PROBLEMS OF REPRESENTATION AND UNIQUENESS FOR FUNCTIONS OF A COMPLEX VARIABLE.

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1°. **Introduction.** This work is essentially a continuation of the function-theoretic developments, which the present author undertook in two extensive memoirs, *in the sequel referred to as* (T_1)[1] *and* (T_2)[2], respectively. In order to save space it will be assumed that the reader is familiar with the main ideas and principles involved in those two works.

We shall first establish a number of theorems relating to the representation of classes of functions (of a complex variable), characterized by certain descriptive properties, by integrals of the form

$$(1^\circ.\,1)\qquad \iint \frac{d\mu(e_\zeta)}{\zeta - z}, \quad \iint \log(\zeta - z)\, d\mu(e_\zeta) \qquad (z = x + iy),$$

[1] W. J. TRJITZINSKY, *Théorie des fonctions d'une variable complexe définies sur des ensembles généraux*, Ann. Ec. Norm., (3), LV, Fasc. 2, pp. 119—191.
[2] W. J. TRJITZINSKY, *Some general developments in the theory of functions of a complex variable*, Acta mathematica, vol. 70 (1938), pp. 63—163.

where $\mu(e)$ is an additive function of sets[1] and $\zeta = \zeta' + i\zeta''$ is the variable with respect to which integration is performed.

Consideration of (T_1), (T_2) and of the present work will make it clear that integrals more complicated than or essentially different from those in $(1^\circ. 1)$ are not necessary, in so far as our main interest is in functions *monogenic* (that is, possessing a unique derivative) in one sense or another over suitable measurable sets.

Results regarding representation of functions as integrals $(1^\circ. 1)$ are involved in Theorems 1. 1, 2. 1, 2. 2, 2. 3, 3. 1. Theorem 2. 1, of which Theorems 2. 2, 2. 3 are Corollaries, relates to classes of functions $C\{\varrho(n)\}$ (Definition 2. 1). The introduction of such classes is made plausible and natural in the light of the developments relating to continuous extension functions and leading to Theorem 1. 2.

In section 4 an investigation is given relating to conditions securing monogeneity and existence of derivatives up to any assigned order: monogeneity on a set that may be without interior points does not necessarily imply analyticity in the set.

Outside of representation theorems, various problems of uniqueness constitute the most important and, incidentally, the most difficult part in a theory, such as developed by the present author. The properties of uniqueness are suggested by various properties of uniqueness which the class of analytic functions possesses. A non analytic class of functions having a particular property of this type could appropriately be termed 'quasi analytic' in a suitable sense. However, traditionally the term 'quasi analytic' has been applied mostly to classes of functions possessing the property of unique determination of the members f of the class, involved, by the values $f^{(\nu)}$ $(\nu = 0, 1, \ldots)$ at a point z_0. Sections 5, 6, 7 relate to uniqueness properties.

Preliminary to the study of uniqueness properties (P), related to sets of positive planar measure, we establish a Theorem 5. 1 on analytic functions, which is along the lines of a similar result due to Beurling and utilized by him in a study of such properties for certain functional classes consisting of limits of rational functions. In Theorem 5. 2 we establish conditions securing properties (P) for limits of analytic functions. The same problem for integrals $(1^\circ. 1)$ is involved in Theorem 6. 1, while for limits of sequences of rational functions, converging uniformly on a closed set G, the problem is treated in Theorem 6. 2.

[1] All the sets mentioned are Lebesgue measurable. Unless the contrary is implied integrations are in the Lebesgue-Stieltjes sense.

Uniqueness properties (C) relating to arcs are studied in section 7. Conditions securing properties (C) for limits of rational functions are given in Theorem 7.1. The method is of the type used for a similar purpose by J. WOLFF. We go further and in Theorem 7.2 establish uniqueness properties related to denumerable sets, the functions studied being expressible as certain limits of rational functions.

The leading idea in (T_1), (T_2) and the present work is that we study functions of a complex variable which are *not necessarily analytic* but which at the same time are sufficiently specialized so as to come within the scope of classical analytic tools (like Lebesgue-Stieltjes integration). It appears that conditions of monogeneity in one sense or another (and, in fact, in a rather generalized sense) over sets in the complex plane, which may be without interior points, are the conditions which give the desired degree of specialization. On the other hand, the classes of functions, so obtained, are of such great vastness and the possibility of classifying these functions according to various uniqueness properties and of subsequently studying them is so wide that there appears to be on hand a very extensive field for new investigation.

It was BOREL who, inspired by Cauchy's point of view in the field of analytic functions, developed a theory of functions, now termed 'Borel monogenic', for which he established a fundamental contour integral formula analogous to Cauchy's integral formula. On the basis of his formula Borel developed a theory of 'Borel monogenic functions; these, incidentally, form a class of functions quasi analytic in the traditional sense. Borel also gave an indication that two-dimensional integrals in place of contour integrals may be fruitful for the further advance of the theory. Under the inspiration of Cauchy and Borel the present author was confirmed in the conviction that the foundation for the theory should be an analogue of Cauchy's integral formula. Hence the representations in terms of two-dimensional Lebesgue-Stieltjes integrals (1°. 1); theorems relating to such representations were already given in (T_1), (T_2). Accordingly, the representation theorems of sections 1, 2, 3 are designed to serve as the basis of our theory, just as Cauchy's fundamental formulas serve effectively as the basis of the theory of analytic functions.

It has been pointed out in (T_1) and (T_2) that functions representable by integrals (1° 1) are also representable, under certain rather wide conditions, in the form

(A) $$\lim_{\nu} f_\nu(z) \qquad (f_\nu(z) \text{ analytic})$$

for z in suitable sets G, which could possibly have no interior points. In (T_2) conditions have been found under which functions of type (1°. 1) are expressible in suitable sets G as

(B) $$\lim_{\nu} \left[\frac{a_{\nu,1}}{z-\alpha_{\nu,1}} + \frac{a_{\nu,2}}{z-\alpha_{\nu,2}} + \cdots + \frac{a_{\nu,m_\nu}}{z-\alpha_{\nu,m_\nu}}\right];$$

under certain other conditions functions (1°. 1) may be expressed as infinite series

(C) $$\sum_{\nu} \frac{a_\nu}{z-\alpha_\nu} \qquad (\Sigma\, |a_\nu| \text{ convergent}).$$

In other words, the various classes of monogenic functions for which representation formulas in terms of integrals (1°. 1) have been established have, for that same reason, also representations (A), (B), (C), valid under suitable conditions. These conditions for (A), (B), (C) are of increasing order of restrictiveness — in the order stated. For this reason, when we study a class of functions with a certain uniqueness property (X) and note that the study can be based on either one or all of the four types of representations,

(1°. 1), (A), (B), (C),

we at the same time observe that the use of these representations is of decreasing degree of desirability in the stated order. This remark is made from the point of view of the generality of the results obtained. The study of functions with a particular uniqueness property requires a specific method. It may happen that the available method does not conveniently apply to integrals (1°. 1); we then apply it, if practicable, to the representation (A); if this is not practicable, we apply it to representation (B), provided the method is suitable for that purpose — and so forth. Also, as is to be expected, in some instances results obtainable, for instance, for (B) will be simpler than those obtainable for (A) and (1°. 1).

The above considerations make apparent the connection between the 'uniqueness' sections 5, 6, 7 and the 'representation' sections 1, 2, 3; these considerations also explain why certain classes of functions have been made a subject of study in sections 4, 5, 6, 7.

In order to follow the terminology in the use of integrations and of additive functions of intervals and of sets reference to a book of S. SAKS[1], in the sequel referred to as (S), will be helpful.

1. **Problems of Representation for General Monogenic Functions of** (T_1) **and Concerning Extension Functions.** In (T_1) we introduced functions which we termed 'general monogenic'. The first development established for such functions was our representation of them with the aid of a double integral. The definition of 'general monogenic' functions as well as this integral representation will be now put on a more rigorous basis. With this purpose in view, let us recall certain pertinent facts about integrals

$$V(M) = \iint_{\Omega} \log\left(\frac{1}{MP}\right) \psi(P)\, d\zeta\, d\eta \qquad (M = (x, y);\ P = (\zeta, \eta)), \tag{1.1}$$

where Ω is a domain (open). If $\psi(P)$ is continuous bounded then $V(M)$ has continuous first order partials which can be obtained by differentiating under the integral sign. If $\psi(P)$ has continuous first order partials or satisfies Hölder's condition or the more general PETRINI[2] conditions then $V_{x,x}$, $V_{y,y}$ exist and $\varDelta V = V_{x,x} + V_{y,y} = -2\pi\psi(M)$. A generalized Laplacian $\varDelta$[3] is an operator with the properties: (1°) if $u_{x,x}$, $u_{y,y}$ exist then $\varDelta u = u_{x,x} + u_{y,y}$; (2°) if $\varDelta u$, $\varDelta v$ exist then $\varDelta(hu + kv) = h\varDelta u + k\varDelta v$; (3°) if u has a maximum at P then $\varDelta u(P) \leqq 0$; (4°) Poisson's formula holds. The latter condition amounts to the assertion that, whenever ψ is continuous in Ω, one has

$$\varDelta\left[\iint_{\gamma} \log \frac{1}{MP} \psi(P)\, d\zeta\, d\eta\right] = -2\pi\psi(M) \tag{1.1a}$$

for M interior a circle γ, whose closure lies in Ω. A simple generalized Laplacian is due to ZAREMBA[4],

$$\varDelta u = \lim_{h \to 0} h^{-2}[u(x+h, y) + u(x-h, y) - 2u(x, y) + u(x, y+h) + u(x, y-h) - 2u(x, y)].$$

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[1] S. SAKS, *Theory of the Integral*, Warszawa-Lwow, 1937.

[2] PETRINI, *Les dérivées premières et secondes du potentiel*, Acta mathematica, vol. 31 (1908), pp. 127—332; also Journ. de Liouville (1909), pp. 127—223.

[3] See BRELOT, Mémorial des Sciences Math., Fasc. XI; in particular, pages 3, 14, 13.

[4] ZAREMBA, *Contribution à la Théorie d'une équation fonctionnelle de la physique*, Rendiconti di Palermo, vol. 19 (1905), pp. 140—150.

Brelot established that if u has a particular generalized Laplacian $\Delta_0 u$, continuous in a domain ω, then all generalized Laplacians of u will coincide with $\Delta_0 u$ in ω. It is also known that Green's formula

$$(1.2) \qquad \iint\limits_D (\varphi \Delta \psi - \psi \Delta \varphi)\, d\zeta\, d\eta + \int \left(\varphi \frac{\partial \psi}{\partial n} - \psi \frac{\partial \varphi}{d n}\right) ds = 0$$

is valid whenever φ, ψ have generalized Laplacians continuous in a domain D_1 containing the closure of D.

Given a function $u(x, y)$, *defined and continuous in a bounded closed set* F, *suppose that the derivatives*

$$(1.3) \qquad u_x,\ u_y,\ u_{x,x},\ u_{x,y},\ u_{y,y}$$

exist in F *in the sense of* Whitney[1] *and are continuous in* F, *while*

$$(1.3\,\mathrm{a}) \qquad u_{x,x} + u_{y,y} = 0 \qquad (\text{in } F).$$

Let Ω be a bounded domain containing F. Let U be a continuous 'extension' of u to Ω with the properties:

$$U = u,\ U_x = u_x,\ U_y = u_y,\ U_{x,x} = u_{x,x},\ U_{x,y} = u_{x,y},\ U_{y,y} = u_{y,y}$$

in F; on the other hand, U, $U_x, \ldots U_{y,y}$ are continuous in Ω.

We form the function

$$\psi = U_{x,x} + U_{y,y};$$

ψ will be continuous in Ω, while

$$\psi = u_{x,x} + u_{y,y} = 0 \qquad (\text{in } F).$$

With Δ denoting, let us say, the generalized Laplacian of Zaremba, Poisson's formula (1.1 a) will hold, yielding

$$(1.4) \qquad \Delta \iint\limits_\gamma \log \frac{1}{MP} \psi(P)\, d\zeta\, d\eta = \begin{cases} -2\pi \psi(M); \\ 0 \quad (\text{in } F). \end{cases}$$

We now form the function

$$(1.4') \qquad h(x, y) = U(x, y) + \frac{1}{2\pi} \iint\limits_\gamma \log \frac{1}{MP} \psi(P)\, d\zeta\, d\eta.$$

[1] H. Whitney, *Analytic extensions of differentiable functions*, Trans. Am. Math. Soc., vol. 36 (1934), pp. 63—89.

On taking Δ of both sides, in consequence of (1.4) it is inferred that

$$\Delta h = \Delta U - \psi(M).$$

Now, by hypothesis, $U_{x,x}$, $U_{y,y}$ exist in Ω. By virtue of condition (1°) one has

$$\Delta U = U_{x,x} + U_{y,y}$$

and, by definition of ψ,

$$\Delta U = \psi;$$

thus

$$\Delta h = 0 \qquad \text{(in } \Omega).$$

Whence h *is harmonic in* Ω.

In consequence of the above considerations we state the following.

Theorem 1. 1. *Suppose* u *is the real part of a general monogenic function; that is,* u *is defined in* F *and has first and second order partials in* F*, as described in the statement with respect to* (1.3), (1.3 a). *One then will have the following representation formulas, for* $(x, y) = M$ *in* F,

$$u(x,y) = h(x,y) - \frac{1}{2\pi}\iint\limits_{\gamma C F} \log \frac{1}{MP}\psi(P)\, d\zeta\, d\eta \tag{1.5}$$

$[P = (\zeta, \eta)]$, *where* $h(x,y)$ *is harmonic in* Ω, γ *is a circle containing* (x,y) *in its interior, while* $\bar{\gamma} < \Omega$; $\psi(P)$ *is continuous in* Ω *and vanishes in* F*; in fact,*

$$\psi(P) = U_{x,x} + U_{y,y},$$

where U *is an extension function of* u.

We say that a function $f(z)$ of a complex variable $z = x + iy$ is *general monogenic in a closed set* F ($<$ a bounded domain Ω) *if*

$$f(z) = u(x,y) + i\, v(x,y),$$

with $u(x,y)$ *from the above theorem and* v *a harmonic conjugate in* F *of* u. The meaning of the latter expression is that v is defined in F and that the derivatives (in the sense of Whitney) v_x, v_y exist in F and that one has

$$v_x = -u_y, \quad v_y = u_x \qquad \text{(in } F).$$

In view of (1.5) *the following is the representation, in* F, *of a function* $f(z)$ *satisfying the above conditions:*

$$f(z) = a(z) + \frac{1}{2\pi}\iint\limits_{\gamma C F} \psi(\zeta, \eta) \log (z - w)\, d\zeta\, d\eta \qquad (w = \zeta + i\eta); \tag{1.6}$$

here $a(z)$ *is a function analytic in* Ω.

We shall have

$$f^{(1)}(z) = a^{(1)}(z) + \frac{1}{2\pi}\iint\limits_{\gamma CF} \frac{\psi(\zeta,\eta)}{z-w}\,d\zeta\,d\eta \tag{1.6 a}$$

at every limiting point of F.

It is of interest to observe the connection between general monogenic functions and subharmonic functions[1]. Let Ω be open connected. It is known that, if at every point M of Ω one has

$$u(M) = -\frac{1}{2\pi}\iint\limits_{\gamma} \log\frac{1}{PM}\psi(P)\,d\zeta\,d\eta \qquad (P=(\zeta,\eta)), \tag{1.7}$$

where circle γ contains M ($\bar{\gamma} < \Omega$) and where $\psi(P) \geqq 0$ is bounded measurable, then $u(M)$ is continuous and subharmonic.

In (1.5) $\psi(P) = U_{x,x} + U_{y,y}$ is continuous; on writing

$$\psi_1(P) = \frac{1}{2}(\psi(P) + |\psi(P)|), \quad \psi_2(P) = \frac{1}{2}(-\psi(P) + |\psi(P)|),$$

one has

$$\psi(P) = \psi_1(P) - \psi_2(P), \quad \psi_1(P) \geqq 0, \quad \psi_2(P) \geqq 0.$$

Accordingly, by (1.4')

$$U(M) = h(x,y) + U_1(M) - U_2(M), \quad U_i(M) = -\frac{1}{2\pi}\iint\limits_{\gamma} \log\frac{1}{MP}\psi_i(P)\,d\zeta\,d\eta$$

(in Ω); here, in consequence of the remark with respect to (1.7), the $U_i(M)$ ($i = 1, 2$) are continuous subharmonic in Ω.

Thus, *the real part $u(x,y)$ of a function general monogenic in F is of the form*

$$u(M) = h(x,y) + U_1(M) - U_2(M) \qquad (M = (x,y)), \tag{1.8}$$

for M in F, where $h(x,y)$ is harmonic and U_1, U_2 are continuous subharmonic in $\Omega > F$.

In the remainder of this section we shall study continuity properties of a continuous extension (over a bounded connected domain K) $f^*(z)$ of $f(z)$ when continuity properties of $f(z)$ (over a closed set $F < K$) are given. This study culminates in Theorem 1.2. This Theorem, as well as the developments leading to it, will make plausible the definition of an extensive class of functions in section 2, a class for which we shall establish an integral representation formula. Without Theorem 1.2 the introduction of the mentioned class of functions might appear quite artifical.

[1] Regarding subharmonic functions see, for instance, Brelot's pamphlet in Actualités Scientifiques et Industrielles, Paris.

For our purpose of the several known methods of construction of continuous extension functions the most suitable one appears to be that of H. BOHR[1]. The pertinent facts with respect to this method may be expressed as follows.

Let $h(z)$ be continuous over F,

$$0 \leqq h(z) \leqq 1 \qquad \text{(in } F\text{)}; \tag{1.9}$$

with $S(z, r)$ denoting a circle of center z and radius r, form the function $\psi(z, r)$ such that

$$\psi(z, r) = \text{upper bound of } h \text{ in } FS(z, r), \tag{1.9'}$$

unless $FS(z, r) = 0$, when $\psi(z, r)$ is defined as zero; on letting

$$\varrho(z) = \text{distance from } z \text{ to } F,$$

one has

$$\varrho(z) \neq 0 \qquad \text{(in } K - F\text{)}.$$

For z in $K - F$ define the function

$$\varphi(z) = \frac{1}{\varrho(z)} \int_{\varrho(z)}^{2\varrho(z)} \psi(z, r)\, dr, \tag{1.9 a}$$

which is continuous in $K - F$; in fact, one has

$$|\varrho(z_1)\varphi(z_1) - \varrho(z)\varphi(z)| \leqq 3\,|z - z_1| \qquad (z, z_1 \text{ in } K - F), \tag{1.9 b}$$

while $\varrho(z)$ is continuous in $K - F$ and is therein different from zero. Also

$$|\varrho(z) - \varrho(z_1)| \leqq |z - z_1| \qquad (z, z_1 \text{ in } K - F). \tag{1.9 c}$$

Finally, there is defined a function

$$H(z) = \begin{cases} h(z) & \text{(in } F\text{)}, \\ \varphi(z) & \text{(in } K - F\text{)} \end{cases} \tag{1.9 d}$$

and it is shown that $H(z)$ is continuous in K; $H(z)$ is accordingly a continuous extension of $h(z)$.

Proceeding with the aid of Bohr's developments, as described above in connection with (1.9)—(1.9 d), we write

$$\varrho(z)(\varphi(z_1) - \varphi(z)) = (\varrho(z_1)\varphi(z_1) - \varrho(z)\varphi(z)) - \varrho(z_1)\varphi(z_1) + \varrho(z)\varphi(z_1);$$

thus in consequence of (1.9 b) and (1.9 c) we obtain

$$|\varrho(z)(\varphi(z_1) - \varphi(z))| \leqq 3\,|z - z_1| + |\varphi(z_1)|\,|\varrho(z) - \varrho(z_1)| \leqq 3\,|z - z_1| + |\varphi(z_1)|\,|z - z_1|$$

[1] See, for instance, CARATHÉODORY, *Vorlesungen über reelle Funktionen*, 1918, pp. 617—620.

for z, z_1 in $K-F$. Inasmuch as $0 \leqq \psi(z, r) \leqq 1$ (cf. (1.9), (1.9')), from (1.9a) it follows that

$$0 \leqq \varphi(z) \leqq 1 \qquad \text{(for } z \text{ in } K-F).$$

Hence

$$|\varphi(z_1)-\varphi(z)| \leqq \frac{4|z-z_1|}{\varrho(z)} \qquad (z, z_1 \text{ in } K-F). \tag{1.10}$$

Inasmuch as $h(z)$ is continuous in F, F being closed, given $\varepsilon(>0)$ there exists $3\eta(>0)$, independent of z, z_0 so that

$$|h(z)-h(z_0)| \leqq \varepsilon \tag{1.11}$$

for z, z_0 in F such that $|z-z_0|<3\eta$. According to Bohr a consequence of such a hypothesis would be

$$|\varphi(z)-h(z_0)| \leqq \varepsilon \qquad (z_0 \text{ a frontier point of } F) \tag{1.11'}$$

for $|z-z_0|<\eta$ and z in $K-F$, which on taking account of (1.9d), (1.11) leads to the conclusion

$$|H(z)-H(z_0)| \leqq \varepsilon \tag{1.11a}$$

for all z such that $|z-z_0|<\eta$ (z_0 a frontier point of F).

Consider the case when F has interior points. Then a point z_0 of F which s not a frontier point is center of a circular domain $S(z_0, \sigma)$ (of radius σ) such that

$$\bar{S}(z_0, \sigma) < F.$$

Inasmuch as F is closed, σ in the latter ralation may be taken as the upper bound of all radii of circulàr domains with center at z_0 and consisting of points of F. If $\sigma \geqq \eta$, then by (1.11)

$$|H(z)-H(z_0)| \leqq \varepsilon \tag{1.12}$$

for all z such that $|z-z_0| \leqq \eta$. Suppose now $\sigma<\eta$. Then in view of (1.11)

$$|H(z)-H(z_0)| \leqq \varepsilon \tag{1.12a}$$

for all z in F such that $|z-z_0| \leqq \eta$. Let z represent a point in $\bar{S}(z_0, \eta)$ in $K-F$ (necessarily $\sigma<|z_0-z| \leqq \eta$). Consider the set $F_{z_0, z}$ of points of F on the segment (z_0, z) and designate by z' the point of $F_{z_0, z}$ nearest to z. Such a point exists because $F_{z_0, z}$ is closed; moreover,

$$|z_0-z'| \geqq \sigma.$$

Necessarily z' will be a frontier point of F, while

$$|z' - z| \leqq \eta - \sigma < \eta.$$

With z' playing the role of z_0 in (1.11 a), we obtain

$$(1.12\,b) \qquad |H(z) - H(z')| \leqq \varepsilon \qquad (z \text{ in } \bar{S}(z_0, \eta), \text{ in } K - F).$$

Now $|H(z') - H(z_0)| \leqq \varepsilon$ in view of (1.12 a). Hence by (1.12 b)

$$(1.12\,c) \qquad |H(z) - H(z_0)| \leqq |H(z) - H(z')| + |H(z') - H(z_0)| \leqq 2\,\varepsilon$$

for z in $K - F$ such that $|z - z_0| \leqq \eta$. Thus in consequence of (1.12 a), (1.12 c) one has

$$(1.12\,d) \qquad |H(z) - H(z_0)| \leqq 2\,\varepsilon \qquad (\text{if } \sigma < \eta)$$

for all z such that $|z - z_0| \leqq \eta$. This, together with (1.12) and (1.11 a), *implies that*

$$(1.13) \qquad |H(z) - H(z_0)| \leqq 2\,\varepsilon \qquad (z_0 \textit{ any point of } F)$$

for all z such that $|z - z_0| < \eta$.

Continuity of $h(z)$ in F may be expressed by saying that there exists a function $r(u)$, independent of z, z_0, continuous for $u > 0$ and such that

$$r(u) \to 0 \qquad (\text{monotonically as } u \to 0),$$

that

$$(1.14) \qquad |h(z) - h(z_0)| \leqq r(|z - z_0|) \qquad (\text{all } z,\, z_0 \text{ in } F).$$

The equation $v = r(u)$ has a unique inverse $u = r_{-1}(v)$, tending to zero monotonically with v. With the aid of the function $r(u)$ we deduce that η (involved in the statement with respect to (1.11)) may be taken as

$$(1.15) \qquad \eta = \frac{1}{3} r_{-1}(\varepsilon).$$

In fact, an inequality

$$|z - z_0| < 3\,\eta = r_{-1}(\varepsilon)$$

would then imply that

$$r(|z - z_0|) \leqq r(r_{-1}(\varepsilon)) = \varepsilon.$$

Inasmuch as (1.11) implies (1.13), the following may now be asserted. *If $h(z)$ satisfies in F a continuity condition* (1.14), *then*

$$(1.16) \qquad |H(z) - H(z_0)| \leqq 2\,r(3\,|z - z_0|) \qquad (z_0 \text{ any point of } F)$$

for all z in K.

To verify this we note that an inequality $|z - z_0| < \eta$ will imply

$$r(3\,|z - z_0|) \leqq r(r_{-1}(\varepsilon)) = \varepsilon$$

so that (1.16) becomes (1.13).

Let z, z_1 be in $K - F$. Designate by z_0 a point in F nearest to z; then $\varrho(z) = |z - z_0|$. We have

$$|H(z) - H(z_1)| \leqq |H(z) - H(z_0)| + |H(z_0) - H(z_1)|$$

and, in view of (1.16).

$$|H(z) - H(z_1)| \leqq 2\,r(3\,|z - z_0|) + 2\,r(3\,|z_1 - z_0|).$$

Since

$$|z_1 - z_0| \leqq |z_0 - z| + |z - z_1|$$

we have

$$|H(z) - H(z_1)| \leqq 2\,r(3\,|z - z_0|) + 2\,r(3\,|z - z_0| + 3\,|z - z_1|) = r.$$

By virtue of (1.9d) and (1.10)

$$|H(z) - H(z_1)| = |\varphi(z) - \varphi(z_1)| \leqq \frac{4\,|z - z_1|}{|z - z_0|} = r_2.$$

Hence

$$(1.17) \qquad |H(z) - H(z_1)| \leqq r_3 \qquad (z, z_1 \text{ in } K - F),$$

where r_3 is defined as the least of the values r_1, r_2.

We define a function $\varrho(u)$ (> 0), for $0 < u \leqq u_0$, so that

$$(1.18) \qquad r^*(u) \equiv r\left(3\frac{u}{\varrho(u)}\right) + r\left(3\frac{u}{\varrho(u)} + 3\,u\right) \geqq 2\,\varrho(u),$$

while

$$\lim \frac{u}{\varrho(u)} = 0 \qquad (\text{as } u \to 0);$$

in consequence of (1.18) we then shall have

$$\lim\ \varrho(u) = 0 \qquad (\text{as } u \to 0).$$

For a fixed pair of points z, z_1 in $K - F$ one has either

$$(1^\circ) \qquad \frac{|z - z_1|}{|z - z_0|} \leqq \varrho(|z - z_1|)$$

or

$$(2^\circ) \qquad |z - z_0| < \frac{|z - z_1|}{\varrho(|z - z_1|)}.$$

On the case (1°), in view of (1.18) one has

$$r_2 \leqq 4\,\varrho(u) \leqq 2\,r^*(u) \qquad (u = |z - z_1|).$$

When (2°) takes place we have

$$r_1 \leqq 2\,r\left(3\frac{u}{\varrho(u)}\right) + 2\,r\left(3\frac{u}{\varrho(u)} + 3\,u\right)$$

and, by (1.18).

$$r_1 \leqq 2\,r^*(u).$$

Hence, in view of the definition of r_3 (in (1.17)), one has $r_3 \leqq 2\,r^*(u)$ and

$$|H(z) - H(z_1)| \leqq 2\,r^*(|z - z_1|) \tag{1.19}$$

for all z, z_1 *in* $K - F$.

Let z, z_1 be any pair of points in K. Then either at least one of the points, say z_1, will be in F and accordingly (1.16) will hold with $z_0 = z_1$; the alternative is the case when z, z_1 are in $K - F$; (1.19) will then take place. *Accordingly*

$$|H(z) - H(z_1)| \leqq r'(|z - z_1|) \tag{1.20}$$

for all z, z_1 *in* K; *here* $r'(u)$ *is defined for* $u > 0$ *as the greater one of the values*

$$2\,r(3\,u),\ 2\,r^*(u). \tag{1.20 a}$$

Consider the case when

$$r(u) = k\,u^{\alpha} \qquad (0 < \alpha \leqq 1). \tag{1.21}$$

Then, in accordance with (1.18), $\varrho(u)$ is to be such that

$$r^*(u) \equiv k\,3^{\alpha}\,u^{\alpha}\,[\varrho^{-\alpha}(u) + (\varrho^{-1}(u) + 1)^{\alpha}] \geqq 2\,\varrho(u).$$

If we take

$$\varrho(u) = b\,u^{\beta} \qquad \left[\beta = \frac{\alpha}{1 + \alpha},\ b > 0\right], \tag{1.22}$$

it can be shown that b can be chosen suitably small so that $r^*(u) \geqq 2\,\varrho(u)$. In fact, one has

$$r^*(u) \equiv k\,3^{\alpha}\,u^{\alpha}\,\frac{1 + (1 + b\,u^{\beta})^{\alpha}}{b^{\alpha}\,u^{\alpha\,\beta}} > 2\,k\,3^{\alpha}\,b^{-\alpha}\,u^{\beta}. \tag{1.22 a}$$

It will suffice to take b so that the last member above is equal to or is greater than $2\,\varrho(u)$ (for $u > 0$). Whence b is to satisfy the inequality

$$b \leqq b' = [3^{\alpha}\,k]^{(1+\alpha)^{-1}}. \tag{1.22 b}$$

With $\varrho(u)$ thus defined, from (1.22a) it is inferred that

$$r^*(u) \leqq k_\alpha u^\beta \qquad \left\{k_\alpha = k\left(\frac{3}{b}\right)^\alpha [1 + (1 + b u_0^\beta)^\alpha]\right\}, \tag{1.22c}$$

where u_0 is from the inequalities

$$0 < u \leqq u_0;$$

one may take u_0 as the diameter of K. By (1.21)

$$2r(3u) = 2k\,3^\alpha u^\alpha. \tag{1.22c'}$$

In view of the statement with respect to (1.20a) the above relations imply that

$$|H(z) - H(z_1)| \leqq k(\alpha)\,|z - z_1|^{\frac{\alpha}{1+\alpha}} \qquad (z, z_1 \text{ in } K), \tag{1.22d}$$

this being a consequence of (1.21); here

$$k(\alpha) = \text{greater one of } 2k_\alpha,\ 2k\,3^\alpha u_0^{\frac{\alpha^2}{1+\alpha}}.$$

If a real valued function $f(z)$ is continuous in F we have

$$|f(z) - f(z_1)| \leqq c(|z - z_1|) \qquad (z, z_1 \text{ in } F), \tag{1.23}$$

where $c(u)$ is modulus of continuity (in F) of $f(z)$. Now

$$-\lambda_1 \leqq f(z) \leqq \lambda_2 \qquad (\text{in } F;\ -\lambda_1 < \lambda_2).$$

On writing

$$h(z) = \frac{f(z) + \lambda_1}{\lambda_1 + \lambda_2},$$

one has

$$0 \leqq h(z) \leqq 1 \qquad (\text{in } F)$$

and

$$|h(z) - h(z_1)| \leqq \frac{1}{\lambda_1 + \lambda_2} c(|z - z_1|) = r(|z - z_1|) \qquad (z, z_1 \text{ in } F), \tag{1.23'}$$

where the last member is introduced in accordance with (1.14). Let $H(z)$ be the extension of $h(z)$, as described above. *The function*

$$F(z) = (\lambda_1 + \lambda_2)\,H(z) - \lambda_1$$

will be an extension for $f(z)$ and will have the property

$$|F(z) - F(z_1)| \leqq (\lambda_1 + \lambda_2)\,r'(|z - z_1|) \qquad (z, z_1 \text{ in } K), \tag{1.23a}$$

where $r'(u)$ is a function constructed on the basis of

$$r(u) = \frac{1}{\lambda_1 + \lambda_2} c(u),$$

as described previously.

The statement with respect to (1.20), (1.20a) implies, of course, more than has been inferred above. By definition of $r^*(u)$, given in (1.18), one has

$$(1.24) \qquad r^*(u) \geqq 2r\left(2\frac{u}{\varrho(u)}\right)$$

for any $\varrho(u)$. Hence the inequality (1.18) will hold if

$$(1.25) \qquad r\left(3\frac{u}{\varrho(u)}\right) \geqq \varrho(u).$$

There is no essential loss of generality in assuming

$$r(u) \leqq 3 \qquad (0 < u \leqq u_0);$$

in fact, this can be always secured by taking $u_0 (> 0)$ suitably small. We then have

$$\frac{3u}{r(u)} \geqq u$$

and, in view of the monotone character of $r(u)$, the function

$$(1.25\,\mathrm{a}) \qquad \varrho(u) = r(u)$$

will be seen to satisfy (1.25). Hence in consequence of (1.18)

$$(1.25\,\mathrm{b}) \qquad r^*(u) \leqq 2r\left(3\frac{u}{\varrho(u)} + 3u\right) = R^*(u) \equiv 2r\left(\frac{3u}{r(u)} + 3u\right).$$

In accordance with the text in connection with (1.20a) one may then define $r(u)$, for $0 < u \leqq u_0$, as the greater of the values

$$2r(3u), \ 2R^*(u).$$

Whence the modulus of continuity involved in (1.20) *may be taken as*

$$(1.25\,\mathrm{c}) \qquad r'(u) = 4r\left(\frac{3u}{r(u)} + 3u\right),$$

which is a perfectly general formula. This is useful only if $u/r(u) \to 0$, as $u \to 0$.

With respect to the above result it is to be noted that in special instances of $r(u)$ better determinations for $r'(u)$ can sometimes be obtained. This is the case in the situation when

$$r(u) = k u^{\alpha} \qquad (0 < \alpha \leqq 1).$$

In fact, if $r(u)$ is such a function, the relation (1.25 c) will yield

$$r'(u) = k_\alpha u^{(1-\alpha)\alpha} \qquad (k_\alpha > 0).$$

This conclusion is not as good as the one obtained previously. However, for small values of α the discrepancy becomes, in a sense, negligible.

Theorem 1. 2. *If a real valued function $f(z)$ has $r(u)$ for its modulus of continuity in a closed set F (F contained in a bounded domain K), then an extension function $F(z)$, continuous over K, may be formed so that the modulus of continuity of $F(z)$ over $\bar{K}$ is of the form*

$$r'(u) = c^* \, r\left(\frac{3u}{r(u)} + 3u\right).$$

In particular, if $r(u) = c^* u^\alpha$ ($0 < \alpha \leqq 1$) one may take

$$r'(u) = c^* u^\beta \qquad \left(\beta = \frac{\alpha}{1+\alpha}\right).$$

1. **Representation Theorems for Continuous Functions.** In the sequel '*intervals*' will signify rectangles, in the complex plane, with sides parallel to the axes. If I is an interval, (I) will denote its boundary (see (S)). *Let K be a non degenerate fixed interval.*

Let $f(z)$ be a complex valued function of which we may think as defined originally only in a closed subset F of K and which is uniformly Lip. 1 in F; thus,

$$|f(z) - f(z')| \leqq A\,|z - z'| \qquad (z, z' \text{ in } F). \tag{2.1}$$

We extend $f(z)$ continuously over K, say by the method of Bohr. Such an extension will still be designated by $f(z)$. By Theorem 1.2 the relation (2.1) will imply

$$|f(z) - f(z')| \leqq A'\,|z - z'|^{\frac{1}{2}} \qquad (z, z' \text{ in } \bar{K}) \tag{2.2}$$

for the extension function. The continuity of $f(z)$, implied by the above, is too weak' to lead us to expect a representation

$$f(z) = h(z) - \frac{1}{2\pi i}\iint\limits_K \frac{d\mu(e_\zeta)}{\zeta - z}, \tag{2.3}$$

where $\mu(e)$ is a complex valued additive function of sets $\{B\}$ (i. e. Borel sets), while $h(z)$ is analytic in K. If, however, we examine in some detail the continuity properties of a Bohr's extension function in the case when (2.1) holds (see the text in section 1 leading up to Theorem 1.2), it is inferred that

$$|f(z)-f(z')| \leqq A_0 |z-z'| \qquad (z \text{ in } F;\ z' \text{ in } \bar{K}) \tag{2.3 a}$$

and

$$|f(z)-f(z')| \leqq b\,n\,|z-z'| \tag{2.3 b}$$

when z and z' are in $\bar{K}-F$ and are such that

$$\delta(z, F) \geqq \frac{1}{n} \quad \delta(z', F) \geqq \frac{1}{n}; \tag{2.4}$$

here the designation

$$\delta(z, F)$$

stands for *the distance from* z *to* F.

Intuitively one would expect that $f(z)$ will be susceptible to the desired representation, provided that the factor $b\,n$ in the second member of (2.3 b) is replaced by $\varrho(n)$, where $\varrho(n)$ tends to $+\infty$, with n, sufficiently slowly. We are accordingly brought to the following precise Definition.

Definition 2.1. *It will be said that* $f(z)$ *is of class* $C\{\varrho(n)\}$ *if* $f(z)$ *is continuous in* $\bar{K}$,

$$|f(z)-f(z')| \leqq A\,|z-z'| \qquad (z \text{ in } F,\ z' \text{ in } \bar{K}) \tag{2.5}$$

and

$$|f(z)-f(z')| \leqq \varrho(n)\,|z-z'| \tag{2.5 a}$$

for z, z' *in* E_n. *In this connection* E_n *is the totality of those points* ζ *in* $\bar{K}$ *for which* $\delta(\zeta, F) \geqq n^{-1}$.

Naturally the sequence $\varrho(n)$ $(n = 1, 2, \ldots)$ may be considered as monotone non decreasing.

Our problem can be now formulated as follows. *To determine a law of increase of the* $\varrho(n)$ *so that the* $f(z)$ *of the corresponding class* $C\{\varrho(n)\}$ *can be represented in the form* (2.3).

Inasmuch as we want to stay within the range of Lebesgue-Stieltjes integration, $\varrho(n)$ will have to be chosen so that the functions of intervals $\mu(I)$, of which the function $\mu(e)$ is the extension, is of bounded variation. At this stage we might remark that whenever a certain property is assigned to a complex valued function of intervals or sets $\{B\}$, it is to be understood that the real and imaginary components of such a function have this property.

We define the function

$$(2.6)\qquad \mu(I) = B(I) + i\,A(I) = \int_{(I)} f(z)\,dz$$

of intervals I in $\bar{K}$. Since $f(z)$ is continuous as a function of z, the additive functions of intervals $\mu(I)$, $B(I)$, $A(I)$, are of course continuous, that is they vanish with the area of I. With $f(z) < C\{\varrho(n)\}$, offhand there is even no assurance that the above functions of intervals are of bounded variation.

In the sequel a helpful role will be played by a paper of J. H. Binney[1], in the sequal referred to as (B), in accordance with which the following may be asserted.

Let K be an interval as before or, more generally, a simply connected bounded domain and consider the equations

$$(1^\circ)\quad \int_s (\Phi(x,y)\,dy + \theta(x,y)\,dx) = A'(\sigma),\quad \int_s (-\theta(x,y)\,dy + \Phi(x,y)\,dx) = B'(\sigma),$$

where s are simple closed rectifiable curves in K with σ denoting the domains interior s; Φ, θ are unknown functions of points, while $A'(\sigma)$, $B'(\sigma)$ are the values on σ of given additive reaal valued functions of $\{B\}$ sets $e < K$. *Only those curves s are considered for which $A'(s) = B'(s) = 0$.* These equations are satisfied by

$$(2^\circ)\qquad \Phi_0(M) = \frac{1}{2\pi}\iint_K \frac{1}{MP}[\cos(MP, y)\,dB'(e_P) - \cos(MP, x)\,dA'(e_P)],$$

$$\theta_0(M) = \frac{1}{2\pi}\iint_K \frac{1}{MP}[\cos(MP, x)\,dB'(e_P) + \cos(MP, y)\,dA'(e_P)]$$

on all those simple closed rectifiable curves s, in K, on which the functions

$$(2.7)\qquad a(z) = \iint_K \frac{1}{|\zeta - z|}\,d\alpha(e_\zeta),\qquad b(z) = \iint_K \frac{1}{|\zeta - z|}\,d\beta(e_\zeta)$$

are L_1 (i. e., Lebesgue integrable) with respect to arc length; here $\alpha(e)$, $\beta(e)$ are additive functions of $\{B\}$-sets,

$$(2.7')\qquad \begin{aligned} \alpha(e) &= \text{total variation of } A'(e),\\ \beta(e) &= \text{toal variation of } B'(e). \end{aligned}$$

[1] J. H. Binney, *An eiliptic system of integral equations ...*, Trans. Am. Math. Soc., vol. 37 (1935), pp. 254—265.

If one puts

$$f(z) = \Phi(x, y) + i\,\theta(x, y), \qquad \mu'(\sigma) = B'(\sigma) + i\,A'(\sigma),$$

equations (1°) are seen to be equivalent to

$$(3^\circ) \qquad \int_s f(z)\,dz = \mu'(\sigma),$$

the solution of which is

$$(4^\circ) \qquad f(z) = h(z) + \frac{1}{2\pi}\iint_K \frac{1}{MP} e^{i\alpha}\,d\mu'(e_P)$$

($h(z)$ analytic in K; $M = (x, y)$), with $\alpha = \angle(MP, g)$.

The above developments of (B) can be modified as follows. Let ζ be a complex variable denoting the point P. One has

$$|\zeta - z| = MP, \; \zeta - z = |\zeta - z|\, e^{i\left(\frac{\pi}{2} - \alpha\right)} = i\,|\zeta - z|\, e^{-i\alpha}.$$

Hence in (4°) we may put

$$MP = -i(\zeta - z)\,e^{i\alpha},$$

obtaining

$$(5^\circ) \qquad f(z) = h(z) - \frac{1}{2\pi i}\iint_K \frac{d\mu'(e_\zeta)}{\zeta - z}$$

as the solution of the equation (3°) (on all curves s as described in connection with (2.7)).

We now go back to our function $\mu = B + iA$ introduced in (2.6). In view of the above and since

$$B, A, \alpha, \beta$$

are set functions vanishing on frontiers of 'intervals', we infer that the equation

$$(6^\circ) \qquad \int_{(I)} Y(z)\,dz = \mu(I)$$

has a solution

$$(7^\circ) \qquad Y(z) = -\frac{1}{2\pi i}\iint_K \frac{d\mu(e_\zeta)}{\zeta - z}$$

on all boundaries (I) *of intervals on which* $a(z)$, $b(z)$ *of* (2.7) *are* L_1 *with respect to arc length.* Suppose the latter condition has been secured for all intervals; then on every frontier (I) (intervals $I < K$) the function $Y(z)$ (7°) and the given function $f(z)$ will satisfy (3°), where $\mu' = \mu$. Accordingly

$$\int_{(I)} (Y(z) - f(z))\, dz = 0 \tag{8°}$$

for all intervals $I < K$. By Morera's theorem and by (7°) *we then would obtain the representation* (2.3).

Consider an interval I with vertices

$$z_0,\ z_0 + b,\ z_0 + b + ia,\ z_0 + ia$$

$(a > 0,\ b > 0)$ where, for instance, $a < b$. Let $n(I)$ be the least integer such that

$$\frac{b}{a} \leqq n(I).$$

We express I as the sum of $n = n(I)$ equal intervals,

$$I = I_1 + I_2 + \cdots + I_n, \tag{2.8}$$

where $I_j (j = 1, \ldots n)$ is the interval with vertices

$$z_0 + (j-1)\frac{b}{n},\ z_0 + j\frac{b}{n},\ z_0 + j\frac{b}{n} + ia,\ z_0 + (j-1)\frac{b}{n} + ia. \tag{2.8'}$$

If $b > a$ we let $m(I)$ be the least integer such that

$$\frac{a}{b} \leqq m(I)$$

and express I as the sum of $m(I)$ equal intervals I_j with vertices

$$z_0 + i(j-1)\frac{b}{m},\ z_0 + b + i(j-1)\frac{b}{m},\ z_0 + b + ij\frac{b}{m},\ z_0 + ij\frac{b}{m} \tag{2.8''}$$

$(j = 1, \ldots m)$. Necessarily

$$n(I),\ m(I) \geqq 2.$$

Whence *the ratio r of the longer side to the shorter for the interval I_j satisfies*

$$2 > r \geqq 1. \tag{2.9}$$

We now introduce the Definition.

Definition 2.2. *Given a figure R (that is, a sum of a finite number of closed intervals), whose component intervals are non overlapping and non degenerate, it will be said that R has been regularized if every component interval, for which the ratio r of the longer side to the shorter exceeds 2, has been broken up into a number of equal intervals for each of which the ratio satisfies* (2.9).

Inasmuch as $\mu(I)$, being continuous as a function of intervals, vanishes on segments parallel to the axes, the component intervals of any figure in this section may be considered to be all non degenerate. With this in mind, we may consider all figures encountered in this section as susceptible to regularization.

Let I be an interval, the length of one of its sides being a and the length of another (non parallel) side being $b \leqq a$, while

$$1 \leqq r = \frac{a}{b} < 2. \tag{2.10}$$

Suppose that for some fixed z' in I one has

$$|f(z) - f(z')| \leqq \varrho |z - z'| \tag{2.10'}$$

for all z in the closed I. On writing

$$f(z) = f(z') + v(z, z')$$

and noting that

$$|v(z, z')| \leqq \varrho |z - z'|,$$

for the indicated values z, z', we infer that

$$|\mu(I)| = |\int_{(I)} f(z)\, dz| = |\int_{(I)} v(z, z')\, dz|$$

$$\leqq \varrho \int_{(I)} |z - z'|\,|dz| \leqq \varrho\, c\,(2a + 2b),$$

where c is the diameter of I. Now by (2.10)

$$c < b\sqrt{5}, \quad c \leqq a\sqrt{2};$$

hence

$$|\mu(I)| < q\varrho |I| \qquad (q = 2(\sqrt{5} + \sqrt{2})), \tag{2.11}$$

where $|I|$ is the area of I; similar inequalities will hold for $|B(I)|$, $|A(I)|$.

If after regularizing a figure R, $\overline{K}$, each of the component closed non overlapping intervals $I_1, \ldots I_\nu$ contains at least one point of F, we have

$$|A(R)|, \ |B(R)|, \ |\mu(R)| < qA|R|. \tag{2.12}$$

In fact, let z_j be a point of F in I_j; then on taking account of (2.5) and of the result (2.11) it is inferred that

$$|\mu(I_j)| < qA|I_j|,$$

$$|\mu(R)| = |\sum_j \mu(I_j)| < qA \sum_j |I_j| = qA|R|.$$

Here $|R|$ denotes measure of R in accordance with the notation in (S).

Similarly, on taking account of (2.5 a) *it is deduced that*

$$(2.12\,a)\qquad |\mu(R)| < q\,\varrho(n)\,|R|$$

for all figures R in E_n.

Let R be a regularized figure in $\bar{\bar{K}}$. We have

$$(2.13)\qquad R = R_0 + R_1,$$

where R_0 is the figure consisting of all those intervals of R which contain at least one point of F on the boundary or in the interior. By (2.12) one will have

$$(2.13\,a)\qquad |\mu(R_0)| < q\,A\,|R_0|.$$

The intervals of the figure R_1 lie, together with their boundaries, in $\bar{\bar{K}} - F$; R_1 is regularized, the decomposition being

$$(2.14)\qquad R_1 = \Sigma\, I_j,$$

where the I_j are intervals. Unless the contrary is stated all the intervals are implied to be closed.

Consider an interval I, $< \bar{\bar{K}} - F$, belonging to a regularized figure. We shall say that I is *divided* when I is bisected by a vertical segment and the two resulting intervals are each bisected, in turn, by horizontal segments. The four intervals so obtained are similar to I; that is, each of them has the same ratio of the longer side to the shorter as is the case for I (a square, if I is a square). It is observed that if some or all the intervals of a regularized figure are divided the resulting figure is regularized. Continuing with the fixed interval I, introduced above, we associate an integer n' such that I has points in $E_{n'}$, but has no points in $E_{n'-1}$. The set $E_{n'-1}$ could possibly be a null set. It will be said that an interval has property $P_{n'}$ if the interval lies in $E_{n'+1} - E_{n'-1}$. If I does not have the property $P_{n'}$ we divide I, obtaining intervals

$$I_1', \; I_1'',$$

where the I_1' have property $P_{n'}$, while the I_1'' do not have this property. It may happen that no I_1'' has points in $E_{n'}$. *One then has a decomposition of I into similar intervals of which some lie in $E_{n'+1} - E_{n'-1}$, while the others lie in $\bar{K} - E_{n'}$;* such a decomposition of I will be termed *'proper'*. If the I_1', I_1'' do not constitute a proper decomposition of I, let the I_1''' designate the intervals amongst the I_1'' which have points in $E_{n'}$. An interval I_1''' will have points in $\bar{K} - E_{n'+1}$. We divide each I_1''' obtaining intervals

$$I_2', \; I_2'',$$

where the I_2' have the property $P_{n'}$, while the I_2'' do not have this property. If no I_2'' has points in $E_{n'}$, the decomposition of I, obtained so far will be proper. If there are I_2'' with points in $E_{n'}$ we let I_2''' denote those intervals of I_2'' which have points in $E_{n'}$. We continue the indicated process of consecutive divisions, at each step applying it to those intervals which do not lie in $E_{n'+1} - E_{n'-1}$ and which at the same have points in $E_{n'}$, until after a finite number of steps we obtain a proper decomposition of I. The fact that a finite number of steps will suffice follows from the circumstance that the intervals that are subjected to divisions are those which at the same time have points in $\bar{K} - E_{n'+1}$ and $E_{n'}$.

Let the

$$I_1^{(1)}, \; I_1^{(2)}$$

be the intervals presenting a proper decomposition of I, where the $I_1^{(1)}$ are in $E_{n'+1} - E_{n'-1}$ and the $I_1^{(2)}$ are in $K - E_{n'}$. We obtain a proper decomposition of each $I_1^{(2)}$ into intervals

$$I_2^{(1)}, \; I_2^{(2)},$$

where the $I_2^{(1)}$ are in $E_{n'+2} - E_{n'}$, while the $I_2^{(2)}$ are in $K - E_{n'-1}$. Continuing this process, after a finite number of steps we obtain a decomposition, valid for any $I < \bar{K} - F$,

$$I = \Sigma\, I_1^{(1)} + \Sigma\, I_2^{(1)} + \Sigma\, I_3^{(1)} + \cdots \tag{2.15}$$

(finite number of terms), where for a fixed j the $I_j^{(1)}$ are intervals lying in

$$E_{n'+j} - E_{n'+j-2};$$

The intervals in the second member in (2.15) are similar to I.

We now turn to the figure R_1, involved in (2.13), (2.14). Since the intervals of R_1 lie in $\bar{K} - F$ and R_1 is regularized we may apply a decomposition (2.15) to each of the component intervals of R_1, obtaining

$$R_1 = R^{(1)} + R^{(2)} + \cdots \tag{2.16}$$

(finite number of terms) where the $R^{(\nu)}$ ($\nu = 1, 2, \ldots$) are non overlapping regularized figures such that

$$R^{(\nu)} < E_{\nu+1} - E_{\nu-1} \qquad (\nu = 1, 2, \ldots), \tag{2.16a}$$

here we put $E_0 =$ null set.

On taking account of the italicized statement with respect to (2. 12 a) we infer that, in view of (2. 16 a),

$$|\mu(R^{(\nu)})| < q\,\varrho(\nu+1)\,|R^{(\nu)}|.$$

Whence by (2. 16)

$$|\mu(R_1)| < q\sum_{\nu}\varrho(\nu+1)\,|R^{(\nu)}|.$$

Consequently from (2. 13) and (2. 13 a) *it is deduced that*

$$|\mu(R)| < q\,A\,|R_0| + q\sum_{\nu}\varrho(\nu+1)\,|R^{(\nu)}| \equiv S(R) \tag{2. 17}$$

for all regularized figures R in $\bar{K}$.

Now

$$m(E_{\nu+1}-E_{\nu-1}) = m(E_{\nu+1}-E_{\nu}) + m(E_{\nu}-E_{\nu-1}).$$

Thus, by (2. 16 a) and (2. 17) formally we obtain

$$\begin{aligned} S(R) &\leqq q\,A\,|K| + q\sum_{\nu=1}^{\infty}\varrho(\nu+1)\,m(E_{\nu+1}-E_{\nu}) \\ &\quad + q\sum_{\nu=1}^{\infty}\varrho(\nu+1)\,m(E_{\nu}-E_{\nu-1}) \leqq q\,(A\,|K| + \varrho(2)\,m\,E_1) + 2\,q\,\Gamma, \end{aligned} \tag{2. 17 a}$$

$$\Gamma = \sum_{\nu=1}^{\infty}\varrho(\nu+2)\,m(E_{\nu+1}-E_{\nu}), \tag{2. 17 b}$$

inasmuch as $\varrho(\nu+1) \geqq \varrho(\nu+2)$. This enables us to formulate *a condition securing bounded variation of the functions of intervals $A(I)$, $B(I)$, $\mu(I)$; this condition is that the series Γ (2. 17 b) be convergent. Herewith we assume that the class $C\{\varrho(n)\}$ of functions under consideration is such that Γ converges.* Accordingly, $a(z)$, $b(z)$ in (2. 7) may be defined with $\alpha(e)$, $\beta(e)$ from (2. 7′), where $A'(e) = A(e)$, $B'(e) = B(e)$, integrations being in the Lebesgue-Stieltjes sense.

We observe that

$$|R_0| \leqq |R|,\quad |R^{(1)}| + |R^{(2)}| + \cdots \leqq |R|. \tag{2. 18}$$

By (2. 17) for any $j > 0$ one has

$$S(R) \leqq q\,A\,|R| + q\sum_{\nu=1}^{j}\varrho(\nu+1)\,|R^{(\nu)}| + G_j(R),$$

where

$$G_j(R) = q\sum_{\nu>j}\varrho(\nu+1)\,|R^{(\nu)}| \leqq q\sum_{\nu>j}\varrho(\nu+1)\,m(E_{\nu+1}-E_{\nu-1}).$$

With the aid of the relation subsequent to (2. 17) it is inferred that

$$|G_j(R)| \leqq 2q \sum_{\nu \geqq j} \varrho(\nu + 2)\, m(E_{\nu+1} - E_\nu)$$

(compare with (2. 17 b)). In view of (2. 18)

$$(2.19)\quad S(R) \leqq qA|R| + q\varrho(j+1)[|R^{(1)}| + |R^{(2)}| + \cdots + |R^{(j)}|] \\ + |G_j(R)| \leqq q(A + \varrho(j+1))|R| + 2q\sum_{\nu \geqq j} \varrho(\nu+2)\, m(E_{\nu+1} - E_\nu)$$

for all figures $R < \bar{K}$, j being at our disposal.

Given $\varepsilon (> 0)$, no matter how small, we choose $j = j(\varepsilon)$ so that

$$2q \sum_{\nu \geqq j(\varepsilon)} \varrho(\nu + 2)\, m(E_{\nu+1} - E_\nu) < \frac{\varepsilon}{2}.$$

We then define $\eta = \eta(\varepsilon)$ so that

$$q[A + \varrho(j(\varepsilon) + 1)]\,\eta < \frac{\varepsilon}{2}.$$

In view of (2. 19) it will follow that

$$S(R) < \varepsilon$$

for all figures R for which $|R| \leqq \eta(\varepsilon)$. Hence by (2. 17) *the function of intervals $\mu(I)$ is absolutely continuous.*

Let

$$p + i\eta,\ h + i\eta \quad (p, h, \eta \text{ real};\ p < h)$$

be a rectilinear segment S' in K and let I_ν (not to be confused with intervals so designated previously) be the interval containing S', whose boundary consists of portions of lines

$$y = \eta \pm \frac{r_0}{\nu},\ x = p - \frac{r_0}{\nu},\ x = h + \frac{r_0}{\nu}$$

($r_0 > 0$, suitably small so that $I_1 < \bar{K}$).

Lemma 2. 1. *With the notation just given in view, one has*

$$\gamma(\zeta) = \int_{S'} \frac{dz}{|z - \zeta|} = \int_p^h \frac{dx}{|z-\zeta|} \leqq c' \log \nu \quad (z = x + i\eta;\ c' > 0)$$

for ζ in $I_\nu - I_{\nu+1}$, $\nu = 2, 3, \ldots$.

To prove this we first note that

$$\gamma(\zeta) = \log \tau(\zeta),\ \tau(\zeta) = \frac{|h + i\eta - \zeta| + \Re(h + i\eta - \zeta)}{|p + i\eta - \zeta| + \Re(p + i\eta - \zeta)} > 1;$$

here $\Re \ldots$ designates 'real part of ...'. Now

$$I_\nu - I_{\nu+1} = I_{\nu,1} + I_{\nu,2} + I_{\nu,3},$$

where $I_{\nu,1}$, $I_{\nu,2}$, $I_{\nu,3}$ are parts of $I_\nu - I_{\nu+1}$ specified by the inequalities

$$p \leqq \Re\zeta \leqq h,\ \Re\zeta > h,\ \Re\zeta < p,$$

respectively.

In $I_{\nu,1}$ we have

$$\gamma(\zeta) = \log \frac{|h + i\eta - \zeta| + \Re(h + i\eta - \zeta)}{|p + i\eta - \zeta| - \Re(\zeta - p - i\eta)} \leqq \log \frac{2|h + i\eta - \zeta|}{|p + i\eta - \zeta^*| - \Re(\zeta - p - i\eta)},$$

where ζ^* is the point on $(I_{\nu+1})$, with $\Re\zeta^* = \Re\zeta$, lying on the same side of the segment $(p + i\eta,\ h + i\eta)$ as ζ; now

$$|p + i\eta - \zeta^*| - R(\zeta - p - i\eta) = [r_0^2(\nu + 1)^{-2} + |R(\zeta - p - i\eta)|^2]^{\frac{1}{2}}$$
$$- R(\zeta - p - i\eta) = \frac{r_0^2}{(\nu+1)^2}\left\{\left[\frac{r_0^2}{(\nu+1)^2} + |R(\zeta - p - i\eta)|^2\right]^{\frac{1}{2}}\right.$$
$$\left. + R(\zeta - p - i\eta)\right\}^{-1} \geqq \frac{c^*}{\nu^2};$$

hence

$$\gamma(\zeta) \leqq c^* \log \nu \quad (\text{in } I_{\nu,1}). \tag{1_0}$$

Here and in the sequel c^* designates generically a positive constant.

In $I_{\nu,2}$ $\left[|\Im\zeta - \eta| \geqq \frac{r_0}{\nu + 1},\ \Im \ldots \text{ denoting imaginary part of } \ldots\right]$ we write

$$\tau(\zeta) = \frac{|h + i\eta - \zeta| - \Re(\zeta - h - i\eta)}{|p + i\eta - \zeta| - \Re(\zeta - p - i\eta)}.$$

Let

$$\zeta^* = \Re\zeta \pm i\frac{r_0}{\nu + 1},$$

where the sign is so chosen that the numbers $\Im\zeta^* - \eta$, $\Im\zeta - \eta$ have the same sign. Then

$$|p + i\eta - \zeta| - R(\zeta - p - i\eta) \geqq |p + i\eta - \zeta^*| - R(\zeta - p - i\eta)$$
$$= \left[(R(\zeta - p - i\eta))^2 + \frac{r_0^2}{(\nu + 1)^2}\right]^{\frac{1}{2}} - R(\zeta - p - i\eta) =$$
$$= \frac{r_0^2}{(\nu + 1)^2}\left\{\left[(R(\zeta - p - i\eta))^2 + \frac{r_0^2}{(\nu + 1)^2}\right]^{\frac{1}{2}} + R(\zeta - p - i\eta)\right\}^{-1} \geqq \frac{c^*}{\nu^2}$$

for ζ in the indicated region. Hence

$$\tau(\zeta) \leqq c^* \nu^2$$

and

$$(2_0) \qquad \gamma(\zeta) \leqq c^* \log \nu \quad \left[\text{in } I_{\nu,2}, \text{ with } |\Im\zeta - \eta| \geqq \frac{r_0}{\nu + 1}\right].$$

For ζ in $I_{\nu,2}$ $\left[\text{with } |\Im\zeta - \eta| < \frac{r_0}{\nu + 1}\right]$ we write

$$\tau(\zeta) = \frac{[(h - \zeta')^2 + (\eta - \zeta'')^2]^{\frac{1}{2}} - (\zeta' - h)}{[(p - \zeta')^2 + (\eta - \zeta'')^2]^{\frac{1}{2}} - (\zeta' - p)},$$

where $\zeta = \zeta' + i\zeta''$. Further, one has

$$\tau(\zeta) = \frac{[(p - \zeta')^2 + (\eta - \zeta'')^2]^{\frac{1}{2}} + (\zeta' - p)}{[(h - \zeta')^2 + (\eta - \zeta'')^2]^{\frac{1}{2}} + (\zeta' - h)},$$

Now

$$\zeta' - h \geqq \frac{r_0}{\nu + 1};$$

thus

$$\tau(\zeta) \leqq \frac{1}{2(\zeta' - h)}\{[(p - \zeta')^2 + (\eta - \zeta'')^2]^{\frac{1}{2}} + (\zeta' - p)\} \leqq c^* \nu$$

and

$$(3_0) \qquad \gamma(\zeta) \leqq c^* \log \nu \quad \left[\text{in } I_{\nu,2}, \text{ with } |\Im\zeta - \eta| < \frac{r_0}{\nu + 1}\right].$$

By similar methods inequalities like (2_0), (3_0) are obtained in $I_{\nu,3}$. Accordingly, we assert that

$$\gamma(\zeta) \leqq c^* \log \nu \qquad (\text{in } I_{\nu,2} + I_{\nu,3}).$$

Together with (1_0) this establishes the Lemma.

As remarked in connection with (7°), (8°) *the desired representation* (2.3) *will take place if* $a(z)$, $b(z)$ *of* (2.7) *are integrable along segments, in* K, *parallel to the axes.* With S' denoting a segment parallel to the axis of reals (see text preceding Lemma 2.1 for notation), we examine conditions under which the integral

$$\int_{S'} a(z)\, dz$$

exists. One has

$$a(z) = a_1(z) + a_2(z),$$

where

$$a_1(z) = \iint\limits_{I_1} \frac{1}{|z-\zeta|}\, d\,\alpha(e\,\zeta), \quad a_2(z) = \iint\limits_{K-I_1} \cdots$$

(the $I_1, I_2, \ldots$ are from the text preceding the Lemma) and

$$a_2(z) \leqq \frac{1}{r_0}\,\alpha(K - I_1) \leqq \frac{1}{r_0}\,\alpha(K) \qquad (z \text{ on } S').$$

Hence $a(z)$ is integrable along S' if and only if

$$\int_p^h a_1(z)\,dx = \iint\limits_{I_1} \gamma(\zeta)\,d\,\alpha(e_\zeta)$$

$(z = x + i\eta)$ exists. The integral in the second member, above, is expressible formally as a series

$$\sum_{\nu=1}^{\infty} \iint\limits_{I_\nu - I_{\nu+1}} \gamma(\zeta)\,d\,\alpha(e_\zeta),$$

which is dominated by the series

$$\lambda_1 + \lambda_2 + \cdots; \quad \lambda_\nu = \varrho_\nu\,\alpha(I_\nu - I_{\nu+1}),$$

where

$$\varrho_\nu = \text{upper bound in } I_\nu - I_{\nu+1} \text{ of } \alpha(\zeta).$$

In view of Lemma 2.1 convergence of

$$S_\alpha = \sum_\nu \alpha(I_\nu - I_{\nu+1}) \log \nu$$

would imply that $a(z)$ is integrable along S'. We carry out similar developments to obtain a condition securing integrability of $a(z)$ along segments (in K) parallel to the axis of imaginaries, as well as to secure analogous conditions for $b(z)$. We accordingly assert that *convergence of the series*

$$S_\alpha,\; S_\beta = \Sigma\,\beta(I_\nu - I_{\nu+1}) \log \nu \tag{2.20}$$

will imply that the functions $a(z)$, $b(z)$ are integrable along segments in K parallel to the axes.

Since

$$|I_\nu - I_{\nu+1}| \leqq \frac{c}{\nu^2} \qquad (c > 0),$$

by (2.19) we obtain

$$(2.21)\qquad S(R) \leqq \frac{qc}{\nu^2}(A + \varrho(j+1)) + 2q\sum_{k \geqq j} \varrho(k+2)\, m(E_{k+1} - E_k) \equiv W(\nu, j)$$

for figures R in $I_\nu - I_{\nu+1}$. On writing

$$A^+(R) = \text{u. b. } A(R'),\quad A^-(R) = \text{u. b. } - A(R')$$

('u. b.' is 'least upper bound') for figures R' in the figure R and noting that by (2.17)

$$|A(R')| \leqq |\mu(R')| \leqq S(R'),$$

one obtains

$$(2.21')\qquad A^+(R),\quad A^-(R) \leqq W(\nu, j)$$

for all figures R in $I_\nu - I_{\nu+1}$. Accordingly for the total variation we obtain

$$\alpha(R) = A^+(R) + A^-(R) \leqq 2\, W(\nu, j)$$

for figures $R < I_\nu - I_{\nu+1}$. A similar inequality will hold for $\beta(R)$. Thus

$$\alpha(I_\nu - I_{\nu+1}),\quad \beta(I_\nu - I_{\nu+1}) \leqq 2\, W(\nu, j).$$

Accordingly, *the series* (2.20) *will converge if*

$$(2.22)\qquad \sum_\nu W(\nu, j) \log \nu \qquad \text{(cf. (2.21))}$$

converges for some choice of j, *depending on* ν.

If (2.22) converges for some $j = j_\nu$, tending to infinity with ν, we obtain the desired representation. We, thus proved the following.

Theorem 2.1. *Suppose* $f(z)$ *is of class* $C\{\varrho(n)\}$, in $\overline{K}$, *in accordance with Definition* 2.1. *If the* $\varrho(n)$ *are such that there exists a sequence of integers*

$$0 < j_1 < j_2 < \cdots,\quad \lim_\nu j_\nu = \infty$$

so that the two series

$$(2.23)\qquad \sum_\nu \varrho(j_\nu + 1)\frac{\log \nu}{\nu^2},\quad \sum_\nu \sum_{k \geqq j_\nu} \varrho(k+2)\, m(E_{k+1} - E_k) \log \nu$$

converge, then $f(z)$ *is representable with the aid of Lebesgue-Stieltjes integration,*

$$(2.24)\qquad f(z) = h(z) - \frac{1}{2\pi i}\iint_K \frac{d\mu(e_\zeta)}{\zeta - z}$$

for z *in* K. *Here* $h(z)$ *is a function analytic in* K *and* $\mu(e)$ *is an additive ab-*

solutely continuous function of sets $\{B\}$; *on intervals* I (*in* K) $\mu(e)$ *coincides with the contour integral*

$$\int_{(I)} f(z)\, dz.$$

The conditions stated with respect to (2. 23) are not as complicated as might appear at the first glance. A simple application is presented by the case, satisfied for an extensive variety of closed sets F, when

$$m(E_{k+1} - E_k) < \frac{c}{k^2}.$$

Theorem 2. 1 will then be applicable for any class $C\{\varrho(n)\}$ for which

$$\varrho(n) \leqq c' n^\alpha \qquad \left(0 < \alpha < \frac{1}{2};\ c' > 0\right),$$

To establish this fact we may employ an auxiliary sequence

$$j_\nu = c_\nu \nu^2 \qquad (0 < c' \leqq c_\nu \leqq c'';\ \nu = 1, 2, \ldots).$$

The following is another consequence of Theorem 2. 1.

Theorem 2. 2. *Suppose* $f(z)$ *is uniformly Lip.* 1 in $\bar{K}$; *that is*,

$$|f(z) - f(z')| \leqq A\,|z - z'| \qquad (z, z' \text{ in } K).$$

Then $f(z)$ *has the integral representation* (2. 24).

In fact, under the above condition $f(z) < C\{\varrho(n)\}$, where the sequence $\{\varrho(n)\}$ is bounded; moreover,

$$\alpha(I_\nu - I_{\nu+1}),\ \ \beta(I_\nu - I_{\nu+1}) \leqq \frac{c^*}{\nu^2};$$

the series S_α, S_β will accordingly converge, leading to the conclusion of the Theorem.

Definition 2. 3. *Let* F *be a perfect bounded set. It will be said that* $f(z)$ *is continuously monogenic* (*c. m.*) *over* F *if at every point* z *of* F $f(z)$ *has a uniquely defined derivative* $f^{(1)}(z)$ (*derivative with respect to* F), *the derivative being continuous over* F.

With $f(z)$ c. m. on F, one has

$$\lim \frac{f(z') - f(z)}{z' - z} = f^{(1)}(z) \qquad (z \text{ in } F)$$

for z' (in F) tending to z. Putting in evidence the real and imaginary parts we write

$$(2.25)\qquad f(z) = u(z) + iv(z), \quad f^{(1)}(z) = u_1(z) + iv_1(z),$$
$$\frac{f(z') - f(z)}{z' - z} - f^{(1)}(z) = \gamma(z', z), \quad \gamma(z', z) = \alpha(z', z) + i\beta(z', z)$$

and obtain

$$|\alpha(z', z)|, \quad |\beta(z', z)| < \varepsilon \qquad (z \text{ in } F)$$

for all z' in F such that

$$|z' - z| \leqq \delta = \delta(z)$$

On writing $z' = x' + iy'$ from the above we obtain

$$(2.26)\qquad u(z') - u(z) = [u_1(z) + \alpha(z', z)](x' - x) - [v_1(z) + \beta(z', z)](y' - y),$$

$$(2.27)\qquad v(z') - v(z) = [v_1(z) + \beta(z', z)](x' - x) + [u_1(z) + \alpha(z', z)](y' - y).$$

The function

$$\gamma(z', z) = \Gamma(x', y', x, y)$$

is defined for (x', y', x, y) in the four dimensional closed set P consisting of points corresponding to all z', z in F, provided we put

$$\gamma(z, z) = \lim \gamma(z', z) \qquad (z \text{ in } F)$$

(for z' tending to z within F) and note that the latter limit is zero. In consequence of the given hypotheses $f(z)$ is continuous in F. Hence from the definition of $\gamma(z', z)$, given in (2.25), it follows that $\gamma(z', z)$ is continuous in z' (in F) and in z (in F). Clearly $\Gamma(x', y', x, y)$ is continuous in (x', y', x, y) over P. Hence this function is bounded in P; thus

$$(2.28)\qquad |\alpha(z', z)|, \quad |\beta(z', z)| \leqq \alpha_1 \qquad (\text{for } z', z \text{ in } F).$$

Inasmuch as, in consequence of continuity of $f^{(1)}(z)$, one has

$$|u_1(z)|, \quad |v_1(z)| < \mu_1,$$

from (2.26), (2.27), (2.28) we now deduce that

$$(2.29)\qquad |u(z') - u(z)|, \quad |v(z') - v(z)| < (\mu_1 + \alpha_1)(|x' - x| + |y' - y|)$$
$$\leqq 2(\mu_1 + \alpha_1)|z' - z| \qquad (z', z \text{ in } F).$$

That is, $u(z)$ *and* $v(z)$ *are uniformly Lip.* 1 *over* F.

Confining our attention for the present to $u(z)$, we observe that (2.26) signifies that $u(z)$ has in F derivatives, in the sense of Whitney,

$$\frac{\partial u(z)}{\partial x} = u_1(z), \quad \frac{\partial u(z)}{\partial y} = -v_1(z),$$

continuous in F. In consequence of certain developments of Whitney there exists a function $U(z)$, continuous in $\bar{K}$, with partial derivatives

$$\frac{\partial U}{\partial x}, \frac{\partial U}{\partial y}$$

continuous in $\bar{K}$, such that

$$U(z) = u(z), \quad \frac{\partial U}{\partial x} = \frac{\partial u}{\partial x}, \quad \frac{\partial U}{\partial y} = \frac{\partial u}{\partial y} \qquad \text{(in } F\text{)}; \tag{2.30}$$

here, as previously, K is an interval or, more generally, a bounded domain containing F. Such a function $U(z)$ is uniformly Lip. 1 over $\bar{K}$. We get a similar result for a continuous extension $V(z)$ (over $\bar{K}$) of $v(z)$. An application of Theorem 2.2 will accordingly yield

Theorem 2.3. *Suppose $f(z)$ is c. m. over a bounded perfect set F in accordance with Definition 2.3. Such a function has in F an integral representation* (2.24).

The points $z = x + iy$ of F which are limit points of points $z' = x' + iy$ of F will be said to have the property (H), while points $z = x + iy$ of F which are limit points of points $z' = x + iy'$ of F will be said to possess the property (V). From (2.26), (2.27) on passing to the limit we obtain

$$\frac{\partial u(z)}{\partial x} = u_1(z), \quad \frac{\partial v(z)}{\partial x} = v_1(z) \tag{1°}$$

at points having property (H), while

$$\frac{\partial u(z)}{\partial y} = -v_1(z), \quad \frac{\partial v(z)}{\partial y} = u_1(z) \tag{2°}$$

at points with property (V). Here the partial derivatives of $u(z)$, $v(z)$ are in the ordinary sense over F. From the above it is concluded that *the* 'Cauchy-Riemann' *equations*

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

hold for every c. m. function $f = u + iv$ at every point z having simultaneously the properties (H), (V).

3. **Problem of Representation for Discontinuous Functions.** We continue to imply that intervals are closed and non degenerate. We shall now consider *functions* $f(z)$ *integrable along all rectilinear segments, in* $\bar{K}$, *parallel to the axes. It is assumed that the set* H *of points of discontinuity of* $f(z)$ *has measure zero.*

With the aid of the integral (2.6) we again define a function

$$\mu(I) = B(I) + iA(I)$$

of intervals I in $\bar{K}$; under the new hypotheses $\mu(I)$ is additive.

We define the function

$$M(z') = \text{u. b. (for } z \text{ in } \bar{K}) \text{ of } \left| \frac{f(z') - f(z)}{z' - z} \right| \tag{3.1}$$

and assume that

$$M(z') < +\infty \tag{3.2}$$

for z' *in* $\bar{K} - H$. It is observed that $M(z')$ is measurable. If we define G_ν as the part of $\bar{K} - H$ such that

$$\nu - 1 \leqq M(z') < \nu, \tag{3.3}$$

it is inferred that

$$\bar{K} = H + G_1 + G_2 + \cdots, \tag{3.4}$$

no two of the sets H, G_1, G_2, ... having points in common.

Let I be an interval; I will contain points of $\bar{K} - H$; more precisely, for some j there will be in I a point z_j belonging to G_j. With z' denoting some point in $(\bar{K} - H)I$ one has

$$|f(z) - f(z')| \leqq M(z')|z - z'|, \quad M(z') < +\infty,$$

for all z in $\bar{K}$. On the basis of this inequality we repeat the reasoning leading from (2.10) to (2.11) and infer that

$$|\mu(I)| < q\,M(z')\,|I| \qquad (q \text{ from } (2.11)), \tag{3.5}$$

provided (2.11) holds.

Let R be any figure, whose regularized (Definition 2.2) decomposition into non overlapping intervals is

$$R = I^{(1)} + I^{(2)} + \cdots + I^{(\sigma)}. \tag{3.6}$$

We designate by

$$I_1, I_2, \ldots I_{\nu_1} \tag{3.6a}$$

all those of the intervals $I^{(j)}$ which have points of G_1. Let

$$(3.6\,\mathrm{b}) \qquad I_{\nu_1+1},\ I_{\nu_1+2},\ \ldots\ I_{\nu_2}$$

be all those of the $I^{(j)}$ which are not included in (3.6 a) and which have points of G_2. Continuing the process of selection in this manner, one breaks up the $I^{(j)}$ $(j = 1, \ldots \sigma)$ uniquely into a number of groups

$$I_{\nu_{s-1}+1},\ I_{\nu_{s-1}+2},\ \ldots\ I_{\nu_s}$$

$(\nu_{s-1} \leqq \nu_s;\ \nu_0 = 0;\ s = 1, 2, \ldots)$, where the intervals just displayed are all those of the $I^{(j)}$ which are not amongst the

$$(3.6\,\mathrm{c}) \qquad I_{\nu_{p-1}+1},\ I_{\nu_{p-1}+2},\ \ldots\ I_{\nu_p} \qquad (p = 1, 2, \ldots, s-1)$$

and which at the same time have points of G_s. Accordingly, from (3.6) one deduces the decomposition

$$(3.7) \qquad R = R^{(1)} + R^{(2)} + \cdots,$$

where

$$(3.7\,\mathrm{a}) \qquad R^{(s)} = I_{\nu_{s-1}+1} + I_{\nu_{s-1}+2} + \cdots + I_{\nu_s}.$$

It is observed that the above is a regularized decomposition of a figure $R^{(s)}$. The figures $R^{(1)}, R^{(2)}, \ldots$ are non overlapping.

Using the fact that every component interval of $R^{(s)}$ has a point of G_s and satisfies (2. 10), from (3. 5) we infer that

$$|\mu(I_j)| < q\, M(z_j)\, |I_j| \qquad (\nu_{s-1} < j \leqq \nu_s),$$

where z_j is a point of G_s. Thus, in view of the definition of G_s, given in connection with (3. 3), from (3. 7 a) one derives

$$|\mu(R^{(s)})| < q\, s\, |R^{(s)}|.$$

Finally, in consequence of (3. 6)

$$(3.8) \qquad |\mu(R)| < q \sum_{s \geqq 1} s\, |R^{(s)}| = Q(R),$$

the number of terms in the last member being finite.

Now $R^{(2)}$ is the sum of intervals (3. 6 b); none of these intervals contains points of G_1, inasmuch as all the intervals having points of G_1 have been included in (3. 6 a). Hence

$$R^{(2)} < \bar{K} - G_1.$$

In general, $R^{(s)}$ is given by the sum (3.7 a), the component intervals having no points of $G_1, G_2, \ldots G_{s-1}$; this property follows from the fact that these intervals are not amongst the (3.6 c), while the intervals (3.6 c) constitute the totality of all $I^{(j)}$ having points of G_p $(p = 1, \ldots, s-1)$. Accordingly

$$R^{(s)} < \bar{K} - (G_1 + \cdots + G_{s-1}). \tag{3.9}$$

Whence

$$|R^{(s)}| \leqq |K| - (m\,G_1 + m\,G_2 + \cdots + m\,G_{s-1}) = r(s) \tag{3.9 a}$$

$(s = 2, 3, \ldots)$ on so far as no two of the G_j have points in common.

Since $mH = 0$ it is noted that

$$\Sigma m\,G_\nu = |K|;$$

accordingly

$$\lim_{s=\infty} r(s) = 0.$$

By (3.8) and (3.9 a)

$$|\mu(R)| < q\,\Gamma, \qquad \Gamma = \sum_{s=1}^{\infty} s\,r(s). \tag{3.10}$$

These considerations lead to the conclusion that *the functions of intervals*

$$B(I), \quad A(I), \quad \mu(I) \tag{3.11}$$

are of bounded variation whenever the series Γ converges. Convergence of Γ is forthwith assumed. We are now able to perform Lebesgue-Stieltjes integration with respect to μ.

In consequence of (3.8) and (3.9 a)

$$Q(R) \leqq q \sum_{s=1}^{\infty} s\,|R^{(s)}| + q\,\Gamma(j), \qquad \Gamma(j) = \sum_{s>j} s\,r(s) \tag{3.12}$$

for all figures $R < \bar{K}$, j being a positive integer at our disposal. Since

$$|R^{(1)}| + \cdots + |R^{(j)}| \leqq |R|$$

from the above we obtain

$$Q(R) \leqq qj\,|R| + q\,\Gamma(j). \tag{3.12 a}$$

In view of (3.12), given $\varepsilon > 0$, no matter how small, one may choose $j = j(\varepsilon)$ so that

$$q\,\Gamma(j(\varepsilon)) < \frac{\varepsilon}{2};$$

we then define $\eta = \eta(\varepsilon)$ by the relation

$$q j(\varepsilon) \eta(\varepsilon) = \frac{\varepsilon}{2};$$

in view of (3. 8) and (3. 12 a) one obtains

$$|\mu(R)| < \varepsilon$$

for figures R in $\bar{\bar{K}}$ for which

$$|R| \leqq \eta(\varepsilon).$$

Hence *the function of intervals* $\mu(I)$, *as well as the corresponding function* $\mu(e)$ *of* $\{B\}$*-sets, is absolutely continuous.*

Let S' be the rectilinear segment and the I_ν be the intervals introduced subsequent to (2. 19). For R in $I_\nu - I_{\nu-1}$ one has

$$|R| \leqq \frac{c}{\nu^2};$$

accordingly, by (3. 12 a)

$$Q(R) \leqq c q j \frac{1}{\nu^2} + q \Gamma(j) = w(\nu, j), \tag{3. 12 b}$$

for figures R in $I_\nu - I_{\nu+1}$. Analogous to (2. 21′) one now has

$$A^+(R), \quad A^-(R) \leqq w(\nu, j) \qquad (R < I_\nu - I_{\nu+1})$$

and, for the total variations of $A(I)$, $B(I)$,

$$\alpha(R), \quad \beta(R) \leqq 2 w(\nu, j)$$

for figures in $I_\nu - I_{\nu+1}$ — in particular for $R = I_\nu - I_{\nu+1}$. By a reasoning of the type used subsequent to (2. 22) we deduce that the desired representation for the case now under consideration will hold if one can choose a sequence $j = j_\nu$,

$$0 < j_1 < j_2 < \cdots,$$

so that the series

$$\sum_\nu w(\nu, j) \log \nu$$

converges. On taking account of (3. 12 b) one obtains the following result.

Theorem 3. 1. *Suppose* $f(z)$ *is a function as described at the beginning of this section up to and including* (3. 2). *Let* G_ν *be the set defined in connection with* (3. 3). *If* $m G_\nu$ *tends to zero with* $\frac{1}{\nu}$ *sufficiently fast so that there exists a sequence of integers* $0 < j_1 < j_2 < \cdots$ *for which the two series*

$$\sum_{\nu} j_\nu \frac{\log \nu}{\nu^2}, \qquad \sum_{\nu} \Big(\sum_{s > j_\nu} s\, r(s) \Big) \log \nu, \tag{3.13}$$

where

$$r(s) = |K| - (m\, G_1 + \cdots + m\, G_{s-1}),$$

converge, then $f(z)$ *has the representation*

$$f(z) = h(z) - \frac{1}{2\pi i} \iint_K \frac{d\mu(e_\zeta)}{\zeta - z} \tag{3.14}$$

for z *in* K *(*$h(z)$ *analytic in* K*;* μ *additive absolutely continuous).*

A simple application is embodied in the following.

Corollary 3.1. *If* $f(z)$ *is a function as described at the beginning of this section and is such that*

$$m\, G_s \leqq \frac{c}{s^\alpha} \qquad (\alpha < 4), \tag{3.15}$$

then $f(z)$ *is representable in the form* (3.14).

To establish this we first note that

$$1 < 2 - \frac{1}{\alpha - 3}.$$

Choose β so that

$$1 < \beta < 2 - \frac{1}{\alpha - 3}$$

By (3.15)

$$r(s) = m\, G_s + m\, G_{s+1} + \cdots < \frac{c'}{s^{\alpha - 1}}$$

and, hence,

$$\sum_{s > j} s\, r(s) < \frac{c''}{j^{\alpha - 3}}. \tag{3.16}$$

We choose integers j_ν subject to relations

$$j_\nu = c_\nu\, \nu^{2 - \beta} \qquad (0 < c_0 \leqq c_\nu \leqq c')$$

and note that

$$j_\nu \frac{\log \nu}{\nu^2} \leqq c_1 \frac{\log \nu}{\nu^\beta};$$

since $\beta > 1$, this implies that the first series (3.13) converges. On the other hand. by (3.16)

$$\Big[\sum_{s > j_\nu} s\, r(s) \Big] \log \nu < c^0\, \nu^{-(2-\beta)(\alpha - 3)} \log \nu;$$

now

$$(2-\beta)(\alpha-3) > 1$$

and, accordingly, the second series (3.13) will be convergent. The Corollary has been proved. It is easy to give examples of functions $f(z)$ satisfying the conditions of the Corollary.

In applying Theorem 3.1 it is useful to note that *the best choice of the* j_ν *is on hand when we require that* j_ν *be the least positive integer such that*

$$r(j_\nu + 1) < \frac{c}{\nu^2}.$$

In fact, when R is a figure in $I_\nu - I_{\nu+1}$ (I_ν introduced subsequent to (3.12a)) and (3.7) is its decomposition we have

$$|R^{(s)}| \leqq \frac{c}{\nu^2}, \qquad |R^{(s)}| \leqq r(s)$$

for $s = 1, 2, \ldots$. The best inequalities, for $R < I_\nu - I_{\nu+1}$, that we obtain from these are

$$|R^{(s)}| \leqq \frac{c}{\nu^2} \quad (s = 1, \ldots j_\nu), \qquad |R^{(s)}| \leqq r(s) \quad (s = j_\nu + 1, \ldots),$$

where j_ν is defined as stated above — a fact which leads to the italicized assertion.

In considering any instances when functions $f(z)$ are representable with the aid of integrals of the form

$$\iint \frac{d\mu(e_z)}{\zeta - z},$$

where $\mu(e)$ is the extension to $\{B\}$-sets of the function of intervals

$$\mu(I) = \int_{(I)} f(z)\, dz,$$

as is the case in Theorems 2.1, 2.2, 2.3, 3.1, for example, the following remarks are in order. *If* $f(z) = u + iv$ *is continuous and is monogenic in the bounded closed set* F *in the sense that except, perhaps, on a denumerable set*

$$\overline{\lim} \frac{1}{|h|} |f(z+h) - f(z)| < \infty \qquad (z, z+h \text{ in } F)$$

and that u, v *satisfy the Cauchy-Riemann conditions almost everywhere in* F, *then*

$$\mu(F) = 0 \tag{3.17}$$

This follows by a theorem of P. T. MAKER[1] which asserts that under the conditions given in the italics, above, one can find a sequence of sets

$$C_n = \sum_m I_{mn},$$

where the I_{mn} are non overlapping closed intervals, so that $C_n > F$ and

$$\lim_n \sum_m \left| \int_{(I_{mn})} f^*(z)\, dz \right| = 0$$

($f^*(z)$ a continuous extension of $f(z)$).

In fact, the latter relation means that

$$\lim_n \mu(C_n) = 0.$$

On the other hand, the C_n can be so chosen that

$$C_1 > C_2 > \cdots; \quad \lim C_n = F.$$

In view of (S; p. 8) this will imply (3.17).

We shall conclude this section with the remark to the effect that *whenever there is a representation on hand of the form*

$$(3.18) \qquad f(z) = h(z) - \frac{1}{2\pi i} \iint_K \frac{d\mu(e_\zeta)}{\zeta - z} \qquad (z \text{ in } K),$$

where $h(z)$ *is analytic in* K *and where* $\mu(e)$ *is an additive function of* $\{B\}$*-sets of bounded variation, the total variations of whose real and imaginary parts vanish on rectilinear segments* l *(in* $\bar{K}$*) parallel to the axes, while*

$$(3.18\text{ a}) \qquad \int_l \left(\iint_K \frac{d\mu(e_\zeta)}{\zeta - z} \right) dz = \iint_K \left(\int_l \frac{dz}{\zeta - z} \right) d\mu(e_\zeta),$$

then necessarily one has

$$\mu(I) = \int_{(I)} f(z)\, dz$$

on intervals $I < K$. In fact, integrating both members of (3.18) around the boundary (I) of any interval in K, by Morera's theorem we obtain

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[1] P. T. MAKER, *The Cauchy theorem for functions on closed sets*, Bull. Am. Math. Soc. (1943), pp. 912—916; see p. 915.

$$\int_{(I)} f(z)\,dz = -\frac{1}{2\pi i}\int_{(I)}\left[\iint_{K}\frac{d\mu(e_\zeta)}{\zeta - z}\right] dz$$

and, in view of (3. 18 a),

$$\int_{(I)} f(z)\,dz = \frac{1}{2\pi i}\iint_{K}\left[\int_{(I)}\frac{dz}{z-\zeta}\right] d\mu(e_\zeta).$$

In consequence of the hypothesis with respect to the total variations of the real and imaginary parts of $\mu(e)$ it is inferred that

$$(3.19)\qquad \int_{(I)} f(z)\,dz = \frac{1}{2\pi i}\iint_{K-(I)}\left[\int_{I}\frac{dz}{z-\zeta}\right] d\mu(e_\zeta).$$

Now

$$\frac{1}{2\pi i}\int_{(I)}\frac{dz}{z-\zeta} = \begin{matrix} 0 \\ 1 \end{matrix} \qquad \begin{matrix} (\zeta \text{ in } K-(I)), \\ (\zeta \text{ interior } I). \end{matrix}$$

Accordingly the conclusion of the above italics will follow from (3. 19).

4. **Questions of Monogeneity and Differentiability.** Consider a family of functions $\{f(z)\}$, with $f(z)$ analytic (uniform) in an open set $O(f)$. Inasmuch as $O(f)$ is not necessarily connected, $f(z)$ may be equal, in a number of subsets of $O(f)$ to a number of distinct analytic functions. Suppose there are points common to all the $O(f)$. *Let F be a closed subset of the set common to all the $O(f)$. Designate by O_δ ($\delta > 0$) the set of points at the distance $< \delta$ from F. For $\delta\,(>0)$ sufficiently small $O_\delta < O(f)$. Let $\delta(f)$ be the upper bound of values δ such that $O_\delta < O(f)$.*

We thus have a closed set F and a family of analytic functions $\{f(z)\}$ on hand; to every function $f(z)$ there corresponds a positive $\delta(f)$ so that $f(z)$ is analytic in the open set $O_{\delta(f)}$.

When

$$\text{l. b. } \delta(f) = \delta_0 > 0 \text{ [1]}$$

the situation is as in the classical theory, inasmuch as in this case all the functions of the family are analytic in the same open set O_δ. It is noted that all the $\delta(f) > 0$; hence the alternative to $\delta_0 > 0$ is

$$\underline{\lim}\ \delta(f) = 0.$$

[1] l. b. means greatest lower bound.

It is precisely the latter case that is of interest from our present point of view. In this case a sequence

$$\delta(f_1) > \delta(f_2) > \ldots; \qquad \lim_{\nu} \delta(f_\nu) = 0$$

can be found. Correspondingly we shall have $f_\nu(z)$ analytic in O_{δ_ν} $(\delta_\nu = \delta(f_\nu))$;

$$O_{\delta_1} > O_{\delta_2} > \ldots; \quad O_{\delta_\nu} > F; \quad \lim O_{\delta_\nu} = F; \quad \bar{O}_{\delta_{\nu+1}} < O_{\delta_\nu}.$$

A sequence $\{f_\nu\}$, *as described above, will be said to belong to* $\{F;\ \delta_\nu\}$; *thus*

$$\{f_\nu\} < \{F;\ \delta_\nu\}.$$

Suppose we turn now to the study of monogeneity in the sense of existence of a unique derivative. In this connection the nature of the set G over which this property holds is of prime importance. If G has no interior points the function in question, monogenic in G, may be non analytic at every point of G (no matter how the function is defined in the complement of G).

Consider a sequence $\{f_\nu\} < \{F;\ \delta_\nu\}$ *(closed* F*). Suppose*

$$f_\nu(\alpha) \to f(\alpha) \qquad (\textit{as } \nu \to \infty;\ \textit{in } F), \tag{4.1}$$

convergence being not necessarily uniform. We shall study differentiability at a point α_0 in F.

It is noted that α_0 is an interior point of the open set $O(\delta_\nu)$ in which $f_\nu(\alpha)$ is analytic. One has

$$f_\nu(\alpha) = f_\nu(\alpha_0) + (\alpha - \alpha_0) f_\nu^{(1)}(\alpha_0) + (\alpha - \alpha_0)^2 c_\nu(\alpha). \tag{4.2}$$

For α in a circle

$$S(\alpha_0,\ r) \qquad (|\alpha - \alpha_0| \leqq r),$$

where r (> 0) is suitably small, $f_\nu(\alpha)$ is analytic and clearly $c_\nu(\alpha)$ is therein analytic. For $|\alpha - \alpha_0| > r$, with α in $O(\delta_\nu)$, on noting that

$$c_\nu(\alpha) = \left[\frac{f_\nu(\alpha) - f_\nu(\alpha_0)}{\alpha - \alpha_0} - f_\nu^{(1)}(\alpha_0)\right] \frac{1}{\alpha - \alpha_0}, \tag{4.2 a}$$

it is observed that $c_\nu(\alpha)$ is analytic, since the functions $f_\nu(\alpha)$, $(\alpha - \alpha_0)^{-1}$ have this property. *Thus* $c_\nu(\alpha)$ *is analytic in* $O(\delta_\nu)$.

We have

$$g_\nu(\alpha) = f_\nu(\alpha) - f_\nu(\alpha_0) \to f(\alpha) - f(\alpha_0) = g(\alpha) \qquad (\text{in } F), \tag{4.3}$$

where $g_\nu(\alpha_0) = g(\alpha_0) = 0$ and $g_\nu(\alpha) < \{F;\ \delta_\nu\}$. Hence there is no loss of generality, if, with α_0 fixed, *we assume that*

(4. 3 a) $$f_\nu(\alpha_0) = f(\alpha_0) = 0.$$

The specific question now raised is as follows. *Under what conditions, given a sequence* (4. 1) (*with* (4. 3 a)), *do we have*

(4. 4) $$f^{(1)}(\alpha_0) = \lim_\nu f_\nu^{(1)}(\alpha_0)?$$

Whenever (4. 4) holds $f(\alpha)$ is monogenic (not necessarily analytic) at α_0, inasmuch as the $f_\nu^{(1)}(\alpha_0)$ being unique numbers, the limit in the second member of (4. 4) will be unique.

For the validity of (4. 4) it is necessary that

(4. 4 a) $$\lim_\nu f_\nu^{(1)}(\alpha_0) = c'$$

should exist. *Accordingly, we may as well assume the latter relation.* The problem now is to find conditions under which $f^{(1)}(\alpha_0) = c'$. By (4. 4 a) and (4. 2 a) there exists a function $c(\alpha)$, defined in F $(\alpha \neq \alpha_0)$ so that

(4. 4 b) $$\lim_\nu c_\nu(\alpha) = c(\alpha) = \left(\frac{f(\alpha)}{\alpha - \alpha_0} - c'\right)\frac{1}{\alpha - \alpha_0}$$

(α in F)[1].

In view of (4. 2)

$$\frac{f_\nu(\alpha)}{\alpha - \alpha_0} = f_\nu^{(1)}(\alpha_0) + (\alpha - \alpha_0)\, c_\nu(\alpha)$$

and in the limit

(4. 5) $$\frac{f(\alpha)}{\alpha - \alpha_0} = c' + (\alpha - \alpha_0)\, c(\alpha) \qquad (\alpha \text{ in } F;\ \alpha \neq \alpha_0).$$

Hence (4. 4) will hold provided

(4. 5 a) $$\lim_{\alpha \to \alpha_0} (\alpha - \alpha_0)\, c(\alpha) = 0 \qquad (\alpha \text{ within } F)^{2}.$$

Accordingly, it is noted that (4. 4 a), (4. 5 a) is sufficient for the validity of (4. 4). Conversely, suppose (4. 4) holds; then (4. 4 a) is asserted and the function $c(\alpha)$ of (4. 4 b) will exist for $\alpha \neq \alpha_0$ in F. The relation (4. 5) could be then expressed in the form

[1] With the aid of Taylor's expansion, applied at α_0 to $f_\nu(\alpha)$, from (4. 2) it is inferred that $2\,c_\nu(\alpha_0) = f_\nu^{(2)}(\alpha_0)$. Existence of $c(\alpha_0)$ is thus contingent on that of $\lim f_\nu^{(2)}(\alpha_0)$ (provided that we define $c(\alpha_0)$ as $\lim\ c_\nu(\alpha_0)$).

[2] The implication, of course, is that α_0 is a limiting point of F.

$$\frac{f(\alpha)-f(\alpha_0)}{\alpha-\alpha_0}=f^{(1)}(\alpha_0)+(\alpha-\alpha_0)\,c(\alpha) \qquad (\alpha \text{ in } F;\ \alpha \neq \alpha_0).$$

The assertion (4.4) implies that the limit of the first member above, as $\alpha \to \alpha_0$ within F, exists and has the value $f^{(1)}(\alpha_0)$. Necessarily (4.5 a) will hold. Accordingly, the following theorem may be stated.

Theorem 4.1. *Consider a sequence* $\{f_\nu(\alpha)\} < \{F;\ \delta_\nu\}$ *converging to a limit* $f(\alpha)$ *in* F. *Let* α_0 *be a fixed point in* F; *assume that* $\lim_\nu f_\nu^{(1)}(\alpha_0)$ *exists. In order that* $f(\alpha)$ *should be monogenic at* α_0 *and*

$$f^{(1)}(\alpha_0)=\lim_\nu f_\nu^{(1)}(\alpha_0)$$

it is necessary and sufficient that $(\alpha-\alpha_0)\,c(\alpha)\to 0$, *as* $\alpha\to\alpha_0$ *within* F. *Here* $c(\alpha)=\lim_\nu c_\nu(\alpha)$, *where* $c_\nu(\alpha)$ *is the function* (4.2 a), *analytic in* $O(\delta_\nu)$.

If a double sequence $\{f_{n,\nu}\}$ $(n, \nu = 1, 2, \ldots)$ is on hand, where the limits

$$\lim_n f_{n,\nu}, \quad \lim_\nu f_{n,\nu}$$

exist, then, as is known,

$$\lim_\nu\,(\lim_n f_{n,\nu}) = \lim_n\,(\lim_\nu f_{n,\nu}) \tag{4.6}$$

if and only if the following holds. Given ε (>0), one can find $m(\varepsilon)$ so that for each $\nu \geqq m(\varepsilon)$ a number k_ν can be found so that

$$|f_{n,\nu}-\lim_\nu f_{n,\nu}|<\varepsilon \qquad (\text{for all } n \geqq k_\nu). \tag{4.6 a}$$

We again consider a sequence $\{f_\nu(\alpha)\} < \{F;\ \delta_\nu\}$ for which $\lim_\nu f_\nu(\alpha)=f(\alpha)$ exists in F. On writing

$$f_{n,\nu}=\frac{f_\nu(\alpha_n)-f_\nu(\alpha_0)}{\alpha_n-\alpha_0} \qquad (\alpha_n \text{ in } F;\ \lim_n \alpha_n=\alpha_0), \tag{4.7}$$

it is observed that

$$\lim_n f_{n,\nu}=f_\nu^{(1)}(\alpha_0),\ \lim_\nu f_{n,\nu}=\frac{f(\alpha_n)-f(\alpha_0)}{\alpha_n-\alpha_0}. \tag{4.7 a}$$

Accordingly (4.6) is equivalent to the relation

$$\lim_\nu f_\nu^{(1)}(\alpha_0)=f^{(1)}(\alpha_0), \tag{4.8}$$

where the derivative of $f(\alpha)$ is to be computed with respect to a sequence α_n (as described in (4.7)).

In view of the statement with respect to (4.6), (4.6a) it is concluded that *for monogeneity of $f(\alpha)$ at α_0 (in F) and in order that* (4.8) *should hold it is necessary and sufficient that for every sequence $\{\alpha_n\} < F$ converging to α_0 the following should hold*:

$$\left| \frac{f_\nu(\alpha_n) - f_\nu(\alpha_0)}{\alpha_n - \alpha_0} - \frac{f(\alpha_n) - f(\alpha_0)}{\alpha_n - \alpha_0} \right| < \varepsilon \tag{4.8a}$$

for all $n \geqq k_\nu$, when ν is any integer such that $\nu \geqq m(\varepsilon)$.

If $\lim\limits_{\nu} f_\nu^{(1)}(\alpha_0) = c'$ exists one may replace (4.8a) by

$$|f_\nu^{(1)}(\alpha_0) + (\alpha_n - \alpha_0)\, c_\nu(\alpha_n) - c' - (\alpha_n - \alpha_0)\, c(\alpha_n)| < \varepsilon.$$

Accordingly, when $\lim\limits_{\nu} f_\nu^{(1)}(\alpha_0) = c'$ exists, *in order that* (4.8) *should hold it is necessary and sufficient to have*

$$|\alpha_n - \alpha_0|\; |c(\alpha_n) - c_\nu(\alpha_n)| < \varepsilon \tag{4.8b}$$

for all $n \geqq k_\nu$ (any $\nu \geqq m(\varepsilon)$).

If we make the convention that $f^{(1)}(\alpha_0)$ is said to exist when

$$\lim_n \frac{f(\alpha_n) - f(\alpha_0)}{\alpha_n - \alpha_0}$$

exists and has the *same* value for *all* sequences $\{\alpha_n\}$ in F and tending to α_0, then *the necessary and sufficient conditions for the validity of* (4.8) *may be given in the form*:

$$|\alpha - \alpha_0|\; |c(\alpha) - c_\nu(\alpha)| < \varepsilon \tag{4.8c}$$

for all α in F with $|\alpha - \alpha_0| \leqq \zeta_\nu$. when ν is any number such that $\nu \geqq m(\varepsilon)$ (still assuming that $\lim\limits_{\nu} f_\nu^{(1)}(\alpha_0) = c'$ exists).

When $\lim\limits_{\nu} f_\nu^{(1)}(\alpha_0)$ exists an obvious *sufficient condition* securing (4.8) is

$$|\alpha - \alpha_0|\; |c_{\nu+p}(\alpha) - c_\nu(\alpha)| < \varepsilon \qquad (p = 1, 2, \ldots) \tag{4.8c'}$$

for all α in F, $|\alpha - \alpha_0| \leqq \zeta_\nu$; any $\nu \geqq m(\varepsilon)$.

On writing

$$r_{\nu,p}(\alpha) = f_{\nu+p}(\alpha) - f_\nu(\alpha) \tag{4.9}$$

in consequence of (4.2a) we obtain

$$(\alpha - \alpha_0)(c_{\nu+p}(\alpha) - c_\nu(\alpha)) = \frac{r_{\nu,p}(\alpha) - r_{\nu,p}(\alpha_0)}{\alpha - \alpha_0} - (f_{\nu+p}^{(1)}(\alpha_0) - f_\nu^{(1)}(\alpha_0)).$$

If $\lim_{\nu} f_{\nu}^{(1)}(\alpha_0)$ exists we can certainly chose $m(\varepsilon)$ so that

$$|f_{\nu+p}^{(1)}(\alpha_0) - f_{\nu}^{(1)}(\alpha_0)| < \frac{\varepsilon}{2} \qquad (\text{all } \nu \geqq m(\varepsilon);\ p = 1, 2, \ldots).$$

Hence, on taking account of the statement with respect to (4.8 c′), it is concluded that (4.8) *is secured provided* $\lim_{\nu} f_{\nu}^{(1)}(\alpha_0)$ *exists and*

$$\left|\frac{r_{\nu,p}(\alpha) - r_{\nu,p}(\alpha_0)}{\alpha - \alpha_0}\right| < \varepsilon \qquad (p = 1, 2, \ldots) \tag{4.9 a}$$

for all α *in* F *for which* $|\alpha - \alpha_0| \leqq \zeta_\nu$, *this being so for every* $\nu \geqq m(\varepsilon)$.

Consider a function

$$f(\alpha) = \iint_K \frac{d\mu}{(z-\alpha)^j} \qquad (\text{integer } j > 0), \tag{4.10}$$

where $\mu = \mu(X)$ is an additive function of sets X (Lebesgue measurable), $\mu \geqq 0$ and $\mu(F) = 0$, F being a closed subset of K. With $\delta_1 > \delta_2 > \ldots$ ($\delta_\nu > 0$; $\lim_{\nu} \delta_\nu = 0$), the open sets $O(\delta_\nu)$ will each contain F and $\lim O(\delta_\nu)$ will be F. The function

$$f_n(\alpha) = \iint_{H_n} \frac{d\mu}{(z-\alpha)^j} \qquad (H_n = K - O(\delta_n)) \tag{4.10 a}$$

will be analytic at every interior point of the complement of H_n (accordingly, in $O(\delta_n)$); morever,

$$H_1 < H_2 < \ldots; \qquad \lim H_\nu = K - F. \tag{4.10 b}$$

Hence

$$\lim_{n} f_n(\alpha) = \iint_{K-F} \frac{d\mu}{(z-\alpha)^j} = f(\alpha) \tag{4.10 c}$$

for α in F, provided that the integral last displayed exists for α in F (which will be assumed to be the case). Clearly

$$\{f_n(\alpha)\} < \{F;\ \delta_\nu\},$$

with $f(\alpha)$ of (4.10) constituting the limiting function (in F) of the sequence. In consequence of (4.2 a) and (4.10 a)

$$(\alpha - \alpha_0)\, c_\nu(\alpha) = \iint_{H_\nu'} \frac{1}{\alpha - \alpha_0}\left[\frac{1}{(z-\alpha)^j} - \frac{1}{(z-\alpha_0)^j}\right] d\mu - j \iint_{H} \frac{d\mu}{(z-\alpha_0)^{j+1}}$$

(α_0 in F; α in $O(\delta_\nu)$). One has

$$(\alpha - \alpha_0)\, c_\nu(\alpha) = \iint_{H_\nu} g(z;\, \alpha, \alpha_0)\, d\mu,$$

where

$$g(z;\, \alpha, \alpha_0) = \frac{(z-\alpha_0)^{j-1} + (z-\alpha_0)^{j-2}(z-\alpha) + \cdots + (z-\alpha)^{j-1}}{(z-\alpha)^j\,(z-\alpha_0)^j} - \frac{j}{(z-\alpha_0)^{j+1}} =$$

$$= \frac{N}{(z-\alpha_0)^j\,(z-\alpha_0)^{j+1}}.$$

On writing $u = z - \alpha_0$, $v = z - \alpha$, it is found that

$$N = (\alpha - \alpha_0)\,[(u^{j-1} + \cdots + v^{j-1}) + v\,(u^{j-2} + \cdots + v^{j-2}) + \cdots + v^{j-1}]$$

$$= (\alpha - \alpha_0)\,[u^{j-1} + 2\,u^{j-2} v + 3\,u^{j-3} v^2 + \cdots + (j-1)\, u\, v^{j-1} + j\, v^{j-1}].$$

Whence

$$c_\nu(\alpha) = \iint_{H_\nu} \frac{h(z;\, \alpha, \alpha_0)\, d\mu}{(z-\alpha)^j\,(z-\alpha_0)^{j+1}}, \tag{4.11}$$

with

$$h(z;\, \alpha, \alpha_0) = (z-\alpha_0)^{j-1} + 2\,(z-\alpha_0)^{j-2}(z-\alpha) + \cdots + j\,(z-\alpha)^{j-1}. \tag{4.11 a}$$

We write

$$c_\nu(\alpha) = \gamma_1(\alpha) + \cdots + \gamma_\nu(\alpha),\ \ \gamma_\nu(\alpha) = c_\nu(\alpha) - c_{\nu-1}(\alpha) \qquad (c_0(\alpha) = 0),$$

and

$$\gamma_\nu(\alpha) = \iint_{H_\nu - H_{\nu-1}} \varrho(z;\, \alpha, \alpha_0)\, d\mu, \quad \varrho(z;\, \alpha, \alpha_0) = \frac{h(z;\, \alpha, \alpha_0)}{(z-\alpha)^j\,(z-\alpha_0)^{j+1}}. \tag{4.11 a'}$$

Inasmuch as for α, α_0 in F and for z in H_ν one has

$$|z-\alpha|,\quad |z-\alpha_0| \geqq \delta_\nu,$$

it is inferred that

$$|\varrho(z;\, \alpha, \alpha_0)| = |(z-\alpha)^{-j}(z-\alpha_0)^{-2} + 2\,(z-\alpha)^{-j+1}(z-\alpha_0)^{-3} + \tag{4.11 a''}$$

$$\cdots + j\,(z-\alpha)^{-1}(z-\alpha_0)^{-j-1}| \leqq \frac{1}{2} j\,(j+1)\,\delta_\nu^{-j-2} \qquad (\alpha, \alpha_0 \text{ in } F;\ z \text{ in } H_\nu).$$

Hence

$$|\gamma_\nu(\alpha)| \leqq b_j\, \delta_\nu^{-j-2}\, \mu(H_\nu - H_{\nu-1}) \qquad \left(b_j = \frac{1}{2} j\,(j+1)\right) \tag{4.11 b}$$

and, consequently,

$$|c_\nu(\alpha)| \leqq b_j \sum_{j=1}^{\nu} \delta_i^{-j-2}\, \mu(H_i - H_{i-j}) \qquad (\alpha, \alpha_0 \text{ in } F;\ H_0 = \text{null set}). \tag{4.11 c}$$

If the series

(4. 12) $$S_j = \sum_{i=1}^{\infty} \delta_i^{-j-2} \mu (H_i - H_{i-1})$$

converges the function $f(\alpha)$ *of* (4. 10) *is monogenic in* F *and, in fact,*

$$f^{(1)}(\alpha) = \iint_K \frac{j\, d\mu}{(z-\alpha)^{j+1}} \qquad \text{(in } F).$$

The condition relating to (4. 12) amounts to the requirement that the 'mass' $\mu (\geqq 0)$ be sufficiently 'rarefied' in the vicinity of the set F. In this connection it is to be noted that $H_i - H_{i-1}$ is the set of all those points of K whose distance ϱ from F satisfies the inequality

$$\delta_i \leqq \varrho < \delta_{i-1}.$$

Apparently there is no essential loss of generality if we take $\delta_i = \frac{1}{i}$.

The condition for monogeneity involved in the italicized statement above has the virtue of simplicity but is otherwise somewhat stringent. The validity of this assertion is established by noting that in consequence of the convergence of (4. 12) the inequalities (4. 11 c) will yield

$$| c_\nu(\alpha) | \leqq b_j S_j \qquad \text{(in } F).$$

Whence the limiting function $c(\alpha)$ will satisfy

$$| c(\alpha) | \leqq b_j S_j \qquad (\alpha \text{ in } F;\ \alpha \neq \alpha_0),$$

which will insure (4. 5 a), thus establishing the desired monogeneity property of $f(\alpha)$ at every limiting point of F.

Inasmuch as convergence of S_j $(j > 1)$ implies that of S_{j-1}, it is concluded that *the function*

(4. 13) $$f(\alpha) = \iint_{K-F} \frac{d\mu}{z-\alpha} \qquad (\mu \geqq 0)$$

is differentiable in F j *times (the derivatives* $f^{(1)}(\alpha), \ldots f^{(j)}(\alpha)$ *being unique in* F *and obtainable by formal derivations), provided* S_j *of* (4. 12) *converges.*

We shall now proceed to lighten the above condition with the aid of a result in (T_2). *Assume merely that* S_{j-1} *converges.* With X denoting any measurable subset of K, we form the additive function of sets (for α in F)

(4. 14) $$\Phi_{j+1}(X) = \iint_X \frac{d\mu}{|z-\alpha|^{j+1}},$$

where μ is an absolutely continuous set function, non negative and vanishing on F. One has

$$\Phi_{j+1}(X) = \sum_{i=1}^{\infty} \iint\limits_{X(H_i - H_{i-1})} \frac{d\mu}{|z-\alpha|^{j+1}}.$$

In view of the inequalities subsequent to (4. 11 a′) it is inferred that

$$\text{(4. 14 a)} \qquad \Phi_{j+1}(X) \leqq \sum_{l=1}^{\infty} \delta_i^{-j-1} \mu(X(H_i - H_{i-1})) \leqq S_{j-1} \qquad (\alpha \text{ in } F).$$

The set function $\Phi_{j+1}(X)$ is, of course, absolutely continuous; we shall establish that the latter property holds uniformly with respect to α (α in F). First it is noted that there exists a function $d(\varepsilon)$, defined for $0 < \varepsilon \leqq \varepsilon_0$ and approaching zero monotonically with ε, such that

$$\text{(4. 15)} \qquad \mu(X) \leqq d(\varepsilon) \qquad (\text{for all } X \text{ with } m\,X \leqq \varepsilon).$$

Convergence of S_{j-1} implies that there exists a monotonically diminishing sequence of positive numbers s_i so that

$$\text{(4. 15 a)} \qquad \mu(H_i - H_{i-1}) \leqq s_i \delta_i^{j+1} \qquad (i = 1, 2, \ldots),$$

while $s_1 + s_2 + \cdots$ converges. With $m\,X \leqq \varepsilon$, in consequence of the above one may assert

$$\text{(4. 15 b)} \qquad \begin{aligned} \mu_i &= \mu(X(H_i - H_{i-1})) \leqq \mu(X) \leqq d(\varepsilon) && (i = 1, \ldots k-1), \\ \mu_i &\leqq \mu(H_i - H_{i-1}) \leqq s_i \delta_i^{j+1} && (i = k, k+1, \ldots), \end{aligned}$$

where k is arbitrary.

By virtue of (4. 14 a) and (4. 15 a), (4. 15 b)

$$\text{(4. 16)} \qquad \begin{aligned} \Phi_{j+1}(X) &\leqq \sum_{i<k} \delta_i^{-j-1} \mu_i + \sum_{i \geqq k} \delta_i^{-j-1} \mu_i \\ &\leqq d(\varepsilon) \sum_{i<k} \delta_i^{-j-1} + \sum_{i \geqq k} s_i = \zeta \qquad (\alpha \text{ in } F). \end{aligned}$$

Now there is no essential loss of generality (for the purposes on hand) to choose the δ_i equal to $\frac{1}{i}$. One then has

$$\text{(4. 16 a)} \qquad \zeta \leqq \gamma_j\, d(\varepsilon)\, k^{j+2} + \sum_{i \geqq k} s_i = \zeta(\varepsilon, k)$$

where γ_j is independent of ε. We choose $k = k(\varepsilon)$ so that

(4.17) $$\lim k(\varepsilon) = \infty, \quad \lim d(\varepsilon) k^{j+2} = 0 \qquad (\text{as } \varepsilon \to 0).$$

Such a choice can be achieved, for instance, by taking

(4.17 a) $$k(\varepsilon) = [d(\varepsilon)^{-\beta}] \qquad \left(\beta = \frac{1}{j+2+\varepsilon'}, \ \varepsilon' > 0\right).$$ [1]

In consequence of (4.17) one clearly has

(4.17 b) $$\zeta(\varepsilon) = \zeta(\varepsilon, k(\varepsilon)) \to 0 \qquad (\text{as } \varepsilon \to 0);$$

moreover, in view of (4.16), (4.16 a),

(4.18) $$\Phi_{j+1}(X) \leqq \zeta(\varepsilon) \qquad (\text{all } \alpha \text{ in } F)$$

for all $X < K$ with $m X \leqq \varepsilon$. Inasmuch as $\zeta(\varepsilon)$ is independent of α, the statement with respect to (4.18) amounts to the assertion that the absolute continuity of $\Phi_{j+1}(X)$ is *uniform* with respect to α (α in F).

If we formally evaluate the derivative of

$$f(\alpha) = \iint\limits_K \frac{d\mu}{(z-\alpha)^j},$$

by differentiating under the integral sign we obtain

$$f^{(1)}(\alpha) = \iint\limits_K \frac{j\, d\mu}{(z-\alpha)^{j+1}}.$$

In view of the preceding convergence of S_{j-1} implies that the absolute continuity of the related set-function[2]

(4.19) $$j\, \Phi^*_{j+1}(X) = \iint\limits_X \frac{j\, d\mu}{(z-\alpha)^{j+1}}$$

is uniform with respect to α (for α in F). In fact,

$$|\Phi^*_{j+1}(X)| \leqq \Phi_{j+1}(X),$$

while the function last displayed satisfies (4.18), (4.17 b). In consequence of statement with respect to (4.19) and on taking account of a result in (T_2) the following is concluded.

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[1] Here [...] signifies greatest integer equal or less than ... .

[2] This set function is complex valued.

If the series S_{j-1} (*cf.* (4.12)) *converges, the function* $f(\alpha)$ (*of* (4.10)) *will be monogenic in* F *and one will have*

$$f^{(1)}(\alpha) = \iint_K \frac{j\, d\mu}{(z-\alpha)^{j+1}} \qquad (in\ F); \tag{4.20}$$

the related set-function (4.19) *will have the property*

$$|j\, \Phi^*_{j+1}(X)| \leqq j\, \zeta(\varepsilon) \qquad (\zeta(\varepsilon) \to 0 \text{ with } \varepsilon) \tag{4.20 a}$$

for all measurable subsets X *of* K *with* $mX \leqq \varepsilon$*; here* $\zeta(\varepsilon)$ *is a function as described in connection with* (4.16 a)—(4.17 b), *provided we take* $\delta_i = \frac{1}{i}$.

Let us investigate the function $\zeta(\varepsilon)$ in the fairly general case when the s_i in (4.15 a; with $\delta_i = i^{-1}$) may be taken as $a\, i^{-1-\tau}$ $(\tau > 0)$. We then will have

$$\sum_{i \geqq k} s_i \leqq a' k^{-\tau} \leqq a_1 d^{\beta\tau}(\varepsilon),$$

provided $k(\varepsilon)$ is defined by (4.17 a). By (4.16 a) the inequality

$$\zeta(\varepsilon) = \zeta(\varepsilon, k(\varepsilon)) \leqq \gamma_j d^{1-\beta(j+2)}(\varepsilon) + a_1 d^{\beta\tau}(\varepsilon)$$

will then follow. On taking ε' in (4.17 a) equal to τ we accordingly obtain

$$\zeta(\varepsilon) \leqq (a_1 + \gamma_j)\, d^{\sigma}(\varepsilon) \qquad \left(\sigma = \frac{\tau}{j+2+\tau};\ d(\varepsilon) \text{ from } (4.15)\right). \tag{4.21}$$

The above inequality, valid when S_{j-1} $\left(with\ \delta_i = \frac{1}{i}\right)$ *converges and* s_i *of* (4.15 a) *is* $a\, i^{-1-\tau}$ $(\tau > 0)$, *has the advantage of simplicity.*

When $\mu \geqq 0$ is absolutely continuous one has

$$\mu(X) = \iint_X q(x)\, dx\, dy \qquad (\text{all measurable } X < K),$$

where $q(z)\,(\geqq 0) < L_1$ over K[1]. This representation enables one to obtain explicit expressions for $d(\varepsilon)$ of (4.15) in some important cases. Thus

$$\mu(X) \leqq \beta\, m X \leqq \beta\, \varepsilon = d(\varepsilon) \qquad (\text{whenever } mX \leqq \varepsilon) \tag{4.22}$$

when $|q(z)| \leqq \beta$ in K. More generally,

---

[1] L_p is the class of functions whose p-th power is Lebesgue integrable.

(4.22 a) $$\mu(X) \leqq \left[\iint_X q^2(z)\,dx\,dy\right]^{\frac{1}{2}} [m\,X]^{\frac{1}{2}} \leqq \beta_2\,\varepsilon^{\frac{1}{2}} = d(\varepsilon) \quad (\text{whenever } m\,X \leqq \varepsilon)$$

in the case when $q(z) < L_2$ in K; one may take

$$\beta_2^2 = \iint_X q^2(z)\,dx\,dy.$$

5. **Uniqueness Properties Related to Sets of Positive Measure.** Such properties have been investigated for contain classes of functions which are limits of rational functions, by A. Beurling[1]. In view of our purposes we shall first establish a result somewhat similar to that involved in Lemma I of (B; p. 201).

We introduce the following Definition.

Definition 5.1. *The frontier of a bounded set will be said to be regular if it consists of a number of simple closed continuous rectifiable curves, the sum of whose lengths is finite.*

Suppose $h(z)$ is analytic uniform in an open bounded set O. Let

(5.1) $$F < O$$

be closed and have a regular frontier. We put

(5.2) $$u(z) = \log|h(z)|.$$

We designate by $\Delta(t)$ the part of F in which $u(z) < t$. On writing

(5.3) $$t^* = \text{u. b. } u(z) \qquad (\text{in } F),$$

it is observed that

(5.3 a) $$\Delta(t) = F \qquad (\text{for all } t > t^*).$$

On letting

(5.4) $$S(t) = m\,\Delta(t)$$

it is noted that $S(t) = m\,F$ $(t \geqq t^*)$.

The frontier of $\Delta(t)$ is regular and is of the form

(5.5) $$A(t) + B(t),$$

where $A(t)$ is a subset of the frontier of F, while $B(t)$ has no points in common with the frontier of F.

[1] A. Beurling, *Sur les fonctions limites quasi analytiques des fractions rationnelles*, Comptes Rendus de huitième Congrès des Math. Scandinaves, 1934, pp. 199—210; in the sequel referred to as (B).

Let $z = x + iy$ designate a point in $B(t)$ and η be the inclination angle of the normal at z directed outwards with respect to $\Delta(t)$. One has

$$u(z) = u(x, y) = t.$$

With $t < t + \lambda < t^*$, we shall have

$$u(z + \sigma e^{i\eta}) = u(x + \sigma \cos \eta,\, y + \sigma \sin \eta) = t + \lambda,$$

provided σ is suitably chosen; $z + \sigma e^{i\eta}$ on $B(t + \lambda)$. Now

$$u(x + \sigma \cos \eta,\ y + \sigma \sin \eta) = u(x, y) + \left(\frac{\partial u}{\partial x} + \varepsilon_1\right) \sigma \cos \eta + \left(\frac{\partial u}{\partial y} + \varepsilon_2\right) \sigma \sin \eta$$

($\varepsilon_1, \varepsilon_2 \to 0$ as $\sigma \to 0$). Accordingly

$$\lambda = \sigma \left(\frac{\partial u}{\partial x} \cos \eta + \frac{\partial u}{\partial y} \sin \eta + \varepsilon\right)$$

($\varepsilon \to 0$ as $\sigma \to 0$). Whence

$$\sigma = \frac{\lambda}{\varepsilon + \dfrac{\partial u}{\partial n}},$$

normal derivation being in the direction outward with respect to $\Delta(t)$; by definition of $\Delta(t)$ one has $\dfrac{\partial u}{\partial n} \geqq 0$.

It is observed that

$$S(t_0 + \lambda_0) - S(t_0) = m(\Delta(t_0 + \lambda_0) - \Delta(t_0))$$

($t_0 < t + \lambda_0 < t^*$) and

$$S(t_0 + \lambda_0) - S(t_0) = \iint ds\, d\sigma = \int\!\!\int \frac{ds\, d\lambda}{\dfrac{\partial u}{\partial n}} = \int\limits_{\lambda=0}^{\lambda_0} \left[\ \int\limits_{B(t_0+\lambda)} \frac{ds}{\dfrac{\partial u}{\partial n}}\right] d\lambda,$$

where ds is differential element of length of arc along the curves constituting $B(t_0 + \lambda)$. Hence

$$\text{(5.6)} \qquad \frac{dS}{dt_0} = \lim_{\lambda_0 \to 0} \frac{1}{\lambda_0} \int\limits_0^{\lambda_0} \left[\ \int\limits_{B(t_0+\lambda)} \left(\frac{\partial u}{\partial n}\right)^{-1} ds\right] d\lambda = \int\limits_{B(t_0)} \frac{ds}{\dfrac{\partial u}{\partial n}}.$$

This formula is in agreement with a similar result in (B).

Now

$$b(t) = \text{length of } B(t) = \int\limits_{B(t)} ds = \int\limits_{B(t)} \left(\frac{\partial u}{\partial n}\right)^{\frac{1}{2}} \left(\frac{\partial u}{\partial n}\right)^{-\frac{1}{2}} ds \leqq \left[\int\limits_{B(t)} \frac{\partial u}{\partial n} ds\right]^{\frac{1}{2}} \left[\int\limits_{B(t)} \frac{ds}{\frac{\partial u}{\partial n}}\right]^{\frac{1}{2}}.$$

Thus, by (5.6)

(5.7) $$\frac{dS}{dt} \geqq \frac{b^2(t)}{\int\limits_{B(t)} \frac{\partial u}{\partial n} ds} \qquad (t < t^*;\ \text{cf. } (5.3)).$$

On writing

$$a(t) = \text{length of } A(t)$$

and *letting* Γ *denote the frontier of* F, we have

$$\text{length of } \Gamma - A(t) = g - a(t),$$

where g is length of Γ. The curves of $A(t) + B(t)$, of total length $a(t) + b(t)$, enclose an area $S(t)$. The curves of $\Gamma - A(t) + B(t)$, of total length $g - a(t) + b(t)$, enclose the set $F - \Delta(t)$, whose area is $mF - S(t)$. It is observed that

$$a(t) + b(t) \geqq A_1(t), \quad g - a(t) + b(t) \geqq A_2(t),$$

where $A_1(t)$ $[A_2(t)]$ is the length of the circumference of a circle of area $S(t)$ $[mF - S(t)]$. Hence

$$b(t) \geqq 2\pi^{\frac{1}{2}} S^{\frac{1}{2}} - a(t),$$

$$b(t) \geqq 2\pi^{\frac{1}{2}} [mF - S]^{\frac{1}{2}} - g + a(t);$$

here $0 \leqq a(t) \leqq g$. Adding these inequalities we obtain

(5.8) $$b(t) \geqq \pi^{\frac{1}{2}} S^{\frac{1}{2}} + \pi^{\frac{1}{2}} [mF - S]^{\frac{1}{2}} - \frac{g}{2} = \gamma(S).$$

We observe that g, being the length of the frontier of the set F is not less than the length of the circumference of a circle of area mF; thus

(5.9) $$g \geqq 2\pi^{\frac{1}{2}} (mF)^{\frac{1}{2}}.$$

The function $\gamma(s)$ vanishes for

(5.9 a) $$2S = mF \pm \left[(mF)^2 - \frac{\lambda^4}{16\pi^2}\right]^{\frac{1}{2}} \qquad (\lambda^2 = g^2 - 4\pi mF).$$

If F is a circular region $\lambda = 0$; in the contrary case $\lambda > 0$ and

$$4\pi m F < g^2.$$

It will be assumed that the set F is such that

$$g^2 < 8\pi m F. \tag{5.10}$$

Under (5.10) one has

$$0 < (mF)^2 - \frac{\lambda^4}{16\pi^2} \leqq (mF)^2$$

and the values S_1, S_2 of S, given in (5.9a), are

$$S_1 = \frac{1}{2} m F - q,\ S_2 = \frac{1}{2} m F + q,\ q = \frac{1}{2}\left[(mF)^2 - \frac{\lambda^4}{16\pi^2}\right]^{\frac{1}{2}}, \tag{5.10a}$$

where

$$0 < q \leqq \frac{1}{2} m F; \tag{5.10b}$$

thus

$$0 \leqq S_1 < \frac{1}{2} m F < S_2 \leqq m F \tag{5.10c}$$

if (5.10) holds; here the equality sign holds only in the case when F is circular.

Now $\gamma(S)$ is monotone increasing for $0 \leqq S \leqq \frac{1}{2} m F$ and is monotone diminishing for $\frac{1}{2} m F \leqq S \leqq m F$. Hence

$$\gamma(S) > 0 \qquad (S_1 < S < S_2).$$

Accordingly, by (5.8)

$$b^2(t) \geqq \gamma^2(S) \qquad (S_1 < S < S_2)$$

and, in view of (5.7),

$$\frac{dS}{dt} \geqq \frac{\gamma^2(S)}{k} \qquad (t < t^*;\ S_1 < S < S_2), \tag{5.11}$$

where k is any constant for which

$$\int\limits_{B(t)} \frac{\partial u}{\partial n}\, ds \leqq k \tag{5.11a}$$

for all t such that $S_1 < S(t) < S_2$.

We shall replace (5.11) by a simplified inequality. For this purpose we express $\gamma(S)$ of (5.8) in the form

(5.12) $$\gamma(S) = \frac{1}{N(S)}(S - S_1)(S_2 - S),$$

where

(5.12 a) $$S_1 = \frac{1}{2}\left\{mF - \left[(mF)^2 - \frac{\lambda^4}{16\pi^2}\right]^{\frac{1}{2}}\right\}, \quad S_2 = \frac{1}{2}\left\{mF + \left[(mF)^2 - \frac{\lambda^4}{16\pi^2}\right]^{\frac{1}{2}}\right\}.$$

A straightforward computation yields

$$N(S) = \left[\pi^{\frac{1}{2}} S^{\frac{1}{2}} + \pi^{\frac{1}{2}}(mF - S)^{\frac{1}{2}} + \frac{g}{2}\right]\left[2\pi(SmF - S^2)^{\frac{1}{2}} + \frac{\lambda^4}{4}\right].$$

Clearly $N(S) > 0$ and attains its maximum for $S = \frac{1}{2} mF$; thus, in view of (5.9 a)

(5.12 b) $$N(S) \leqq c(F) = \left[(2\pi mF)^{\frac{1}{2}} + \frac{g}{2}\right]\frac{g^2}{4}.$$

By the above

$$\gamma(S) \geqq \frac{1}{c(F)}(S - S_1)(S_2 - S).$$

We accordingly replace (5.11) by the inequality

(5.13) $$\frac{dS}{dt} \geqq A(F)(S - S_1)^2(S_2 - S)^2, \quad A(F) = \frac{1}{k c^2(F)} \qquad \text{(cf. (5.11 a), (5.12 b))}$$

valid for

$$t < t^*, \quad S_1 < S < S_2.$$

Now by (5.12 a) and (5.10 a)

$$S_2 - S_1 = 2q.$$

On taking account of (5.10 a)—(5.10 c) we conclude that

(5.14) $$\frac{dS}{dt} \geqq A(F)q^2(S - S_1)^2$$

for

(5.14 a) $$t < t^*, \quad S_1 < S \leqq \frac{1}{2} mF$$

and

(5.15) $$\frac{dS}{dt} \geqq A(F)q^2(S_2 - S)^2$$

for

(5.15 a) $$t < t^*, \quad \frac{1}{2} mF \leqq S < S_2.$$

Consider now the case when (5.14a) holds. In consequence of (5.14)

$$\frac{1}{q^2 A(F)}(S - S_1)^{-2} dS = (1 + p(t))\,dt,$$

where $p(t) \geqq 0$, is some function defined for values of t for which (5.14a) holds. Let t_1 correspond to S_1; integrating we obtain

$$-\frac{1}{q^2 A(F)}(S - S_1)^{-1} = t - \varrho(t),$$

where

$$\varrho(t) = c - \int_{t_1}^{t} p(t)\,dt \qquad (c \text{ a constant}).$$

Clearly $\varrho(t)$ *is monotone non increasing for* $t \geqq t_1$. We have

$$(5.16) \qquad S - S_1 = \frac{1}{q^2 A(F)} \frac{1}{\varrho(t) - t}.$$

Since under (5.14a) one has

$$S - S_1 \leqq \frac{1}{2} m F - S_1 = q$$

it follows that

$$(5.16\,a) \qquad \varrho(t) \geqq t + \frac{1}{q^3 A(F)}$$

for the range of values t for which (5.14a) holds. *Suppose now that for a value* α, *such that*

$$(5.17) \qquad S_1 \leqq S(\alpha) \leqq \frac{1}{2} m F,$$

we have

$$(5.17\,a) \qquad S(\alpha) - S_1 \geqq \frac{1}{q^2 A(F)} \frac{1}{\varrho_0 - \alpha},$$

where ϱ_0 *is a number such that*

$$\varrho_0 > \alpha + \frac{1}{q^3 A(F)}.$$

Let β be such that

$$(5.18) \qquad \alpha < \beta \leqq \varrho_0 - \frac{1}{q^3 A(F)}, \quad S(\beta) \leqq \frac{1}{2} m F.$$

By (5.16) and (5.17a)

$$S(\alpha) - S_1 = \frac{1}{q^2 A(F)} \frac{1}{\varrho(\alpha) - \alpha} \geqq \frac{1}{q^2 A(F)} \frac{1}{\varrho_0 - \alpha}.$$

Hence

$$\varrho(\alpha) \leqq \varrho_0;$$

since $\varrho(t)$ is monotone non increasing for the range under consideration, in view of (5.18) one has

$$\varrho(\beta) \leqq \varrho(\alpha) \leqq \varrho_0.$$

Accordingly (5.16), with $t = \beta$, will yield the inequality

(5.19) $$S(\beta) - S_1 \geqq \frac{1}{q^2 A(F)} \frac{1}{\varrho_0 - \beta},$$

as a consequence of (5.17), (5.17a); this discussion has been for the case (5.14a).

Consider now the situation corresponding to (5.15a). By (5.15) one has

$$\frac{1}{q^2 A(F)}(S_2 - S)^{-2}\, dS = (1 + p(t))\, dt \qquad (p(t) \geqq 0).$$

On letting t_2 denote the value of t corresponding to S_2 and integrating it is inferred that

$$\frac{1}{q^2 A(F)} \frac{1}{S_2 - S} = t - \varrho(t), \qquad \varrho(t) = c + \int_t^{t_2} p(t)\, dt,$$

where c is a constant and $\varrho(t)$ is monotone non increasing $(t \leqq t_2)$. Corresponding to (5.16) we now have

(5.16′) $$S_2 - S = \frac{1}{q^2 A(F)} \frac{1}{t - \varrho(t)};$$

moreover,

(5.16a′) $$\varrho(t) \leqq t - \frac{1}{q^3 A(F)}$$

under (5.15a). Suppose that for a value α, such that

(5.17′) $$\frac{1}{2} m F \leqq S(\alpha) \leqq S_2$$

the inequality

(5.17a′) $$S_2 - S(\alpha) \leqq \frac{1}{q^2 A(F)} \frac{1}{\alpha - \varrho_0}$$

takes place, with

$$\varrho_0 < \alpha - \frac{1}{q^3 A}.$$

Let β be such that

(5.18′) $$\alpha < \beta, \quad S(\beta) \leqq S_2.$$

Then by virtue of (5. 16′), (5. 17 a′)

$$S_2 - S(\alpha) = \frac{1}{q^2 A(F)} \frac{1}{\alpha - \varrho(\alpha)} \leqq \frac{1}{q^2 A(F)} \frac{1}{\alpha - \varrho_0};$$

whence $\varrho(\alpha) \leqq \varrho_0$ and, in view of the monotone character of $\varrho(\alpha)$

$$\varrho(\beta) \leqq \varrho(\alpha) \leqq \varrho_0.$$

Hence in consequence of (5. 16′), where $t = \beta$, it is concluded that

$$S_2 - S(\beta) \leqq \frac{1}{q^2 A(F)} \frac{1}{\beta - \varrho_0}. \tag{5.19'}$$

Suppose (5. 17 a) holds for a value α. One may then assert (5. 19) for

$$\beta = \beta_0 = \varrho_0 - \frac{1}{q^2 A(F)}, \tag{5.20}$$

obtaining

$$S(\beta_0) \geqq \frac{1}{2} m F.$$

We now may use β_0 as α in (5. 17 a′); in fact, on writing

$$q_0 = \beta_0 + \frac{1}{q^2 A(F)} \frac{1}{S(\beta_0) - S_2} \tag{5.20 a}$$

it is deduced that

$$q_0 < \beta_0 - \frac{1}{q^3 A(F)}$$

and that

$$S(\beta_0) - S_2 = \frac{1}{q^2 A} \frac{1}{q_0 - \beta_0}.$$

With the aid of (5. 19′) it is then inferred that

$$S(\beta) \geqq S_2 - \frac{1}{q^2 A(F)} \frac{1}{\beta - q_0} \tag{5.20 b}$$

for all $\beta > \beta_0$, whenever (5. 17 a) holds for a value α.

The above results may be formulated as follows.

Theorem 5. 1. *Suppose $h(z)$ is analytic uniform in an open bounded set O containing a closed set F, whose frontier Γ is regular (Definition 5. 1); g = length of Γ. Let $\Delta(t)$ be the part of F in which*

$$u(z) = \log |h(z)| < t$$

and put $S(t) = m\Delta(t)$. *Designate by* $B(t)$ *the part of the frontier of* $\Delta(t)$ *having no points in common with* Γ. *It is assumed that* F *is such that* g^2 *is less than* $8\pi m F$. *The following notation is introduced*

$$(5.21)\qquad \lambda^2 = g^2 - 4\pi m F (\geqq 0),\quad q^2 = \frac{1}{4}\left[(mF)^2 - \frac{\lambda^4}{16\pi^2}\right],$$

$$S_1 = \frac{1}{2} m F - q,\quad S_2 = \frac{1}{2} m F + q,\quad c(F) = \left[(2\pi m F)^{\frac{1}{2}} + \frac{g}{2}\right]\frac{g^2}{4},$$

$$A(F) = \frac{1}{k c^2(F)},\quad \beta_0 = \varrho_0 - \frac{1}{q^3 A(F)}\qquad \text{(cf. (5.22));}$$

here k *is any constant equal to or greater than the integral over* $B(t)$ *of* $\frac{\partial u}{\partial n} ds$ (*normal derivation outward with respect to* $\Delta(t)$) *when* t *is such that* $S_1 < S(t) < S_2$.

If for some α *one has*

$$(5.22)\qquad \frac{1}{2} m F \geqq S(\alpha) \geqq S_1 + \frac{1}{q^2 A}\frac{1}{\varrho_0 - \alpha}\qquad \left[\textit{some } \varrho_0 > \alpha + \frac{1}{q^3 A(F)}\right],$$

then for every β *such that* $\alpha < \beta \leqq \beta_0$ *we have*

$$(5.22\text{a})\qquad S(\beta) \geqq S_1 + \frac{1}{q^2 A(F)}\frac{1}{\varrho_0 - \beta},$$

while for every $\beta > \beta_0$

$$(5.22\text{b})\qquad S(\beta) \geqq S_2 - \frac{1}{q^2 A(F)}\frac{1}{\beta - \varrho_0}\qquad \left[q_0 = \beta_0 + \frac{1}{q^2 A(F)}\frac{1}{S(\beta_0) - S_2}\right].$$

If, on the other hand, for some α *we have*

$$(5.23)\qquad S(\alpha) \geqq S_2 - \frac{1}{q^2 A(F)}\frac{1}{\alpha - \varrho_0}\qquad \left[\textit{some } \varrho_0 < \alpha - \frac{1}{q^3 A(F)}\right]^1,$$

then for every $\beta > \alpha$ *one has*

$$(5.23\text{a})\qquad S(\beta) \geqq S_2 - \frac{1}{q^2 A(F)}\frac{1}{\beta - \varrho_0}.$$

Let G be a closed set, contained in a bounded domain K,

$$(5.24)\qquad m G > 0.$$

Let open sets O_ν, $> G$, be such that

$$(5.24\text{a})\qquad K > Q_1 > Q_2 > \cdots,\quad O_\nu > \bar{O}_{\nu+1},\qquad \lim O_\nu = G;$$

[1] This condition will imply that $S(\alpha) > \frac{1}{2} m F$.

moreover, $\bar{O}_\nu (\nu = 1, 2, \ldots)$ *is to have a regular frontier.* For a given G and K the sets O_ν, as just described, can be always constructed.

Suppose $f_n(z)$ is analytic in O_n; then, incidentally,

$$(5.25) \qquad |f_n(z)| \leqq M_n \qquad (\text{in } \bar{O}_{n+1}).$$

Assume, for the present, that the sequence $\{f_n(z)\}$ of functions of the above description converges uniformly in G; there exists then a function $f(z)$ such that

$$(5.25\,\text{a}) \qquad |f(z) - f_n(z)| < \varepsilon_n \qquad (z \text{ in } G;\ \lim \varepsilon_n = 0).$$

Let us consider consequences of the relation

$$(5.26) \qquad f(z) = 0 \qquad (\text{in } H_0),$$

where $H_0 < G$, while

$$0 < m H_0 < m G;$$

in view of (5.25 a)

$$(5.26\,\text{a}) \qquad |f_n(z)| < \varepsilon_n \qquad (z \text{ in } H_0).$$

We shall apply Theorem 5.1 with $f_n(z)$, O_n, $\bar{O}_{n+1}$ playing the role of $h(z)$, O, F, respectively. On taking account of that Theorem the following notation is introduced:

$$\Gamma_n = \text{frontier of } \bar{O}_{n+1}, \quad g_n = \text{length of } \Gamma_n,$$

$\Delta_n(t) =$ part of $\bar{O}_{n+1}$ in which $u_n(z) = \log |f_n(z)| < t$, $S_n(t) = m\,\Delta_n(t)$; $B_n(t)$ is to be the part of the frontier of $\Delta_n(t)$ having no points in common with Γ_n; *it is assumed that*

$$(5.27) \qquad g_n^2 < 8\pi m\, O_{n+1};$$

$$\lambda_n^2 = g_n^2 - 4\pi m\, O_{n+1}, \quad q_n^2 = \frac{1}{4}\left[(m\, O_{n+1})^2 - \frac{\lambda_n^4}{16\pi^2}\right],$$

$$S_{1,n} = \frac{1}{2} m\, O_{n+1} - q_n, \quad S_{2,n} = \frac{1}{2} m\, O_{n+1} + q_n;$$

moreover,

$$(5.27') \qquad c_n = \left[(2\pi m\, O_{n+1})^{\frac{1}{2}} + \frac{g_n}{2}\right]\frac{g_n^2}{4}, \qquad A_n = \frac{1}{k_n c_n^2},$$

where k_n is any constant such that

$$(5.27\,\text{a}) \qquad \int_{B_n(t)} \frac{\partial u}{\partial n}\, d s \leqq k_n \qquad (u = u_n(z))$$

(derivations outward with respect to $\Delta_n(t)$) *for all t such that* $S_{1,n} < m\,\Delta_n(t) < S_{2,n}$.

Inasmuch as $m\,O_{n+1} > m\,G$ it is observed that (5.27) holds when

(5.28) $$g_n^2 \leqq 8\,\pi\,m\,G.$$

Now, in view of (5.26 a)

(5.29) $$u_n(z) < \log \varepsilon_n = \alpha_n \qquad \text{(in } H_0\text{)}.$$

Clearly

$$H_0 < \Delta_n(\alpha_n);$$

whence

(5.29 a) $$S_n(\alpha_n) \geqq m\,H_0.$$

We wish first to restrict H_0 so that a relation corresponding to (5.22) holds; accordingly it will be required that

(5.30) $$m\,H_0 \geqq S_{1,n} + \frac{1}{q_n^2\,\Lambda_n}\,\frac{1}{\varrho_{0,n} - \alpha_n} \qquad \left[\text{some } \varrho_{0,n} > \alpha_n + \frac{1}{q_n^2\,\Lambda_n}\right];$$

(5.30) with the equality sign would yield

(5.30 a) $$\varrho_{0,n} = \alpha_n + \frac{1}{q_n^2\,\Lambda_n(m\,H_0 - S_{1,n})} > \alpha_n + \frac{1}{q_n^2\,\Lambda_n};$$

that is,

$$S_{1,n} < m\,H_0 < q_n + S_{1,n}$$

and, in view of (5.27),

(5.30 b) $$\frac{1}{2}m\,O_{n+1} - q_n < m\,H_0 < \frac{1}{2}m\,O_{n+1}.$$

If

$$\underline{\lim}\; q_n = 0$$

the above could not hold unless $m\,H_0 = \frac{1}{2}m\,G$. Accordingly, in order that the part of Theorem 5.1 corresponding to (5.22), (5.22 a) should be applicable one should have

$$\underline{\lim}\; q_n = q > 0\,^1;$$

that is (cf. (5.27)),

(5.31) $$q_n^2 = \frac{1}{64\,\pi^2}\left[16\,\pi^2 (m\,O_{n+1})^2 - (g_n^2 - 4\,\pi\,m\,O_{n+1})^2\right] \to q > 0$$

as $n = n_j \to \infty$. Now $m\,O_{n+1} \to m\,G$; hence g_{n_j} tends to a limit g; in view of (5.27)

(5.32) $$\lim g_{n_j}^2 = g^2 \leqq 8\,\pi\,m\,G$$

and

[1] The q_n are positive.

$$q^2 = \frac{1}{64\,\pi^2}[16\,\pi^2 (m\,G)^2 - (g^2 - 4\,\pi\,m\,G)^2] > 0. \tag{5.32 a}$$

Now $g_n \geqq$ length of circumference of a circle of area $m\,O_{n+1}$; thus

$$g_n^2 \geqq 4\,\pi\,m\,O_{n+1}$$

and

$$g^2 - 4\,\pi\,m\,G \geqq 0.$$

Hence the relation $q > 0$ implies that

$$(0 \leqq)\, g^2 - 4\,\pi\,m\,G < 4\,\pi\,m\,G\,[1]; \tag{5.32 a'}$$

it is noted that (5.32 a) *and* (5.32 a′) *are equivalent.*

For the present we shall assume that $q > 0$ *and shall consider the case when*

$$\frac{1}{2}\,m\,G - q < m\,H_0 < \frac{1}{2}\,m\,G. \tag{5.33}$$

Unless the contrary is implied in the sequel we take $n = n_j \geqq n_0$. In view of the above (5.33) implies that (5.30 b) and, hence, (5.30) (whith the equality sign) will hold for $n = n_j \geqq n_0$ (n_0 suitably great). Either $S_n(\alpha_n) > \frac{1}{2}\,m\,O_{n+1}$ or, in consequence of (5.29 a).

$$\frac{1}{2}\,m\,O_{n+1} \geqq S_n(\alpha_n) \geqq S_{1,\,n} + \frac{1}{q_n^2\,\Lambda_n}\,\frac{1}{\varrho_{0,\,n} - \alpha_n}, \tag{5.34}$$

where $\varrho_{0,\,n}$ is from (5.30 a). By Theorem 5.1 (cf. (5.22), (5.22 a)) the above implies

$$S_n(\beta) \geqq S_{1,\,n} + \frac{1}{q_n^2\,\Lambda_n}\,\frac{1}{\varrho_{0,\,n} - \beta} \tag{5.34 a}$$

for all β such that

$$\alpha < \beta \leqq \beta_{0,\,n} = \varrho_{0,\,n} - \frac{1}{q_n^3\,\Lambda_n}; \tag{5.35}$$

from this it follows that

$$\frac{1}{2}\,m\,O_{n+1} \geqq S_{1,\,n} + \frac{1}{q_n^2\,\Lambda_n}\,\frac{1}{\varrho_{0,\,n} - \beta}.$$

Hence (5.35) *will imply* (5.34 a) *in any case* (provided (5.33) holds). In particular, on letting in (5.34 a) $\beta = \beta_{0,\,n}$ one obtains

$$S_n(\beta_{0,\,n}) \geqq \frac{1}{2}\,m\,O_{n+1}. \tag{5.35'}$$

[1] The relation $q = 0$ would imply $g^2 - 4\,\pi\,m\,G = 4\,\pi\,m\,G$ and conversely.

This implies that

(5.36) $$u_n(z) < \varrho_{0,n} - \frac{1}{q_n^3 \Lambda_n}$$

in a set H_n $(= \Delta_n(\beta_{0,n}))$ such that

(5.36a) $$H_0 < H_n < \bar{O}_{n+1}, \quad m\, H_n \geqq \frac{1}{2} m\, O_{n+1} > \frac{1}{2} m\, G.$$

Consider sets $H_n' = G\, H_n$. We have

$$m\, H_n' = m\, H_n - m(H_n - G), \quad m(H_n - G) \leqq m(O_{n+1} - G).$$

Hence, in view of (5.36a) and since $m(O_{n+1} - G) \to 0$,

(5.37) $$m\, H_n \geqq m\, H_n - m(O_{n+1} - G) > \frac{1}{2} m\, G - \varepsilon \qquad (n \geqq n_0 = n(\varepsilon))$$

where ε (> 0) can be chosen arbitrarily small; moreover,

(5.37a) $$H_0 < H_n' < G.$$

We shall have

(5.38) $$H_{n+m}' < H_n'$$

if for every ζ of G for which $u_{n+m} < \beta_{0,n+m}$, i. e.

(5.38a) $$|f_{n+m}(\zeta)| < e^{\beta_{0,n+m}},$$

one has $u_n < \beta_{0,n}$, that is to say

$$|f_n(\zeta)| < e^{\beta_{0,n}}.$$

Now

$$|f_n(\zeta)| \leqq |f_{n+m}(\zeta)| + |f(\zeta) - f_{n+m}(\zeta)| + |f_n(\zeta) - f(\zeta)|$$

and, by (5.38a) and (5.25c),

$$|f_n(\zeta)| < e^{\beta_{0,n+m}} + \varepsilon_{n+m} + \varepsilon_n.$$

Thus (5.38) will hold provided

$$e^{\beta_{0,n+m}} + \varepsilon_{n+m} + \varepsilon_n \leqq e^{\beta_{0,n}}.$$

Suppose now that for every n $(\geqq n_0)$ an m (> 0) can be found so that

(5.39) $$\varepsilon_{n+m} + \varepsilon_n \leqq e^{\beta_{0,n}} - e^{\beta_{0,n+m}}.$$

There exists then an infinite sequence of integers

$$n_0 < n_1 < n_2 < \dots$$

(subsequence of the original sequence so designated) so that

$$(5.40) \qquad H'_{n_1} > H'_{n_2} > \dots .$$

In view of (5.37), (5.37 a) the set

$$(5.41) \qquad H_\varepsilon = H'_{n_1} H'_{n_2} \dots = \lim H'_{n_j}$$

will have the properties

$$(5.41\text{ a}) \qquad H_0 < H_\varepsilon < G, \qquad m H_\varepsilon \geqq \frac{1}{2} m G - \varepsilon.$$

By (5.36) it will follow that

$$(5.42) \qquad f(z) = 0 \qquad (\text{in } H_\varepsilon)$$

if (5.39) holds and if

$$(5.42\text{ a}) \qquad \lim \left(\varrho_{0,n} - \frac{1}{p_n^3 A_n} \right) = -\infty .$$

Consider a sequence ε_j (> 0),

$$\varepsilon_1 > \varepsilon_2 > \dots, \qquad \lim \varepsilon_j = 0.$$

Under (5.33), (5.39) *and* (5.42 a) *vanishing of* $f(z)$ *in* H_0 *will imply*

$$(5.43) \qquad f(z) = 0 \qquad (\textit{in } H^*),$$

where

$$(5.43\text{ a}) \qquad H^* = H_{\varepsilon_1} + H_{\varepsilon_2} + \dots, \quad H_0 < H^* < G, \quad m H^* \geqq \frac{1}{2} m G.$$

Let us first find conditions under which (5.42 a) holds. By (5.27′)

$$(5.44) \qquad \frac{1}{A_n} \leqq \lambda_0 k_n \qquad (\lambda_0 = \text{an upper bound of } c_n^2).$$

Hence in view of (5.30 a), (5.29)

$$(5.44\text{ a}) \qquad \varrho_{0,n} - \frac{1}{q_n^3 A_n} = \log \varepsilon_n + \frac{\frac{1}{2} m O_{n+1} - m H_0}{q_n^3 A_n (m H_0 - S_{1,n})} \leqq$$

$$\log \varepsilon_n + \frac{\lambda_1 k_n}{q_n^3 \left(m H_0 - \frac{1}{2} m O_{n+1} + q_n \right)}$$

where

$$\lambda_1 = \left(\frac{1}{2} m O_{n_0+1} - m H_0 \right) \lambda_0 .$$

By virtue of (5.31) and (5.33)

(5.44 b) $$\frac{1}{q_n^3} \frac{\lambda_1}{m\, H_0 - \frac{1}{2} m\, O_{n+1} + q_n} \to \frac{1}{q^3} \frac{\lambda_1}{m\, H_0 - \frac{1}{2} m\, G + q} = q',$$

as $n = n_j \to \infty$, where

(5.44 b') $$q' > \frac{\lambda_1}{q^4}.$$

Whence

$$\beta_{0,n} = \varrho_{0,n} - \frac{1}{q_n^3 A_n} < \log \varepsilon_n + \tau_0 k_n \qquad (n_j \geqq n_0)$$

where τ_0 *is any number exceeding* q' (n_0 suitably great). Consequently (5.42 a) holds if the second member above tends to $-\infty$. *Accordingly*, (5.42 a) *takes place whenever*

(5.45) $$\varepsilon_n = \varphi(n)\, \tau^{k_n} \qquad (\varphi(n) > 0, \quad 0 < \tau < 1),$$

where $\varphi(n)$ *is any function tending to zero and* $\tau = \exp. (-\tau_0)$.

By (5.44 a) and, in view of the preceding, exp. $\beta_{0,n} > \varepsilon_n$. Hence

$$\zeta_n = e^{\beta_{0,n}} - \varepsilon_n > 0.$$

Thus, inasmuch as $\lim \beta_{0,n} = 0$ (cf. (5.42 a)), we shall have

$$\varepsilon_{n_{j+1}} + e^{\beta_{0,n_{j+1}}} \leqq \zeta_{n_j} \qquad (j = 1, 2, \ldots),$$

provided one replaces the original sequence $\{n_j\}$, if necessary, by a suitable subsequence[1]. The latter inequalities, however, imply the validity of (5.39) with

$$n = n_j, \quad n + m = n_{j+1} \qquad (j = 1, 2, \ldots).$$

Whence, under (5.32 a'), (5.33) *and* (5.45), *vanishing of* $f(z)$ *in* H_0 *implies*

(5.46) $$f(z) = 0 \qquad (\text{in } H^*;\ H_0 < H^* < G),$$

where $m\, H^* \geqq \frac{1}{2} m\, G$.

We shall engage now in a type of 'quasi analytic continuation'. We let H^* play the role of H and consider first the case when

(5.47) $$m\, H^* = \frac{1}{2} m\, G.$$

---

[1] The same notation is retained for the latter as for the original.

One may apply the part of the Theorem relating to (5.22), (5.22 b)[1]. In place of (5.29 a) we have

$$(5.48)\qquad S_n(\alpha_n) \geqq m\,H^* = \frac{1}{2}\,m\,G.$$

One may have $S_n(\alpha_n) > \frac{1}{2}\,m\,O_{n+1}$; in the contrary case we put

$$(5.49)\qquad \left[\frac{1}{2}\,m\,O_{n+1} \geqq \right] S_n(\alpha_n) \geqq \frac{1}{2}\,m\,G = S_{1,n} + \frac{1}{q_n\,A_n}\,\frac{1}{\varrho_{0,n} - \alpha_n},$$

which gives

$$(5.49\text{ a})\qquad \varrho_{0,n} = \alpha_n + \frac{1}{q_n^2\,A_n\left(\frac{1}{2}\,m\,G - S_{1,n}\right)} > \alpha_n + \frac{1}{q_n^3\,A_n},$$

provided

$$0 < \frac{1}{2}\,m\,G - \frac{1}{2}\,m\,O_{n+1} + q_n < q_n,$$

that is, if

$$(5.49\text{ b})\qquad \frac{1}{2}\,m\,O_{n+1} - \frac{1}{2}\,m\,G < q_n;$$

now, the latter inequality clearly holds for $n = n_j \geqq n_0$ (n_0 suitably great), inasmuch as (5.32 a′) (implying (5.31)) has been assumed. With (5.49), (5.49 a) corresponding to (5.22), the implication of (5.22 a) would be

$$(5.50)\qquad S_n(\beta) \geqq S_{2,n} - \frac{1}{q_n^2\,A_n(\beta - p_n)},$$

where

$$(5.50\text{ a})\qquad p_n = \beta_{0,n} + \frac{1}{q_n^2\,A_n}\,\frac{1}{S_n(\beta_{0,n}) - S_{2,n}},$$

for all

$$(5.50\text{ b})\qquad \gamma_n \geqq \beta > \beta_{0,n} = \varrho_{0,n} - \frac{1}{q_n^3\,A_n} = \alpha_n + \frac{\frac{1}{2}(m\,O_{n+1} - m\,G)}{q_n^3\,A_n\left(\frac{1}{2}\,m\,G - \frac{1}{2}\,m\,O_{n+1} + q_n\right)}.$$

Since the last member above exceeds α_n we have $\beta_{0,n} > \alpha_n$ and

$$(5.50\text{ b}')\qquad S_n(\beta_{0,n}) \geqq S_n(\alpha_n) \geqq \frac{1}{2}\,m\,G,\quad S_n(\beta_{0,n}) - S_{2,n} \geqq \frac{1}{2}\,m\,G - \frac{1}{2}\,m\,O_{n+1} - q_n.$$

---

[1] q of the theorem is not to be confused with q of (5.32 a).

In view of the latter inequality and of the one preceding (5.49 b) one has

$$\frac{1}{\Lambda_n q_n^3} > \beta_{0,n} - p_n.$$

We take

$$(5.51) \qquad \gamma_n = p_n + \frac{1}{\Lambda_n q_n^2 \varepsilon} \qquad (0 < \varepsilon < q).$$

Clearly $\gamma_n > \beta_{0,n}$. In (5.50) we put $\beta = \gamma_n$, obtaining

$$(5.52) \qquad S_n(\gamma_n) \geqq \frac{1}{2} m\, G + q - \varepsilon.$$

Whence, if $S_n(\alpha_n) \leqq \frac{1}{2} m\, O_{n+1}$,

$$(5.53) \qquad u_n(z) < p_n + \frac{1}{\varepsilon \Lambda_n q_n^2}$$

in a set $H_n (= \Delta_n(\gamma_n))$ *such that*

$$(5.53\text{ a}) \qquad H_0 < H_n < \bar{O}_{n+1}, \qquad m\, H_n \geqq \frac{1}{2} m\, G + q - \varepsilon.$$

Consider now the case when $m\, H^* = \frac{1}{2} m\, G$, while $S_n(\alpha_n) > \frac{1}{2} m\, O_{n+1}$. With (5.23) in view (where F is replaced by $\bar{O}_{n+1}$), we write

$$(5.54) \qquad S_n(\alpha_n) = S_{2,n} - \frac{1}{q_n^2 \Lambda_n} \frac{1}{\alpha_n - p_{0,n}};$$

here

$$(5.54\text{ a}) \qquad p_{0,n} = \alpha_n - \frac{1}{q_n^2 \Lambda_n} \frac{1}{q_n + \frac{1}{2} m\, O_{n+1} - S_n(\alpha_n)} < \alpha_n - \frac{1}{q_n^3 \Lambda_n}.$$

Accordingly, by (5.23 a).

$$S_n(\beta) \geqq S_{2,n} - \frac{1}{q_n^2 \Lambda_n} \frac{1}{\beta - p_{0,n}}$$

for all $\beta > \alpha_n$. Now, in view of (5.50 b) $\gamma_n > \alpha_n$. Hence one may take $\beta = \gamma_n$, obtaining

$$S_n(\gamma_n) \geqq S_{2,n} - \frac{1}{\delta_n}$$

where

$$\delta_n = q_n^2 \Lambda_n (\gamma_n - p_{0,n}) = \frac{1}{S_n(\beta_{0,n}) - \frac{1}{2} m\, O_{n+1} - q_n} + \frac{1}{\varepsilon}$$

$$+ \frac{1}{q_n + \frac{1}{2} m\, O_{n+1} - S_n(\alpha_n)} + \frac{1}{2} \frac{m\, O_{n+1} - m\, G}{q_n \left(\frac{1}{2} m\, G - \frac{1}{2} m\, O_{n+1} + q_n \right)} > \frac{1}{\varepsilon},$$

provided $S_n(\alpha_n) < S_{n,2}$. Hence

$$S_n(\gamma_n) > \frac{1}{2} m\, O_{n+1} + q_n - \varepsilon \tag{5.55}$$

if $S_n(\alpha_n) < S_{n,2}$; clearly (5.55) holds in the contrary case as well. Let $q > \varepsilon_0 > \varepsilon$, where ε_0 may be as near to ε as desired; (5.55) implies

$$S_n(\gamma_n) > \frac{1}{2} m\, G + q - \varepsilon_0 \qquad (n = n_j \geqq n_0), \tag{5.55 a}$$

where n_0 is suitably great. *In view of* (5.52) *it is observed that* (5.55 a) *is satisfied for all* $n_j \geqq n_0$ *when* $m H^* = \frac{1}{2} m\, G$, *whether*

$$S_n(\alpha_n) > \frac{1}{2} m\, O_{n+1}$$

or not. In place of (5.53), (5.53 a) we may assert that *the relation* $m H^* = \frac{1}{2} m\, G$ *implies that*

$$u_n(z) < p_n + \frac{1}{\varepsilon_0 \Lambda_n q_n^2} \tag{5.56}$$

in a set H_n *such that*

$$H_0 < H_n < \bar{O}_{n+1}, \qquad m H_n \geqq \frac{1}{2} m\, G + q - \varepsilon_0 \tag{5.56 a}$$

for all $n = n_j \geqq n_0$, *where* $\varepsilon_0 (> 0)$ *is arbitrarily small.*

Continuing the discussion begun with (5.46) we are brought to the consideration of the case when

$$\frac{1}{2} m\, G + q > m H^* > \frac{1}{2} m\, G. \tag{5.57}$$

Corresponding to (5.29 a) one now has

$$S_n(\alpha_n) \geqq m H^* > \frac{1}{2} m\, G.$$

and

$$S_n(\alpha_n) \geqq S_{2,n} - \frac{1}{q_n^2 \Lambda_n} \frac{1}{\alpha_n - p^{0,n}}, \tag{5.58}$$

where

(5.58 a) $$p^{0,n} = \alpha_n - \frac{1}{q_n^2 A_n} \cdot \frac{1}{q_n + \frac{1}{2} m O_{n+1} - m H^*} < \alpha_n - \frac{1}{q_n^3 A_n}$$

$(n = n_j \geqq n_0)$. Whence, in consequence of (5.23 a)

$$S_n(\beta) \geqq S_{2,n} - \frac{1}{q_n^2 A_n} \frac{1}{\beta - p_n} \qquad (\beta > \alpha_n).$$

In particular, with γ_n from (5.51), one obtains

$$S_n(\gamma_n) \geqq S_{2,n} - \frac{1}{\delta(n)},$$

where $\delta(n)$ is δ_n with $S_n(\alpha_n)$ replaced by $m H^*$; thus, in view of the inequalities preceding (5.55), we have

$$\delta(n) > \frac{1}{\varepsilon} \qquad (n = n_j \geqq n_0).$$

In establishing this use has been made of the first inequality (5.57). *Whence* (5.55) *and, finally,* (5.55 a) *will hold for all* $n = n_j \geqq n_0$. Accordingly, it is concluded that *the relation*

(5.59) $$\frac{1}{2} m G + q > m H^* \geqq \frac{1}{2} m G$$

implies that

(5.59 a) $$u_n(z) < p_n + \frac{1}{\varepsilon_0 A_n q_n^2} = \zeta(n) \qquad \text{(cf. (5.50 a), (5.50 b))}$$

in a set H_n *such that*

(5.59 b) $$H_0 < H_n < \bar{O}_{n+1}, \qquad m H_n \geqq \frac{1}{2} m G + q - \varepsilon_0.$$

Suppose that the relation

(5.60) $$\lim \zeta(n_j) = -\infty$$

has been secured. Then by (5.59 a) it will follow that

(5.60 a) $$f(z) = 0 \qquad (\text{in } H(\varepsilon_0)),$$

where

(5.60 a′) $$H_0 < H(\varepsilon_0) = \prod_j H_{n_j} < G.$$

By a reasoning employed before we infer that in consequence of (5.60) one has

$$H_{n_1} > H_{n_2} > \cdots$$

for a suitable subsequence of $\{n_j\}$. By (5.59 b) the latter inclusion relations lead to the conclusion that

$$m\,H(\varepsilon_0) \geqq \frac{1}{2} m\,G + q - \varepsilon_0. \tag{5.60 a''}$$

Suppose that for values n belonging to an infinite subsequence of $\{n_j\}$

$$S_n(\beta_{0,n}) \geqq S_{2,n}. \tag{5.61}$$

We then shall have

$$S_n(\beta_{0,n}) \geqq \frac{1}{2} m\,G + q - \varepsilon_0 \qquad (n \geqq n_0); \tag{5.61 a}$$

that is,

$$u_n(z) < \beta_{0,n} \qquad (\text{in } H_n),$$

where H_n satisfies (5.59 b). Repeating the reasoning from (5.59 a) to (5.60 a''), with $\zeta(n)$ replaced by (5.59 a) *we deduce that* (5.60 a)—(5.60 a'') *will hold if*

$$\underline{\lim}\, \beta_{0,n} = -\infty. \tag{5.61 b}$$

If the statement made with respect to (5.61) does not hold we have

$$S_n(\beta_{0,n}) < S_{2,n} \tag{5.62}$$

for an infinite sequence of values $n = n_j$. By (5.59 a), (5.50 a), (5.50 b) one then will obtain

$$\zeta(n) = \beta_{0,n} + \frac{1}{q_n^2 A_n}\frac{1}{S_n(\beta_{0,n}) - S_{2,n}} + \frac{1}{\varepsilon_0 q_n^2 A_n}$$

$$< \alpha_n + \frac{1}{q_n^2 A_n}\zeta_n + \frac{1}{\varepsilon_0 q_n^2 A_n}, \qquad \zeta_n = \frac{\frac{1}{2}(m\,O_{n+1} - m\,G)}{q_n\left(\frac{1}{2}\,m\,G - \frac{1}{2}\,m\,O_{n+1} + q_n\right)}.$$

Now

$$0 < \zeta_n, \qquad \lim \zeta_n = 0.$$

Whence

$$\zeta(n) < \alpha_n + \frac{1 + \varepsilon_0\varepsilon'}{\varepsilon_0 q_n^2 A_n} \qquad (n \geqq n_0),$$

where by a suitable choice of n_0 one may make ε' (> 0) arbitrarily small. Since $q_n \to q > 0$, by virtue of (5.44) it is found that

$$\zeta(n) < \alpha_n + \frac{1 + \varepsilon_0\varepsilon'}{\varepsilon_0 q_n^2}\lambda_0 k_n \leqq \alpha_n + \frac{\lambda^*}{\varepsilon_0}k_n \qquad (n \geqq n_0) \tag{5.62 a}$$

where

$$\lambda^* > \frac{\lambda_0}{q^2}; \tag{5.62 a'}$$

here λ^* may be chosen arbitrarily near to $\lambda_0 q^{-2}$ (if n_0 is chosen suitably great). We further infer that (5.60) will take place if

$$\alpha_n + \frac{\lambda^*}{\varepsilon_0} k_n \to -\infty,$$

that is, provided

(5.63) $$\varepsilon_n = \varphi(n)\, \omega_{\varepsilon_0}^{k_n} \qquad (n_j \geqq n_0),$$

where $\varphi(n) (> 0)$ is any function tending to zero with n^{-1} and where

(5.63 a) $$0 < \omega_{\varepsilon_0} = \exp.\left\{-\frac{\lambda^*}{\varepsilon_0}\right\} < 1.$$

By (5.50 b) and (5.44)

$$\beta_{0,n} \leqq \alpha_n + \sigma(n)\lambda_0 k_n, \qquad 0 < \sigma(n) \to 0 \quad (\text{as } n_j \to \infty).$$

Hence

$$\beta_{0,n} \leqq \alpha_n + \varepsilon^* \lambda_0 k_n \qquad (\varepsilon^* > 0;\ n \geqq n_0);$$

here, with n_0 suitable chosen ε^* may be taken arbitrarily small. Clearly

$$\beta_{0,n} \leqq \alpha_n + \frac{\lambda^*}{\varepsilon_0} k_n.$$

Thus (5.63) (for all $n \geqq n_0$), (5.63 a) will imply (5.61 b) in the case when the statement with respect to (5.61) holds. Accordingly, *if* $f(z) = 0$ *in* H_0 $(H_0 < G)$, *where* H_0 *satisfies* (5.33), *then* $f(z) = 0$ *in* $H(\varepsilon_0)$, *where* $H(\varepsilon_0)$ *satisfies* (5.60 a'), (5.60 a''), *provided* (5.63) *holds for all* $n \geqq n_0$.

If

(5.64) $$\varepsilon_n = \varphi(n)\, \tau(n)^{k_n} \qquad (0 < \tau(n)),$$

where $\varphi(n)$ is as in (5.63) and $\tau(n)$ is any function tending to zero with n^{-1}, then we shall have the following consequences. Let ε_j (> 0) be a sequence such that

$$\varepsilon_0 > \varepsilon_j, \qquad \lim \varepsilon_n = 0.$$

We shall have

$$\varepsilon_n = \varphi_j(n)\, \omega_{\varepsilon_j}^{k_n} \qquad (\text{all } n \geqq n(j)),$$

where

$$\omega_{\varepsilon_j} = \exp.\left\{-\frac{\lambda^*}{\varepsilon_j}\right\}, \qquad \varphi_j(n) \leqq \varphi(n) \qquad (n \geqq n(j))$$

(suitable $n(j)$). Hence in consequence of the above italics

$$f(z) = 0 \qquad (\text{in } H(\varepsilon_j)),$$

where

$$H_0 < H(\varepsilon_j) \lessdot G, \qquad m\, H(\varepsilon_j) \geqq \frac{1}{2}\, m\, G + q - \varepsilon_j.$$

Clearly $f(z)$ will vanish in the set

$$H^* = \Sigma H(\varepsilon_j).$$

We have

$$m H^* \geqq m H(\varepsilon_j) \geqq \frac{1}{2} m G + q - \varepsilon_j \qquad (j = 1, 2, \ldots);$$

whence

$$m H^* \geqq \frac{1}{2} m G + q. \tag{5.64 a}$$

We are now ready to formulate the following.

Theorem 5. 2. *Let* $\{f_n(z)\}$ *be a sequence of analytic functions, whose limit in* G *is* $f(z)$, *as described from* (5. 24) *to* (5. 25 a). *With* g_n *denoting the length of the frontier* Γ_n *of* O_{n+1}, *it is assumed that*

$$g_n^2 \leqq k_0 < 8 \pi m G \tag{5.65}$$

(*necessarily* $g_n^2 \geqq 4 \pi m O_{n+1} > 4 \pi m G$). *For a suitable sequence* $\{n = n_j\}$ *we have* $\lim g_n = g \leqq k_0$. *Under* (5. 65)

$$q^2 = \frac{1}{64 \pi^2} [16 \pi^2 (m G)^2 - (g^2 - 4 \pi m G)^2] > 0.$$

We consider consequences of the vanishing of $f(z)$ *in* H_0, *where* $H_0 < G$ *and*

$$\frac{1}{2} m G - q < m H_0 < \frac{1}{2} m G.$$

If

$$\varepsilon_n = \varphi(n) \tau^{k_n} \qquad (0 < \tau < 1; \text{ cf. } (5.27 \text{ a})), \tag{5.65 a}$$

where $\varphi(n)\ (>0) \to 0$ *with* n^{-1} *and* $\tau = \exp.(-\tau_0)$ (τ_0 *introduced subsequent to* (5. 44 b), (5.44 b′)), *then* $f(z)$ *is zero in a set* H^* *such that*

$$H_0 < H^* < G, \qquad m H^* \geqq \frac{1}{2} m G.$$

If, on the other hand,

$$\varepsilon_n = \varphi(n) \omega^{k_n} \qquad (n \geqq n_0;\ n_0 \text{ suitably great}), \tag{5.65 b}$$

where $\varphi(n)\,(>0) \to 0$ *with* n^{-1} *and*

$$\omega = \omega_\varepsilon = \exp.\left\{-\frac{\lambda^*}{\varepsilon}\right\} \qquad (\text{cf. } (5.62 \text{ a}'), (5.44)),$$

with $0 < \varepsilon < q$, *then* $f(z) = 0$ *in a set* $H(\varepsilon)$ *such that*

$$H_0 < H(\varepsilon) < G, \qquad m H(\varepsilon) \geqq \frac{1}{2} m G + q - \varepsilon.$$

If, finally, one has

(5.65 c) $$\varepsilon_n = \varphi(n)\,\tau(n)^{k_n} \qquad (\varphi(n),\ \tau(n) > 0)$$

where $\varphi(n)$ and $\tau(n)$ tend to zero with n^{-1}, then $f(z)$ will vanish in a set H^* such that

$$H_0 < H^* < G, \qquad m H^* \geqq \frac{1}{2} m G + q.$$

In the above k_n is from (5.27 a).

6. **Uniqueness Properties (Continued).** Consider

(6.1) $$f(z) = \iint\limits_K \frac{d\mu(e_\zeta)}{\zeta - z},$$

where $\mu(e)$ is an additive function of Lebesgue measurable sets, not necessarily absolutely continuous and possibly complex valued; thus

$$\mu(e) = (\mu_1' - \mu_2') + i\,(\mu_1'' - \mu_2'');$$

here $\mu_1', \mu_2', \mu_1'', \mu_2''$ are non negative additive functions of sets. Suppose $\mu_1', \ldots \mu_2''$ are zero in a bounded closed set $G < K$, $m G > 0$; then

(6.1 a) $$f(z) = \iint\limits_{K-G} \frac{d\mu(e_\zeta)}{\zeta - z}.$$

Let $r_0 > 0$; designate by $O\left(\frac{r_0}{n}\right)$ the set of points (in K) at distance less than $\frac{r_0}{n}$ from G. As is possible, we form an open set O_n so that

$$O\left(\frac{r_0}{n+1}\right) < O_n < O\left(\frac{r_0}{n}\right)$$

and so that the frontier $F(O_n)$ of O_n is *regular* and is such that

$$F(O_n) < O\left(\frac{r_0}{n}\right) - \bar{O}\left(\frac{r_0}{n+1}\right).$$

We shall have

$$\bar{O}_n < O\left(\frac{r_0}{n}\right) < O_{n-1}.$$

Moreover, $\lim O_n = G$. The sets G, O_n satisfy the conditions (5.24), (5.24 a). We now note that the function

(6.2) $$f_n(z) = \iint\limits_{H_n} \frac{d\mu(e_\zeta)}{\zeta - z} \qquad (H_n = K - O_n)$$

is analytic in O_n.

On writing

$$g_n = \text{length of } F(O_{n+1}), \tag{6.3}$$

we assume that

$$g_n^2 \leqq k_0 < 8\pi m G. \tag{6.3 a}$$

Whether such an inequality can be secured, for a suitable choice of the O_n, or not rests in intrinsic properties of G[1].

With a view to applying Theorem 5.2 we seek to secure inequalities corresponding to (5.25 a).

For ζ in H_i and z in G one has

$$|\zeta - z| > \frac{r_0}{i+1};$$

now

$$O_n = (H_{n+1} - H_n) + (H_{n+2} - H_{n+1}) + \cdots;$$

since

$$f(z) - f_n(z) = \iint_{K-H_n} \cdots = \iint_{O_n} \cdots = \sum_{i \geqq n} \iint_{H_{i+1}-H_i} \frac{d\mu}{\zeta - z}$$

one has

$$|f(z) - f_n(z)| \leqq \sum_{i \geqq n} \left| \iint_{H_{i+1}-H_i} \frac{d\mu}{\zeta - z} \right| \leqq \sum_{i \geqq n} \left\{ \left| \iint \frac{d\mu_1'}{\zeta - z} \right| \right.$$
$$\left. + \left| \iint \frac{d\mu_2'}{\zeta - z} \right| + \left| \iint \frac{d\mu_1''}{\zeta - z} \right| + \left| \iint \frac{d\mu_2''}{\zeta - z} \right| \right\}$$

and, finally, in view of the inequality for $|\zeta - z|$

$$|f(z) - f_n(z)| \leqq \frac{1}{r_0} \sum_{i \geqq n} (i+2) \{\mu_1'(H_{i+1} - H_i) + \cdots + \mu_2''(H_{i+1} - H_i)\}$$

in G. Hence

$$|f(z) - f_n(z)| \leqq r(n) = \varepsilon_n = \frac{1}{r_0} \sum_{i \geqq n} (i+2)\, \mu^*(H_{i+1} - H_i) \tag{6.4}$$

in G, where $\mu^* = \mu_1' + \mu_2' + \mu_1'' + \mu_2''$.

The rarefication of mass μ (more precisely, of μ^*) in the vicinity of G will be supposed to be such that the series in the second member above converges. We then shall have $f_n \to f$ uniformly in G.

---

[1] It can be shown that such sets G, with $mG > 0$ and having no interior points, exist.

In agreement with the text subsequent to (5.26a) k_n is a number for which the inequality

$$\int_{B_n(t)} \frac{\partial u_n}{\partial n} ds \leqq k_n \tag{6.5}$$

holds when t is such that $S_{1,n} < m\,\Delta_n(t) < S_{2,n}$; here $u_n(z) = \log|f_n(z)|$ and $B_n(t)$ is the part of the frontier of $\Delta_n(t)$ having no points in common with $F(O_{n+1})$; in this connection $\Delta_n(t)$ is the part of $\bar{O}_{n+1}$, where $u_n(z) < t$.

For ζ in H_n and z in $\bar{O}_{n+1}$ the inequality

$$|\zeta - z| \geqq \lambda_n > 0$$

will hold, where λ_n is distance between the frontiers of O_{n+1} and O_n. The distance between the frontiers of $O\left(\frac{r_0}{n}\right)$, $O\left(\frac{r_0}{n+1}\right)$ is

$$\frac{r_0}{n(n+1)}.$$

By choosing O_n *with its regular frontier suitably near to the frontier of* $O\left(\frac{r_0}{n}\right)$ (doing this for $n = 1, 2, \ldots$) *we arrange to have*

$$\lambda_n \geqq \frac{r_1}{n(n+1)} \qquad (0 < r_1 < r_0),$$

where r_1 is as near to r_0 as desired. Accordingly

$$|\zeta - z| \geqq \frac{r_1}{n(n+1)} \geqq \frac{r'}{n^2} \qquad (\zeta \text{ in } H_n,\ z \text{ in } \bar{O}_{n+1};\ 0 < r' < r_1). \tag{6.6}$$

With α denoting the angle made by the normal direction (bound outward with respect to $\Delta_n(t)$), one has

$$\frac{\partial u_n}{\partial n} = \frac{\partial u_n}{\partial x}\cos\alpha + \frac{\partial u_n}{\partial y}\sin\alpha = \frac{1}{|f_n|}\left\{\frac{\partial}{\partial x}|f_n|\cos\alpha + \frac{\partial}{\partial y}|f_n|\sin\alpha\right\}$$

so that

$$0 \leqq \frac{\partial u_n}{\partial n} \leqq \frac{1}{|f_n|}\left\{\left|\frac{\partial}{\partial x}|f_n|\right| + \left|\frac{\partial}{\partial y}|f_n|\right|\right\}.$$

On writing $\mu' = \mu_1' - \mu_2'$, $\mu'' = \mu_1'' - \mu_2''$ and

$$f_n = \alpha_n + i\beta_n, \quad \zeta = \zeta' + i\zeta''$$

we obtain

$$\alpha_n = \iint_{H_n} \frac{(\zeta' - x)\, d\mu' + (\zeta'' - y)\, d\mu''}{|\zeta - z|^2}, \qquad \beta_n = \iint_{H_n} \frac{-(\zeta'' - y)\, d\mu' + (\zeta' - x)\, d\mu''}{|\zeta - z|^2}.$$

Whence

$$\frac{\partial}{\partial x}|f_n| = \frac{1}{|f_n|}\left(\alpha_n \frac{\partial \alpha_n}{\partial x} + \beta_n \frac{\partial \beta_n}{\partial x}\right), \quad \frac{\partial}{\partial y}|f_n| = \frac{1}{|f_n|}\left(\alpha_n \frac{\partial \alpha_n}{\partial y} + \beta_n \frac{\partial \beta_n}{\partial y}\right),$$

where

$$(6.7) \qquad \frac{\partial \alpha_n}{\partial x} = \iint_{H_n} \frac{(\zeta' - x)^2 - (\zeta'' - y)^2}{|\zeta - z|^4} d\mu' + \iint_{H_n} \frac{2(\zeta' - x)(\zeta'' - y)}{|\zeta - z|^4} d\mu'',$$

$$\frac{\partial \beta_n}{\partial x} = -\iint_{H_n} \frac{2(\zeta' - x)(\zeta'' - y)}{|\zeta - z|^4} d\mu' + \iint_{H_n} \frac{(\zeta' - x)^2 - (\zeta'' - y)^2}{|\zeta - z|^4} d\mu'';$$

there are similar formulas for the derivatives with respect to y. Accordingly, in view of (6.6) and (6.2)

$$|\alpha_n|,\ |\beta_n| \leqq |f_n| \leqq \frac{n^2}{r'} \mu^*(H_n) \leqq l_0 n^2 \qquad \left(l_0 = \frac{1}{r'} \mu^*(K)\right)$$

and, in consequence of (6.7) and since $|\zeta' - x|,\ |\zeta'' - y| \leqq |\zeta - z|$

$$\left|\frac{\partial \alpha_n}{\partial x}\right|,\ \left|\frac{\partial \beta_n}{\partial x}\right|,\ \left|\frac{\partial \alpha_n}{\partial y}\right|,\ \left|\frac{\partial \beta_n}{\partial y}\right| \leqq l_1 n^4 \qquad (l_1 > 0)$$

for z in $\bar{O}_{n+1}$; here l_0, l_1 are independent of n. Whence

$$\left|\frac{\partial}{\partial x}|f_n|\right|,\ \left|\frac{\partial}{\partial y}|f_n|\right| \leqq \frac{1}{|f_n|} l_0 l_1 n^6$$

and

$$(6.8) \qquad \frac{\partial u_n}{\partial n} \leqq \frac{1}{|f_n^2|} 2 l_0 l_1 n^6 = e^{-2t} 2 l_0 l_1 n^6 \qquad (\text{on } B_n(t)).$$

Hence, in view of (6.5), *one may take*

$$(6.9) \qquad k_n \geqq 2 l_0 l_1 n^6 e^{-2t} b_n(t)$$

(t *such that* $S_{1,n} < m\, \Delta_n(t)$), *where* $b_n(t)$ *is length of* $B_n(t)$. In any case we arrange to have

$$k_1 \leqq k_2 \leqq \ldots .$$

With the aid of (6.4) and of Theorem 5.2 the following result may be now deduced.

Theorem 6.1. *Let*

$$f(z) = \iint\limits_K \frac{d\mu(e_\zeta)}{\zeta - z}$$

be a function as described in connection with (6.1), (6.1 a). *We note the notation leading up to the hypothesis* (6.3 a) *and consider consequences of the relation*

$$f(z) = 0 \qquad \left(in\ H_0;\ \frac{1}{2} m\, G - q < m\, H_0 < \frac{1}{2} m\, G\right).$$

If

$$\mu^*(H_{i+1} - H_i) \leqq \frac{s_i}{i} \tau^{k_i} \qquad (s_i > 0;\ 0 < \tau < 1), \tag{6.10}$$

where $s_1 + s_2 + \ldots$ *converges, and when the constant* τ *satisfies the conditions of Theorem* 5.2 (*as relating to the present case*), *then*

$$f(z) = 0 \qquad \left(in\ H^*;\ H_0 < H^* < G;\ m\, H^* \geqq \frac{1}{2} m\, G\right).$$

If (6.10) *holds with* τ *replaced by* ω_ε (ω_ε *formed in agreement with Theorem* 5.2) *then*

$$f(z) = 0 \qquad \left(in\ H(\varepsilon);\ H_0 < H(\varepsilon) < G;\ m\, H(\varepsilon) \geqq \frac{1}{2} m\, G + q - \varepsilon\right).$$

On the other hand, if

$$\mu^*(H_{i+1} - H_i) \leqq \frac{s_i}{i} \tau(i)^{k_i} \qquad (0 < \tau(i) < 1) \tag{6.10 a}$$

where $\tau(i) \to 0$ *monotonically with* i^{-1}, *then*

$$f(z) = 0 \qquad \left(in\ H^*;\ H_0 < H^* < G;\ m\, H^* \geqq \frac{1}{2} m\, G + q\right).$$

We turn now back to Theorem 5.2. It is recalled that $f_n(z)$ is analytic in $O_n > \bar{O}_{n+1}$. Let $W_n > O_n$, be the set of all points in the complex plane at which $f_n(z)$ is analytic (uniform).

We designate by $\Delta_n^*(t)$ the part of W_n in which $u_n(z) < t$. Clearly

$$\Delta_n^*(t)\, \bar{O}_{n+1} = \Delta_n(t). \tag{6.11}$$

Suppose the frontier Γ_n^* *of* $\Delta_n^*(t)$ *has no points in common with the frontier of* W_n *for values* t *for which*

$$S_{1,n} < m\, \Delta_n < S_{2,n}.$$

The part Γ_n^* in $\bar{O}_{n+1}$ is identical with $B_n(t)$. With normal derivation outwards with respect to $\Delta_n^*(t)$, one has

$$\frac{\partial u_n}{\partial n} \geqq 0 \qquad \text{(on } \Gamma_n^*\text{)}.$$

Since

$$\int\limits_{\Gamma_n^*(t)} \frac{\partial u_n}{\partial n} d s = 2 \pi z_n,$$

where z_n *is the number of zeros of* $f_n(z)$ *in* $\Delta_n^*(t)$ (for values t as specified above) we shall have

$$\int\limits_{B_n(t)} \frac{\partial u_n}{\partial n} d s \leqq \int\limits_{\Gamma_n^*(t)} \frac{\partial u_n}{\partial n} d s = 2 \pi z_n.$$

Hence *one may replace* k_n *by* $2 \pi z_n$, *whenever the italicized statement subsequent to* (6.11) *holds.*

When $f_n(z)$ is rational of degree $\leqq n$[1] with poles not in G, sets O_ν, as described in section 5, can be always constructed. The set W_n will consist of the whole plane excepting the poles of $f_n(z)$; the totality of the latter points will constitute the frontier of W_n. Since $u_n(z) = \log |f_n(z)|$ is positively infinite in the vicinity of the poles, every set $\Delta_n^*(t)$ and its closeure $\bar{\Delta}_n^*(t)\,(t < +\infty)$ contains no points of the frontier of W_n. Hence, in the case under consideration, the italicized statement subsequent to (6.11) will certainly hold. Thus, *Theorem* 5.2 *may be restated, when the* $f_n(z)$ *are rational functions of degree* $\leqq n$, *with*

$$k_n = 2 \pi n. \tag{6.12}$$

Consider the number τ_0 involved in Theorem 5.2. We have, in accordance with the italics subsequent to (5.44b′),

$$\tau_0 > \frac{\lambda_1}{q^4}, \qquad \lambda_1 = \left(\frac{1}{2}\, m\, O_{n_0+1} - m\, H_0\right) \lambda_0,$$

where (cf. (5.44))

$$\lambda_0 \geqq c_n^2.$$

Since $m\, O_n \to m\, G$ one may take

$$\lambda_1 = \frac{1}{2} (m\, G)\, \lambda_0.$$

---

[1] I. e., f_n is ratio of polynomials of degree $\leqq n$.

By (5.65)

$$g_n^2,\ g^2 \leqq k_0 < 8\pi m G;$$

hence, in view of (5.27'), we have

$$c_n \leqq 4\sqrt{2}\,\pi^{\frac{3}{2}}(m\,G)^{\frac{3}{2}} \qquad (n \geqq n_0;\ n_0 \text{ suitably great})$$

so that it would suffice to take

$$\lambda_1 = 16\,\pi^3 (m\,G)^4.$$

Whence the condition of the Theorem will be satisfied if

(6.13) $$\tau_0 > \tau', \qquad \tau' = \frac{1}{q^4}\,16\,\pi^3(m\,G)^4.$$

In view of the above we may take

$$\lambda_0 = 32\,\pi^3(m\,G)^3.$$

On taking note of (5.62a') it is possible to choose λ^* as any number such that

(6.13a) $$\lambda^* > \lambda', \qquad \lambda' = \frac{1}{q^2}\,32\,\pi^3(m\,G)^3.$$

We shall introduce the Definition.

Definition 6.1. *A function $f(z)$ will be said to belong to the class $C(G;\gamma)$ $(0 < \gamma < 1)$, provided $f(z)$ is limit of a sequence of functions $f_n(z)$, rational of degree $\leqq n$,*

(6.14) $$|f(z) - f_n(z)| < \varphi(n)\,\gamma^{2\pi n} \qquad (in\ G),$$

the poles of $f_n(z)$ lying exterior the closed bounded set G $(m\,G > 0)$ and $\varphi(n)$ (> 0) being any function[1] *tending to zero with n^{-1}.*

A class $C(G;\gamma)$ will be said to be regular if open sets O_n with regular frontiers can be formed, so that $f_n(z)$ has no poles in O_n and so that $O_\nu > \bar{O}_{\nu+1}$, $O_\nu > G$, $\lim O_\nu = G$, while

$$g_n^2 \leqq k_0 < 8\,\pi\,m G,$$

where g_n is length of the frontier of O_{n+1}[2]; the above being true for all sequences $\{f_n(z)\}$ associated with the class.

Suppose $f = \lim f_n$, $h = \lim h_n$ belong to a regular class $C(G;\gamma)$. Then for the function

[1] The function $\varphi(n)$ is allowed to be different for different members of the class.

[2] The property of regularity of $C(G;\gamma)$ is essentially a property of G.

$$(6.14') \qquad \psi = f - h = \lim \psi_{2n}, \qquad \psi_{2n} = f_n - h_n$$

we shall have

$$|\psi - \psi_{2n}| \leqq \varphi_1(n)\gamma^{2\pi n} + \varphi_2(n)\gamma^{2\pi n} \qquad \text{(in } G\text{)},$$

where $\varphi_1(n)$, $\varphi_2(n)$ are certain positive functions tending to zero with n^{-1}. The ψ_{2n} being rational of degree $\leqq 2n$, in view of

$$(6.15) \qquad |\psi - \psi_{2n}| \leqq \varphi(n)(\gamma^{\frac{1}{2}})^{2\pi 2n} \qquad (\lim \varphi(n) = 0; \text{ in } G),$$

it is inferred that ψ belongs to a regular class $C(G;\gamma^{\frac{1}{2}})$.

If $f = h$ in a set H_0 as described in Theorem 5.2, that is, if

$$\psi = 0 \qquad \text{(in } H_0\text{)},$$

then (6.15) will imply $\psi = 0$ in H^* (H^* as introduced preceding (5.65 b)), provided

$$\gamma^{\frac{1}{2}} = e^{-\tau_0}, \qquad \tau_0 > \tau' \qquad \text{(cf. (6.13))};$$

that is, *provided γ is less than* exp. $(-2\tau')$. We shall have $\psi = 0$ in a set $H(\varepsilon)$ (as in the Theorem), in the case when

$$\gamma = \exp.\left\{-\frac{2\lambda^*}{\varepsilon}\right\}, \qquad \lambda^* > \lambda' \qquad \text{(cf. (6.13 a))};$$

the latter condition holds when

$$\gamma < \exp.\left\{-\frac{2\lambda'}{\varepsilon}\right\}.$$

It will be said that $f(z) = \lim f_n(z)$, where $f_n(z)$ is rational of degree $\leqq n$, belongs to the class $C_0(G)$, if

$$(6.16) \qquad |f - f_n| \leqq \varepsilon_n = \varphi(n)\tau(n)^{2\pi n} \qquad (in\ G),$$

where $\varphi(n)(> 0)$, $\tau(n)(> 0) \to 0$ with n^{-1}.

In this connection $\varphi(n)$, $\tau(n)$ may be different for different members of the class $C_0(G)$[1].

Suppose $f = \lim f_n$, $h = \lim h_n < C_0(G)$; thus

$$|f - f_n| \leqq \varphi_1(n)(\tau_1(n))^{2\pi n}, \qquad |h - h_n| \leqq \varphi_2(n)(\tau_2(n))^{2\pi n} \qquad \text{(in } G\text{)},$$

where $\varphi_1(n)$, $\varphi_2(n)$, $\tau_1(n)$, $\tau_2(n) \to 0$ with n^{-1}. For the functions ψ, ψ_{2n} of (6.14') we shall now have

[1] One may, of course, write n in place of $2\pi n$ in (6.16); the notation in use is in agreement with (5.65 c).

$$|\psi - \psi_{2n}| \leqq \varphi_1(n)(\tau_1(n))^{2\pi n} + \varphi_2(n)(\tau_2(n))^{2\pi n}.$$

Let $\varphi(n), \tau^2(n)$ be such that

$$\varphi_1(n),\ \varphi_2(n) \leqq \varphi(n); \qquad \tau_1(n),\ \tau_2(n) \leqq \tau^2(n),$$

while $\varphi(n),\ \tau(n) \to 0$ with n^{-1}. One then may assert that

$$|\psi - \psi_{2n}| \leqq 2\varphi(n)(\tau^2(n))^{2\pi n} = 2\varphi(n)(\tau(n))^{2\pi 2n} \tag{6.17}$$

(in G). Thus vanishing of ψ in H_0 (H_0 as in Theorem 5.2) will imply (in consequence of (6.17)) that ψ will vanish in H^*, that is

$$f = h \qquad \text{(in } H^*\text{)},$$

where H^* is the set referred to at the end of Theorem 5.2.

We are now ready to formulate the following.

Theorem 6.2. *If $f, h <$ regular $C(G;\gamma)$ (Definition 6.1), then the relation*

$$f = h \qquad (in\ H_0;\ H_0 < G), \tag{6.18}$$

where

$$\frac{1}{2} m\, G - q < m\, H_0 < \frac{1}{2} m\, G, \tag{6.18a}$$

will have the following consequences.

If

$$0 < \gamma < e^{-2\tau'}, \qquad \tau' = \frac{1}{q^4} 16\pi^3 (m\, G)^4,$$

then $f = h$ in a set H^* such that

$$H_0 < H^* < G, \qquad m\, H^* \geqq \frac{1}{2} m\, G.$$

If

$$0 < \gamma < \exp.\left\{-\frac{2\lambda'}{\varepsilon}\right\} \qquad \left[0 < \varepsilon < q;\ \lambda' = \frac{1}{q^2} 32\pi^3 (m\, G)^2\right],$$

then $f = h$ in a set $H(\varepsilon)$ such that

$$H_0 < H(\varepsilon) < G, \qquad m\, H^* \geqq \frac{1}{2} m\, G + q - \varepsilon.$$

Finally, if $f, h <$ regular $C_0(G)$ (cf. italics in connection with (6.16)) then (6.18), (6.18a) will imply that $f = h$ in a set H^* such that

$$H_0 < H^* < G, \qquad m\,H^* \geqq \frac{1}{2} m\,G + q.$$

The above Theorem gives conditions under which certain classes of functions, consisting of limits of rational functions, are quasi-analytic in the sense of unique determination of the members of the class by their values on sets of positive measure H_0. We note that $m\,H_0$ is required to be suitably great before the Theorem can be applied; moreover, unique determination is secured not necessarily almost everywhere in G.

7. Unique Determination by Values on an Arc and on Denumerable Sets.

Uniqueness properties, related to arcs Γ for functions of the general form (6. 1) have been studied with considerable detail in (T_1), (T_2). The results therein obtained are of a rather complicated character. Accordingly, it appears to be of interest to obtain simpler results for the important specialized case of functions, which are limits of rational functions. For this purpose one may apply with good effect a suitable adaptation of the very elegant method of J. Wolff[1], utilized by that write in the study of functions of the form

$$\sum_k \frac{A_k}{z - \alpha_k}.$$

Direct application of this method to functions (6. 1) does not appear to be convenient.

We now consider functions $f(z)$, *such that*

$$(7.1) \qquad |f(z) - f_n(z)| \leqq \varepsilon_n \qquad (in\ G;\ \lim \varepsilon_n = 0),$$

where

$$(7.1\,a) \qquad f_n(z) = \sum_{k=1}^{n} \frac{A_{n,k}}{z - \alpha_{n,k}}.$$

Here the $\alpha_{n,k}$ $(k = 1, \ldots n)$, *all finite and distinct, are outside the closed bounded set* G; *moreover,*

$$(7.1\,a') \qquad |\alpha_{n,k}| \leqq R;$$

the latter is not a very essential restriction.

[1] J. Wolff, *Généralisation d'un théorème de M. Carleman sur les séries de fractions rationnelles*, Comptes Rendus, t. 202 (1936), pp. 551—553, in the sequel referred to as (W).

It is recalled that, in accordance with (T_2; § 5), certain wide classes of functions are representable as described above.

Let Γ be a simple continuous rectifiable arc in G. As in (W), we introduce a conformal transformation

$$\zeta = \zeta(z) = \zeta_1 (z - z_0) + \zeta_2 (z - z_0)^2 + \cdots \qquad (\zeta_1 \neq 0) \tag{7.2}$$

(the series here converges in a vicinity of z_0), which maps the region consisting of the z-plane, with Γ deleted, on the circular domain

$$S \qquad (|\zeta| < 1);$$

z_0 is to denote a point of G not on Γ. We have

$$z = z(\zeta) = z_0 + z_1 \zeta + z_2 \zeta^2 + \cdots \qquad \left(z_1 = \frac{1}{\zeta_1}\right) \tag{7.2 a}$$

convergent for $|\zeta| < 1$. On writing

$$f_n(z) = F_n(\zeta), \tag{7.3}$$

it is observed that $F_n(\zeta)$ is analytic in S, except for simple poles at

$$\beta_{n,k} = \zeta(\alpha_{n,k}), \quad |\beta_{n,k}| < 1 \qquad (k = 1, \ldots n). \tag{7.3'}$$

As in (W), application of Jensen's formula will yield

$$\log |F_n(0)| = \log |f_n(z_0)| \leqq \frac{1}{2\pi} \int_0^{2\pi} \log |F_n(e^{i\theta})| \, d\theta + \sum_1^n \log \frac{1}{|\beta_{n,k}|}. \tag{7.4}$$

We wish to examine consequences of the relation

$$f(z) = 0 \qquad (\text{on } \Gamma). \tag{7.5}$$

In view of (7.1) the above will imply

$$|f_n(z)| = |F_n(e^{i\theta})| \leqq \varepsilon_n \qquad (z \text{ on } \Gamma).$$

Hence (7.4) yields

$$\log |f_n(z_0)| \leqq \log \varepsilon_n + k_n, \tag{7.6}$$

where k_n is any number such that

$$\sum_1^n \log \frac{1}{|\beta_{n,k}|} \leqq k_n. \tag{7.6 a}$$

We shall obtain $f(z_0) = 0$ whenever

$$\underline{\lim} \, (\log \varepsilon_n + k_n) = -\infty;$$

that is, when

$$(7.7)\qquad \underline{\lim}\, e^{k_n} \varepsilon_n = 0.$$

In consequence of (7.2) and (7.3′)

$$\left|\frac{1}{\beta_{n,k}}\right| \leqq \frac{A_1(z_0)}{|\alpha_{n,k} - z_0|} \qquad (A_1(z_0) > 0)$$

and

$$(7.8)\qquad \sum_k \log\left|\frac{1}{\beta_{n,k}}\right| \leqq n\,A(z_0) + \sum_k \log \frac{1}{|\alpha_{n,k} - z_0|},$$

where $A(z_0)$ is independent of n.

Let $\psi(n)(> 0)$ *be a function such that* $\lim \psi(n) = +\infty$ *(as* $n \to \infty$). An infinite sequence $n = n_j$ $(n_1 < n_2 < \cdots)$ can be found such that

$$(7.9)\qquad \sum \frac{1}{\psi(n_j)} \leqq \varepsilon_0 \qquad (\varepsilon_0 > 0),$$

where ε_0 may be assigned beforehand as small as desired. Designate by $S(c, r)$ a circular domain $|z - c| < r$. *Suppose* G *is such that a portion*

$$G\,S\left(c, \frac{1}{4}\right)$$

of G, *having a positive planar measure, exists.* We designate by

$$\sum_{k=1}^{n}{}', \quad \sum_{k=1}^{n}{}''$$

summations extended over those $\alpha_{n,k}$ which lie in $S\left(c, \frac{1}{2}\right)$ and outside of $S\left(c, \frac{1}{2}\right)$, respectively. When a point $\alpha_{n,k}$ is in $S\left(c, \frac{1}{2}\right)$ one has

$$(7.9')\qquad 0 < \iint_{S(c, \frac{1}{2})} \log \frac{1}{|z - \alpha_{n,k}|}\, dx\, dy < \int_0^{2\pi}\int_0^1 \log \frac{1}{r}\, r\, dr\, d\theta = M;$$

hence

$$\iint_{S(c,\frac{1}{2})} \sum_{k=1}^{n}{}' \log \frac{1}{|z - \alpha_{n,k}|}\, dx\, dy < n\,M.$$

Let E_j be the part of $S\left(c, \frac{1}{2}\right)$ in which

$$\sum_{k=1}^{n}{}' \log \frac{1}{|z - \alpha_{n,k}|} > n_j M \psi(n_j).$$

Necessarily

$$m E_j \leqq \frac{1}{\psi(n_j)}.$$

In fact, if the contrary were the case one would have

$$\iint\limits_{S(c,\frac{1}{2})} \ldots dx\,dy \geqq \iint\limits_{E_j} \sum_{k=1}^{n}{}' \log \frac{1}{|z - \alpha_{n,k}|} dx\,dy \geqq n_j M \psi(n_j)\, m E_j > n_j M;$$

this is in disagreement with a previous inequality. Now

$$\sum_{k=1}^{n_j}{}' \log \frac{1}{|z - \alpha_{n,k}|} \leqq n_j M \psi(n_j) \qquad \left(\text{in } H_j = S\left(c, \frac{1}{2}\right) - E_j\right),$$

where

$$m H_j \geqq \frac{\pi}{4} - \frac{1}{\psi(n_j)}.$$

Form the set

$$E^* = \Pi H_j = S\left(c, \frac{1}{2}\right) - \Sigma E_j.$$

Clearly

$$(7.10) \qquad E^* < S\left(c, \frac{1}{2}\right), \quad m E^* \geqq \frac{\pi}{4} - \Sigma m E_j \geqq \frac{\pi}{4} - \Sigma \frac{1}{\psi(n_j)} \geqq \frac{\pi}{4} - \varepsilon_0$$

(cf. (7.9)) and

$$(7.10\text{a}) \qquad \sum_{k=1}^{n_j}{}' \log \frac{1}{|z - \alpha_{n,k}|} \leqq n_j M \psi(n_j) \qquad (\text{in } E^*).$$

In agreement with the statement relating to (7.9) we choose $\{n_j\}$ so that

$$(7.11) \qquad (0 <)\, \varepsilon_0 < m\, G\, S\left(c, \frac{1}{4}\right).$$

On writing

$$H^* = E^* S\left(c, \frac{1}{4}\right),$$

and letting C denote complements of sets with respect to $S\left(c, \frac{1}{2}\right)$ we obtain

$$H^* = S\left(c, \frac{1}{2}\right) - C E^* - C S\left(c, \frac{1}{4}\right).$$

By (7. 10), (7. 11)

$$m\,CE^* = \frac{\pi}{4} - m\,E^* \leqq \varepsilon_0 < m\,G\,S\left(c, \frac{1}{4}\right);$$

moreover,

$$(7.11') \qquad m H^* \geqq \frac{\pi}{4} - m\,CE^* - m\,C\,S\left(c, \frac{1}{4}\right) \geqq \frac{\pi}{4} - \varepsilon_0 - \left(\frac{\pi}{4} - \frac{\pi}{16}\right) = \frac{\pi}{16} - \varepsilon_0;$$

thus

$$m\,H^* > \frac{\pi}{16} - m\,G\,S\left(c, \frac{1}{4}\right);$$

in fact, by suitable choice of (n_j) we may get $m\,H^*$ as near as desired to $\frac{\pi}{16}$. Let C now denote complements of sets with respect to $S\left(c, \frac{1}{4}\right)$. Since $H^* < S\left(c, \frac{1}{4}\right)$ one has

$$G\,H^* = G\,S\left(c, \frac{1}{4}\right) H^* = S\left(c, \frac{1}{4}\right) - C\left(G\,S\left(c, \frac{1}{4}\right)\right) - C\,H^*$$

and

$$m\,(G\,H^*) \geqq \frac{\pi}{16} - m\,C\left(G\,S\left(c, \frac{1}{4}\right)\right) - m\,C\,H^*$$

$$= \frac{\pi}{16} - \left[\frac{\pi}{16} - m\left(G\,S\left(c, \frac{1}{4}\right)\right)\right] - \left[\frac{\pi}{16} - m\,H^*\right] = m\left(G\,S\left(c, \frac{1}{4}\right)\right) + m\,H^* - \frac{\pi}{16}.$$

Hence, in view of (7. 11′),

$$(7.12) \qquad m\,(G\,H^*) \geqq m\,G\,S\left(c, \frac{1}{4}\right) - \varepsilon_0, \quad G\,H^* < S\left(c, \frac{1}{4}\right), \quad G\,H^* < E^*;$$

in $G\,H^*$ (7. 10 a) will hold for $j = 1, 2, \ldots$.

For z in $G\,H^*$ and $\alpha_{n,k}$ outside $S\left(c, \frac{1}{2}\right)$ we have $|z - \alpha_{n,k}| > \frac{1}{4}$; thus

$$\log \frac{1}{|z - \alpha_{n,k}|} < \log 4$$

and

$$\sum_{k=1}^{n_j}{}'' \log \frac{1}{|z - \alpha_{n,k}|} < n_j \log 4.$$

Hence in consequence of (7. 10 a) (valid in $G\,H^*$) *it is deduced that*

(7.13) $$\sum_{k=1} \log \frac{1}{|z-\alpha_{n_j,k}|} < n_j \log 4 + n_j M \psi(n_j) \qquad (j = 1, 2, \ldots)$$

for z in GH^* (cf. (7.12)).

In consequence of the above and of (7.8) the number k_{n_j} introduced in (7.6a) may be taken as

(7.14) $$k_{n_j} = n_j A(z_0) + n_j \log 4 + n_j M \psi(n_j),$$

when z_0 is a point in GH^*. Accordingly

$$e^{k_n} \varepsilon_n \leqq \varepsilon_n e^{\mu n \psi(n)} \qquad (n = n_j \geqq n(z_0)),$$

where μ is any number exceeding M and $n(z_0)$ is suitably great. Hence (7.7) will hold and, thus, one will have

(7.15) $$f(z) = 0 \qquad (\text{in } GH^*),$$

if

(7.15a) $$\varepsilon_n = \varphi(n) e^{-\mu n \psi(n)} \qquad (\varphi(n) > 0;\ \lim \varphi(n) = 0).$$

Here μ is independent of z_0. Inasmuch as $\psi(n) (> 0)$, with $\lim \psi(n) = +\infty$, is arbitrary, *μ may be absorbed* in $\psi(n)$. This may have effect on the choice of (n_j) necessary for the validity of (7.12) — a circumstance of no importance in the final statement made in connection with (7.15), (7.15a).

In (7.12) $\varepsilon_0 (> 0)$ is arbitrarily small. We take a sequence

$$\varepsilon_{0,m} > 0, \quad \lim \varepsilon_{0,m} = 0.$$

Correspondingly, in consequence of (7.15a) it is inferred that $f(z) = 0$ in a set GH_m^* such that

$$m(GH_m^*) \geqq m\, G\, S\left(c, \frac{1}{4}\right) - \varepsilon_{0,m}, \quad GH_m^* < S\left(c, \frac{1}{4}\right),$$

whence one will have $f(z) = 0$ in

$$G_c = \sum_m GH_m^*, \quad G_c < G\, S\left(c, \frac{1}{4}\right);$$

here

$$m\, G\, S\left(c, \frac{1}{4}\right) \geqq m\, G_c \geqq m\, G\, H_m^* \geqq m\, G\, S\left(c, \frac{1}{4}\right) - \varepsilon_{0,m}$$

$(m = 1, 2, \ldots)$. Hence

$$m\, G_c = m\, G\, S\left(c, \frac{1}{4}\right).$$

that is, (7.15 a) implies that $f(z)=0$ almost everywhere in $G\,S\left(c, \frac{1}{4}\right)$. The set G being bounded, there exists a finite number of points c_j so that

$$G < \sum_{j=1}^{q} S\left(c_j, \frac{1}{4}\right), \quad G = \sum_{j} G\,S\left(c_j, \frac{1}{4}\right).$$

For a fixed j the set $G\,S\left(c_j, \frac{1}{4}\right)$ will have positive planar measure, when one may assert that (7.15 a) holds almost everywhere in this set; the alternative is

$$m\,G\,S\left(c_j, \frac{1}{4}\right) = 0.$$

Theorem 7.1. *Consider functions $f(z)$ of the type described in connection with (7.1), (7.1 a), G being closed bounded. If*

$$\varepsilon_n = \varphi(n)\, e^{-n\psi(n)} \qquad (n = 1, 2, \ldots)$$

where $\varphi(n) > 0$, $\psi(n) > 0$ and

$$\varphi(n), \quad \frac{1}{\psi(n)} \to 0 \qquad (as\ n \to \infty),$$

then vanishing of $f(z)$ on a simple continuous rectifiable arc $\Gamma < G$ will imply vanishing of $f(z)$ everywhere in G.

Vanishing of $f(z)$ almost everywhere in G follows by the preceding developments. Now, $f(z)$ is continuous. Hence the part of G in which $f=0$ is closed. The set $G_0 < G$ in which $f \neq 0$ has zero measure and must be open with respect to G; G_0 can have no points interior with respect to G since if it had one would have $m\,G_0 > 0$. Whence G_0 is a null set. This completes the proof of the Theorem.

If we denote by $C(G)$ the class of functions $f(z)$ which are of the type considered in the above Theorem, it is noted that the class is additive. In fact, suppose

$$|f - f_n| < \varphi_1(n)\, e^{-n\psi_1(n)}, \quad |g - g_n| < \varphi_2(n)\, e^{-n\psi_2(n)}$$

in G, where f_n, g_n are of the form (7.1 a) (poles not in G), while $\varphi_1, \varphi_2, \psi_1, \psi_2$ are positive and

$$\varphi_1(n), \varphi_2(n), \frac{1}{\psi_1(n)}, \frac{1}{\psi_2(n)} \to 0 \qquad (\text{as } n \to \infty).$$

Consider the function

$$q = c_1 f + c_2 g \qquad (c_1, c_2 \text{ constants}).$$

On writing

$$q_{2n} = c_1 f_n + c_2 g_n$$

it is observed that

$$|q - q_{2n}| = |c_1 (f - f_n) + c_2 (g - g_n)| < |c_1| \varphi_1(n) e^{-n\psi_1(n)} + |c_2| \varphi_2(n) e^{-n\psi_2(n)}$$

(in G). We introduce positive functions $\varphi_0(n)$, $\psi_0(n)$ such that

$$\varphi_1(n),\ \varphi_2(n) \leqq \varphi_0(2n), \quad \psi_0(2n) \leqq \psi_1(n),\ \psi_2(n).$$
$$\lim \varphi_0(2n) = 0, \qquad \lim \psi_0(2n) = +\infty.$$

It is observed that

$$|q - q_{2n}| < \varphi(2n) e^{-2n\psi(2n)} \qquad \text{(in } G),$$

where

$$\varphi(2n) = (|c_1| + |c_2|)\varphi_0(2n), \quad \psi(2n) = \frac{1}{2}\psi_0(2n);$$

whence $q < C(G)$.

In consequence of the Theorem *the class* $C(G)$ *is 'quasi analytic'* in the sense that the members of the class are determined uniquely in G by the functional values on any continuous rectifiable arc $\Gamma < G$.

We continue with functions $f(z)$ as described in connection with (7.1), (7.1 a). *Let* $z_\infty, z_1, z_2, \ldots$ *be a denumerably infinite set of points on the arc* Γ, *with* z_∞ *as the sole limiting point;*

$$\lim_{\nu} z_\nu = z_\infty. \tag{7.16}$$

Conditions will be found under which vanishing of $f(z)$ on $\{z_n\}$ will imply vanishing of $f(z)$ on a more extensive set. *Let* ν_m *be the smallest integer such that*

$$|z_\infty - z_\nu| < \frac{1}{m} \qquad (\text{all } \nu \geqq \nu_m) \tag{7.16 a}$$

and designate by Γ_m the portion of Γ consisting of points ζ such that $|z_\infty - z| \leqq \leqq \frac{1}{m}$. If $z\,(\neq z_n)$ is any point on Γ_m, a point $z_{\nu'}$ can be found so that $\nu' \geqq \nu_m$, while

$$|z - z_{\nu'}| \leqq |z - z_\nu| \qquad (\text{all } \nu \geqq \nu_m). \tag{7.16 b}$$

The end points of Γ_m and the z_ν $(\nu \geqq \nu_m)$ divide Γ_m into a denumerably infinite sequence of adjacent subarcs δ_m; *we let*

$$\lambda_m = \text{u. b. } l_m \qquad (\text{fixed } m) \tag{7.16 c}$$

where l_m is the distance between the end points of an arc δ_m. In view of (7.16b)

(7.16d) $$|z - z_{\nu'}| \leqq \lambda_m.$$

Clearly $\lim_m \lambda_m = 0$; in fact,

(7.16e) $$\lambda_m \leqq \frac{2}{m}.$$

Since we have assumed that $f = 0$ on $\{z_n\}$ it will follow that

(7.17) $$|f_n(z_\nu)| \leqq \varepsilon_n \qquad (\nu = 1, 2, \ldots$$

and, for z on Γ_m,

$$|f_n(z)| = |f_n(z_{\nu'})| + |f_n(z) - f\;(z_{\nu'})_1| \leqq \varepsilon_n + \mu_n(\lambda_m),$$

where μ_n is a modulus of continuity of $f_n(\zeta)$ on Γ (or in G). In view of (7.1a) one may take

(7.17') $$\mu_n(\lambda) = M_n\lambda, \quad M_n = \frac{1}{\delta_n^2}\sum_{k=1}^{n}|A_{n,k}|$$

where δ_n is distance (necessarily positive) *from the set of points*

$$\alpha_{n,1}, \ \alpha_{n,2}, \ \ldots \ \alpha_{n,n}$$

to G. We have

(7.18) $$|f_n(z)| \leqq \varepsilon_n + M_n\lambda_m \leqq \varepsilon_n + \frac{2}{m}M_n \qquad (\text{on } \Gamma_m).$$

We take m sufficiently great so that Γ_m is a single simple arc. In place of (7.2) introduce a transformation

$$\zeta = \zeta_m(z), \quad \zeta_m(z_0) = 0,$$

z_0 being a fixed point in G not on Γ, mapping the region consisting of the z-plane, with Γ_m deleted, conformally on the circular domain $S(|\zeta| < 1)$. We repeat the developments given subsequent to (7.2a), with $\zeta(z) = \zeta_m(z)$, $\beta_{n,k} = \beta_{n,k,m}$ and

$$f_n = f_n(z_m) = F_{n,m}(\zeta),$$

where

$$z = z_m(\zeta) = z_0 + z_{1,m}\zeta + \zeta_{2,m}\zeta + \cdots \quad \left(z_{1,m} = \frac{1}{\zeta_{1,m}}\right)$$

is the inverse of the mapping function $\zeta_m(\zeta)$, whose analytic element at z_0 may be written as

$$\zeta_m(z) = \zeta_{1,m}(z - z_0) + \zeta_{2,m}(z - z_0) + \cdots \quad (\zeta_{1,m} \neq 0).$$

In view of (7.18)

$$|F_{n,m}(e^{i\theta})| \leqq \varepsilon_n + M_n \lambda_m;$$

we choose $m = m_m$ *so that* $\lim\limits_{n} M_n \lambda_m = 0$; accordingly, by virtue of an analogue to (7.4)

$$\log |f_n(z_0)| \leqq \log(\varepsilon_n + M_n \lambda_m) + k_{n,m}, \tag{7.19}$$

where $k_{n,m}$ is any number such that

$$\sum_1^n \log \frac{1}{|\beta_{n,k,m}|} \leqq k_{n,m}.$$

Inasmuch as $\beta_{n,k,m} = \zeta_m(\alpha_{n,k})$ it is noted that

$$\left|\frac{1}{\beta_{n,k,m}}\right| \leqq \frac{A_m(z_0)}{|\alpha_{n,k} - z_0|} \qquad (A_m(z_0) > 0); \tag{7.20}$$

this inequality corresponds to the one preceding (7.8). Repeating the steps leading to (7.14) we obtain, as a possible choice,

$$k_{n,m} = n A_m(z_0) + n \log 4 + n M \psi(n)$$

for a suitable infinite sequence $n = n_j$. In view of (7.19) it is accordingly inferred that

$$f(z_0) = 0 \qquad (\text{in } GH^*) \tag{7.21}$$

if

$$\theta(n) = \log(\varepsilon_n + M_n \lambda_{m_n}) + n A_{m_n}(z_0) + n \log 4 + n M \psi(n) \to -\infty$$

as $n \to \infty$. Now

$$\theta(n) \leqq \theta^*(n) = \log(\varepsilon_n + M_n \lambda_{m_n}) + n A_{m_n}(z_0) + \mu n \psi(n) \tag{7.21'}$$

$(\mu > M;\ n \geqq n_0)$. Hence (7.21) will hold if $\theta^*(n) \to -\infty$; that is, if

$$\varepsilon_n + M_n \lambda_{m_n} \leqq 2\,\varphi(n)\, e^{-\mu n \psi(n)}\, e^{-n A_{m_n}(z_0)}$$

where $\varphi(n)\,(>0) \to 0$. It will be convenient to choose m_n so that

$$M_n \lambda_{m_n} \leqq \varepsilon_n. \tag{7.22}$$

Absorbing μ in $\psi(n)$ (inasmuch as $\psi(n) \to +\infty$, $\psi(n) > 0$, this is possible)[1] we conclude that, under (7.22), it is sufficient to have

$$\varepsilon_n \leqq \varphi(n)\, e^{-n[\psi(n) + A_{m_n}(z_0)]}. \tag{7.23}$$

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[1] Note the remark subsequent to (7.15 a).

On the basis of (7.23) *a reasoning of the type used subsequent to* (7.15) *will lead to the inference that*

$$f(z) = 0 \qquad (in\ G) \tag{7.23'}$$

in consequence of the relations $f(z_\nu) = 0 \quad (\nu = 1, 2, \ldots)$.

We shall obtain some essential information regarding $A_m(z_0)$ in the important case when Γ is a rectilinear interval. *It will suffice to take for Γ a subinterval of the axis of reals, say* $(-a, a)$ $(a > 0)$, with $z_\infty = 0$. We consider the transformation

$$w = w(z) = \frac{z}{a} - \left[\left(\frac{z}{a}\right)^2 - 1\right]^{\frac{1}{2}} = \frac{1}{a}[z - (z^2 - a^2)^{\frac{1}{2}}]; \tag{7.24}$$

in this connection we introduce a cut along Γ and take the determination of the square root which is positive for real $z > a$. The function $w(z)$ maps the z-plane, with Γ deleted, on $|w| < 1$. The points $z = \infty$ and $w = 0$ will correspond. The function

$$\zeta = \zeta(z) = \frac{w - w_0}{\bar{w}_0 w - 1}, \tag{7.24 a}$$

where $w = w(z)$, $w_0 = w(z_0)$ (z_0 not on Γ), thought of as a function of w, transforms the circular domain $|w| < 1$ onto the circular domain $|\zeta| < 1$, while $\zeta = 0$ for $w = w_0$. If one thinks of ζ as a function of z, it is noted that (7.24 a) transforms the z-plane, with Γ deleted, on the circular domain $|\zeta| < 1$, while the point $z = z_0$ goes into $\zeta = 0$ $(z = \infty \to \zeta = w_0)$. We have

$$\frac{a}{\zeta(z)} = \frac{[\bar{z}_0 - (\bar{z}_0^2 - a^2)^{\frac{1}{2}}][z - (z^2 - a^2)^{\frac{1}{2}}] - a^2}{[z - (z^2 - a^2)^{\frac{1}{2}}] - [z_0 - (z_0^2 - a^2)^{\frac{1}{2}}]}.$$

The denominator in the second member above is equal to

$$z - z_0 - \frac{z^2 - z_0^2}{(z^2 - a^2)^{\frac{1}{2}} + (z_0^2 - a^2)^{\frac{1}{2}}}.$$

Hence

$$\frac{-a(z - z_0)}{\zeta(z)} = \frac{[\bar{z}_0 - (\bar{z}_0^2 - a^2)^{\frac{1}{2}}][z - (z^2 - a^2)^{\frac{1}{2}}] - a^2}{[z_0 - (z_0^2 - a^2)^{\frac{1}{2}}] + [z - (z^2 - a^2)^{\frac{1}{2}}]}[(z_0^2 - a^2)^{\frac{1}{2}} + (z^2 - a^2)^{\frac{1}{2}}].$$

With z, z_0 not on Γ suppose that

$$\frac{a}{|z_0|}, \ \frac{a}{|z|} \leqq \sigma, \tag{7.25}$$

where $\sigma (0 < \sigma < 1)$ is such that

(7.25 a) $$T(\sigma) = \sum_{j=1}^{\infty} | C^{\frac{1}{2}}_{j+1} | (2j + 1) \sigma^{2j} \leqq \frac{1}{4}.$$

Under (7.25)

$$z - (z^2 - a^2)^{\frac{1}{2}} = \frac{a^2}{z} \sum_{j=0}^{\infty} C^{\frac{1}{2}}_{j+1} (-1)^j \left(\frac{a}{z}\right)^{2j}$$

and

$$D = [z_0 - (z_0^2 - a^2)^{\frac{1}{2}}] + [z - (z^2 - a^2)^{\frac{1}{2}}] = \sum_{j=0}^{\infty} C^{\frac{1}{2}}_{j+1} (-1)^j \frac{z^{2j+1} + z_0^{2j+1}}{z_0^{2j+1} z^{2j+1}} a^{2j+2};$$

hence

$$D = D_0 (z + z_0) \frac{a^2}{z_0 z}; \quad D_0 = \sum_{j=0}^{\infty} C^{\frac{1}{2}}_{j+1} (-1)^j \frac{a^{2j}}{z_0^{2j} z^{2j}} (z^{2j} - z^{2j-1} z_0 + \cdots + z_0^{2j});$$

here

$$D_0 = \frac{1}{2} + D_1, \quad | D_1 | = \left| \sum_{j=1}^{\infty} C^{\frac{1}{2}}_{j+1} (-1)^j a^{2j} \left[\frac{1}{z_0^{2j}} - \frac{1}{z_0^{2j-1} z} + \cdots + \frac{1}{z^{2j}} \right] \right| \leqq T(\sigma);$$

accordingly, by (7.25 a) $| D_0 | \geqq \frac{1}{4}$ and

(7.26) $$\left| \frac{1}{D} \right| \leqq \frac{4 | z_0 z |}{| z + z_0 | a^2}.$$

On the other hand,

(7.26 a) $$N^* = (z_0^2 - a^2)^{\frac{1}{2}} + (z^2 - a^2)^{\frac{1}{2}} = z_0 + z + \sum_{1}^{\infty} C^{\frac{1}{2}}_{j} (-1)^j a^{2j} \frac{z^{2j-1} + z_0^{2j-1}}{z_0^{2j-1} z^{2j-1}} = (z_0 + z) Q,$$

where

(7.26 a′) $$| Q | \leqq \left| 1 + \sum_{j=1}^{\infty} C^{\frac{1}{2}}_{j} (-1)^j a^{2j} \left[\frac{1}{z_0^{2j-1} z} - \frac{1}{z_0^{2j-2} z^2} + \cdots + \frac{1}{z_0 z^{2j-1}} \right] \right| \leqq 1 + \sum_{j=1}^{\infty} | C^{\frac{1}{2}}_{j} | (2j - 1) \sigma^{2j} = q_0.$$

Finally, in consequence of the formula subsequent to (7.25 a)

(7.26 b) $$N = [\bar{z}_0 - (\bar{z}_0^2 - a^2)^{\frac{1}{2}}] [z - (z^2 - a^2)^{\frac{1}{2}}] - a^2 = - a^2 + a^4 T_1,$$

where

(7.26 b′) $$| T_1 | \leqq \frac{1}{| z z_0 |} t_1^2, \quad t_1 = \sum_{j=0}^{\infty} | C^{\frac{1}{2}}_{j+1} | \sigma^{2j}.$$

From (7.26)—(7.26b) it is deduced that

$$\frac{a\,|z-z_0|}{|\zeta(z)|}=\left|\frac{NN^*}{D}\right|\leqq 4\,|z_0 z|\,q_0\left[1+a^2\frac{t_1^2}{|z z_0|}\right]\leqq 4\,q_0\,t_0\,|z_0 z|,$$

where

(7.26b'') $$t_0=1+\sigma^2 t_1^2.$$

Replacing Γ by $\Gamma_m\left(-\frac{1}{m}\leqq z\leqq\frac{1}{m}\right)$ and putting

$$a=\frac{1}{m},\quad z=\alpha_{n,k},\quad \zeta(z)=\zeta_m(z),$$

in accordance with previous notation, we obtain

(7.27) $$\frac{1}{|\beta_{n,k,m}|}=\frac{1}{|\zeta_m(\alpha_{n,k})|}\leqq\frac{4\,q_0\,t_0\,|z_0\,\alpha_{n,k}|}{|\alpha_{n,k}-z_0|}\,m\leqq\frac{|z_0|\,q\,m}{|\alpha_{n,k}-z_0|}$$

($q=4\,q_0\,t_0\,R$; cf. (7.1a')), provided conditions corresponding to (7.25) hold, that is, if

(7.27a) $$m=m_n\geqq\frac{1}{\sigma\,|z_0|},\quad m\geqq\frac{1}{\sigma\,|\alpha_{n,k}|}\qquad (k=1,\ldots).$$

By virtue of (7.27) and (7.20) *we may take*

(7.28) $$A_m(z_0)=|z_0|\,q\,m$$

whenever Γ is a rectilinear segment (as assumed previously).

With respect to (7.27a) we note that, inasmuch as we take m_n so that $\lim m_n=+\infty$, the first condition is not essential since it will be satisfied for $n\geqq n_0$ ($n_0=n_0(z_0)$ suitably great); the second condition (7.27a) will hold if

(7.27a') $$m_n\geqq\frac{1}{\sigma\,\delta_n}$$

(see text subsequent to (7.17)), inasmuch $z=0$ is in G and $|\alpha_{n,k}|\geqq\delta_n$. If $\overline{\lim}\,\delta_n>0$, $f(z)$ will be analytic at $z=0$ and so, naturally, will have the desired uniqueness property; that is, vanishing of $f(z)$ on $\{z_\nu\}$ will imply vanishing of $f(z)$ in the connected domain containing $z=0$. Accordingly there is a problem on hand only if $\lim\delta_n=0$. In view of (7.27a') we shall choose m_n so that

(7.27a'') $$\frac{1}{\sigma\delta_n}\leqq m_n\leqq\frac{\sigma_2}{\delta_n}\qquad\left(\sigma_2>\frac{1}{\sigma}\right).$$

By (7.21'), (7.17'), (7.28) we have

$$\theta^*(n) = \log(\varepsilon_n + h_n \delta_n^{-2} \lambda_{m_n}) + n|z_0| q m_n + \mu n \psi(n), \qquad h_n = \sum_{k=1}^{n} |A_{n,k}|\cdot$$

Now z_0 being a point of the bounded set G one has $|z_0| q \leqq q^0$. Thus in view of (7.27 a'')

(7.29) $$\theta^*(n) \leqq \log(\varepsilon_n + h_n \delta_n^{-2} \lambda_{m_n}) + n q^* \frac{1}{\delta_n} + \mu n \psi(n) \qquad (q^* = q^0 \sigma_2).$$

On the basis of the text leading to (7.23) we conclude that (7.23) holds if the second member, above, tends to $-\infty$ as $n \to \infty$; that is, if

$$\varepsilon_n + h_n \delta_n^{-2} \lambda_{m_n} \leqq \varphi_0(n) e^{-n q^* \frac{1}{\delta_n} - \mu n \psi(n)}, \qquad \varphi_0(n) \to 0.$$

These considerations lead to the conditions

(7.30) $$\lambda_{m_n} \leqq \varphi_1(n) \frac{1}{h_n} \delta_n^2 e^{-n q^* \frac{1}{\delta_n} - n \psi(n)},$$

(7.30 a) $$\varepsilon_n \leqq \varphi_2(n) e^{-n q^* \frac{1}{\delta_n} - n \psi(n)},$$

where $\varphi_1(n)$, $\varphi_2(n)$ *are positive and tend to zero with* $\frac{1}{n}$ (μ has been absorbed in $\psi(n)$). We sum the above developments as follows.

Theorem 7.2. *Let* $f(z) = \lim f_n(z)$ *be a function as described in connection with* (7.1), (7.1 a). *Suppose* G *contains an interval* Γ *on which there is a denumerable infinity of points* $\{z_\nu\}$, *such-that* $\lim z_\nu = z'$. *Designate by* Γ_m *the portion of* Γ *for which* $|z - z'| \leqq \frac{1}{m}$ (*integer* $m > 0$). *Associated with the set* $\{z_\nu\}$ *there exists a set of numbers* λ_m,

$$\lambda_m = \text{u. b. } l_\nu \qquad (\nu \geqq \nu_m),$$

where the $\{l_\nu\}$ $(\nu \geqq \nu_m)$ *are the lengths of the non overlapping intervals into which* Γ_m *is divided by the* z_ν *lying in* Γ_m. *Let* δ_n *designate the distance from the set of points* $\alpha_{n,k}$ $(k = 1, \ldots n)$ *to* G. *We assume that the more interesting and important case when* $\lim \delta_n = 0$ *is on hand. With* $\{m_n\}$ *denoting a sequence of integers such that* $\sigma^{-1} \delta_n^{-1} \leqq m_n \leqq \sigma_2 \delta_n^{-1} \left(\sigma_2 > \frac{1}{\sigma}\right)$, *it is observed that under conditions* (7.30), (7.30 a) *vanishing of* $f(z)$ *on* $\{z_\nu\}$ *will imply that*

$$f(z) = 0 \qquad (\textit{in } G);$$

in this connection $q^*(7.29)$, $\sigma(7.25\,\mathrm{a})$[1] *are positive constants defined as described previously, while* $\psi(n)$ (>0) *is any function tending to* $+\infty$ *with* n;

$$h_n = \sum_k |A_{n,k}|.$$

Note. It is observed that (7.30) expresses a 'metric' property of the set $\{z_\nu\}$ in the vicinity of its limiting point z'; (7.30a) relates, of course, to the speed of convergence of $\{f_n(z)\}$ in G.

In the case when the $A_{n,k}$, $\alpha_{n,k}$ are independent of n we have

$$(7.31) \qquad A_{n,k} = A_k, \quad \alpha_{n,k} = \alpha_k, \quad f(z) = \sum_1^\infty \frac{A_k}{z - \alpha_k}.$$

Let $\{\delta_k\}$ $(\delta_1 \geqq \delta_2 \geqq \cdots)$ be a sequence of positive numbers such that the distance from α_k to G is equal or greater than δ_k. We then may take

$$(7.31\,\mathrm{a}) \qquad \varepsilon_n = \sum_{k>n} \frac{|A_k|}{\delta_k},$$

the supposition being that the series

$$\sum_k \frac{|A_k|}{\delta_k}$$

converges. In (7.30), (7.30a) δ_n may be given the meaning assigned above; moreover, since

$$h_n \leqq \sum_k |A_k| = h,$$

h_n *in* (7.30) *may be absorbed in* $\varphi_1(n)$. Finally, *in view of the Theorem the uniqueness property, involved, will take place for functions* $f(z)$ *of the form* (7.31), *provided* (7.30) *holds* (as stated above) *and*

$$|A_k| \leqq s_k \delta_k \exp.\left\{-kq^* \frac{1}{\delta_k} - k\psi(k)\right\} \qquad (k = 1, 2, \ldots),$$

where the s_k (>0) *are such that* $s_1 + s_2 + \cdots$ *converges.*

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[1] We may take $q^* = 4q_0(1 + \sigma^2 t_1^2) RD$, where q_0, t_1 are from (7.26 a′), (7.26 b′) and D is the diameter of G.