corollary to Lemma 5.

COROLLARY 2. *There exists a constant A, depending only on n and θ, such that*

$$\langle \tau, \nu \rangle \leqslant A \min\{\langle \tau, \bar{\nu}^+\rangle, \langle \tau, \bar{\nu}^-\rangle\}$$

for each unit vector τ on $\partial\Gamma_x(\theta, e_n)$, where $\bar{\nu}^+ = \nu^+/|\nu^+|$ and $\bar{\nu}^- = \nu^-/|\nu^-|$.

Proof. We show that $\langle \tau, \bar{\nu}^+\rangle \geqslant A\langle \tau, \nu\rangle$. It is enough to consider τ in a small neighborhood of ϱ^+.

Since, simultaneously,

$$\alpha(\nu, e_n) \leqslant \delta \quad \text{and} \quad \alpha(\nu, \varrho^+) \geqslant \tfrac{1}{2}\pi - K\mu^+,$$

the component ν_1 of ν in the (e_n, ϱ^+)-plane satisfies

$$\alpha(\nu_1, \bar{\nu}^+) \leqslant \overline{K}\mu^+,$$

and the normal component ν_2 satisfies

$$|\nu_2| \leqslant (C\delta\mu^+)^{1/2}.$$

On the other hand, a vector

$$\tau = \lambda_1 e_n + \lambda_2 \varrho^+ + \lambda_3 g,$$

with $g \perp (e_n, \varrho^+)$, satisfies

$$\alpha(\lambda_1 e_n + \lambda_2 \varrho^+, \varrho^+) \leqslant c\delta\lambda_3^2$$

for small values of λ_3, since τ is on $\partial\Gamma(\theta, e_n)$. Thus,

$$\langle \bar{\nu}^+, \tau\rangle \geqslant \bar{c}(\mu^+ + \delta\lambda_3^2)$$

and

$$\langle \nu - \bar{\nu}^+, \tau\rangle \leqslant \overline{K}\mu^+ + \lambda_3(C\delta\mu^+)^{1/2} \leqslant \widetilde{C}(\mu^+ + \delta\lambda_3^2).$$

Since $\langle \nu, \tau\rangle = \langle \nu - \bar{\nu}^+, \tau\rangle + \langle \bar{\nu}^+, \tau\rangle$, the conclusion follows. □

4. Interior gain process in space

By the result of §3, a common enlarged cone of monotonicity for u in space $\Gamma_x(\nu, \bar{\theta})$ exists on both sides of the free boundary. As it was remarked at the end of §2, this increased cone of monotonicity is valid only in a parabolic region away from the free boundary.

Now, if we are willing to give up a portion of our $\Gamma_x(\nu, \bar{\theta})$ we can always have a gain valid in a hyperbolic region. This can be achieved in the following manner.

For any unit vector $\bar{\tau}\in\Gamma_x(\nu,\bar{\theta})$, $D_{\bar{\tau}}u\geqslant 0$ is equivalent to $D_\tau u\geqslant(\beta/\alpha)D_{e_n}u$ where $\bar{\tau}=\alpha\tau-\beta e_n$, $\tau\in\Gamma_x(e_n,\theta)$, $|\tau|=1$ with $1\leqslant\alpha\leqslant\sin(2\bar{\theta}-\theta)/\sin\theta$, $\beta\geqslant 0$.

Note that if the cones touch, then $\beta=0$, $\alpha=1$. So if we delete a small portion of $\Gamma_x(e_n,\theta)$ about the contact line, we can have

$$\frac{\beta}{\alpha}\geqslant c\delta$$

where c depends on the size of the deleted neighborhood and $\delta=\frac{1}{2}\pi-\theta$ is the defect angle in space. Thus, the inequality $D_\tau u\geqslant c\delta D_{e_n}u$ holds in $B_{1/8}\left(\pm\frac{3}{4}e_n\right)\times\left(0,\frac{1}{2}a^2\right)$. We shall show that it propagates in time to an interval of order δ/μ, where μ is the defect angle in time, that is, $\mu=\frac{1}{2}\pi-\theta^t$, where θ^t is the opening of the section of the cone of monotonicity in the (e_n,e_t)-plane. (Although we start with a circular cone, so that $\delta=\mu$, in the iterative process we will use later the defect angle in space goes to zero much faster than the one in time, and therefore we have to assume $\delta\leqslant\mu$.)

This is the content of the following lemma.

LEMMA 5. *Let u be a viscosity solution of a free boundary problem in Q_1. Let δ and μ be the defect angles in space and time, respectively, with $0<\delta\leqslant\mu<\frac{1}{2}\pi$. If for $\tau\in\Gamma_x(e_n,\theta)$,*

$$D_\tau u(x,t)\geqslant c\delta D_{e_n}u(x,t)$$

for every $(x,t)\in\left(B_{1/8}\left(\frac{3}{4}e_n\right)\cup B_{1/8}\left(-\frac{3}{4}e_n\right)\right)\times\left(0,\frac{1}{4}a^2\right)$ $(a=\min(a_1,a_2))$, then there exist constants $\bar{c}$ and C such that

$$D_\tau u(x,t)\geqslant \bar{c}\delta D_{e_n}u(x,t)$$

for every $(x,t)\in\left(B_{1/8}\left(\frac{3}{4}e_n\right)\cup B_{1/8}\left(-\frac{3}{4}e_n\right)\right)\times(-C\delta/\mu,C\delta/\mu)$.

Proof. Let $\gamma:=ae_n+be_t$, where $a^2+b^2=1$ and $e_t=(0,...,0,1)$, be perpendicular to the axis of symmetry of the full cone of monotonicity. Note that

$$|D_\gamma u(x,t)|\leqslant C\mu D_{e_n}u(x,t)$$

for any (x,t) where the derivatives exist. By interior a priori estimates and Theorem A,

$$|D_{\gamma\tau}u(x,t)|\leqslant C\mu D_{e_n}u(x,t)$$

for any (x,t) staying uniformly away from the parabolic boundary and free boundary.

Now, for any $p=(x,0)$ with $x\in B_{1/8}\left(\pm\frac{3}{4}e_n\right)$, we have

$$D_\tau u(p+k\gamma)\geqslant c\delta D_{e_n}u(p)+kD_{\gamma\tau}u(p+\bar{k}\gamma)$$

for some $0<|\bar{k}|<|k|$. Using the above estimate for the second derivatives as well as Theorem A, we establish that

$$D_\tau u(p+k\gamma) \geqslant (c\delta - C|k|\mu) D_{e_n} u(p+k\gamma).$$

Choosing $|k|<\frac{1}{2}(c/C)(\delta/\mu)$ and using Harnack's inequality for $D_\tau u$, we see that

$$D_\tau u(q) \geqslant c\delta D_{e_n} u(q)$$

where q belongs, at least, to $B_{1/8}(\pm\frac{3}{4}e_n)\times(-C\delta/\mu, C\delta/\mu)$. □

The set of the directions in Lemma 5, the directions of the original cone $\Gamma_x(\theta, e_n)$, and linear combinations of them, is easily seen to contain a cone $\Gamma_x(\bar{\theta}, \nu)$, where u is monotone, such that

$$\frac{\frac{1}{2}\pi-\bar{\theta}}{\frac{1}{2}\pi-\theta} < 1 - c(\theta, n)\cdot\delta.$$

5. Interior gain process in time

The method for the improvement of the cone in space in the previous section fails if we try to enlarge the cone in time. We will proceed in a different way, by first estimating in terms of δ/μ the oscillation of $D_n u$ ($D_n = D_{e_n}$), pointwise in the interior and in measure on the free boundary.

Let e_n be the projection in space of the axis of the monotonicity cone whose defect angle in time is $\mu = \frac{1}{2}\pi - \theta^t$; that is, there exist $A, B \in \mathbf{R}$, $A \leqslant B$ such that $B - A \approx \mu$ and

$$A \leqslant -\frac{D_t u^+(x,t)}{D_n u^+(x,t)} \left(-\frac{Du_t^-(x,t)}{Du_n^-(x,t)}\right) \leqslant B \quad (D_n = D_{e_n})$$

for (x,t) everywhere not on the free boundary and almost everywhere on the free boundary.

Suppose, now, that the defect angle in space is δ and satisfies $\delta < \delta/\mu \ll \mu$. Then we have the following:

LEMMA 6. *Let u be a viscosity solution of our free boundary value problem. Then*

$$u(x,t) = u(\pm\tfrac{3}{4}e_n, 0) + \alpha_\pm(x_n \pm \tfrac{3}{4}) + \alpha_\pm O(\delta/\mu)$$

for every $(x,t) \in B_{1/8}(\pm\frac{3}{4}e_n)\times(-C\delta/\mu, C\delta/\mu)$, where $\alpha_\pm := D_n u(\pm\frac{3}{4}e_n, 0)$.

Proof. For any $x \in B_{1/8}(\frac{3}{4}e_n)$, we have ($D_j = D_{e_j}$, $D_{ij} = D_{e_i e_j}$)

$$u(x,t) = u(\tfrac{3}{4}e_n, t) + D_n u(\tfrac{3}{4}e_n, t)(x_n - \tfrac{3}{4}) + \sum_{i=1}^{n-1} D_i u(\tfrac{3}{4}e_n, t) x_i$$
$$+ \int_0^1 \int_0^s \sum_{i,j=1}^n D_{ij} u((1-r)\tfrac{3}{4}e_n + rx, t)(x - \tfrac{3}{4}e_n)_i (x - \tfrac{3}{4}e_n)_j \, dr\, ds.$$

We estimate each term on the right, separately. For the first one, we write

$$u\big(\tfrac{3}{4}e_n,t\big)=u\big(\tfrac{3}{4}e_n,0\big)+\int_0^t D_t u\big(\tfrac{3}{4}e_n,s\big)\,ds.$$

Since$|D_t u|\leqslant cD_n u$, by using Theorem A, we have, for $|t|<C\delta/\mu$,

$$u\big(\tfrac{3}{4}e_n,t\big)=u\big(\tfrac{3}{4}e_n,0\big)+D_n u\big(\tfrac{3}{4}e_n,0\big)O(\delta/\mu).$$

The second term can be written as

$$D_n u\big(\tfrac{3}{4}e_n,t\big)=D_n u\big(\tfrac{3}{4}e_n,0\big)+\int_0^t D_{nt} u\big(\tfrac{3}{4}e_n,s\big)\,ds.$$

Using interior estimates and Theorem A, we see that $|D_t u|\leqslant cD_n u$ implies

$$\big|D_{nt}u\big(\tfrac{3}{4}e_n,t\big)\big|\leqslant CD_n u\big(\tfrac{3}{4}e_n,0\big).$$

Therefore, for $|t|<C\delta/\mu$, we have

$$D_n u\big(\tfrac{3}{4}e_n,t\big)=D_n u\big(\tfrac{3}{4}e_n,0\big)+D_n u\big(\tfrac{3}{4}e_n,0\big)O(\delta/\mu).$$

Since for $i=1,\dots,n-1$, $|D_i u|\leqslant c\delta D_n u$, by the previous estimate we have, for $|t|<C\delta/\mu$,

$$D_i u\big(\tfrac{3}{4}e_n,t\big)=D_n u\big(\tfrac{3}{4}e_n,0\big)O(\delta).$$

Finally, for the fourth term we observe that if p stays away from the free boundary then, by interior estimates as well as Theorem A, $|D_i u(p)|\leqslant c\delta D_n u(p)$, $i=1,\dots,n-1$, implies

$$|D_{ij}u(p)|\leqslant C\delta D_n u(p),\quad i=1,\dots,n-1,\ j=1,\dots,n.$$

This in turn implies, using the equation $\Delta u=a_1 D_t u$,

$$|D_{nn}u(p)|\leqslant C\delta D_n u(p)+ca_1\mu D_n u(p).$$

By hypothesis, $a_1\ll\delta$ and therefore, by Theorem A, we obtain

$$|D_{ij}u(p)|\leqslant C\delta D_n u\big(\tfrac{3}{4}e_n,0\big),\quad i,j=1,\dots,n.$$

Collecting all the estimates above we arrive at the desired result for the positive side of u. In a similar fashion we prove the result for the negative side, too. $\square$

We need to know, now, the size of the set of free boundary points where $D_n u^+$ is close to α_+. We do this by estimating their difference in the L^2-sense at all time levels.

LEMMA 7. *If F_t denotes the free boundary of u at each time level, then for all $|t|<C\delta/\mu$,*

$$\fint_{B_{1/8}(0)\cap F_t} |D_n u^{\pm}(x,t)-\alpha_{\pm}|^2\, dS \leqslant \alpha_{\pm}^2 O(\delta/\mu)$$

where $\alpha_{\pm}:=D_n u^{\pm}\left(\pm\frac{3}{4}e_n, 0\right)$.

Proof. We prove only the estimate for the positive part since the negative part can be done in a similar way. We do this in two steps.

Step 1: To prove

$$\left|\fint_{B_{1/8}(0)\cap F_t} (D_n u^+ - \alpha_+)\, dS\right| \leqslant \alpha_+ O(\delta/\mu).$$

Set $D:=\left\{(x',x_n)\in\mathbf{R}^{n-1}\times\mathbf{R} : |x'|<\frac{1}{8}, |x_n|<\frac{3}{4}\right\}\cap\{u<0\}$. By Theorem A, after rescaling if necessary, the function $w_+:=u+cu^{1+\varepsilon}$ is subharmonic in D and $c<\delta$, for some positive ε (see Lemma 5 in [ACS]). Then for almost all $|t|<C\delta/\mu$,

$$0\leqslant \int_D \Delta w_+(x,t)\, dx = \int_{\partial D} D_\nu w_+(x,t)\, dS_x$$

or

$$\int_{\bar{D}\cap F_t} D_{\nu_{\mathrm{int}}} u^+\, dS \leqslant \int_S D_\nu w_+\, dS + \int_T D_n w_+\, dS$$

where

$$S:=\left\{|x'|=\tfrac{1}{8}, |x_n|<\tfrac{3}{4}\right\}\cap\{u>0\},\quad T:=\left\{|x'|<\tfrac{1}{8}, x_n=\tfrac{3}{4}\right\}\cap\{u>0\}.$$

Since $|D_j w_+|\leqslant|D_j u|(1+c\delta)$, $j=1,...,n$, $|D_i u|\leqslant c\delta D_n u$, $i=1,...,n-1$, and $D_{\nu_{\mathrm{int}}}u^+\geqslant D_n u^+$, using Lemma 6 we have

$$\int_{\bar{D}\cap F_t} D_n u^+\, dS \leqslant \int_S C\delta D_n u\, dS + \int_T \alpha_+(1+O(\delta/\mu))\, dS$$

or

$$\int_{\bar{D}\cap F_t} D_n u^+\, dS \leqslant C\delta \int_{\partial S\cap\{x_n=3/4\}} u\, dS + \int_T \alpha_+(1+O(\delta/\mu))\, dS.$$

Hence, by Theorem A and Lemma 6, we obtain

$$\fint_{\bar{D}\cap F_t} (D_n u^+ - \alpha_+)\, dS \leqslant \alpha_+ O(\delta/\mu).$$

Similarly, using $w_-:=u-cu^{1+\varepsilon}$, we obtain

$$\fint_{\bar{D}\cap F_t} (D_n u^+ - \alpha_+)\, dS \geqslant -\alpha_+ O(\delta/\mu).$$

Step 2: To prove

$$\left|\fint_{B_{1/10}(0)\cap F_t}((D_n u^+)^2-\alpha_+^2)\,dS\right| \leqslant \alpha_\pm^2 O(\delta/\mu).$$

For any $0\leqslant r\leqslant \frac{1}{8}$, set

$$D_r := \{(x',x_n)\in \mathbf{R}^{n-1}\times\mathbf{R} : |x'|<r,\ |x_n|<\tfrac{3}{4}\}\cap\{u>0\},$$
$$S_r := \{|x'|=r,\ |x_n|<\tfrac{3}{4}\}\cap\{u>0\},\quad T_r=\{|x'|<r,\ x_n=\tfrac{3}{4}\}\cap\{u>0\}.$$

Again, as in Step 1, for almost all $|t|<C\delta/\mu$ we have

$$\int_{D_r}\nabla(D_n w_+)\nabla w_+\,dx \leqslant \int_{D_r}(\nabla(D_n w_+)\nabla w_+ + D_n w_+\Delta w_+)\,dx = \int_{\partial D_r} D_n w_+ D_\nu w_+\,dS$$

or

$$\int_{\bar D_r\cap F_t} D_n u^+ D_{\nu_{\rm int}} u^+\,dS + \frac{1}{2}\int_{D_r} D_n(|\nabla w_+|^2)\,dx \leqslant \int_{S_r} D_n w_+ D_\nu w_+\,dS + \int_{T_r}(D_n w_+)^2\,dS.$$

Integrating the second term on the left with respect to x_n and using $D_{\nu_{\rm int}}u^+\geqslant D_n u^+$ we obtain

$$\frac{1}{2}\int_{\bar D_r\cap F_t}(D_n u^+)^2\,dS \leqslant \int_{S_r} D_n w_+ D_\nu w_+\,dS + \frac{1}{2}\int_{T_r}(D_n w_+)^2\,dS.$$

Observe that $|D_i w_+|\leqslant c\delta D_n w_+$ for $i=1,\dots,n-1$, since $|D_i u|\leqslant c\delta D_n u$ for $i=1,\dots,n-1$. Therefore,

$$\int_{\bar D_r\cap F_t}(D_n u^+)^2\,dS \leqslant C\delta\int_{S_r}(D_n w_+)^2\,dS + \int_{T_r}(D_n w_+)^2\,dS.$$

Integrate the above from $\frac{1}{10}$ to $R\,(\leqslant\frac{1}{8})$, to get

$$\int_{1/10}^{R}\int_{\bar D_r\cap F_t}(D_n u^+)^2\,dS\,dr \leqslant C\delta\int_{1/10}^{R}\int_{S_r}(D_n w_+)^2\,dS\,dr + \int_{1/10}^{R}\int_{T_r}(D_n w_+)^2\,dS\,dr$$

or

$$\left(R-\tfrac{1}{10}\right)\int_{\bar D_{1/10}\cap F_t}(D_n u^+)^2\,dS \leqslant C\delta\int_{1/10}^{R}\int_{S_r}|\nabla w_+|^2\,dS\,dr + \left(R-\tfrac{1}{10}\right)\int_{T_R}(D_n w_+)^2\,dS.$$

By integrating by parts the first term on the right and using the subharmonicity of w_+, we obtain

$$\int_{\bar D_{1/10}\cap F_t}(D_n u^+)^2\,dS$$
$$\leqslant \frac{C\delta}{R-\frac{1}{10}}\left\{\int_{S_R} w_+ D_\nu w_+\,dS + \int_{S_{1/10}} w_+ D_\nu w_+\,dS + \int_{T_R-T_{1/10}} w_+ D_n w_+\,dS\right\} + \int_{T_R}(D_n w_+)^2\,dS.$$

Since $|D_i w_+|\leqslant|D_i u|(1+c\delta)$ for $i=1,...,n$ and $|D_j u|\leqslant c\delta D_n u$ for $j=1,...,n-1$, the above can be written as

$$\int_{\bar{D}_{1/10}\cap F_t}(D_n u^+)^2\,dS\leqslant\frac{C\delta^2(1+c\delta)}{R-\frac{1}{10}}\left\{\int_{S_R}\tfrac{1}{2}D_n(u^2)+\int_{S_{1/10}}\tfrac{1}{2}D_n(u^2)\right\}$$
$$+\frac{C\delta(1+c\delta)}{R-\frac{1}{10}}\int_{T_R-T_{1/10}}u_+D_n u+\int_{T_R}(D_n u)^2\,dS.$$

Integrating the first two terms on the right with respect to x_n and using Theorem A we have

$$\int_{\bar{D}_{1/10}\cap F_t}(D_n u^+)^2\,dS\leqslant\frac{C_1\delta^2}{R-\frac{1}{10}}\left\{\int_{\partial S_R\cap\{x_n=3/4\}}(D_n u)^2+\int_{\partial S_{1/10}\cap\{x_n=3/4\}}(D_n u)^2\right\}$$
$$+\frac{C\delta}{R-\frac{1}{10}}\int_{T_R-T_{1/10}}(D_n u)^2+\int_{T_R}(D_n u)^2\,dS.$$

By Lemma 5, we have

$$⨍_{\bar{D}_{1/10}\cap F_t}((D_n u^+)^2-\alpha_+^2)\,dS\leqslant\left(\frac{C\delta^2}{R-\frac{1}{10}}+\left(\frac{C\delta}{R-\frac{1}{10}}+1+O(\delta/\mu)\right)|T_R-T_{1/10}|\right)\alpha_+^2.$$

Therefore, by choosing $R-\frac{1}{10}\sim\delta$, we have

$$⨍_{\bar{D}_{1/10}\cap F_t}((D_n u^+)^2-\alpha_+^2)\,dS\leqslant\alpha_+^2 O(\delta/\mu).$$

If we use w_- instead of w_+ in the above calculations the proof of Step 2 is completed.

Combining Step 1 and Step 2 we arrive at the desired result. □

Now, we are able to show that the cone of monotonicity can be increased in time, simultaneously, on both sides of the free boundary if we stay away from the free boundary. More precisely:

LEMMA 8. *If $G(\alpha_+,\alpha_-,e_n)\geqslant -b:=-\frac{1}{2}A+B$ (or $G(\alpha_+,\alpha_-,e_n)\leqslant -b$), then there exists $c,\bar{c}>0$ such that if δ is small, $\delta\leqslant c\bar{\mu}^3$,*

$$-\frac{D_t u}{D_n u}\leqslant B-c\mu\quad\left(or\ -\frac{D_t u}{D_n u}\geqslant A+c\mu\right)$$

for

$$(x,t)\in\left(B_{1/8}(\tfrac{3}{4}e_n)\cup B_{1/8}(-\tfrac{3}{4}e_n)\right)\times(-C\delta/\mu,C\delta/\mu).$$

Proof. Observe that, since u satisfies $\Delta u+a_1 D_t u=0$ and $\Delta u+a_2 D_t u=0$ in $\Omega^+=(B_1\times(-1,1))\cap\{u>0\}$ and $\Omega^-=(B_1\times(-1,1))\cap\{u<0\}$, respectively, so does $D_t u+BD_n u$ (or $D_t u+AD_n u$).

For each $(x,t)\in B_{1/8}\times(-C\delta/\mu, C\delta/\mu)$ let $R_t=\Omega^+\cap(-a_1+t,t)$ and $\omega^{(x,t)}$ be the caloric measure in R_t evaluated at (x,t).

Now, on the free boundary, almost everywhere with respect to surface measure, we have

$$\frac{D_t u^+}{D_n u^+}=\frac{D_t u^+}{D_\nu u^+}(1+O(\delta))=(1+O(\delta))G(\nu, D_\nu u^+, D_\nu u^-).$$

By Lemma 7, if

$$\Sigma_t=\{p\in F\cap \bar{R}_t : D_n u^\pm(p)=\alpha_\pm(1+O(\delta^{1/3}))\}$$

then $|\Sigma_t|\geqslant\frac{1}{2}|F\cap\bar{R}_t|$ for any $t\in(-C\delta/\mu, C\delta/\mu)$.

Since (the restriction of) $\omega^{(x,t)}$ on $F\cap\bar{R}_t$ is an A_∞ weight with respect to surface measure, we have

$$\omega^{(x,t)}(\Sigma_t)\geqslant c.$$

On the other hand, on Σ_t,

$$\begin{aligned} G(\nu, D_\nu u^+, D_\nu u^-)&=G(e_n,\alpha_+,\alpha_-)+O(\delta^{1/3})\geqslant -b+O(\delta^{1/3})\\ &\geqslant -B+c\mu+O(\delta^{1/3})\geqslant -B+\bar{c}\mu. \end{aligned}$$

Therefore, we can write

$$(D_t u+BD_n u)(x,t)=\int_{\partial_p R_t}(D_t u+BD_n u)\,d\omega^{(x,t)}\geqslant \bar{c}\mu\alpha_+\omega^{(x,t)}(\Sigma_t)\geqslant C\mu\alpha_+.$$

Since $\delta/\mu\ll\mu$ and $Du_n(x,t)=\alpha_+(1+O(\delta/\mu))$ in $B_{1/8}\left(\frac{3}{4}e_n\right)\times(-\alpha_1+t,t)$, we obtain finally

$$(D_t u+BD_n u)(x,t)\geqslant C\mu D_n u.$$

Using Ω^- instead of Ω^+ we obtain the same inequality for

$$(x,t)\in B_{1/8}\left(-\tfrac{3}{4}e_n\right)\times(-C\delta/\mu, C\delta/\mu).$$

Similarly, we prove the complementary statement, too. □

6. Families of continuous perturbations

In this section we construct, starting from a viscosity solution, a particular family of subcaloric functions. These auxiliary functions play a major role in carrying to the free boundary the interior gains, by means of a powerful topological method first introduced by Caffarelli in [C1].

LEMMA 9. *Let u be a viscosity solution of a free boundary problem in a domain $D\subset\mathbf{R}^{n+1}$ and monotone increasing in every direction $\sigma\in\Gamma(\theta_0,e_n)$. Suppose φ is a C^2 function such that $1\leqslant\varphi\leqslant 2$ and satisfies*

$$D_t\varphi\geqslant 0,\quad \Delta\varphi-c_1\frac{\partial\varphi}{\partial t}-C\frac{|\nabla_x\varphi|^2}{\varphi}-c_2|\nabla_x\varphi|\geqslant 0$$

in a domain $D'\subset D$ with $\operatorname{dist}(D,D')\geqslant d>0$ *for some positive constants $c_1,c_2,C>1$ depending on n,θ_0,d. Then*

$$v(x,t):=\sup_{B_{\varphi(x,t)}(x,t)}u(y,s)$$

is subcaloric in $\{v>0\}\cap D'$ and in $\{v<0\}\cap D'$.

Proof. It is enough to show that the expression

$$\liminf_{r\downarrow 0}\left\{\frac{2(n+2)}{r^2}\fint_{B_r(\xi)}[v(x,t)-v(\xi,r)]\,dx\right\}-a_j\limsup_{h\to 0}\frac{1}{h}[v(\xi,\tau+h)-v(\xi,\tau)]\qquad(*)$$

($j=1,2$), is non-negative for every $(\xi,\tau)\in D'$. For simplicity let's assume that $(\xi,\tau)=(0,0)$ and $v(0,0)>0$. To estimate the first term by below we proceed as in [C1]. Choose the system of coordinates so that

(1) $v(0,0)=u(\varphi(0,0)\bar\nu)$ where $\bar\nu=\varepsilon e_n+\delta\varepsilon_{n+1}$, $|\bar\nu|=1$, $\varepsilon>0$,

(2) $\nabla_x\varphi(0,0)=\alpha e_1+\beta e_n$.

By definition of v,

$$v(x,0)\geqslant u(y(x),\varphi(0,0)\delta)$$

for $y(x):=x+\sqrt{\varphi^2(x,0)-\varphi^2(0,0)\delta^2}\cdot\nu_x/|\nu_x|$ where

$$\nu_x:=e_n+\frac{\beta x_1-\alpha x_n}{\varepsilon^2\varphi(0,0)}e_1+\frac{\gamma}{\varepsilon\varphi(0,0)}\sum_{i=2}^{n-1}x_ie_i$$

with γ solving $(1+\gamma)^2=(1+\beta/\varepsilon)^2+\alpha^2/\varepsilon^2$. Expand and collect terms to get

$$y(x)=y^*(x)+q(x)e_n+O(|x|^2)\mu+O(|x|^3)e$$

where

$$y^*(x):=\varepsilon\varphi(0,0)e_n+x+\frac{1}{\varepsilon}(\beta x_1-\alpha x_n)e_n+\frac{1}{\varepsilon}(\alpha x_1+\beta x_n)e_n+\gamma\sum_{i=2}^{n-1}x_ie_i,$$

$$q(x):=\frac{1}{2\varepsilon}D_{ij}\varphi(0,0)x_ix_j-\frac{(\beta x_1-\alpha x_n)^2}{2\varepsilon^3\varphi(0,0)}-\frac{\gamma^2\sum_{i=2}^{n-1}x_i}{2\varepsilon\varphi(0,0)}-\frac{\delta^2(\alpha x_1+\beta x_n)^2}{2\varepsilon^3\varphi(0,0)}$$

and $\mu, e\in\mathbf{R}^n$, $|\mu|=1$, $|e|=1$, $\mu\cdot e_n=0$. By the choice of γ we observe that

$$\frac{1}{1+\gamma}(y^*(x)-y(0))$$

is a rotation in the (e_1,e_n)-plane. Therefore

$$\lim_{r\downarrow 0}\frac{n}{\omega_n r^{n+2}}\int_{B_r(0)}(u(y^*(x),\varphi(0,0)\delta)-u(\varphi(0,0)\bar{\nu}))\,dx=\frac{(1+\gamma)^2}{2(n+2)}\Delta u(\varphi(0,0)\bar{\nu}).$$

Also,

$$\begin{aligned}
&\lim_{r\downarrow 0}\frac{1}{r^2}\fint_{B_r(0)}[u(y(x),\varphi(0,0)\delta)-u(y^*(x),\varphi(0,0)\sigma)]\,dx\\
&=\lim_{r\downarrow 0}\fint_{B_r(0)}|\nabla_x u(\varphi(0,0)\bar{\nu})|\{q(x)+O(|x^3|)\}\,dx\\
&=|\nabla_x u(\varphi(0,0)\bar{\nu})|\left\{\frac{1}{2\varepsilon(n+2)}\left[\Delta\varphi(0,0)-\left(\frac{\beta^2+\alpha^2}{\varepsilon^2}+\frac{\delta^2(\alpha^2+\beta^2)}{\varepsilon^2}+(n-2)\gamma^2\right)\frac{1}{\varphi(0,0)}\right]\right\}\\
&\geqslant\frac{1}{2\varepsilon(n+2)}|\nabla_x u(\varphi(0,0)\bar{\nu})|\left[\Delta\varphi(0,0)-\frac{(1+\delta^2)+(n-2)}{\varepsilon^2}\cdot\frac{|\nabla_x\varphi(0,0)|^2}{\varphi(0,0)}\right].
\end{aligned}$$

Therefore

$$\begin{aligned}
&\liminf_{r\downarrow 0}\left\{\frac{2(n+2)}{r^2}\fint_{B_r(0)}(v(x,0)-v(0,0))\,dx\right\}\\
&\quad\geqslant\lim_{r\downarrow 0}\frac{2(n+2)}{r^2}\fint_{B_r(0)}[u(y(x),\varphi(0,0)\delta)-u(\varphi(0,0)\bar{\nu})]\,dx\\
&\quad=\lim_{r\downarrow 0}\frac{2(n+2)}{r^2}\fint_{B_r(0)}[u(y(x),\varphi(0,0)\delta)-u(y^*(x),\varphi(0,0)\delta)]\,dx\\
&\qquad+\lim_{r\downarrow 0}\frac{2(n+2)}{r^2}\fint_{B_r(0)}[u(y^*(x),\varphi(0,0)\delta)-u(\varphi(0,0)\nu)]\,dx\\
&\quad\geqslant\frac{1}{\varepsilon}|\nabla_x u(\varphi(0,0)\bar{\nu})|\left[\Delta\varphi(0,0)-\frac{n-\varepsilon^2}{\varepsilon^2}\cdot\frac{|\nabla_x\varphi(0,0)|^2}{\varphi(0,0)}\right]+(1+\gamma)^2\Delta u(\varphi(0,0)\bar{\nu}).
\end{aligned}$$

Now the second term of the expression $(*)$ is easily seen to be bounded by

$$a_1\left\{|\nabla_x u(\varphi(0,0)\bar{\nu})|\varepsilon\frac{\partial\varphi(0,0)}{\partial t}+\frac{\partial u(\varphi(0,0)\bar{\nu})}{\partial t}\left(1+\delta\frac{\partial\varphi(0,0)}{\partial t}\right)\right\}.$$

Hence, using the fact that u satisfies $\Delta u=a_1u_t$, the expression $(*)$ becomes greater or equal to

$$\begin{aligned}
&\frac{1}{\varepsilon}|\nabla_x u(\varphi(0,0)\bar{\nu})|\left[\Delta\varphi(0,0)-a_1\varepsilon^2\frac{\partial\varphi}{\partial t}-\frac{n-\varepsilon^2}{\varepsilon^2}\cdot\frac{|\nabla_x\varphi(0,0)|^2}{\varphi(0,0)}\right]\\
&\qquad+a_1u_t(\varphi(0,0)\bar{\nu})\left[\frac{2\beta}{\varepsilon}+\frac{\alpha^2+\beta^2}{\varepsilon^2}-\delta\frac{\partial\varphi(0,0)}{\partial t}\right].
\end{aligned}$$

Since u is monotone in every $\sigma\in\Gamma(\theta_0, e_n)$, $|u_t|\leqslant\cot\theta_0|\nabla_x u|$ and $\varepsilon\geqslant\sin 2\theta_0$. Therefore, since $1\leqslant\varphi\leqslant 2$, the above is

$$\geqslant\frac{1}{\varepsilon}|\nabla_x u(\varphi(0,0)\bar{\nu})|\left\{\Delta\varphi - c_1\frac{\partial\varphi}{\partial t} - C\frac{|\nabla\varphi|^2}{\varphi} - c_2|\nabla\varphi|\right\}$$

where c, c_2 depend only on θ_0 while C depends on θ_0 and n. □

LEMMA 10. *Let u be a viscosity solution to a free boundary problem in Q_2 and*

$$v(x,t) := \sup_{B_{\varphi(x,t)}(x,t)} u(y,s)$$

with φ as in Lemma 9 and $|\nabla\varphi|<1$. If $(x_0,t_0)\in\partial\{v>0\}\cap Q_{3/2}$, $(y_0,s_0)\in F$ and $(y_0,s_0)\in\partial B_{\varphi(x_0,t_0)}(x_0,t_0)$ then

(i) *$\partial\{v>0\}$ has a tangent ball at (x_0,t_0) from the right (i.e., there exists $B^{n+1}\subset\{v>0\}$ such that $\bar{B}^{(n+1)}\cap\partial\{v>0\}=\{(x_0,t_0)\}$),*

(ii) *if $\partial\{u>0\}$ is a Lipschitz graph and $|\nabla\varphi|$ is small enough (depending on the Lipschitz constant L of $\partial\{u>0\}$), the set $\partial\{v>0\}$ is a Lipschitz graph with Lipschitz constant*

$$L'\leqslant L + C\sup|\nabla\varphi|,$$

(iii) *if near (y_0,s_0), at s_0-level, u has the asymptotic expansion*

$$u(y,s_0) = \alpha_+\langle y-y_0,\nu\rangle^+ - \alpha_-\langle y-y_0,\nu\rangle^- + o(|y-y_0|)$$

where $\nu=(y_0-x_0)/|y_0-x_0|$, then near (x_0,t_0), at t_0-level,

$$v\langle x,t_0\rangle \geqslant \alpha_+\left\langle x-x_0, \nu+\frac{\varphi(x_0,t_0)}{|y_0-x_0|}\nabla_x\varphi\right\rangle^+ - \alpha_-\left\langle x-x_0, \nu+\frac{\varphi(x_0,t_0)}{|y_0-x_0|}\nabla_x\varphi\right\rangle^- + o(|x-x_0|).$$

Proof. The proofs of (i) and (ii) are similar to those of Lemma 10 of [C1].

To prove (iii), for x near x_0 set $y=x+\nu\bar{\varphi}(x)$ where $\bar{\varphi}(x):=\sqrt{\varphi^2(x,t_0)-(s_0-t_0)^2}$. Then $v(x,t_0)\geqslant u(y,s_0)$. Hence, since

$$\bar{\varphi}(x)-\bar{\varphi}(x_0) = x-x_0+\frac{\varphi(x_0,t_0)}{|y_0-x_0|}\nabla_x\varphi(x_0,t_0)+o(|x-x_0|)$$

we have, by Theorem B,

$$v^+(x,t_0)\geqslant\alpha_+\left\langle x-x_0, \nu+\frac{\varphi(x_0,t_0)}{|y_0-x_0|}\nabla_x\varphi(x_0,t_0)\right\rangle^+ + o(|x-x_0|)$$

and

$$v^-(x,t_0)\leqslant\alpha_-\left\langle x-x_0, \nu+\frac{\varphi(x_0,t_0)}{|y_0-x_0|}\nabla_x\varphi(x_0,t_0)\right\rangle^- + o(|x-x_0|).$$ □

7. Propagation to the free boundary

In this section we show how an interior gain in the aperture of the cone of monotonicity can be carried to the free boundary.

LEMMA 11. *Let $0<T_0\leqslant T$ and $C>1$. There exist positive constants $\overline{C}$, k and h_0, depending only on C and T_0 such that, for any h, $0<h<h_0$, there exists a family of C^2 functions φ_η, $0\leqslant\eta\leqslant1$, defined in*

$$D:=\left[B_1\setminus\left\{\overline{B}_{1/8}\left(\tfrac{3}{4}e_n\right)\cup\overline{B}_{1/8}\left(-\tfrac{3}{4}e_n\right)\right\}\right]\times(-T,T),$$

such that

(i) $1\leqslant\varphi_\eta\leqslant1+\eta h$ *in* D,
(ii) $\Delta\varphi_\eta-c_1D_t\varphi_\eta-C|\nabla_x\varphi_\eta|^2/\varphi_\eta-c_2|\nabla_x\varphi_\eta|\geqslant0$ *in* D,
(iii) $\varphi_\eta\equiv1$ *outside* $B_{8/9}\times\left(-\frac{7}{8}T,T\right)$,
(iv) $\varphi_\eta(x,t)\geqslant1+k\eta h$ *for* $(x,t)\in B_{1/2}\times\left(-\frac{1}{2}T,\frac{1}{2}T\right)$,
(v) $|\nabla\varphi_\eta|\leqslant\overline{C}\eta h$ *in* D,
(vi) $D_t\varphi_\eta\geqslant0$ *in* D,

provided that c_1 and c_2 are small enough.

Proof. Set $\psi_\eta=-1+\varphi_\eta^{1-C}$. If $C>1$ and ψ_η satisfies

$$\Delta\psi_\eta-c_1D_t\psi_\eta-c_2|\nabla_x\psi_\eta|\leqslant0 \tag{1}$$

then φ_η satisfies (ii). Now, if $c_1,c_2>0$ are small enough, it is easily seen that there exist functions ψ_η satisfying (1) in D such that

$$\begin{cases} -a\eta h\leqslant\psi_\eta\leqslant0 & \text{in } D,\\ \psi_\eta\equiv0 & \text{outside } B_{9/10}\times\left(-\frac{7}{8}T,T\right),\\ \psi_\eta\leqslant-bk\eta h & \text{in } \overline{B}_{1/2}\times\left[-\frac{1}{2}T,\frac{1}{2}T\right],\\ |\nabla\psi_\eta|\leqslant\widetilde{C}\eta h & \text{in } D,\\ D_t\psi_\eta\leqslant0 & \text{in } D,\end{cases}$$

where h is small and $a,b,\widetilde{C}$ ($b<a$) are chosen so that

$$1-bk\eta h\leqslant(1+k\eta h)^{1-C},\quad 1-a\eta h\leqslant(1+\eta h)^{1-C}\quad\text{and}\quad \widetilde{C}(C-1)<2^C\overline{C}. \qquad \square$$

LEMMA 12 (propagation lemma). *Let $u_1\leqslant u_2$ be two viscosity solutions of our free boundary problem in Q_2 with $F(u_2)$, the free boundary of u_2, Lipschitz continuous through the origin. Assume further that*

$$v_\varepsilon(x,t):=\sup_{B_\varepsilon^{(n+1)}(x,t)}u_1\leqslant u_2(x,t)$$

for $(x,t)\in B_1\times(-T,T)$, *and, for some* h *small,*

$$u_2(x,t)-v_{(1+h\sigma)\varepsilon}(x,t)\geqslant C\sigma\varepsilon u_2\big(\tfrac{3}{4}e_n,0\big)\qquad \forall(x,t)\in B_{1/8}\big(\tfrac{3}{4}e_n\big)\times(-T,T)\subset\{u_2>0\},$$
$$u_2(x,t)-v_{(1+h\sigma)\varepsilon}(x,t)\geqslant -C\sigma\varepsilon u_2\big(-\tfrac{3}{4}e_n,0\big)\quad \forall(x,t)\in B_{1/8}\big(-\tfrac{3}{4}e_n\big)\times(-T,T)\subset\{u_2<0\}.$$

Then, if $\varepsilon>0$ *and* $h>0$ *are small enough, there exists* c, $0<c<1$, *such that*

$$v_{(1+ch\sigma)\varepsilon}(x,t)\leqslant u_2(x,t)\quad \textit{in } B_{1/2}\times\big(-\tfrac{1}{2}T,\tfrac{1}{2}T\big).$$

Proof. We construct a continuous family of functions $\bar v_\eta\leqslant u_2$ for $0\leqslant\eta\leqslant1$ such that $\bar v_1\geqslant v_{(1+ch\sigma)\varepsilon}$ in $B_{1/2}\times\big(-\frac{1}{2}T,\frac{1}{2}T\big)$. For that purpose set

$$\bar v_\eta(x,t):=v_\eta(x,t)+C\sigma\varepsilon w(x,t)$$

where

$$v_\eta(x,t):=\sup_{B^{(n+1)}_{\varepsilon\varphi\sigma\eta}(x,t)} u_1(y,s)$$

and w_η is a continuous function in

$$D:=\big[B_{9/10}(0)\setminus\{B_{1/8}\big(\tfrac{3}{4}e_n\big)\cup B_{1/8}\big(-\tfrac{3}{4}e_n\big)\}\big]\times\big(-\tfrac{9}{10}T,\tfrac{9}{10}T\big)$$

defined as follows:

$$\begin{cases}\Delta w-a_1D_tw=0 & \text{in } D\cap\{u_2>0\},\\ \Delta w-a_2D_tw=0 & \text{in } D\cap\{u_2<0\},\\ w=0 & \text{on } D\cap\{u_2=0\},\\ w=u_2\big(\tfrac{3}{4}e_n,0\big) & \text{on } \partial B_{1/8}\big(\tfrac{3}{4}e_n\big)\times\big(-\tfrac{9}{10}T,\tfrac{9}{10}T\big),\\ w=-u_2\big(-\tfrac{3}{4}e_n,0\big) & \text{on } \partial B_{1/8}\big(-\tfrac{3}{4}e_n\big)\times\big(-\tfrac{9}{10}T,\tfrac{9}{10}T\big),\\ w\equiv0 & \text{on } \partial_pD\setminus\{\partial\big(B_{1/8}\big(\tfrac{3}{4}e_n\big)\cup B_{1/8}\big(-\tfrac{3}{4}e_n\big)\big)\times\big(-\tfrac{9}{10}T,\tfrac{9}{10}T\big)\big\}.\end{cases}$$

We prove now that the set of η's for which $\bar v_\eta\leqslant u_2$ is open and closed. First we notice that, by hypothesis and maximum principle, this set contains $\eta=0$, and so is nonempty. Also, it is easy to see that this set is closed. To show that it is open, assuming that $\bar v_{\eta_0}\leqslant u_2$ for some $\eta_0\in[0,1]$, it is enough to show that $D\cap\{\bar v_{\eta_0}>0\}$ is compactly contained in $D\cap\{u_2>0\}$. Suppose not, then there exists $(x_0,t_0)\in F(\bar v_{\eta_0})\cap F(u_2)\cap D$. About such a point, by Lemma 10, we have

$$v_{\eta_0}(x,t_0)\geqslant\alpha_+^*\langle x-x_0,\nu^*\rangle^+-\alpha_-^*\langle x-x_0,\nu^*\rangle^-+o(|x-x_0|)$$

where

$$\alpha_+^* = \alpha_+^{(1)}|\tau^*|, \quad \alpha_-^* = \alpha_-^{(1)}|\tau^*|, \quad \nu^* = \frac{\tau^*}{|\tau^*|},$$

$$\tau^* := \nu^{(1)} + \frac{\varepsilon^2 \varphi_{\sigma\eta}(x_0, t_0)}{|y_0 - x_0|} \nabla_x \varphi_{\sigma\eta}(x_0, t_0), \quad \nu^{(1)} := \frac{y_0 - x_0}{|y_0 - x_0|}$$

and

$$\frac{s_0 - t_0}{|y_0 - x_0|} =: \frac{\beta_+^{(1)}}{\alpha_+^{(1)}} \leqslant G(\nu^{(1)}, \alpha_+^{(1)}, \alpha_-^{(1)}).$$

Also, by Theorem B,

$$u_2(x, t_0) = \alpha_+^{(2)} \langle x - x_0, \nu^{(2)} \rangle^+ - \alpha_-^{(2)} \langle x - x_0, \nu^{(2)} \rangle^- + o(|x - x_0|)$$

where $\nu^{(2)} = \nu^*$ and

$$G(\nu^{(2)}, \alpha_+^{(2)}, \alpha_-^{(2)}) \leqslant \frac{\beta_+^{(2)}}{\alpha_+^{(2)}} := \left(\frac{s_0 - t_0}{|y_0 - x_0|} + \frac{\varepsilon^2 \varphi_{\sigma\eta}(x_0, t_0)}{|y_0 - x_0|} D_t \varphi_{\sigma\eta_0}(x_0, t_0) \right) |\tau^*|^{-1}.$$

Since $F(u_2)$ is Lipschitz, by Corollary 1 [ACS], we have $w/u_2 \geqslant C$ in $\{u_2 > 0\}$ and $w/(-u_2) \geqslant C$ in $\{u_2 < 0\}$, strictly away from the parabolic boundary of D. Hence, in the hyperplane $t = t_0$, near x_0, we have

$$\bar{v}_{\eta_0}(x, t_0) \geqslant \bar{\alpha}_+ \langle x - x_0, \nu^* \rangle^+ - \bar{\alpha}_- \langle x - x_0, \nu^* \rangle^- + o(|x - x_0|)$$

where

$$\bar{\alpha}_+ = \alpha_+^* + C\sigma\varepsilon\alpha_+^{(2)}, \quad \bar{\alpha}_- = \alpha_-^* - C\sigma\varepsilon\alpha_-^{(2)}.$$

Now, by Theorem B,

$$\begin{aligned} G(\nu^{(2)}, \alpha_+^{(2)}, \alpha_-^{(2)}) \leqslant \frac{\beta_+^{(2)}}{\alpha_+^{(2)}} &\leqslant \left(\frac{\beta_+^{(1)}}{\alpha_+^{(1)}} + C\sigma\varepsilon h \right) (1 + C\sigma\varepsilon h) \\ &\leqslant G(\nu^{(1)}, \alpha_+^{(1)}, \alpha_-^{(1)}) + \bar{C}\sigma\varepsilon h. \end{aligned}$$

Since

$$\alpha_+^{(1)} \leqslant \bar{\alpha}_+ - C\sigma\varepsilon\alpha_+^{(2)} + C\sigma\varepsilon h, \quad \alpha_-^{(1)} \geqslant \bar{\alpha}_- + C\sigma\varepsilon\alpha_-^{(2)} - C\sigma\varepsilon h,$$

and by hypothesis, $\partial G / \partial \alpha_+ > c^* > 0$, $\partial G / \partial \alpha_- < -c^*$, we have

$$G(\nu^{(1)}, \alpha_+^{(1)}, \alpha_-^{(1)}) < G(\nu^{(1)}, \bar{\alpha}_+ - C\sigma\varepsilon\alpha_+^{(2)} + C\sigma\varepsilon h, \bar{\alpha}_- + C\sigma\varepsilon\alpha_-^{(2)} - C\sigma\varepsilon h).$$

By the mean value theorem applied on each of its arguments and the fact that G is Lipschitz continuous in all of its arguments, we obtain

$$G(\nu^{(1)},\alpha_+^{(1)},\alpha_-^{(1)})<G(\nu^{(2)},\bar{\alpha}_+,\bar{\alpha}_-)+C|\nu^{(2)}-\nu^{(1)}|-c^*C\sigma\varepsilon(\alpha_+^{(2)}+\alpha_-^{(2)})+C\sigma\varepsilon h.$$

Since, by hypothesis, $\alpha_+^{(2)}+\alpha_-^{(2)}\geqslant m>0$ and $|\nu^{(2)}-\nu^{(1)}|\leqslant C\sigma\varepsilon h$, we get

$$\leqslant G(\nu^{(2)},\alpha_+,\bar{\alpha}_-)-(c^*m-h)C\sigma\varepsilon.$$

Thus, choosing $h<c^*m$, we have

$$G(\nu^{(2)},\alpha_+^{(2)},\alpha_-^{(2)})<G(\nu^{(2)},\bar{\alpha}_+,\bar{\alpha}_-).$$

On the other hand, $u_2-\bar{v}_{\eta_0}\geqslant 0$ and it is a supercaloric in $\{\bar{v}_{\eta_0}>0\}$. This implies, $\alpha_-^{(2)}\leqslant\bar{\alpha}_-$ and, by Hopf's maximum principle, $\alpha_+^{(2)}>\bar{\alpha}_+$. Therefore, $G(\nu^{(2)},\alpha_+^{(2)},\alpha_-^{(2)})>G(\nu^{(2)},\alpha_+^{(2)},\alpha_-^{(2)})$ by the strict monotonicity of G. We have reached a contradiction that concludes the proof. □

8. Regularization in space

In this section we will apply the results of §7 in order to show that the free boundary is a C^1 domain in space. This amounts to say that the defect angle in space is as small as we want so that we can proceed to apply the results in §5 to achieve regularity in time and improve the regularity in space. The symbol $\Gamma(\nu,\theta^x,\theta^t)$ will denote an elliptic cone with axis ν and aperture θ^x in space and θ^t in time.

LEMMA 13. *Let u be a viscosity solution to a free boundary problem in $Q_1=B_1\times(-1,1)$, monotone increasing in every direction $\tau\in\Gamma(e_n,\theta,\theta^t)$ for some $0<\theta_0\leqslant\theta^t\leqslant\theta<\frac{1}{2}\pi$. Then there exist positive constants $c,\bar{c}$ and a unit vector ν_1 such that, in $B_{1/2}\times(-1,1)$, the function $u_1(x,t)=u(x,\bar{c}\delta^2 t)$ is monotone increasing along every direction $\tau\in\Gamma(\nu_1,\theta_1,\theta_1^t)$ with*

$$\theta_1\geqslant\theta+c\delta^3,\quad \theta_1^t\geqslant\theta_0.$$

Proof. By Corollary 2 in §3 there exists a unit vector ν and $b<1$ such that, for $\theta^*:=\frac{1}{2}\pi-b\left(\frac{1}{2}\pi-\theta\right)$, u is monotone increasing in every spatial direction $\tau_x\in\Gamma_x(\nu,\theta^*)\supset\Gamma_x(e_n,\theta)$ for every $(x,t)\in B_{1/8}\left(\pm\frac{3}{4}e_n\right)\times\left(-\frac{1}{16}a^2,\frac{1}{16}a^2\right)$. Consider all spatial unit directions $\tau_x\in\Gamma_x(e_n,\theta)-\mathcal{N}$ where $\mathcal{N}$ denotes a neighborhood of the line of touching (if they touch), i.e., of the set $\partial\Gamma_x(e_n,\theta)\cap\partial\Gamma_x(\nu,\theta^*)$. Then, by Lemma 4, there exist positive constants $C,\tilde{c}$, depending also on the site of the deleted neighborhood $\mathcal{N}$, such that

$$D_{\tau_x}u(x,t)\geqslant\tilde{c}\delta D_{e_n}u(x,t)$$

for every $(x,t)\in B_{1/8}\left(\pm\frac{3}{4}e_n\right)\times(-C\delta/\mu, C\delta/\mu)$ where $\mu=\frac{1}{2}\pi-\theta_0$. Choose now a number $c_1\ll 1$, set $\bar{c}:=(C/\mu)c_1$ and perform a dilation in time of order $\bar{c}\delta^2$, that is, set $u_1(x,t):=u(x,\bar{c}\delta^2 t)$. Observe that the coefficient of the t-derivatives of u_1 in the equation as well as in the free boundary condition is multiplied by the factor $1/\bar{c}\delta^2$. Also, the region $B_1\times(-C\delta/\mu, C\delta/\mu)$ is mapped to $B_1\times(-1/c_1\delta, 1/c_1\delta)$ and the cone $\Gamma(e_n,\theta,\theta^t)$ is transformed into an elliptic cone which certainly contains a circular cone $\bar{\Gamma}(e_n,\theta)$.

Now, for every direction in space and time $\varrho\in\bar{\Gamma}(e_n,\theta)$ of the form $\varrho=\lambda_1\sigma+\lambda_2 e_t$ with σ a unit vector in $\Gamma_x(e_n,\theta)-\mathcal{N}$, $\lambda_1^2+\lambda_2^2=1$ and $|\lambda_2|\leqslant\frac{1}{2}\cdot\lambda_1 c_2/\bar{c}\delta$, we have

$$D_\varrho u_1(x,t)\geqslant c\delta D_{e_n}u_1(x,t) \tag{$*$}$$

$\forall(x,t)\in B_{1/8}\left(\pm\frac{3}{4}e_n\right)\times(-1/c_1\delta, 1/c_1\delta)$ since $D_t u_1\leqslant\bar{c}\delta^2 D_{e_n}u_1$. Let now $\tau\in\bar{\Gamma}\left(e_n,\frac{1}{2}\theta\right)$ be a small vector of the form $\tau=\eta\varrho$ $(0<\eta\ll 1)$ and $(x_0,t_0)\in B_{1/8}\left(\pm\frac{3}{4}e_n\right)\times(-1/c_1\delta, 1/c_1\delta)$. Then, for any $(y,s)\in B_\varepsilon^{(n+1)}(x_0,t_0)$ where $\varepsilon=|\tau|\sin\frac{1}{2}\theta$, we have

$$u_1(y,s):=u_1((y,s)-\bar{\tau})=u_1((x_0,t_0)-\tau)=u_1(x_0,t_0)-D_\tau u_1(x^*,t^*)|\tau|$$

$(\bar{\tau}=\tau+(x_0-y,t_0-s))$. Then by $(*)$ and the fact that $D_{e_n}u\sim u/d_{x,t}$ we obtain

$$\bar{v}_\varepsilon(x_0,t_0):=\sup_{B_\varepsilon^{(n+1)}(x_0,t_0)} u_1(y,s)\leqslant u_1(x_0,t_0)-c\delta\varepsilon u_1(x_0,t_0).$$

Furthermore by Lemma 12 there exists an $h>0$ and a $C>0$ such that

$$\bar{v}_{(1+h\delta)\varepsilon}(x,t)\leqslant u_1(x,t)-C\delta\varepsilon u_1(x_0,t_0)$$

for every (x,t) in a parabolic neighborhood of (x_0,t_0). But, since for

$$(x_0,t_0)\in B_{1/8}\left(\tfrac{3}{4}e_n\right)\times\left(-\frac{1}{c_1\delta},\frac{1}{c_1\delta}\right)\quad\text{and}\quad(x_0,t_0)\in B_{1/8}\left(-\tfrac{3}{4}e_n\right)\times\left(-\frac{1}{c_1\delta},\frac{1}{c_1\delta}\right)$$

we have $u_1(x_0,t_0)\sim u_1\left(\frac{3}{4}e_n,0\right)$ and $u_1(x_0,t_0)\sim u_1\left(-\frac{3}{4}e_n,0\right)$, respectively, we can write

$$\bar{v}_{(1+h\delta)\varepsilon}(x,t)\leqslant u_1(x,t)-C\delta\varepsilon u_1\left(\tfrac{3}{4}e_n,0\right)\quad\forall(x,t)\in B_{1/8}\left(\tfrac{3}{4}e_n\right)\times\left(-\frac{1}{c_1\delta},\frac{1}{c_1\delta}\right)$$

and

$$\bar{v}_{(1+h\delta)\varepsilon}(x,t)\leqslant u_1(x,t)-C\delta\varepsilon u_1\left(-\tfrac{3}{4}e_n,0\right)\quad\forall(x,t)\in B_{1/8}\left(-\tfrac{3}{4}e_n\right)\times\left(-\frac{1}{c_1\delta},\frac{1}{c_1\delta}\right).$$

Since $u_1(x,t)$ and $\bar{u}_1(x,t):=u_1((x,t)-\tau)$ are two viscosity solutions in the region $B_1\times(1/c_1\delta, 1/c_1\delta)$ and satisfy the hypotheses of the propagation lemma, taking into account the modification in the free boundary relation due to the time scaling, we have for a small positive constant c,

$$v_{(1+ch\delta^3)\varepsilon}(x,t) \leqslant u_1(x,t)$$

for every $(x,t) \in B_{1/2} \times (-1/2c_1\delta, 1/2c_1\delta)$. (Notice that in order to apply the propagation lemma the coefficient of $D_t u_1$ must be kept small, but as it has been remarked in §2 it can be made as small as we like.)

The last inequality implies that, along any direction of the form $\tau + (1+ch\delta^3)\varepsilon\nu$ where $\nu \in \mathbf{R}^{n+1}$, $|\nu|=1$, u_1 is monotone increasing.

The convex envelope of this family of directions and the cone $\bar{\Gamma}(e_n,\theta)$ is readily seen to contain a new cone $\Gamma(\nu_1,\theta_1,\theta_1^t)$ with $\theta_1^t \geqslant \theta_0$ and

$$\theta_1 - \theta \geqslant c\delta^3. \qquad \square$$

THEOREM 1. *Let u be a viscosity solution in $Q_1 = B_1 \times (-1,1)$. Then for each time level t, the transition surface $F_t := F \times \{t\}$ is a C^1 surface.*

Proof. This is proved essentially by an iteration of the previous lemma. Set $u_1(x,t) = u(x, \bar{c}\delta_0^2 t)$ where $\delta_0 = \delta$, $\bar{c}$ and δ as in Lemma 13.

By this lemma, u_1 is monotone increasing along any direction belonging to a cone $\Gamma(\nu_1,\theta_1,\theta_1^t)$ with $\theta_1^t \geqslant \theta_0$ and $\delta_1 = \frac{1}{2}\pi - \theta_1 \leqslant \delta_0 - c\delta_0^3$ in $B_{1/2} \times (-1,1)$. Suppose now that $u_k = u_k(x,t)$, $k \geqslant 1$, satisfies the hypotheses of Lemma 13 in a cone $\Gamma(\nu_k,\theta_k,\theta_k^t)$ with $\theta_k^t \geqslant \theta_0$ for (x,t) in $B_{1/2} \times (-1,1)$. Set

$$u_{k+1}(x,t) := u_k(2^{-r_k}x, \bar{c}2^{-r_k}\delta_k^2 t) \cdot 2^{r_k}$$

where $\delta_k = \frac{1}{2}\pi - \theta_k$ and r_k is a large integer chosen such that $2^{-r_k} < \bar{c}\delta_k^2$.

Then, by Lemma 13, u_{k+1} is monotone in a cone $\Gamma(\nu_{k+1},\theta_{k+1},\theta_{k+1}^t)$ with $\theta_{k+1}^t \geqslant \theta_0$ and

$$\delta_{k+1} \leqslant \delta_k - c\delta_k^3 \quad \text{in } B_{1/2} \times (-1,1).$$

This recurrence relation implies $\delta_k \to 0$ as $k \to +\infty$ and the conclusion follows. $\square$

9. Regularization in space-time

The fact that the defect angle in space, by the previous section, can be made as small as we prefer, allows us to use the results of §5. The delicate balance between the defect angle in space and the defect angle in time will give us a modulus of continuity in time and an improved one in space. As we have already said in §5, defect angle in time equal to μ means that, for some A, B with $0 < B - A \leqslant c\mu$, u is monotone increasing along the directions $e_t + Be_n$ and $-e_t - Ae_n$. To enlarge the cone in time, we have to lower B or increase A.

LEMMA 14 (basic iteration). *Let u be a viscosity solution in $B_1 \times (-1,1)$ of a free boundary problem with the following properties*:

(i) *u is monotone increasing in any direction of a space cone*

$$\Gamma_x(e_n,\theta), \quad 0<\theta_0 \leqslant \theta < \tfrac{1}{2}\pi.$$

(ii) *There exist constants $\bar{c}_1$ (positive) and A, B such that u is monotone increasing along the directions $e_t + Be_n$ and $-e_t - Ae_n$ with*

$$0 < B-A \leqslant \bar{c}_1 \mu.$$

Then, if $\delta := \frac{1}{2}\pi - \theta \ll \mu^3$, there exist constants c_1, c_2, C (positive) and A_1, B_1 depending only on n, θ_0, and a spatial unit vector ν_1, such that, in $B_{1/2} \times (-C\delta/2\mu, C\delta/2\mu)$,

(a) *u is monotone increasing in any direction $\tau \in \Gamma_x(\nu_1, \theta_1)$ with*

$$\tfrac{1}{2}\pi - \theta_1 := \delta_1 \leqslant \delta - c_1 \frac{\delta^2}{\mu},$$

(b) *u is monotone increasing along the directions $e_t + B_1 \nu_1$ and $-e_t - A_1 \nu_1$ with*

$$0 < B_1 - A_1 \leqslant \bar{c}_1 \mu_1 \quad \textit{and} \quad \mu_1 \leqslant \mu - c_2 \delta.$$

Proof. We first perform a dilation in time of size δ/μ and set $w(x,t) = u(x, \delta t/\mu)$. After this dilation the coefficient of the t-derivative in the heat equation as well as in the free boundary condition is multiplied by μ/δ. Also, the regions $B_1 \times (-1,1)$ and $B_{1/8}(\pm\frac{3}{4}e_n) \times (-C\delta/\mu, C\delta/\mu)$ are mapped to $B_1 \times (-\mu/\delta, \mu/\delta)$ and $B_{1/8}(\pm\frac{3}{4}e_n) \times (-C, C)$, respectively.

The condition (ii) becomes: w is monotone increasing along the directions

$$e_t + \frac{\delta}{\mu} B e_n \quad \text{and} \quad -e_t - \frac{\delta}{\mu} A e_n$$

with

$$0 < B - A \leqslant \bar{c}_1 \delta.$$

To enlarge the cone in space and prove (a), consider the spatial vectors $\tau \in \Gamma_x(e_n, \theta - \delta)$ with $|\tau| \ll \delta$, and let

$$\varepsilon = |\tau| \sin \delta.$$

If we define $w_1(x,t) = w(x-\tau, t)$, clearly we have

$$\sup_{B_\varepsilon(x)} w_1(y,t) \leqslant w(x,t) \quad \forall (x,t) \in B_{1-\varepsilon} \times \left(-\frac{\mu}{\delta}, \frac{\mu}{\delta}\right).$$

We proceed now as in Corollary 1 (§2) and then as in Lemma 5, i.e., by deleting, say, $\frac{1}{4}$ of the original cone, containing a fixed neighborhood of both the generatrices of $\Gamma_x(e_n,\theta)$ opposite to $\nabla u(\pm\frac{3}{4}e_n,-\frac{1}{2}a^2)$ (see §3), we conclude that for any other unit vector σ in the remaining cone $\Gamma_x^{\frac{1}{2}}(e_n,\theta)$, we have

$$D_\sigma w \geqslant c\delta D_n w \quad \text{in } B(\pm\tfrac{3}{4}e_n)\times(-C,C). \tag{$*$}$$

This inequality extends to derivatives along directions having a t-component of order δ. In fact, if for the spatial unit vector σ, $(*)$ holds, then, for $\lambda_1^2+\lambda_2^2=1$, since $|D_t w|\leqslant \tilde{c}(\delta/\mu)D_n w$, we have

$$\lambda_1 D_\sigma w+\lambda_2 D_t w \geqslant \left(c\lambda_1\delta-\tilde{c}\lambda_2\frac{\delta}{\mu}\right)D_n w \geqslant \widetilde{C}\delta D_n w$$

as long as $|\lambda_2|\leqslant\delta$, since $\delta/\mu\ll 1$. As a consequence, if ϱ is any unit vector in $\mathbf{R}^{n+1}$ and $\bar{\tau}=\tau+\varepsilon\varrho$ in $B_{1/8}(\pm\frac{3}{4}e_n)\times(-C,C)$, respectively, we can write

$$w((x,t)-\bar{\tau})-w(x,t)=-D_{\bar{\tau}}w(\tilde{x},\tilde{t})\leqslant -c\delta\varepsilon D_n w(x,t)\leqslant -c\varepsilon\delta w^{\pm}(\pm\tfrac{3}{4}e_n,0)$$

(since $|\bar{\tau}|\geqslant c\varepsilon$ and by Theorem A). Therefore

$$v_\varepsilon(x,t):=\sup_{B_\varepsilon^{(n+1)}(x,t)} w_1(y,s)\leqslant w(x,t)-c\varepsilon\delta w^{\pm}(\pm\tfrac{3}{4}e_n,0),$$

respectively, in $B_{1/8}(\pm\frac{3}{4}e_n)\times(-C,C)$. Now, as in Lemma 2, we obtain, for a small $\bar{h}$, in the same set,

$$v_{(1+\bar{h}\delta)\varepsilon}\leqslant w-c\varepsilon\delta w^{\pm}(\pm\tfrac{3}{4}e_n,0).$$

Hence w satisfies the hypotheses of the propagation lemma with $T=C$, and $h=\bar{h}\delta/\mu$ (taking into account the effect of the time-dilation).

We conclude, for a small c, that in $B_{1/2}\times(-\frac{1}{2}C,\frac{1}{2}C)$,

$$v_{(1+c\bar{h}\delta^2/\mu)\varepsilon}\leqslant w.$$

This amounts to say that w is monotone increasing along all the directions of the form

$$\bar{\tau}=\tau+\left(1+c\bar{h}\frac{\delta^2}{\mu}\right)\varepsilon\varrho, \tag{$**$}$$

in $B_{1/2}\times(-\frac{1}{2}C,\frac{1}{2}C)$.

The convex envelope of the old cone $\Gamma_x^{\frac{1}{2}}(e_n,\theta)$ and the set of directions $(**)$ contains a new cone (in space) $\Gamma_x(\nu_1,\theta_1)$ such that

$$\theta_1-\theta=\bar{c}\bar{h}\frac{\delta^2}{\mu},$$

which, after rescaling back in time, corresponds to (a) with $c_1=\bar{c}\bar{h}$.

To prove (b) notice first that the new axis ν_1 in space is shifted (with respect to e_n) of order δ in a spatial direction octogonal to e_n. Since $\delta\ll\mu^3$ we can use the results of Lemma 8 that, in the present situation, read

$$D_t w+\frac{\delta}{\mu}BD_n w\geqslant c\delta D_n w \quad \text{or} \quad -D_t w-A\frac{\delta}{\mu}D_n w\geqslant c\delta D_n w.$$

Suppose for simplicity that the left inequality holds and call $\bar{\varrho}, \bar{\varrho}_1$, respectively, the unit vectors in the direction $e_t+(\delta/\mu)Be_n$ and $e_t+(\delta/\mu)B\nu_1$. Then it is easy to see that, for $\lambda_1^2+\lambda_2^2=1$, if e^1 is any spatial direction orthogonal to e_n, we have

$$\lambda_1 D_{\bar{\varrho}_1}w+\lambda_2 D_{e^1}w\geqslant\left(c\delta-\tilde{c}\frac{\delta^2}{\mu}B-\tilde{c}_1\delta^2\right)D_n w\geqslant\widetilde{C}\delta D_{\nu_1}w$$

in $B_{1/8}(\pm\frac{3}{4})\times(-C,C)$, as long as $|\lambda_2|\leqslant 2\delta$ since $D_{\bar{\varrho}_1}w\geqslant cD_{\bar{\varrho}}w-\tilde{c}(\delta^2/\mu)BD_n w$. Let now ϱ^* denote the direction below $\bar{\varrho}_1$ (with respect to time) in the (e_t,ν_1)-plane, which makes an angle δ with $\bar{\varrho}_1$. For any small vector τ in the ϱ^* direction, set again $\varepsilon=|\tau|\sin\delta$, $w_1(x,t)=w((x,t)-\tau)$.

Then

$$v_\varepsilon(x,t)\leqslant w(x,t) \quad \text{in } B_{1-\varepsilon}\times\left(-\frac{\mu}{\delta}+\varepsilon, \frac{\mu}{\delta}-\varepsilon\right).$$

Proceeding exactly as before, we conclude that

$$v_{(1+\bar{c}\bar{h}\delta^2/\mu)\varepsilon}(x,t)\leqslant w(x,t) \quad \text{in } B_{1/8}\times\left(-\tfrac{1}{2}C, \tfrac{1}{2}C\right).$$

This implies that, in the same set, w is monotone increasing along the direction

$$e_t+\left(\frac{\delta}{\mu}B-\bar{c}\bar{h}\frac{\delta^2}{\mu}\right)\nu_1,$$

that is, rescaling back in time, u is monotone increasing along the direction

$$e_t+(B-\bar{c}\bar{h}\delta)\nu_1.$$

Therefore (b) holds with $B_1=B-\bar{c}h\delta$, $A_1=A$ and $B_1-A_1=B-A-\bar{c}\bar{h}\delta\leqslant\bar{c}_1(\mu-c_2\delta)$ so that the proof is complete. □

Proof of the main theorem. By the results in the previous section, for λ large enough, the function $u_\lambda=u(\lambda x,\lambda t)/\lambda$, that we call again u, falls under the hypotheses of Lemma 14. Now we proceed inductively, by applying Lemma 14 to

$$u_k(x,t)=2^k u(2^{-k}x, 2^{-k}t), \quad k\geqslant 1.$$

In this way we define a sequence of space cones $\Gamma_x(\nu_k, \theta_k)$ with axis ν_k and opening θ_k, and sequences $\{A_k\}, \{B_k\}, \{\delta_k\}, \{\mu_k\}$ with the following properties: in

$$B_{2^{-k}} \times \left(-\frac{C\delta_k}{2^k \mu_k}, \frac{C\delta_k}{2^k \mu_k} \right),$$

(a) u is monotone increasing in any (spatial) direction belonging to $\Gamma_x(\nu_k, \theta_k)$,

(b) u is monotone increasing along the directions $e_t + B_k \nu_k$ and $-e_t - A_k \nu_k$ where

$$0 < B_k - A_k \leqslant \bar{c}_1 \mu_k,$$

(c) the sequences $\{\delta_k\}$ and $\{\mu_k\}$ satisfy the recurrent relations

$$\begin{aligned} &\delta_{k+1} \leqslant \delta_k - c_1 \frac{\delta_k^2}{\mu_k} \quad \left(\delta_k = \tfrac{1}{2}\pi - \theta_k\right), \\ &\mu_{k+1} \leqslant \mu_k - c_2 \delta_k, \end{aligned}$$

as long as $\delta_k \ll \mu_k^3$.

From (c) we easily obtain the asymptotic behavior

$$\delta_k \sim \frac{c_1(\eta)}{k^{3/2-\eta}}, \quad \mu_k \sim \frac{c_2(\eta)}{k^{1/2-\eta}},$$

for any small $\eta > 0$. Then using (a) and (b) the assertion (1) of the main theorem follows immediately.

To prove (2), notice that for each time level $t_0 \in (-1, 1)$, $\Omega^\pm \cap \{t = t_0\}$ is a Liapunov–Dini domain. Since u_t is bounded, the results of K.-O. Widman [W] apply and therefore $\nabla_x u^\pm$ are continuous up to the free boundary at each level of time. Finally, using the free boundary condition the proof is easily completed. □

10. A counterexample

In this section, we present a counterexample in which the free boundary of the solution to a two-phase Stefan problem remains only Lipschitz for some interval of time, in spite of the two phases having non-zero temperature. "Waiting time" phenomena in the behavior of solutions to degenerate parabolic equations for which some regularizing effect takes a while to occur, have been extensively studied for the "gas flow in porous media" equation

$$u_t = \Delta u^m.$$

They reflect the fact that the free boundary (or degeneracy boundary) has some sort of hyperbolic behavior.

LEMMA 15. *In* $\mathbf{R}^3$, *let* $D_{\lambda,\mu}$ *denote the domain*

$$D_{\lambda,\mu}=\{x_3\geqslant r(\lambda\cos\lambda\theta-\mu)\}$$

(*with* r,θ *polar coordinates in the* (x_1,x_2)*-plane and* λ *integer*).

Let

$$h_{\lambda,\mu}=\varrho^\alpha f_{\lambda,\mu}(\sigma)$$

be the positive harmonic function in $D_{\lambda,\mu}$, *homogeneous of degree* $\alpha(\lambda,\mu)$ *vanishing on* $\partial D_{\lambda,\mu}$.

Then for $\lambda>\lambda_0$, $|\mu|<1$,

$$\alpha\geqslant 3$$

$(\varrho=(|x|,\sigma=|x|/|x|,x=(x_1,x_2,x_3))$.

Proof. Recall that h is constructed by separation of variables by computing $\Delta h=\varrho^{\alpha-2}[(\alpha(\alpha-1)+2\alpha)f+\Delta_\sigma f]$ with Δ_σ the Laplace–Beltrami operator on the unit sphere ∂B_1 of $\mathbf{R}^3$. Thus f is chosen to be the first eigenfunction of $-\Delta_\sigma$ for the (spherical) domain

$$D_{\lambda,\mu}\cap\partial B_1=:S_{\lambda,\mu},$$

and α must be chosen so that $\alpha(\alpha+1)$ equals the first eigenvalue of $-\Delta_\sigma$ on $S_{\lambda,\mu}$.

But, if we fix $|\mu|\leqslant 1$ and let λ go to infinity, the domain $D_{\lambda,\mu}$ becomes very narrow, and the first eigenvalue, i.e., $\alpha(\alpha+1)$, tends to infinity. (For instance, any disc of radius c/λ cuts the complement of $S_{\lambda,\mu}$ in a fixed proportion, and thus the Raleigh quotient

$$\frac{\int f^2}{\int|\nabla f|^2}\leqslant\frac{c}{\lambda}.)$$

This proves the lemma. □

LEMMA 16. *Let*

$$\begin{cases} w(x,t)=(1+Mt)h_{\lambda_0+1,t}(x) & \textit{on } D_{\lambda_0+1,t},\\ w(x,t)=0 & \textit{otherwise.}\end{cases}$$

Then, for ε *small and* M *properly chosen,* $w(x,t)$ *is a supersolution of the Stefan problem on* $B_\varepsilon(0)\times[0,\varepsilon]$.

Proof. We must show that the speed of the free boundary, V_ν, is larger than $-w_\nu$ (ν pointing towards the zero region), and that $Hw=\Delta w-D_tu\leqslant 0$. The function h is really well-defined up to a normalization, say $\|f\|_{L^2}=1$. Under this normalization $h_{\lambda,\mu}$ is smooth with respect to λ,μ since $D_{\lambda,\mu}$ changes smoothly (in fact, analytically) in λ,μ.

Since $D_{\lambda_0+1,t}$ is increasing in t, by choosing M large we can make $(1+Mt)h_{\lambda_0+1,t}$ monotone in t.

Therefore
$$Hw = -D_t(1+Mt)h_{\lambda_0+1,t} < 0.$$

Now, on the free boundary,
$$-w_\nu = (1+Mt)\varrho^{\alpha(\lambda_0+1,t)} f_{\nu^*}(\sigma)$$

where $f_{\nu^*}(\sigma)$ denotes the (smooth) normal derivative of f along $\partial S_{\lambda_0+1,t}$. Thus, since for t small, $\alpha(\lambda_0+1,t)\geqslant 3$,
$$-w_\nu(x,t) \leqslant C\varrho^3 = C|x|^3.$$

But, the speed of the free boundary $D_{\lambda_0+1,t}$ is the ratio between the time and space components of the gradient of
$$g(x,t) = x_3 - r[(\lambda_0+1)\cos(\lambda_0+1)\theta - t].$$

Since g is homogeneous of degree 1,
$$V_\nu = \frac{r}{|\nabla_x g|} \geqslant Cr$$

and therefore
$$V_\nu \geqslant -w_\nu$$

for ϱ, t small. □

Remark. The same construction can be made in $D^c_{\lambda,0}$, the complement of $D_{\lambda,0}$, since $D^c_{\lambda,0}$ is obtained by a rigid motion, R, from $D_{\lambda,0}$. We call w^+ the function constructed in Lemma 16 and w^- the subsolution obtained by changing $D_{\lambda,0}$ by $D^c_{\lambda_0}$ and setting $w^-(x) = -w^+(Rx)$.

As a consequence we have

THEOREM 2 (counter example). *Let u be the solution of the two-phase Stefan problem in $B_\varepsilon \times [0,\varepsilon]$, with initial and lateral data u satisfying*
$$w^- \leqslant u \leqslant w^+.$$

Then, the free boundary, F, for u is contained in
$$\{x_n \geqslant r[(\lambda_0+1)\cos(\lambda_0+1)\theta - t]\} \cap \{x_n \leqslant r[(\lambda_0+1)\cos(\lambda_0+1)\theta + t]\}.$$

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