Acta Math., 166 (1991), 121–161

Almost commuting matrices and a quantitative version of the Brown–Douglas–Fillmore theorem

by

I. DAVID BERG and KENNETH R. DAVIDSON

University of Illinois
Urbana, IL, U.S.A.

University of Waterloo
Waterloo, Ontario, Canada

1. Introduction

In this paper we give a constructive proof of the following theorem, Theorem 1.1, of Brown, Douglas and Fillmore. Our proof yields a quantiative version, Theorem 1.2, depending on the norm of the self-commutator and subject to a natural resolvent condition.

THEOREM 1.1 (BDF). *Let T be an operator on a Hilbert space $\mathcal{H}$ such that T^*T-TT^* is compact, and such that the Fredholm index* $\operatorname{ind}(T-\lambda)=0$ *whenever this is defined* $(\lambda\notin\sigma_e(T))$. *Then there is a compact operator K such that $T-K$ is normal.*

Our quantitative version yields an estimate of $\|K\|$ in terms of the homogeneous quantity $\|T^*T-TT^*\|^{1/2}$ provided the spectrum of T is in a natural quantitative sense close to the essential spectrum $\sigma_e(T)$. Indeed, if N is normal, and $\|T-N\|<\varepsilon$, then

$$\|(T-\lambda I)^{-1}\| < (\operatorname{dist}(\lambda, \sigma(N))-\varepsilon)^{-1}.$$

So it is reasonable to assume this inequality when $\operatorname{dist}(\lambda, \sigma(N))>\varepsilon$.

THEOREM 1.2. *Given a compact subset X of the plane, there is a continuous positive real-valued function f_X defined on $[0,\infty)$ such that $f(0)=0$ with the following property:*

Let T be essentially normal and satisfy the BDF *hypotheses:*

(i) $\sigma_e(T)=X$,

(ii) $\operatorname{ind}(T-\lambda I)=0$ for all $\lambda\notin X$.

Furthermore let T satisfy the quantitative hypotheses:

(iii) $\|T^*T-TT^*\|^{1/2}<\varepsilon$.

(iv) $\|(T-\lambda I)^{-1}\|<(\operatorname{dist}(\lambda, X)-\varepsilon)^{-1}$ *for all λ such that* $\operatorname{dist}(\lambda, X)>\varepsilon$.

Then there is a compact operator K such that $\|K\|<f_X(\varepsilon)$ *and such that* $T-K$ *is a normal operator with spectrum X.*

One reason for the significance of the BDF theorem is the connection demonstrated between operator algebras and algebraic topology. Given any essentially normal operator T, its image $\pi(T)$ in the Calkin algebra is normal with spectrum $X=\sigma_e(T)$. So then $C^*(\pi(T))$ is isomorphic to $C(X)$ by mapping T to the identity function z. The inverse map is a monomorphism τ of $C(X)$ into the Calkin algebra. An equivalence relation is put on such monomorphisms, namely unitary equivalence: $\tau_1 \sim \tau_2$ if there is a unitary U such that $\tau_2(f)=\pi(U)\tau_1(f)\pi(U)^*$ for f in $C(X)$. The set of equivalence classes is denoted $\mathrm{Ext}(X)$. This set has a binary operation given by $[\tau_1]+[\tau_2]=[\tau_1\oplus\tau_2]$ and this is shown to be a group. Indeed, the definition of $\mathrm{Ext}(X)$ in this context extends readily to any compact metric space. Brown, Douglas and Fillmore prove that $X\to\mathrm{Ext}(X)$ is a covariant functor from the category of compact metric spaces with continuous maps into the category of abelian groups. Moreover, it is shown that $\mathrm{Ext}(X)$ is a homology theory with Bott periodicity and certain pairings with topological K-theory.

These connections with algebraic topology, especially the higher dimensional phenomena, will not be examined in this paper except for a few comments in § 6. Our purpose is to show that in the case of planar sets (where the topology is easily understood), the results of BDF can be explained in a purely operator theoretic way. Furthermore, this approach yields more information about how the desired compact perturbation is obtained. However, it is worth pursuing one other aspect of the Ext functor. The Fredholm index is a continuous homomorphism of the invertible elements of the Calkin algebra into the integers. Thus it naturally induces a map from $\mathrm{Ext}(X)$ into $\mathrm{Hom}(\pi_1(X), \mathbf{Z})$. That is, given τ in $\mathrm{Ext}(X)$ and any homotopy class $[f]$ of invertible elements of $C(X)$, one has

$$(\operatorname{ind}\tau)[f]=\operatorname{ind}\tau(f).$$

In the case of planar sets, $\pi^1(X)$ is an abelian group generated by $[z-\lambda_i]$ where $\{\lambda_i\}$ is a set of points, one from each bounded component of the complement of X in the plane. Thus, if τ corresponds to an essentially normal operator T via $\tau(z)=\pi(T)$, then $\operatorname{ind}\tau$ is determined by the integers $n_i=\operatorname{ind}(T-\lambda_i I)$. For planar sets, it was shown [12] that ind is an isomorphism. Showing that ind is surjective is an easy consequence of the Berger–Shaw Theorem [9]. Thus, the BDF theorem for indices other than 0 is an immediate corollary of Theorem 1.1 and can be stated:

COROLLARY 1.3. *Let S and T be two essentially normal operators such that $\sigma_e(S)=\sigma_e(T)$ and $\operatorname{ind}(S-\lambda I)=\operatorname{ind}(T-\lambda I)$ for all $\lambda\notin\sigma_e(S)$. Then there is a unitary operator U and a compact operator K so that $T=USU^*+K$.*

Our approach has its roots in two papers by the first author [4, 5] in which constructive techniques were developed to prove our Theorem 1.2 in the special case in which T is the direct sum of a normal operator and a weighted shift. The problem was then to free the applications of these methods from an apparent dependence on the special character of weighted shifts. A paper of the second author, *Almost commuting Hermitian matrices,* [18], improved the methods and developed an approach which proves sufficiently general to lead to our results.

In that paper [18], the existence of our proof of BDF for the disk was announced but the case of the annulus, which, of course, we solve here, was listed as open.

The problem principally addressed in that paper, [18], is still an important open question.

Problem 1.4. Given $\varepsilon>0$, is there a $\delta>0$ so that whenever T is a norm 1, finite rank operator such that $\|T^*T-TT^*\|<\delta$, there is a normal operator N satisfying $\|T-N\|<\varepsilon$?

This problem is unresolved; however in [18] an absorption theorem is proved which turns out to be an acceptable substitute.

THEOREM 1.5 ([18]). *Given a matrix T in $\mathcal{M}_n(\mathbf{C})$, there is a normal matrix N in $\mathcal{M}_n(\mathbf{C})$ with $\|N\|\leq\|T\|$, and a normal matrix M in $\mathcal{M}_{2n}(\mathbf{C})$ such that*

$$\|T\oplus N-M\|\leq 75\|T^*T-TT^*\|^{1/2}.$$

The key to constructing normal approximants without solving problem 1.4 will be an adaptation of the method developed in [18]. Given an operator T with small self-commutator, we develop a block tridiagonal form for T based on short spectral intervals of $\operatorname{Re} T$ or of $|T|$ as the situation demands. A simple argument of [12] shows that we can extract from T sufficient approximate eigenvectors to construct a normal operator N so that T is close to unitarily equivalent to $T\oplus N$. The normal N is shown to be close to a tridiagonal operator which mimics T on long strings of consecutive blocks. We then construct a small perturbation of $T\oplus N$ that intertwines T and N along the string of blocks on which T and N agree. This permits us to "uncouple" $T\oplus N$ into a direct sum where each summand is supported on two of these strings of consecutive blocks. We will insure that each summand will be close to a normal by insuring that either the real part or the absolute value of each summand will be constant and hence $T \oplus N$ and consequently T will be close to a normal operator. A refinement of the method will insure compactness of the difference operator if T has a compact self-commutator.

An important step in our construction is that of establishing that an essentially normal operator, that is, an operator with a compact self-commutator, with no index obstruction is quasidiagonal. We recall that an operator T is called *quasidiagonal* if there is an increasing sequence P_n of finite rank projections with $\sup P_n = I$ such that $\lim_{n\to\infty} \|P_n T - TP_n\| = 0$. It is easy to show that if T is quasidiagonal, then for every $\varepsilon > 0$, there is a compact K with $\|K\| < \varepsilon$ so that $T - K \cong \Sigma_{n=1}^{\infty} \oplus T_n$, where the latter operator is the direct sum of finite rank operators. The set of quasidiagonal operators is closed, invariant under compact perturbations, and if T is quasidiagonal, every element of $C^*(T)$ is quasidiagonal. We know that every normal is quasidiagonal by our extension of the Weyl–von Neumann theorem [3].

We will see that the lack of an indicial obstruction allows the quasidiagonality of an essentially normal operator, T, yielding $T' = T + K$ for small compact K where $T' = \Sigma \oplus T_n$ and where $\lim_{n\to\infty} \|[T_n^*, T_n]\| = 0$. Now if we knew problem 1.5 had a positive solution we would instantly obtain a normal operator $T'' = \Sigma \oplus N_n$ where $T_n - N_n$ was small provided $[T_n^*, T_n]$ was small, yielding the compactness of $T - T''$. Unfortunately, we do not have a solution to problem 1.5, and we have to work harder.

Once we have established quasidiagonality, Theorem 1.5 yields Theorem 1.2 for the disc case ($R_1 = 0$) fairly easily [18]. In order to solve the general problem, it is necessary to control the spectrum of the normal matrix N. This and the proof of quasidiagonality are the main steps in the proof. These two steps are very closely related and for the most part are done at the same time, with few differences between the finite and infinite dimensional versions. As our spectrum becomes uglier, our estimates become worse so violently that the reader may replace every constant by M and lose little. This is due to the use of conformal mapping techniques. It would be more satisfactory to be able to deal with an operator without having to distort the spectrum into a manageable shape since this would almost surely lead to better norm estimates.

Our results were described briefly in our Bulletin announcement [7]. A more detailed but still essentially non-computational guide to the present paper is contained in section 2.

2. Notation and outline

All Hilbert spaces in this paper are separable or finite dimensional. The space of bounded linear operators on $\mathscr{H}$ will be denoted $\mathscr{B}(\mathscr{H})$, or possibly $\mathscr{M}_n$ if $\mathscr{H}$ is n-dimensional. The ideal of compact operators is $\mathscr{K}$, and π is the quotient map of $\mathscr{B}(\mathscr{H})$ onto the Calkin algebra $\mathscr{B}(\mathscr{H})/\mathscr{K}$. The spectrum of an operator T is $\sigma(T)$, and $\sigma_e(T)$ denotes $\sigma(\pi(T))$. The Fredholm index of T is defined, if $\pi(T)$ is invertible, by $\operatorname{ind} T = \operatorname{nul} T - \operatorname{nul} T^*$, where $\operatorname{nul} T$ is the dimension of the kernel of T. The commutator of two operators is $[A,B]=AB-BA$. Given a normal operator N, the spectral measure of N will be denoted $E_N(\cdot)$, and $E_N(\sigma)$ is the orthogonal projection associated with the set σ. $\mathrm{Nor}(\mathscr{H})$, or just Nor, will denote the set of normal operators on $\mathscr{H}$. Given a subspace M, we let $P(M)$ denote the (orthogonal) projection onto M.

A normal operator is called *diagonal* if it has an orthonormal basis of eigenvectors. There will be frequent need for our extension to normal operators of the Weyl–von Neumann theorem, sometimes called the Weyl–von Neumann–Berg theorem [3], which states that given a normal operator N and a positive $\varepsilon>0$, there is a compact operator K with $\|K\|<\varepsilon$ such that $N-K$ is a diagonal normal operator.

An operator T is called *almost normal* if $T=N+K$ for some normal N and compact K.

An operator T is called *essentially normal* if $[T^*,T]$ is compact. Equivalently, T is essentially normal if $\pi(T)$ is a normal element of the Calkin algebra. We recall that BDF shows that lack of an index obstruction is both necessary and sufficient for essential normality to imply almost normality. In [12] Lemma 2.2, an important absorption principle is established for essentially normal operators:

LEMMA 2.1. *Let T be essentially normal and let $\varepsilon>0$ be given. Suppose that N is a normal operator such that $\sigma(N)$ is contained in $\sigma_e(T)$. Then there is a compact operator K with $\|K\|<\varepsilon$ such that $T-K\cong T\oplus N$ (we will often write simply $T\oplus N \doteqdot T$).*

This result is not particularly difficult, but it is of central importance. The key point is that if λ belongs to $\sigma_e(T)$, then $T-\lambda I$ is not bounded below. So an orthonormal sequence x_n such that $\lim_{n\to\infty}\|(T-\lambda I)x_n\|=0$ is obtained. The compactness of $[T^*,T]$ implies that $\{x_n\}$ is an approximate eigenvector of T^* for $\bar{\lambda}$ as well. Do this for a dense set $\{\lambda_k\}$ in $\sigma_e(T)$, and one can obtain an "approximate summand" which is a diagonal normal operator M with $\sigma(M)=\sigma_e(M)=\sigma_e(T)$. The rest follows from the extended Weyl–von Neumann theorem.

Now a few comments on our approach. In [12], the main technical device is cutting the spectrum in two. Our approach differs significantly in that we never cut through a hole. The most important step is the construction for the annulus, which is our method for dealing with a hole in the spectrum. An essentially normal operator of index zero with spectrum an annulus is approximated by a direct sum of multiples of unitaries. Intuitively, the annulus is approximated by a bunch of thinner annuli. After that, the finitely connected case is reduced to the annulus case by showing that it is possible to cut the spectrum with a line that does not cut any holes. So, an n-holed spectrum is cut into n pieces conformally equivalent to an annulus. An approximation of an arbitrary essentially normal operator is obtained by "fattening" the spectrum of a normal summand to a nice set. Thereby, essentially normal operators with zero index data are shown to be quasidiagonal. The same methods are repeated on finite dimensional blocks to show that an absorption theorem analogous to Theorem 1.5 holds for general sets X. From this, Theorem 1.1 and 1.2 are deduced.

Our proofs often appear dauntingly elaborate and computational. We can, however, first give the reader a reasonable conceptual guide through the ideas of the proof. Much of the technical awkwardness in later sections arises from a quest for the right order of magnitude of the error. This is omitted from our overview here.

Our most important technique is "intertwining". An operator T is *tridiagonal* with respect to a decomposition $H=\Sigma\oplus\mathcal{H}_n$ (finite or infinite) if its operator matrix (T_{ij}) has $T_{ij}=0$ if $|i-j|\geq 2$. Suppose T and S are tridiagonal on $\mathcal{H}=\Sigma\oplus\mathcal{H}_n$ and $\mathcal{K}=\Sigma\oplus\mathcal{K}_n$ respectively, and $T_{ij}=S_{ij}$ for $1\leq i+j\leq 2N+1$. Then

$$T\oplus S\left|\sum_{n=1}^{N}\oplus(\mathcal{H}_n\oplus\mathcal{K}_n)=\left(T\left|\sum_{n=1}^{N}\oplus\mathcal{H}_n\right)^{(2)}.\right.\right.$$

Let $\mathcal{H}_n'$ be the image of the subspace $\mathcal{H}_n\oplus 0$ of $\mathcal{H}_n\oplus\mathcal{K}_n$ after a rotation through an angle of $n\pi/2N$ by a 2×2 scalar valued matrix. Then $T\oplus S$ almost leaves

$$\sum_{n\leq 0}\oplus\mathcal{H}_n\oplus\sum_{n=1}^{N}\oplus\mathcal{H}_n'\oplus\sum_{n>N}\oplus\mathcal{K}_n$$

invariant, and similarly for its orthogonal complement. The restriction of $T\oplus S$ to this summand looks like T at the beginning, has their common structure of S and T in the middle, and looks like S on the tail section. The reverse holds for the complementary summand. So a small perturbation of $T\oplus S$ intertwines the two summands along their common section.

This technique is particularly useful when S "stops" soon, i.e. acts on $\Sigma_{n=0}^{N}\oplus\mathcal{H}_n$ itself or perhaps $\Sigma_{n=-1}^{N+1}\oplus\mathcal{H}_n$. For then, the intertwining has succeeded in splitting T into two summands. We shall say that T has been *uncoupled*, or that the head and tail sections of T have been uncoupled. We next observe that this uncoupling procedure can be implemented on many different sections of T without increasing the norm of the perturbation. This is because the perturbations take place on pairwise orthogonal pieces, and T does not map anything in one of these blocks into any other. Thus this method can be used to split T into many direct summands.

Let A be a self-adjoint operator, and let B be an operator such that $[A,B]$ is compact and $\|[A,B]\|\leqslant\varepsilon^2$. Use the spectrum of A to split $\mathcal{H}$ into a direct sum $\Sigma\oplus\mathcal{H}_n$ corresponding to the spectral subspaces of A for intervals $[n\varepsilon,(n+1)\varepsilon)$. With respect to this decomposition, B has a matrix (B_{ij}). A simple computation shows that B_{ij} is compact and $\|B_{ij}\|\leqslant\varepsilon/|i-j|-1$ for $|i-j|\geqslant 2$. Hence the tridiagonal part B' of B differs from B by a small compact operator. (A more delicate argument shows that $\|B-B'\|\leqslant C\varepsilon$ independent of the number of summands, which is important to get best possible estimates.)

Now suppose T is an operator with small self-commutator. Even if this commutator is not compact, the approach of Lemma 2.1 approximates T by an operator unitarily equivalent to $T\oplus N$ where N is normal and $\sigma(N)=\sigma_e(N)=\sigma_e(T)$. Write $T=A+iB$ as the sum of its real and imaginary parts. Decompose $\mathcal{H}$ as in the previous paragraph, and modify B to a tridiagonal B'. If $\sigma_e(T)$ contains an interval $\{\lambda+it\colon|t|\leqslant\|B\|\}$, we can use the intertwining technique to split T approximately into summands corresponding to spectrum in $\{z\colon\operatorname{Re}z\geqslant\lambda\}$ and $\{z\colon\operatorname{Re}z\leqslant\lambda\}$ respectively as follows. Take an integer $n\doteqdot\varepsilon^{-1/2}$ and consider the string of $n+1$ blocks $\mathcal{S}=\Sigma_{i=0}^{n}\oplus\mathcal{H}_i$ corresponding to the spectrum of A centered about λ. Let B_0 be the compression of B' to $\mathcal{S}$. The operator $N_0=\lambda I+iB_0$ is normal with spectrum $\{\lambda+it\colon|t|\leqslant\|B\|\}$, which is a direct summand of N, and is very close ($O(\varepsilon^{1/2})$) to the compression of $A+iB_0$ to $\mathcal{S}$. So

$$T\doteqdot T\oplus N\doteqdot T\oplus N_0\oplus N_0'\doteqdot(A+iB')\oplus(A+iB'|\mathcal{S})\oplus N_0'.$$

Now B' and B_0 agree on the string $\mathcal{S}$, so we can use the intertwining to uncouple B'. A moment's reflection on how this decomposition works shows that A is not perturbed at all. So T has likewise been uncoupled. This will be referred to as the *cutting lemma*.

Now consider the case of an annulus $A=\{z\colon 1\leqslant|z|\leqslant R\}$, and an operator T with $\|T\|=R$, $\|T^{-1}\|=1$, $\sigma_e(T)=A$, and $\|[T,T^*]\|\leqslant\varepsilon^2$. Let $T=UP$ be the polar decomposition of T. Then T is normal if and only if P and U commute. Moreover $\|[U,P^2]\|=$

$\|[T, T^*]\,U\| \leq \varepsilon^2$. So split $\mathscr{H}$ into summands corresponding to spectral intervals of P^2 of length ε. A small perturbation of U to U' makes it tridiagonal. We wish to uncouple U' into direct summands corresponding to small spectral intervals of P^2 (roughly $\varepsilon^{1/2}$; that corresponds to strings of length about $\varepsilon^{-1/2}$). The procedure is much like the one described in the previous paragraph except that the compression of U' to a string $\mathscr{S}=\Sigma_{k=0}^{n}\oplus\mathscr{H}_k$ of blocks will not be unitary or even close to unitary. Care must be taken to overcome this by modifying the compression to a unitary U_0 which is still tridiagonal and still agrees with U' except on a few blocks at each end of the string. We accomplish this by restricting U' to the interior of the string $\mathscr{S}'=\Sigma_{k=2}^{n-2}\oplus\mathscr{H}_k$ on which $U'|\mathscr{S}'$ is almost isometric with range in $\mathscr{S}$. In the finite dimensional case, one keeps track of kernels and ranges and finds that the complement of the range of $U'|\mathscr{S}'$ in $\mathscr{H}_0\oplus\mathscr{H}_1$ is precisely $\dim(\mathscr{H}_0\oplus\mathscr{H}_1)$, and similarly for $\mathscr{H}_{n-1}\oplus\mathscr{H}_n$. So one extends $U'|\mathscr{S}'$ to U_0 so as to be both almost unitary and tridiagonal. In the infinite dimensional case, we have domain and ranges always of infinite dimension; so a similar extension of $U'|\mathscr{S}'$ is obtained.

This accomplishes the construction of a unitary U which is (i) tridiagonal, (ii) splits as a direct sum $U=\Sigma\oplus U_n$ where each U_n acts on a string $\mathscr{S}_n$ of length $\varepsilon^{-1/2}$, and (iii) U_n agrees with U' on the interior string $\mathscr{S}_n'$ of $\mathscr{S}_n$. Let λ_n^2 be the midpoint of the spectral interval of P^2 corresponding to the string $\mathscr{S}_n$. Then $N_0=\Sigma\oplus\lambda_n U_n$ is normal, and is a direct summand of a normal N with $\sigma(N)=A$. Also N_0 is close to UP. So

$$T \doteqdot T\oplus N \doteqdot T\oplus N_0\oplus N_0' \doteqdot U'P\oplus UP\oplus N_0' \cong (U'\oplus U)\,P^{(2)}\oplus N_0'.$$

The uncoupling procedure on each string $\mathscr{S}_n$ splits $U'\oplus U$ into a direct sum $\Sigma\oplus W_n$ of almost unitaries, each of which is supported on a spectral interval for P^2 of length $2\varepsilon^{1/2}$. It is now a simple matter to replace P^2 by a scalar on each of these blocks by a perturbation of $\varepsilon^{1/2}$, and modify W_n to be bonafide unitaries. Their product is a normal operator close ($O(\varepsilon^{1/2})$) to T.

Now if $[T^*, T]$ is compact, this procedure as it stands may not yield a compact perturbation. To overcome this, we first establish that T is quasidiagonal. In fact, we will do this subject only to the hypotheses: $[T^*, T]$ is compact, $\sigma_e(T)=A$, and $\operatorname{ind} T=0$. From the polar decomposition $T=UP$, we obtain $\operatorname{ind} U=0$ and $\sigma_e(P)=[1, R]$. So we find a compact perturbation $T_0=U_0P_0$ where U_0 is unitary, $I\leq P_0\leq RI$, and P_0 is diagonal with respect to a basis $\{l_n, n\geq 1\}$. Given $\varepsilon>0$, it is a routine exercise to split this basis into blocks $\mathscr{H}_k=\operatorname{span}\{l_i: n_{k-1}<i\leq n_k\}$ so that U_0 differs from its tridiagonal part U_1 by a compact operator of norm at most ε.

Write $U_1=(U_{ij})$ and $P=\operatorname{diag}(D_j)$. Since $[P^2, U_1]=(D_i^2\,U_{ij}-U_{ij}D_j^2)$ is compact, all terms are small ($<\varepsilon$) once $i, j \geq N$, say. Replace P by another positive operator $Q=\operatorname{diag}(E_j)$ where $E_j=I$ for $j\leq N$, $E_j=D_j$ for $j\geq 2N$, and $E_j=t_j I+(1-t_j)D_j$ for j in between and t_j slowly tapers from 1 down to 0. A simple computation shows that $\|[Q^2, U_0]\|\leq C\varepsilon$. The spectrum of Q is still $[1,R]$ and $S_\varepsilon=U_0 Q$ is essentially normal with $\|S_\varepsilon\|=R$, $\|S_\varepsilon^{-1}\|=1$ and $\sigma_e(S_\varepsilon)=A$. Since $\|[S_\varepsilon, S_\varepsilon^*]\|=\|[Q^2, U_0]\|$, our distance estimate shows that $\operatorname{dist}(S_\varepsilon, \text{Normal})=O(\varepsilon^{1/2})$. But each $S_\varepsilon-T$ is compact, thus T is norm limit of operators which are normal plus compact, and, a fortiori, quasidiagonal. Hence T is quasidiagonal.

Now decompose $T\cong\Sigma_{n\geq 1}\oplus T_n\oplus N$ on $\Sigma_{n\geq 1}\oplus\mathcal{H}_n\oplus\mathcal{H}'$ where T_n act on finite dimensional spaces, $\|T_n\|\leq R$, $\|T_n^{-1}\|\leq 1$, and N is a normal with $\sigma(N)=A$. Since $[T^*, T]=\Sigma_{n\geq 1}\oplus[T_n^*, T_n]$ is compact, we have $\|[T_n^*, T_n]\|=\varepsilon_n^2$ tends to 0 as n increases. Apply the distance estimate to each summand in turn. We obtain normal operators N_n and M_n on $\mathcal{H}_n$ and $\mathcal{H}_n\oplus\mathcal{H}_n$ respectively so that $\|T_n\oplus N_n-M_n\|=O(\varepsilon_n^{1/2})$ tends to zero. Now $N\doteqdot\Sigma_{n\geq 1}\oplus N_n\oplus N$, hence

$$\begin{aligned} T &\doteqdot \sum_{n\geq 1}\oplus T_n\oplus N \doteqdot \sum_{n\geq 1}\oplus(T_n\oplus N_n)\oplus N \\ &= \sum_{n\geq 1}\oplus M_n\oplus N-\sum_{n\geq 1}\oplus(M_n-T_n\oplus N_n)\doteqdot N-K \end{aligned}$$

where $K=\Sigma_{n\geq 1}\oplus(M_n-T_n\oplus N_n)$ is compact. Moreover, if $\|[T^*, T]\|=\varepsilon^2$, we can arrange that $\varepsilon_n\leq\varepsilon$ for all n. So we obtain a compact perturbation K so that $T-K$ is normal and $\|K\|$ is small if $\|[T^*, T]\|$ is small. In our more precise calculations, we will actually achieve $\|K\|\leq 100\|[T^*, T]\|^{1/2}$. Similar arguments yield the result when $\sigma(T)$ is a disc or a rectangle, using the real part instead of absolute value.

Now turn to the general case of an essentially normal operator T with zero index data, and $\sigma_e(T)=X$. By the absorption lemma, $T\doteqdot T\oplus N$ where N is normal and $\sigma(N)=\sigma_e(N)=X$. Given $\varepsilon>0$, there is a nice region X_ε with $X\subset X_\varepsilon\subset\{x: \operatorname{dist}(x, X)<\varepsilon\}$, where by nice we mean a finitely connected region with piecewise analytic Jordan curves as boundary, and allowing conformal maps as necessary. There is a normal operator N_ε with $\sigma(N_\varepsilon)=X_\varepsilon$ and $\|N-N_\varepsilon\|<\varepsilon$. Thus T is the norm limit of operators $T_\varepsilon\doteqdot T\oplus N_\varepsilon$. If it can be shown that each T_ε is quasidiagonal, then so is T.

Fix T_ε. By a theorem of Apostol [1], one can assume that T has been replaced by a very small compact perturbation of itself so that $\sigma(T)$ is contained in X_ε except for finitely many isolated eigenvalues. Clearly, these can be removed by a compact perturbation, so we ignore them. Let f be a conformal map in a neighborhood of X_ε

which carries each component of X_ε onto a rectangle with rectangular holes parallel to the axes. Then $S=f(T_\varepsilon)$ is an essentially normal operator with $\sigma(S)=\sigma_e(S)=f(X_\varepsilon)$. Use the cutting lemma finitely many times to uncouple S into a direct sum $S\doteq\Sigma_{k=1}^n\oplus S_k$ where each S_k is essentially normal and has spectrum equal to a rectangle R_k with or without a rectangular hole. Each R_k is conformally equivalent to an annulus or a disc, so another conformal map $g_k(S_k)$ is an essentially normal operator with zero index data and annular spectrum. Thus $g_k(S_k)$ is normal plus compact. This implies that each S_k, and thus S, is normal plus compact. In turn $T_\varepsilon=f^{-1}(S)$ is normal plus compact, and a fortiori, quasidiagonal. Thus T is quasidiagonal.

Write $T\doteq\Sigma_{n\geqslant 1}\oplus T_n$ where T_n act on finite dimensional spaces $\mathscr{H}_n$. Study the method of the previous paragraph to see what the construction does to the summands T_n in $T_\varepsilon=\Sigma_{n\geqslant 1}\oplus T_n\oplus N_\varepsilon$. One notices that the functional calculus preserves direct sums, so $f(T_\varepsilon)=\Sigma_{n\geqslant 1}\oplus f(T_n)\oplus f(N_\varepsilon)$. Next, the cutting lemma works by first decomposing $\mathscr{H}$ via the spectral measure of $\operatorname{Re} f(T_\varepsilon)=\Sigma\oplus\operatorname{Re} f(T_n)\oplus\operatorname{Re} f(N_\varepsilon)$, which also respects the summands. Then it adds on a normal summand produced on the same space. In other words, a finite rank normal summand is adjoined to each finite rank summand, thereby allowing an uncoupling of each summand. The same procedure occurs again when the conformal map carries each uncoupled piece onto an annulus or disc. Again the perturbations respect the block decomposition. Consequently, the procedure yields normal matrices N_n and M_n with spectrum in X_ε so that $K=\Sigma_{n\geqslant 1}\oplus(M_n-T_n\oplus N_n)$ is compact. Moreover,

$$T_\varepsilon\doteq\sum_{n\geqslant 1}\oplus(T_n\oplus N_n)\oplus N_\varepsilon=\sum_{n\geqslant 1}\oplus M_n\oplus N_\varepsilon+K.$$

It follows that $\|M_n-T_n\oplus N_n\|$ tends to 0 as n increases.

Now for any $\varepsilon>0$, $\|M_n-T_n\oplus N_n\|<\varepsilon$ for n greater than some integer n_ε. There are normal matrices M_n' and N_n' with spectrum in X so that $\|N_n-N_n'\|<\varepsilon$ and $\|M_n-M_n'\|<\varepsilon$. Hence $\|M_n'-T_n\oplus N_n'\|<3\varepsilon$ for all $n\geqslant n_\varepsilon$. But $\varepsilon>0$ is arbitary. Thus it is possible to choose M_n' and N_n' so that $\lim_{n\to\infty}\|M_n'+T_n\oplus N_n'\|=0$. Thus

$$T\doteq\sum_{n\geqslant 1}\oplus T_n\oplus N\doteq\sum_{n\geqslant 1}\oplus(T_n\oplus N_n')\oplus N\doteq\sum_{n\geqslant 1}\oplus M_n'\oplus N+K\doteq N+K'$$

where K' and $K=\Sigma_{n\geqslant 1}\oplus(M_n'-T_n\oplus N_n')$ are compact. This establishes the BDF theorem. Moreover, a norm estimate is obtained for each nice region X_ε. The fact that one can choose ε to best advantage for each summand separately allows us to compute a uniform norm estimate for the region X as well.

3. The approximation techniques

In this section, the principal technical tool will be described. This is a refinement of the interchange technique for weighted shifts [4] (see also [19, 21, 30]) to arbitrary "tridiagonal" operators. This method was developed in [18].

An operator T is *tridiagonal* with respect to a decomposition of $\mathcal{H}=\Sigma_{n\in I}\oplus\mathcal{H}_n$ if its corresponding matrix (T_{ij}), where T_{ij} maps $\mathcal{H}_j$ into $\mathcal{H}_i$, has $T_{ij}=0$ if $|i-j|>1$. The index set I may be a finite or infinite set of integers. It is worth pointing out that every operator T in $\mathcal{B}(\mathcal{H})$ is tridiagonal with respect to some decomposition $\mathcal{H}=\Sigma_{n=0}^{\infty}\oplus\mathcal{H}_n$ in which every $\mathcal{H}_0$ has finite dimension. Furthermore, $\mathcal{H}_0$ is arbitrary. Indeed, choose an orthonormal basis $\{e_n, n\geq 1\}$, and define

$$\mathcal{H}_{n+1}=\operatorname{span}\left\{\sum_{j=0}^{n}\oplus\mathcal{H}_j, T\mathcal{H}_n, T^*\mathcal{H}_n, e_{n+1})\right\}\ominus\sum_{j=0}^{n}\oplus\mathcal{H}_n$$

for $n\geq 0$. The verification is routine.

There will be a need for the following variant.

LEMMA 3.1. *Let A and B be operators, and suppose A is self-adjoint. Given $\varepsilon>0$, there are compact operators K_1 and K_2 with $\|K_i\|<\varepsilon$ and a decomposition of $\mathcal{H}$ as $\Sigma_{n\geq 0}\oplus\mathcal{H}_n$ into finite dimensional blocks so that $A-K_1$ is block diagonal, and $B-K_2$ is tridiagonal.*

Proof. First, choose K_1 with $\|K_1\|<\varepsilon$ so that $A-K_1=D$ is diagonal. Let $\{e_n, n\geq 0\}$ be an orthonormal basis of eigenvectors of D. Let $\mathcal{H}_0=\operatorname{span}\{e_0\}$. Suppose $\mathcal{H}_0,\dots,\mathcal{H}_n$ have been chosen so that each $\mathcal{H}_j$ is the span of finitely many of the vectors $\{e_j\}$, $\Sigma_{j=0}^{n}\oplus\mathcal{H}_j$ contains $\operatorname{span}\{e_0,\dots,e_n\}$, and

$$\left\|\left(\sum_{j=0}^{k+1}P_j\right)^{\perp}BP_k\right\|+\left\|P_kB\left(\sum_{j=0}^{k+1}P_j\right)^{\perp}\right\|<\varepsilon/2^k \quad \text{for} \quad 0\leq k\leq n-1,$$

where P_j is the orthogonal projection onto $\mathcal{H}_j$. Now, choose $\mathcal{H}_{n+1}$ to be of the same form, containing e_{n+1} if possible, so that the above norm condition is satisfied for $k=n$ as well. Then one defines

$$K_2=\sum_{k\geq 0}\left(\sum_{j=0}^{k+1}P_j\right)^{\perp}BP_k+P_kB\left(\sum_{j=0}^{k+1}P_j\right)^{\perp}.$$

Clearly $\|K_2\|<\varepsilon$, K_2 is compact, and $B-K_2$ is tridiagonal. □

First, a weak version of our method is described. This has the advantage of being intuitively clear. The stronger version is based on the same principle, but is technically somewhat more difficult.

LEMMA 3.2. *Let $A=(A_{ij})$ and $B=(B_{ij})$ be tridiagonal with respect to $\Sigma_{i=0}^{N}\oplus\mathcal{H}_i$, and suppose that $A_{ij}=B_{ij}$ except for $i=j=0$ and $i=j=N$. Let C and D be tridiagonal with $C_{ij}=D_{ij}=A_{ij}$ for $1\leqslant i+j\leqslant 2N-1$, and $C_{00}=A_{00}$, $D_{00}=B_{00}$, $C_{NN}=B_{NN}$, and $D_{NN}=A_{NN}$. Then there is a unitary U such that $\|A\oplus B-U^*(C\oplus D)U\|<(\pi/N)\|A\|$.*

Remark 3.3. This lemma probably sounds rather useless as it stands, so some comment is needed. Think of A as a doubly infinite tridiagonal, and B as a finite tridiagonal agreeing with A on entries (i,j) with $1\leqslant i+j\leqslant 2N-1$. Clumping all the $\Sigma_{i\leqslant 0}\oplus\mathcal{H}_i$ into the initial block and $\Sigma_{i\geqslant N}\oplus\mathcal{H}_i$ into the Nth block yields the situation in the lemma. The operator C is tridiagonal agreeing with A up to the $(N-1)$st block but finishing off like B; and D starts like B and looks like the "tail" of A on $\Sigma_{i>0}\oplus\mathcal{H}_i$.

This will be used as a perturbation result. Namely, $A\oplus B$ is close to the operator $C\oplus D$. This will be thought of as the operation of splitting off the "head" of A from its "tail" into two direct summands by a small change in norm. It is easy to see that if A and B are self-adjoint or unitary, then so are C and D.

Finally, an important observation to make is that the perturbation is made only on the blocks $(\Sigma_{i=1}^{N-1}\oplus\mathcal{H}_i)^{(2)}$. As a result, this splitting procedure may be effected at many points along the length of A (by tacking on various summands B_j) without any addition of norm of the errors.

Proof. Represent the domain of $A\oplus B$ as $\Sigma_{j=0}^{N}\oplus\mathcal{H}_j^{(2)}$. Consider the projection $P=\Sigma_{j=0}^{N}\oplus P_j$ where P_j acts on $\mathcal{H}_j^{(2)}$ by the matrix

$$\begin{bmatrix} c_j^2 I & c_j s_j I \\ c_j s_j I & s_j^2 I \end{bmatrix}$$

where $c_j=\cos(j\pi/2N)$ and $s_j=\sin(j\pi/2N)$. Note that

$$P_0=\begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}, \quad P_N=\begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix},$$

and that P_j commutes with

$$\begin{pmatrix} A_{jj} & 0 \\ 0 & A_{jj} \end{pmatrix}.$$

Thus P_j commutes with

$$\begin{pmatrix} A_{jj} & 0 \\ 0 & B_{jj} \end{pmatrix} \quad \text{for} \quad 0 \leqslant j \leqslant N.$$

Hence $[A \oplus B, P]$ is a matrix with entries

$$P_i \begin{pmatrix} A_{ij} & 0 \\ 0 & A_{ij} \end{pmatrix} - \begin{pmatrix} A_{ij} & 0 \\ 0 & A_{ij} \end{pmatrix} P_j \quad \text{for} \quad |i-j| = 1$$

and zero otherwise. But P_j "commutes" with

$$\begin{pmatrix} A_{ij} & 0 \\ 0 & A_{ij} \end{pmatrix},$$

so

$$\|P_i A_{ij}^{(2)} - A_{ij}^{(2)} P_j\| \leqslant \|A\| \left\| \begin{bmatrix} c_j^2 & c_j s_j \\ c_j s_j & s_j^2 \end{bmatrix} - \begin{bmatrix} c_i^2 & c_i s_i \\ c_i s_i & s_i^2 \end{bmatrix} \right\|$$
$$= \|A\| \left\| \begin{bmatrix} -s_1^2 & c_1 s_1 \\ c_1 s_1 & s_1^2 \end{bmatrix} \right\| = s_1 \|A\| < \frac{\pi}{2N} \|A\|.$$

Thus $[A \oplus B, P]$ is the sum of its first superdiagonal and first subdiagonal, and so has norm at most $(\pi/N)\|A\|$.

The unitary taking $\Sigma_{j=0}^{N} \oplus \mathscr{H}_j^{(2)}$ onto itself is given by $U = \Sigma_{j=0}^{N} \oplus U_j$ where

$$U_j = \begin{bmatrix} c_j I & -s_j I \\ s_j I & c_j I \end{bmatrix}$$

is rotation through the angle $j\pi/2N$. This takes the domain of A onto the range of P, and the domain of B onto the range of $P^{\perp}$. The operators $A \oplus B$ and $U^*(C \oplus D)U$ differ only on the two off-diagonals as above. The $(j, j+1)$ term of $A \oplus B$ is $A_{j,j+1}^{(2)}$, whereas the $(j, j+1)$ term of $U^*(C \oplus D)U$ is $U_j^* A_{j,j+1}^{(2)} U_{j+1} = A_{j,j+1}^{(2)} U_1$. Since $\|U_1 - I^{(2)}\| = 2\sin(\pi/4N) < \pi/2N$, it follows that

$$\|A \oplus B - U^*(C+D)U\| < \frac{\pi}{N} \|A\|. \qquad \square$$

In the strengthened version of Lemma 3.2 which we now address there is no apparent hypothesis of tridiagonalization for B. This, however, will be guaranteed by a hypothesis of almost-commutativity with A. We state this as a separate lemma.

This useful lemma, which will provide the desired tridiagonal decompositions, is essentially Lemma 3.1 of [18]. The proof of this lemma was based on the methods of [10]. If A is a self adjoint operator and

$$a_0 \leq -\|A\| < a_1 < \ldots < a_n \quad (\text{where } \|A\| < a_n)$$

is a partition of the spectrum of A into intervals $I_j = [a_j, a_{j+1})$, then A has a diagonal form $A = \Sigma_{j=0}^{n-1} \oplus A_j$ where A_j is the restriction to $E_A(I_j) = E_j$. Any other operator B has a matrix $B = (B_{ij})$ with respect to this decomposition. Let B' be its "tridiagonal part", namely the matrix (B'_{ij}) where $B'_{ij} = B_{ij}$ if $|i-j| \leq 1$ and $B'_{ij} = 0$ for $|i-j| > 1$.

LEMMA 3.4. *Let A, B and B' be as above. Suppose that $a_{j+1} - a_j \geq \varepsilon$ for $0 \leq j < n$ and $\|[A, B]\| < \delta$. Then $\|B - B'\| < 12\delta/\varepsilon$.*

Proof. The commutator $[A, B]$ has matrix entries $A_i B_{ij} - B_{ij} A_j$. Thus $[A, B']$ is the tridiagonal part of $[A, B]$. Hence $\|[A, B']\| < 2\delta$. (To see this, note that the 2×2 blocks $\|(E_j + E_{j+1})[A, B](E_j + E_{j+1})\| < \delta$.)

Hence $\|[A, B-B']\| < 3\delta$. By Lemma 3.1 of [18], it follows that

$$\operatorname{dist}(B-B', \{A\}') < 6\delta/\varepsilon,$$

where $\{A\}'$ is the commutant of A. This consists of diagonal operators C. Thus if $\|B - B' - C\| < 6\delta/\varepsilon$, it follows that $\|C\| < 6\delta/\varepsilon$. Hence $\|B - B'\| < 12\delta/\varepsilon$. □

Now for the stronger version, which is essentially Corollary 4.3 of [18].

LEMMA 3.5. *Let $A = A_1 \oplus A_2 \oplus A_3$ be a self adjoint operator on $H_1 \oplus H_2 \oplus H_3$ such that $A_1 \leq tI \leq A_2 \leq (t+\varepsilon)I \leq A_3$. Let B be a operator with matrix $[B_{ij}]_{1 \leq i,j \leq 3}$ satisfying $\|[A, B]\| < \delta$. Suppose that D is a tridiagonal operator of the form*

$$D = \begin{pmatrix} D_{11} & B_{12} & 0 \\ B_{21} & B_{22} & B_{23} \\ 0 & B_{32} & D_{33} \end{pmatrix}.$$

Then there is a projection of the form $P = (0 \oplus I) \oplus P_2 \oplus (I \oplus 0)$ acting on $H_1^{(2)} \oplus H_2^{(2)} \oplus H_3^{(2)}$ with

$$P_2 = \begin{pmatrix} S^2 & SC \\ SC & C^2 \end{pmatrix}$$

where S and C are positive operators in $C^*(A_2)$ such that $S^2+C^2=I$ and $\|[B\oplus D, P]\| \leq 61\delta/\varepsilon$. Furthermore, if $[A,B]$ is compact, then $[B\oplus D, P]$ is compact.

If K_1 and K_3 are subspaces of H_1 and H_3, let D' be the compression of D to $K_1\oplus H_2\oplus K_3$ and let P' be the restriction of P to this subspace. Then $\|[B\oplus D', P']\| \leq 61\delta/\varepsilon$ and $[B\oplus D', P']$ is compact when $[B,A]$ is compact. In particular, this holds when $K_1=K_3=\{0\}$ and $D'=[B_{22}]$.

Proof. Let $X=\sin^2(\frac{1}{2}\pi\varepsilon^{-1}(A_0-t))$ and let $S=\sin(\frac{1}{2}\pi X)$ and $C=\cos(\frac{1}{2}\pi X)$. With P_2 and P defined as in the statement of the lemma, compute

$$[B\oplus D, P] = \begin{pmatrix} 0 & B_{12}^{(2)}(P_2-(0\oplus I)) & B_{13}\oplus 0 \\ ((0\oplus I)-P_2)B_{21}^{(2)} & [B_{22}^{(2)}, P_2] & ((I\oplus 0-P_2)B_{23}^{(2)} \\ -B_{31}\oplus 0 & B_{32}^{(2)}((I\oplus 0)-P_2) & 0 \end{pmatrix}.$$

Following the proof of Corollary 4.3 of [18], one obtains $\|B_{13}\|<\delta/\varepsilon$, $\|B_{31}\|<\delta/\varepsilon$ and $\|[B_{22}^{(2)}, P_2]\| \leq 2(1+e^{\pi})\,\delta/\varepsilon$. Also,

$$P_2-(0\oplus I) = \begin{pmatrix} S^2 & SC \\ SC & -S^2 \end{pmatrix} = \begin{pmatrix} S & 0 \\ 0 & S \end{pmatrix}\begin{pmatrix} S & C \\ C & -S \end{pmatrix}.$$

Hence,

$$\|B_{12}^{(2)}(P_2-(0\oplus I))\| \leq \|B_{12}S\| \left\| \begin{pmatrix} S & C \\ C & -S \end{pmatrix} \right\| \leq 2\pi\delta/\varepsilon.$$

Likewise, the 21, 23 and 32 entries are bounded by $2\pi\delta/\varepsilon$. Totalling yields the estimate $61\delta/\varepsilon$ for $\|[B\oplus D, P]\|$ as in [18].

When $[A,B]$ is compact, $[0\oplus S\oplus I, B]$ is compact since $0\oplus S\oplus I$ is a continuous function of A. Hence $B_{12}S$ and SB_{21} are compact. Similarly it follows that all the matrix entries of $[B\oplus D, P]$ are compact.

If Q is the projection onto $K_1\oplus H_2\oplus K_3$, then since $QP=PQ$, one has $[B\oplus D', P']=Q[B\oplus D, P]\,Q|QH$. So the results extend to these compressions. □

Remark 3.6. The main advantage of this lemma over the previous one is that the interchange can be made in one block provided the block is "parameterized" by the spectrum of a self-adjoint operator which almost commutes with B. Apply Lemma 3.5 with $D=B_{22}$ and let P be the projection obtained. Let B_+ and B_- be the compressions of $B\oplus B_{22}$ to $P\mathcal{H}$ and $P^{\perp}\mathcal{H}$ respectively. Then $\|B\oplus B_{22}-U(B_+\oplus B_-)U^*\| \leq 61\delta/\varepsilon$ where U is the natural unitary taking $P\mathcal{H}\oplus P^{\perp}\mathcal{H}$ onto $\mathcal{H}\oplus\mathcal{H}_0$ given by the matrix

$$\begin{pmatrix} I & 0 & 0 & 0 \\ 0 & C & -S & 0 \\ 0 & S & C & 0 \\ 0 & 0 & 0 & I \end{pmatrix}.$$

Furthermore, since P commutes with $A \oplus A_0$, one readily verifies that $(A \oplus A_0)|P\mathscr{H} \cong A_+ \oplus A_0$ and $A \oplus A_0|P\mathscr{H} \cong A_- \oplus A_0$ where these pairings are given by the same unitary U. Thus, $B \oplus B_{22}$ has been split into two summands corresponding to a split of $A \oplus A_0$ into two summands, one with spectrum in $(-\infty, t+\varepsilon]$ and the other in $[t, \infty)$. This will enable us to cut the spectrum of an operator with small self commutator.

This section will conclude with a proof of BDF for the disc which is independent of [12]. The proof relies heavily on the absorption principle of Theorem 1.5. In [18], Theorem 6.1 this is proved subject to the hypothesis that T is quasidiagonal. Quasidiagonality will be proved by using techniques developed in this section.

THEOREM 3.7. *Let T be essentially normal with $\sigma_e(T) = D = \{\lambda \in \mathbf{C} : |\lambda| \le R\}$. Then there is a compact operator K such that $T-K$ is normal. If $\|T\| = R$, then one may take $\|K\| \le 75 \|T^*T - TT^*\|^{1/2}$.*

Proof. Since $\|T\|_e = R$, there is a compact perturbation of T satisfying $\|T-K\| = R$. So it will be assumed that $\|T\| = R$. It suffices to prove that T is quasidiagonal. Make a (small) compact perturbation of T so that $T-K \cong T \oplus N$ where N is normal with spectrum D. Write $T = A + iB$ as the sum of its real and imaginary parts. Use Lemma 3.1 to make (small) compact perturbations of A and B so that with respect to a decomposition $\mathscr{H} = \sum_{k \ge 0} \oplus P_k \mathscr{H}$, A is block diagonal and B is tridiagonal.

Let B' be the tridiagonal operator with the same matrix entries as B except that $B'_{2^n, 2^n+1} = 0 = B'_{2^n+1, 2^n}$. Then $T' = A + iB'$ is block diagonal since $F_n = \sum_{k=0}^{2^n} P_k$ commutes with T' for every $n \ge 0$. So

$$[T'^*, T'] = 2i[A, B'] = 2i \sum_{n \ge 1} (F_n - F_{n-1}) [A, B] (F_n - F_{n-1}).$$

It follows that T' is essentially normal. By [18], Theorem 6.1, there is a compact operator K' so that $T_0 = T' - K'$ is normal.

Hence $T - K \cong T \oplus N \doteq T \oplus (T_0 \oplus N) \doteq T \oplus T_0$. So there is a compact perturbation of T unitarily equivalent to $T \oplus T'$. The two tridiagonal operators B and B' agree on $(F_{n+1} - F_n)\mathscr{H}$ which consists of 2^n blocks. By Lemma 3.2, there is a projection P_n such that $F_n \oplus 0 < P_n < F_{n+1} \oplus (F_{n+1} - F_n)$ such that $\|[B \oplus B', P_n]\| < 2^{-n}\pi$. The form of P_n given in the proof shows that P_n commutes with $A\oplus A$. Let $Q_n=P_n+(0\oplus F_n)$. This is a sequence of finite rank projections such that $\lim_{n\to\infty}\|[T\oplus T', Q_n]\|=0$ and Q_n tends to the identity in the strong operator topology. Thus $T\oplus T'$, and consequently T, is quasidiagonal. □

Remark 3.8. This proof works equally well if the spectrum of T is a rectangle. In fact, the constants are somewhat better because the constant in the absorption lemma obtained in [18] Theorem 4.4 for the rectangle case is $25\sqrt{2}<36$. The rectangle version will be used in section 5.

4. The annulus

The purpose of this section is to prove Theorem 1.2. We have described an estimate-free version of the proof in section 2. The reader should keep it in mind as we work through the technical details that allow a uniform estimate of the right order of magnitude. These estimates will also provide independent estimates for the disk using techniques somewhat different from those of [18].

LEMMA 4.0. *Let U be a tridiagonal, invertible operator with respect to $\mathcal{H}=\Sigma_{i=0}^{\infty}\oplus E_i$, where E_i are either all finite dimensional or all infinite dimensional. Then there is a tridiagonal, invertible operator V which leaves $S_k=\Sigma_{i=7k}^{7k+6}\oplus E_i$ invariant for $k\geq 0$ and agrees with U on $\Sigma_{i=7k+2}^{7k+4}\oplus E_i$ for $k\geq 0$. Moreover, if U is within ε of a unitary, so is V.*

Proof. We deal with the finite dimensional situation first. Set

$$M=U\sum_{i=0}^{5}\oplus E_i \quad\text{and}\quad N=U\sum_{i=6}^{\infty}\oplus E_i.$$

These subspaces are algebraic complements since U is invertible. Moreover, M is contained in S_0, and N is contained in $\Sigma_{i=5}^{\infty}\oplus E_i$. Let $N'=N\cap E_5\oplus E_6$. This is a complement to M in S_0. To see this, note that each vector x in S_0 decomposes as $x=m+n$ with $m\in M$ and $n\in N$. Since $M\in S_0$, one has n belonging to $N\cap S_0=N'$.

Set $L=U^{-1}N'$, and note that this is a subspace of $U^{-1}N=\Sigma_{i=6}^{\infty}\oplus E_i$. Now

$$\dim L=\dim N'=\dim S_0-\dim M=\dim S_0-\dim\sum_{i=0}^{5}\oplus E_i=\dim E_6.$$

Let J be an isometry of S_0 onto $\Sigma_{i=0}^{5}\oplus E_i\oplus L$ which is the identity on $\Sigma_{i=0}^{5}\oplus E_i$. Set $V|S_0=UJ|S_0$. Then V maps S_0 onto itself, agrees with U on $\Sigma_{i=0}^{5}\oplus E_i$, and maps E_6 onto

N'. Thus V is tridiagonal. Moreover, if U is within ε of unitary, so is $V|S_0$, since for unit x in S_0, recalling that for all unit x in $\mathscr{H}$ we have $|\,\|Ux\|-1|<\varepsilon$, we see that $|\,\|Vx\|-1|<\varepsilon$.

In the same way, we can modify U on E_{7k} and E_{7k+6} to obtain a tridiagonal invertible operator $V|S_k$ which agrees with U on $\Sigma_{i=7k+1}^{7k+5}\oplus E_i$.

We describe the construction for E_7. Let $\tilde{S}_0=\Sigma_{i=7}^{\infty}\oplus E_i$. Now let

$$M=U\sum_{i=8}^{\infty}\oplus E_i \quad\text{and}\quad N=U\sum_{i=0}^{7}\oplus E_i.$$

As before M and N are algebraic complements, $M\subset\tilde{S}_0$ and $N\subset\Sigma_{i=0}^{8}\oplus E_i$. Let $N'=N\cap E_7\oplus E_8=N\cap\tilde{S}_0$, a complement to M in $\tilde{S}_0$. Let $L=U^{-1}N'$ a subspace of $\Sigma_{i=0}^{8}\oplus E_i$. We now see $\dim L=\dim E_7$. To see this recall $U\Sigma_{i=0}^{7}\oplus E_i\subset\Sigma_{i=0}^{8}\oplus E_i$. But the projection of $U\Sigma_{i=0}^{7}\oplus E_i$ on $\Sigma_{i=0}^{6}\oplus E_i$ has the dimension of $\Sigma_{i=0}^{6}\oplus E_i$, and so

$$\dim N'=\dim\left(\left(U\sum_{i=0}^{7}\oplus E_i\right)\cap\left(\sum_{i=0}^{6}\oplus E_i\right)^{\perp}\right)=\dim U\sum_{i=0}^{7}\oplus E_i-\dim\sum_{i=0}^{6}\oplus E_i=\dim E_7.$$

Again let J be an isometry of $\tilde{S}_0$ onto $\Sigma_{i=8}^{\infty}\oplus E_i\oplus L$ which is the identity on $\Sigma_{i=8}^{\infty}\oplus E_i$. Set $V|\tilde{S}_0=UJ|\tilde{S}_0$. Then V maps $\tilde{S}_0$ onto itself, agrees with U on $\Sigma_{i=8}^{\infty}\oplus E_i$ and maps E_7 onto N'. If U is within ε of unitary so is $V|\tilde{S}_0$.

Since the correction to U made for E_6 and that for E_7 had both orthogonal ranges and orthogonal domains, V is within ε of unitary on U and has S_0 as a direct summand. We now continue on E_{7+6} and so on in the same manner.

When E_i are infinite, we proceed as above with

$$M=U\sum_{i=0}^{4}\oplus E_i,\quad N=U\sum_{i=5}^{\infty}\oplus E_i\quad\text{and}\quad N'=N\cap S_0.$$

Again N' is a complement of M in S_0. The extra blocks ensure that N' is infinite dimensional. As above, take an isometry J of $E_5\oplus E_6$ onto $L=U^{-1}N'$, with the further stipulation that JE_6 is contained in $U^{-1}(N\cap E_5\oplus E_6)$ which is infinite dimensional. Then $V|S_0=UJ$ will indeed be tridiagonal. The rest of the argument is the same. □

LEMMA 4.1. *Let T be an invertible operator on $\mathscr{H}$ (of finite or infinite dimension). Suppose that $\|T\|\leqslant R$ and $\|T^{-1}\|\leqslant 1$. Then there is a normal operator N on $\mathscr{H}$ with $\|N\|\leqslant R$ and $\|N^{-1}\|\leqslant 1$ so that*

$$\operatorname{dist}((T\oplus N),\operatorname{Nor}(\mathscr{H}\oplus\mathscr{H}))\leqslant 95\|T^*T-TT^*\|^{1/2}.$$

Proof. Let $\varepsilon=\|T^*T-TT^*\|^{1/2}$. If $\varepsilon\geq 0.05$, apply Theorem 1.5 to obtain a normal N_0 with $\|N_0\|\leq R$ and $\operatorname{dist}(T\oplus N_0, \mathrm{Nor})\leq 75\varepsilon$. There is a normal N such that $\|N\|\leq R$, $\|N^{-1}\|\leq 1$, and $\|N-N_0\|\leq 1\leq 20\varepsilon$. So $\operatorname{dist}(T\oplus N, \mathrm{Nor})\leq 95\varepsilon$.

Let us assume $\varepsilon<0.05$.

Let $T=UP$ be the polar decomposition of T. In case $\mathscr{H}$ is infinite dimensional, we require that the essential spectrum of P be $[1,R]$. This is easily achieved by replacing T by $T\oplus N$ where N is normal and has spectrum the whole annulus. This doesn't affect the norm of the self commutator. Let $a_n=1+4n\varepsilon$ for $n\geq 0$, and define $E_n=E_P[a_n, a_{n+1})$. With respect to this decomposition, $P^2=\Sigma_{n\geq 0}\oplus P_n^2$ and $U=[U_{ij}]$. Let U' be the tridiagonal part of U. Then let $T'=U'P$ be the tridiagonal part of T. Since $\|P^2U-UP^2\|=\varepsilon^2$, and since $a_{n+1}^2-a_n^2=4\varepsilon(a_{n+1}+a_n)>8a_n\varepsilon>8\varepsilon$, it follows from Lemma 3.6 that $\|U-U'\|\leq\frac{3}{2}\varepsilon$. It will be established that $\|T-T'\|\leq\frac{11}{2}\varepsilon$ as well.

Group together strings of blocks corresponding to the spectral projections $F_k=E_p[2^k, 2^{k+1}), k\geq 0$. First tridiagonalize U with respect to this coarse decomposition to obtain U''. Since $\|UP^2-P^2U\|=\varepsilon^2$ and the spectral gaps are $3\cdot 2^{2k}$, the total perturbation of U is a compact operator with norm at most

$$\|U-U''\|\leq\sum_{k=0}^{\infty}\|(U-U'')F_k\|\leq\frac{\varepsilon^2}{3\cdot 4}+\frac{\varepsilon^2}{3\cdot 16}+\sum_{k=2}^{\infty}\frac{\varepsilon^2}{3\cdot 4^{k-1}}<\frac{2\varepsilon^2}{9}.$$

Moreover, the total perturbation of T to $T''=U''P$ is a compact operator of norm at most

$$\|T-T''\|\leq\sum_{k=0}^{\infty}\|(U-U'')PF_k\|\leq\frac{\varepsilon^2}{3\cdot 4}\cdot 2+\frac{\varepsilon^2}{3\cdot 16}\cdot 4+\sum_{k=2}^{\infty}\frac{\varepsilon^2}{3\cdot 4^{k-1}}\cdot 2^{k+1}<\varepsilon^2.$$

Now consider $T''F_k$ and its factorization $(U''F_k)(PF_k)$, and tridiagonalize this with respect to the finer partition into intervals $E_P(a_n, a_{n+1})$, where $2^k\leq a_n<2^{k+1}$. The spectral gap for P^2 is at least $(2^k+4\varepsilon)^2-2^{2k}>2^{k+3}\varepsilon$. Since $U''F_k=(F_{k-1}+F_k+F_{k+1})\,UF_k$, we see that $\|[U''F_k, P^2]\|\leq\|[U,P^2]\|=\varepsilon^2$. So by Lemma 3.4, the tridiagonal part U' of U (or U') satisfies

$$\|(U'-U'')F_k\|<\frac{12\varepsilon^2}{2^{k+3}\varepsilon}=3\cdot 2^{-k-1}\varepsilon$$

and with $T'=U'P$,

$$\|(T'-T'')F_k\|\leq\|(U'-U'')F_k\|\,2^{k+1}<3\varepsilon.$$

As $(T'-T'')F_k=(F_{k-1}+F_k+F_{k+1})(T'-T'')F_k$, every third term has orthogonal range as well domain. So summing over all $k=i\ (\mathrm{mod}\,3)$, $i=0,1,2$ yields an operator of norm at most 3ε. These three terms have orthogonal domains, hence their sum has norm at most $3\sqrt{3}\,\varepsilon$. Hence $\|T-T''\|<3\sqrt{3}\,\varepsilon+\varepsilon^2<\frac{11}{2}\varepsilon$.

Also notice that $\|[U',P^2]\|\leq 2\varepsilon^2$ for this is the tridiagonalization of $[U,P^2]$. The obvious estimate is $3\varepsilon^2$ but we obtained $2\varepsilon^2$ by noting that a tridiagonal breaks up as the sum of disjoint 2×2 blocks (norm ε^2), and that of the remaining off diagonal terms which have orthogonal domains and ranges (norm ε^2). This refined estimate was pointed out to us by M. D. Choi.

Split the blocks into strings $\mathcal{S}_k$ of length seven. We apply Lemma 4.0 to obtain an (almost) unitary V' which leaves each $\mathcal{S}_k$ invariant, is tridiagonal, and agrees with U' on the middle three blocks of each $\mathcal{S}_k$.

Now we can intertwine U' and V' on the middle block of each string $\mathcal{S}_k$ using Lemma 3.5 and Remark 3.6. Because V' is reduced by each $\mathcal{S}_k$, the perturbation W' is decomposed as a direct sum $W'=\Sigma_{k\geq 0}\oplus W_k'$ where W_k' is supported on a subspace of $\mathcal{S}_{k-1}\oplus\mathcal{S}_k$. The perturbation required is estimated by Lemma 3.5. The spectral gap of the middle block of $\mathcal{S}_k$ is

$$(a_{7k+2}+4\varepsilon)^2-a_{7k+2}^2>8a_{7k+2}\,\varepsilon.$$

The commutator is at most $2\varepsilon^2$, so

$$\|W'-U'\oplus V'|\mathcal{S}_k\|<\frac{61\cdot 2\varepsilon^2}{8a_{7k+2}\,\varepsilon}=\frac{61\varepsilon}{4a_{7k+2}}.$$

Now we can define the normal operators N and M, which are going to be roughly $V'P$ and $W'P^{(2)}$ respectively. Let P_0 be the positive operator which on each string $\mathcal{S}_k$ is a scalar equal to the midpoint of the spectral interval for $\mathcal{S}_k$. (Precisely, if D_k is the projection onto $\mathcal{S}_k$ and $\lambda_k=1+(28k+14)\,\varepsilon$ is the midpoint, then $P_0=\Sigma_{k\geq 0}\lambda_k D_k$.) Clearly, $\|P_0-P\|\leq 14\varepsilon$. Let V be the unitary in the polar decomposition of V'. Now by Lemma 3.4, U' differs from being isometric on $\mathcal{S}_k$ by at most

$$\frac{12\varepsilon^2}{8\varepsilon a_{7k}}=\frac{3}{2}\varepsilon a_{7k}^{-1}.$$

Hence $V'|\mathcal{S}_k$ differs from $V|\mathcal{S}_k$ by this amount. Now $N=VP_0$ is normal, and

$$\begin{aligned}\|N-V'P\|&\leq\|(V-V')P_0\|+\|P_0-P\|\\&<\max_k\frac{3}{2}\frac{\varepsilon}{a_{7k}}\cdot(a_{7k}+14\varepsilon)+14\varepsilon<\frac{31}{2}\varepsilon+21\varepsilon^2.\end{aligned}$$

The normal M is defined in the same manner; W is the unitary part of W' and P_1 is a positive operator which is constant on each summand of W with scalar value a_{7k}. As this kth summand is supported in $\mathcal{S}_{k-1}\oplus\mathcal{S}_k$, one has $\|P\oplus P-P_1\|\leq 28\varepsilon$. Set $M=WP_1$. Now W' is bounded below on $\mathcal{S}_k$ by the same $1-3\varepsilon/2a_{7k}$ estimate as U' and V'. So $M-W'(P\oplus P)$ is computed by maximizing over these summands:

$$\begin{aligned}\|M-W'(P\oplus P)\| &\leq \max_{k\geq 1}\|(W_k-W_k')P_1\|+\|P_1-P\oplus P\|\\ &= \max_{k\geq 1}\frac{3\varepsilon}{2a_{7k-7}}a_{7k}+28\varepsilon<29\frac{1}{2}\varepsilon+42\varepsilon^2.\end{aligned}$$

Now, since $\|U'\oplus V'-W'|\mathcal{S}_k\|<61\varepsilon(4a_{7k+2})^{-1}$, one gets

$$\|(U'\oplus V'-W')(P\oplus P)|\mathcal{S}_k\|<\frac{61\varepsilon}{4a_{7k+2}}a_{7k+7}\leq\frac{61}{4}\varepsilon+305\varepsilon^2.$$

The intertwining of U' and V' to obtain W' occurs on the middle blocks of each $\mathcal{S}_k$. Hence $U'\oplus V'-W'$ is the orthogonal direct sum of $(V'\oplus V'-W')|\mathcal{S}_k$. Since $P\oplus P$ also leaves $\mathcal{S}_k$ invariant we obtain

$$\|(U'\oplus V'-W')(P\oplus P)\|\leq\frac{61}{4}\varepsilon+305\varepsilon^2.$$

So

$$\begin{aligned}\|T\oplus N-M\| &\leq \|T\oplus N-T'\oplus V'P\|+\|(U'\oplus V'-W')(P\oplus P)\|+\|W'(P\oplus P)-M\|\\ &< \max\left\{\frac{11}{2}\varepsilon,\frac{31}{2}\varepsilon+21\varepsilon^2\right\}+\left(\frac{61}{4}\varepsilon+305\varepsilon^2\right)+\left(29\frac{1}{2}\varepsilon+42\varepsilon^2\right)\\ &< 61\varepsilon+368\varepsilon^2\leq 80\varepsilon.\end{aligned}$$

□

Remark 4.2. We call attention to the effect of the zero index. One might ask if U being unitary, as opposed to merely isometric, is essential here. Yes it is. The very first computation relating $[T^*,T]$ and $[U,P^2]$ relies on this. Furthermore, if $\operatorname{ind} T\neq 0$ and $[T^*,T]$ is small, then P cannot be bounded below by one. Indeed, $P\geq I$ forces $\|[T^*,T]\|\geq 1$ in this case.

COROLLARY 4.2. *Let T be an invertible operator on $\mathcal{H}$ such that $\|T\|\leq R_2$ and $\|T^{-1}\|\leq R_1^{-1}$. Let N be normal with spectrum $A=\{\lambda\in\mathbf{C}: R_1\leq|\lambda|\leq R_2\}$. Then*

$$\operatorname{dist}(T\oplus N,\operatorname{Nor})\leq 95\|T^*T-TT^*\|^{1/2}.$$

Proof. The main point is that $\|T^*T-TT^*\|^{1/2}$ is homogeneous of order one. So T can be scaled so that $R_1=1$. Hence Lemma 4.1 applies. The other point is that the extended Weyl–von Neumann theorem [3] guarantees that any normal with spectrum A will suffice. □

THEOREM 4.3. *Let T be an invertible operator on $\mathcal{H}$ such that $\|T\|\le R_2$, $\|T^{-1}\|\le R_1^{-1}$, and $\sigma_e(T)=A=\{\lambda\in\mathbf{C}: R_1\le|\lambda|\le R_2\}$. Then there is a normal N with spectrum A such that*

$$\|T-N\|\le 100\|T^*T-TT^*\|^{1/2}.$$

Proof. Let $\varepsilon>\|T^*T-TT^*\|^{1/2}$. For each λ in A, either $T-\lambda$ or $(T-\lambda)^*$ is not bounded below or has infinite dimensional kernel. Following [12, Lemma 2.1], it is possible, for $\{\lambda_n\}$ dense in A, to extract an orthonormal sequence e_n so that (i) the projection P onto $\operatorname{span}\{e_n, n\ge 1\}$ satisfies $\|[T,P]\|\le\varepsilon$, and (ii) the normal operator given by $Ne_n=\lambda_n e_n$ on $P\mathcal{H}$ satisfies $\|PT|P\mathcal{H}-N\|\le\varepsilon$. Indeed, we simply choose our e_n so that either $(T-\lambda_n)e_n$ or $(T-\lambda_n)^*e_n$ is as small as desired and so that $\{e_n\}$ is not just orthonormal but so that for $n\ne m$ we have $\langle Te_n, Te_m\rangle$, $\langle Te_n, T^*e_m\rangle$ and $\langle T^*e_n, T^*e_m\rangle$ similarly as small as desired. Thus there is an operator T' of the form $T'=S\oplus N$ with $\|T'-T\|\le 2\varepsilon$. But T' is unitarily equivalent to $S\oplus N\oplus N=T'\oplus N$. Thus there is an operator T'' unitarily equivalent to $T\oplus N$ such that $\|T-T''\|\le 4\varepsilon$. By Corollary 4.2, $\operatorname{dist}(T'', \mathrm{Nor})\le 95\varepsilon$. Hence there is a normal M so that

$$\|T-M\|<\|T-T''\|+\|T''-M\|<100\varepsilon. \qquad \square$$

THEOREM 4.4. *Let T be a quasidiagonal, essentially normal operator such that $\|T\|=R_2$, $\|T^{-1}\|=R_1^{-1}$, and $\sigma_e(T)=A=\{\lambda: R_1\le|\lambda|\le R_2\}$. Then there is a compact operator K such that $T-K$ is normal with spectrum A and $\|K\|\le 100\|T^*T-TT^*\|^{1/2}$.*

Proof. Given any $\delta>0$, there is a compact operator K_1 with $\|K_1\|<\delta$ so that $T-K_1$ is unitarily equivalent to $T\oplus N$ where N is normal and has spectrum A [12, Lemma 2.2]. Since T is quasidiagonal, there is another compact operator K_2 with $\|K_2\|<\delta$ so that

$$T' = T-K_1-K_2=\sum_{n=1}^{\infty}\oplus T_n\oplus N$$

where each T_n acts on a finite dimensional space, and satisfies $\|T_n\|\le R_2$, $\|T_n^{-1}\|\le R_1^{-1}$, and $\|[T_n^*, T_n]\|^{1/2}\le\|[T^*,T]\|^{1/2}+\delta=\varepsilon+\delta$. Furthermore, $\lim_{n\to\infty}\|[T_n^*, T_n]\|=0$ since $[T'^*, T']$ is compact.

By Lemma 4.1, there are finite rank normal matrices N_n and M_n so that

$T_n \oplus N_n - M_n = K'_n$ satisfies $\|K'_n\| \leqslant 100 \|[T_n^*, T_n]\|^{1/2}$. By the extended Weyl–von Neumann theorem,

$$N \cong \left(\sum_{n=1}^{\infty} \oplus N_n \right) \oplus N + K_3$$

where $\|K_3\| < \delta$, and K_3 is compact. So

$$\begin{aligned} T' - (0 \oplus K_3) &\cong \sum_{n=1}^{\infty} \oplus (T_n \oplus N_n) \oplus N \\ &= \left(\sum_{n=1}^{\infty} \oplus M_n \oplus N \right) + \left(\sum_{n=1}^{\infty} \oplus K'_n \oplus 0 \right). \end{aligned}$$

Let $K_4 = \Sigma_{n=1}^{\infty} \oplus K'_n \oplus 0$. Then there is a compact operator K so that $T - K$ is normal with spectrum A, and

$$\begin{aligned} \|K\| &\leqslant \|K_1\| + \|K_2\| + \|K_3\| + \|K_4\| \\ &\leqslant 3\delta + 95 \max \|[T_n^*, T_n]\|^{1/2} < 95\varepsilon + 95\delta < 100\varepsilon, \end{aligned}$$

provided δ is chosen sufficiently small. □

Now we show that the zero index data guarantees quasidiagonality.

THEOREM 4.5. *Let T be an essentially normal operator with $\sigma_e(T) = A = \{\lambda \in \mathbf{C} : 1 \leqslant |\lambda| \leqslant R\}$ and* $\operatorname{ind} T = 0$. *Then T is quasidiagonal.*

Proof. We proceed as outlined in section 2, merely providing more detail. The polar decomposition is $T = UP$, and the hypotheses guarantee that $\operatorname{ind} U = 0$ and $\sigma_e(P) = [1, R]$. So by a compact perturbation of T, one can arrange that U is unitary and $I \leqslant P \leqslant RI$. Moreover, the generalized Weyl–von Neumann theorem allows us also (with this compact perturbation) to assume that P is diagonalized with respect to a basis $\{e_n, n \geqslant 1\}$. Let $\varepsilon > 0$ be given. Split this basis into finite dimensional blocks $\mathcal{H}_k = \operatorname{span}\{e_i : n_{k-1} < i \leqslant n_k\}$ so that the tridiagonal part U_0 of U differs from U by a compact operator of norm at most ε. With respect to this block decomposition, write $U_0 = (U_{ij})$ and $P = \operatorname{diag}(D_j)$. Let P_n be the projection onto the sum of the first n blocks.

The commutator $[P^2, U_0] = (D_i^2 U_{ij} - U_{ij} D_j^2)$ is compact. So one can choose n so large that $\|P_n^{\perp} [P^2, U_0]\| < \varepsilon$ and $n^{-1} < \varepsilon$. Let $Q = \operatorname{diag}(E_j)$ where $E_j = I$ for $1 \leqslant j \leqslant n$,

$E_j=(t_jI+(1-t_j)D_j^2)^{1/2}$ for $n<j<2n$, where $t_j=2-j/n$, and $E_j=D_j$ for $j\geq 2n$. Then $Q-P$ is compact, and

$$\begin{aligned}[Q^2, U_0] &= (E_i^2 U_{ij}-U_{ij}E_j^2)\\ &= P_{2n}^{\perp}[P^2, U_0]+(P_{2n}-P_n)\left((1-t_i)(D_i^2 U_{ij}-U_{ij}D_j^2)+\frac{i-j}{n}U_{ij}D_j^2\right).\end{aligned}$$

Thus

$$\begin{aligned}\|[Q^2, U]\| &\leq 2R^2\|U-U_0\|+\|P_n^{\perp}[P^2, U_0]\|+\frac{2}{n}\|P^2\|\\ &< 2R^2\varepsilon+\varepsilon+2R^2\varepsilon = C\varepsilon.\end{aligned}$$

That is, Q tapers so slowly from the identity to P that its commutator with U is locally no worse than that of P for large coordinates and almost 0 for all earlier coordinates. Hence $T_\varepsilon=UQ$ is essentially normal with $\|T_\varepsilon\|\leq R$, $\|T_\varepsilon^{-1}\|\leq 1$ and $\|[T_\varepsilon^*, T_\varepsilon]\|<C\varepsilon$. By Theorem 4.3, $\operatorname{dist}(T_\varepsilon, \mathrm{Nor})$ tends to zero as ε tends to 0. But each T_ε differs from T by a compact operator, whence T is the limit of operators which are normal plus compact. Such operators are quasidiagonal, so T is also quasidiagonal. □

THEOREM 4.6. *Let T be an essentially normal operator with $\sigma_e(T)=A=\{\lambda\in\mathbf{C}: R_1\leq|\lambda|\leq R_2\}$ and* $\operatorname{ind} T=0$. *Then there is a compact operator K such that $T-K$ is normal. If, furthermore, $\|T\|=R_2$ and $\|T^{-1}\|=R_1^{-1}$, then one can arrange that*

$$\|K\|\leq 100\,\|[T^*, T]\|^{1/2}.$$

Proof. Let $T=UP$ be the polar decomposition. Since $\operatorname{ind} T=0$, we may assume that U is modified to be unitary. The essential spectrum $\sigma_e(P)$ equals $[R_1, R_2]$. Thus there is a compact, self-adjoint operator C so that $\sigma(P-C)$ equals $[R_1, R_2]$. Hence $T'=U(P-C)$ is a compact perturbation of T satisfying $\|T'\|\leq R_2$ and $\|(T')^{-1}\|=R_1^{-1}$. By Theorem 4.5, T' is quasidiagonal. Hence Theorem 4.4 applies. □

This concludes the proof of BDF for the annulus. Since $\operatorname{Ext}(X)$ is a homeomorphism invariant, the case of any region homeomorphic to an annulus follows. We include it for the reader's convenience.

COROLLARY 4.7. *Let T be essentially normal with $\sigma_e(T)$ homeomorphic to an annulus, and* $\operatorname{ind}(T-\lambda I)=0$ *for λ in $\mathbf{C}\setminus\sigma_e(T)$. Then T is normal plus compact.*

Proof. Let h be a homeomorphism of $\sigma_e(T)$ onto an annulus A. Then $h(\pi(T))$ is a normal element of the Calkin algebra with spectrum A and index zero. By Theorem 4.6, $h(\pi(T))=\pi(N)$ for some normal operator N with spectrum A. Hence

$$\pi(T)=h^{-1}(\pi(N))=\pi(h^{-1}(N)).$$

Thus $h^{-1}(N)$ is the required normal, and $K=T-h^{-1}(N)$ is compact. □

Remark 4.8. Already, the ideal control on K appears to be lost. Even the stringent condition that $\sigma_e(T)$ be a spectral set for T seems as if it may not be enough. Corollary 4.7 could be pushed further, but the proofs would be the same as the general arguments to follow.

5. The general case

The second key step in the proof is to apply the techniques of section 3 to "cutting" the spectrum.

LEMMA 5.1 (the Cutting lemma). *Suppose T is essentially normal, $\|\operatorname{Im} T\|=1$, and $\sigma_e(T)$ contains $R_\delta=\{\alpha+i\beta: 0\leqslant\alpha\leqslant\delta$ and $|\beta|\leqslant 1\}$. Let $\varepsilon=\|T^*T-TT^*\|^{1/2}$. Then there are essentially normal operators T_1 and T_2 such that* $\operatorname{Re} T_1\leqslant\delta$, $\operatorname{Re} T_2\geqslant 0$, *and $T\cong T_1\oplus T_2+K$ where K is compact and $\|K\|\leqslant 67\max\{\varepsilon,\varepsilon^2/\delta\}$. Furthermore, $\sigma_e(T_1)=\{\lambda\in\sigma_e(T)\colon \operatorname{Re}\lambda\leqslant\delta\}$ and $\sigma_e(T_2)=\{\lambda\in\sigma_e(\tau)\colon \operatorname{Re}\lambda\geqslant 0\}$.*

Proof. Let $\eta=\min\{\delta,\varepsilon\}$. Write $T=A+iB$ where $A=\operatorname{Re} T$ and $B=\operatorname{Im} T$. Let $E_{-1}=E_A(-\infty,0)$, $E_0=E_A[0,\eta)$, and $E_1=E_A[\eta,\infty)$. Then with respect to the decomposition $\mathcal{H}=\sum_{i=-1}^{1}\oplus\mathcal{H}_i$ where $\mathcal{H}_i=E_i\mathcal{H}$, one has $A=\sum_{i=-1}^{1}\oplus A_i$ and B has a 3×3 matrix form (B_{ij}).

Let N_0 be a normal operator with spectrum

$$R_\eta=\{\alpha+i\beta: 0\leqslant\alpha\leqslant\eta, |\beta|\leqslant 1\}.$$

Let $N_1=(A_0+iB_{00})\oplus N_0^{(2)}$. This is an essentially normal operator with spectrum R_η, and $\|[N_1^*,N_1]=\|E_0[T^*,T]E_0\|\leqslant\varepsilon^2$. By Theorem 3.7 and the subsequent remark, it follows that there is a compact operator K_1 with $\|K_1\|<36\varepsilon$ such that $N_2=N_1-K_1$ is normal with spectrum R_η.

By the absorption principle (Lemma 2.1), T has a small compact perturbation unitarily equivalent to $T\oplus N_2$. Thus there is a compact operator K_2 with $\|K_2\|<36\varepsilon$ such that $T-K_2\cong T\oplus N_1$. This acts on $(\mathcal{H}_{-1}\oplus\mathcal{H}_0\oplus\mathcal{H}_1)\oplus(\mathcal{H}_0\oplus\mathcal{H}^{(2)})$ which can be rearranged as $(\mathcal{H}_{-1}\oplus\mathcal{H}_0^{(2)}\oplus\mathcal{H}_1)\oplus\mathcal{H}\oplus\mathcal{H}$. With respect to this decomposition

$$T\oplus N_1\cong\left[\begin{bmatrix} A_{-1} & 0 & 0\\ 0 & A_0^{(2)} & 0\\ 0 & 0 & A_1\end{bmatrix}+i\begin{bmatrix} B_{-1-1} & [B_{-10}\,0] & B_{-11}\\ \begin{bmatrix}B_{0-1}\\ 0\end{bmatrix} & B_{00}^{(2)} & \begin{bmatrix}B_{01}\\ 0\end{bmatrix}\\ B_{1-1} & [B_{10}\,0] & B_{11}\end{bmatrix}\right]\oplus N_0\oplus N_0.$$

Let P be the projection given by $P=(0\oplus P_0\oplus I)\oplus I\oplus 0$ where

$$P_0=\begin{pmatrix} \sin^2\frac{\pi}{2}D_0 & \sin\frac{\pi}{2}D_0\cos\frac{\pi}{2}D_0\\ \sin\frac{\pi}{2}D_0\cos\frac{\pi}{2}D_0 & \cos^2\frac{\pi}{2}D_0\end{pmatrix}\quad\text{and}\quad D_0=\sin^2\frac{\pi}{2\eta}A_0.$$

By Lemma 3.4, $\|[T\oplus N_1,P]\|<61\varepsilon^2/2\eta$ (since $\|[\operatorname{Re}T,\operatorname{Im}T]\|=\frac{1}{2}\|[T^*,T]\|$). Let $\tilde{P}$ be the projection on $\mathcal{H}$ obtained from the unitary identifying $T\oplus N_2$ and T. Then $[T,\tilde{P}]$ is compact by Lemma 3.5, and

$$\begin{aligned}\|[T,\tilde{P}]\|&\le\|[T\oplus N_1,P]\|+\|[K_2,\tilde{P}]\|\\ &<61\varepsilon^2/2\eta+36\varepsilon<67\max\{\varepsilon,\varepsilon^2/\eta\}.\end{aligned}$$

Let $T_2=\tilde{P}T|\tilde{P}\mathcal{H}$ and $T_1=\tilde{P}^\perp T|\tilde{P}^\perp\mathcal{H}$. Then $T-T_1\oplus T_2$ is compact of norm $\|[T,\tilde{P}]\|$. From the construction,

$$\operatorname{Re}T_1\le A_{-1}\oplus A_0^{(2)}\oplus\operatorname{Re}N_0\le\eta I$$

and similarly, $\operatorname{Re}T_2\ge 0$. Clearly, $\sigma_e(T)$ is the union of $\sigma_e(T_i)$. As N_0 is a direct summand of both T_i, R_η is contained in both of $\sigma_e(T_i)$. So T_1 and T_2 have the desired properties. □

If $\sigma(T)=\sigma_e(T)$, this argument does not show that $\sigma(T_i)=\sigma_e(T_i)$. However, because the perturbation is small, this cannot introduce isolated eigenvalues far from $\sigma_e(T)$. Hence Apostol's theorem [1] allows us to make another small perturbation to arrange $\sigma(T_i)=\sigma_e(T_i)$ if desired.

The following lemma is proved the same way.

LEMMA 5.2 (Cutting lemma II). *Suppose T is a matrix with $\|\operatorname{Im}T\|\le 1$ and $\|T^*T-TT^*\|^{1/2}=\varepsilon$. Then there is a normal matrix N with $\sigma(N)$ contained in*

$R_\delta=\{\alpha+i\beta: 0\leqslant\alpha\leqslant\delta, |\beta|\leqslant 1\}$ and matrices T_1 and T_2 with $\operatorname{Re} T_1\leqslant\delta I$ *and* $\operatorname{Re} T_2\geqslant 0$ *and a unitary U such that*

$$\|T\oplus N-U(T_1\oplus T_2)U^*\|\leqslant 67\max\{\varepsilon, \varepsilon^2/\delta\}.$$

We observe that we could have still a third cutting lemma using annuli for exchange and so cutting the spectrum into a component inside a disk and a component outside a slightly smaller disk. We could use this split to deal with disconnected components of essential spectrum, but we have chosen to use a theorem of Apostol [1] which allows us to find corresponding disconnected components of the spectrum itself, instead. Since all holes of index 0 can be removed by small compact perturbations [1], we can assume there are no such holes. So for essentially normal operators with zero index data we can assume components of $\sigma_e(T)$ are components of $\sigma(T)$. Hence they can split out by circuit integrals using the Riesz functional calculus and dealt with independently. Thus we will assume all our sets in $\sigma_e(T)$ are connected for our proofs.

We are now ready to obtain quantitative versions of BDF. The remaining problem is that of reducing the pathology of the spectrum so as to be amenable to attack with the few tools we have developed. This is conceptually easier than we might guess. First, in Theorem 5.3, we show that an essentially normal operator with no index obstructions is quasidiagonal. Because this is a norm closed property only gross properties such as large holes could offer difficulties and we can handle those. Recall that for this result constants are irrelevant.

Now we are confronted with an infinite set of finite blocks to be subjected to normal dilations and here we make the general observation that the principal role the spectrum plays is to provide us with eigenvectors for an almost matching normal operator suitable for direct summing on each finite dimensional block. (This is, incidentally, one source of difficulty in the case of a finite dimensional space. Whence come the summands?). So, if a fatter spectrum provides us with a suitable normal summand on one of our finite blocks, the genuine spectrum provides the same summand with a finite dimensional error bounded by the difference in the spectra.

For a nice spectrum close to the genuine spectrum, a conformal mapping to a spectrum amenable to our techniques produces an arbitrarily small error ε for all blocks sufficiently far out. But once we have obtained our ε error on a block we have that ε plus the spectral distances forever, no matter how bad the conformal estimates are for later, tighter approximations of the spectrum. Hence the succession of conformal maps, each with small error only for very large index blocks, actually provides a uniformly small compact error, yielding our quantitative BDF theorem.

THEOREM 5.3. *Let T be an essentially normal operator such that* $\operatorname{ind}(T-\lambda)=0$ *for all* $\lambda\notin\sigma_e(T)$. *Then T is the limit of operators in* $\mathrm{Nor}+\mathcal{K}$, *and hence is quasi-diagonal.*

Proof. By Apostol's theorem [1], we can improve T by a compact perturbation so that $\sigma(T)=\sigma_e(T)$. Let $\varepsilon>0$ be given. Let $X=\sigma_e(T)$ and let X_ε be a finite union of finitely connected regions each with smooth boundary consisting of disjoint Jordan curves such that X is contained in the interior of X_ε, and X_ε is contained in $\{\lambda: \operatorname{dist}(\lambda, X)<\varepsilon\}$. Let N be a normal operator with $\sigma(N)=\sigma_e(N)=X$; and let N_ε be a normal operator with $\sigma(N_\varepsilon)=\sigma_e(N_\varepsilon)=X_\varepsilon$ such that $\|N-N_\varepsilon\|<\varepsilon$. By the absorption principle, there is a small compact operator K and a unitary operator U such that $T=U(T\oplus N)U^*+K$. Thus

$$\|T-U(T\oplus N_\varepsilon)U^*\|\leq\|N-N_\varepsilon\|+\|K\|\leq\varepsilon$$

for appropriate choice of K. Hence T is the limit of operators T_ε unitarily equivalent to $T\oplus N_\varepsilon$. The operator T_ε is essentially normal with "nice" spectrum X_ε, and $\operatorname{ind}(T_\varepsilon-\lambda)=0$ for $\lambda\notin X_\varepsilon$. Thus it suffices to prove the theorem for "nice" spectra. As we have remarked, we can assume a nice spectrum has one component.

Let T be essentially normal with nice essential spectrum X, and let $t=\pi T$. Let h be a homeomorphism of X onto a rectangle with parallel rectangular holes R. (For example, if X has p holes, take

$$R=\{(x,y): 0\leq x\leq p, |y|\leq 1\}\setminus\bigcup_{j=1}^{p}\{(x,y): |x-(j-1/2)|<1/4, |y|<1/2\}.)$$

Then $s=h(t)$ is essentially normal with spectrum R and $\operatorname{ind}(s-\lambda)=0$ for $\lambda\notin R$. Let S be any operator such that $\pi(S)=s$ and $\|\operatorname{Im} S\|=\|\operatorname{Im} s\|$.

Apply Lemma 5.1 to S $(n-1)$ times to obtain a compact perturbation of S unitarily equivalent to $\Sigma_{j=1}^{p}\oplus S_j$ where each S_j is essentially normal with spectrum equal to a rectangle with a rectangular hole. By Corollary 4.7, each S_j is normal plus compact. Thus, there is a normal operator M such that $\pi(M)=s$ and $\sigma(M)=\sigma_e(M)=R$. Then $h^{-1}(M)$ is normal and $\pi(h^{-1}(M))=h^{-1}(s)=t=\pi T$. Thus T is a normal plus compact operator. □

To prove the BDF theorem, it is necessary to examine how the preceding proof works on individual summands of T. This has to be done with operators, not in the Calkin algebra. So the homeomorphism h will be taken to be analytic in a neighborhood of the spectrum so that the Riesz functional calculus is available.

5.4. *Proof of Theorem* 1.1 (BDF). Let T be essentially normal with $\sigma_e(T)=X$ and $\operatorname{ind}(T-\lambda)=0$ for all $\lambda\notin X$. By the previous theorem, T is quasidiagonal. Hence there is a (small) compact perturbation K so that

$$T'\cong T-K=\sum_{n=1}^{\infty}\oplus T_n\oplus N$$

where T_n act on finite dimensional spaces $\mathcal{H}_n$, and N is normal with $\sigma(N)=\sigma_e(N)=X$. Let X_ε be a nice region containing X as in the previous proof. Then we can further take X_ε so that the interior of X_ε is conformally equivalent to the interior of a rectangle R with n parallel slits. (As conceivable worst $X_\varepsilon\setminus X$ has a small hole which maps to the infinite component of R complement.) Let f be the conformal map. Choose a region S such that $f(X)\subset S\subset\bar{S}\subset R$ and so that $\bar{S}$ is a rectangle with p parallel rectangles removed. Let $Y_\varepsilon=f^{-1}(\bar{S})$. Let N_ε be a normal operator such that $\sigma(N_\varepsilon)=\sigma_e(N_\varepsilon)=Y_\varepsilon$.

The spectrum of T_n is contained in Y_ε except for finitely many n; otherwise we would have a cluster point of eigenvalues for orthogonal eigenvectors, hence a point of essential spectrum outside X. Extend f arbitrarily to a neighborhood of $Y_\varepsilon\cup\bigcup_{n\geq 1}\sigma(T_n)$ so that f takes $\sigma(T')$ into S. Let $T_\varepsilon=\Sigma_{n=1}^{\infty}\oplus T_n\oplus N_\varepsilon$, and let $S_\varepsilon=f(T_\varepsilon)$. The operator S_ε is given by the Riesz functional calculus, and one readily obtains

$$S_\varepsilon=\sum_{n=1}^{\infty}\oplus f(T_n)\oplus f(N_\varepsilon)=\sum_{n=1}^{\infty}\oplus S_n\oplus M.$$

The operator $M=f(N_\varepsilon)$ is normal with $\sigma(M)=\sigma_e(M)=\bar{S}$. Since S_ε is essentially normal, $\varepsilon_n=\|[S_n^*,S_n]\|^{1/2}$ tends to zero as n tends to infinity.

The region $\bar{S}$ is the union of p regions A_j, $1\leqslant j\leqslant p$, each of which is a rectangle with a rectangular hole. The regions A_j and A_{j+1} intersect in a rectangle R_j, $1\leqslant j\leqslant p-1$. Let $\delta>0$ be the minimal width of the R_j's. Let the projection of A_j onto the x-coordinate be $[\lambda_j,\mu_j]$. Apply the Cutting lemma 5.2 to each summand S_n in turn $(p-1)$ times. That is, one obtains normal matrices $M_{n,j}$ with spectrum contained in R_j, $1\leqslant j\leqslant p-1$ so that $S_n\oplus\Sigma_{j=1}^{p-1}\oplus M_{n,j}$ can be approximated by an operator of the form $\Sigma_{j=1}^{p}\oplus S_{n,j}$ with $\lambda_jI\leqslant\operatorname{Re}S_{n,j}\leqslant\mu_jI$. Notice that the error incurred in making this perturbation is the maximum of the $(p-1)$ errors since these changes are on orthogonal pieces (see Remark 3.3). Thus, if one writes $M_n=\Sigma_{j=1}^{p-1}\oplus M_{n,j}$, we have

$$\left\| S_n\oplus M_n-\sum_{j=1}^{p}\oplus S_{n,j}\right\| \leq 67\max\{\varepsilon_n, \varepsilon_n^2/\delta\}.$$

For n sufficiently large, this error is $67\varepsilon_n$. (Incidentally, the proof of Lemma 5.2 shows that M_n acts on a space of dimension no larger than that on which S_n acts.)

By the extended Weyl–von Neumann theorem, M has a small compact perturbation unitarily equivalent to $\Sigma_{n\geq 1}\oplus M_n\oplus M$. So

$$S_\varepsilon\cong\sum_{n=1}^{\infty}\oplus(S_n\oplus M_n)\oplus M-K_0$$

where K_0 is compact and $\|K_0\|\leq\varepsilon_0=\|[S_\varepsilon^*, S_\varepsilon]\|^{1/2}$. Let

$$K_1=\sum_{n=1}^{\infty}\oplus\left(S_n\oplus M_n-\sum_{j=1}^{p}\oplus S_{n,j}\right)\oplus 0.$$

This is compact, and $\|K_1\|\leq 67\max\{\varepsilon_0, \varepsilon_0^2/\delta\}$. Let $K=K_0+K_1$. Then

$$\|K\|\leq 68\max\{\varepsilon_0, \varepsilon_0^2/\delta\}$$

and

$$S_\varepsilon-K\cong\sum_{j=1}^{p}\oplus\left(\sum_{n\geq 1}\oplus S_{n,j}\right)\oplus M.$$

Now $M=\Sigma_{j=1}^{p}\oplus M_j$ is normal with spectrum A_j. Let $S_{\varepsilon,j}=\Sigma_{n\geq 1}\oplus S_{n,j}\oplus M_j$. Then $S_\varepsilon-K\cong\Sigma_{j=1}^{p}\oplus S_{\varepsilon,j}$ and each $S_{\varepsilon,j}$ is an essentially normal, quasidiagonal operator with $\lambda_j I\leq\operatorname{Re}S_{\varepsilon,j}\leq\mu_j I$. Thus $\sigma_e(S_{\varepsilon,j})$ is contained in the strip $\{z\in\mathbf{C}:\lambda_j\leq\operatorname{Re}z\leq\mu_j\}$ intersect $\sigma_e(S)$, which is A_j. Because $S_{\varepsilon,j}$ contains M_j as a summand, $\sigma_e(S_{\varepsilon,j})=A_j$.

Repeat this procedure with the operator $S_{\varepsilon,j}$ and a conformal map onto an annulus. Then by the proof of Theorem 4.4, one obtains a compact perturbation of $S_{\varepsilon,j}$ which is normal with spectrum A_j *and*, as above, this compact perturbation is achieved by attaching to $S_{n,j}$ a normal summand $M_{n,j}$ with

$$\lim_{n\to\infty}\|(S_{n,j}\oplus M_{n,j})-N_{n,j}\|=0.$$

There is a compact operator C_j of norm at most ε_0 so that

$$S_{\varepsilon,j}-C_j=\sum_{n\geq 1}\oplus(S_{n,j}\oplus M_{n,j})\oplus M_j.$$

Define $C_j'=\Sigma_{n\geq 1}\oplus(S_{n,j}\oplus M_{n,j}-N_{n,j})$. This is compact, and

$$S_{\varepsilon,j}-C_j-C_j'=\sum_{n\geq 1}\oplus N_{n,j}\oplus M_j$$

which is normal.

Recombining all the terms, one obtains normal operators $M_n=\Sigma_{j=1}^{p}\oplus M_{n,j}$ and $N_n=\Sigma_{j=1}^{p}\oplus N_{n,j}$ with spectrum contained in $\tilde{S}$ so that

$$\lim_{n\to\infty}\|S_n\oplus M_n-N_n\|=0.$$

Consequently, as f is conformal from a neighborhood of Y_ε onto a neighborhood of $\tilde{S}$

$$\lim_{n\to\infty}\|T_n\oplus f^{-1}(M_n)-f^{-1}(N_n)\|=0.$$

The operators $f^{-1}(M_n)$ and $f^{-1}(N_n)$ are normal with spectrum in Y_ε. This (finally) recaptures the proof of Theorem 5.3 with the important additional information that the compact perturbation is obtained by adding on to each summand T_n a normal summand $A_n=f^{-1}(M_n)$ so that $T_n\oplus A_n$ is close to a normal B_n for n large. (Furthermore, A_n acts on a space of dimension no larger than the domain of T_n.)

Choose a sequence ε_k decreasing to zero. For each k, obtain normal matrices $A_{n,k}$ and $B_{n,k}$ with spectra in Y_{ε_k} so that

$$\lim_{n\to\infty}\|T_n\oplus A_{n,k}-B_{n,k}\|=0.$$

Choose an increasing sequence of integers N_k such that for all $n\geq N_k$

$$\|T_n\oplus A_{n,k}-B_{n,k}\|\leq\varepsilon_k.$$

Choose normal matrices A_n and B_n with spectra in X so that

$$\|A_{n,k}-A_n\|<\varepsilon_k\quad\text{and}\quad\|B_{n,k}-B_n\|<\varepsilon_k\quad\text{for}\quad N_k\leq n<N_{k+1}.$$

So then $\|T_n\oplus A_n-B_n\|\leq\varepsilon_k$ for $N_k\leq n<N_{k+1}$. For $1\leq n<N_1$, choose B_n arbitrarily and A_n to

be vacuous. No good norm estimate holds for these terms. Finally, $T'=\Sigma_{n\geq 1}\oplus T_n\oplus N$ is a (small) compact perturbation of an operator unitarily equivalent to

$$T''=\Sigma_{n\geq 1}\oplus(T_n\oplus A_n)\oplus N.$$

Let $K=\Sigma_{n\geq 1}\oplus(T_n\oplus A_n-B_n)\oplus 0$. This is compact since $\lim_{n\to\infty}\varepsilon_n=0$, and so $T''-K=\Sigma_{n\geq 1}\oplus B_n\oplus N$ is normal with spectrum X. □

To get a quantitative version of BDF, it now suffices to examine the way the compact perturbation is obtained in the previous proof. It is the fact that each summand T_n in the block diagonal form of T is perturbed *individually* by adding on a normal summand that makes this possible.

Definition 5.5. Let X be a compact subset of $\mathbf{C}$, and let $\mathrm{Nor}(X)$ be the set of normal matrices with spectrum contained in X. For $t\geq 0$, let X_t denote $\{\lambda\in\mathbf{C}: \mathrm{dist}(\lambda, X)\leq t\}$. Let $\mathcal{S}_t(X)$ denote the set of matrices T such that (i) $\|[T^*,T]\|^{1/2}\leq t$ and (ii) $\|(T-\lambda I)^{-1}\|\leq \mathrm{dist}(\lambda, X_t)^{-1}$ for all $\lambda\notin X_t$. Define a function $f_X\colon[0,\infty)\to[0,\infty)$ by

$$f_X(t)=\sup_{T\in\mathcal{S}_t(X)}\ \inf_{N\in\mathrm{Nor}(X)} \mathrm{dist}(T\oplus N, \mathrm{Nor}).$$

LEMMA 5.6 (Absorption principle). $\lim_{t\to 0^+} f_X(t)=0$.

Proof. Choose T_n in $\mathcal{S}_{1/n}$ so that $\mathrm{dist}(T\oplus N, \mathrm{Nor})>\frac{1}{2}f_X(1/n)$. Let $T=\Sigma_{n=1}^{\infty}\oplus T_n\oplus N$. This is essentially normal and quasidiagonal. Furthermore, $\sigma_e(T)$ equals X since if $\mathrm{dist}(\lambda, X)>1/k$,

$$S_\lambda=\sum_{n=1}^{k-1}\oplus 0\oplus\sum_{n\geq k}(T_n-\lambda I)^{-1}\oplus(N-\lambda I)^{-1}$$

is bounded by k^2+k and is an inverse modulo $\mathcal{K}$ of $T-\lambda I$. From the proof of 5.4, one obtains normal summands N_n of N so that $\mathrm{dist}(T_n\oplus N_n, \mathrm{Nor})$ tends to zero. Hence $\lim_{n\to\infty} f_X(1/n)=0$. Since f_X is monotone, $\lim_{t\to 0} f_X(t)=0$. □

5.7. *Proof of Theorem* 1.2. Let T be an essentially normal operator with $\sigma_e(T)=X$ and $\mathrm{ind}(T-\lambda I)=0$ for $\lambda\notin X$. By Theorem 5.3, T is quasidiagonal. Given any $\delta>0$, there is a compact K_0 with $\|K_0\|<\delta$ so that

$$T-K_0\cong\sum_{n\geq 1}\oplus T_n\oplus N$$

where N is a diagonal normal operator with $\sigma(N)=\sigma_e(N)=X$, each T_n is finite rank and T_n belongs to $\mathcal{S}_\varepsilon$ for all $n\geq 1$. From the proof of 5.4, and Lemma 5.6, one obtains finite dimensional summands N_n of N and normal matrices M_n so that $\|T_n\oplus N_n-M_n\|<f_X(\varepsilon)$ and $\lim_{n\to\infty}\|T_n\oplus N_n-M_n\|=0$. Thus one obtains a compact operator K so that $T-K$ is normal, and

$$\|K\|<\delta+\sup_n\|T_n\oplus N_n-M_n\|<f_X(\varepsilon)$$

if δ is sufficiently small. □

This section is concluded with the easy proof of the classification of essentially normal operators up to compalence.

THEOREM 5.8 ([12]). ind: $\mathrm{Ext}(X)\to\mathrm{Hom}(\pi^1(X),\mathbf{Z})$ *is an isomorphism. Consequently, if S and T are essentially normal operators such that* $\sigma_e(S)=\sigma_e(T)=X$, *and* $\mathrm{ind}(S-\lambda I)=\mathrm{ind}(T-\lambda I)$ *for all* $\lambda\notin X$, *then there is a compact operator K such that* $S\cong T-K$.

Proof. It is possible to write down explicit operators to show that ind is surjective. See [12], in the remarks following § 11.5, where this is done by using the Berger–Shaw trace identity [9]. Thus it suffices to prove that ind is injective. So suppose S and T satisfy the hypotheses. Let R be an essentially normal operator with $\sigma_e(R)=X$ and $\mathrm{ind}(R-\lambda I)=-\mathrm{ind}(S-\lambda I)$ for all $\lambda\notin X$. Such an R exists by surjectivity (or, one can take R to be the real transpose of S [11]). The operators $S\oplus R$ and $R\oplus T$ are essentially normal and have zero index data. By Theorem 1.1, $S\oplus R\sim N\sim R\oplus T$. Hence by the absorption principle 2.1,

$$S\sim S\oplus N\sim S\oplus R\oplus T\sim N\oplus T\sim T. \qquad \square$$

6. Remarks and problems

6.1. The big open question in finite dimensions in this area is Problem 1.4. This may be rephrased as:

Problem 6.1. For each $\varepsilon>0$, is there a $\delta>0$ so that if A and B are Hermitian matrices of norm one with $\|[A,B]\|<\delta$, then there are commuting Hermitian matrices A_1 and B_1 with $\|A-A_1\|<\varepsilon$ and $\|B-B_1\|<\varepsilon$?

This problem was attacked in [18, 26, 27], but it has proven to be very resistant. An example in section 2 of [18] provides self-adjoint matrices A_n and normal matrices B_n such that $\|A_n\|=\|B_n\|=1$ and $\lim_{n\to\infty}\|[A_n, B_n]\|=0$, yet $\{A_n, B_n\}$ is bounded away from commuting pairs $\{A, B\}$ in which $A=A^*$ uniformly for $n\geq 3$ (by 0.04). Voiculescu [32, 33] has other examples of this phenomenon. In particular, he exhibits pairs of unitaries with small commutator which are far from commuting unitaries. Terry Loring [24] has a K-theoretic appoach to Voiculescu's example.

In view of our results, it would seem natural to try to extract from $T=A+iB$ an approximate summand which is nearly normal. If it is big enough, perhaps it can be recombined by the absorption technique. But how does one obtain such a summand?

In spite of all related negative examples, there does not seem to be any good reason for conjecturing the answer to the Hermitian case. Or rather, the authors of this paper each have a conjecture but they are contradictory. Voiculescu remarks that the known examples referred to above exploit the non-trivial cohomology of the 2-torus and the 2-sphere; and thus perhaps are not a good indicator of the Hermitian case which corresponds to the disc.

There was a related question about arbitrary matrices. Namely, if A and B are norm one matrices and $\|[A, B]\|$ is sufficiently small, are A and B closed to a commuting pair? In this case, there is a counterexample. Let A_n be a diagonal normal given by $A_n e_k=(k/n)e_k$ for $0\leq k\leq n$, and let B_n be the Jordan block $B_n e_k=e_{k+1}, 0\leq k\leq n, B_n e_n=0$. It is easy to verify that $\|[A_n, B_n]\|=1/n$. The argument of [16] mentioned above shows that no commuting pair (A, B) with $A=A^*$ is close to (A_n, B_n). M. D. Choi [15] by an elegant, quite different, argument has recently shown that there is no such commuting pair, even without the requirement $A=A^*$. Recently, Exel and Loring [34] have found a very simple winding number argument that gives the same result for Voiculescu's unitaries.

6.2. For the purpose of getting a quantitative version of BDF, Problem 1.4 can be replaced by a quest for a better absorption result. The strongest possible version is the following.

Problem 6.2. Is there a universal constant C so that: if T is any matrix, and $\|[T^*, T]\|^{1/2}=t$, then there is a normal matrix N with $\sigma(N)$ contained in $\{\lambda: \|(T-\lambda I)^{-1}\|\geq t^{-1}\}$ such that $\operatorname{dist}(T\oplus N, \mathrm{Nor})\leq Ct$?

This would imply that for an essentially normal operator T with zero index data, $\operatorname{dist}(T, \mathrm{Nor})\leq C\|[T^*, T]\|^{1/2}$. It would be promising even to get a constant C_X for nice regions X so that the function $f_X(t)\leq C_X t$. At the least, one should be able to weaken the resolvent condition (ii) and obtain an explicit formula for f_X.

One of the difficulties is finding the appropriate analogue of the condition

$\|T\| \leq R_2$ and $\|T^{-1}\| \leq R_1^{-1}$ for the annulus. This condition is equivalent to the conditions that $D_{R_2} = \{\lambda \in \mathbf{C}: |\lambda| \leq R_2\}$ and $D'_{R_1} = \{\lambda \in \mathbf{C}: |\lambda| \leq R_1\} \cup \{\infty\}$ are spectral sets for T. But this does not seem to help us even for sets with one hole. The reason is that working with the Riesz functional calculus is unwieldy. To progress, it seems likely that one has to deal with the spectrum without deforming it.

6.3. It is an interesting problem to try to compute the distance of an operator T to the set of normal operators. Two obstructions are immediately apparent (i) $\|[T^*, T]\|^{1/2}$ and (ii) $\Omega = \{\lambda \in \mathbf{C}: \operatorname{ind}(T-\lambda) \neq 0\}$. To measure the latter quantity, let $\delta(T) = \sup \{\operatorname{dist}(\lambda, \mathbf{C} \setminus \Omega)\}$. It is not necessary to worry about those λ such that $T-\lambda I$ is semi-Fredholm of index $\pm\infty$. As in the proof of Theorem 4.3, for each λ in $\sigma_e(T)$ it is possible to choose an orthonormal sequence which is an approximate eigenvector for T with eigenvalue λ, or, is an approximate eigenvalue for T^* with eigenvalue $\bar{\lambda}$. As in the proof, a perturbation of order $O(\|[T^*, T]\|^{1/2})$ yields a normal direct summand N with $\sigma(N) = \sigma_e(N) = \sigma_e(T)$. Moreover this same argument works for $\{\lambda: \|\pi(T-\lambda I)^{-1}\| \geq \varepsilon^{-1}\}$ where $\varepsilon = \|[T^*, T]\|^{1/2}$, and this may be much larger set. Now a perturbation of size $\delta(T)$ yields a normal summand N_0 which has expanded the spectrum of N to include all of Ω. A small compact perturbation will remove all holes of index zero [1]. Thus the problem is reduced to considering operators $T_0 \cong T_0 \oplus N_0$ which are biquasitriangular, $\sigma(N_0) = \sigma_e(N_0) = \sigma(T_0) \setminus \sigma_0(T_0)$ where $\sigma_0(T_0)$ is a finite set of isolated eigenvalues of finite multiplicity, and $\|(T_0 - \lambda I)^{-1}\|_e = O(\operatorname{dist}(\lambda, \sigma_e(T))^{-1})$ as λ approaches $\sigma_e(T)$. As the isolated eigenvalues may cause special problems, we suggest first trying to solve the problem without them. The "staircase" form of biquasitriangular operator [20] may be of use. Formally, we propose:

Problem 6.3. Suppose $T \cong T \oplus N$ is a biquasitriangular operator, and N is normal such that $\sigma(N) = \sigma_e(N) = \sigma(T)$. Furthermore, suppose that $\|(T-\lambda I)^{-1}\| = \operatorname{dist}(\lambda, \sigma(T))^{-1}$. Is $\operatorname{dist}(T, \mathrm{Nor})$ bounded by a function of $\|[T^*, T]\|$?

There is an unsatisfactory aspect even with the quantitative properties of $\|T^*T - TT^*\|^{1/2}$. This is because $\operatorname{dist}(T, \mathrm{Nor})$ may be much less than this quantity. For example, a bilateral weighted shift U with all weights 1 except one $1+\varepsilon$ has $\|[U, U^*]\|^{1/2} = \varepsilon^{1/2}$ whereas $\operatorname{dist}(U, \mathrm{Nor}) = \varepsilon/2$. On the other hand, if S is the unilateral shift, an easy computation shows that

$$\operatorname{dist}((I+\varepsilon S) \oplus N, \mathrm{Nor}) \geq \varepsilon/5 = \frac{1}{5} \|[I+\varepsilon S, I+\varepsilon S^*]\|^{1/2}$$

for every normal operator N.

What is a good lower bound for dist(T, Nor)? It is clear that

$$\frac{1}{2}\sup_{\|x\|=1} |\,\|Tx\|-\|T^*x\|\,|$$

is such a bound. If $T=UP$ is the polar decomposition, is $\frac{1}{2}\|PU-UP\|$ a lower bound?

6.4. One of the important consequences of the BDF theorem is that the set Nor$+\mathcal{K}$ is norm closed. This is an immediate corollary since any operator in the closure is essentially normal and has zero index data. It follows that Nor$+\mathcal{K}$ is the intersection of the set of essentially normal operators and the set of quasidiagonal operators, both of which are closed. Thus an answer to Problem 6.2 would yield a more direct proof.

This interesting fact breaks down in higher dimensions ($n\geq 4$). In particular, there are subsets X of $\mathbf{C}^2$ such as the suspended solenoid [12] for which there are commuting pairs (A_n, B_n) of normal operators with joint spectrum equal to X and compact operators K_n and L_n such that $A=\lim_{n\to\infty} A_n+K_n$ and $B=\lim_{n\to\infty} B_n+L_n$ exists, yet there are no compact operators K and L so that $(A-K, B-L)$ is a pair of commuting normal operators. Salinas [28] identifies the closure of this set of commuting normal operators plus compacts as the set of quasidiagonal, essentially normal operators. His methods have a similar flavor to our own, in that he shows how explicit approximations can be made by dilating finite dimensional, completely positive maps (which is related to our absorption method). Also related are O'Donovan's ideas in his paper [25] on the relation between quasidiagonal and essentially normal operators, and Arveson's approach [2] to Voiculescu's theorem [31].

This indicates that the absorption phenomenon breaks down by dimension 4. In $\mathbf{R}^3$, L. Brown has shown [11] that the index map is still an isomorphism. So there is a possibility of using constructive methods there. However, a partial breakdown already occurs.

6.5. Consider the case of the 2-sphere S^2 imbedded into $\mathbf{R}\times\mathbf{C}$ as the surface of a cylinder $S=(0,1)\times T\cup\{0,1\}\times D$ where $D=\{\lambda\in\mathbf{C}: |\lambda|\leq 1\}$ and T is the boundary of D. Let (A_n, B_n) be the almost commuting pairs of [16] mentioned in section 6.1. Let $A=\Sigma_{n\geq 1}\oplus A_n$ and $B=\Sigma_{n\geq 1}\oplus B_n$. Then $[A, B]$ is compact, and the joint spectrum of the commuting, normal pair $(\pi(A), \pi(B))$ is S. Now $\mathrm{Ext}(S^2)=0$, so there is a compact perturbation of (A, B) to a commuting normal pair. We demonstrate this explicitly.

Recall that A_n is diagonal given by $A_n e_k=(k/n^2)\, e_k$, for $0\leq k\leq n^2$ and B_n is a weighted shift $B_n e_k=b_k e_{k+1}$, for $0\leq k\leq n^2$ where $b_k=(k+1)/n$ for $0\leq k<n$, $b_k=1$ for $n\leq k\leq n^2$, and $b_k=$

$(n^2-k)/n$ for $n^2-n<k\leqslant n^2$. So that on the span$\{e_k: n\leqslant k\leqslant n^2-n\}$, B_n behaves like the shift. Consider the pair $(A_n\oplus A_m, B_n\oplus B_m^*)$ with $m\geqslant n$. Using the interchange technique, which was originally developed precisely for weighted shifts [4] [32] (also [20]) one can approximate $B_n\oplus B_m^*$ by a direct sum of normal operators N_j acting on a subspace of

$$E_{A_n\oplus A_m}\left[\frac{j-1}{n},\frac{j+1}{n}\right].$$

The self-adjoint pair $A_n\oplus A_m$ is perturbed by an error of $O(1/n)$ to be scalar $((j/n)I)$ on the domain of each normal N_j. Thus $(A_n\oplus A_m, B_n\oplus B_m^*)$ is close to a commuting normal pair. Since

$$(A\oplus A, B\oplus B^*)\cong\sum_{n\geqslant 1}(A_n\oplus A_n, B_n\oplus B_n^*),$$

it is a compact perturbation of a commuting normal pair (H,N). By the absorption lemma, (A,B) is unitarily equivalent to a compact perturbation of $(A\oplus H, B\oplus N)$. So one has

$$\begin{aligned}(A,B)&\doteqdot(A\oplus H, B\oplus N)\doteqdot(A\oplus A\oplus A, B\oplus B\oplus B^*)\\ &\approx\left(\sum_{n\geqslant 1}\oplus A_n\oplus A_{2n-1}\oplus A_n\oplus A_{2n}, \sum_{n\geqslant 1}\oplus B_n\oplus B_{2n-1}^*\oplus B_n\oplus B_{2n}^*\right).\end{aligned}$$

But the pairs $(A_n\oplus A_{2n-1}\oplus A_n\oplus A_{2n}, B_n\oplus B_{2n-1}^*\oplus B_n\oplus B_{2n}^*)$ can be perturbed to a normal pair by a compact change of norm $O(1/n)$. Thus $(A,B)\sim(H,N)$.

Let Nor_2 denote the set of commuting normal pairs. The above construction also yields the fact that

$$\lim_{n\to\infty}\mathrm{dist}\{(A_n\oplus H, B_n\oplus N), \mathrm{Nor}_2\}=0.$$

So it is curious that the absorption phenomenon fails. Let $\mathscr{S}$ denote the set of commuting pairs of normal matrices with spectrum contained in the cylinder S. Then

$$\liminf_{n\to\infty}\inf_{(H',N')\in S}\mathrm{dist}\{(A_n\oplus H', B_n\oplus N'), \mathrm{Nor}_2\}=\delta>0.$$

The reason is that the proof in [18] that (A_n, B_n) is far from commuting is not compromised by (H',N'). In order to overcome the dimension argument, it is necessary to go to infinite dimensions.

This indicates two related problems of a general nature:

Problem 6.4. Let $\mathcal{S}_t(X)$ be the set defined in section 5.5. Let N be a normal operator with $\sigma(N)=\sigma_e(N)=X$. Define

$$g_X(t) = \sup_{t\in\mathcal{S}_t} \operatorname{dist}(T\oplus N, \mathrm{Nor}).$$

Is $g_X(t)=f_X(t)$? Or is at least $f_X(t)=O(g_X(t))$ as $t\to 0$?

Problem 6.5. If T is essentially normal, let

$$d_K(T, \mathrm{Nor}) = \inf\{\|K\|: K\in\mathcal{K} \text{ and } T-K \text{ is normal}\}.$$

Is $d_K(T, \mathrm{Nor})\leq C \operatorname{dist}(T, \mathrm{Nor})$ for some universal constant C?

Returning to the case of the two sphere, it is possible to give a proof that $\mathrm{Ext}(S)=0$ using our methods. If (A, B) is an essentially normal pair with joint essential spectrum S, then block diagonalize A using a partition of $\sigma(A)=[0, 1]$ into small intervals. Then B will be tridiagonal except for a compact, and on the "middle" of this tridiagonal part, B will be close to isometric. So there will be a unitary V which acts on the middle seven blocks and is close to B on the middle three except for a compact operator. Then $(A, B)\doteq(A\oplus\frac{1}{2}I, B\oplus V)$ and the intertwining technique splits this into two summands with joint spectra equal to the two halves of S. These are normal plus compact by the disc case. The conclusion is that (A, B) is the limit of normal plus compact operators, and thus is quasidiagonal. Now one proves an absorption theorem by generalizing the construction used for the example.

6.6. Is there more that can be said about essentially normal operators with non-zero index? A strong conjecture is given in [5]. A somewhat weaker version is the following.

Problem 6.6. Let X be a compact subset of $\mathbf{C}$. Is there a continuous function $h_X(t)$ such that $h_X(0)=0$ and if: S and T are essentially normal with $\sigma_e(S)=\sigma_e(T)=X$, $\mathrm{ind}(T-\lambda)=\mathrm{ind}(S-\lambda)$ for $\lambda\notin X$, $\|[T^*, T]\|\leq t^2$ and $\|[S^*, S]\|\leq t^2$, plus some resolvent condition, there is a unitary U and a compact operator K and finite dimensional normal operators N and M such that

$$S\oplus N = U(T\oplus M)U^*+K \quad \text{and} \quad \|K\|\leq h_X(t)?$$

6.7. There is an infinite dimensional analogue of Problem 6.1 which we state in a rather vague way.

Problem 6.7. Give reasonable conditions on a pair of operators A, B so that

(a) if $\|[A,B]\|$ is small, then (A,B) is close to a commuting pair, and

(b) if $[A,B]$ is compact, then (A,B) has a compact perturbation to a commuting pair.

The authors of this paper have been particularly interested in this problem. In [8], it is shown that if $T=A+iB$ is essentially normal, then (A,B) has a compact perturbation to a commuting pair only if $\mathrm{ind}(T-\lambda)=0$ for all $\lambda\notin\sigma_e(T)$, in which case the desired perturbation is provided by BDF. In [6], the situation of a pair (S,T) is studied in which S is the unilateral shift and T is a weighted shift on the same basis. A complete analysis is given for T with real weights. Often, the desired results hold. However, it is shown that $[S,T]$ may be small and compact yet be far from a commuting pair even when there is no index obstruction. The case of complex weights is still open in general, although some of the questions can be answered using the results of [16]. In this paper, one considers the problem of perturbing an essentially commuting pair (N,M) of essentially normal operators to a "doubly-commuting" pair (that is N' commutes with M' and M'^*). This problem was solved when the joint spectrum $X=\sigma_e(N,M)$ was fairly nice by identifying a certain subgroup of $\mathrm{Ext}(X)$. In [17], this idea was extended to a non-commutative setting (N,Q) where N is essentially normal and Q is quasinilpotent. Although a fair bit of pathology was identified, the positive results are limited. In those positive cases, quasidiagonality played a useful role. So the methods of this paper may be useful.

Quasidiagonality of weighted shifts was analyzed by Russell Smucker [30] using an interchange technique to extract direct summands. This may be an appropriate place to mention that though the paper appeared in 1982 it was based on Smucker's 1973 Indiana University dissertation research. Smucker and we (Berg) developed our interchange techniques independently.

6.8. A set of $\mathcal{S}$ of operators is jointly quasitriangular if there is an increasing sequence of finite rank projections P_n with s-lim$P_n=I$ and $\lim_{n\to\infty}\|P_n^\perp SP_n\|=0$ for all S in $\mathcal{S}$.

Problem 6.8. Let A and B be essentially commuting, essentially normal operators. Determine necessary and sufficient conditions for $\{A,B\}$ to be jointly quasitriangular.

This amounts to determining a subsemigroup $\mathrm{Ext}_{qt}(X)$ of $\mathrm{Ext}(X)$ where X is the joint essential spectrum of (A, B). However it is not a C^* question since it is the quasitriangularity of the *non self-adjoint* algebra generated by A and B that is in question. This problem was solved for subsets of a cylinder $S^1 \times [0, 1]$ by Kaplan [23], and also studied in [21]. Salinas [29] showed that methods of several complex variables can be used. We feel that the constructive methods developed here may have a role to play as well.

References

[1] Apostol, C., The correction by compact perturbations of the singular behaviour of operators. *Rev Roumaine Math. Pures et Appl.*, 21 (1976), 249–265.

[2] Arveson, W., Notes on extensions of C^* algebras. *Duke Math. J.*, 44 (1977), 329–355.

[3] Berg, I. D., An extension of the Weyl–von Neumann theorem to normal operators. *Trans. Amer. Math. Soc.*, 160 (1971), 365–371.

[4] — On approximation of normal operators by weighted shifts. *Michigan Math. J.*, 21 (1974), 377–383.

[5] — Index theory for perturbations of direct sums of normal operators and weighted shifts. *Canad. J. Math.*, 30 (1978), 1152–1165.

[6] — On operators which almost commute with the shift. *J. Operator Theory*, 11 (1984), 365–377.

[7] Berg, I. D. & Davidson, K. R., Almost commuting matrices and the Brown–Douglas–Fillmore theorem. *Bull. Amer. Math. Soc.*, 16 (1987), 97–100.

[8] Berg, I. D. & Olsen, C. L., A note on almost commuting operators. *Proc. Roy. Irish Acad. Sect. A*, 81 (1981), 43–47.

[9] Berger, C. A. & Shaw, B. I., Self commutators of multicyclic operators are always trace class. *Bull. Amer. Math. Soc.*, 78 (1973), 1194–1199.

[10] Bhatia, R., Davis, C. & McIntosh, A., Perturbation of spectral subspaces and solution of linear operator equations. *Linear Algebra Appl.*, 52/53 (1983), 45–67.

[11] Brown, L. G., Extensions and the structure of C^* algebras. *Sympos. Math.*, 20. Rome, 1977.

[12] Brown, L. G., Douglas, R. G. & Fillmore, P. A., Unitary equivalence modulo the compact operators and extensions of C^* algebras, in *Proceedings of a Conference on Operator Theory*. Lecture Notes in Mathematics no. 345, pp. 58–128. Springer-Verlag, Berlin–New York–Heidelberg, 1973.

[13] — Extensions of C^* algebra operators with compact self-commutators, and K-homology. *Bull. Amer. Math. Soc.*, 79 (1973), 973–978.

[14] — Extensions of C^* algebras and K-homology. *Ann. of Math.*, 105 (1977), 265–324.

[15] Choi, M. D., Almost commuting matrices need not be nearly commuting. *Proc. Amer. Math. Soc.*, 102 (1988), 529–533.

[16] Davidson, K. R., Lifting commuting pairs of C^*-algebras. *J. Funct. Anal.*, 48 (1982), 20–42.

[17] — Essentially spectral operators. *Proc. London Math. Soc.*, 46 (1983), 547–560.

[18] — Almost commuting Hermitian matrices. *Math. Scand.*, 56 (1985), 222–240.

[19] — Berg's technique and irrational rotation algebras. *Proc. Roy. Irish Acad. Sect. A*, 84 (1984), 117–123.

[20] Foias, C., Pearcy, C. & Voiculescu, D., The staircase representation of biquasitriangular operators. *Michigan Math. J.*, 22 (1975), 343–352.

[21] GOHKALE, S., Joint quastriangularity of cylinder supported pairs of operators. To appear in *Rev. Roumaines Math. Pures et Appl.*
[22] HERRERO, D., Unitary orbits of power partial isometries and approximation by block diagonal nilpotents, in *Topics in Modern Operator Theory*. Birkhäuser, Basel 1981, pp. 171–210.
[23] KAPLAN, G., *Joint quasitriangularity of essentially normal essentially commuting operators*. Thesis, SUNY Stony Brook, 1979.
[24] LORING, T., *The torus and non-commutative topology*. Thesis, University of California, Berkeley, 1986.
[25] O'DONOVAN, D., Quasidiagonality in the Brown–Douglas–Fillmore theory. *Duke Math. J.*, 44 (1977), 767–776.
[26] PEARCY, C. & SHIELDS, A., Almost commuting matrices. *J. Funct. Anal.*, 339 (1979), 332–338.
[27] PHILLIPS, J., Nearest normal approximation for certain operators. *Proc. Amer. Math. Soc.*, 67 (1977), 236–240.
[28] SALINAS, N., Homotopy invariance of Ext(A). *Duke Math. J.*, 44 (1977), 777–794.
[29] — Quasitriangularity of C^* algebras and problems on joint quasitriangularity of operators. *J. Operator Theory*, 10 (1983), 167–205.
[30] SMUCKER, R., Quasidiagonal weighted shifts. *Pacific J. Math.*, 98 (1982), 173–181.
[31] VOICULESCU, D., A non commutative Weyl–von Neumann theorem. *Rev. Roumaine Math. Pures et Appl.*, 21 (1976), 97–113.
[32] — Remarks on the singular extension in the C^*-algebra of the Heisenberg group. *J. Operator Theory*, 5 (1981), 147–170.
[33] — Asymptotically commuting finite rank unitaries without commuting approximants. *Acta Sci. Math.*, 451 (1983), 429–431.
[34] EXEL, R. & LORING, T., Asymptotically commuting unitary matrices. *Proc. Amer. Math. Soc.*, 106 (1989), 913–915.

Received July 15, 1987
Received in revised form February 27, 1989