THE CLOSEST PACKING OF CONVEX TWO-DIMENSIONAL DOMAINS, CORRIGENDUM

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Sometime ago I published an account(1), in outline, of certain results, which had been anticipated and largely superseded by work of L. Fejes Tóth(2). I now find that it is necessary to correct one of the results.

Let K be an open convex two-dimensional set. A system $K+\mathbf{a}_1$, $K+\mathbf{a}_2$, ... of translates of K by vectors $\mathbf{a}_1$, $\mathbf{a}_2$, ... is called a packing, if no two of the sets have any point in common. Let $d(K)$ denote the lower bound of the determinants of the lattices Λ, with the property that the system of translates of K by the vectors of Λ forms a packing.

My 1951 paper only proves

THEOREM 1a. *Let K and S be any open bounded convex sets with areas $a(K)$ and $a(S)$. Let K be symmetrical. If n sets K can be packed into S (with $n \geqslant 1$), then*

$$(n-1)\, d(K) + a(K) \leqslant a(S).$$

It incorrectly claims to prove such a result without the supposition that K should be symmetrical. No restriction to symmetrical sets is needed in Theorem 2, nor in the main conclusion that it is impossible to find a packing of similarly orientated congruent convex domains, which is closer than the closest lattice packing of the domains.

The error arises in the proof of Lemma 5; there is no justification for the assertion that it is permissible to suppose that the point $\frac{1}{2}(\mathbf{c}+\mathbf{d})$ coincides with the origin, since in this lemma a change of origin changes the area of the polygon Π. It is easy to see that this movement of the origin increases the area of Π by $\frac{1}{2}|\mathbf{b}-\mathbf{a}|\cdot(h_2-h_1)$,

(1) *Acta Mathematica*, 86 (1951), 309–321.
(2) *Acta Sci. Math. (Szeged)*, 12 (1950), 62–67.

where $|\mathbf{b}-\mathbf{a}|$ is the length of the segment $\mathbf{a}\mathbf{b}$ and h_1, h_2 are the perpendicular distances from the origin, supposed in K, to the tac-lines to K parallel to the line $\mathbf{a}\mathbf{b}$.

Consequently, in Lemma 5, the inequality (8) needs to be replaced by

$$(m+\tfrac{1}{2}n-\tfrac{1}{2})\,d(K)\leqslant a(\Pi)+\tfrac{1}{2}\,|\mathbf{b}-\mathbf{a}|\,(h_2-h_1). \tag{8 a}$$

In order to state the appropriately revised version of Theorem 1, it is convenient to introduce the mixed area of two convex domains. For our purpose, it is sufficient to define the mixed area $a(P, K)$ of two convex domains P and K, by writing

$$a(P+K)=a(P)+2\,a(P,K)+a(K),$$

where $P+K$ denotes the vector sum of the sets P and K. We indicate how the following result may be established.

THEOREM 1 b. *Let K and P be any open bounded convex sets. If $\mathbf{a}_1, \ldots, \mathbf{a}_n$ are n points of P (with $n\geqslant 1$), and the sets $K+\mathbf{a}_1, \ldots, K+\mathbf{a}_n$ form a packing, then*

$$(n-1)\,d(K)\leqslant a(P)+a(P,K)+a(P,-K). \tag{2 b}$$

Note that when K is symmetrical this inequality reduces to that of Theorem 1 a.

If Q is any convex set strictly included in P, we have

$$a(Q)+a(Q,K)+a(Q,-K)<a(P)+a(P,K)+a(P,-K);$$

it follows, essentially as in the 'proof' of Theorem 1 given before, that we may suppose that P is minimal. As before this implies that the sets $K+\mathbf{a}_1, K+a_2, \ldots, K+\mathbf{a}_n$ must touch each other in the way described. This enables us to apply the corrected form of Lemma 5 and to obtain the inequality (2 b) without difficulty.

Recently N. Oler (1) and H. L. Davies (2) have obtained results on the packing of convex symmetrical domains which are significantly better than Theorem 1 a. While it should be possible to use these results to obtain an improvement of Theorem 1 b, it does not seem to be easy to combine elegance with refinement.

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(1) To appear in Acta Mathematica, vol. 105.

(2) Personal communication.