LINEAR DIFFERENCE EQUATIONS WITH ARBITRARY REAL SPANS.

BY

W. T. MARTIN[1]

in CAMBRIDGE, MASS.

I. Introduction.

The object of many investigations has been the study of difference equations of the forms

$$(1.1) \qquad \sum_{s=0}^{m} c_s(x) F(x+s) = G(x)$$

and

$$(1.2) \qquad F_j(x+1) = \sum_{k=1}^{p} c_{jk}(x) F_k(x) + G_j(x), \qquad j = 1, \ldots, p,$$

in which the coefficients are functions of the real variable x asymptotically constant,

$$(1.3) \qquad \lim_{x=+\infty} c_s(x) = c_s, \qquad s = 0, 1, \ldots, m,$$

and

$$(1.4) \qquad \lim_{x=+\infty} c_{jk}(x) = c_{jk}, \qquad j, k = 1, \ldots, p.$$

Bochner[2] has considered the more general equations

$$(1.5) \qquad \sum_{s=0}^{m} c_s(x) F(x+\delta_s) = G(x)$$

---

[1] National Research Fellow.

[2] S. BOCHNER, *Math. Zeitschrift* 33 (1931), pp. 426—450. Hereafter we shall refer to this paper as »I». BOCHNER gives other references to the literature.

and

$$F_j(x + \omega_j) = \sum_{k=1}^{p} c_{jk}(x) F_k(x) + G_j(x), \qquad j = 1, \ldots, p, \tag{1. 6}$$

in which the spans δ_s and ω_j are any positive numbers without any arithmetical restrictions. Bochner has developed the theory of the equation (1. 5) and has stated the problem for the system (1. 6) and for still more general systems.

In the investigation of the equation (1. 1) with the special spans one considers the *periodic* function

$$e(t) = c_0 + c_1 e^t + \cdots + c_m e^{mt} \tag{1. 7}$$

and uses the Dirichlet developments of $1/e(t)$ in strips parallel to the imaginary axis in which $e(t)$ is non-vanishing. The theory for this case is simplified by the introduction of a new variable $\tau = e^t$ and by the consideration of the polynomial

$$\varepsilon(\tau) = c_0 + c_1 \tau + \cdots + c_m \tau^m. \tag{1. 8}$$

The Laurent developments of $1/\varepsilon(\tau)$ in annular rings replace the Dirichlet developments of $1/e(t)$ in strips. In the investigation of the equation (1. 5) with the general spans no such simplification is possible. In this case Bochner has studied the *almost periodic* function

$$E(t) = c_0 e^{\delta_0 t} + c_1 e^{\delta_1 t} + \cdots + c_m e^{\delta_m t} \tag{1. 9}$$

and its reciprocal $1/E(t)$ in strips parallel to the imaginary axis in which $E(t)$ is non-vanishing. By use of the almost periodicity of $1/E(t)$ in these strips and by use of its generalized Dirichlet series, Bochner has developed the theory for the equation (1. 5).

In the study of the equation (1. 1) certain of the solutions are characterized by the behavior of the limit value

$$\lim_{x=+\infty} \frac{F(x + 1)}{F(x)} \tag{1. 10}$$

or, if this is not existent, of

$$\limsup_{x=+\infty} |F(x)|^{1/x} = \limsup_{x=+\infty} e^{x^{-1} \log |F(x)|}. \tag{1. 11}$$

Bochner has characterized certain of the solutions of (1. 5) by means of

$$\liminf_{x=-\infty} x^{-1} \log |F(x)|, \quad \limsup_{x=+\infty} x^{-1} \log |F(x)|, \tag{1.12}$$

and certain others by the existence of the limit value

$$\lim_{x=+\infty} F(x). \tag{1.13}$$

In this paper we shall study difference equations of the form

$$\sum_{s=0}^{m} c_s(z) F(z + \delta_s) = G(z) \tag{1.14}$$

and systems of difference equations of the form

$$\sum_{k=1}^{p} \left\{ \sum_{s=0}^{m_{jk}} c_s^{jk}(z) F_k(z + \delta_s^{jk}) \right\} = G_j(z), \qquad j = 1, \ldots, p, \tag{1.15}$$

in which the spans δ_s and δ_s^{jk} are any real non-negative numbers without any arithmetical restrictions and the coefficients $c_s(z)$ and $c_s^{jk}(z)$ are analytic functions of the complex variable z asymptotically constant in certain sectors. The methods developed by Bochner are applicable to the study of the equation (1.14) and are used in this paper. In part II we study the equation (1.14) with constant coefficients c_s. Using the method of successive approximations and applying the results of part II, we develop in part III the theory of the equation (1.14) with asymptotically constant coefficients. In part IV we study the system (1.15) with constant coefficients c_s^{jk}. Subjecting the system to certain negative restrictions, we use the symbolic treatment of systems of equations developed by Bochner[1] and Carmichael.[2] By use of this symbolic treatment we reduce the problem of studying the system (1.15) with constant coefficients to the problem of studying p equations of the form (1.14) with constant coefficients. The methods which enable us to pass from the theory of the single equation with constant coefficients to the single equation with asymptotically constant coefficients enable us to pass from the theory of the system with constant coefficients to the system with asymptotically constant coefficients. Using these methods, we treat in part V the system (1.15) with asymptotically constant coefficients.

---

[1] S. Bochner, *Math. Annalen* 104 (1933), pp. 579—587. Hereafter we shall refer to this paper as »II».

[2] R. D. Carmichael, *Trans. Am. Math. Soc.* 35 (1933), pp. 1—28.

II. The Equation with Constant Coefficients.

2. 1. In this part we consider the equation

$$\sum_{s=0}^{m} c_s F(z + \delta_s) = G(z), \tag{2. 1}$$

in which the coefficients c_s are any complex constants and the spans δ_s are any non-negative numbers ordered as follows:

$$0 \leqq \delta_0 < \delta_1 < \delta_2 < \cdots < \delta_m.$$

It is convenient to assume that $m \geqq 1$ and $c_0 \neq 0$, $c_m \neq 0$. Under these hypotheses the exponential polynomial

$$E(t) = \sum_{s=0}^{m} c_s e^{\delta_s t} \tag{2. 2}$$

for complex values of t, $t = u + iv$, has infinitely many zeros.[1] We shall state here various facts relating to the function $E(t)$ and its reciprocal $1/E(t)$. These facts are all given by Bochner (I, pp. 434—435) and are restated here for future reference.

1°. The zeros of $E(t)$ all lie in a bounded strip $u_0 < u < u_1$. If we consider on the u-axis the set of the real parts of all the zeros of $E(t)$ and annex its limit points then we obtain a set m whose complementary set on the u-axis is composed of at most denumerably many open intervals, among them being two half-lines. Following Bochner's terminology, we denote the left half-line by J_0, the right half-line by J_1, the remaining intervals in any order by $J_2, J_3, \ldots$ The boundaries of the interval J_σ we will denote by $\underline{b}_\sigma$ and $\bar{b}_\sigma$ (therefore $\underline{b}_0 = -\infty$, $\bar{b}_1 = +\infty$), and by J_σ we will understand not only the »interval»

$$\underline{b}_\sigma < u < \bar{b}_\sigma, \tag{2. 3}$$

but also the »strip» (2. 3), that is, the totality of all complex numbers t, whose real part u verifies the relation (2. 3).

---

[1] The fact that $E(t)$ has infinitely many zeros is easily proved by use of the theory developed by A. PRINGSHEIM, *Math. Annalen* 58 (1904), pp. 257—342. The zeros of such functions have been studied by various writers. Cf. S. BOCHNER, I, p. 434, and R. E. LANGER, *Bull. Amer. Math. Soc.* 37 (1931), pp. 213—219. Other references are given in these papers.

2°. We consider the numbers representable in the form

$$n_0 \delta_0 + n_1 \delta_1 + \cdots + n_m \delta_m \tag{2. 4}$$

with integral coefficients n_s. We order them in a fixed sequence and denote them by

$$\lambda_0, \lambda_1, \lambda_2, \ldots; \quad \lambda_0 = 0. \tag{2. 5}$$

3°. The function $1/E(t)$ is an analytic almost periodic function in the strip J_σ and as such possesses a generalized Dirichlet expansion of the form

$$\frac{1}{E(t)} = \sum_{n=0}^{\infty} \gamma_{\lambda_n}^{\sigma} e^{\lambda_n t} \qquad \underline{b}_\sigma < u < \overline{b}_\sigma, \tag{2. 6}$$

where the exponents λ_n are the numbers in (2. 5) and the $\gamma_{\lambda_n}^{\sigma}$ are uniquely determined constants. The series (2. 6) converges absolutely in J_σ.

4°. From the identity

$$1 = E(t) \sum_{n=0}^{\infty} \gamma_{\lambda_n}^{\sigma} e^{\lambda_n t}$$

follow the useful relations

$$\sum_{s=0}^{m} c_s \gamma_{\lambda_n - \delta_s}^{\sigma} = \begin{cases} 1 & \text{for } n = 0, \\ 0 & \text{for } n \neq 0. \end{cases} \tag{2. 7}$$

5°. For the cases $\sigma = 0, 1$ the series (2. 6) are »one-sided», that is,

$$\gamma_{\lambda_n}^{0} = 0 \quad \text{for} \quad \lambda_n \leqq -\delta_0, \tag{2. 8}$$

$$\gamma_{\lambda_n}^{1} = 0 \quad \text{for} \quad \lambda_n \geqq -\delta_m. \tag{2. 9}$$

2. 2. Let $G(z)$ be an analytic function of the complex variable z, $z = x + iy = \varrho e^{i\theta}$. If $G(z)$ is regular in a strip $-\infty < x < +\infty$, $\alpha \leqq y \leqq \beta$, then we shall briefly say that $G(z)$ is regular in the strip (α, β). If $G(z)$ is regular in a strip (α, β) and such that real numbers $\underline{b}$, $\overline{b}$, A exist of such a sort that

$$|G(z)| \leqq A e^{\underline{b} x}, \qquad x \leqq 0, \qquad \alpha \leqq y \leqq \beta,$$

and

$$|G(z)| \leqq A e^{\overline{b} x}, \qquad x \geqq 0, \qquad \alpha \leqq y \leqq \beta,$$

then we shall simply write

$$|G(z)| \leqq A (e^{\underline{b} x}, e^{\overline{b} x}), \qquad \alpha \leqq y \leqq \beta. \tag{2. 10}$$

We note that the relation (2. 10) implies that

$$\underline{b} \leqq \liminf_{x=-\infty} x^{-1} \log |G(x+iy_0)|, \quad \limsup_{b=+\infty} x^{-1} \log |G(x+iy_0)| \leqq \bar{b},$$

for every y_0 in the interval $\alpha \leqq y_0 \leqq \beta$.

Lemma 2. 1. *Let us consider an interval J_σ and an analytic function $G(z)$ regular in a strip (α, β). If $G(z)$ is such that* (2. 10) *holds where $\underline{b}_\sigma < \underline{b} \leqq \bar{b} < \bar{b}_\sigma$, then the series*

$$\Gamma^\sigma G = \sum_{n=0}^{\infty} \gamma^\sigma_{\lambda_n} G(z+\lambda_n) \tag{2. 11}$$

converges absolutely and uniformly in every finite region contained within the strip (α, β) and represents a solution $F(z)$ of (2. 1) *which is analytic in the strip (α, β) and which is such that*

$$|F(z)| \leqq A'(e^{\underline{b}x}, e^{\bar{b}x}), \qquad \alpha \leqq y \leqq \beta, \tag{2. 12}$$

where $A' = 3AC$ and C is a constant independent of $G(z)$.

The equation (2. 1) *has only one analytic solution $F(z)$ which verifies a relation of the form* (2. 12); *in fact, it has only one analytic solution for which the relations*

$$\underline{b}_\sigma < \liminf_{x=-\infty} x^{-1} \log |F(x+iy_0)|, \quad \limsup_{x=+\infty} x^{-1} \log |F(x+iy_0)| < \bar{b}_\sigma, \tag{2. 13}$$

hold for a fixed value of y_0, $\alpha \leqq y_0 \leqq \beta$.[1]

For the absolute convergence of (2. 11) we use the relation (2. 10) and the absolute convergence of the series (2. 6). Let us denote by C the sum[2]

$$C = \sum_{\lambda_n < 0} |\gamma_{\lambda_n}| e^{\lambda_n \underline{b}} + \sum_{\lambda_n \geqq 0} |\gamma_{\lambda_n}| e^{\lambda_n \bar{b}}.$$

First let us assume that $x > 0$. Let us write (for fixed x)

$$\sum_{n=0}^{\infty} |\gamma_{\lambda_n} G(z+\lambda_n)| \equiv \sum_{\lambda_n \geqq 0} + \sum_{-x \leqq \lambda_n < 0} + \sum_{\lambda_n < -x} \equiv \sum\nolimits_1 + \sum\nolimits_2 + \sum\nolimits_3.$$

Then

[1] This lemma is analogous to and was suggested by Lemmas 1 and 2 of S. Bochner, I, pp. 433—435. Its proof is similar to the proofs of those lemnas.

[2] When there is no chance for ambiguity we shall simply write γ_{λ_n} for $\gamma^\sigma_{\lambda_n}$.

$$\sum_1 \leqq A \sum_{\lambda_n \geqq 0} |\gamma_{\lambda_n}| e^{\bar{b}(x+\lambda_n)} \leqq A\, C e^{\bar{b}x},$$

$$\sum_2 \leqq A \sum_{-x \leqq \lambda_n < 0} |\gamma_{\lambda_n}| e^{\bar{b}(x+\lambda_n)} \leqq A\, e^{\bar{b}x} \sum_{\lambda_n < 0} |\gamma_{\lambda_n}| e^{\bar{b}\lambda_n} \leqq A\, C e^{\bar{b}x},$$

$$\sum_3 \leqq A \sum_{\lambda_n < -x} |\gamma_{\lambda_n}| e^{\underline{b}(x+\lambda_n)} \leqq A\, C e^{\underline{b}x} \leqq A\, C e^{\bar{b}x}.$$

For the case $x < 0$ the proof is analogous and for $x = 0$ it is even simpler. We omit these parts. The series in (2. 11) then converges absolutely for all z in (α, β). If z is restricted to any finite region contained within the strip (α, β) then the uniform convergence of (2. 11) follows at once from the bounds just derived. Consequently the sum function $F(z)$ is analytic in the strip (α, β) and it verifies (2. 12).

That the function $F(z)$ defined by (2. 11) is a solution of (2. 1) follows at once from (2. 7) and the absolute convergence of the series (2. 11). Writing

$$F(z + \delta_s) = \sum_{n=0}^{\infty} \gamma_{\lambda_n} G(z + \lambda_n + \delta_s), \qquad s = 0, 1, \ldots, m,$$

substituting these series into (2.1), combining equal arguments of $G(\xi)$ and using the relations (2. 7), we obtain

$$\sum_{s=0}^{m} c_s F(z + \delta_s) = \sum_{n=0}^{\infty} G(z + \lambda_n) \sum_{s=0}^{m} c_s \gamma_{\lambda_n - \delta_s} = G(z).$$

For the uniqueness of the solution let us assume that $F_*(z)$ is an analytic function which verifies the relations in (2. 13). Then it verifies a relation of the form

$$(2.\ 14) \qquad |F_*(x + i y_0)| \leqq A(y_0)(e^{\underline{b}' x}, e^{\bar{b}' x}), \qquad \underline{b}_\sigma < \underline{b}' \leqq \bar{b}' < \bar{b}_\sigma.$$

If $F_*(z)$ is a solution of (2. 1) we may write

$$F(x + i y_0) = \sum_{n=0}^{\infty} \gamma_{\lambda_n} G(x + i y_0 + \lambda_n) = \sum_{n=0}^{\infty} \gamma_{\lambda_n} \sum_{s=0}^{m} c_s F_*(x + i y_0 + \delta_s + \lambda_n).$$

In view of (2. 14) the double series is absolutely convergent and may be rearranged

$$F(x+iy_0)=\sum_{n=0}^{\infty}F_*(x+iy_0+\lambda_n)\sum_{s=0}^{m}c_s\gamma_{\lambda_n-\delta_s}$$

and from (2.7) we see that $F(x+iy_0)=F_*(x+iy_0)$. Hence $F(z)\equiv F_*(z)$.

2.3. The following lemma gives a class of functions verifying bounds of the form (2.10):

Lemma 2.2. *Let $G(z)$ be an analytic function in a sector $S(-\delta<\theta<\pi+\delta$ $(0<\delta))$ and such that*

$$\limsup_{\varrho=\infty}\varrho^{-1-\eta}\log|G(\varrho e^{i\theta})|=0 \tag{2.15}$$

uniformly in $\theta(-\delta<\theta<\pi+\delta)$, for every positive value of η but not for any negative value. Define

$$h(\theta|G)=\limsup_{\varrho=\infty}\varrho^{-1}\log|G(\varrho e^{i\theta})|. \tag{2.16}$$

If $h(0|G)$ and $h(\pi|G)$ are finite and if for some value θ' $(0<\theta'<\pi)$ a constant h' exists such that $h(\theta'|G)\leqq h'$, then for every pair of positive numbers ε and β there exists a constant $A=A(\varepsilon,\beta)$ such that

$$|G(z)|\leqq A(\overline{e^{-h(\pi|G)-\varepsilon x},\ e^{h(0|G)+\varepsilon x}}),\qquad 0\leqq y\leqq\beta. \tag{2.17}$$

Let $H(\theta)$ be the function of the form $a\cos\theta+b\sin\theta$ which takes the values $h(0|G)$, h' at 0, θ' respectively. From the theory developed by Phragmén and Lindelöf[1] it follows that for every positive number ζ a constant $R=R(\zeta)$ (independent of θ) exists such that

$$|G(\varrho e^{i\theta})|<e^{\overline{H(\theta)+\zeta}\varrho},\quad \varrho>R,\qquad 0\leqq\theta\leqq\theta'.$$

Using the continuity of $H(\theta)$ and the fact that $H(0)=h(0|G)$, we obtain the part of the bound in (2.17) which holds for $x\geqq 0$. By means of a similar argument and by use of a function $H_1(\theta)=a_1\cos\theta+b_1\sin\theta$ which is such that $H_1(\theta')=h'$, $H_1(\pi)=h(\pi|G)$, we obtain the part of the relation (2.17) which holds for $x<0$. We omit the details here.

By an analogous argument we prove the following lemma:

[1] E. Phragmén and E. Lindelöf, *Acta math.* 31 (1908), pp. 381—406. See, in particular, pp. 391—395.

Lemma 2. 3. *Let $G(z)$ be an analytic function regular in $S_0\left(-\delta \leqq \theta \leqq \delta\left(0 < \delta \leqq \frac{\pi}{2}\right), |z| \geqq R\right)$ and such that* (2. 15) *holds uniformly in $\theta\,(-\delta \leqq \theta \leqq \delta)$, for every positive value of η, but not for any negative value. If $h(\theta|G)$ defined in* (2. 16) *is finite in the interval $-\delta_1 < \theta < \delta_1$, where $0 < \delta_1 \leqq \delta$, then for every positive number ε and for every pair of real numbers α, β $(\alpha < \beta)$ a constant $A = A(\varepsilon; \alpha, \beta)$ exists such that*

$$|G(z)| \leqq A\, e^{\overline{h(0|G)+\varepsilon x}}, \quad z \text{ in } S_0, \qquad \alpha \leqq y \leqq \beta.$$

The following theorem is an immediate consequence of Lemmas 2. 1 and 2. 2.

Theorem 2. 1. *Let us consider an interval J_σ and a function $G(z)$ satisfying the hypotheses of Lemma 2. 2. If in addition*

$$\underline{b}_\sigma < -h(\pi|G) \leqq h(0|G) < \bar{b}_\sigma, \tag{2. 18}$$

then the function $F(z) = \Gamma^\sigma G$ defined in (2. 11) *is a solution of* (2. 1) *which is analytic in the upper half-plane $0 \leqq y < \infty$ and which is such that for every pair of positive numbers ε, β there exists a constant $A = A'(\varepsilon, \beta)$ of such a sort that*

$$|F(z)| \leqq A'\,(e^{\overline{-h(\pi|G)-\varepsilon x}},\ e^{\overline{h(0|G)+\varepsilon x}}), \qquad 0 \leqq y \leqq \beta. \tag{2. 19}$$

The equation (2. 1) *has one analytic solution which verifies a relation of the form* (2. 19); *in fact, it has only one analytic solution for which*

$$\underline{b}_\sigma < -h(\pi|F), \qquad h(0|F) < \bar{b}_\sigma.\text{[1]} \tag{2. 20}$$

By means of Lemmas 2. 1 and 2. 3 we prove at once the following theorem[2]:

Theorem 2. 2. *Let us consider a function $G(z)$ satisfying the hypotheses of Lemma 2. 3. If in addition $h(\theta|G)$ is such that*

$$h(0|G) < \bar{b}_0, \tag{2. 21}$$

then the function $F(z)$ defined by the series

---

[1] We note that (2. 19) and (2. 18) imply (2. 20).

[2] It is to be observed that the interval J_0 is used in Theorem 2. 2. The theorem treats functions $G(z)$ analytic in $S_0\left(-\delta \leq \theta \leqq \delta\left(0 < \delta \leqq \frac{\pi}{2}\right), |z| > R\right)$, this being possible since $\gamma^0_{\lambda_n} = 0$ for $\lambda_n \geq -\delta_0$ (see (2. 8)). For the case when σ is zero, it is clear that Lemma 2. 1 may be appropriately modified to treat functions $G(z)$ analytic in S_0.

$$(2.\ 22) \qquad \Gamma^0 G = \sum_{n=0}^{\infty} \gamma_{\lambda_n}^0 G(z + \lambda_n)$$

is a solution of (2. 1) *which is analytic for all z such that $z - \delta_0$ is in S_0. For every positive number ε and for every pair of real numbers α, β ($\alpha < \beta$) there exists a constant $A' = A'(\varepsilon;\ \alpha, \beta)$ of such a sort that*

$$(2.\ 23) \qquad |F(z)| \leqq A' e^{\overline{h(0|G)+\varepsilon x}},\ \alpha \leqq y \leqq \beta,\ z - \delta_0 \ in\ S_0.$$

The equation (2. 1) *has only one analytic solution which satisfies a relation of the form* (2. 23); *in fact, it has only one analytic solution for which $h(0|F) < \bar{b}_0$.*

A similar theorem holds for functions possessing in S_1 ($\pi - \delta \leqq \theta \leqq \pi + \delta$, $|z| > R$) the character the function $G(z)$ in Theorem 2. 2 possesses in S_0. The condition $\underline{b}_1 < -h(\pi|G)$ replaces the condition (2. 21). For this case the series $\Gamma^1 G$ is used. We omit the statement of this theorem.

We next treat the case where $G(z)$ is a function of finite *exponential type* q, that is, where $G(z)$ is such that

$$\limsup_{n=\infty} |G^{(n)}(0)|^{1/n} = q$$

holds.[1] Let $G(z)$ be any such function and define $h(\theta|G)$ as in (2. 16). For later use we state here the following results given by Pólya (*loc. cit.*, pp. 571—585):

---

[1] Cf. S. PINCHERLE, *Acta math.* 38 (1926), pp. 279—304 (first published in 1888); G PÓLYA, *Math. Zeitschrift* 29 (1929), pp. 549—640. Pólya shows that there is an intimate relationship between a function $G(z)$ of exponential type and its *Borel transform* $g(t) = \Sigma_0^\infty G^n(0) t^{-n-1}$. In Theorem 2. 3 and Corollary 2. 1 we state certain relations which exist between the Borel transform $g(t)$ of the given function $G(z)$ in (2. 1) and the Borel transform $f(t)$ of a particular solution $F(z)$ of (2. 1). We state these relations in the present paper merely for the sake of completeness, relations of this sort not being peculiar to equations of the form (2. 1) but holding for (being implicitly contained in the theory of) more general equations such as difference equations with arbitrary complex spans and differential equations of infinite order.

If the function $G(z)$ in equation (2. 1) is of exponential type and if $F(z)$ is a solution of (2. 1) which is of exponential type in a sector $S_0 \left(-\zeta \leqq \arg z \leqq \zeta,\ 0 < \zeta \leqq \frac{\pi}{2}\right)$ then the function $F(z)$ is of exponential type in the entire plane. BOCHNER suggested the truth and the manner of proof of this statement to me. For the proof of the statement it is sufficient (and also necessary) to show that if there exist positive constants p and M such that

$$|G(z)| < M e^{p|z|},\ \text{all (finite)}\ z; \qquad |F(z)| < M e^{p|z|},\ z \text{ in } S_0;$$

then there exist constants p' and M' such that

$$(*) \qquad |F(z)| < M' e^{p'|z|},\ \text{all}\ z.$$

(a) $h(\theta|G)$ is defined and continous for all real θ;

(b) $h\left(\theta + \frac{\pi}{2}\middle| G\right) + h\left(\theta - \frac{\pi}{2}\middle| G\right) > 0$, and hence $h(0|G) > -h(\pi|G)$;

(c) the *Borel transform* $g(t)$ of $G(z)$,

$$g(t) = \sum_{n=0}^{\infty} G^{(n)}(0)\, t^{-n-1} \tag{2. 24}$$

is analytic for all t exterior to the convex region K, called the *conjugate diagram* of $G(z)$, where K is the point set defined in the following manner: the point $t = u + iv$ belongs to K if and only if for all real values of θ

$$u \cos\theta - v \sin\theta - h(\theta|G) \leqq 0; \tag{2. 25}$$

(d) if we denote by C_ε the boundary of the convex region K_ε, where K_ε is the point set

$$u \cos\theta - v\sin\theta - h(\theta|G) - \varepsilon \leqq 0, \qquad \varepsilon > 0, \tag{2. 26}$$

We note that this includes the case when $G(z)$ is identically zero. Without loss of generality we take $c_0 = 1$ and $\delta_0 = 0$. From the relation

$$F(z) = G(z) - \sum_{s=1}^{m} c_s F(z + \delta_s) \tag{**}$$

we easily verify that $F(z)$ is an integral function. In order to see that $F(z)$ is of exponential type we consider a sequence of sectors S_n $(-\zeta \leqq \arg(z + n\delta_1) \leqq \zeta)$, $n = 0, 1, 2, \ldots$ Let us assume that for every point z_n in a given sector S_n we have

$$|F(z_n)| < M(1 + P + \cdots + P^n)\, e^{p|z_n|},\ z_n \text{ in } S_n,\ \left(P = \sum_{s=1}^{m} |c_s| e^{\delta_s p}\right). \tag{***}$$

If z_{n+1} is any point in S_{n+1} then each of the points $z_{n+1} + \delta_s$, $s = 1, \ldots, m$, is in S_n and applying the relations (***) and (**), we see that (***) holds with n replaced throughout by $n+1$. The relation (***) holds when n is zero and hence by induction, we see that it holds for $n = 0, 1, 2, \ldots$ If $P < 1$, then the facts that the relations (***) hold for every n and that every (finite) point z is in some sector S_n imply that the relation (*) already holds with $M' = M/(1 - P)$ and $p' = p$. If z_{n+1} is a point in S_{n+1} and not in S_n, than we see at once that $|z_{n+1}| > n\delta_1 \sin\zeta$. If $P = 1$, then

$$1 + P + \cdots + P_{n+1} = n + 2 < e \cdot e^n < e \cdot e^{|z_{n+1}|/(\delta_1 \sin\zeta)},$$

and hence (*) holds with $M' = Me$ and $p' = p + 1/(\delta_1 \sin\zeta)$. Finally, if $P > 1$, then

$$1 + P + \cdots + P^{n+1} = \frac{P^{n+2} - 1}{P - 1} < \frac{P^2}{P - 1} e^{n \log P} < \frac{P^2}{P - 1} e^{|z_{n+1}|(\log P)/(\delta_1 \sin\zeta)},$$

and hence the relation (*) holds with $M' = M P^2/(P - 1)$ and $p' = p + (\log P)/(\delta_1 \sin\zeta)$.

then

$$G(z) = \frac{1}{2\pi i}\int_{C_\varepsilon} e^{zt} g(t)\, dt; \tag{2. 27}$$

(e) $-q \leqq h(\theta | G) \leqq q$, the maximum q being assumed for at least one value of $\theta\,(0 \leqq \theta < 2\pi)$.

We next state and prove a theorem relating to the equation (2. 1) when $G(z)$ is a function of finite exponential type.

Theorem 2. 3. *Let us consider an interval J_σ and a function $G(z)$ of exponential type q. Let us assume that the relations* (2. 18) *hold. Then the function $F(z)$ defined by* (2. 11) *is an analytic solution of* (2. 1) *for which the relations* (2. 20) *hold and is the only analytic solution of* (2. 1) *for which the relations* (2. 20) *do hold.*

The function $F(z)$ is of exponential type q, and indeed

$$h(\theta | F) \equiv h(\theta | G). \tag{2. 28}$$

If $f(t)$ and $g(t)$ denote the Borel transforms of $F(z)$ and $G(z)$ respectively, then not only do $f(t)$ and $g(t)$, together with their analytic continuations, have the same singular points but also they possess the same Riemann surface.

The facts that the function $F(z)$ defined by (2. 11) is an analytic solution of (2. 1) of the character described in (2. 20) and that it is unique in these respects follow at once from Lemma 2. 1. (It is easily verified that $G(z)$ satisfies the hypotheses of Lemma 2. 1 in every strip (α, β).) For the remainder of the theorem we note that in view of (2. 18) the function $E(t)$ has no zero within or on the boundary of the conjugate diagram K of $G(z)$. Let ε be a positive constant such that

$$\underline{b}_\sigma < -h(\pi | G) - \varepsilon, \quad h(0 | G) + \varepsilon < \bar{b}_\sigma, \tag{2. 29}$$

and let C_ε be the contour defined in (d) for this ε. Then the function

$$F_*(z) = \frac{1}{2\pi i}\int_{C_\varepsilon} \frac{e^{zt}}{E(t)} g(t)\, dt, \tag{2. 30}$$

easily seen to be an (analytic) solution of (2. 1), verifies an inequality of the form

$$|F_*(\varrho e^{i\theta})| < M_\varepsilon \left| e^{(u+iv)\varrho e^{i\theta}} \right| = M_\varepsilon\, e^{(u\cos\theta - v\sin\theta)\varrho}$$

where $t = u + iv$ is on C_ε. From this inequality and the inequality (2. 26) we see that

$$(2.\ 31) \qquad |F_*(\varrho e^{i\theta})| < M_\varepsilon e^{\overline{h(\theta|G)+\varepsilon\varrho}},$$

and from this we see that $F_*(z)$ verifies relations of the form (2. 20). In view of the part of the theorem already proved this implies that $F_*(z) \equiv F(z)$. The function $F(z)$ is obviously of exponential type not exceeding q and from (2. 31) it follows that $h(\theta|F) \leqq h(\theta|G)$.[1] One can easily show by direct means that $F(z)$ is of exponential type precisely q and that $h(\theta|F) \equiv h(\theta|G)$, but since the type q and the function $h(\theta)$ of a function of exponential type depend only upon the singularities of the Borel transform these facts are consequences of the properties of $f(t)$ and $g(t)$ next to be proved. Writing $F(z)$ in the form

$$F(z) = \frac{1}{2\pi i}\int_{C_\varepsilon} e^{zt} f(t)\,dt$$

and using the fact that $F(z)$ is a solution of (2. 1), we see that

$$(2.\ 32) \qquad \sum_{s=0}^{m} c_s F(z+\delta_s) - G(z) = \frac{1}{2\pi i}\int_{C_\varepsilon} e^{zt}[E(t)f(t) - g(t)]\,dt \equiv 0.$$

If we write

$$(2.\ 33) \qquad E(t)f(t) - g(t) = I(t),$$

then we see that (2. 32) implies that $I(t)$ is analytic for all t within and on C_ε. But $f(t)$, $g(t)$ and $E(t)$ are analytic outside C_ε and consequently $I(t)$ is an integral function. Accordingly, the functions $f(t)$ and $g(t)$ have the same singular points except possibly at the points within or on C_ε at which $E(t)$ vanishes. But in view of (2. 29), $E(t)$ does not vanish within or on C_ε and hence $f(t)$ and $g(t)$ possess the same singular points. The relation (2. 33) implies that these singular points are of the same nature and that $f(t)$ and $g(t)$ possess the same Riemann surface. Pólya, (*loc. cit.*), has shown that the conjugate diagram of a function of exponential type is completely determined by the singularities of the Borel transform of the function and, in turn, that the function $h(\theta)$ is uniquely determined by the conjugate diagram. The type q of a function of exponential type is obviously determined by the function $h(\theta)$. From these facts and the

[1] This inequality also implies that $F(z)$ is of exponential type not exceeding q.

fact that $f(t)$ and $g(t)$ possess the same singular points we see that (2. 28) holds, that $F(z)$ is of exponential type q, and that the functions $F(z)$ and $G(z)$ possess the same conjugate diagram. This completes the proof of Theorem 2. 3.

We have the following corollary to Theorem 2. 3:

Corollary 2. 1. *If $E(t)$ has no zero for $|t| \leqq q$, then the function $F(z)$ defined by* (2. 11) *is the only solution of exponential type not exceeding q. If $E(t)$ has zeros of multiplicities s_μ at the points a_μ, $\mu = 1, \ldots, k$, $|a_\mu| \leqq q$, then every solution of* (2. 1) *of exponential type not exceeding q is expressible in the form*

$$F_1(z) = F(z) + \sum_{\mu=1}^{k} \sum_{\nu=0}^{s_\mu - 1} d_{\mu\nu} z^\nu e^{a_\mu z},$$

where $F(z)$ is the function defined by (2. 11) *and the $d_{\mu\nu}$ are arbitrary constants. The function $F(z)$ is the only solution of* (2. 1) *whose conjugate diagram is the same as that of $G(z)$. If $F_1(z)$ is any solution, other than $F(z)$, of* (2. 1) *of exponential type not exceeding q, then the conjugate diagram of this solution $F_1(z)$ is larger than the conjugate diagram of the function $G(z)$, every point of the conjugate diagram of $G(z)$ being contained within or on the boundary of the conjugate diagram of $F_1(z)$. The function $f_1(t)$ has a singular point at every point at which $g(t)$ has one and moreover, it has at least one additional singular point.*

The first two statements in the above corollary are classic results in the theory of difference equations. They are easily proved by means of contour integrals. We omit the proofs here. For the remaining parts of the corollary we note that the Borel transform $f_1(t)$ of $F_1(z)$ is expressible in the form

$$f_1(t) = f(t) + \sum_{\mu=1}^{k} \sum_{\nu=0}^{s_\mu - 1} d_{\mu\nu} \frac{\nu!}{(t - a_\mu)^{\nu+1}}.$$

The function $f(t)$ is analytic outside the conjugate diagram of $G(z)$ and all the points a_μ lie outside this conjugate diagram. The conjugate diagram of $F_1(z)$ must contain within or on its boundary all the singular points of $f_1(t)$. Since $f(t)$ and $g(t)$ have the same singular points we see that if $F_1(z)$ is any solution, other than $F(z)$, then $f_1(t)$ has every singular point which $g(t)$ has and in addition it has at least one other; in fact, it has a singular point at each of the points a_μ for which at least one of the constants $d_{\mu 0}, \ldots, d_{\mu, s_\mu - 1}$ is different from zero. This shows that the conjugate diagram of $G(z)$ is smaller than that of $F_1(z)$. This completes the proof of Corollary 2. 1.

2. 4. In preparation for the treatment of a certain class of meromorphic functions we prove the following lemma:

Lemma 2. 4. *Let us consider an interval J_σ and a meromorphic function $G(z)$ possessing a pole of order $h+1$ $(h \geqq 0)$ at $z = b = \alpha + i\beta$. Let $G(z)$ be such that*

$$(z-b)^{h+1} G(z) = J(z) \tag{2. 34}$$

where $J(z)$ is a function of finite exponential type of such a sort that

$$\underline{b}_\sigma < -h(\pi|J), \qquad h(0|J) < \bar{b}_\sigma. \tag{2. 35}$$

Then the equation (2. 1) *has the solutions*

$$\begin{aligned} F_1(z) &= \Gamma^\sigma G, & I_1(-\infty < y < \beta), \\ F_2(z) &= \Gamma^\sigma G, & I_2(\beta < y < +\infty). \end{aligned} \tag{2. 36}$$

Each function $F_\mu(z)$, $\mu = 1, 2$, is analytic in the associated strip I_μ and is such that

$$\underline{b}_\sigma < \liminf_{x=-\infty} x^{-1} \log |F_\mu(x+iy_\mu)|, \quad \limsup_{x=+\infty} x^{-1} \log |F_\mu(x+iy_\mu)| < \bar{b}_\sigma, \tag{2. 37}$$

where y_μ is a particular value of y in the interval I_μ.

The equation (2. 1) *has only one analytic solution which verifies a relation of the form* (2. 37) *for a particular value of y_μ in I_μ.*

If $\{\lambda_{n_j}\}$ denotes the subsequence of $\{\lambda_n\}$ for which $\gamma^\sigma_{\lambda_n} \neq 0$, then each of the functions $F_1(z)$ and $F_2(z)$ has singular points at the closure of the set of points $\{b - \lambda_{n_j}\}$. If the λ_{n_j} are everywhere dense on the real axis, then the line $y = \beta$ forms a natural boundary for each of the functions $F_1(z)$ and $F_2(z)$. If no λ_{n_j} lies in the open interval $\Lambda_1 < \lambda < \Lambda_2$, the function $F_1(z)$ remains regular on the segment $\alpha - \Lambda_1 < x < \alpha - \Lambda_1$, $y = \beta$ and $F_1(z) \equiv F_2(z)$. If several gaps occur among the λ_{n_j}, $F_1(z)$ can be continued over each corresponding gap on the line $y = \beta$, the continuation always leading to the same function $F(z)$. Isolated singularities are poles of order $h+1$ and correspond to isolated λ_{n_j}.[1]

[1] In another connection S. Bochner and F. Bohnenblust, *Ann. of Math.* (2) 35 (1934), pp. 152—161, have obtained theorems which suggested the results contained in the final paragraph of Lemma 2. 4. The method used here in studying the singularities of $F_1(z)$ and $F_2(z)$ was suggested by Bochner. For results of this sort see also E. Goursat, *Bull. Sci. Math.* (2) 11 (1887), pp. 109—114.

That the series $\Gamma^\sigma G$ converges absolutely in each of the strips I_1 and I_2 and that the convergence is uniform in every finite part of either strip are immediate consequences of the relations (2. 34) and (2. 35). Indeed we may write, for example,

$$(2.\ 38)\quad |F_\mu(z)| = \left|\sum_{n=0}^{\infty} \gamma_{\lambda_n} \frac{J(z+\lambda_n)}{(z-b+\lambda_n)^{h+1}}\right| \leqq \frac{1}{|y-\beta|^{h+1}} \sum_{n=0}^{\infty} |\gamma_{\lambda_n}|\,|J(z+\lambda_n)|,\ z \text{ in } I_\mu,$$

and we may apply the results of Theorem 2. 3 to the latter series. The functions $F_1(z)$ and $F_2(z)$ are clearly solutions of (2. 1) and are analytic in the strips I_1 and I_2 respectively. That each function $F_\mu(z)$, $\mu = 1, 2$, verifies a relation of the form (2. 37) and that it is unique in this respect we easily see as in the proofs of Theorem 2. 3 and Lemma 2. 1.

We proceed now with the proof of the third part of the lemma. Let C be a circle of radius R, center a, containing none of the points $b=\lambda_{n_j}, j=0, 1, 2, \ldots$, in its interior and passing through a single one, $b-\lambda_{n_k}$, so that $|a-b+\lambda_{n_k}| = R$, $|a-b+\lambda_{n_j}| > R$, $j \neq k$. If C' is any circle of radius $R' < R$, center a, then it is clear that the series $\Gamma^\sigma G$ converges absolutely and uniformly in C' and hence the series $\Gamma^\sigma G$ represents an analytic function regular in the interior of the circle C. We shall show that the function represented by this series in C has a singularity at the point $b-\lambda_{n_k}$. Let L denote the interior of the line segment joining the point a to the point $b-\lambda_{n_k}$. Then if z is any point on L, we have $|z-b+\lambda_{n_k}| < |z-b+\lambda_{n_j}|$, $j \neq k$. Writing

$$(2.\ 39)\quad (z-b+\lambda_{n_k})^{h+1}\,\Gamma^\sigma G = \sum_{j=0}^{\infty} \gamma_{\lambda_{n_j}} J(z+\lambda_{n_j}) \left(\frac{z-b+\lambda_{n_k}}{z-b+\lambda_{n_j}}\right)^{h+1},\ z \text{ in } C,$$

and using the fact that the series

$$\sum_{j=0}^{\infty} \gamma_{\lambda_{n_j}} J(z+\lambda_{n_j})$$

converges absolutely and uniformly in every finite region, we see that

$$\lim_{z=b-\lambda_{n_k},\ z \text{ on } L} (z-b+\lambda_{n_k})^{h+1}\,\Gamma^\sigma G = \gamma_{\lambda_{n_k}} J(b).$$

Since $\gamma_{\lambda_{n_k}} J(b) \neq 0$, the point $b-\lambda_{n_k}$ is a singular point of the function represented within C by $\Gamma^{\sigma} G$. If λ_{n_k} is isolated the point $b - \lambda_{n_k}$ is clearly a pole of order $h + 1$ of the function represented by $\Gamma^{\sigma} G$. If a gap occurs among the λ_{n_j} we may take as the point a any interior point of the corresponding gap on the line $y = \beta$ and we see that $F_1(z)$ remains regular on this gap on the line $y = \beta$ and that $F_1(z) \equiv F_2(z)$. Since each of the functions $F_1(z)$ and $F_2(z)$ has singular points at the closure of the set $\{b - \lambda_{n_j}\}$, the line $y = \beta$ is a natural boundary for each of the functions $F_1(z)$ and $F_2(z)$ if the points λ_{n_j} are dense on the real axis. This completes the proof of Lemma 2.4.

On applying Lemmas 2.1 and 2.4, we immediately prove the following theorem:

Theorem 2.4. *Let us consider an interval J_σ and a set of functions $G_0(z)$, $G_1(z), \ldots, G_k(z)$ of finite exponential type such that*

$$\underline{b}_\sigma < -h(\pi | G_p), \quad h(0 | G_p) < \bar{b}_\sigma, \qquad p = 0, 1, \ldots, k. \tag{2.40}$$

Let $b_1 = \alpha_1 + i\beta_1, \ldots, b_k = \alpha_k + i\beta_k$ be any k points of the plane and let us denote by $\beta_{p_1}, \ldots, \beta_{p_a}$ the distinct β's ordered as follows: $\beta_{p_1} < \beta_{p_2} < \cdots < \beta_{p_a}$. Define $\beta_{p_0} = -\infty$, $\beta_{p_{a+1}} = +\infty$. If we write

$$G(z) = G_0(z) + \sum_{p=1}^{k} \frac{G_p(z)}{(z - b_p)^{m_p + 1}}, \tag{2.41}$$

where the m_p are non-negative integers, then the equation (2.1) has the solutions

$$F_\mu(z) = \Gamma^\sigma G, \quad I_\mu(\beta_{p_\mu} < y < \beta_{p_{\mu+1}}), \qquad \mu = 0, 1, \ldots, a. \tag{2.42}$$

Each solution $F_\mu(z)$ is analytic in the associated strip I_μ and is such that

$$\underline{b}_\sigma < \liminf_{x=-\infty} x^{-1} \log |F_\mu(x + i y_\mu)|, \quad \limsup_{x=+\infty} x^{-1} \log |F_\mu(x + i y_\mu)| < \bar{b}_\sigma, \tag{2.43}$$

where y_μ is any fixed value of y in the interval I_μ.

The equation (2.1) has only one analytic solution which verifies relations of the form (2.43) for a particular value of y_μ in the interval I_μ.

The results contained in the third part of Lemma 2.4 enable us to discuss the singularities of the functions $F_\mu(z)$. Before studying this problem we consider a special example which illustrates how certain of the singularities may cancel.

Let us consider the equation

$$(2.44) \qquad F(z+1) - 2F(z) = G(z).$$

For $G(z)$ denoting successively the functions

$$\frac{1}{z+1}, \quad -\frac{2}{z}, \quad \frac{1}{z+1} - \frac{2}{z},$$

the equation (2. 44) has the solutions

$$\Gamma^0\left(\frac{1}{z+1}\right) = -\sum_{n=0}^{\infty} \frac{1}{2^{n+1}} \frac{1}{z+n+1}, \quad \Gamma^0\left(-\frac{2}{z}\right) = \sum_{n=0}^{\infty} \frac{1}{2^n} \frac{1}{z+n}, \quad \Gamma^0\left(\frac{1}{z+1} - \frac{2}{z}\right) = \frac{1}{z}.$$

The solution of the first equation has poles at $z = -1, -2, -3, \ldots$; the solution of the second equation has poles at $z = 0, -1, -2, \ldots$; while the solution of the third equation has but a single pole, namely $z = 0$.

We return to the case considered in Theorem 2. 4. Let us assume that $G_p(b_p) \neq 0$, $p = 1, \ldots, k$, so that the points b_p are actually poles of orders $m_p + 1$ of $G(z)$. Let $\{\lambda_{n_j}\}$ be the subsequence of $\{\lambda_n\}$ for which $\gamma_{\lambda_n}^{\sigma} \neq 0$. For definiteness we consider the line $y = \beta_1$, where $\beta_1 \leqq \beta_p$, $p = 2, \ldots, k$, and we assume that $b_1, \ldots, b_r$ are the b's whose imaginary parts are β_1. We distinguish between two cases.

(1). No two of the points $b_1, \ldots, b_r$ differ by a λ_n. In this case each of the functions $F_0(z)$ and $F_1(z)$ has singularities at the closures of the sets of points $\{b_1 - \lambda_{n_j}\}, \ldots, \{b_r - \lambda_{n_j}\}$. If one or more gaps on the line $y = \beta_1$ occur among these sets then both $F_0(z)$ and $F_1(z)$ are regular on each such gap and $F_0(z) = F_1(z)$. If $F_0(z)$ $(\equiv F_1(z))$ possesses an isolated singularity on $y = \beta_1$, the singularity is a pole.

(2). Some of the points $b_1, \ldots, b_r$ do differ by a λ_n. If $b_1 - b_2 = \lambda_n$ but if $m_1 \neq m_2$, then again there is no cancellation of singularities (in so far as the singularities arise from the poles b_1, b_2). Hence let us assume that

$$b_2 = b_1 + \lambda_{k_2}, \quad b_3 = b_1 + \lambda_{k_3}, \quad \ldots, \quad b_{\varrho_0} = b_1 + \lambda_{k_{\varrho_0}},$$

and that $m_1 = m_2 = \cdots = m_{\varrho_0}$. If we write

$$H_n = \gamma_{\lambda_n} G_1(b_1) + \gamma_{\lambda_n + \lambda_{k_2}} G_2(b_2) + \cdots + \gamma_{\lambda_n + \lambda_{\varrho_0}} G_{\varrho_0}(b_{\varrho_0}), \qquad n = 0, 1, 2, \ldots,$$

and if we denote by $\{H_{n^j}\}$ the subsequence of $\{H_n\}$ for which $H_n \neq 0$, then we see that the singularities on the line $y = \beta_1$ arising from the poles $b_1, \ldots, b_{\varrho_0}$

are the points which consist of the closure of the set $\{b_1 - \lambda_n i\}$. The singularities arising from the poles $b_{\varrho_0+1}, \ldots, b_r$ do not cancel any of the singularities arising from the poles $b_1, \ldots, b_{\varrho_0}$. If there are any intervals on the line $y=\beta_1$ on which no singularity occurs then, as in (1), $F_0(z) \equiv F_1(z)$.

The behavior on each of the other lines is similar in every respect.

Finally we note that if $E(t)$ is such that for some σ the relation $\underline{b}_\sigma < 0 < \overline{b}_\sigma$ holds[1], then Theorem 2.4 applies to the case where $G(z)$ is any rational function and the remarks following Theorem 2.4 give us the nature of the singularities of the solutions.

2.5. We next prove the following theorem:

Theorem 2.5. *Let $E(t)$ be such that $\overline{b}_0 > 0$. Let $G(z)$ be analytic in $S\left(-\delta \leqq \theta \leqq \delta \left(0 < \delta \leqq \frac{\pi}{2}\right), \ |z| > R\right)$ and such that*

$$\lim_{z=\infty} G(z) = a, \qquad z \text{ in } S. \tag{2.45}$$

Then the function $F(z)$ defined by the series $\Gamma^0 G$ (for $|z|$ sufficiently large) is analytic for all values of z for which $z - \delta_0$ is in S and it represents a solution of the equation (2.1) for which

$$\lim_{z=\infty} F(z) = a \sum_{n=0}^{\infty} \gamma_{\lambda_n}^0, \qquad z \text{ in } S. \tag{2.46}$$

There is only one analytic solution of (2.1) for which the limit as $z \to \infty$ in S exists.

If, moreover, $G(z)$ possesses the asymptotic expansion

$$G(z) \sim a_0 + \frac{a_1}{z} + \frac{a_2}{z^2} + \cdots, \qquad z \text{ in } S, \tag{2.47}$$

then the function $F(z) = \Gamma^0 G$ possesses the asymptotic expansion

$$F(z) \sim b_0 + \frac{b_1}{z} + \frac{b_2}{z^2} + \cdots, \qquad z \text{ in } S, \tag{2.48}$$

where

---

[1] The condition $\underline{b}_\sigma < 0 < \overline{b}_\sigma$ for some σ is equivalent to the condition that $|E(iv)| \geqq \eta > 0$ for $-\infty < v < +\infty$. Cf. BOCHNER, I, p. 440.

$$(2.49) \qquad b_0 = a_0 \sum_{n=0}^{\infty} \gamma_{\lambda_n}^0, \quad b_\nu = \sum_{p=1}^{\nu} a_p \binom{\nu - 1}{p - 1} \sum_{n=0}^{\infty} \gamma_{\lambda_n}^0 (-\lambda_n)^{\nu - p}.$$

For every positive ε there exists a quantity $\varrho_0 = \varrho_0(\varepsilon)$ such that

$$|G(z) - a| < \varepsilon, \quad |z| > \varrho_0, \qquad z \text{ in } S.$$

Using (2.8), we see that

$$\left|F(z) - a \sum_n \gamma_{\lambda_n}^0\right| = \left|\sum_n \gamma_{\lambda_n}^0 [G(z + \lambda_n) - a]\right| < \varepsilon \sum_n |\gamma_{\lambda_n}^0|, \ |z - \delta_0| > \varrho_0, \qquad z \text{ in } S.$$

It follows that (2.46) holds and moreover the series $\Gamma^0 G$ converges absolutely for all z in S for which $|z - \delta_0| > \varrho_0$. In every finite subregion of this region the convergence is uniform and hence the sum-function $F(z)$ is analytic for all z in S for which $|z - \delta_0| > \varrho_0$. It is easily verified that $F(z)$ is a solution of (2.1). Analytically continuing the solution by means of the relation

$$(2.50) \qquad F(z) = -c_0^{-1} \left[\sum_{s=1}^{m} c_s F(z + \delta_s - \delta_0) + G(z - \delta_0)\right],$$

we see that $F(z)$ is analytic for all values of z for which $z - \delta_0$ is in S.

For the uniqueness let $F_*(z)$ be any analytic solution of (2.1) for which the limit as $z \to \infty$ in S exists. Writing

$$F(z) = \sum_n \gamma_{\lambda_n}^0 G(z + \lambda_n) = \sum_n \gamma_{\lambda_n}^0 \sum_s c_s F_*(z + \delta_s + \lambda_n),$$

rearranging the (absolutely convergent) double series and using (2.7), we see that $F(z) \equiv F_*(z)$.

Let us now assume that $G(z)$ possesses the asymptotic expansion (2.47). In the present paragraph ν will denote a fixed positive integer and only values of z which are in S will be considered. For every positive number ε there exists a $\varrho = \varrho(\varepsilon)$ such that

$$(2.51) \qquad |z + \lambda_n|^\nu \left| G(z + \lambda_n) - a_0 - \frac{a_1}{z + \lambda_n} - \cdots - \frac{a_\nu}{(z + \lambda_n)^\nu} \right| < \varepsilon, \quad |z + \lambda_n| < \varrho.$$

If we write

$$b_{0,n} = a_0, \quad b_{j,n} = \sum_{p=1}^{j} a_p \binom{j - 1}{p - 1} (-\lambda_n)^{j-p}, \qquad j = 1, \ldots, \nu,$$

then by (2.49) we see that

$$(2.52)\quad |z^\nu|\left|F(z)-b_0-\frac{b_1}{z}-\cdots-\frac{b_\nu}{z^\nu}\right| \leqq \sum_{n=0}^{\infty}|\gamma^0_{\lambda_n}||z^\nu|\left|G(z+\lambda_n)-b_{0,n}-\cdots-\frac{b_{\nu,n}}{z^\nu}\right|.$$

By use of the inequalities (2.51) and (2.52) we verify that

$$\lim_{z=\infty}|z^\nu|\left|F(z)-b_0-\frac{b_1}{z}-\cdots-\frac{b_\nu}{z^\nu}\right|=0,$$

and hence $F(z)$ possesses the asymptotic developement (2.48).

An analogous theorem holds for the case when $E(t)$ is such that $b_1 < 0$ and where the sector S of Theorem 2.5 is replaced by a sector S_1 $(\pi-\delta\leqq\theta\leqq\pi+\delta$, $|z|>R)$. Because of its similarity to Theorem 2.5 we omit its statement.

III. The Equation with Asymptotically Constant Coefficients.

3.1. In this part we shall consider the equation

$$(3.1)\qquad \sum_{s=0}^{m} c_s(z)\,F(z+\delta_s)=G(z),$$

where the coefficients $c_s(z)$, analytic functions in a given sector, are asymptotically constant in this sector. Denoting by c_s the asymptotic values approached by the functions $c_s(z)$, we continue to use the notation of section 2.1 relating to the function $E(t)$, etc.[1] We use the following abbreviations due to Bochner:

$$(3.2)\qquad \varDelta F\equiv\sum_{s=0}^{m}c_s F(z+\delta_s),\quad \Psi(F)\equiv\sum_{s=0}^{m}(c_s-c_s(z))\,F(z+\delta_s).$$

We next prove the following theorem:

Theorem 3.1. *Let us consider the equation* (3.1) *where the* $c_s(z)$ *are analytic in* $S\left(-\delta\leqq\theta\leqq\delta\left(0<\delta\leqq\frac{\pi}{2}\right),\ |z|>R\right)$ *and such that*

[1] In section 2.1 we assumed that $c_0\neq 0$, $c_m\neq 0$ and that otherwise the c_s were arbitrary complex constants. In the theorems which follow in this part (Theorems 3.1 and 3.2) it is not necessary that c_m be different from zero. In these theorems the series $\Gamma^0 G$ occurs and in their proofs the non-vanishing of c_0 is definitely used. For analogous theorems where the series $\Gamma^1 G$ would occur c_0 might be zero and c_m would be assumed different from zero.

$$\lim_{z=\infty} c_s(z) = c_s, \qquad s = 0, 1, \ldots, m, \qquad z \text{ in } S, \tag{3.3}$$

where $c_0 \neq 0$, *and let* $c_0(z)$ *be non-vanishing in* S. *If* $G(z)$ *is analytic in* S *and is such that for every pair of real numbers* α, β $(\alpha < \beta)$ *a constant* $A = A(\alpha, \beta)$ *exists of such a sort that*

$$|G(z)| \leqq A e^{bx}, \qquad z \text{ in } S, \qquad \alpha \leqq y \leqq \beta, \tag{3.4}$$

where $b < \bar{b}_0$, *then by successive solutions of the equations*

$$\begin{aligned} \varDelta H^0 &= G, \\ \varDelta H^{\nu+1} &= \Psi(H^\nu), \qquad \nu = 0, 1, 2, \ldots, \end{aligned} \tag{3.5}$$

we are led to a function

$$F(z) = H^0(z) + H^1(z) + H^2(z) + \cdots \tag{3.6}$$

which is a solution of the equation (3.1). *The function* $F(z)$ *is analytic for all* z *such that* $z - \delta_0$ *is in* S *and moreover it is such that*

$$|F(z)| \leqq A' e^{bx}, \qquad z - \delta_0 \text{ in } S, \qquad \alpha \leqq y \leqq \beta, \tag{3.7}$$

where A' *is a constant.*

There exists only one analytic solution of (3.1) *which verifies a relation of the form* (3.7) *with an exponent* $b < \bar{b}_0$.[1]

We define successively the functions $H^0, H^1, H^2, \ldots$, by means of

$$H^0 = \Gamma^0 G, \qquad H^{\nu+1} = \Gamma^0 \Psi(H^\nu), \qquad \nu = 0, 1, 2, \ldots$$

By a slight modification of Lemma 2.1 we see that

$$|H^0| \leqq A\, C e^{bx}, \qquad z - \delta_0 \text{ in } S, \qquad \alpha \leqq y \leqq \beta,$$

where C is a constant independent of $G(z)$. For every positive ε there exists an x_ε such that $|c_s - c_s(z)| < \varepsilon$ for z in S and $|z| > x_\varepsilon$. Hence

$$|\Psi(H^0)| \leqq \varepsilon A\, C \sum_{s=0}^{m} e^{\delta_s b} = A(\varepsilon\, C D) e^{bx}, \quad z \text{ in } S, \quad |z| > x_\varepsilon, \quad \alpha \leqq y \leqq \beta,$$

and we may form the series $\Gamma^0 \Psi(H^0)$. By induction we easily verify that

---

[1] The methods used in the proof of this theorem are similar to those used by Bochner, I, pp. 443—444.

$$(3.8)\qquad |H^{\nu}| \leqq A\,C(\varepsilon\,C D)^{\nu} e^{bx}, \quad z-\delta_0 \text{ in } S, \quad |z-\delta_0| > x_{\varepsilon}, \quad \alpha \leqq y \leqq \beta,$$
$$\nu = 0, 1, 2, \ldots$$

Denoting by ε_0 a positive constant such that $\varepsilon_0\,C D < 1$, we see by (3.8) that the series (3.6) converges absolutely and uniformly in every finite region contained in that part of S for which $|z-\delta_0| > x_{\varepsilon_0}$, $\alpha \leqq y \leqq \beta$. Since α and β are arbitrary real numbers ($\alpha < \beta$) we see that the sum-function $F(z)$ is analytic in that part of S for which $|z-\delta_0| > x_{\varepsilon_0}$. From (3.6) and (3.8) we see that

$$(3.9)\qquad |F(z)| \leqq A_1 e^{bx}, \qquad z \text{ in } S, \qquad |z-\delta_0| > x_{\varepsilon_0}, \qquad \alpha \leqq y \leqq \beta,$$

where $A_1 = A\,C/(1 - C D)$. By means of the equations (3.5) we easily verify that the function $F(z)$ in (3.6) is a solution of (3.1). Using a relation of the form (2.50) with c_s replaced by $c_s(z-\delta_0)$, we continue the solution analytically and obtain a function $F(z)$ analytic for all z for which $z-\delta_0$ is in S. In view of (3.9) and the analyticity of $F(z)$ for $z-\delta_0$ in S we see that $F(z)$ verifies a relation of the form (3.7). This completes the proof of the first part of Theorem 3.1.

In order to prove the uniqueness we assume that $F_*(z)$ is an analytic solution of (3.1) which verifies a relation of the form (3.7). We determine successively the functions J^0, J^1, J^2, ..., as »the» solutions of the equations

$$(3.10)\qquad \Delta J^0 = \Psi(F_*),$$

$$(3.11)\qquad \Delta J^{\nu+1} = \Psi(J^{\nu}), \qquad \nu = 0, 1, 2, \ldots$$

In the same manner in which we derived the relations (3.8) we derive the relations

$$(3.12)\qquad |J^{\nu}| \leqq A'\,C(\varepsilon_0\,C D)^{\nu} e^{bx}, \qquad \begin{array}{ll} z \text{ in } S, & \nu = 0, 1, 2, \ldots, \\ |z-\delta_0| > x_{\varepsilon_0}, & \alpha \leqq y \leqq \beta. \end{array}$$

From the relations (3.12) we see that we may write $F_*(z)$ in the form

$$(3.13)\qquad F_* = H_*^0 + H_*^1 + H_*^2 + \cdots, \quad z \text{ in } S, \quad |z-\delta_0| > x_{\varepsilon_0}, \qquad \alpha \leqq y \leqq \beta,$$

where

$$H_*^0 = F_* - J^0, \qquad H_*^{\nu+1} = J^{\nu} - J^{\nu+1}, \qquad \nu = 0, 1, 2, \ldots$$

The functions H_*^{ν} are solutions of the system of equations (3.5) and moreover they satisfy bounds of the form (3.8). By Lemma 2.1 we see that these facts imply that

$$H_*^\nu = H^\nu, \qquad \nu = 0, 1, 2, \ldots,$$

and hence

$$F_* = \sum_{\nu=0}^{\infty} H^\nu = F.$$

This completes the proof of Theorem 3.1.

In an analogous manner we obtain a similar theorem treating the case where the functions $c_s(z)$ and $G(z)$ are analytic in $S_1 (\pi - \delta \leqq \theta \leqq \pi + \delta, |z| > R)$ and where the functions $c_s(z)$ are asymptotically constant in S_1. Because of its similarity to Theorem 3.1 we omit its statement.

We next state and prove a theorem analogous to Theorem 2.5.

Theorem 3.2. *Let the functions $c_s(z)$ in (3.1) be analytic in*

$$S\left(-\delta \leqq \theta \leqq \delta \left(0 < \delta \leqq \frac{\pi}{2}\right), |z| > R\right)$$

and such that the relations in (3.3) hold where $c_0 \neq 0$, and let $c_0(z)$ be non-vanishing in S. Let $E(t)$ be such that $\bar{b}_0 > 0$. Let $G(z)$ be analytic in S and such that

$$\lim_{z=\infty} G(z) = a, \qquad z \text{ in } S. \tag{3.14}$$

Then by successive solutions of the equations (3.5) we are led to a function $F(z)$ defined by (3.6) which is a solution of the equation (3.1). This function is analytical for all z for which $z - \delta_0$ is in S and moreover it is such that

$$\lim_{z=\infty} F(z) = a \sum_{n=0}^{\infty} \gamma^0_{\lambda_n}, \qquad z \text{ in } S. \tag{3.15}$$

There is only one analytic solution of (3.1) for which the limit as $z \to \infty$ in S exists.

If moreover the functions $G(z)$ and $c_s(z)$ possess the asymptotic expansions

$$G(z) \sim a_0 + \frac{a_1}{z} + \frac{a_2}{z^2} + \cdots, \qquad z \text{ in } S, \tag{3.16}$$

$$c_s(z) \sim c_s - \frac{c_1^s}{z} - \frac{c_2^s}{z^2} - \cdots, \qquad z \text{ in } S, \qquad s = 0, 1, \ldots, m, \tag{3.17}$$

then the functions $H^\nu(z)$ possess asymptotic expansions of the form

$$(3.\ 18)\qquad H^\nu(z) \sim \frac{b_\nu^\nu}{z^\nu} + \frac{b_{\nu+1}^\nu}{z^{\nu+1}} + \frac{b_{\nu+2}^\nu}{z^{\nu+2}} + \cdots, \quad z \text{ in } S, \qquad \nu = 0, 1, 2, \ldots,$$

and the function $F(z)$ possesses the asymptotic expansion

$$(3.\ 19)\qquad F(z) \sim b_0 + \frac{b_1}{z} + \frac{b_2}{z^2} + \cdots, \quad z \text{ in } S,$$

where

$$(3.\ 20)\qquad \begin{aligned} b_0 &= b_0^0 = a_0 \sum_{n=0}^{\infty} \gamma_{\lambda_n}^0, \\ b_\nu &= b_\nu^0 + b_\nu^1 + \cdots + b_\nu^\nu, \qquad \nu = 1, 2, 3, \ldots \end{aligned}$$

We first prove the first part of the theorem. We define successively the functions H^0, H^1, H^2, ..., by means of

$$H^0 = \Gamma^0 G, \qquad H^{\nu+1} = \Gamma^0 \Psi(H^\nu).$$

By Theorem 2. 5 we see that

$$(3.\ 21)\qquad \lim_{z=\infty} H^0(z) = a \sum_{n=0}^{\infty} \gamma_{\lambda_n}^0, \qquad z \text{ in } S,$$

and hence for every positive number ζ_0 there exists a constant $p_0 = p_0(\zeta_0)$ such that

$$(3.\ 22)\qquad \left| H^0(z) - a \sum_n \gamma_{\lambda_n}^0 \right| < \zeta_0, \quad z \text{ in } S, \qquad |z| > p_0.$$

In view of (3. 3) and (3. 21) we see that for every $\zeta > 0$ there exists a $p = p(\zeta)$ such that

$$|\Psi(H^0)| < \zeta, \qquad z \text{ in } S, \qquad |z| > p,$$

and from this it follows that

$$|H^1(z)| < \zeta \sum_n |\gamma_{\lambda_n}^0| = \zeta\, d, \qquad z \text{ in } S, \qquad |z - \delta_0| > p.$$

For every $\varepsilon > 0$ there exists a $\varrho = \varrho(\varepsilon)$ such that

$$|c_s - c_s(z)| < \varepsilon, \qquad z \text{ in } S, \qquad |z| > \varrho,$$

and consequently we see that

$$|\Psi(H^1)| < \varepsilon(m + 1)\, d\, \zeta, \qquad z \text{ in } S, \qquad |z| > (p, \varrho).$$

By induction we easily verify that

$$|H^{\nu+1}(z)| < \varepsilon^{\nu}(m+1)^{\nu} d^{\nu+1}\zeta, \qquad \nu = 0, 1, 2, \ldots, \qquad z \text{ in } S, \qquad |z-\delta_0| > (p, \varrho),$$

and hence the series (3.6) converges absolutely for all z in that part of S for which $|z| > (p, \varrho(\varepsilon_0))$, where ε_0 is a positive constant such that $\varepsilon_0(m+1)d < 1$. If z is restricted to any finite part of the above region the convergence of (3.6) is uniform and the function $F(z)$ is thus seen to be analytic for z in that part of S for which $|z| > (p, \varrho(\varepsilon_0))$. As in the proof of Theorem 3.1 we see that $F(z)$ is a solution of (3.1) and that it is analytic for all z such that $z-\delta_0$ is in S. In order to prove that (3.15) holds we write

$$\Big|F(z) - a\sum_n \gamma^0_{\lambda_n}\Big| \leqq \Big|H^0(z) - a\sum_n \gamma^0_{\lambda_n}\Big| + |H^1(z)| + |H^2(z)| + \cdots < \zeta_0 + \\ + \frac{d\zeta}{1-\varepsilon_0(m+1)d},$$

for z in S and $|z|$ sufficiently large. Since a relation of this form holds for every positive ζ_0 and ζ we see that (3.15) holds.

Noting that any analytic function for which the limit as $z \to \infty$ in S exists satisfies a relation of the form (3.7), we see that the uniqueness of this solution is a consequence of Theorem 3.1.

Next let us assume that $G(z)$ and the $c_s(z)$ possess the asymptotic expansions (3.16) and (3.17). By Theorem 2.5 we see that

$$H^0(z) \sim b^0_0 + \frac{b^0_1}{z} + \frac{b^0_2}{z^2} + \cdots, \qquad z \text{ in } S,$$

where $b^0_0 = a\sum \gamma^0_{\lambda_n}$. If we assume that for a fixed integer ν the function $H^{\nu}(z)$ possesses an asymptotic expansion of the form (3.18) then in view of (3.17) we see that the function $\Psi(H^{\nu}(z))$ possesses an asymptotic expansion of the form

$$\Psi(H^{\nu}(z)) \sim \frac{d^{\nu}_{\nu+1}}{z^{\nu+1}} + \frac{d^{\nu}_{\nu+2}}{z^{\nu+2}} + \cdots, \qquad z \text{ in } S.$$

Applying Theorem 2.5 to the equation $\Delta H^{\nu+1} = \Psi(H^{\nu})$, we see that the function $H^{\nu+1}(z)$ has an asymptotic expansion of the form (3.18) with ν replaced throughout by $\nu+1$. The induction argument is complete and we see that each of the functions $H^0(z)$, $H^1(z)$, $H^2(z)$, ..., does possess an asymptotic expansion of the form (3. 18). In order to show that $F(z)$ possesses the asymptotic expansion (3. 19) we consider, for μ a fixed non-negative integer, the expression

$$(3.23)\quad z^{\mu}\left(F(z)-b_0-\cdots-\frac{b_\mu}{z^\mu}\right)=z^{\mu}\sum_{j=0}^{\mu}\left(H^j(z)-\frac{b_j^j}{z^j}-\cdots-\frac{b_\mu^j}{z^\mu}\right)+z^{\mu}\sum_{k=1}^{\infty}H^{\mu+k}(z).$$

From the definition of an asymptotic expansion we see that the relations (3. 18) imply that

$$(3.24)\qquad \lim_{z=\infty} z^{\mu}\left(H^j(z)-\frac{b_j^j}{z^j}-\cdots-\frac{b_\mu^j}{z^\mu}\right)=0,\qquad z \text{ in } S,\qquad j=0,1,\ldots,\mu,$$

and

$$(3.25)\qquad \lim_{z=\infty} z^{\mu}H^{\mu+1}(z)=0,\qquad z \text{ in } S.$$

Making an induction argument of the form made in the first paragraph following the statement of Theorem 3. 2, we show that for every pair of positive numbers ε and ζ there exist constants $\varrho(\varepsilon)$ and $p(\zeta)$ such that

$$(3.26)\qquad |z^{\mu}H^{\mu+k}(z)|<\varepsilon^{k-1}(m+1)^{k-1}d^{k-1}\zeta,\quad z \text{ in } S,$$
$$|z|>(\varrho(\varepsilon),p(\zeta)),\qquad k=0,1,2,\ldots$$

Letting ε be a quantity ε_0 such that $0<\varepsilon_0(m+1)d<1$ and using the fact that (3. 26) holds for every positive ζ, we see that (3. 26) implies

$$(3.27)\qquad \lim_{z=\infty} z^{\mu}\sum_{k=1}^{\infty}H^{\mu+k}(z)=0,\qquad z \text{ in } S.$$

From (3. 23), (3. 24) and (3. 27) we see that

$$(3.28)\qquad \lim_{z=\infty} z^{\mu}\left(F(z)-b_0-\cdots-\frac{b_\mu}{z^\mu}\right)=0,\qquad z \text{ in } S,$$

and since this is true for every non-negative integer μ the function $F(z)$ has the asymptotic expansion (3. 19). This completes the proof of Theorem 3. 2.

We omit the statement of an analogous theorem relating to the case where the functions are analytic in $S_1(\pi-\delta\leqq\theta\leqq\pi+\delta,\ |z|>R)$.

IV. The System with Constant Coefficients.

4. 1. In developing the theory of systems of difference we shall use a symbolic notation of the sort used by Bochner and Carmichael.[1]

Let us consider a difference-expression

$$\Delta F \equiv \sum_{s=0}^{m} c_s F(z + \delta_s) \tag{4. 1}$$

with complex constants c_s and non-negative δ_s, $0 \leqq \delta_0 < \delta_1 < \delta_2 < \cdots < \delta_m$. We associate with such a difference-expression as its characteristic function $E(t)$ the function

$$E(t) = \sum_{s=0}^{m} c_s e^{\delta_s t}. \tag{4. 2}$$

To every function of the form (4. 2) there corresponds a unique difference-expression (4. 1). In view of this one-to-one correspondence we shall represent the difference-expression (4. 1) also by the symbol

$$E[F]. \tag{4. 3}$$

If $E_1(t)$ and $E_2(t)$ represent two characteristic functions then the relations

$$E_1[E_2[F]] = E_1 E_2[F] = E_2[E_1[F]] \tag{4. 4}$$

hold, where by $E_1 E_2(t)$ we mean the ordinary product $E_1(t) E_2(t)$ of the functions $E_1(t)$ and $E_2(t)$.

Let us consider p^2 difference-expressions

$$\Delta_{jk} F \equiv \sum_{s=0}^{m_{jk}} c_s^{jk} F(z + \delta_s^{jk}), \qquad j, k = 1, \ldots, p, \tag{4. 5}$$

and their characteristic functions

$$E_{jk}(t) = \sum_{s=0}^{m_{jk}} c_s^{jk} e^{\delta_s^{jk} t}, \tag{4. 6}$$

[1] S. Bochner, II; R. D. Carmichael, *loc. cit.* The symbolic notation which we use in this section is a special case of that used by Bochner in his treatment of difference-differential equations. The laws of combination of the symbolic operators given in the present paper in (4. 4), (4. 12), (4. 13), etc., are given by Bochner, II, pp. 582—583 and by Carmichael, pp. 2—4.

where the m_{jk} are arbitrary non-negative integers, the δ_s^{jk} are real non-negative numbers, and the c_s^{jk} are complex constants, the sets c_s^{jk} and δ_s^{jk} being subject to the restriction that the functional determinant

$$E(t) \equiv |E_{jk}(t)| \tag{4. 7}$$

shall not vanish identically in t,

$$|E(t)| \not\equiv 0. \tag{4. 8}$$

We shall under these conditions consider the non-singular[1] system of difference equations

$$\sum_{k=1}^{p} \Delta_{jk} F_k = G_j(z), \qquad j = 1, \ldots, p, \tag{4. 9}$$

which we may also write in the form

$$\sum_{k=1}^{p} E_{jk}[F_k] = G_j, \qquad j = 1, \ldots, p. \tag{4. 10}$$

Denoting by $e_{jk}(t)$ the cofactor of $E_{jk}(t)$ in $E(t)$, we have

$$\sum_{j=1}^{p} e_{j\mu}(t) E_{jk}(t) = \begin{cases} E(t), & k = \mu, \\ 0 \ , & k \neq \mu. \end{cases} \tag{4. 11}$$

The $e_{jk}(t)$ being characteristic functions of difference-expressions, we can for every j »multiply» the relation (4. 10) by $e_{j\mu}$, where μ is any fixed integer, $1 \leqq \mu \leqq p$.[2] Performing these »multiplications», we have

$$\sum_{k=1}^{p} e_{j\mu} E_{jk}[F_k] = e_{j\mu}[G_j], \qquad j = 1, \ldots, p. \tag{4. 12}$$

Summing the expressions (4. 12) as to j from 1 to p, we obtain

$$E[F_\mu] = \sum_{j=1}^{p} e_{j\mu}[G_j], \qquad \mu = 1, \ldots, p. \tag{4. 13}$$

---

[1] CARMICHAEL calls non-singular a system of difference equations for which the associated functional determinant is not identically zero.

[2] Cf. BOCHNER, II, p. 583.

Every set of solutions of the system (4. 10) is a set of solutions of the system (4. 13).[1] We shall in this part obtain an appropiate set of solutions of (4. 13) which is also a set of solutions of (4. 10).

The determinant $E(t)$ in (4. 7) may be written in the form (4. 2) where the c_s are non-vanishing complex constants, the δ_s are distinct real non-negative constants, and m is a non-negative integer. If m is zero the problem is trivial as we see from (4. 13). Henceforth we shall assume that m is greater than zero and under this assumption we may use the notation of section 2. 1 relating to the function $E(t)$, its zeros, the generalized Dirichlet expansions of its reciprocal, etc.

4. 2. We next prove a lemma which is the analogue of Lemma 2. 1.

Lemma 4. 1 (2. 1)′. *Let us consider an interval J_σ and p analytic functions $G_1(z), \ldots, G_p(z)$ regular in a strip (α, β). If the $G_j(z)$ are such that*

$$|G_j(z)| \leqq A(e^{\underline{b}x}, e^{\bar{b}x}), \qquad \alpha \leqq y \leqq \beta, \qquad j = 1, \ldots, p, \tag{4. 14}$$

where $\underline{b}_\sigma < \underline{b} \leqq \bar{b} < \bar{b}_\sigma$, then the non-singular system (4. 10) *has the set of solutions*

$$F_\mu(z) = \Gamma^\sigma \sum_{j=1}^{p} e_{j\mu}[G_j], \qquad \mu = 1, \ldots, p. \tag{4. 15}$$

The functions $F_1(z), \ldots, F_p(z)$ are analytic in the strip (α, β) and are such that

$$|F_\mu(z)| \leqq A'(e^{\underline{b}x}, e^{\bar{b}x}), \qquad \alpha \leqq y \leqq \beta, \qquad \mu = 1, \ldots, p, \tag{4. 16}$$

where $A' = 3AC'$ and C' is a constant independent of the $G_j(z)$.

The system (4. 10) *has only one set of analytic solutions $F_1(z), \ldots, F_p(z)$ for which relations of the form* (4. 16) *hold; in fact, it has only one set for which the relations*

$$\underline{b}_\sigma < \liminf_{x=-\infty} x^{-1} \log |F_\mu(x + iy_0)|, \quad \limsup_{x=+\infty} x^{-1} \log |F_\mu(x + iy_0)| < \bar{b}_\sigma, \tag{4. 17}$$

where $\mu = 1, \ldots, p$, hold for a particular value of y_0 in the interval $\alpha \leqq y_0 \leqq \beta$.

We first show that it is possible to form the series (4. 15). Using the facts that the $e_{jk}(t)$ are characteristic functions of difference-expressions and that the relations (4. 14) hold, we see at once that the functions

[1] CARMICHAEL, *loc. cit.* p. 5, gives an example showing that the converse does not hold.

$$(4.\ 18)\qquad \sum_{j=1}^{p} e_{j\mu}[G_j], \qquad \mu = 1, \ldots, p,$$

verify relations of the form

$$(4.\ 19)\qquad \left|\sum_{j=1}^{p} e_{j\mu}[G_j]\right| \leqq A\, B(e^{bx},\ e^{\bar{b}x}), \qquad \alpha \leqq y \leqq \beta, \qquad \mu = 1, \ldots, p,$$

where B is any constant such that

$$(4.\ 20)\qquad B \geqq \sum_{j=1}^{p} e_{j\mu}(\bar{b}), \qquad \mu = 1, \ldots, p.$$

Since the relations (4. 19) hold we may apply Lemma 2. 1 to each of the equations (4. 13) and we see that the functions in (4. 15) form a set of solutions of the system (4. 13), that these functions are analytic in the strip (α, β) and that they satisfy relations of the form (4. 16) with $C' = B\,C$. We next show that these functions furnish a set of solutions of the system (4. 10). Writing

$$\sum_{\mu=1}^{p} E_{j\mu}[F_\mu] = \sum_{\mu=1}^{p} E_{j\mu}\left[\sum_{n=0}^{\infty} \gamma^{\sigma}_{\lambda_n} \sum_{k=1}^{p} e_{k\mu}[G_k(z+\lambda_n)]\right], \qquad j = 1, \ldots, p,$$

performing an interchange of order of summation, collecting terms (these processes being valid because of (4. 14), (4. 15) and (4. 19)) and using (4. 11), we see that

$$\sum_{\mu=1}^{p} E_{j\mu}[F_\mu] = \sum_{n=0}^{\infty} \gamma^{\sigma}_{\lambda_n} E[G_j(z+\lambda_n)] = \sum_{n=0}^{\infty} G_j(z+\lambda_n) \sum_{s=0}^{m} c_s \gamma^{\sigma}_{\lambda_n-\vartheta_s}, \qquad j = 1, \ldots, p.$$

From the relations (2. 7) it follows that the functions $F_1(z), \ldots, F_p(z)$ constitute a set of solutions of the system (4. 10). This completes the proof of the first part of the lemma. For the uniqueness we note that every set of solutions of the system (4. 10) is also a set of solutions of the system (4. 13). The uniqueness follows immediately by an application of Lemma 2. 1 to each of the equations in (4. 13). This completes the proof of Lemma 4. 1.[1]

[1] In the proof of Lemma 4. 1 we have used not the fact that the functions $G(z)$ satisfy relations of the form (4. 14) but the fact that the functions (4. 18) satisfy relations of the sort given in (4. 14) and (4. 19). If, instead of assuming (4. 14), we merely assume (4. 19) we arrive at the conclusion that the functions $F(z)$ satisfy (4. 16) and, from these relations and (4. 10), we see that the functions $G(z)$ do satisfy relations of the form (4. 14).

Theorem 4. 1 (2. 1)'. *Let us consider an interval* J_σ *and* p *functions* $G_1(z)$, $\ldots$, $G_p(z)$, *each satisfying the hypotheses of Lemma* 2. 2. *If in addition*

$$\underline{b}_\sigma < -h(\pi|G_j), \qquad h(0|G_j) < \bar{b}_\sigma, \qquad j = 1, \ldots, p, \tag{4. 21}$$

then the non-singular system (4. 10) *has the set of solutions* (4. 15). *The functions* $F_1(z)$, $\ldots$, $F_p(z)$ *are analytic in the upper half-plane* $0 \leqq y < +\infty$ *and are such that for every positive number* β *and for every pair of numbers* $\underline{b}$ *and* $\bar{b}$ $(\underline{b} \leqq \bar{b})$ *for which*

$$\underline{b}_\sigma < \underline{b} < -h(\pi|G_j), \qquad h(0|G_j) < \bar{b} < \bar{b}_\sigma, \qquad j = 1, \ldots, p, \tag{4. 22}$$

there exists a constant $\tilde{A} = \tilde{A}(\beta; \underline{b}, \bar{b})$ *of such a sort that*

$$|F_\mu(z)| \leqq \tilde{A}(e^{\underline{b}x}, e^{\bar{b}x}), \qquad 0 \leqq y \leqq \beta, \qquad \mu = 1, \ldots, p. \tag{4. 23}$$

The system (4. 10) *has only one set of analytic solutions for which relations of the form* (4. 23) *hold; in fact, it has only one set of analytic solutions for which*

$$\underline{b}_\sigma < -h(\pi|F_\mu), \qquad h(0|F_\mu) < \bar{b}_\sigma, \qquad \mu = 1, \ldots, p. \tag{4. 24}$$

This theorem is an immediate consequence of Lemmas 2. 2 and 4. 1.

Slightly modifying Lemma 4. 1, when σ is zero, to treat functions analytic in sectors for $|z| > R$ and applying this modification and Lemma 2. 3, we obtain the following theorem:

Theorem 4. 2 (2. 2)'. *Let us consider* p *functions* $G_1(z)$, $\ldots$, $G_p(z)$, *each satisfying the hypotheses of Lemma* 2. 3. *If in addition*

$$h(0|G_j) < \bar{b}_0, \qquad j = 1, \ldots, p, \tag{4. 25}$$

then the non-singular system (4. 10) *has the set of solutions*

$$F_\mu(z) = \Gamma^0 \sum_{j=1}^{p} e_{j\mu}[G_j], \qquad \mu = 1, \ldots, p. \tag{4. 26}$$

The functions $F_1(z)$, $\ldots$, $F_p(z)$ *are analytic for* $z - \delta_0$ *in* S_0 *and moreover they are such that for every pair of real numbers* α, β $(\alpha < \beta)$ *and for every number* b *for which*

$$h(0|G_j) < b < \bar{b}_0, \qquad j = 1, \ldots, p,$$

there exists a constant $\tilde{A} = \tilde{A}(\alpha, \beta; b)$ *of such a sort that*

$$(4.\ 27) \qquad |F_\mu(z)| \leqq \tilde{A}\, e^{bx}, \qquad \alpha \leqq y \leqq \beta, \qquad z - \delta_0 \textit{ in } S_0 \qquad \mu = 1, \ldots, p.$$

The system (4. 10) *has only one set of analytic solutions for which relations of the form* (4. 27) *hold; in fact, it has only one set of analytic solutions for which* $h(0|F_\mu) < \bar{b}_0$, $\mu = 1, \ldots, p$.

We next treat the case where the functions $G_1(z), \ldots, G_p(z)$ are of finite exponential type. We have the following theorem:

Theorem 4. 3 (2. 3)'. *Let us consider an interval* J_σ *and* p *functions* $G_1(z), \ldots, G_p(z)$, *each of exponential type not exceeding* q. *Let us assume that the* $G(z)$ *are such that the relations* (4. 21) *hold. Then the non-singular system* (4. 10) *has the set of solutions* (4. 15). *The functions* $F_1(z), \ldots, F_p(z)$ *are such that the relations* (4. 24) *hold.*

The system (4. 10) *has only one set of analytic solutions for which the relations* (4. 26) *hold.*

Each of the functions $F_1(z), \ldots, F_p(z)$ *is of exponential type not exceeding* q. *When one at least of the functions* $G_1(z), \ldots, G_p(z)$ *is of exponential type* q, *then at least one of the solutions* $F_1(z), \ldots, F_p(z)$ *is of exponential type* q.[1]

The functions in (4. 18) are of exponential type not exceeding q and their associated functions $h(\theta)$ satisfy relations of the form (4. 21) whenever the functions $G_1(z), \ldots, G_p(z)$ possess these properties. Consequently we may apply Lemma 4. 1 and Theorem 2. 3 and we immediately obtain the first two parts of Theorem 4. 3, together with the fact that the functions $F_1(z), \ldots, F_p(z)$ are of exponential type not exceeding q. We see the truth of the final statement in Theorem 4. 3 by noting that if each of the functions $F_1(z), \ldots, F_p(z)$ is of exponential type less than q then the equations (4. 10) imply that each of the functions $G_1(z), \ldots, G_p(z)$ is of exponential type less than q. This completes the proof of Theorem 4. 3.

Using the results of section 2. 4, we can treat the case when the functions $G_1(z), \ldots, G_p(z)$ belong to the class of meromorphic functions considered in section 2. 4. Furthermore, if $E(t)$ is such that for some σ the relation $\underline{b}_\sigma < 0 < \bar{b}_\sigma$

[1] The results contained in the final paragraph of Theorem 4. 3 were suggested by similar results obtained by CARMICHAEL, *loc. cit.*, p. 7.

holds, we can treat the case when the functions $G_1(z), \ldots, G_p(z)$ are any rational functions. We omit the statements of these results.

We conclude this part with a theorem relating to the case when the functions $G_1(z), \ldots, G_p(z)$ are asymptotically constant in a given sector. As in section 2.5 we impose an additional restriction on the function $E(t)$.

Theorem 4.4 (2.5)$'$. *Let $E(t)$ be such that $\bar{b}_0 > 0$. Let $G_1(z), \ldots, G_p(z)$ be analytic in $S\left(-\delta \leqq \theta \leqq \delta\left(0 < \delta \leqq \frac{\pi}{2}\right),\ |z| > R\right)$ and such that*

$$\lim_{z=\infty} G_j(z) = a_{0j}, \qquad z \text{ in } S, \qquad j = 1, \ldots, p. \tag{4.28}$$

Then the functions $F_1(z), \ldots, F_p(z)$ defined by (4.26) (for $|z|$ sufficiently large in S) are analytic for all values of z for which $z - \delta_0$ is in S and they represent a set of solutions of the non-singular system (4.10) for which

$$\lim_{z=\infty} F_\mu(z) = \sum_{j=1}^{p} e_{j\mu}(0)\, a_{0j} \sum_{n=0}^{\infty} \gamma_{\lambda_n}^{0}, \qquad z \text{ in } S, \qquad \mu = 1, \ldots, p. \tag{4.29}$$

There is only one set of analytic solutions of (4.10) for which the limit of each function as $z \to \infty$ in S exists.

If, moreover, the functions $G_1(z), \ldots, G_p(z)$ possess the asymptotic expansions

$$G_j(z) \sim a_{0j} + \frac{a_{1j}}{z} + \frac{a_{2j}}{z^2} + \cdots, \qquad z \text{ in } S, \qquad j = 1, \ldots, p, \tag{4.30}$$

then the functions in (4.18) and the functions $F_1(z), \ldots, F_p(z)$ possess asymptotic expansions in S. If we write

$$\sum_{j=1}^{p} e_{j\mu}[G_j] \sim \alpha_{0\mu} + \frac{\alpha_{1\mu}}{z} + \frac{\alpha_{2\mu}}{z^2} + \cdots, \qquad z \text{ in } S, \qquad \mu = 1, \ldots. p, \tag{4.31}$$

then the asymptotic expansions of the functions $F_1(z), \ldots, F_p(z)$ have the form

$$F_\mu(z) \sim b_{0\mu} + \frac{b_{1\mu}}{z} + \frac{b_{2\mu}}{z^2} + \cdots, \qquad z \text{ in } S, \qquad \mu = 1, \ldots, p, \tag{4.32}$$

where

$$b_{0\mu} = \alpha_{0\mu} \sum_{n=0}^{\infty} \gamma_{\lambda_n}^{0} = \sum_{j=1}^{p} e_{j\mu}(0)\, a_{0j} \sum_{n=0}^{\infty} \gamma_{\lambda_n}^{0}, \qquad \mu = 1, \ldots, p, \tag{4.33}$$

and

$$(4.\ 34) \qquad b_{\nu\mu} = \sum_{\varrho=1}^{\nu} \alpha_{\varrho\mu} \binom{\nu - 1}{\varrho - 1} \sum_{n=0}^{\infty} \gamma^{0}_{\lambda_n} (-\lambda_n)^{\nu-\varrho}, \qquad \nu = 1, 2, \ldots; \ \mu = 1, \ldots, p.$$

From (4. 28) we see that each of the functions in (4. 18) also possesses a limit as $z \to \infty$ in S. As a consequence we may apply Theorem 2. 5 to each of the equations (4. 13) and we see that the functions $F_1(z), \ldots, F_p(z)$ are analytic for all z for which $z - \delta_0$ is in S and that they satisfy (4. 29). It is easily verified as in Lemma 4. 1 that these functions constitute a set of solutions of the system (4. 10) and that they form the only analytic set for which the relations (4. 29) hold, in fact, that they form the only set of analytic solutions for which the limit of each function exists as $z \to \infty$ in S. If (4. 30) holds, then each of the functions in (4. 18) possesses an asymptotic expansion in S and, applying again Theorem 2. 5 to each of the equations (4. 13), we see that each of the functions $F_1(z), \ldots, F_p(z)$ possesses an asymptotic expansion of the form (4. 32).

V. The System with Asymptotically Constant Coefficients.

5. 1. In this part we shall consider the system of difference equations

$$(5.\ 1) \qquad \sum_{k=1}^{p} \left\{ \sum_{s=0}^{m_{jk}} c_s^{jk}(z) F_k(z + \delta_s^{jk}) \right\} = G_j(z), \qquad j = 1, \ldots, p,$$

in which the coefficients are asymptotically constant in a given sector and the δ_s^{jk} are non-negative numbers ordered as follows:

$$(5.\ 2) \qquad 0 \leqq \delta_0^{jk} < \delta_1^{jk} < \delta_2^{jk} < \cdots < \delta_{m_{jk}}^{jk}, \qquad j, k = 1, \ldots, p.$$

Denoting by c_s^{jk} the asymptotic values of the functions $c_s^{jk}(z)$, we associate with the system (5. 1) the functions $E_{jk}(t)$ and $E(t)$ defined by (4. 6) and (4. 7). We consider only those systems for which the determinant $E(t)$ does not vanish identically in t and we write $E(t)$ in the form (4. 2), where the c_s are non-zero complex constants, the δ_s are non-negative numbers and m is a non-negative integer. The problem being trivial if m is zero, we assume that m is greater than zero. We will use the symbolic treatment of section 4. 1 and the notation of section 2. 1 relating to the function $E(t)$, etc. We will also use the following abbreviations:

$$\Psi_{jk}(F) \equiv \sum_{s=0}^{m_{jk}} (c_s^{jk} - c_s^{jk}(z)) F(z + \delta_s^{jk}), \qquad j, k = 1, \ldots, p. \tag{5.3}$$

5.2. We next prove the following theorem:

Theorem 5.1 (3.1)$'$. *Let us consider the system* (5.1) *where the* $c_s^{jk}(z)$ *are analytic in* $S\left(-\delta \leqq \theta \leqq \delta \left(0 < \delta \leqq \frac{\pi}{2}\right), |z| > R\right)$ *and such that*

$$\lim_{z=\infty} c_s^{jk}(z) = c_s^{jk}, \quad z \text{ in } S, \qquad s = 0, 1, \ldots, m_{jk};\ j, k = 1, \ldots, p. \tag{5.4}$$

Let us assume that the system (5.1) *is such that*

$$\delta_0^{jk} = 0, \qquad j, k = 1, \ldots, p, \tag{5.5}$$

that the functional determinant

$$\Delta(z) \equiv |c_0^{jk}(z)|$$

is non-vanishing in S *and that the determinant*

$$\Delta = |c_0^{jk}|$$

is different from zero.[1] *If* $G_1(z), \ldots, G_p(z)$ *are analytic in* S *and such that for every pair of real numbers* $\alpha, \beta\,(\alpha < \beta)$ *there exists a constant* $A = A(\alpha, \beta)$ *of such a sort that we have*

$$|G_j(z)| \leqq A\, e^{bx}, \quad z \text{ in } S, \quad \alpha \leqq y \leqq \beta, \qquad j = 1, \ldots, p, \tag{5.6}$$

where $b < \bar{b}_0$, *then by successive solutions of the systems*

$$\sum_{k=1}^{p} E_{jk}[H_k^0] = G_j(z), \qquad j = 1, \ldots, p, \tag{5.7}$$

$$\sum_{k=1}^{p} E_{jk}[H_k^{\nu+1}] = \sum_{k=1}^{p} \Psi_{jk}(H_k^{\nu}), \qquad j = 1, \ldots, p, \qquad \nu = 0, 1, 2, \ldots, \tag{5.8}$$

we are led to the functions

$$F_\mu(z) = H_\mu^0(z) + H_\mu^1(z) + H_\mu^2(z) + \cdots, \qquad \mu = 1, \ldots, p, \tag{5.9}$$

[1] In view of (5.5) the condition that Δ be different from zero implies that the function $E(t)$ defined in (4.7) does not vanish identically in t.

which constitute a set of solutions of the system (5. 1). *The functions* $F_1(z), \ldots, F_p(z)$ *are analytic in* S *and are such that there exists a constant* $\tilde{A} = \tilde{A}(\alpha, \beta)$ *of such a sort that we have*

$$(5.\ 10) \qquad |F_\mu(z)| \leqq \tilde{A}\, e^{bx}, \qquad z \text{ in } S, \qquad \alpha \leqq y \leqq \beta. \qquad \mu = 1, \ldots, p.$$

There exists only one set of analytic solutions of (5. 1) *which satisfies relations of the form* (5. 10) *with an exponent* $b < \bar{b}_0$.

For the proof we use Theorem 3. 1 and Lemma 4. 1. As usual, we make a slight modification in Lemma 4. 1 to treat the case when the region S is used, such a modification being possible since we are using the interval J_0. By this modified form of Lemma 4. 1 we see that the functions

$$(5.\ 11) \qquad H_\mu^0(z) = \Gamma^0 \sum_{j=1}^{p} e_{j\mu}[G_j], \qquad \mu = 1, \ldots, p,$$

are analytic in S and that they constitute a set of solutions of the system (5. 7) of such a sort that we have

$$(5.\ 12) \qquad |H_\mu^0(z)| \leqq A\, B\, C e^{bx}, \qquad z \text{ in } S, \qquad \alpha \leqq y \leqq \beta, \qquad \mu = 1, \ldots, p.$$

Recalling the form of $\Psi_{jk}(F)$ and using (5. 4) and (5. 12), we see that for every positive number ε there exists an x_ε such that we have

$$(5.\ 13) \qquad \left| \sum_{k=1}^{p} \Psi_{jk}(H_k^0) \right| \leqq A\, B\, C e^{bx}(\varepsilon D), \quad z \text{ in } S, \quad \alpha \leqq y \leqq \beta, \qquad |z| > x_\varepsilon,$$

where D is any constant such that

$$(5.\ 14) \qquad D \geqq \sum_{k=1}^{p} \sum_{s=0}^{m_{jk}} e^{\delta_s^{jk}}, \qquad j = 1, \ldots, p.$$

Proceeding as before, we obtain a set of solutions $H_1^1(z), \ldots, H_p^1(z)$ of the system (5. 8) for $\nu = 0$, each function of which is analytic in that part of S for which $|z| > x_\varepsilon$ and each one of which satisfies a relation of the form

$$(5.\ 15) \qquad |H_\mu^1(z)| \leqq A\, B\, C e^{bx}(\varepsilon\, B\, C D), \qquad z \text{ in } S, \qquad \alpha \leqq y \leqq \beta, \qquad |z| > x_\varepsilon.$$

Continuing this process, we see that the functions $H_1^{\nu+1}(z), \ldots, H_\nu^{\nu+1}(z)$ are analytic in S for $|z| > x_\varepsilon$, that they form a set of solutions of (5. 8) and that they satisfy the relations

$$(5.\ 16)\quad \left|H_\mu^{\nu+1}(z)\right| \leqq A\,B\,C\,e^{bx}(\varepsilon\,B\,C\,D)^{\nu+1}, \qquad z \text{ in } S, \qquad \alpha \leqq y \leqq \beta,$$
$$|z| > x_\varepsilon, \qquad \nu = 0,\ 1,\ 2, \ldots$$

If ε_0 is a positive constant such that $\varepsilon_0 B\,C\,D < 1$, then the series in (5. 9) converge absolutely and uniformly in every finite subregion of that part of S for which $\alpha \leqq y \leqq \beta$, $|z| > x_{\varepsilon_0}$, and the sum-functions $F_1(z), \ldots, F_p(z)$ are analytic in that part of S for which $|z| > x_{\varepsilon_0}$. Moreover these functions verify the relations

$$(5.\ 17)\qquad |F_\mu(z)| \leqq A\ C'\,e^{bx}, \qquad z \text{ in } S, \qquad \alpha \leqq y \leqq \beta, \qquad |z| > x_{\varepsilon_0},$$

where $C' = B\,C/(1 - \varepsilon_0 B\,C\,D)$. By means of these properties and equations (5. 7) and (5. 8) we see at once that the functions $F_1(z), \ldots, F_p(z)$ form a set of solutions of the system (5. 1). Using the relations in (5. 1) and (5. 5), we may write

$$\sum_{k=1}^{p} c_0^{jk}(z)\,F_k(z) = G_j(z) + \sum_{k=1}^{p}\sum_{s=1}^{m_{jk}} c_s^{jk}(z)\,F_k(z + \delta_s^{jk}), \qquad j = 1, \ldots, p.$$

Using the fact that the functional determinant $\Delta(z)$ is non-vanishing in S, we see that we can continue the solutions analytically and accordingly, we see that the functions $F_1(z), \ldots, F_p(z)$ are analytic for all z in S. Since they are analytic in S and since they satisfy the relations (5. 17) for $|z| > x_{\varepsilon_0}$ they satisfy relations of the form (5. 10).

In order to prove the uniqueness we proceed as we did in the proof of Theorem 3. 1. We assume that the functions $F_{1*}(z), \ldots, F_{p*}(z)$ constitute a set of analytic solutions of the system (5. 1) verifying relations of the form (5. 10). We determine successively the sets of functions

$$J_\mu^0(z),\ J_\mu^1(z),\ J_\mu^2(z), \ldots, \qquad \mu = 1, \ldots, p,$$

as »the» sets of solutions of the systems

$$(5.\ 18)\qquad \sum_{k=1}^{p} E_{jk}[J_k^0] = \sum_{k=1}^{p} \Psi_{jk}(F_{k*}), \qquad j = 1, \ldots, p,$$

$$(5.\ 19)\qquad \sum_{k=1}^{p} E_{jk}[J_k^{\nu+1}] = \sum_{k=1}^{p} \Psi_{jk}(J_k^\nu), \quad j = 1, \ldots, p, \qquad \nu = 0,\ 1,\ 2, \ldots$$

In the same manner in which we derived the relations (5. 15) and (5. 16) we derive the relations

$$(5.20)\qquad \left|J_\mu^\nu(z)\right| \leqq \tilde{A}\,B\,C\,e^{bx}(\varepsilon_0 B\,C\,D)^\nu, \qquad z \text{ in } S, \qquad \alpha \leqq y \leqq \beta,$$
$$|z| > x_{\varepsilon_0}, \qquad \nu = 0, 1, 2, \ldots,$$

Accordingly, we may write the functions $F_{1*}(z), \ldots, F_{p*}(z)$ in the form

$$(5.21)\qquad F_{\mu*} = H^0_{\mu*} + H^1_{\mu*} + H^2_{\mu*} + \cdots, \quad z \text{ in } S, \quad |z| > x_{\varepsilon_0}, \quad \mu = 1, \ldots, p,$$

where

$$(5.22)\qquad H^0_{\mu*} = F_{\mu*} - J^0_\mu, \quad H^{\nu+1}_{\mu*} = J^\nu_\mu - J^{\nu+1}_\mu, \quad \mu = 1, \ldots, p; \qquad \nu = 0, 1, 2, \ldots$$

The sets of functions $H^\nu_{\mu*}, \ldots, H^\nu_{p*}$ are sets of solutions of the systems (5. 7) and (5. 8) and moreover they verify bounds of the form (5. 15) and (5. 16). By Lemma 4. 1 we see that

$$H^\nu_{\mu*} = H^\nu_\mu, \qquad \mu = 1, \ldots, p; \qquad \nu = 0, 1, 2, \ldots,$$

and hence

$$F_{\mu*} = \sum_{\nu=0}^{\infty} H^\nu_\mu = F_\mu, \qquad \mu = 1, \ldots, p.$$

This completes the proof of Theorem 5. 1.

There is an analogous theorem when the region S is replaced by a region $S_1\left(\pi - \delta \leqq \theta \leqq \pi + \delta\left(0 < \delta \leqq \frac{\pi}{2}\right), |z| > R\right)$. Because of its similarity to Theorem 5. 1 we omit its statement.

The theorem which we next prove is analogous to Theorems 3. 2 and 4. 4.

Theorem 5. 2 (3. 2)$'$. *Let the functions $c_s^{jk}(z)$ in the system (5. 1) be analytic in $S\left(-\delta \leqq \theta \leqq \delta\left(0 < \delta \leqq \frac{\pi}{2}\right), |z| > R\right)$ and such that (5. 4) holds. Let the system (5. 1) be such that (5. 5) holds, such that the functional determinant $\Delta(z)$ is non-vanishing in S, such that the determinant Δ is different from zero, and such that $\bar{b}_0 > 0$. If the functions $G_1(z), \ldots, G_p(z)$ are analytic in S and such that*

$$(5.23)\qquad \lim_{z=\infty} G_j(z) = a_{0j}, \qquad z \text{ in } S, \qquad j = 1, \ldots, p,$$

then by successive solutions of the systems (5. 7) and (5. 8) we are led to the functions $F_1(z), \ldots, F_p(z)$ defined by (5. 9) which constitute a set of solutions of the system (5. 1). The functions $F_1(z), \ldots, F_p(z)$ are analytic in S and are such that

$$(5.\,24)\qquad \lim_{z=\infty} F_\mu(z) = \sum_{j=1}^{p} e_{j\mu}(0)\, a_{0j} \sum_{n=0}^{\infty} \gamma^0_{\lambda_n}, \quad z \text{ in } S, \qquad \mu = 1, \ldots, p.$$

There is only one set of analytic solutions of the system (5. 1) *for which the limit of each function exists as* $z \to \infty$.

If moreover the functions $G_j(z)$ *and* $c^{jk}_s(z)$ *possess the asymptotic expansions*

$$(5.\,25)\qquad G_j(z) \sim a_{0j} + \frac{a_{1j}}{z} + \frac{a_{2j}}{z^2} + \cdots, \quad z \text{ in } S, \qquad j = 1, \ldots, p,$$

$$(5.\,26)\qquad c^{jk}_s(z) \sim c^{jk}_s - \frac{c^{jk}_{1s}}{z} + \frac{c^{jk}_{2s}}{z^2} - \cdots, \quad z \text{ in } S, \qquad \begin{array}{l} s = 0, 1, \ldots, m_{jk}, \\ j, k = 1, \ldots, p, \end{array}$$

then the functions $H^\nu_1(z), \ldots, H^\nu_p(z)$ *possess asymptotic expansions of the form*

$$(5.\,27)\qquad H^\nu_\mu(z) \sim \frac{b^\nu_{\mu,\nu}}{z^\nu} + \frac{b^\nu_{\mu,\nu+1}}{z^{\nu+1}} + \frac{b^\nu_{\mu,\nu+2}}{z^{\nu+2}} + \cdots, \quad z \text{ in } S, \qquad \begin{array}{l} \mu = 1, \ldots, p, \\ \nu = 0, 1, 2, \ldots, \end{array}$$

and the functions $F_1(z), \ldots, F_p(z)$ *possess the asymptotic expansions*

$$(5.\,28)\qquad F_\mu(z) \sim b_{\mu 0} + \frac{b_{\mu 1}}{z} + \frac{b_{\mu 2}}{z^2} + \cdots, \quad z \text{ in } S, \qquad \mu = 1, \ldots, p,$$

where

$$(5.\,29)\qquad b_{\mu 0} = b^0_{\mu 0} = \sum_{j=1}^{p} e_{j\mu}(0)\, a_{0j} \sum_{n=0}^{\infty} \gamma^0_{\lambda_n}, \qquad \mu = 1, \ldots, p,$$

$$(5.\,30)\qquad b_{\mu\nu} = b^0_{\mu\nu} + b^1_{\mu\nu} + \cdots + b^\nu_{\mu\nu}, \qquad \mu = 1, \ldots, p; \ \nu = 1, 2, \ldots$$

The sets of functions $H^0_\mu(z)$, $H^1_\mu(z)$, $H^2_\mu(z)$, $\ldots$, $\mu = 1, \ldots, p$, defined as »the» sets of solutions of the systems (5. 7) and (5. 8), are also sets of solutions of the systems

$$(5.\,31)\qquad E[H^0_\mu] = \sum_{j=1}^{p} e_{j\mu}[G_j], \qquad \mu = 1, \ldots, p,$$

$$(5.\,32)\qquad E[H^{\nu+1}_\mu] = \sum_{j=1}^{p} e_{j\mu}\left[\sum_{k=1}^{p} \Psi_{jk}(H^\nu_k)\right], \qquad \mu = 1, \ldots, p; \ \nu = 0, 1, 2, \ldots$$

From (5. 23) and from the form of the operators e_{jk} we see that

$$(5.\,33)\qquad \lim_{z=\infty} e_{j\mu}[G_j] = a_{0j}\, e_{j\mu}(0), \qquad z \text{ in } S,$$

and hence by Theorem 3. 2 we see that

$$\lim_{z=\infty} H_\mu^0(z) = \sum_{j=1}^{p} a_{0j}\, e_{j\mu}(0) \sum_{n=0}^{\infty} \gamma_{\lambda n}^0, \qquad z \text{ in } S, \qquad \mu = 1, \ldots, p. \tag{5.34}$$

From the form of $\Psi_{jk}(F)$ and from (5.4) and (5.34) we easily verify that

$$\lim_{z=\infty} e_{j\mu}\left[\sum_{k=1}^{p} \Psi_{jk}(H_k^0)\right] = 0, \qquad z \text{ in } S, \qquad \mu = 1, \ldots, p, \tag{5.35}$$

and as before we have

$$\lim_{z=\infty} H_\mu^1(z) = 0, \qquad z \text{ in } S, \qquad \mu = 1, \ldots, p. \tag{5.36}$$

By continuing this argument, we obtain the relations

$$\lim_{z=\infty} H_\mu^{\nu+1}(z) = 0, \quad z \text{ in } S, \qquad \mu = 1, \ldots, p;\ \nu = 0, 1, 2, \ldots, \tag{5.37}$$

and by making an analysis of the manner in which these limits are approached (similar to the argument made in the proof of the first part of Theorem 3.2), we show not only that the series

$$\sum_{\nu=0}^{\infty} H_\mu^{\nu+1}(z), \qquad \mu = 1, \ldots, p, \tag{5.38}$$

converge absolutely for $|z|$ sufficiently large in S but also that

$$\lim_{z=\infty} \sum_{\nu=0}^{\infty} H_\mu^{\nu+1}(z) = 0, \qquad z \text{ in } S, \qquad \mu = 1, \ldots, p. \tag{5.39}$$

The convergence of each of the series (5.38) is uniform in every finite subregion of the region considered. Hence the functions $F_1(z), \ldots, F_p(z)$, defined by (5.9), are analytic for $|z|$ sufficiently large in S, and since (5.34) and (5.39) hold, these functions satisfy (5.24). That they form a set of solutions of the system (5.1) and that the set is unique follow from Theorem 5.1 since any set of analytic functions $F_1(z), \ldots, F_p(z)$, for which the limit of each function exists as $z \to \infty$ in S, satisfies relations of the form (5.10) for $0 < b < \bar{b}_0$. From Theorem 5.1 it follows that the functions $F_1(z), \ldots, F_p(z)$ are analytic in S.

Finally we assume that (5.25) and (5.26) hold. Applying Theorem 2.5 to each of the equations (5.31), we see that each of the functions $H_1^0(z), \ldots, H_p^0(z)$ possesses an asymptotic expansion in S. Using Theorem 2.5 and the form of the operators e_{jk}, we can easily compute the form of these asymptotic expansions. We omit the computation and simply write them in the form (5.27), for $\nu = 0$. Using (5.9) and (5.32) and applying successively the preceding argument, we see that the functions $H_1^\nu(z), \ldots, H_p^\nu(z)$, $\nu = 1, 2, 3, \ldots$, possess asymptotic expansions in S and that these expansions have the forms indicated in (5.27). In order to see that the functions $F_1(z), \ldots, F_p(z)$ have the asymptotic expansions denoted in (5.28) we write, for h a fixed non-negative integer,

$$(5.40) \qquad z^h\left\{F_\mu(z) - b_{\mu 0} - \cdots - \frac{b_{\mu h}}{z^h}\right\} = z^h \sum_{l=0}^{h} \left\{H_\mu^l(z) - \frac{b_{\mu l}^l}{z^l} - \cdots - \frac{b_{\mu h}^l}{z^h}\right\} + z^h \sum_{n=1}^{\infty} H_\mu^{h+n}(z), \qquad \mu = 1, \ldots, p.$$

The relations (5.27) imply (by definition) that

$$(5.41) \qquad \lim_{z=\infty} z^h \sum_{l=0}^{\infty} \left\{H_\mu^l(z) - \frac{b_{\mu l}^l}{z^l} - \cdots - \frac{b_{\mu h}^l}{z^h}\right\} = 0, \quad z \text{ in } S, \qquad \mu = 1, \ldots, p,$$

$$(5.42) \qquad \lim_{z=\infty} z^h H_\mu^{h+n}(z) = 0, \qquad z \text{ in } S, \qquad \begin{matrix} \mu = 1, \ldots, p, \\ n = 1, 2, 3, \ldots \end{matrix}$$

Making an argument of the form made in the proof of Theorem 3.2, we see that the limits in (5.42) are such that

$$(5.43) \qquad \lim_{z=\infty} z^h \sum_{n=1}^{\infty} H_\mu^{h+n}(z) = 0, \qquad z \text{ in } S, \qquad \mu = 1, \ldots, p.$$

From (5.40), (5.41) and (5.43) we have

$$(5.44) \qquad \lim_{z=\infty} z^h\left\{F_\mu(z) - b_{\mu 0} - \cdots - \frac{b_{\mu h}}{z^h}\right\} = 0, \quad z \text{ in } S, \qquad \mu = 1, \ldots, p.$$

Since these relations hold for every non-negative integer h, the functions $F_1(z), \ldots, F_p(z)$ possess the asymptotic expansions (5.28), where the coefficients are given in (5.29) and (5.30). This completes the proof of Theorem 5.2.

An analogous theorem holds when the region S is replaced by a region $S_1\left(\pi - \delta \leqq \theta \leqq \pi + \delta \left(0 < \delta \leqq \frac{\pi}{2}\right), \; |z| > R\right)$. Because of its similarity to Theorem 5.2 we omit its statement.

Princeton University and The Institute for Advanced Study, Princeton, New Jersey, February 16, 1935.