ON THE THEORY OF HARMONIC FUNCTIONS OF SEVERAL VARIABLES

II. Behavior near the boundary

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Contents

Introduction (1)

This paper is a continuation of the series begun in [9]. Here, as in the previous paper, we are concerned with the following problem: To extend, as far as possible to the general case of several variables, properties of harmonic functions in two variables which result from their close connection to analytic functions in one variable.

(1) The main results of this paper were announced in abstracts no. 566–35 and 566–36, Notices of the A.M.S., 1960.

10 – 61173060. *Acta mathematica*. 106. Imprimé le 20 décembre 1961.

We shall be concerned with the local behavior of harmonic functions near the boundary. To explain the main ideas of this paper we begin by recalling some results from the classical case.

There we deal with a function $u(x, y)$ harmonic in the upper-half plane $y > 0$. We are concerned with the behavior of $u(x, y)$ near the x-axis, or more precisely, near a general measurable set E located on the x-axis. The study of this behavior is intimately related with that of the conjugate function $v(x, y)$, and thus the analytic function $F(z) = u + iv$, $z = x + iy$. A basic concept in this connection is that of a "non-tangential" limit at a point $(x, 0)$ located on the x-axis. The results of the "local theory" in the classical setup which concern us are then: (1)

(A) *$u(x, y)$ has a non-tangential limit for a.e. $x \in E$ if $u(x, y)$ is non-tangentially bounded for a.e. $x \in E$.*

(B) *If $u(x, y)$ has a non-tangential limit for a.e. $x \in E$ then the same is true for $v(x, y)$, and conversely.* (2)

The property of having a non-tangential limit (or more generally of being non-tangentially bounded) is of an elusive nature and thus difficult to pin down analytically. It is therefore desirable to reexpress this property in a more tractable but logically equivalent form. This restatement may be accomplished from results of Marcinkiewicz and Zygmund and Spencer. We shall use the following definition. $\Gamma(x_0)$ will denote a standard triangular neighborhood which lies in the upper half plane and whose vertex is at the point $(x_0, 0)$. More precisely,

$$\Gamma(x_0) = \{(x, y): |x - x_0| < \alpha y, \ 0 < y < h\}$$

for two fixed constants α and h. We then define the so-called area integral

$$A(x_0) = \iint\limits_{\Gamma(x_0)} \left\{ \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial x}\right)^2 \right\} dx\,dy$$

with represents the area (points counted according to their multiplicity) of the image of $\Gamma(x_0)$ under $F(z) = u + iv$. The theorem of Marcinkiewicz, Zygmund and Spencer can be stated, in this context, as follows:

(C) *$u(x, y)$ has a non-tangential limit for a.e. $x \in E$ if and only if the area integral $A(x)$ is finite for a.e. $x \in E$.*

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(1) We use the abbreviation a.e. throughout to mean "almost every" or "almost everywhere" with respect to Lebesgue measure.

(2) These and other results of the classical theory may be found in [12, Chap. 14], where references to the other original works may also be found.

It should be noted that in this form, the proposition (C) implies (B), because

$$\left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial x}\right)^2 = \left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2,$$

by the Cauchy–Riemann equations. We add here that the concept of "non-tangential" limits and the corresponding notion of non-tangential boundedness are basic for the conclusions (A), (B), and (C). For example, the approach to the boundary by the normal direction only would not do as a substitute notion. (1)

We now turn to the situation in any number of variables. The generalization of (A) to harmonic functions of several variables has been known for some time, see [1].

It is the purpose of this paper to obtain the extension of theorems (B) and (C) to several variables.

We begin by considering the extension of (C). If $u(X, y)$, $X = (x_1, x_2, \ldots, x_n)$ is harmonic in the upper half space $y > 0$, as a function of the $n+1$ variables (X, y), then we set

$$A(X_0) = \iint_{\Gamma(X_0)} y^{1-n} |\nabla u|^2 \, dX \, dy, \tag{*}$$

where $|\nabla u|^2 = |\partial u/\partial y|^2 + \sum_{k=1}^{n} |\partial u/\partial x_k|^2$, and $\Gamma(X_0)$ is the truncated "cone"

$$\{(X, y): |X - X_0| < \alpha y, \ 0 < y < h\},$$

for fixed α and h. In the following theorem E denotes an arbitrary measurable subset of E_n, where E_n is considered as the boundary hyper-plane of our half-space.

THEOREM 1. *In order that $u(X, y)$ have a non-tangential limit for a.e. $X \in E$, it is necessary and sufficient that the generalized area integral, $A(X)$, be finite for a.e. $X \in E$.*

The proof of the theorem, which is contained in section 4, is based on the elementary lemmas of Section 3. The necessity of the finiteness of the integral in (*) was previously known, see [2]. The method we use leads to a simplification of the proof of that part of the theorem. The sufficiency, which is our principal object, makes use of some similar ideas, but is more difficult. We add two remarks: (*a*) A different approach leading to the proof of Theorem 1 was found independently by Calderon (*b*). The generalized area integral was considered in a different context by us in [8].

By the use of Theorem 1 we can obtain a generalization of proposition (B) to

(1) See in particular the example in [11], Chap. 14, p. 204.

any number of variables. For this purpose let us recall the system of harmonic functions considered in Chapter I. The so-called Riesz system is made up of $n+1$ functions, $u, v_1, v_2, \ldots, v_n$, satisfying

$$\frac{\partial u}{\partial y}+\sum_{k=1}^{n}\frac{\partial v_k}{\partial x_k}=0, \quad \frac{\partial u}{\partial x_k}=\frac{\partial v_k}{\partial y}, \quad \frac{\partial v_k}{\partial x_j}=\frac{\partial v_j}{\partial x_k}.$$

It can be characterized alternatively as arising as the gradient of a harmonic function $H(x; y)$; that is

$$(u, v_1, v_2, \ldots, v_n)=\left(\frac{\partial H}{\partial y}, \frac{\partial H}{\partial x_1}, \frac{\partial H}{\partial x_2}, \ldots, \frac{\partial H}{\partial x_n}\right).$$

Our generalization of Chapter II can then be stated as follows (see Section 7).

THEOREM 4. *If u has a non-tangential limit for a.e. $X \in E$, then so do the conjugates $v_1, v_2, \ldots, v_n$, and conversely.*

It must be remarked that this theorem does *not* follow directly from Theorem 1, as in the case $n=1$. This is due to the fact that if $n>1$ then there is no simple appropriate relation between $\sum_{k=1}^{n}|\nabla v_k|^2$ and $|\nabla u|^2$.

Thus an extra step is needed to deduce Theorem 4. This step is given in Section 5, and it allows us to obtain a wide generalization of Theorem 4. The nature of this generalization may be understood as follows. The system of harmonic functions satisfying the M. Riesz equations above represents one possible extension of the Cauchy-Riemann equations to several variables. There are other generalizations—although less direct—which are of importance. Some of these systems are discussed in Sections 7 and 8. A systematic discussion of these extensions cannot be given here, but will be the subject of a future paper in this series. Without discussing the general problem exhaustively, we can give a definition of conjugacy—which although tentative in nature—is significant technically in view of its inclusiveness and its applicability.

We shall say the harmonic function $u(X, y)$ *is conjugate to* $v(X, y)$ if there exists a positive integer r and a differential polynomial $P(D)$ homogeneous of degree r in $\partial/\partial y, \partial/\partial x_1, \ldots, \partial/\partial x_2$ (with constant coefficients) so that u and v are related by

$$\frac{\partial^r u}{\partial y^r}=P(D)\,v.$$

This definition can be extended to the case when u and v are respectively vectors of harmonic functions with k and m components, and $P(D)$ is then a $k\times m$ matrix whose entries are differential operators of the type described. Our generalization of Theorem 4 is then (see Section 6).

THEOREM 3. *If* $v(X, y)$ *has a non-tangential limit for a.e.* $X \in E$ *then so does* $u(X, y)$.

Examples illustrating this notion of conjugacy and Theorem 3, are given in Section 7. In Section 8 the meaning of this conjugacy is further examined in terms of harmonic functions which are Poisson integrals. It then turns out that this notion is equivalent with that arising from singular integrals (i.e., generalization of the Hilbert transform) whose "symbols", when restricted to the unit sphere, are (harmonic) polynomials. This fact is summarized in Theorem 7 of Section 8 below.

We wish now to discuss briefly the possibility of further extensions of the above. The first generalization is immediate: we need not assume that our functions are defined and harmonic in the entire upper-half space, but only in an appropriate region about our set E. For example, we could restrict our consideration to the "cylinder" $\{X, y): X \in E_1, 0 < y < h_2\}$ where E_1 is the set of all points at distance not greater than h_1 from E, and h_1, h_2 are two fixed positive constants. In all our proofs below we actually do not go outside such a cylinder, and we shall therefore assume once and for all that all our theorems are considered with this slight unstated generalization in mind.

Our sets E lie on the boundary, which is a hyper-plane ($y = 0$). It would be desirable to extend these results by considering non-tangential behavior for sets lying on more general hyper-surfaces. Presumably this could be done without too much difficulty if the bounding hyper-surface were smooth enough. It would be of definite interest, however, to allow the most general bounding hyper-surface for which non-tangential behavior is meaningful. Hence, extension of these results to the case when the bounding surfaces are, for example, of class C^1 would have genuine merit. Whether this can be done is an open problem.

Chapter I

The main purpose of this chapter is the proof of Theorem 1 in Section 4. Section 1 contains various definitions and statements of known facts. Section 2 deals with a technical device useful for the proof of Theorem 1. Section 3 contains several lemmas needed in the proof of the theorem.

1. Preliminaries

We shall follows as far as possible the notation of the previous paper in this series, which we now summarize.

E_n will denote the Euclidean space of n dimensions. Points in this space will be denoted by capital letters X, X_0, Y, Z, and in coordinate notation we will set $X=(x_1, x_2, \ldots, x_n) \ldots$ etc. E_{n+1}^+ will denote the Euclidean $n+1$ dimensional upper half space: Its points will be denoted by the pair (X, y), where $X \in E_n$ and $0<y<\infty$. For $X \in E_n$, we have $(X, 0) \in E_{n+1}$, thus we consider E_n as embedded in E_{n+1} as the boundary hyper-plane of E_{n+1}^+.

We shall also use the following convention. Integrals over an $(n+1)$ dimensional subspace of E_{n+1}^+ will be denoted by double integrals, such as $\iint(\cdot)\,dx\,dy$. If we integrate over an n-dimensional subset, such as over E_n, we shall indicate this by a single integral like $\int(\cdot)\,dX$.

$m(E)$ will denote the n-dimensional measure of a set in E_n (all sets occurring will be assumed to be measurable). σ and σ' will denote points in E_{n+1}^+, and Σ will denote a sphere whose center is σ.

Let X_0 denote a point in E_n. We denote by $\Gamma(X_0)$ the interior of a truncated cone in E_{n+1}^+ with vertex at X_0. Thus

$$\Gamma(X_0)=\{(X, y):\ |X-X_0|<\alpha y,\ 0<y<h\},$$

for some fixed α and h. When we wish to indicate the parameters α and h we shall write

$$\Gamma(X_0)=\Gamma(X_0;\, \alpha, h).$$

In what follows we shall refer to the interior of truncated cones simply as *cones*.

For any set $E \subset E_n$, and α and h fixed we shall associate a region R in E_{n+1}^+. The region R is the union of all cones $\Gamma(X_0;\, \alpha, h)$ where X_0 ranges over the points of E. Thus

$$R=\bigcup_{X_0 \in E} \Gamma(X_0;\, \alpha, h).$$

The following two lemmas are known and we take them for granted. The first is of an elementary character; the second, however, is deep. (1)

LEMMA 1. *Let $u(X, y)$ be continuous in E_{n+1}^+. Suppose we are given a bounded set $E_0 \subset E_n$ with the following property. Whenever $X_0 \in E_0$, $u(X, y)$ is bounded as (X, y) ranges in some cone $\Gamma(X_0)$. (The shape of the cone and bound may depend on X_0.) For any $\varepsilon>0$, then there exists a closed subset E, $E \subset E_0$. so that*

(1) The first lemma is contained, although not stated explicitly, in [1]. The second lemma, in a more general form, is the main result of that paper.

(1) $m(E_0 - E) < \varepsilon$.

(2) *If* α *and* h *are fixed,* $u(X, y)$ *is uniformly bounded in* $R = \bigcup_{X_0 \in E} \Gamma(X_0; \alpha, h)$.

For the statement of the next lemma we shall need the following definition. If $u(X, y)$ is defined in E^+_{n+1}, we shall say that it has a *non-tangential limit* l at the point X_0, $(X_0 \in E_n)$, if for every fixed α, $u(X, y) \to l$, as $y \to 0$, with $|X - X_0| < \alpha y$.

LEMMA 2. *Suppose that* $u(X, y)$ *is harmonic (as a function of the* $n+1$ *variables) in* E^+_{n+1}, *and that for every point* X_0 *belonging to a set* $E \subset E_n$, $u(x, y)$ *is bounded in a cone* $\Gamma(X_0)$ *whose vertex is at* X_0. *Then* $u(X, y)$ *has a non-tangential limit for a.e. point* $X_0 \in E$.

2. Regularization of the region R

Given a closed bounded subset E of E_n and fixed positive quantities α and h we associate with it, as before, the open region $R = \bigcup_{X_0 \in E} \Gamma(X_0; \alpha, h)$. It is to be noted that the region R is not necessarily connected. [1]

We add a marginal comment. This type of region has been considered for some time in the study of non-tangential behavior of harmonic functions, especially when $n = 1$. In that case the boundary of R is a rectifiable curve and thus the study of harmonic functions in R is greatly facilitated by the use of conformal transformations of R. [2] Needless to say, these considerations are not applicable in the general case.

The boundary B of R consists of two pieces, $B = B^1 \cup B^2$. To describe them we introduce the distance function $d(X, E)$ = distance of X from E. Then B^1 is the "surface" $y = \alpha^{-1} \cdot d(X, E)$, lying over those points X so that $d(X, E) < \alpha h$. B^2 is that portion of the hyperplane, $y = h$, lying over those X for which $d(X, E) \leqslant \alpha h$.

A basic step in the argument that follows is the application of Green's Theorem to certain integrals extended over the region R. This requires that we approximate our given region by a family of smooth regions for which Green's Theorem is applicable. This is accomplished in the lemma below.

LEMMA 3. *There exists a family of regions* R_ε, $\varepsilon > 0$, *with the following properties.*

(1) $R_\varepsilon \subset R$

(2) $R_{\varepsilon_1} \subset R_{\varepsilon_2}$, *if* $\varepsilon_2 < \varepsilon_1$

(3) $R_\varepsilon \to R$ *as* $\varepsilon \to 0$ (i.e., $\bigcup R_\varepsilon = R$)

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[1] Even though it may be made connected by an inessential modification.

[2] See footnote (1) on p. 138.

(4) *the boundary* B_ε *of* R_ε *is at a positive distance from* E_n $(y=0)$, *and consists of two pieces* B_ε^1 *and* B_ε^2 *so that*

(5) B_ε^1 *is a portion of the surface* $y=\alpha^{-1}\cdot\delta_\varepsilon(X)$, *where* $\left|\dfrac{\partial\,\delta_\varepsilon(X)}{\partial x_k}\right|\leqslant 1$, $\varepsilon>0$, $k=1,\ldots,n$, *and* $\delta_\varepsilon(X)\in C^\infty$.

(6) B_ε^2 *is a portion of the hyperplane* $y=h$.

Proof. Let $\delta(X)=d(X,E)$ when $d(X,E)\leqslant h$, otherwise let $\delta(X)=h$. Then $\delta(X)$ is defined on all of E_n, and as is easily seen satisfies the Lipschitz condition

$$|\delta(X)-\delta(Y)|\leqslant|X-Y|.$$

Let $\varphi_\eta(X)$ be a C^∞ "approximation to the identity". It may be constructed as follows. Take $\varphi(X)\in C^\infty$, $\varphi(X)\geqslant 0$, $\varphi(X)=0$ if $|X|\geqslant 1$, and

$$\int_{E_n}\varphi(X)\,dX=1.$$

Set $\varphi_\eta(X)=\eta^{-n}\varphi(X/\eta)$. Let $f_\eta(X)=\int_{E_n}\delta(X-Y)\,\varphi_\eta(Y)\,dY$. Then by the usual arguments, $f_\eta(X)\in C^\infty$, and $f_\eta(X)\to\delta(X)$ uniformly as $\eta\to 0$. Let $\eta=\eta(\varepsilon)$ be so small so that

$$|f_\eta(X)-\delta(X)|<\varepsilon,\quad\text{and set }\ \delta_\varepsilon(X)=f_\eta(X)+2\varepsilon.$$

Taking a subset of the collection $\{\delta_\varepsilon(X)\}$ (with possible reindexing of the subscript ε) we obtain

(a) $\delta_\varepsilon(X)>\delta(X)$

(b) $\delta_{\varepsilon_1}(X)\geqslant\delta_{\varepsilon_2}(X)$, if $\varepsilon_2>\varepsilon_1$

(c) $\delta_\varepsilon(X)\to\delta(X)$.

Define now the regions R_ε to be

$$R_\varepsilon=\{(X,y):\ \delta_\varepsilon(X)<\alpha y,\ 0<y<h\}.$$

In view of the fact that $\delta(X)=\min\{d(X,E),h\}$, (a), (b) and (c) imply conclusions (1), (2) and (3) of the lemma.

The boundary B_ε of R_ε is the union of two sets, B_ε^1 and B_ε^2:

$$B_\varepsilon^1=\{(X,y):\ \alpha y=\delta_\varepsilon(X),\ 0<y<h\}$$

and

$$B_\varepsilon^2=\{(X,y):\ y=h,\ \delta_\varepsilon(X)\leqslant\alpha h\}.$$

Clearly B^1_ε is a portion of the smooth surface $\alpha y = \delta_\varepsilon(X)$, while B^2_ε is a portion of the hyper-plane $y = h$. In fact, the region R_ε consists exactly in the set of points lying above B^1_ε and below B^2_ε.

In order to conclude the proof of the lemma it remains to be shown that

$$\left|\frac{\partial \delta_\varepsilon(X)}{\partial x_k}\right| \leqslant 1.$$

In view of the definition of $\delta_\varepsilon(X)$ it is sufficient to prove a similar inequality for

$$f_\eta(X) = \int_{E_n} \delta(X - Y)\,\varphi_\eta(Y)\,dY.$$

Now,
$$f_\eta(X_1) - f_\eta(X_2) = \int [\delta(X_1 - Y) - \delta(X_2 - Y)]\,\varphi_\eta(Y)\,dY.$$

Hence $|f_\eta(X_1) - f_\eta(X_2)| \leqslant |X_1 - X_2| \int \varphi_\eta(Y)\,dY = |X_1 - X_2|$ owing to the fact that $\delta(X)$ satisfies the above discussed Lipschitz condition. Therefore

$$\left|\frac{\partial f_\eta(X)}{\partial x_k}\right| \leqslant 1,$$

and hence
$$\left|\frac{\partial \delta_\varepsilon(X)}{\partial x_k}\right| \leqslant 1 \quad \text{q.e.d.}$$

3. Basic lemmas

In all that follows $|\nabla u|$ will denote

$$|\nabla u| = \left\{\left|\frac{\partial u}{\partial y}\right|^2 + \sum_{k=1}^{n}\left|\frac{\partial u}{\partial x_k}\right|^2\right\}^{\frac{1}{2}}.$$

We let β, α, k, and h be given positive quantities with $\beta > \alpha$, and $k > h$.

LEMMA 4. *Let $u(X, y)$ be harmonic in the cone $\Gamma(X_0; \beta, k)$ and suppose that $|u(X, y)| \leqslant 1$ there. Then*

$$y\,|\nabla u| \leqslant A \quad \textit{in the cone} \quad \Gamma(X_0; \alpha, h),$$

where $A = A(\beta, \alpha, k, h)$ depends only on the indicated parameters but not on X_0 or u. (1)

(1) The constants A, a, c, ... need to be the same in different contexts.

Proof. We shall need the following fact: If u is harmonic in a sphere Σ of radius one in E_{n+1}, and its absolute value is bounded by one there, then the value of $|\nabla u|$ at the center σ of Σ is bounded by fixed constant A, which does not depend on u. This may be read off from the familiar Poisson integral representation of harmonic functions in a sphere in terms of their boundary values. Alternatively, we may use the following indirect argument. Assuming the contrary, there would then exist a sequence u_n of functions harmonic in Σ and bounded by one in absolute value so that $|\nabla u_n(\sigma)| \to \infty$. By a well-known property of harmonic functions, we can select a subsequence of the u_n which converge together with all derivatives uniformly on every closed set interior to Σ. This is a contradiction and proves the existence of the required A.

If now Σ is the sphere of radius ϱ and $|u|$ is still bounded by one there, then $|\nabla u(\sigma)| \leqslant A/\varrho$. This follows from our previous observation by making a change of scale which expands each coordinate by a factor of ϱ.

We now consider $u(X, y)$ which is harmonic in the cone

$$\Gamma(\beta, k) = \{(X, y): |X - X_0| < \beta y, \ 0 < y < h\}.$$

Let (X, y) be any point in the smaller cone $\Gamma(\alpha, h)$. Notice that since $\alpha < \beta$, and $h < k$, there exists a fixed constant $c > 0$, so that the sphere of radius cy whose center is (X, y) lies entirely in $\Gamma(\beta, k)$.

We now apply the previous fact to the case where Σ is the sphere of radius cy whose center σ is (X, y), and obtain

$$|\nabla u(X, y)| \leqslant A/cy, \quad (X, y) \in \Gamma(\alpha, h)$$

that is,
$$y\,|\nabla u(X, y)| \leqslant A/c, \quad \text{for} \quad (X, y) \in \Gamma(\alpha, h), \quad \text{q.e.d.}$$

LEMMA 5. *Suppose that $u(X, y)$ is harmonic in the cone $\Gamma(X_0; \beta, k)$ and*

$$\iint_{\Gamma(X_0;\, \beta,\, k)} y^{1-n}\, |\nabla u|^2\, dX\, dy \leqslant 1.$$

Then
$$y\,|\nabla u(X, y)| \leqslant A \quad \text{in} \quad \Gamma(X_0; \alpha, h),$$

$\alpha < \beta$, $h < k$. The constant A depends only on α, β, h and k and not on u or X_0.

Proof. Let Σ denote a sphere located in E_{n+1}^{+}, and let σ denote its center. Then by the mean-value theorem

$$\frac{\partial u}{\partial x_k}(\sigma) = \frac{1}{|\Sigma|} \iint_{\Sigma} \frac{\partial u}{\partial x_k}\, dX\, dy, \quad k = 0, 1, \ldots, n,$$

where $y = x_0$, and $|\sum|$ denotes the $n+1$ dimensional volume of $\sum$. Hence, by Schwarz's inequality

$$\left|\frac{\partial u(\sigma)}{\partial x_k}\right|^2 \leqslant \frac{1}{|\sum|} \iint\limits_{\Sigma} \left|\frac{\partial u}{\partial x_k}\right|^2 dX\,dy.$$

Adding, we obtain
$$|\nabla u(\sigma)|^2 \leqslant \frac{1}{|\sum|} \iint\limits_{\Sigma} |\nabla u|^2 dX\,dy.$$

Arguing as in the proof of the previous lemma, we take $\sigma = (X, y)$ to be any point in $\Gamma(X_0; \alpha, h)$; then if Σ is the sphere of radius cy whose center is σ, $\Sigma \subset \Gamma(X_0; \beta, k)$. Notice that $|\sum| = cy^{n+1}$. We therefore have

$$|\nabla u(\sigma)|^2 = |\nabla u(X, y)|^2 \leqslant \frac{y^{-n-1}}{c_1} \iint\limits_{\Sigma} |\nabla u|^2\, dX\,dy$$
$$\leqslant c_2 y^{-2} \iint\limits_{\Sigma} y^{-n+1} |\nabla u|^2 dX\,dy \leqslant c_2 y^{-2} \iint\limits_{\Gamma} y^{-n+1} |\nabla u|^2 dX\,dy,$$

where $\Gamma = \Gamma(X_0; \beta, k)$. This proves the lemma.

4. The generalized area theorem

The theorem which we shall prove can be formulated as follows.

THEOREM 1. *Let $u(X, y)$ be harmonic in E^+_{n+1}.*

(a) *Suppose that for every point X_0 belonging to a set E, $u(X, y)$ is bounded in a cone $\Gamma(X_0)$ whose vertex is X_0. Then the generalized area integral* (1)

$$\iint\limits_{\Gamma(X_0)} y^{1-n} |\nabla u|^2 dX\,dy \tag{4.1}$$

is finite for a.e. $X_0 \in E$

(b) *Conversely, suppose that for every $X_0 \in E$, the integral* (4.1) *is finite, then $u(X, y)$ has a non-tangential limit for a.e. $X_0 \in E$.*

Proof. We consider first part (a).

We may assume, without loss of generality, that the set E has finite measure, and by the use of Lemma 1, neglecting a set of arbitrarily small measure, we may

(1) We use the terminology of "generalized area integral" although (4.1) when $n > 1$, no longer can be interpreted as an actual area or volume.

also assume that E is closed and bounded on that $u(X, y)$ is uniformly bounded in the region.

$$\tilde{R} = \bigcup_{X_0 \in E} \Gamma(X_0; \beta, k) \tag{4.2}$$

whatever fixed β and k we choose.

We shall show that

$$A(X_0) = \iint_{\Gamma(X^0)} y^{1-n} |\nabla u|^2 \, dX \, dy \quad \text{is finite for a.e. } X_0 \in E,$$

where $\Gamma(X_0) = \Gamma(X_0; \alpha, h)$ and h are fixed quantities chosen once and for all, and taken so that $\beta > \alpha$, $k > h$.

Let

$$R = \bigcup_{X_0 \in E} \Gamma(X_0 \ \alpha, h) \tag{4.3}$$

and thus $R \subset \tilde{R}$, and hence u is uniformly bounded in R also. In order to show that $A(X_0) < \infty$, for a.e. $X_0 \in E$, it suffices to show that

$$\int_E A(X_0) \, dX_0 < \infty.$$

Let $\Psi(X_0; X, y)$ be the characteristic function of $\Gamma(X_0; \alpha, h)$. That is,

$$\Psi(X_0; X, y) = 1 \quad \text{if} \quad |X - X_0| < \alpha y \quad \text{and} \quad 0 < y < h,$$

otherwise $\Psi(X_0; X, y) = 0$.

We must, therefore, show that

$$\iint_R \left\{ \int_E \Psi(X_0; X, y) \, dX_0 \right\} y^{1-n} |\nabla u(X, y)|^2 \, dX \, dy$$

is finite However $\displaystyle\int_E \Psi(X_0; X, y) \, dX_0 \leqslant \int_{|X_0 - X| < \alpha y} dX_0 = c\, y^n.$

Thus it suffices to show that

$$\iint_R y \, |\nabla u(X, y)|^2 \, dX \, dy < \infty, \tag{4.4}$$

where R is as in (4.3).

We shall transform the integral (4.4) by Green's theorem. In order to do this we shall use the approximating smooth regions R_ε discussed in Section 2. By the properties listed in Lemma 3 it may be seen that (4.4) is equivalent with

$$\iint_{R_\varepsilon} y\,|\nabla u(X, y)|^2\, dX\, dy \leqslant c < \infty, \tag{4.5}$$

where the constant c is independent of ε.

Since the region R_ε has a sufficiently smooth boundary B_ε we apply to it Green's theorem in the form

$$\int_{B_\varepsilon} \left(G\frac{\partial F}{\partial n_\varepsilon} - F\frac{\partial G}{\partial n_\varepsilon}\right) d\tau_\varepsilon = \iint_{B_\varepsilon} (G\Delta F - F\Delta G)\, dX\, dy.$$

Here $\partial/\partial\eta_\varepsilon$ indicates the directional derivative along the outward normal to B_ε. $d\tau_\varepsilon$ is the element of "area" of B_ε.

In the above formula we take $F = u^2$, and $G = y$. A simple calculation shows that $\Delta(u^2) = 2|\nabla u|^2$, since u is harmonic, while it is clear that $\Delta(y) = 0$. Therefore, we obtain

$$\int_{B_\varepsilon} \left(y\frac{\partial u^2}{\partial n_\varepsilon} - u^2\frac{\partial y}{\partial n_\varepsilon}\right) = 2\iint_{R_\varepsilon} y\,|\nabla u|^2\, dX\, dy.$$

It is therefore sufficient to prove that

$$\int_{B\varepsilon} \left(y\frac{\partial u^2}{\partial n_\varepsilon} - u^2\frac{\partial y}{\partial n_\varepsilon}\right) d\tau_\varepsilon \leqslant c < \infty. \tag{4.6}$$

Notice that $B_\varepsilon \subset R \subset \tilde{R}$. Hence, u and therefore u^2 is bounded uniformly there. Moreover $|\partial y/\partial n_\varepsilon| \leqslant 1$; notice also that $\partial u^2/\partial n = 2u\,\partial u/\partial n$. Thus

$$\left|y\frac{\partial u^2}{\partial n_\varepsilon}\right| \leqslant 2|u|\cdot y\cdot\left|\frac{\partial u}{\partial n_\varepsilon}\right| \leqslant 2|u|\cdot y\cdot|\nabla u|;$$

therefore by Lemma 3 $y\,\partial u^2/\partial n_\varepsilon$ is uniformly bounded in $B_\varepsilon \subset R = \bigcup_{X_0\in E}\Gamma(X_0;\, \alpha, h)$, because u is bounded in $\tilde{R} = \bigcup_{X_0\in E}\Gamma(X_0;\, \beta, k)$.

Hence the integral in (4.6) is uniformly bounded by a constant multiple of $\int_{B_\varepsilon} d\tau_\varepsilon$.

However,
$$\int_{B_\varepsilon} d\tau_\varepsilon = \int_{B_\varepsilon^1} d\tau_\varepsilon + \int_{B_\varepsilon^2} d\tau_\varepsilon.$$

Now B_ε^1 is a portion of the surface $y = \alpha^{-1} \cdot \delta_\varepsilon(X)$. Therefore

$$d\tau_\varepsilon = \left(1 + \alpha^{-2} \cdot \sum_{k=1}^{n} \left(\frac{\partial \delta_\varepsilon(X)}{\partial x_k}\right)^2\right)^{\frac{1}{2}} dX$$

there. However, $|\partial \delta_\varepsilon(X)/\partial x_k| \leqslant 1$. Thus $d\tau_\varepsilon \leqslant (1 + n\alpha^{-2})^{\frac{1}{2}} dX$. Also B_ε^2 is a portion of the hyper-plane $y = h$. Since both B_ε^1 and B_ε^2 are included in a fixed sphere, it follows that $\int_{B_\varepsilon} d\tau_\varepsilon$ is uniformly bounded. This proves (4.5), and hence part (a) of the theorem.

We now pass to the proof of part (b). We temporarily relable the set on which the integral (4.1) is finite by calling it E_0. By simple arguments we may reduce the hypotheses to:

(1) $\iint_{\Gamma(X_0;\,\beta,\,k)} y^{1-n} |\nabla u|^2 \, dX\, dy$ is uniformly bounded as X_0 ranges over E_0, where β and k are some fixed positive quantities,

(2) the set E_0 is bounded.

Given now any $\eta > 0$, we may pick a *closed* set E, $E \subset E_0$ which satisfies the following two additional properties

(3) $m(E_0 - E) < \eta$

(4) there exists a fixed ϱ_η, so that

$$m(\{Y: |X - Y| \leqslant \varrho\} \cap E_0) \geqslant \tfrac{1}{2} \cdot m\{Y: |X - Y| \leqslant \varrho\}, \quad \text{if } X \in E,\ 0 < \varrho < \varrho_\eta.$$

This can be done as follows. Almost every $X \in E_0$ is a point of density of E_0; then for such X we have

$$\lim_{\varrho \to 0} \frac{m(\{Y: |X - Y| < \varrho\} \cap E_0)}{m(\{Y: |X - Y| \leqslant \varrho\}} = 1.$$

Hence a simple argument shows that for any η we can find an appropriate subset E of E_0 to satisfy (3) and (4).

We now fix the set E found in this way. It will suffice to show that $u(X, y)$ has a non-tangential limit for a.e. $X \in E$. (Thus at the conclusion of the proof we let $m(E_0 - E) \to 0$.)

First step

We consider the region

$$R = \bigcup_{X_0 \in E} \Gamma(X_0;\, \alpha, h),$$

where $\alpha < \beta$, $h < k$, and B is the boundary. The first step in the proof of part (*b*) will be to show, in effect,

$$\int_B |u|^2\, d\tau < \infty. \tag{4.7}$$

Of course, (4.7) as it stands is not meaningful, because u is not defined for all of B and neither is the element of "area" $d\tau$.

To bypass these technical difficulties we consider again the approximating regions R_ε with their boundaries B_ε discussed in section 2, and we show that

$$\int_{B\varepsilon} |u|^2\, d\tau_\varepsilon \leqslant c < \infty, \tag{4.7*}$$

where the constant c is independent of ε.

The proof of (4.7*) is in some ways a reversal of the argument used to prove part (a). We begin by showing that

$$\iint_R y\, |\nabla u|^2\, dX\, dy < \infty. \tag{4.8}$$

This is done as follows. By (1) we have

$$\iint_{\Gamma(X_0;\, \beta, k)} y^{1-n}\, |\nabla u|^2\, dX\, dy \leqslant A < \infty, \quad X_0 \in E_0.$$

Integrating over E_0 we obtain

$$\iiint \chi_{E_0}(X_0)\, \Psi(X_0;\, X, y)\, y^{1-n}\, |\nabla u(X, y)|^2\, dX\, dy\, dX_0 < \infty. \tag{4.9}$$

Here χ_{E_0} is the characteristic function of E_0, and Ψ is the characteristic function of the cone $\Gamma(X_0;\, \beta, k)$. We shall show that

$$\int_{E_0} \Psi(X_0;\, X, y)\, dX_0 \geqslant c\, y^n, \tag{4.10}$$

where $(X, y) \in R$, $c > 0$.

Recall that $R=\bigcup_{Z\in E}\Gamma(Z;\,\alpha,h)$. Thus $(X,y)\in R$ means that there exists a $Z\in E$ so that $|X-Z|<\alpha y$, $0<y<h$. Since Ψ is the characteristic function of the set where $|X-X_0|<\beta y$, $0<y<k$, we see that

$$\int_{E_0}\Psi(X_0;\,X,y)\,dX_0\geqslant\int_{E_0\cap\{|X_0-Z|<(\beta-\alpha)y\}}dX_0.$$

Since $Z\in E$, an application of (4) shows that the second integral exceeds $c\,y^n$, (if $0<y<h$), for some appropriate constant c, $c>0$. This proves (4.10). Applying (4.10) in (4.9) proves (4.8). We now replace (4.8) by an equivalent statement

$$\iint_{R_\varepsilon}y\,|\nabla u|^2\,dX\,dy\leqslant c<\infty. \tag{4.11}$$

R_ε are the approximating regions of R, and c is independent of ε.

We now transform (4.11) by Green's theorem—as in the proof (a) of the theorem. We obtain

$$0\leqslant\int_{B_\varepsilon}\left(y\frac{\partial u^2}{\partial n_\varepsilon}-u^2\frac{\partial y}{\partial n_\varepsilon}\right)d\tau_\varepsilon\leqslant c<\infty. \tag{4.12}$$

Now the boundary B_ε is the union of two parts B_ε^1 and B_ε^2. However, B_ε^2 is a portion of the hyperplane $y=h$ (and thus at a positive distance from the boundary hyperplane $y=0$).

Moreover, B_ε^2 is contained in a fixed sphere. Thus the total contribution of the integral (4.12) over B_ε^2 is uniformly bounded. We therefore have

$$\left|\int_{B_\varepsilon^1}\left(y\frac{\partial u^2}{\partial n_\varepsilon}-u^2\frac{\partial y}{\partial n_\varepsilon}\right)d\tau_\varepsilon\right|\leqslant c<\infty. \tag{4.13}$$

We claim that $\partial y/\partial n_\varepsilon\leqslant-\alpha(\alpha^2+n)^{-\frac{1}{2}}$. In fact, $\partial/\partial n_\varepsilon$ is the (outward) normal derivative to the surface whose equation is $F_\varepsilon(X,y)\equiv\alpha y-\delta_\varepsilon(X)=0$. A set of (unnormalized) direction numbers for this direction is

$$\left(\frac{-\partial F_\varepsilon}{\partial y},\frac{\partial F_\varepsilon}{\partial x_1},\frac{\partial F_\varepsilon}{\partial x_2},\ldots,\frac{\partial F_\varepsilon}{\partial x_n}\right)=\left(-\alpha,\frac{-\partial\delta_\varepsilon}{\partial x_1},\frac{-\partial\delta_\varepsilon}{\partial x_2},\ldots,\frac{-\partial\delta_\varepsilon}{\partial x_n}\right).$$

However, $|\partial\delta_\varepsilon(X)/\partial x_k|\leqslant 1$. This shows that $\partial y/\partial n_\varepsilon\leqslant-\alpha(\alpha^2+n)^{-\frac{1}{2}}$.

But $\left| y \dfrac{\partial u^2}{\partial n} \right| = \left| 2\, u \cdot y \cdot \dfrac{\partial u}{\partial n} \right| \leqslant 2\, |u| \cdot y \cdot |\nabla u|$. Moreover by Lemma 4 $y\, |\nabla u|$ is bounded in $\bigcup_{X_0 \in E} \Gamma(X_0;\, \alpha, h)$ and hence in B^1_ε. Combining these facts in (4,13) we obtain

$$\int_{B^1_\varepsilon} |u|^2\, d\tau_\varepsilon \leqslant c_1 \int_{B^1_\varepsilon} |u|\, d\tau_\varepsilon + c_2 \leqslant c_1 \left(\int_{B^1_\varepsilon} |u|^2\, d\tau_\varepsilon \right)^{\frac{1}{2}} \left(\int_{B^1_\varepsilon} d\tau \right)^{\frac{1}{2}} + c_2.$$

We have seen in the proof of part (a) that $\int_{B^1_\varepsilon} d\tau_\varepsilon$ is uniformly bounded. Letting

$$J^2_\varepsilon = \int_{B^1_\varepsilon} |u|^2\, d\tau_\varepsilon,$$

we obtain

$$J^2_\varepsilon \leqslant c_3\, J_\varepsilon + c_2.$$

Since c_3 and c_2 are independent of ε, we then have that J_ε is uniformly bounded hence

$$\int_{B^1_\varepsilon} |u|^2\, d\tau_\varepsilon \leqslant c < \infty. \tag{4.14}$$

Second step

We next seek to majorize the function $u(X, y)$ by another, $v(X, y)$, whose non-tangential behavior is known to us. We proceed as follows. The "surface" B^1_ε is a portion of the surface

$$y = \alpha^{-1} \cdot \delta_\varepsilon(X).$$

Let $f_\varepsilon(X)$ be the function defined on $y = 0$, whose values are the projection on $y = 0$ of the value of $u(X, y)$ on B^1_ε, and otherwise zero. That is,

$$f_\varepsilon(X) = u(X, \alpha^{-1}\, \delta_\varepsilon(X)),$$

for those $(X, 0)$ lying below B^1_ε, otherwise $f(X) = 0$. We claim

$$\int_{E_n} |f_\varepsilon(X)|^2\, dX \leqslant c < \infty. \tag{4.15}$$

In fact, since $d\tau_\varepsilon \geqslant dX$, we have

$$\int_{E_n} |f_\varepsilon(X)|^2\, dX \leqslant \int_{B^1_\varepsilon} |u|^2\, d\tau_\varepsilon \leqslant c < \infty,$$

by (4.14).

We now let $v_\varepsilon(X, y)$ be the harmonic function which is the Poisson integral of the function $|f_\varepsilon(X)|$. Thus

$$v_\varepsilon(X, y) = \int_{E_n} P(X - Z, y) \, |f_\varepsilon(Z)| \, dZ,$$

where $P(X, y)$ is the Poisson kernel

$$P(X, y) = c_n^{-1} \cdot \frac{y}{(y^2 + |X|^2)^{\frac{1}{2}(n+1)}}.$$

(For the needed properties of Poisson integrals, see the previous paper in this series, Section 3.)

We shall show that there exists two constants c_1 and c_2 so that

$$|u(X, y)| \leqslant c_1 v_\varepsilon(X, y) + c_2, \quad (X, y) \in R_\varepsilon. \tag{4.16}$$

By the maximum principle for harmonic functions it is sufficient to show that the inequality (4.16) holds for (X, y) belonging to the boundary B_ε. Now $B_\varepsilon = B_\varepsilon^1 \cup B_\varepsilon^2$, where B_ε^2 is a subset of the hyperplane $y = h$, lying in a fixed sphere. Since $v_\varepsilon(X, y) \geqslant 0$, we can satisfy (4.16) on B_ε^2 by choosing c_2 large enough (and independent of ε).

It remains to consider $(X, y) \in B_\varepsilon^1$. Let us call $\sigma = (X, y)$. Since

$$B_\varepsilon^1 \subset R = \bigcup_{X_0 \in E} \Gamma(X_0; \alpha, h),$$

we can find a constant $c > 0$, with the following property: The sphere $\sum$ whose center is $\sigma = (X, y)$, and whose radius to cy lies entirely in $\bigcup_{X^0 \in E} \Gamma(X_0; \beta^*, k^*)$, where $\alpha < \beta^* < \beta$, and $h < k^* < k$. Recall that the cones $\Gamma(X_0; \beta, k)$ have the property that

$$\iint_{\Gamma(X_0;\, \alpha,\, k)} y^{1-n} |\nabla u|^2 \, dX \, dy \leqslant A < \infty, \quad X_0 \in E.$$

Making use of Lemma 5, it follows that

$$y |\nabla u| \leqslant A_1 < \infty \quad \text{for} \quad (X, y) \in \bigcup_{X_0 \in E} \Gamma(X_0; \beta^*, k^*). \tag{4.17}$$

Let now σ' be another point in the sphere $\sum$, and let l be the line segment joining σ' with σ. Then

$$|u(\sigma') - u(\sigma)| \leqslant |\sigma' - \sigma| \sup_l |\nabla u|.$$

Since, however, $|\sigma' - \sigma| \leqslant$ radius of $\sum = cy$, it follows from (4.17) that

$$|u(\sigma') - u(\sigma)| \leqslant A, \quad \text{if} \quad \sigma' \in \sum. \tag{4.18}$$

Let S = that portion of the surface B_ε^1 which lies in the sphere $\sum$.
Let $|S|$ denote its area

$$|S| = \int_{\Sigma \cap B_\varepsilon^1} d\tau_\varepsilon.$$

Since $d\tau_\varepsilon \geqslant dX$, and $\sum$ is a sphere of radius cy, we obtain after a simple geometric argument,

$$|S| \geqslant a y^n,$$

where a is an appropriate constant, $a > 0$. Using (4.18) we obtain

$$|u(\sigma)| \leqslant \frac{1}{|S|} \cdot \int_S |u(\sigma')| \, d\tau_\varepsilon(\sigma') + A.$$

In view of our definition of $f_\varepsilon(X)$, our estimate for $|S|$, and the fact that $d\tau_\varepsilon \leqslant (1 + n\alpha^{-2})^{\frac{1}{2}} dX$ (see the proof of part (a)), we then get

$$|u(\sigma)| \leqslant b y^{-n} \int_{|X-Z|<cy} |f_\varepsilon(Z)| \, dZ + A.$$

The Poisson kernel has the property that

$$P(X, y) \geqslant (b/c_1) y^{-n} \quad \text{for} \quad |X| < cy,$$

where c_1 is an appropriate constant. We therefore obtain

$$|u(\sigma)| = |u(X, y)| \leqslant c_1 \int P(X - Z, y) |f_\varepsilon(Z)| \, dZ + A \quad \text{for} \quad (X, y) \in B_\varepsilon^1.$$

This proves our desired estimate (4.16) on B_ε^1. We have already remarked that on B_ε^2 it is semi-trivial. Hence, we have the estimate on B_ε and therefore on R_ε. Thus (4.16) is completely proved.

Because of the uniform estimate (4.15) on the norms of $f_\varepsilon(X)$, we can select a subsequence $\{|f_{\varepsilon_k}(X)|\}$ of the functions $\{|f_\varepsilon(X)|\}$ which converge to $|f(X)| \in L^2(E_n)$, weakly in L^2.

Let $v(X, y) = \int P(X-Z, y)\,|f(Z)|\,dZ$ be the Poisson integral of $|f|$. Then for each (X, y), $y > 0$, $v_{\varepsilon_k}(X, y) \to v(X, y)$. Since $R_\varepsilon \to R$, we then have

$$|u(X, y)| \leqslant c_1 v(X, y) + c_2, \ (X, y) \in R. \tag{4.19}$$

This is the decisive majorization of $u(X, y)$.

Final step

Because of the known behavior of Poisson integrals near the boundary, we can assert that $v(X, y)$ is bounded non-tangentially for almost every X_0 in E_n. More precisely, for a.e. $X_0 \in E_n$, (X, y) is bounded in the cone $\Gamma(X_0) = \Gamma(X_0; \alpha, h)$. (For these facts see I, Section 3.)

Because $R = \bigcup_{X_0 \in E} \Gamma(X_0; \alpha, h)$, and (4.19), it follows that for a.e. $X_0 \in E$, $u(X, y)$ is bounded in $\Gamma(X_0; \alpha, h)$. In view of Lemma 2, this shows that u has a non-tangential limit for a.e. $X_0 \in E$. This concludes the proof of the theorem.

Chapter II

The main purpose of this chapter will be the proof of Theorem 3 in Section 6. Actually this will be an easy result of Theorem 1 proved in the previous chapter and an auxiliary result, Theorem 2, which is contained in Section 5.

5. An Auxiliary theorem

Theorem 1 we have just proved is useful because—disregarding sets of measure zero—it shows that the existence of non-tangential limits for harmonic functions is equivalent with the finiteness of certain integrals. In many cases these integrals are easier to deal with. We shall see that this is the case in the following theorem which is of particular interest in terms of its applications considered in the following paragraphs.

In what follows $u(X, y)$ will denote a vector of k components

$$(u_1(X, y), u_2(X, y) \ldots, u_k(X, y)),$$

where each component is harmonic. Similarly $v(X, y)$ will denote a vector of m components ($k \neq m$, in general), each component being harmonic. We shall set

$$|u| = \left(\sum_{i=1}^{k} |u_i|^2\right)^{\frac{1}{2}},$$

similarly for v. When we say that $u(X, y)$ has a non-tangential limit at a given point X_0, we shall mean that each component has, etc.

THEOREM 2. *Let $u(X, y)$, $v(X, y)$ be harmonic in the cone $\Gamma(X_0; \beta, k)$. Let $P(D)$ be a $k \times m$ matrix, each of whose entries is a homogeneous differential operator in $\partial/\partial y$, $\partial/\partial x_1, \ldots, \partial/\partial x_n$ of degree r, with constant coefficients. Assume that u and v are related by*

$$\frac{\partial^r u}{\partial y^r}(X, y) = P(D) v. \tag{5.1}$$

Assume also that
$$\iint_{\Gamma(X_0; \beta, k)} y^{1-n} |v|^2 \, dX \, dy < \infty.$$

Then if $\alpha < \beta$, $h < k$, we can conclude that

$$\iint_{\Gamma(X_0; \alpha, h)} y^{1-n} |u|^2 \, dX \, dy < \infty.$$

The equation (5.1) may be viewed as a relation between a harmonic function and its conjugates, in its most general form. Examples and interpretations of (5.1) will be discussed in Sections 7 and 8.

Before proving the theorem we derive from it a particular consequence of interest.

COROLLARY. *Let the cones $\Gamma(X_0; \beta, k)$ and $\Gamma(X_0; \alpha, h)$ be as in the above theorem. Suppose that $H(X, y)$ is harmonic in the cone $\Gamma(X_0; \beta, k)$.*

(a) *if*
$$\iint_{\Gamma(X_0; \beta, k)} y^{1-n} \left|\frac{\partial H}{\partial y}\right|^2 dX \, dy < \infty.$$

then
$$\iint_{\Gamma(X_0; \alpha, h)} y^{1-n} \left|\frac{\partial H}{\partial x_j}\right|^2 dX \, dy < \infty, \; j = 1, 2, \ldots, n.$$

(b) *If each of the integrals*

$$\iint_{\Gamma(X_0; \beta, k)} y^{1-n} \left|\frac{\partial H}{\partial x_j}\right|^2 dX \, dy < \infty, \; j = 1, 2, \ldots, n;$$

then,
$$\iint_{\Gamma(X_0; \alpha, h)} y^{1-n} \left|\frac{\partial H}{\partial y}\right|^2 dX \, dy < \infty.$$

Part (a) of the corollary is proved by taking $u = \partial H/\partial x_j$, $v = \partial H/\partial y$; then (5.1) becomes $\partial u/\partial y = \partial v/\partial x_j$. To prove (*b*) we take

$$u = \partial H/\partial y, \text{ and } v = (v_1, v_2, \ldots, v_n) = (\partial H/\partial x_j, \ldots, \partial H/\partial x_n).$$

Then (5.1) becomes $\partial u/\partial y = -(\partial v_1/\partial x_1 + \partial v_2/\partial x_2 \ldots + \partial v_n/\partial x_n)$.

The meaning of this corollary in connection with Theorem 1 is clear. In order to prove that $u(X, y)$ has a non-tangential limit a.e. in a set $E \in E_n$, it suffices to prove that either

$$\iint_{\Gamma(X_0)} y^{1-n} \left|\frac{\partial u}{\partial y}\right|^2 dX\,dy < \infty, \quad \text{for a.e. } X_0 \in E,$$

or

$$\iint_{\Gamma(X_0)} y^{1-n} \left\{\sum_{j=1}^{n} \left|\frac{\partial u}{\partial x_j}\right|^2\right\} dX\,dy < \infty, \quad \text{for a.e. } X_0 \in E.$$

This is a definite strengthening of Theorem 1, part (b).

For the case $n = 1$ this fact has already found application in certain problems in one real variable, see Stein and Zygmund [10]. In that case $(n = 1)$ the corollary follows from a theorem of Friedrichs, see [4]. As far as the case of general n is concerned, Friedrichs has proved in [5] a generalization of his previous result. But this does not overlap with our result for general n.

The proof of Theorem 2 will require two preliminary lemmas of an elementary character.

LEMMA 6. *Let* $\Phi(s) = \int_s^\infty \frac{\varphi(t)}{t}\,dt$. *Then*

$$\int_0^\infty |\Phi(s)|^2\,ds \leqslant 4 \int_0^\infty |\varphi(t)^2\,dt.$$

This is a well-known inequality of Hardy, see [6].

LEMMA 7. *Let* $0 \leqslant a_0 \leqslant a < \infty$, *and*

$$F(s) = \int_{as}^{b} f(t)\,dt.$$

Then

$$\int_0^b |F(s)|^2 s\,ds \leqslant 4 a_0^{-2} \int_0^b |f(t)|^2 t^3\,dt. \tag{5.2}$$

Proof. First, if $\Phi(s) = \int_{as}^{\infty} \frac{\varphi(t)}{t} dt$, then $\int_0^{\infty} |\Phi(s)|^2 ds \leq 4a^{-1} \int_0^{\infty} |\varphi(t)|^2 dt$. This follows from Lemma 6 by the change of variable $s \to as$. Next

$$|s^{\frac{1}{2}} F(s)| = \left| s^{\frac{1}{2}} \int_{as}^{b} f(t)\, dt \right| \leq a^{\frac{1}{2}} \int_{as}^{b} t^{\frac{1}{2}} |f(t)|\, dt = a^{\frac{1}{2}} \int_{as}^{b} t^{\frac{3}{2}} |f(t)| \frac{dt}{t}.$$

Applying the previous inequality proves the lemma.

We now come to the proof of Theorem 2. We shall assume for simplicity that the vertex is $X_0 = 0$; this involves no loss of generality. We then relable the cones $\Gamma(X_0; \beta, k)$ and $\Gamma(X_0; \alpha, h)$ as $\Gamma(\beta, k)$ and $\Gamma(\alpha, h)$ respectively.

Now let ϱ denote that segment of a ray passing through the origin and lying in the cone $\Gamma(\alpha, h)$. Let s be the parametrization of the segment ϱ according to its length, with $s = 0$ corresponding to the origin. With $u(X, y)$ given, we shall define $u_\varrho(s)$ by, $u_\varrho(s) =$ restriction of $u(X, y)$ to the ray segment ϱ.

We shall show that

$$\int_0^h s\, |u_\varrho(s)|^2 ds \leq A < \infty, \tag{5.3}$$

where the bound is independent of the ray ϱ lying in $\Gamma(\alpha, h)$. If we prove this inequality, then an integration of it over all ϱ of the type specified will then prove our theorem. We therefore turn to the proof of (5.3).

By (5.1) we obtain

$$u(X, y) = \frac{-1}{(r-1)!} \int_y^h (y-\tau)^{r-1} [P(D)\, v(X, \tau)]\, d\tau + R. \tag{5.4}$$

Since R involves only the values of $u(X, y)$, $\partial u(X, y)/\partial y$, $\ldots$, $\partial^{r-1}(u(X, y))/\partial y^{r-1}$ at $y = h$, this term is uniformly bounded.

We now examine the term $P(D)\, v(X, \tau)$ in (5.4). We use again a fact used several times before: We can find a constant c, $c > 0$, so that if $\Sigma(X, \tau)$ is the sphere whose center is (X, τ), with $(X, \tau) \in \Gamma(\alpha, h)$ and whose radius is $c\tau$, then $\Sigma \subset \Gamma(\beta, k)$. We fix this constant c in the rest of this proof. We also need the following fact. Let $P(D)$ be a fixed matrix of differential polynomials, homogeneous of degree r, and let Σ be the sphere whose center is σ and radius in δ. Then (if $v(X, y)$ is harmonic)

$$|P(D)\, v(\sigma)| \leq A\, \delta^{-\frac{1}{2}(n+1+2r)} \Big(\iint_{\Sigma} |v|^2 dX\, dy \Big)^{\frac{1}{2}}. \tag{5.5}$$

This may be seen as follows. First consider the case when Σ is of radius one. Notice that the class of harmonic functions which satisfy $\iint_{\Sigma} |v|^2\,dX\,dy \leqslant 1$, are uniformly bounded on interior compact subsets, by the mean value property. It then follows by an indirect argument (of the type used in the proof of Lemma 4) that there exist a constant A so that $|P(D)\,v(\sigma)| \leqslant A$, for this class of functions. The general inequality (5.5) then follows by a homogeneity argument which involves stretching each component by the factor δ. Alternatively, (5.5) can be proved directly from the Poisson integral representation for spheres.

We then take (5.5) with Σ being the sphere $\Sigma(X, \tau)$ whose center is (X, τ) and whose radius is $c\tau$, and substitute this estimate in (5.4). This gives

$$|u(X, y)| \leqslant B \int_{y}^{h} \tau^{-\frac{1}{2}(n+3)} \left\{ \iint_{\Sigma(X,\tau)} |v|^2\,dX\,dy \right\}^{\frac{1}{2}} d\tau + A. \tag{5.6}$$

We now call S_τ the "layer" in the cone $\Gamma(\beta, k)$ contained between $\tau - c\tau$ and $\tau + c\tau$; i.e.,

$$S_\tau = \{(X, y)\colon\ |X| < \beta y,\ \tau - c\tau < y < \tau + c\tau\}.$$

Then clearly, since $\Sigma(X, \tau) \subset \Gamma(\beta, k)$ we have $\Sigma(X, \tau) \subset S_\tau$, and therefore

$$\iint_{\Sigma} |v|^2\,dX\,dy \leqslant \iint_{S_\tau} |v|^2\,dX\,dy = J_\tau.$$

Next, call θ the angle that the ray ϱ makes with the y-axis. Since the ray is contained in the cone $|X| < \alpha y$, $0 < y$, it follows that $1 \geqslant \cos\theta \geqslant a_0 = (1+\alpha^2)^{-\frac{1}{2}} > 0$. Notice that $y = s\cos\theta$, where s is the parameter of arc-length along ϱ. Recalling the definition of $u_\varrho(s)$, (5.6) gives

$$|u_\varrho(s)| \leqslant B \int_{s\cos\theta}^{h} \tau^{-\frac{1}{2}(n+3)} J_\tau^{\frac{1}{2}}\,d\tau + A. \tag{5.7}$$

We now invoke (5.2) of Lemma 7. We therefore obtain

$$\int_0^h s\,|u_\varrho(s)|^2\,ds \leqslant B' \int_0^h \tau^{-n} J_\tau\,d\tau + A.$$

However,

$$\int_0^h \tau^{-n} J_\tau \, d\tau = \int_0^h \tau^{-n} \left\{ \iiint_{S_\tau} |v|^2 \, dX \, dy \right\} d\tau \leqslant \iint_{\Gamma(\beta, k)} |v|^2 \left\{ \int \tau^{-n} \chi(\tau, X, y) \, d\tau \right\} dX \, dY,$$

where $\chi(\tau, X, y)$ is the characteristic function of the layer S_τ in $\Gamma(\beta, k)$.

But $$\int \tau^{-n} \chi(\tau, X, y) \, d\tau = \int_{\left\{\substack{\tau - c\tau < y < \tau + c\tau \\ 0 < y < k}\right.} \tau^{-n} \, d\tau \leqslant \int_{y/1+c}^{y/1-c} \tau^{-n} \, d\tau = c_1 y^{-n+1}.$$

This shows that

$$\int_0^h s \, |u_\varrho(s)|^2 \, ds \leqslant B_1 \iint_{\Gamma(\beta, k)} y^{-n-1} |v|^2 \, dX \, dy + B_2,$$

which proves (5.3).

Integrating (5.3) over all ray's ϱ lying in $\Gamma(\alpha, h)$ shows the finiteness of the integral

$$\iint_{\tilde{\Gamma}} y^{1-n} |u|^2 \, dX \, dy.$$

Here $\tilde{\Gamma}$ is that portion of the cone $|X| < \alpha y$, in the upper half-space, which is truncated by the sphere $|X|^2 + y^2 = h^2$. This "cone" differs from our original cone $\Gamma(\alpha, h)$ by a set which lies at a positive distance from the exterior of $\Gamma(\beta, k)$. Since u was assumed harmonic in $\Gamma(\beta, k)$ it is certainly bounded in $\Gamma(\alpha, h) - \tilde{\Gamma}$. Thus the integral over $\Gamma(\alpha, h)$ is also finite. This concludes the proof of theorem.

6. General theorems about non-tangential limits of conjugate functions

We now come to the principal result of this paper.

u and v will denote, as in the previous section, vectors of harmonic functions of k and m components respectively.

THEOREM 3. *Let $u(X, y)$ and $v(X, y)$ be harmonic in E^+_{n+1}. Suppose that they satisfy the relation*

$$\frac{\partial^r u}{\partial y^r} = P(D) v, \tag{6.1}$$

where $P(D)$ is a $k \times m$ matrix whose entries are differential polynomials (with constant coefficients) homogeneous of degree r, $r \geqslant 1$. Suppose that for a given set E, $E \subset E_n$, $v(X, y)$

has a non-tangential limit for every $X \in E$. *Then* $u(X, y)$ *has a non-tangential limit for a.e.* $X \in E$.

Proof. In order to show that $u(X, y)$ has a non-tangential limit a.e. in E it suffices, by part (b) of Theorem 1, to show that

$$\iint\limits_{\Gamma(X_0;\, \alpha,\, h)} y^{1-n} |\nabla u|^2 \, dX\, dy < \infty \quad \text{for a.e. } X_0 \in E.$$

However, by part (*a*) of that theorem is follows that

$$\iint\limits_{\Gamma(X_0;\, \beta,\, k)} y^{1-n} |\nabla v|^2 \, dX\, dy < \infty \quad \text{for a.e. } X_0 \in E.$$

Let now U denote the vector whose components are $(\partial u/\partial y, \partial u/\partial x_1, \dots, \partial u/\partial x_n)$. Actually since u itself is a vector of k components, then U is a vector of $k(n+1)$ components, suitable arranged. Similarly let $V = (\partial v/\partial y, \partial v/\partial x_1, \dots, \partial v/\partial x_n)$ be the indicated vector, of $m(n+1)$ components. If we differentiate the relation (6.1) successively with respect to $\partial/\partial y, \partial/\partial x_1, \dots, \partial/\partial x_n$ then we obtain the relation

$$\frac{\partial^r U}{\partial y^r} = \tilde{P}(D)\, V. \tag{6.2}$$

Here $\tilde{P}(D)$ is a $k(n+1) \times m(n+1)$ matrix which consists of $n+1$ copies of $P(D)$ arranged down the diagonal. Or put in another way, the matrix $\tilde{P}(D)$ is the tensor product of $P(D)$ by I_{n+1}, where I_{n+1} is the $(n+1)\times(n+1)$ identity matrix.

Notice that by our definitions $|V| = |\nabla v|$ and $|U| = |\nabla u|$. We are thus in a position to apply Theorem 2, with (6.2) in place of (5.1). From the finiteness of

$$\iint\limits_{\Gamma(X_0;\, \beta,\, k)} y^{1-n} |\nabla v|^2 \, dX\, dy = \iint\limits_{\Gamma(X_0;\, \beta,\, k)} y^{1-n} |V|^2 \, dX\, dy$$

follows the finiteness of

$$\iint\limits_{\Gamma(X_0;\, \alpha,\, h)} y^{1-n} |\nabla u|^2 \, dX\, dy = \iint y^{1-n} |U|^2 \, dX\, dy;$$

and thus by what has been said above, we obtain the proof of the theorem.

Chapter III

7. Various examples

We consider first the generalization of the Cauchy-Riemann equations studied in paper I, and also discussed in the introduction of the present paper. Let us change the notation slightly (making it symmetric in all variables) by calling $y=x_0$. Thus the underlying variables are $x_0, x_1, \ldots, x_n$. Similarly let us call $u_0=u$, and $u_1=v_1$, $u_2=v_2, \ldots, u_n=v_n$. Then the equations become

$$\left.\begin{aligned} &\sum_{k=0}^{n} \frac{\partial u_k}{\partial x_k}=0, \\ &\frac{\partial u_k}{\partial x_j}=\frac{\partial u_j}{\partial x_k}, \quad o \leqslant j,\ k \leqslant n. \end{aligned}\right\} \tag{7.1}$$

We then have

THEOREM 4. *Let $u_0, u_1, \ldots, u_n$ the system of functions satisfying* (7.1).

(a) *Suppose that u_0 has a non-tangential limit for each point $(x_1, x_2, \ldots, x_n)$ belonging to a set $E \subset E_n$ (= the hyperplane $x_0=0$). Then for a.e. $(x_1, x_2, \ldots, x_n) \in E$, the same is true for each u_k, $k=1, 2, \ldots, n$.*

(b) *Conversely, suppose that $u_1, u_2, \ldots, u_n$ each have non-tangential limits in a se $E \subset E_n$. Then for a.e. point in E the same is true for u_0.*

Proof. To prove (a) we use $\partial u_k/\partial y=\partial u_0/\partial x_k$. For (b) use

$$\partial u_0/\partial y = -(\partial u_1/\partial x_1+\partial u_2/\partial x_2 + \ldots + \partial u_n/\partial x_n).$$

Thus an application of Theorem 3 proves the theorem immediately.

It is evident that this theorem generalizes the corresponding classical result for analytic functions of Privalov and Plessner.

It may be seen that this theorem ist best possible in the following sense.

(a) If we want we existence of non-tangential limits for the $n+1$ components $u_0, u_1, \ldots, u_n$ (a.e. on a set E) by assuming it for only one of them, then this one must be u_0.

(b) However, if we do not make any assumptions on u_0, we must assume that the remaining n components, $u_1, u_2, \ldots, u_n$, have non-tangential a.e. in E in order to obtain the conclusion for all the $n+1$ components. To show this consider an $F(z)=u_0(x_1+iy)+iu_1(x_1+iy)$ which is analytic for $y>0$, but does not have boundary values for $y=0$. Then the set $(u_0, u_1, 0, 0, \dots, 0)$ satisfies (7.1) but does not have non-tangential limits.

It is to be recalled that the system (7.1) is locally equivalent with one arising out of a single harmonic function $H(X, y)$ via

$$u_j=\frac{\partial H}{\partial x_j}, \ x_0=y, \ 0 \leqslant j \leqslant n. \tag{7.1*}$$

The system (7.1) (or alternatively (7.1*)) may be thought of as the most direct generalization of the Cauchy-Riemann equations. However, there are notions of "conjugacy" which have no direct analogue to the classical case but which are nevertheless of interest in higher dimensions. A systematic approach to the possible notions of conjugacy (i.e., appropriate generalizations of the Cauchy-Riemann equations) involves the study of how these systems transforms under rotations—and thus is intimately connected with the theory of representations of the group of rotations in $n+1$ variables. This problem will be treated in a future paper of Guido Weiss and the author. Here we shall consider only briefly some of the possible systems which arise.

For every integer r we shall consider the "gradient of order r"—that is, the system of harmonic functions obtained from a single harmonic function $H(x_1, x_2, \dots, x_n, y)$ as follows:

$$\left\{\frac{\partial^r H(x_1, x_2, \dots, y)}{\partial y^{i_0} \partial x_1^{i_1} \dots \partial x_n^{i_n}}, \quad (i_0+i_1+ \dots +i_n=r)\right\}. \tag{7.2}$$

This system may also be characterized by a set of equations like (7.1. We now state the following theorem which generalizes Theorem 4.

THEOREM 5. *Suppose that for each point $(x_1, x_2, \dots, x_n)$ belonging to a set $E \subset E_n$, the function $\partial^r H/\partial y^r$ has a non-tangential limit. Then the same is true a.e. in E for each other derivative of order r (i.e., other component of (7.2)), and conversely.*

This theorem, like Theorem 4, is an immediate consequence of the general Theorem 3.

Remark. While this theorem is clearly a generalization of Theorem 4, it has only a secondary interest relative to Theorem 4. This is because the assumptions of the converse are to a large measure redundant. This may already be seen in the case $r=2$, $n=2$. The existence of the non-tangential limits of each of the following three sets of components implies that a.e. all the other second order components have non-tangential limits:

(1) $\dfrac{\partial^2 H}{\partial y^2}\left(= -\dfrac{\partial^2 H}{\partial x_1^2} - \dfrac{\partial^2 H}{\partial x_2^2}\right)$

(2) $\dfrac{\partial^2 H}{\partial y\, \partial x_1}$ and $\dfrac{\partial^2 H}{\partial y\, \partial x_2}$

(3) $\dfrac{\partial^2 H}{\partial x_1 \partial x_2}$ and $\dfrac{\partial^2 H}{\partial x_1^2} - \dfrac{\partial^2 H}{\partial x_2^2}$.

That (1) does is of course contained in Theorem 5. From Theorem 4 it may be shown that the existence of non-tangential limits of (2) also implies the other second-order components. (3) may be proved by similar arguments.

This leads us to the following general question. Suppose $P_1(D)$, $P_2(D)$ ... $P_k(D)$ are k given differential polynomials in $\partial/\partial y$, $\partial/\partial x_1$, ..., $\partial/\partial x_n$, homogeneous of degree r.

Question: What conditions must be imposed on $P_1, P_2, \dots, P_k$, so that they are *determining* in the following sense: If H is harmonic in $(x_1, x_2, x_3, \dots, x_n, y)$ and $P_1(D)H$, $P_2(D)H$, ..., $P_k(D)H$ have non-tangential limits in E, then a.e. in E so does any derivative of order r of H.

We shall now attempt to answer this question.

Suppose that $P(D)$ is a homogeneous polynomial of degree r in $\partial/\partial y$, $\partial/\partial x_1$, ..., $\partial/\partial x_n$, we shall consider with in the associated polynomial, $p(X)$, which is a polynomial of the n variables $x_1, x_2, \dots, x_n$ of degree $\leqslant r$.

First, there exists a homogeneous polynomial of degree r in $\partial/\partial y$, $\partial/\partial x_1$, ..., $\partial/\partial x_n$, $P^*(D)$, so that

(a) $P^*(D)H = P(D)H$, whenever H is harmonic,

(b) $P^*(D)$ is of degree $\leqslant 1$ in $\partial/\partial y$.

$P^*(D)$ is obtained from $P(D)$ by replacing $\partial^j/\partial y^j$ by $(-)^{\frac{1}{2}j}\,(\partial^2/\partial x_1^2 + \partial^2/\partial x_2^2 \dots + \partial^2/\partial x_n^2)^{\frac{1}{2}j}$ if j is even; and replacing $\partial^j/\partial x^j$ by $\partial/\partial y\,(-)^{\frac{1}{2}(j-1)}(\partial^2/\partial x_1^2 + \dots + \partial^2/\partial x_n^2)^{\frac{1}{2}(j-1)}$, if j is odd. Thus

$$P^*(D) = s\left(\frac{\partial}{\partial X} + \frac{\partial}{\partial y}\, t\left(\frac{\partial}{\partial X}\right)\right),$$

where s and t are respectively homogeneous polynomials of degrees r and $r-1$, and $\partial/\partial X = (\partial/\partial x_1, \partial/\partial x_2, \dots, \partial/\partial x_n)$.

We then define $p(X)$ by

$$p(X) = s(X) + t(X).$$

Examples of this definition are as follows:

(a) If $P(D)=\partial^r/\partial x_j^r$, then the associated polynomial, $p(X)$, is x_j^r.

(b) If $P(D)=\partial^r/\partial y^r$, then $p(X)=(-)^{\frac{1}{2}r}(x_1^2+x_2^2 \dots + x_n^2)^{\frac{1}{2}r}$, if r is even;

$$p(X)=(-)^{\frac{1}{2}(r-1)}(x_1^2+x_2^2 \dots + x_n^2)^{\frac{1}{2}(r-1)})$$

if r is odd.

We can now obtain the following result which is a refinement of Theorem 5.

THEOREM 6. *A sufficient condition that the k differential polynomials $P_1(D)$, $P_2(D)$, ..., $P_k(D)$ are determining is that the common complex zeroes of the k associated polynomials $p_1(X)$, $p_2(X)$, ..., $p_k(X)$ satisfy $x_1^2+x_2^2+x_3^2 \dots + x_n^2=0$.*

We list two immediate consequences of interest.

COROLLARY 1. *The existence of the non-tangential limits in a set E of the components*

$$\frac{\partial^r H}{\partial x_1^r}, \frac{\partial^r H}{\partial x_2^r}, \dots, \frac{\partial^r H}{\partial x_n^r},$$

implies the existence of non-tangential limits a.e. in E of all the other derivatives of order r of H.

COROLLARY 2. *Let $H(x_1, x_2, y)$ be harmonic in E_3^+. Suppose that the two functions v_1 and v_2, defined by*

$$v_1=\sum_{j\text{ even}}(-)^{\frac{1}{2}j}C_j^r\frac{\partial^r H}{\partial x_1^{r-j}\partial x_2^j},\quad v_2=\sum_{j\text{ odd}}(-)^{\frac{1}{2}(j-1)}C_j^r\frac{\partial^r H}{\partial x_1^{r-j}\partial x_2^j},$$

have non-tangential limits the in E. (C_j^r are binomial coefficients.) Then a.e. in E the same is true of all the other r-th derivatives of H.

The significance of the type of conjugacy arising in Corollary 2 will be discussed further in Section 8.

We shall need the following lemma.

LEMMA 8. *Let I be an ideal in the ring of all polynomials in $x_1, x_2, \dots, x_n$. Suppose that $f(X)$ is a given polynomial. Then a sufficient condition that there exists an integer N so that $(f(X))^N \in I$ is that the (complex) zeroes of f be contained in the common zeroes of I.*

The condition is evidently necessary. This is the Hilbert "Nullstellensatz" for the complex number field. See e.g. [12], § 79.

Proof of the theorem. Let I be the ideal generated by $p_1(X)$, $p_2(X)$ ... $p_k(X)$. Take $f(X)=x_1^2+x_2^2 \dots + x_n^2$ in the above lemma. Because of our assumptions on the

$p_j(X)$ it follows that there exists an integer N, and polynomials $q_1(X), q_2(X), \ldots, q_k(X)$ so that

$$\sum_{j=1}^{k} q_j(X)\, p_j(X) = (x_1^2 + x_2^2 \ldots + x_n^2)^N. \tag{7.3}$$

Recall that

$$p_j(X) = s_j(X) + t_j(X),$$

where the s_j are homogeneous of degree r, and t_j are homogeneous of degree $r-1$. It should be noted from this that $2N \geqslant r-1$ and that we may assume the degrees of the $q_j(X)$ are $\leqslant 2N - r + 1$. Write now

$$q_j(X) = \sum_e q_{j,e}(X),$$

where $q_{j,e}(X)$ is homogeneous of degree e. Thus we have

$$\sum_j \left(\sum_e q_{j,e}(X)\right)(s_j(X) + t_j(X)) = (x_1^2 + x_2^2 \ldots + x_n^2)^N.$$

Making the substitution $\{x_j\} \to \{\xi^{-1} x_j\}$, we get

$$\sum_j \left(\sum_e q_{j,e}(X)\,\xi^{-e}\right)(s_j(X)\,\xi^{-r} + t_j(X)\,\xi^{-r+1}) = \xi^{-2N}(x_1^2 + x_2^2 \cdots + x_n^2)^N,$$

and hence

$$\sum_j \left(\sum_e q_{j,e}(X)\,\xi^{2N-e}\right)(s_j(X) + t_j(X)\,\xi) = \xi^r (x_1^2 + x_2^2 \ldots + x^2)^N. \tag{7.4}$$

Since the degree of $q_j(X)$ is $\leqslant 2N - r + 1$, it follows that $e \leqslant 2N - r + 1$ (and since $r \geqslant 1$, then $2N - e \geqslant 0$.

Now in the above polynomial identity, substitute for $X = (x_1, x_2, \ldots, x_n)$, $(\partial/\partial x_1, \partial/\partial x_2, \ldots, \partial/\partial x_n)$; and for ξ substitute $\partial/\partial y$. (7.4) so transformed and applied to a harmonic function H, gives

$$\left(\sum_j Q_j(D)\, P_j^*(D)\right) H = (-)^N \frac{\partial^{2N+r} H}{\partial y^{2N+r}}, \tag{7.5}$$

where

$$P_j^*(D) = s_j\left(\frac{\partial}{\partial X}\right) + \frac{\partial}{\partial y} t_j\left(\frac{\partial}{\partial X}\right)$$

and

$$Q_j(D) = \sum_e q_{j,e}\left(\frac{\partial}{\partial X}\right) \cdot \left(\frac{\partial}{\partial y}\right)^{2N-e},$$

This shows that the $Q_j(D)$ are homogeneous in $\partial/\partial y, \partial/\partial x_1, \ldots, \partial/\partial x_n$ of degree $2N$.

Let now $u = \partial^r H/\partial y^r$ and $v_j = P_j(D)\, H$. Then (7.5) becomes

$$\frac{\partial^{2N} u}{\partial y^{2N}} = (-)^N [Q_1(D)\, v_1 + Q_2(D)\, v_2 \ldots + Q_n(D)\, v_n]. \tag{7.6}$$

This is exactly of the form (6.1) appearing in Theorem 3 (here $2N=r$). By our assumption $(v_1, v_2, \dots, v_k)=(P_1(D)H, P_2(D)H, P_k(D)H)$ have non-tangential limits in E. Thus by Theorem 3 so has $u=\partial^r H/\partial y^r$ a.e. in E. The proof of the theorem is completed by appealing to Theorem 5.

We add one final remark. It is possible that the condition that the associated polynomials vanish jointly only on the set $x_1^2+x_2^2 \dots + x_n^2=0$, is not the best possible. In fact, it may be conjectured that a necessary and sufficient condition that the polynomials $P_1(D), \dots, P_k(D)$ are determining (in the sense defined above) is that the only *real* common zero of the associated polynomials be the origin. The proof of this latter assertion, if true, would seem to be beyond the methods of this section.

8. Relations with generalized Hilbert transforms

Up to now we have considered harmonic functions defined in E_{n+1}^+ (or smaller subsets) and have studied relations of conjugacy given by differential equations like (6.1). We want now to investigate further this meaning of conjugacy in terms of harmonic functions which are Poisson integrals (of their boundary values in E_n). This will allow us to understand relation (6.1) in terms of (a) the boundary values of the harmonic functions, and (b) the Fourier transforms of these boundary values.

Let us consider first the classical case ($n=1$). See, e.g., [11], Chap. 5. Let $u(x, y)$ be the Poisson integral of a function $f(x)$ belonging to, say, $L^2(-\infty, \infty)$. [1] Then

$$u(x, y)=\frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{y}{y^2+(x-z)^2} f(z)\,dz.$$

Moreover, the conjugate function $v(x, y)$ (which is determined up to an additive constant) is again the Poisson integral of a function, $g(x)$, belonging to $L^2(-\infty, \infty)$. $g(x)$ and $f(x)$ are related by the Hilbert transform

$$g(x)\equiv\frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{f(z)}{z-x}\,dz, \tag{8.1}$$

the integral existing a.e. in the principal value sense. If we denote by $\hat{f}$ and $\hat{g}$ respectively the Fourier transforms of f and g, then we have

$$\hat{g}(x)=-i\,\text{sign}\,(x)\,\hat{f}(x). \tag{8.2}$$

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(1) The limitation to functions in L^2 is made only for the sake of convenience and is not necessary. Many other classes of functions would do.

We can take (8.1) (or alternatively (8.2)), which expresses the classical relation of conjugacy in terms of boundary values, as our starting point.

The transformation (8.1) has a well-known generalization to n dimensions. If we use the notation $X=(x_1, x_2, \dots, x_n)$, $Z=(z_1, z_2, \dots, z_n)$, then we consider transformation $f \to T(f)$ defined by

$$T(f)(X) \equiv g(X) = a f(X) + \int_{E_n} \frac{\Omega(X-Z)}{|X-Z|^n} f(Z)\, dZ. \tag{8.1*}$$

Here $\Omega(Z)$ is a function which is homogeneous of degree zero (that is, depends only on the direction of the vector Z) and has the further property that its mean-value on the unit sphere vanishes; a is a constant. The integral exists a.e. in the principal value sense, if we restrict Ω and f appropriately; moreover, if Ω satisfies certain minimal restrictions (e.g., Ω is bounded), the transformation is a bounded operator on $L^2(E_n)$. For those Ω, the Fourier transform

$$a + \lim_{\varepsilon \to 0} \int_{\varepsilon < |Z| < 1/\varepsilon} \frac{\Omega(Z)}{|Z|^n} e^{-iX \cdot Z}\, dZ = \hat{\Omega}(X)$$

exists for every X, is bounded and is homogeneous of degree zero. Moreover, if $\hat{f}$ and $\hat{g}$ denote the Fourier transforms of f and g respectively, then

$$\hat{g}(X) = \hat{\Omega}(X) \hat{f}(X). \tag{8.2*}$$

Thus (8.1*) and (8.2*) are clearly the generalizations of (8.1) and (8.2). The function $\hat{\Omega}(X)$ is sometimes referred to as the *symbol* of the transformation (8.1*).

Since $\Omega(X)$ is a function which is completely determined by its values on the unit sphere, we expand it in spherical harmonics. That is, we have

$$\Omega(X) = \Omega_1(X) + \Omega_2(X) + \dots + \Omega_N(X) + \dots, \tag{8.3}$$

where $\Omega_N(x)$ has the following properties:

(a) $|X|^N \Omega_N(X)$ is a homogeneous polynomial of degree N.

(b) $|X|^N \Omega_N(X)$ is harmonic, in the n variables $X=(x_1, x_2, \dots, x_n)$.

(c) The expansion (8.3) converges to Ω in L^2 norm of the unit sphere.

We now define $\hat{\Omega}_N(X)$ as the Fourier transform of $\Omega_N(X)/|X|^n$ (as in the formula defining $\hat{\Omega}(X)$ above, with $a=0$), then we have

$$\hat{\Omega}_N(X) = \gamma_n^{(N)} \Omega_N(X). \tag{8.4}$$

Here $\gamma_n^{(N)}$ is a non-zero constant depending only on the degree N and the dimension. It does not otherwise depend on Ω_N. From this we can easily conclude the kernel Ω of the transformation (8.1*) is a (spherical harmonic) polynomial if and only if $\hat{\Omega}$ has the same property. The results in the foregoing paragraphs related to the transformation (8.1*) may be found in, e.g. [3], where there are other references.

Let us now return to harmonic functions in E_{n+1}^+, more properly those which are Poisson integrals.

Let $u(X, y)$ be the Poisson integral a function $f(X) \in L^2(E_n)$. Since $u(X,y) \to f(X)$ in the L^2 norm as $y \to 0$, we shall denote by $u(X, 0) = f(X)$. Similarly let $v(X, y)$ be the Poisson integral of $v(X, 0) \in L^2(E_n)$. We let $\hat{u}(X)$, $\hat{u}(X, y)$, $\hat{v}(X)$, and $\hat{v}(X)$ denote the Fourier transforms of $u(X, 0)$, $u(X, y)$, $v(X, 0)$, and $v(X, y)$ respectively (as functions of X, for each fixed y). Then, as is well known,

$$\hat{u}(X, y) = e^{-y|X|}\hat{u}(X); \quad \hat{v}(X, y) = e^{-y|X|}\hat{v}(X). \tag{8.5}$$

We now assume that $u(X, 0)$ and $v(X, 0)$ are related by (8.1*); that is

$$u(X, 0) = a\,v(X, 0) + \int_{E_n} \frac{\Omega(X-Z)}{|X-Z|^n} v(z, 0)\,dz. \tag{8.6}$$

This means that the Poisson integrals are related similarly,

$$u(X, y) = a\,v(X, y) + \int_{E_n} \frac{\Omega(X-Z)}{|X-Z|^n} v(Z, y)\,dZ, \tag{8.7}$$

and their Fourier transforms are related by

$$\hat{u}(X, y) = \hat{\Omega}(X)\,\hat{v}(X, y). \tag{8.8}$$

By what has been said above it is not difficult to see that indeed (8.6), (8.7), and (8.8) are fully equivalent.

From now on it will be convenient to adopt the following notation. j will stand for a multi-index of n components. Thus $j = (j_1, j_2, \dots, j_n)$. The symbol X^j will stand for the monomial $x_1^{j_1}, x_2^{j_2} \dots x_n^{j_n}$, and $|j|$ will stand for its degree, $|j| = j_1 + j_2 + \dots + j_n$. Similarly $\partial^{|j|}/\partial X^j$ will stand for the differential monomial $(\partial/\partial x_2)^{j_1} (\partial/\partial x_2)^{j_2} \dots (\partial/\partial x_n)^{j_n}$.

Reverting to our discussion, let us take the case where $\Omega(X) = \Omega_N(X)$, and $a = 0$. Then $|X|^N \hat{\Omega}_N(X)$ is a homogeneous polynomial of degree N, and hence in our notation we write

$$\hat{\Omega}_N(X) = |X|^{-N} \sum_{|j|=N} a_j X^j$$

and then
$$\hat{u}(X, y) = \{|X|^{-N} \sum_{|u|=N} a_j X^j\} \hat{v}(X, y). \tag{8.9}$$

Let us now recall that the Fourier transform of $\partial^{|j|} v(X, y)/\partial X^j$ is $(i)^N X^j \hat{v}(X, y)$, $|j| = N$.

Moreover, in view of (8.5) the Fourier transform of $\partial^N u(X, y)/\partial y^N$ is

$$(-|X|)^N \hat{u}(X, y).$$

This shows that the relation (8.9) is equivalent with

$$\frac{\partial^N u(X, y)}{\partial y^N} = P_N(D) v(X, y), \tag{8.10}$$

where
$$P_N(D) = i^{-N} \sum_{|j|=N} a_j \frac{\partial^{|j|}}{\partial X^j}.$$

More generally, suppose that on the unit sphere $\hat{\Omega}$ is a spherical harmonic polynomial of degree r. Then

$$\hat{\Omega}(X) = a + \hat{\Omega}_1(X) + \dots + \hat{\Omega}_r(X).$$

It then follows by the same argument that the relation (8.7) or (8.6) (for this $\hat{\Omega}$) is the equivalent with

$$\frac{\partial^r u(X, y)}{\partial y^r} = P(D) v(X, y), \tag{8.11}$$

where
$$P(D) = a\left(\frac{\partial}{\partial y}\right)^r + \sum_{N=1}^{r} \left\{i^{-N} \sum_{|j|=N} a_j \frac{\partial^{|j|}}{\partial X^j}\right\} \left(\frac{\partial}{\partial y}\right)^{r-N}. \tag{8.12}$$

We thus see that whenever Ω is a spherical harmonic polynomial (or what amounts to the same thing, $\hat{\Omega}$ is a spherical harmonic polynomial), then the transformation (8.1*) when expressed in terms of Poisson integrals can be written in the form (8.11).

Let us now consider the converse. We thus have two harmonic functions $u(X, y)$ and $v(X, y)$ which are Poisson integrals of $u(X, 0)$, $(X, 0)$ respectively (both being in $L^2(E_n)$ and are related by (8.11). (Notice that (8.12) represents the most general differential polynomial homogeneous of degree r in $\partial/\partial y$, ∂x_1, …, $\partial/\partial x_n$.) We then have

$$\hat{u}(X, y) = \{a + \sum_{N=1} |X|^{-N} \sum_{|j|=N} a_j X^j\} \hat{v}(X, y).$$

Now it is known that any polynomial, e.g.,

$$\sum_{|j|=N} a_j X^j$$

is equal on the unit sphere to a harmonic polynomial. Thus

$$a + \sum_{N=1} |X|^{-N} \{ \sum_{|j|=N} a_j X^j \} = a' + \hat{\Omega}_1(X) + \dots + \hat{\Omega}_r(X)$$

(for appropriate Ω_N's) on the unit sphere. Since both sides are homogeneous of degree zero, the above identity holds everywhere. We therefore have

$$\hat{u}(X) = \{a' + \hat{\Omega}_1(X) + \dots + \hat{\Omega}_r(X)\} \hat{v}(X)$$

and hence
$$u(X, 0) = a' v(X, 0) + \int_{E_n} \frac{\Omega(X-Z)}{|X-Z|^n} v(Z, 0)\, dZ,$$

where Ω restricted to the unit sphere in a harmonic polynomial. This proves the converse.

A similar situation holds if we replace the relation (8.11) by one among vectors, as we have done in the above sections.

We discuss briefly two exemples.

(a) The notion of M. Riesz conjugacy, contained in equations (7.1) leads to the generalization of the Hilbert transform whose "symbol" is the vector $-i(x_1/|X|, x_2/|X|, \dots, x_n/|X|)$. The n component transformations (corresponding to (8.1*) with $a=0$) are then the so-called Riesz transforms. These transformations were discussed in paper I. For more details, see also [7].

(b) We next consider the notion of conjugacy implicit in Corollary 2 of Theorem 6. Since $X = (x_1, x_2)$ it is convenient to use polar coordinates, $x_1 + i x_2 = \varrho e^{i\theta}$. Because $\hat{\Omega}(X)$ is a function homogeneous of degree zero, we can consider it as a function of θ, and write it as $\hat{\Omega}(\theta)$, $0 \leqslant \theta \leqslant 2\pi$. Of special interest is the case

$$\hat{\Omega}(\theta) = e^{ir\theta} \quad \text{or} \quad e^{-ir\theta}, \quad r \text{ a positive integer.}$$

Since
$$e^{ir\theta} = (\cos\theta + i \sin\theta)^r = \left(\frac{x_1}{|X|} + i \frac{x_2}{|X|}\right)^r,$$

we have
$$e^{ir\theta} = |X|^{-r} \{ \sum_{j \text{ even}} (-)^{\frac{1}{2}j} C_j^r x_1^j x_2^{r-j} + i \sum_{j \text{ odd}} (-)^{\frac{1}{2}(j-1)} C_j^r x_1^j x_2^{r-j} \}.$$

Thus for $\hat{\Omega}(\theta) = e^{ir\theta}$, we have

$$\frac{\partial^r u}{\partial y^r} = P(D) r,$$

where
$$P(D) = \left[-i \left(\frac{\partial}{\partial x_1} + i \frac{\partial}{\partial x_2} \right) \right]^r.$$

However, if H is harmonic in (x_1, x_2, y) and $u = \partial^r H / \partial y^r$, then

$$\frac{\partial^r u}{\partial y^r} = \frac{\partial^{2r} H}{\partial y^{2r}} = (-)^r \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}\right)^r H = (-)^r \left(\frac{\partial}{\partial x_1} + i \frac{\partial}{\partial x_2}\right)^r \left(\frac{\partial}{\partial x_1} - i \frac{\partial}{\partial x_2}\right)^r H.$$

But
$$\left(\frac{\partial}{\partial x_1} - i \frac{\partial}{\partial x_2}\right)^r H = v_1 - i v_2,$$

where
$$v_1 = \sum_{j \text{ even}} (-)^{\frac{1}{2} j} C_j^r \frac{\partial^r H}{\partial x_1^{r-j} \partial x_2^j}$$

$$v_2 = \sum_{j \text{ odd}} (-)^{\frac{1}{2}(j-1)} C_j^r \frac{\partial^r H}{\partial x_1^{r-j} \partial x_2^j}.$$

Thus if $v = -i(v_1 - i v_2)$, $u = \partial^r H / \partial y^r$ we have $\partial^r u / \partial y^r = [-i(\partial/\partial x_1 + \partial/\partial x_2)]^r v$, as above.

Also u and v are related by

$$u(X, y) = \int_{E_2} \frac{\Omega(X - Z)}{|X - Z|^2} v(Z, y)\, dZ$$

with $\gamma_1^{|r|} \Omega(\varrho e^{i\theta}) = e^{ir\theta}$.

It should be noted that the transformation T with this kernel is unitary (on $L^2(E_2)$) and its inverse is obtained by replacing r with $-r$.

We now summarize the discussion in this section in the following theorem.

THEOREM 7. *Let $u(X, y)$, $v(X, y)$ be two harmonic functions in E_{n+1}^+, which are Poisson integrals of $u(X, 0)$, $v(X, 0) \in L^2(E_n)$, respectively.* (1) *Then the following three statements are equivalent*

(1) *$u(X, y)$ and $v(X, y)$ are related by*

$$\frac{\partial^r u(X, y)}{\partial y^r} = P(D)\, v(X, y),$$

where $P(D)$ is a differential polynomial in $\partial/\partial y, \partial/\partial x_1, \ldots, \partial/\partial x_n$ homogeneous of degree r.

(2) $$u(X, 0) = a\, v(X, 0) + \int_{E_n} \frac{\Omega(X - Z)}{|X - Z|^n} v(Z, 0)\, dZ,$$

where Ω is homogeneous of degree 0, and coincides on the unit sphere with a harmonic polynomial.

(1) See the footnote on p. 168.

$$\hat{u}(X, 0) = \{\hat{\Omega}(X)\}\, \hat{v}(X, 0), \tag{3}$$

where $\hat{\Omega}$ *is homogeneous of degree zero and coincides on the unit sphere with a harmonic polynomial, and* $\hat{u}(X, 0)$, $\hat{v}(X, 0)$ *denote respectively the Fourier transforms of* $u(X, 0)$, $v(X, 0)$.

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