THE D-NEUMANN PROBLEM

BY

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Introduction

In his paper [15], D. C. Spencer has given a canonical procedure for associating to a differential operator $\mathcal{D}$ a sequence of differential operators

$$0 \longrightarrow \theta \longrightarrow \underline{C}^0 \xrightarrow{D} \underline{C}^1 \xrightarrow{D} \cdots \xrightarrow{D} \underline{C}^n \longrightarrow 0, \tag{1}$$

which, in the sense of formal exactness, resolves the sheaf θ of germs of solutions to the homogeneous equation $\mathcal{D}s=0$. In the case where $\mathcal{D}$ is elliptic Spencer has proposed a certain boundary value problem for the purpose of studying the cohomology of (1), or more precisely, the cohomology of

$$\Gamma(\Omega, C^0) \xrightarrow{D} \Gamma(\Omega, C^1) \xrightarrow{D} \cdots \xrightarrow{D} \Gamma(\Omega, C^n) \longrightarrow 0, \tag{2}$$

where Ω is a compact manifold-with-boundary and $\Gamma(\Omega, C^i)$ is the space of smooth sections of C^i over Ω. This boundary value problem, the D-Neumann problem, is the topic of this paper.

The D-Neumann problem is closely related to the existence problem for elliptic differential equations; a theorem due to D. G. Quillen asserts that the exactness of (2) at $\Gamma(\Omega, C^1)$ is equivalent to the existence of global solutions on Ω to the equation $\mathcal{D}s=t$, where t satisfies the appropriate compatibility conditions in the overdetermined case. Similarly the exactness of (1) at $\underline{C}^1$ is equivalent to the local solvability of $\mathcal{D}s=t$.

Our efforts here yield some sufficient conditions for the solvability of the D-Neumann problem. These take the form of sufficient (and, for the most part, necessary) conditions for certain a priori estimates to hold. However, the conditions which have wider scope are also more difficult to interpret; in general, the problem of interpreting the conditions remains.

Chapter I begins with the definition of the jet bundles and a description of the jet representation of differential operators. In section 5 we define the first Spencer sequences, and we establish the stability of these sequences in section 7. The results in section 8 prove the theorem of Quillen quoted above. Almost all of the definitions and propositions in Chapter I can be found in Quillen [12]; however, to suit our purposes, we give different proofs in several cases. In particular, the proof of the main proposition in section 7 is due to the author.

In Chapter II we construct the second Spencer sequence and show that its cohomology is isomorphic to the stable cohomology of the first Spencer sequences. We also show that the associated symbol sequence is exact whenever the original operator is elliptic. We then state the D-Neumann problem and give a priori estimates which imply its solvability. Following along lines suggested by the work of Hörmander [7], we reduce the study of these estimates to the study of similar estimates on the boundary. By taking advantage of certain formal properties of the Spencer sequence we obtain estimates which are simpler than those given in [7]. At the end of Chapter II we discuss sufficient conditions for the solvability of the D-Neumann problem.

In Chapter III we consider the restriction of the Spencer sequence to the boundary and define the D_b-problem. In section 3 we give a condition under which the solvability of the D_b-problem implies the solvability of the D-Neumann problem. In section 5 we discuss the D_b-problem corresponding to the Cauchy–Riemann equation; our results here agree with those of Kohn–Rossi [11].

Chapter I. The first Spencer sequence

0. Notation

Whenever the word "smooth" occurs in connection with manifolds, bundles, or maps of these objects, it is to be interpreted as meaning differentiable of class C^∞. If E is a vector bundle, we denote by $\underline{E}$ the sheaf of germs of smooth sections of E. Unless stated otherwise, a bundle map between two bundles over the same manifold M is assumed to cover the identity map M; we do not require that a bundle map have constant rank. In this chapter we consider only vector bundles whose fibers are real vector spaces; this is only for the sake of being definite, and all the results hold for complex vector bundles also.

By multi-index in n variables we mean an n-tuble $\alpha=(\alpha_1, ..., \alpha_n)$ of non-negative integers. We write $|\alpha|=\alpha_1+...+\alpha_n$ and $\alpha!=\alpha_1!\cdot ... \cdot\alpha_n!$. If f is a smooth function on $\mathbf{R}^n$, we write $\partial_\nu f$ for the partial derivative of f with respect to the ν-th coordinate place; we also write $\partial_\alpha=\partial_1^{\alpha_1}\cdot ... \cdot\partial_n^{\alpha_n}$ for a multi-index α. If $x=(x^1, ..., x^n)\in\mathbf{R}^n$, then $x^\alpha=(x^1)^{\alpha_1}\cdot ... \cdot(x^n)^{\alpha_n}$. Multi-indices will be used similarly in other contexts.

1. The jet bundles

Let $E\to M$ be a real vector bundle over a smooth n-manifold M, let $x\in M$, and let μ be a non-negative integer. We denote by $S_x(E)$ the space of smooth local sections of E at x. We define an equivalence relation on $S_x(E)$ by calling two local sections $s_1, s_2\in S_x(E)$ equivalent if they agree up to order μ at x. This means that for every smooth curve $\varphi:\mathbf{R}\to M$ with $\varphi(0)=x$ and for every smooth function $\psi:E\to\mathbf{R}$ which is linear on each fiber the derivatives

$$\frac{d^k}{dt^k}\psi\circ(s_2-s_1)\circ\varphi(t)$$

should vanish at $t=0$ for $k=0, 1, ..., \mu$. We denote the set of equivalence classes by $J_\mu(E)_x$ and write $J_\mu(E)$ for the disjoint union, over all x in M, of the sets $J_\mu(E)_x$. $J_\mu(E)$ is called the set of μ-jets of sections of E; the equivalence class in $J_\mu(E)_x$ of a local section s is called the μ-jet of s at x.

Our aim in this section is to show that $J_\mu(E)$ is a vector bundle over M. Indeed, there is an obvious projection $J_\mu(E)\to M$, and it is easy to see that the fiber $J_\mu(E)_x$ inherits from $S_x(E)$ the structure of a vector space over $\mathbf{R}$. However, before putting additional structure on $J_\mu(E)$, it is convenient to establish the functorial properties of J_μ.

Let $E_1\to M_1$ be another vector bundle, let $g: M\to M_1$ be a diffeomorphism, and let $h: E\to E_1$ be a bundle map which covers g. The mapping from $S_x(E)$ to $S_{g(x)}(E_1)$ defined by

$$s\to h\circ s\circ g^{-1}$$

is compatible with the equivalence relation defining the jets and with the operations of addition and scalar multiplication. Thus we obtain a mapping

$$J_\mu(g, h): J_\mu(E)\to J_\mu(E_1) \tag{1.1}$$

which maps the fiber over x linearly into the fiber over $g(x)$. If $M=M_1$, $E=E_1$, and if g and h are the identity maps, then $J_\mu(g, h)$ is the identity map. If $E_2\to M_2$ is a third bundle and if $g_1: M_1\to M_2$ and $h_1: E_1\to E_2$ have the same properties as g and h, then

$$J_\mu(g_1\circ g, h_1\circ h) = J_\mu(g_1, h_1)\circ J_\mu(g, h).$$

Finally, we note that for an open subset U of M we have $J_\mu(E)_U = J_\mu(E_U)$; that is, J_μ commutes with the operation of restricting to U.

We now give $J_\mu(E)$ a bundle structure in the special case where M is an open subset G of $\mathbf{R}^n$ and $E\to M$ is the trivial bundle $G\times\mathbf{R}^m\to G$. For $x=(x^1, \ldots, x^n)$ in G an element s of $S_x(G\times\mathbf{R}^m)$ can be considered as a local $\mathbf{R}^m$-valued function

$$s(y) = (s^1(y), \ldots, s^m(y))$$

defined for $y=(y^1, \ldots, y^n)$ in a neighborhood of x. The section s is equivalent, in the sense of μ-jets at x, to a unique $\mathbf{R}^m$-valued polynomial in $(y-x)$, namely, the truncated Taylor series

$$\textstyle\sum_{|\alpha|\leqslant\mu}(y-x)^\alpha\partial_\alpha s(x)/\alpha!$$

of s at x. If we denote by F_μ^m the vector space of all $\mathbf{R}^m$-valued polynomials of degree at most μ in the indeterminate $X=(X^1, \ldots, X^n)$, then we obtain a bijective map from $J_\mu(G\times\mathbf{R}^m)$ to $G\times F_\mu^m$ which sends the μ-jet of s at x to the pair

$$(x, \textstyle\sum_{|\alpha|\leqslant\mu}X^\alpha\partial_\alpha s(x)/\alpha!).$$

This mapping is linear on the fibers and thus defines a bundle structure on $J_\mu(G\times\mathbf{R}^m)$. If σ is the μ-jet of s at x, then we write $\sigma_\alpha=\partial_\alpha s(x)$ so that the mapping $J_\mu(G\times\mathbf{R}^m)\to G\times F_\mu^m$ is given by

$$\sigma \to (x, \textstyle\sum_{|\alpha| \leqslant \mu} \sigma_\alpha X^\alpha / \alpha!) \tag{1.2}$$

for σ in the fiber over x. The vectors $\sigma_\alpha \in \mathbf{R}^m$ are called the components of σ.

Now let g be a diffeomorphism of G with another open subset G_1 of $\mathbf{R}^n$, let $(x, e) \to (g(x), h_x e)$ give a bundle map h from $G \times \mathbf{R}^m$ to $G_1 \times \mathbf{R}^k$, and consider the map

$$\mathbf{J}_\mu(g, h): J_\mu(G \times \mathbf{R}^m) \to J_\mu(G_1 \times \mathbf{R}^k).$$

An element σ of $J_\mu(G \times \mathbf{R}^m)_x$ is the μ-jet at x of the section

$$s(y) = \textstyle\sum_{|\alpha| \leqslant \mu} (y - x)^\alpha \sigma_\alpha / \alpha!;$$

$\tau = J_\mu(g, h)\, \sigma$ is the μ-jet at $g(x)$ of the section $h \circ s \circ g^{-1}$, and thus

$$\tau_\beta = \textstyle\sum_{|\alpha| \leqslant \mu} \partial_\beta ((g^{-1}(y) - x)^\alpha h_{g^{-1}(y)}\, \sigma_\alpha / \alpha!)\big|_{y = g(x)}. \tag{1.3}$$

The map $J_\mu(g, h)$ is thus given by

$$(x, \textstyle\sum_{|\alpha| \leqslant \mu} \sigma_\alpha X^\alpha / \alpha!) \to (g(x), \textstyle\sum_{|\beta| \leqslant \mu} \tau_\beta X^\beta / \beta!), \tag{1.4}$$

where τ_β is defined by (1.3). It is clear that $J_\mu(g, h)$ is smooth, and thus it is a bundle map from $J_\mu(G \times \mathbf{R}^m)$ to $J_\mu(G_1 \times \mathbf{R}^k)$.

We are now ready to put a bundle structure on $J_\mu(E)$. Let U be an open subset of M which is diffeomorphic to an open subset G of $\mathbf{R}^n$, and assume that E_U can be trivialized by a mapping $E_U \to U \times \mathbf{R}^m$. Composing with the diffeomorphism $U \to G$, we obtain a bundle isomorphism

$$\begin{array}{ccc} E_U & \xrightarrow{h} & G \times \mathbf{R}^m \\ \downarrow & & \downarrow \\ U & \xrightarrow{g} & G \end{array} \tag{1.5}$$

We thus obtain a commutative diagram

$$\begin{array}{ccc} J_\mu(E)_U & \xrightarrow{J_\mu(g,h)} & J_\mu(G \times \mathbf{R}^m) \\ \downarrow & & \downarrow \\ U & \xrightarrow{g} & G \end{array} \tag{1.6}$$

where $J_\mu(g, h)$ is bijective since $J_\mu(g, h)^{-1} = J_\mu(g^{-1}, h^{-1})$ exists. We use this diagram to transfer the bundle structure of $J_\mu(G \times \mathbf{R}^m)$ to $J_\mu(E)_U$ and claim that the structure so obtained is independent of the choice of g and h. Indeed, if

$$\begin{array}{ccc} E_U & \xrightarrow{h_1} & G_1 \times \mathbf{R}^m \\ \downarrow & & \downarrow \\ U & \xrightarrow{g_1} & G_1 \end{array}$$

is another bundle isomorphism, then $J_\mu(g, h)\circ J_\mu(g_1, h_1)^{-1} = J_\mu(g\circ g_1^{-1}, h\circ h_1^{-1})$ is a bundle isomorphism of $J_\mu(G_1\times \mathbf{R}^m)$ with $J_\mu(G\times\mathbf{R}^m)$ so that the bundle structures on $J_\mu(E)_U$ obtained by transfering are the same. If U ranges through a covering of M by coordinate disks, the resulting diagrams (1.6) thus define a bundle structure on $J_\mu(E)$. A local coordinate in $J_\mu(E)$ is obtained by composing the bundle map (1.6) with the bundle map described by (1.2); in such a local coordinate a jet σ will have components $\sigma_\alpha\in\mathbf{R}^m$ for $|\alpha|\leqslant\mu$. The coordinate changes for $J_\mu(E)$ are expressed by (1.3) and (1.4). Note finally that the mapping (1.1) is smooth and hence a bundle map.

2. An exact sequence

Let $E\to M$ be a real vector bundle with fiber dimension m. Since the equivalence relation defining the $(\mu+1)$-jets of sections at $x\in M$ is stronger than the one defining the μ-jets, there exists a mapping

$$\pi: J_{\mu+1}(E)\to J_\mu(E) \tag{2.1}$$

which sends the $(\mu+1)$-jet at x of a local section s to the μ-jet of s at x. In a local coordinate, π is given by

$$(x, \textstyle\sum_{|\alpha|\leqslant\mu+1} X^\alpha\sigma_\alpha/\alpha!)\to(x, \textstyle\sum_{|\alpha|\leqslant\mu} X^\alpha\sigma_\alpha/\alpha!). \tag{2.2}$$

It is clear that π is smooth and hence is a bundle map.

Denote by $S^{\mu+1}T^*$ the $(\mu+1)$-fold symmetric product of the cotangent bundle T^* of M, and define a mapping

$$i: S^{\mu+1}T^*\otimes E\to J_{\mu+1}(E) \tag{2.3}$$

as follows. If $\omega\in(S^{\mu+1}T^*\otimes E)_x$ has the form $\omega=v^1\odot v^2\odot\ldots\odot v^{\mu+1}\otimes e$, then choose a local section s of E with $s(x)=e$ and local functions $f^1, \ldots, f^{\mu+1}$, vanishing at x, such that $df^\nu=v^\nu$ at x. Then define $i\omega$ to be the $(\mu+1)$-jet at x of the section $f^1\cdot\ldots\cdot f^{\mu+1}\cdot s$. It is easily verified that the definition of $i\omega$ depends only on ω and that i can be extended to all of $(S^{\mu+1}T^*\otimes E)_x$ by linearity. In a local coordinate i is given by

$$(x, \textstyle\sum_{|\alpha|=\mu+1}(dx)^\alpha\otimes\sigma_\alpha/\alpha!)\to(x, \textstyle\sum_{|\alpha|=\mu+1} X^\alpha\sigma_\alpha/\alpha!). \tag{2.4}$$

Thus i is a bundle map, and in view (2.2) the sequence

$$0\longrightarrow S^{\mu+1}T^*\otimes E\xrightarrow{\ i\ } J_{\mu+1}(E)\xrightarrow{\ \pi\ } J_\mu(E)\longrightarrow 0 \tag{2.5}$$

is exact. It follows immediately from the definitions that the sequence (2.4) is natural in the following sense.

PROPOSITION 2.1. *Let E and F be two bundles over M, and let $h:E\to F$ be a bundle map which covers the identity map of M. Then the diagram*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & S^{\mu+1}T^*\otimes E & \longrightarrow & J_{\mu+1}(E) & \longrightarrow & J_\mu(E) & \longrightarrow & 0 \\ & & \downarrow 1\otimes h & & \downarrow j_{\mu+1}(1,h) & & \downarrow j_\mu(1,h) & & \\ 0 & \longrightarrow & S^{\mu+1}T^*\otimes F & \longrightarrow & J_{\mu+1}(F) & \longrightarrow & J_\mu(F) & \longrightarrow & 0 \end{array} \tag{2.6}$$

commutes.

3. Differential operators

Let E and F be real vector bundles over M with fiber dimensions m and k respectively. A differential operator of order μ from E to F is defined to be a sheaf map $\mathcal{D}:\underline{E}\to\underline{F}$ which in local coordinates is given by a differential operator of order μ in the usual sense. This means that for a local coordinate $x=(x^1, \ldots, x^n)$ defined on an open subset U of M and for trivializations $E_U\to U\times\mathbf{R}^m$ and $F_U\to U\times\mathbf{R}^k$ there should exist smooth $(k\times m)$-matrix-valued functions A^α, $0\leqslant|\alpha|\leqslant\mu$, defined on U such that

$$\mathcal{D}s(x)=\textstyle\sum_{|\alpha|\leqslant\mu}A^\alpha(x)\,\partial_\alpha s(x) \tag{3.1}$$

for a section $s(x)=(s^1(x), \ldots, s^m(x))$ of $U\times\mathbf{R}^m\approx E_U$.

To provide an example we construct a canonical differential operator

$$j_\mu:\underline{E}\to\underline{J_\mu(E)}.$$

To a section s of E we assign the section of $J_\mu(E)$ which maps x to the μ-jet of s at x. The sheaf map thus defined is denoted by j_μ. In local coordinates j_μ sends the section $s(x)=(s^1(x), \ldots, s^m(x))$ of E to the section $\sigma(x)$ of $J_\mu(E)$ with components $(\sigma^1_\alpha(x), \ldots, \sigma^m_\alpha(x))=\sigma_\alpha(x)=\partial_\alpha s(x)$, $|\alpha|\leqslant\mu$. Thus j_μ is a differential opeator of order μ on E.

PROPOSITION 3.1. *To each differential operator $\mathcal{D}:\underline{E}\to\underline{F}$ of order μ there corresponds a bundle map $\varrho_\mu(\mathcal{D}):J_\mu(E)\to F$ such that the diagram*

$$\begin{array}{ccc} \underline{J_\mu(E)} & \xrightarrow{\varrho_\mu(\mathcal{D})} & \underline{F} \\ \uparrow j_\mu & & \uparrow 1 \\ \underline{E} & \xrightarrow{\mathcal{D}} & \underline{F} \end{array} \tag{3.2}$$

commutes. Moreover, the correspondence $\mathcal{D}\to\varrho_\mu(\mathcal{D})$ is one-to-one and onto the set of bundle maps from $J_\mu(E)$ to F.

Proof. Let $\mathcal{D}$ be a differential operator of order μ from E to F. In view of the local representations (3.1) of $\mathcal{D}$ the mapping

$$S_x(E)\ni s\to\mathcal{D}s(x)\in F_x$$

is compatible with the equivalence relation defining the jets and thus induces a mapping $\varrho_\mu(\mathcal{D}): J_\mu(E) \to F$. If $\mathcal{D}$ is given by (3.1) in some local coordinate, then $\varrho_\mu(\mathcal{D})$ is given by

$$\varrho_\mu(\mathcal{D})\,\sigma = \sum_{|\alpha|\leqslant\mu} A^\alpha(x)\,\sigma_\alpha, \quad \sigma \in J_\mu(E)_x, \tag{3.3}$$

in the same coordinate. The mapping $\varrho_\mu(\mathcal{D})$ is smooth and thus is a bundle map. The assertions in the proposition are now easily verified.

Following Quillen's presentation [12], we illustrate the correspondence $\mathcal{D} \to \varrho_\mu(\mathcal{D})$ with several examples which will be used in the sequel.

Example 3.1. We let $\mathcal{D}: \underline{E} \to \underline{F}$ be a differential operator of order μ and form the differential operator $j_\nu \mathcal{D}: \underline{E} \to \underline{J_\nu(F)}$, which has order $\mu+\nu$. The bundle map $\varrho_{\mu+\nu}(j_\nu \mathcal{D})$ from $J_{\mu+\nu}(E)$ to $J_\nu(F)$ is called the ν-th prolongation of $\varrho_\mu(\mathcal{D})$. By Prop. 3.1 the diagram

$$\begin{array}{ccc} \underline{J_{\mu+\nu}(E)} & \xrightarrow{\varrho_{\mu+\nu}(j_\nu \mathcal{D})} & \underline{J_\nu(F)} \\ \uparrow j_{\mu+\nu} & & \uparrow j_\nu \\ \underline{E} & \xrightarrow{\mathcal{D}} & \underline{F} \end{array} \tag{3.4}$$

commutes. We use this diagram to compute $\varrho_{\mu+\nu}(j_\nu \mathcal{D})$ in a local coordinate. Given $\sigma \in J_{\mu+\nu}(E)_x$ we consider the section

$$s(y) = \sum_{|\alpha|\leqslant\mu+\nu} (y-x)^\alpha \sigma_\alpha/\alpha!$$

of E, which is defined for $y=(y^1, \ldots, y^n)$ in some neighborhood of $x=(x^1, \ldots, x^n)$. Now $j_{\mu+\nu}s$ is a section of $J_{\mu+\nu}(E)$ through σ, and thus by (3.4) $\tau = \varrho_{\mu+\nu}(j_\nu \mathcal{D})\sigma$ is equal to the value of $j_\nu \mathcal{D}s$ at x; that is,

$$\tau_\beta = \partial_\beta \mathcal{D}s(x) = \partial_\beta\big(\sum_{|\alpha|\leqslant\nu} \sum_{|\gamma|\leqslant\mu} A_\gamma(y)\,\sigma_{\alpha+\gamma}(y-x)^\alpha/\alpha!\big)\big|_{y=x} \tag{3.5}$$

for $|\beta| \leqslant \nu$ if $\mathcal{D}$ is given by (3.1) in the local coordinate. If either the coefficient matrices $A_\gamma(y)$ are constant, or if σ is in the kernel of $\pi: J_{\mu+\nu}(E) \to J_{\mu+\nu-1}(E)$, then (3.5) simplifies to

$$\tau_\beta = \sum_{|\gamma|\leqslant\mu} A_\gamma(x)\,\sigma_{\beta+\gamma}, \quad |\beta| \leqslant \nu. \tag{3.6}$$

Example 3.2. If G is an open subset of $\mathbf{R}^n$, then ∂_ν is a first order differential operator from the trivial bundle $G \times \mathbf{R}^m$ to itself. We will denote $\varrho_1(\partial_\nu)$ and all of its prolongations by δ_ν. By (3.6) if $\sigma \in J_{\mu+1}(G \times \mathbf{R}^m)$, then

$$(\delta_\nu \sigma)_\beta = \sigma_{\beta+1_\nu}, \quad |\beta| \leqslant \mu, \tag{3.7}$$

where 1_ν is the multi-index with 1 in the ν-th place and 0's elsewhere. More generally for a bundle $E \to M$ and a coordinate neighborhood $U \subset M$ one can define $\delta_\nu: J_{\mu+1}(E)_U \to J_\mu(E)_U$ in terms of the coordinate on U and a trivialization $E_U \to U \times \mathbf{R}^m$. The definition depends, of course, on the choice of coordinate and trivialization.

Example 3.3. In view of (3.2) we have $\varrho_\mu(j_\mu)=\mathbf{1}$ for the canonical operator $j_\mu:\underline{E}\to\underline{J_\mu(E)}$. In local coordinates we have

$$j_\mu s(x)=\sum_{|\alpha|\leqslant\mu}X^\alpha\partial_\alpha s(x)/\alpha!;$$

thus by (3.6) the ν-th prolongation of $\varrho_\mu(j_\mu)$, which we denote by $\varphi:J_{\mu+\nu}(E)\to J_\nu(J_\mu(E))$, is given by

$$(\varphi\sigma)_\beta=\sum_{|\alpha|\leqslant\mu}X^\alpha\sigma_{\alpha+\beta}/\alpha!,\quad |\beta|\leqslant\nu, \tag{3.8}$$

in local coordinates. It is clear that φ is injective.

Example 3.4. Let $h:E\to F$ be a bundle map which covers the identity map of M; passing to sheaves, we obtain a differential operator $h:\underline{E}\to\underline{F}$ of order 0. Note that $J_0(E)=E$ and that $\varrho_0(h)=h$. Comparing (1.3) with (3.5) we see that the ν-th prolongation is the map $J_\nu(h)=J_\nu(\mathbf{1},h)$ defined near (1.1).

Example 3.5. The identity map $\mathbf{1}$ of $\underline{E}$ may be viewed as a first order differential operator from E to E. From (3.6) we see that $\varrho_{1+\nu}(j_\nu\mathbf{1})$ is the map $\pi:J_{\nu+1}(E)\to J_\nu(E)$.

The following technical result will be useful later on.

PROPOSITION 3.2. *Let E_1, E_2, E_3 be bundles, and let $\mathcal{D}_1:\underline{E}_1\to\underline{E}_2$ and $\mathcal{D}_2:\underline{E}_2\to\underline{E}_3$ be differential operators of order μ and λ respectively. Then*

$$\varrho_{\mu+\lambda+\nu}(j_\nu\mathcal{D}_2\mathcal{D}_1)=\varrho_{\lambda+\nu}(j_\nu\mathcal{D}_2)\,\varrho_{\mu+\lambda+\nu}(j_{\nu+\lambda}\mathcal{D}_1) \tag{3.9}$$

for $\nu=0,1,\ldots$

Proof. From the definitions we compute that $j_{\mu+\lambda+\nu}$ followed by the map on the right side of (3.9) is equal to $\varrho_{\lambda+\nu}(j_\nu\mathcal{D}_2)j_{\nu+\lambda}\mathcal{D}_1=j_\nu\mathcal{D}_2\mathcal{D}_1$; the identity (3.9) now follows from the uniqueness assertion in Prop. 3.1.

4. Properties of differential operators

Let $\mathcal{D}:\underline{E}\to\underline{F}$ be a differential operator of order μ_0; for each $\mu\geqslant\mu_0$ we define subsets $R_\mu=R_\mu(j_{\mu-\mu_0}\mathcal{D})$ and $g_\mu=g_\mu(j_{\mu-\mu_0}\mathcal{D})$ of $J_\mu(E)$ by the exact sequences

$$0\longrightarrow R_\mu\longrightarrow J_\mu(E)\xrightarrow{\varrho_\mu}J_{\mu-\mu_0}(F)$$

$$0\longrightarrow g_\mu\longrightarrow R_\mu\xrightarrow{\pi}J_{\mu-1}(E),$$

where $\varrho_\mu=\varrho_\mu(j_{\mu-\mu_0}\mathcal{D})$ and π is the restriction of $\pi:J_\mu(E)\to J_{\mu-1}(E)$. We claim that $\pi\varrho_{\mu+1}=\varrho_\mu\pi$; indeed, by Prop. 3.2

$$\pi\varrho_{\mu+1}=\varrho_{\mu+1-\mu_0}(j_{\mu-\mu_0}\mathbf{1})\,\varrho_{\mu+1}(j_{\mu+1-\mu_0}\mathcal{D})=\varrho_{\mu+1}(j_{\mu-\mu_0}\mathcal{D})=\varrho_\mu(j_{\mu-\mu_0}\mathcal{D})\,\varrho_{\mu+1}(j_\mu\mathbf{1})=\varrho_\mu\pi.$$

Thus for each $\mu \geqslant \mu_0$ we have an exact commutative diagram

$$\begin{array}{ccccccc} & & 0 & & 0 & & 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & g_{\mu+1} & \longrightarrow & S^{\mu+1}T^* \otimes E & \xrightarrow{\varrho_{\mu+1}} & S^{\mu+1-\mu_0}T^* \otimes F \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & R_{\mu+1} & \longrightarrow & J_{\mu+1}(E) & \xrightarrow{\varrho_{\mu+1}} & J_{\mu+1-\mu_0}(F) \\ & & \downarrow \pi & & \downarrow \pi & & \downarrow \pi \\ 0 & \longrightarrow & R_\mu & \longrightarrow & J_\mu(E) & \xrightarrow{\varrho_\mu} & J_{\mu-\mu_0}(F) \\ & & & & \downarrow & & \downarrow \\ & & & & 0 & & 0 \end{array} \tag{4.1}$$

where the columns on the right are instances of (2.5).

Definition. The differential operator $\mathcal{D}$ is said to be regular if (i) for each $\mu \geqslant \mu_0$, g_μ and R_μ are vector bundles over M and (ii) if for each $\mu \geqslant \mu_0$ the map $\pi: R_{\mu+1} \to R_\mu$ is surjective.

The condition (i) in the definition requires that the maps ϱ_μ and $\varrho_\mu | S^\mu T^* \otimes E$ have constant rank; the condition (ii) is more subtle and requires that the homogeneous equation $\mathcal{D}s = 0$ be completely solvable in a formal sense. For example, if the data E, F, M, $\mathcal{D}$ are all real analytic, then (ii) implies that every $\sigma \in R_\mu$ can be prolonged to a formal power series solution of the equation $\mathcal{D}s = 0$.

PROPOSITION 4.1. *Let $\mathcal{D}$ be as above, let $\mu \geqslant \mu_0$, and let $\mathcal{D}_1$ be the operator $j_{\mu-\mu_0}\mathcal{D}: \underline{E} \to \underline{J_{\mu-\mu_0}(F)}$ of order μ. Then for each $v \geqslant \mu$ we have*

$$R_v(j_{v-\mu}\mathcal{D}_1) = R_v(j_{v-\mu_0}\mathcal{D}). \tag{4.2}$$

In particular, if $\mathcal{D}$ is regular then so is $\mathcal{D}_1$.

Proof. By Prop. 3.2 we have

$$\varrho_v(j_{v-\mu}\mathcal{D}_1) = \varrho_v(j_{v-\mu}j_{\mu-\mu_0}\mathcal{D}) = \varrho_{v-\mu_0}(j_{v-\mu}j_{\mu-\mu_0})\,\varrho_v(j_{v-\mu_0}\mathcal{D}) = \varphi\varrho_v(j_{v-\mu_0}\mathcal{D}),$$

where $\varphi: J_{v-\mu_0}(F) \to J_{v-\mu}(J_{\mu-\mu_0}(F))$ is the map defined in Example 3.3. Since φ is injective, (4.2) holds.

For a non-zero cotangent vector $\xi \in T_y^*$ we define the symbol $s_\xi = s_\xi(\mathcal{D})$ of $\mathcal{D}$ at ξ to be the composition

$$E_y \longrightarrow (S^{\mu_0}T^* \otimes E)_y \xrightarrow{i} J_{\mu_0}(E)_y \xrightarrow{\varrho_{\mu_0}} F_y,$$

where i is the injection (2.3) and the first map on the left is defined by $e \to \xi^{\mu_0} \otimes e/\mu_0!$.

If $\xi = \xi_1 dx^1 + \ldots + \xi_n dx^n$ in a local coordinate $x = (x^1, \ldots, x^n)$, then

$$\xi^{\mu_0} = \sum_{|\alpha| = \mu_0} (dx)^\alpha \xi^\alpha (|\alpha|!/\alpha!);$$

thus by (2.4) and (3.3) we have

$$s_\xi(\mathcal{D})\, e = \sum_{|\alpha| = \mu_0} A_\alpha(y)\, \xi^\alpha e, \tag{4.3}$$

where the A_α's are the coefficient matrices for $\mathcal{D}$ in the local coordinate x.

Definition. A non-zero cotangent vector ξ is said to be characteristic for $\mathcal{D}$ if the symbol $s_\xi(\mathcal{D})$ is not injective. $\mathcal{D}$ is said to be elliptic if $s_\xi(\mathcal{D})$ is injective for each non-zero cotangent vector ξ.

PROPOSITION 4.2. *With the notation of Prop. 4.1, ξ is characteristic for $\mathcal{D}$ if and only if it is characteristic for $\mathcal{D}_1$. In particular, $\mathcal{D}$ is elliptic if and only if $\mathcal{D}_1$ is.*

Proof. Let $x = (x^1, ..., x^n)$ be a local coordinate defined on a neighborhood of $y \in M$, and let $\xi = \xi_1 dx^1 + ... + \xi_n dx^n$ be a non-zero cotangent vector at y. Let $e \in E_y$ and let $\tau = s_\xi(\mathcal{D}_1)\, e \in J_{\mu-\mu_0}(F)_y$. In view of (3.6) and (4.3) we have:

$$\tau_\beta = \begin{cases} \xi^\beta \sum_{|\gamma| = \mu_0} A_\gamma(y)\, \xi^\gamma e = \xi^\beta s_\xi(\mathcal{D})\, e, & |\beta| = \mu - \mu_0, \\ 0, \quad |\beta| < \mu - \mu_0. \end{cases} \tag{4.4}$$

The proposition now follows.

Definition. The differential operator $\mathcal{D}$ is said to be under-determined (resp. determined) if for each $y \in M$ there exists $\xi \in T^*_y$ such that $s_\xi(\mathcal{D}): E_y \to F_y$ is surjective (resp. bijective).

PROPOSITION 4.3. *If $\mathcal{D}$ is under-determined, then $\mathcal{D}$ is regular and the mappings*

$$\varrho_\mu(j_{\mu-\mu_0}\mathcal{D}): J_\mu(E) \to J_{\mu-\mu_0}(F) \tag{4.5}$$

are surjective for $\mu \geqslant \mu_0$.

Proof. Assume for the moment that the maps

$$\varrho_\mu: S^\mu T^* \otimes E \to S^{\mu-\mu_0} T^* \otimes F \tag{4.6}$$

are known to be surjective for $\mu \geqslant \mu_0$. The surjectivity of (4.6) for $\mu = \mu_0$ implies the surjectivity of (4.5) for $\mu = \mu_0$, and using diagram (4.1) we can conclude by induction that (4.5) is surjective for all $\mu \geqslant \mu_0$. It follows from the surjectivity of (4.5) and (4.6) that the corresponding kernels R_μ and g_μ are vector bundles, and by chasing diagram (4.1) we can infer the surjectivity of the maps $\pi: R_{\mu+1} \to R_\mu$, $\mu \geqslant \mu_0$.

Thus it suffices to show that the maps (4.6) are surjective. To do this we work in the fibers over $y \in M$ and with a local coordinate $x=(x^1, \ldots, x^n)$ defined near y. By hypothesis $s_\xi(\mathcal{D})$ is surjective for some $\xi_0 \in T^*_y$ and thus by continuity, for all $\xi \neq 0$ in some cone C containing ξ_0. Since the range of (4.6) contains the range of $s_\xi(j_{\mu-\mu_0}\mathcal{D})$, it follows from (4.4) that $\varrho_\mu(S^\mu T^* \otimes E)_y$ contains all elements of $(S^{\mu-\mu_0}T^* \otimes F)_y$ of the form

$$\sum_{|\beta|=\mu-\mu_0} \xi^\beta (dx)^\beta \otimes f/\beta! \tag{4.7}$$

where $f \in F_y$ and $\xi = \xi_1 dx^1 + \ldots + \xi_n dx^n \in C$. Since $\varrho_\mu(S^\mu T^* \otimes E)_y$ is trivially a closed subset of $(S^{\mu-\mu_0}T^* \otimes F)_y$, it must contain all partial derivatives of (4.7) with respect to the ξ_ν's evaluated at points ξ in C. Taking the β-th derivative at 0, we see that $(dx)^\beta \otimes f$ is in the range of (4.6) for each $|\beta| = \mu - \mu_0$ and each $f \in F_y$. The proof is now complete.

5. The first Spencer sequence

Let E be a vector bundle as before. Following R. Bott's presentation [1] we consider the diagram

$$\begin{tikzcd}
0 \arrow[r] & T^* \otimes J_{\mu+1}(E) \arrow[r] \arrow[d, "1\otimes\pi"] & J_1(J_{\mu+1}(E)) \arrow[r, "\pi'"] \arrow[d, "J_1(\pi)"] & J_{\mu+1}(E) \arrow[r] \arrow[d, "\pi"] \arrow[dl, "\varphi"'] & 0 \\
0 \arrow[r] & T^* \otimes J_\mu(E) \arrow[r] & J_1(J_\mu(E)) \arrow[r, "\pi''"] & J_\mu(E) \arrow[r] & 0.
\end{tikzcd} \tag{5.1}$$

The map φ is the injection given in Example 3.3, and the rest of the diagram is an instance of the exact commutative diagram (2.6); we have used primes to distinguish the various π's. It follows from Prop. 3.2 that $\pi''\varphi = \pi$. Indeed, denoting by $\mathbf{1}$ the identity map on $J_\mu(E)$, we find that $\pi = \varrho_{\mu+1}(j_\mu) = \varrho_{\mu+1}(\mathbf{1}j_\mu) = \varrho_1(\mathbf{1})\varrho_{\mu+1}(j_1 j_\mu) = \pi''\varphi$. Thus if $k = J_1(\pi) - \varphi\pi'$, we have $\pi'' k = \pi'' J_1(\pi) - \pi''\varphi\pi' = \pi\pi' - \pi\pi' = 0$; and by exactness we may consider k as a map from $J_1(J_{\mu+1}(E))$ to $T^* \otimes J_\mu(E)$. The sheaf map kj_1 is thus a first order differential operator from $J_{\mu+1}(E)$ to $T^* \otimes J_\mu(E)$, which we denote by D.

We now compute D locally. Let $x = (x^1, \ldots, x^n)$ be a coordinate on an open disk $U \subset M$, and let $E_U \to U \times \mathbf{R}^m$ be a trivialization. An element τ of $J_1(J_{\mu+1}(E))$ is now identified with a polynomial

$$\tau = \tau_0 + \sum_1^n Y^\nu \tau_\nu,$$

where $Y^1, \ldots, Y^n$ are indeterminates and each $\tau_\nu = \sum_{|\alpha| \leqslant \mu+1} X^\alpha \tau_{\nu,\alpha}/\alpha!$ is an $\mathbf{R}^m$-valued polynomial in the indeterminate $X = (X^1, \ldots, X^n)$. By (3.8) we have

$$\varphi\pi'\tau = \varphi\tau_0 = \sum_{|\alpha|\leqslant\mu} X^\alpha \tau_{0,\alpha}/\alpha! + \sum_1^n Y^\nu \left(\sum_{|\alpha|\leqslant\mu} X^\alpha \tau_{0,\alpha+1_\nu}/\alpha!\right) = \pi\tau_0 + \sum_1^n Y^\nu \delta_\nu \tau_0,$$

where δ_ν is the map in Example 3.2 defined by the coordinate and trivialization chosen above. By (1.3) and (1.4) we have $J_1(\pi)\tau = \pi\tau_0 + \sum_1^n Y^\nu \pi\tau_\nu$ so that $J_1(\pi)\tau - \varphi\pi'\tau = \sum_1^n Y^\nu(\pi\tau_\nu - \delta_\nu\tau_0)$ and $k\tau \in T^* \otimes J_\mu(E)$ is equal to $\sum_1^n dx^\nu \otimes (\pi\tau_\nu - \delta_\nu\tau_0)$. Thus

$$D\sigma(x) = \sum_1^n dx^\nu \otimes (\pi\partial_\nu\sigma(x) - \delta_\nu\sigma(x)) \tag{5.2}$$

for sections σ of $J_{\mu+1}(E)_U$.

We now imbed our discussion of D in the following:

PROPOSITION 5.1. *For each pair of non-negative integers i, μ there exists a first order differential operator*

$$D: \underline{\Lambda^i T^* \otimes J_{\mu+1}(E)} \to \underline{\Lambda^{i+1} T^* \otimes J_\mu(E)} \tag{5.3}$$

such that

(i) *$D(\xi \wedge \sigma) = d\xi \wedge \pi\sigma + (-1)^j \xi \wedge D\sigma$ for each section σ of $\Lambda^i T^* \otimes J_{\mu+1}(E)$ and each j-form ξ on M,*

(ii) *$\underline{E} \xrightarrow{j_{\mu+1}} \underline{J_{\mu+1}(E)} \xrightarrow{D} \underline{T^* \otimes J_\mu(E)}$ is an exact sequence,*

(iii) *$D^2 = 0$.*

Moreover, (i) *and* (ii) *determine D uniquely.*

Proof. We first verify the uniqueness statement. Suppose there are two operators D satisfying (i) and (ii) and denote their difference by K; we must show that $K = 0$. By (i) $K(\xi \wedge \sigma) = (-1)^j \xi \wedge K\sigma$ for j-forms ξ; so it suffices to show that K annihilates sections of $J_{\mu+1}(E)$. Again by (i) K is linear over the functions and therefore must be a differential operator of order 0. Thus K operates on elements of $J_{\mu+1}(E)$, and it suffices to show that K annihilates $J_{\mu+1}(E)$. Now if $\sigma \in J_{\mu+1}(E)_x$, we can find a section s of E such that $j_{\mu+1}s(x) = \sigma$. By (ii) we have $K\sigma = (Kj_{\mu+1}s)(x) = 0$. Therefore the uniqueness statement holds.

A similar argument shows that (i) and (ii) imply (iii). From (i) we compute that $D^2(\xi \wedge \sigma) = D(d\xi \wedge \pi\sigma) + (-1)^j D(\xi \wedge D\sigma) = (-1)^{j+1} d\xi \wedge D\pi\sigma + (-1)^j d\xi \wedge \pi D\sigma + \xi \wedge D^2\sigma = \xi \wedge D^2\sigma$. Thus it suffices to show that D^2 annihilates sections of $J_{\mu+1}(E)$. Since D^2 is linear over functions and $D^2 j_{\mu+1} = 0$ we can conclude as before that $D^2 = 0$.

It remains to establish the existence of D satisfying (i) and (ii). Because of the uniqueness statement, it suffices to prove existence locally. In a local coordinate $x = (x^1, ..., x^n)$ we define

$$D\sigma(x) = \sum_1^n dx^\nu \wedge (\pi\partial_\nu\sigma(x) - \delta_\nu\sigma(x)) \tag{5.4}$$

for local sections σ of $\Lambda^i T^* \otimes J_{\mu+1}(E)$. By (5.2) this agrees with our previous definition in the case $i = 0$, and (i) clearly holds. To verify (ii) we let σ be a section of $J_{\mu+1}(E)$ satisfying $D\sigma = 0$. Then by (5.2) we have $\partial_\nu\sigma = \delta_\nu\sigma$ in each local coordinate, and thus by (3.7) we have

$\sigma_{\alpha+1_\nu}=\partial_\nu\sigma_\alpha$. It follows by repeated use of this identity that $\sigma_\alpha=\partial_\alpha\sigma_0$ for each $|\alpha|\leqslant\mu+1$, and thus $\sigma=j_{\mu+1}\sigma_0$. By reversing the argument just given one sees that $Dj_\mu=0$. Thus (ii) holds and the proof is complete.

From the operators (5.3) we may form a sequence

$$0 \longrightarrow \underline{E} \xrightarrow{j_{\mu+n}} \underline{J_{\mu+n}(E)} \xrightarrow{D} \underline{T^*\otimes J_{\mu+n-1}(E)} \xrightarrow{D} \cdots \xrightarrow{D} \underline{\Lambda^n T^*\otimes J_\mu(E)} \longrightarrow 0 \tag{5.5}$$

for each $\mu\geqslant 0$. The following proposition, together with (3.4), shows that these sequences are compatible with the jet representation of differential operators.

PROPOSITION 5.2. *Let $\mathcal{D}:\underline{E}\to\underline{F}$ be a differential operator of order μ_0. Then for $\mu\geqslant\mu_0$ the diagram*

$$\begin{array}{ccc} \underline{\Lambda^i T^*\otimes J_{\mu+1}(E)} & \xrightarrow{\varrho_{\mu+1}} & \underline{\Lambda^i T^*\otimes J_{\mu+1-\mu_0}(F)} \\ \downarrow D & & \downarrow D \\ \underline{\Lambda^{i+1} T^*\otimes J_\mu(E)} & \xrightarrow{\varrho_\mu} & \underline{\Lambda^{i+1} T^*\otimes J_{\mu-\mu_0}(F)} \end{array} \tag{5.6}$$

commutes, where we have written ϱ_μ for $1\otimes\varrho_\mu(j_{\mu-\mu_0}\mathcal{D})$ and $\varrho_{\mu+1}$ for the corresponding expression.

Proof. Write $K=\varrho_\mu D-D\varrho_{\mu+1}$. By (i) of Prop. 5.1 and by (4.1) we have $K(\xi\wedge\sigma)=d\xi\wedge(\varrho_\mu\pi-\pi\varrho_{\mu+1})\sigma+(-1)^j\xi\wedge K\sigma=(-1)^j\xi\wedge K\sigma$ for j-forms ξ. Thus it suffices to prove the proposition in the case $i=0$. Now K is linear over functions, and by (ii) of Prop. 5.1 we have $Kj_{\mu+1}=\varrho_\mu Dj_{\mu+1}-D\varrho_{\mu+1}j_{\mu+1}=0-Dj_{\mu+1-\mu_0}\mathcal{D}=0$. Arguing as in the proof of Prop. 5.1 we conclude that $K=0$.

If $\mathcal{D}:\underline{E}\to\underline{F}$ is a differential operator of order μ_0, then Prop. 5.2 shows that D restricts to a differential operator from $\Lambda^i T^*\otimes\underline{R_{\mu+1}}$ to $\Lambda^{i+1}T^*\otimes\underline{R_\mu}$ for $\mu\geqslant\mu_0$. Thus for $\mu\geqslant\mu_0$ we obtain a sequence

$$0 \longrightarrow \theta \xrightarrow{j_{\mu+n}} \underline{R_{\mu+n}} \xrightarrow{D} \underline{T^*\otimes R_{\mu+n-1}} \xrightarrow{D} \cdots \xrightarrow{D} \underline{\Lambda^n T^*\otimes R_\mu} \longrightarrow 0, \tag{5.7$_\mu$}$$

where θ is the subsheaf $j_{\mu+n}^{-1}(\underline{R_{\mu+n}})$ of $\underline{E}$. From (3.4) we see that θ is the sheaf of germs of solutions s to the homogeneous equation $\mathcal{D}s=0$, and by (ii) of Prop. 5.1 the sequence (5.7) is exact at $\underline{R_{\mu+n}}$. The sequences (5.7) are called the first Spencer sequences for $\mathcal{D}$ (see Spencer [15]).

The fundamental problem, of course, is to determine when the Spencer sequences are exact. From experience with the Dolbeault sequence in complex analysis one would expect a wealth of applications in the case where exactness holds. Indeed, an important theorem of D. G. Quillen (see Prop. 8.3 below) shows that the exactness of (5.7) implies the local solvability of the inhomogeneous equation $\mathcal{D}s=t$, and in D. C. Spencer's deformation theory of pseudogroup structures (see [15] and [16]) the exactness of (5.7) plays an important role.

6. The trivial operator

Let E be a vector bundle over M, and consider the trivial differential operator which maps each section of E into the zero section of E. For this operator we have $R_\mu = J_\mu(E)$ for $\mu \geqslant 0$, $g_\mu = G_\mu = \ker\,(J_\mu(E) \xrightarrow{\pi} J_{\mu-1}(E))$, and $g_0 = G_0 = E$. Thus the first Spencer sequences are the sequences (5.5). From Prop. 5.2 (with $\mathcal{D} = \mathbf{1}_E$, considered as a first order operator) we obtain a commutative diagram

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & & 0 \arrow[d] & \\
 & 0 \arrow[r] & \underline{G_\mu} \arrow[r, "-\delta"] \arrow[d] & \underline{G^1_{\mu-1}} \arrow[r, "-\delta"] \arrow[d] & \cdots \arrow[r, "-\delta"] & \underline{G^n_{\mu-n}} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \underline{E} \arrow[r] \arrow[d, "\mathbf{1}"] & \underline{J_\mu(E)} \arrow[r, "D"] \arrow[d, "\pi"] & \underline{J^1_{\mu-1}(E)} \arrow[r, "D"] \arrow[d, "\pi"] & \cdots \arrow[r, "D"] & \underline{J^n_{\mu-n}(E)} \arrow[r] \arrow[d, "\pi"] & 0 \\
0 \arrow[r] & \underline{E} \arrow[r] & \underline{E_{\mu-1}(E)} \arrow[r, "D"] \arrow[d] & \underline{J^1_{\mu-2}(E)} \arrow[r, "D"] \arrow[d] & \cdots \arrow[r, "D"] & \underline{J^n_{\mu-n-1}(E)} \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 & & 0 &
\end{tikzcd} \tag{6.1}$$

with exact columns. Here we have made the abbreviations $J^i_\mu(E) = \Lambda^i T^* \otimes J_\mu(E)$ and $G^i_\mu = \Lambda^i T^* \otimes G_\mu$, and we have written $-\delta$ for the restriction of D to $\underline{G^i_\mu}$. It follows from Prop. 5.1 (i) that δ is linear over functions and thus must be a bundle map; in fact, from (5.4) we see that in a local coordinate

$$\delta\sigma = \textstyle\sum_1^n dx^\nu \wedge \delta_\nu \sigma \tag{6.2}$$

for elements σ in G_μ. Since δ_ν is the formal jet representation for differentiation with respect to x^ν, (6.2) suggests that δ be interpreted as formal exterior differentiation (see Spencer [14]).

Our aim here is to establish the exactness of the first Spencer sequences for the trivial operator. The diagram (6.1) leads us to the following proposition.

PROPOSITION 6.1. *If we interpret G^i_μ as 0 whenever $\mu < 0$, then for each $\mu \geqslant 1$ the sequence*

$$0 \longrightarrow G_\mu \xrightarrow{\ \delta\ } G_{\mu-1} \xrightarrow{\ \delta\ } \cdots \xrightarrow{\ \delta\ } G_{\mu-n} \xrightarrow{\ \delta\ } 0$$

is exact.

Proof. It suffices to work in the fibers over a single point and with a local coordinate $x=(x^1, \ldots, x^n)$. Thus for the purposes of this proof we may interpret G^i_μ as the space of forms $\sigma=\sum_{|I|=i}\sigma_I dx^I$, where σ_I is a homogeneous $\mathbf{R}^m$-valued polynomial of degree μ in $x=(x^1, \ldots, x^n)$. The proof is by induction on n.

If $n=1$, then in view of (6.2) we are to verify that the map $\delta_1: G_\mu \to G_{\mu-1}$ is bijective if $\mu \geqslant 1$. The injectivity is immediate, and surjectivity holds even for $n \geqslant 1$, $\sigma \in G_{\mu-1}$ being the image under δ_n of the element

$$\delta_{-n}\,\sigma = \sum_{|\alpha|=\mu-1} X^{\alpha+1_\nu}\sigma_\alpha/(\alpha+1_\nu)! \tag{6.3}$$

in G_μ.

Now assume that the proposition holds when n is replaced by $n-1$, and let $\sigma \in G^k_{\mu-k}$ satisfy $\delta\sigma=0$. We may assume that $\mu \geqslant k$ since otherwise there is nothing to prove. Thus by the remark above we can choose $\tau \in G^{k-1}_{\mu+1-k}$ such that $\delta_n \tau = \sigma \overline{\wedge}\, \partial/\partial x^n$. Now consider $\sigma_1 = \sigma - \delta\tau = (dx^n \wedge \sigma)\overline{\wedge}\,\partial/\partial x^n + dx^n \wedge (\sigma \overline{\wedge}\,\partial/\partial x^n) - dx^n \wedge \delta_n \tau - \sum_1^{n-1} dx^\nu \wedge \delta_\nu \tau = (dx^n \wedge \sigma)\overline{\wedge}\,\partial/\partial x^n - \sum_1^{n-1} dx^\nu \wedge \delta_\nu \tau$. $\sigma_1 \overline{\wedge}\,\partial/\partial x^n = 0$; and since $0=\delta\sigma=\delta\sigma_1 = dx^n \wedge \delta_n \sigma_1 +$ (terms independent of dx^n), we must have $\delta_n \sigma_1 = 0$. Having seen that σ_1 is independent of x^n and dx^n, we use the inductive hypothesis to conclude that σ_1, and hence σ, is in the range of δ. The proof is complete.

PROPOSITION 6.2. *If we interpret $J^i_\mu(E)$ as 0 whenever $\mu<0$, then for each $\mu \geqslant 0$ the sequence*

$$0 \longrightarrow \underline{E} \longrightarrow \underline{J_\mu(E)} \xrightarrow{D} \underline{J^1_{\mu-1}(E)} \xrightarrow{D} \cdots \xrightarrow{D} \underline{J^n_{\mu-n}(E)} \longrightarrow 0$$

is exact.

Proof (by induction on μ): For $\mu=0$ the sequence reduces to $0 \to \underline{E} \xrightarrow{1} \underline{E} \to 0$. The inductive step is given by Prop. 6.1 and diagram 6.1.

Prop. 6.2 states that the Spencer sequences for the trivial operator are exact. Note that the arguments leading to Prop. 6.2 do not carry over immediately to the case of an arbitrary differential operator $\mathcal{D}: \underline{E} \to \underline{F}$ of order μ_0. For example, the map $\delta_n: g_{\mu+1} \to g_\mu$, discussed in the proof of Prop. 6.1, is not always surjective; and even assuming the analogue of Prop. 6.1, the induction used to prove Prop. 6.2 does not always have a trivial beginning. In the next two sections we will discuss the analogues of Props. 6.1 and 6.2 in the general case.

At several points in the discussion to follow it will be convenient to consider first order operators only. The following proposition shows that this can be done without loss of generality.

PROPOSITION 6.3. *Let $\mathcal{D}:\underline{E}\to\underline{F}$ have order μ_0 and let $\mu\geqslant\mu_0$. For each $\nu\geqslant 1$ denote by R'_ν the image of $R_{\mu+\nu}(j_{\mu+\nu-\mu_0}\mathcal{D})$ under the injection $\varphi:J_{\mu+\nu}(E)\to J_\nu(J_\mu(E))$. Then there exists a first order differential operator $\mathcal{D}':\underline{J_\mu(E)}\to\underline{F}'$ such that*

$$R_\nu(j_{\nu-1}\mathcal{D}')=R'_\nu,\quad \nu=1,2,\ldots. \tag{6.4}$$

Proof. Assume at first that $\mathcal{D}$ is the trivial operator on E, and write Q for the quotient bundle $J_1(J_\mu(E))/\varphi(J_{\mu+1}(E))$. For $\mathcal{D}'$ we choose the differential operator $K:\underline{J_\mu(E)}\to\underline{Q}$ defined by

$$K=kj_1,$$

where $k:J_1(J_\mu(E))\to Q$ is the natural map. It follows immediately that

$$R_1(K)=\varphi(J_{\mu+1}(E))$$

so that (6.4) holds for $\nu=1$. Thus $0=\varrho_1(K)\varrho_{\mu+1}(j_1j_\mu)=\varrho_{\mu+1}(Kj_\mu)$, and hence $Kj_\mu=0$. This shows that j_μ maps $\underline{E}$ into the kernel θ of K. Also $\varrho_\nu(j_{\nu-1}K)\varphi=\varrho_\nu(j_{\nu-1}K)\varrho_{\mu+\nu}(j_\nu j_\mu)=\varrho_{\mu+\nu}(j_{\nu-1}Kj_\mu)=0$ so that φ maps $J_{\mu+\nu}(E)$ into $R_\nu(j_{\nu-1}K)$. To prove that

$$R_2(j_1K)=\varphi(J_{\mu+2}(E))$$

we may now use the following diagrams which commute by Prop. 5.2:

$$\begin{tikzcd}
0 \arrow[r] & \theta \arrow[r, "j_1"] & \underline{R_1(K)} \arrow[r, "D"] & \underline{J_\mu(E)\otimes T^*} \\
0 \arrow[r] & \underline{E} \arrow[r, "j_{\mu+1}"] \arrow[u, "j_\mu"] & \underline{J_{\mu+1}(E)} \arrow[r, "D"] \arrow[u, "\varphi"] & \underline{J_\mu(E)\otimes T^*}, \arrow[u, "1"]
\end{tikzcd}$$

$$\begin{tikzcd}
0 \arrow[r] & \theta \arrow[r, "j_2"] & \underline{R_2(j_1K)} \arrow[r, "D"] & \underline{R_1(K)\otimes T^*} \arrow[r, "D"] & \underline{J_\mu(E)\otimes\Lambda^2T^*} \\
0 \arrow[r] & \underline{E} \arrow[r, "j_{\mu+2}"] \arrow[u, "j_\mu"] & \underline{J_{\mu+2}(E)} \arrow[r, "D"] \arrow[u, "\varphi"] & \underline{J_{\mu+1}(E)\otimes T^*} \arrow[r, "D"] \arrow[u, "\varphi"] & \underline{J_\mu(E)\otimes\Lambda^2T^*}. \arrow[u, "1"]
\end{tikzcd}$$

In the first diagram the bottom row is exact, and φ is an isomorphism. It follows by diagram chasing that j_μ gives an isomorphism of $\underline{E}$ onto θ. Thus, in the second diagram, we may use the exactness of the bottom row to conclude by diagram chasing that $R_2(j_1K)=\varphi(J_{\mu+2}(E))$. The proof of

$$R_\nu(j_{\nu-1}K)=\varphi(J_{\mu+\nu}(E)) \tag{6.5}$$

for $\nu\geqslant 3$ now follows by an inductive argument based on the diagram

$$\begin{tikzcd}
0 \arrow[r] & \theta \arrow[r] & \underline{R_\nu(j_{\nu-1}K)} \arrow[r] & \underline{R_{\nu-1}(j_{\nu-2}K)\otimes T^*} \arrow[r] & \underline{R_{\nu-2}(j_{\nu-3}K)\otimes\Lambda^2T^*} \\
0 \arrow[r] & \underline{E} \arrow[r] \arrow[u, "j_\mu"] & \underline{J_{\mu+\nu}(E)} \arrow[r] \arrow[u, "\varphi"] & \underline{J_{\mu+\nu-1}(E)\otimes T^*} \arrow[r] \arrow[u, "\varphi"] & \underline{J_{\mu+\nu-2}(E)\otimes\Lambda^2T^*}. \arrow[u, "\varphi"]
\end{tikzcd}$$

In the case of a general differential operator we write ϱ for $\varrho_\mu(j_{\mu-\mu_0}\mathcal{D})$ and define $\mathcal{D}'$ to be the operator

$$K \oplus j_1\varrho : \underline{J_\mu(E)} \to Q \oplus \underline{J_1(J_{\mu-\mu_0}(F))}.$$

The first summand in $\mathcal{D}'$ has already been discussed; as to the second, we claim that the diagram

$$\begin{array}{ccc}
J_{\mu+\nu}(E) & \xrightarrow{\varrho_{\mu+\nu}(j_{\nu+\mu-\mu_0}\mathcal{D})} & J_{\nu+\mu-\mu_0}(F) \\
\Big| & & \downarrow \varphi \\
\varphi \Big| & & J_\nu(J_{\mu-\mu_0}(F)) \\
\downarrow & & \downarrow \varphi \\
J_\nu(J_\mu(E)) & \xrightarrow{\varrho_\nu(j_{\nu-1}j_1\varrho)} & J_{\nu-1}(J_1(J_{\mu-\mu_0}(F)))
\end{array}$$

commutes. Indeed, by Prop. 3.2 $\varrho_\nu(j_{\nu-1}j_1\varrho)\varphi = \varrho_\nu(j_{\nu-1}j_1\varrho)\varrho_{\mu+\nu}(j_\nu j_\mu) = \varrho_{\mu+\nu}(j_{\nu-1}j_1\varrho j_\mu) = \varrho_{\mu+\nu}(j_{\nu-1}j_1 j_{\mu-\mu_0}\mathcal{D}) = \varrho_\nu(j_{\nu-1}j_1)\varrho_{\nu+\mu-\mu_0}(j_\nu j_{\mu-\mu_0})\varrho_{\mu+\nu}(j_{\nu+\mu-\mu_0}\mathcal{D}) = \varphi\varphi\varrho_{\mu+\nu}(j_{\nu+\mu-\mu_0}\mathcal{D})$. To establish (6.4) we let $\sigma \in J_\nu(J_\mu(E))$ be in the kernel of $\varrho_\nu(j_{\nu-1}\mathcal{D}') = \varrho_\nu(j_{\nu-1}K) \oplus \varrho_\nu(j_{\nu-1}j_1\varrho)$. By (6.5) we have $\sigma = \varphi\tau$ for some $\tau \in J_{\mu+\nu}(E)$. Thus, using the preceding diagram, we have $0 = \varrho_\nu(j_{\nu-1}j_1\varrho)\varphi\tau = \varphi\varphi\,\varrho_{\mu+\nu}(j_{\nu+\mu-\mu_0}\mathcal{D})\tau$. Since the maps φ are injective, $\tau \in R_{\mu+\nu}(j_{\nu+\mu-\mu_0}\mathcal{D})$ and $\sigma \in R'_\nu$. Therefore $R_\nu(j_{\nu-1}\mathcal{D}') \subset R'_\nu$. Be reversing the argument we obtain the opposite inclusion, and the proof is complete.

Prop. 6.3 states that φ restricts to an isomorphism of $R_{\nu+\mu}(j_{\nu+\mu-\mu_0}\mathcal{D})$ with $R_\nu(j_{\nu-1}\mathcal{D}')$. Since $\varphi\pi = \pi\varphi$ by Prop. 3.2, φ restricts further to an isomorphism of $g_{\nu+\mu}(j_{\nu+\mu-\mu_0}\mathcal{D})$ with $g_\nu(j_{\nu-1}\mathcal{D}')$. Since φ commutes with the maps δ_k defined by a local coordinate, these isomorphisms are compatible with δ and δ_k. We will use this fact in the next section.

7. Stability of the first Spencer sequences

Let $\mathcal{D}: \underline{E} \to \underline{F}$ be a regular differential operator of order μ_0, and let $R_\mu, g_\mu, \varrho_\mu, \mu \geqslant \mu_0$, be the corresponding data as previously defined. Combining Prop. 5.2 with diagram (6.1) we obtain for each $\mu \geqslant \mu_0$ a commutative diagram

$$\begin{array}{ccccccccccccc}
 & & & & 0 & & 0 & & & & 0 & & \\
 & & & & \downarrow & & \downarrow & & & & \downarrow & & \\
 & & 0 & \longrightarrow & \underline{g_{\mu+n+1}} & \xrightarrow{-\delta} & \underline{g^1_{\mu+n}} & \xrightarrow{-\delta} & \cdots & \xrightarrow{-\delta} & \underline{g^n_{\mu+1}} & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & & & \downarrow & & \\
0 & \longrightarrow & \theta & \longrightarrow & \underline{R_{\mu+n+1}} & \xrightarrow{D} & \underline{R^1_{\mu+n}} & \xrightarrow{D} & \cdots & \xrightarrow{D} & \underline{R^n_{\mu+1}} & \longrightarrow & 0 \\
 & & & & \downarrow \pi & & \downarrow \pi & & & & \downarrow \pi & & \\
0 & \longrightarrow & \theta & \longrightarrow & \underline{R_{\mu+n}} & \xrightarrow{D} & \underline{R^1_{\mu+n-1}} & \xrightarrow{D} & \cdots & \xrightarrow{D} & \underline{R^n_\mu} & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & & & \downarrow & & \\
 & & & & 0 & & 0 & & & & 0 & &
\end{array} \tag{7.1}$$

where the columns are exact because $\mathcal{D}$ is regular.

Let $U \subset M$ be a coordinate disk with coordinate $x=(x^1, \ldots, x^n)$, and choose a trivialization $E_U \to U \times \mathbf{R}^m$. In terms of this data we have maps $\delta_\nu : J_{\mu+1}(E)_U \to J_\mu(E)_U$, $\nu = 1, \ldots, n$. Note that if $\sigma \in g_{\mu+1}$, then $\delta_\nu \sigma = \delta\sigma \,\overline{\wedge}\, \partial/\partial x^\nu \in g_\mu$. We will use the following notation for various objects P (e.g., for one of the bundles $(g_\mu)_U$ or for a fiber $(g_\mu)_y$, $y \in U$):

$$\begin{aligned} P^{(n)} &= P, \\ P^{(\nu)} &= \{\sigma \in P \mid \delta_n \sigma = \ldots = \delta_{\nu+1}\sigma = 0\}, \quad 1 \leqslant \nu < n, \\ P^{(0)} &= 0. \end{aligned} \tag{7.3}$$

Definition. The local coordinate x is said to be $\mathcal{D}$-regular at $y \in U$ if the maps

$$\delta_\nu : (g_{\mu+1})_y^{(\nu)} \to (g_\mu)_y^{(\nu)} \tag{7.4}$$

are surjective for each $\mu \geqslant \mu_0$ and each $1 \leqslant \nu \leqslant n$. The operator $\mathcal{D}$ is called involutive if there is a $\mathcal{D}$-regular coordinate at each $y \in M$.

Using (1.3) one can check that the choice of trivialization does not effect the $\mathcal{D}$-regularity of coordinate at a point; we decline to prove this fact, however, as it will not be used in an essential way. Note that involutiveness is exactly the property required to carry out the induction used in the proof of Prop. 6.1. Thus we have:

PROPOSITION 7.1. *If $\mathcal{D}$ is involutive, then for each $\mu \geqslant \mu_0$ the sequence*

$$0 \longrightarrow g_{\mu+n} \xrightarrow{\delta} g^1_{\mu+n-1} \xrightarrow{\delta} \cdots \xrightarrow{\delta} g^n_\mu \longrightarrow 0 \tag{7.5}$$

is exact.

The usefulness of Prop. 7.1 is extended by the following proposition.

PROPOSITION 7.2. *There exists an integer $\mu_1(n, m, \mu_0)$, depending only on $n = \dim M$, $m =$ fiber dim E, and $\mu_0 =$ order of $\mathcal{D}$, such that for each $\mu \geqslant \mu_1(n, m, \mu_0)$ the operator $j_{\mu-\mu_0}\mathcal{D}$ is involutive.*

Postponing the proof for a moment, we use Prop. 4.1 to reformulate Prop. 7.2 as follows.

PROPOSITION 7.2′. *For each $y \in M$ there exists a coordinate such that the maps* (7.4) *are surjective for each $\mu \geqslant \mu_1(n, m, \mu_0)$ and each $1 \leqslant \nu \leqslant n$.*

Also using Prop. 4.1 we combine Props. 7.1 and 7.2 to obtain:

PROPOSITION 7.3. *For each $\mu \geqslant \mu_1(n, m, \mu_0)$ the sequence* (7.5) *is exact.*

Diagram (7.1) and the preceeding proposition now establish the stability of the first Spencer sequences:

PROPOSITION 7.4. *If $\mathcal{D}$ is regular, then for each $\mu \geqslant \mu_1(n, m, \mu_0)$ the sequence* $(5.7)_\mu$ *has the same cohomology as the sequence* $(5.7)_{\mu+1}$.

We now turn to the proof of Prop. 7.2′, which will occupy several pages. Recalling the remaks at the end of section 6, we may assume that $\mu_0 = 1$; we may also work in the fibers over a single point $y \in M$. As in section 6, we will write $G_0 = E$ and $G_\mu = \ker(J_\mu(E) \xrightarrow{\pi} J_{\mu-1}(E))$ for $\mu \geqslant 1$.

LEMMA 7.5. *For each $\mu \geqslant 1$ the maps*

$$\delta_\nu : (g_{\mu+1})_y^{(\nu)} \to (g_\mu)_y^{(\nu)}, \quad \nu = 1, \ldots, n \tag{7.6}$$

are surjective if and only if the maps

$$\varrho_{\mu+1} : (G_{\mu+1})_y^{(\nu)} \to (\varrho_{\mu+1}(G_{\mu+1})_y)^{(\nu)} \quad \nu = 1, \ldots, n-1 \tag{7.7}$$

are surjective.

Proof. Suppressing the subscript y, we consider for $0 \leqslant \nu \leqslant n-1$ the following exact commutative diagram:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & g_{\mu+1}^{(\nu)} \arrow[r] \arrow[d] & G_{\mu+1}^{(\nu)} \arrow[r] \arrow[d] & \varrho_{\mu+1}(G_{\mu+1}^{(\nu+1)})^{(\nu)} \arrow[d] & \\
0 \arrow[r] & g_{\mu+1}^{(\nu+1)} \arrow[r] \arrow[d] & G_{\mu+1}^{(\nu+1)} \arrow[r] \arrow[d] & \varrho_{\mu+1}(G_{\mu+1}^{(\nu+1)}) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & g_{\mu}^{(\nu+1)} \arrow[r] & G_{\mu}^{(\nu+1)} \arrow[r] \arrow[d] & \varrho_{\mu}(G_{\mu}^{(\nu+1)}) \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 &
\end{tikzcd}$$

By diagram chasing we find that the surjectivity of the maps (7.6) is equivalent to $\varrho_{\mu+1}(G_{\mu+1}^{(\nu)}) = \varrho_{\mu+1}(G_{\mu+1}^{(\nu+1)})^{(\nu)}$ for $1 \leqslant \nu \leqslant n-1$. Thus if the maps (7.6) are surjective, then $\varrho_{\mu+1}(G_{\mu+1}^{(\nu)}) = \varrho_{\mu+1}(G_{\mu+1}^{(\nu+1)})^{(\nu)} = [\varrho_{\mu+1}(G_{\mu+1}^{(\nu+2)})^{(\nu+1)}]^{(\nu)} = \varrho_{\mu+1}(G_{\mu+1}^{(\nu+2)})^{(\nu)} = \ldots = \varrho_{\mu+1}(G_{\mu+1})^{(\nu)}$, and the maps (7.7) are surjective. Conversely, if the maps (7.7) are surjective, then for $1 \leqslant \nu \leqslant n-1$ we have $\varrho_{\mu+1}(G_{\mu+1}^{(\nu)}) \subset \varrho_{\mu+1}(G_{\mu+1}^{(\nu+1)})^{(\nu)} \subset \varrho_{\mu+1}(G_{\mu+1})^{(\nu)} \subset \varrho_{\mu+1}(G_{\mu+1}^{(\nu)})$, and the maps (7.6) are surjective.

LEMMA 7.6. *If the maps* (7.6) *are surjective for $\mu = r$, then they are surjective for $\mu = r+1$ and hence for all $\mu \geqslant r$.*

Proof. By Lemma 7.5 the maps (7.7) are surjective for $\mu = r$. We show by induction on ν that they are surjective for $\mu = r+1$. The inductive step follows from the five lemma and the following diagram.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & G_{r+2}^{(\nu-1)} & \longrightarrow & G_{r+2}^{(\nu)} & \xrightarrow{\delta_\nu} & G_{r+1}^{(\nu)} & \longrightarrow & 0 \\
 & & \downarrow \varrho_{r+2} & & \downarrow \varrho_{r+2} & & \downarrow \varrho_{r+1} & & \\
0 & \longrightarrow & \varrho_{r+2}(G_{r+2})^{(\nu-1)} & \longrightarrow & \varrho_{r+2}(G_{r+2})^{(\nu)} & \xrightarrow{\delta_\nu} & \varrho_{r+1}(G_{r+1})^{(\nu)} & \longrightarrow & 0
\end{array}$$

Note that the left column consists of zeros when $\nu=1$; thus the induction has a beginning.

Our proof of Prop. 7.2′ (in the case $\mu_0=1$) is by induction on n. If $n=0$, the proposition holds trivially; assume that it holds when n is replaced by $n-1$. Set $a=\mu_1(n-1,m,1)$ and $c=m\binom{a+n}{n-1}+a+3$. Among all coordinates at y choose a coordinate $x=(x^1,\dots,x^n)$ which maximizes

$$\sum_{\mu=a+1}^{c}\dim \delta_n^{c-\mu}(g_c)_y. \tag{7.8}$$

Since the data $(g_\mu)_y$ depend only on the first order coefficients of $\mathcal{D}$ at y, we may assume that $\mathcal{D}$ is given by

$$\mathcal{D}s=\sum_1^n A_\nu\partial_\nu s$$

in the coordinate x, where A_ν is a constant $k\times m$ matrix. Since the data $(g_\mu)_y^{(n-1)}$ correspond to the operator $\sum_1^{n-1}A_\nu\partial_\nu$ in $n-1$ variables, the inductive hypothesis permits us to assume that (7.6) is surjective for each $1\leqslant\nu\leqslant n-1$ and each $\mu\geqslant a$.

Suppressing the subscript y, we claim that

$$g_\mu^{(n-1)}\subset\delta_n^{c-\mu}(g_c) \tag{7.9}$$

for $\mu=a+1, a+2, \dots, c$. Otherwise, we can choose an integer μ in this range and an element $\sigma\in g_\mu^{(n-1)}$ such that $\sigma\notin\delta_n^{c-\mu}(g_c)$. By hypothesis we can choose $\eta\in g_c^{(n-1)}$ such that $\delta_{n-1}^{c-\mu}\eta=\sigma$. We also choose $\eta_1,\dots,\eta_N\in g_c$ such that $\delta_n^{c-\mu}\eta_1,\dots,\delta_n^{c-\mu}\eta_N$ form a basis of $\delta_n^{c-\mu}(g_c)$. Now consider the new coordinate $x'_{n-1}=x_{n-1}-tx_n$, $x'_\nu=x_\nu$ for $\nu\neq n-1$. Corresponding to differentiation with respect to x'_n we have the map $\delta'_n=\delta_n+t\delta_{n-1}$. For small t the elements $(\delta'_n)^{c-\mu}\eta_1,\dots,(\delta'_n)^{c-\mu}\eta_N$ are linearly independent and span a subspace which converges to $\delta_n^{c-\mu}(g_c)$ as $t\to 0$. Since $\sigma\notin\delta_n^{c-\mu}(g_c)$, σ must be linearly independent of $(\delta'_n)^{c-\mu}\eta_1,\dots,(\delta'_n)^{c-\mu}\eta_N$ for small $t>0$. Since $\sigma=t^{\mu-c}(\delta'_n)^{c-\mu}\eta$ is in $(\delta'_n)^{c-\mu}(g_c)$, this means that $\dim(\delta'_n)^{c-\mu}(g_c)>\dim\delta_n^{c-\mu}(g_c)$ for small $t>0$. By semi-continuity this is also true for the other summands in (7.8), only with "$>$" replaced by "$\geqslant$". Thus we have obtained a contradiction to the maximal property of x.

Now let $a+1\leqslant\mu<c$, and let $\sigma\in g_\mu$. We write σ as a polynomial

$$\sigma=\sum_0^\mu\sigma_\nu X_n^{\mu-\nu}/(\mu-\nu)!, \tag{7.10}$$

where each σ_ν is a homogeneus $\mathbf{R}^m$-valued polynomial of degree ν in $X_1, \ldots, X_{n-1}$. A straightforward calculation shows that

$$\varrho_\mu \sigma = \sum_{\nu=0}^{\mu-1} (\varrho_{\nu+1} \sigma_{\nu+1} + A_n \sigma_\nu) X_n^{\mu-\nu-1}/(\mu-\nu-1)!, \tag{7.11}$$

where $A_n \sigma_\nu$ is the polynomial obtained by applying A_n to each coefficient of the polynomial σ_ν. We extend the usual inner product in $\mathbf{R}^m$ to $\mathbf{R}^m$-valued polynomials by $\langle \sum X^\alpha \tau_\alpha, \sum X^\beta \zeta_\beta \rangle = \sum_\alpha \langle \tau_\alpha, \zeta_\alpha \rangle$ and let τ_{1+a} be the orthogonal projection of σ_{1+a} on $g_{1+a}^{(n-1)}$. By the preceeding paragraph there exists $\tau \in g_{\mu+1}$ such that $\delta_n^{\mu-a} \tau = \tau_{1+a}$. Thus if we expand $\sigma' = \sigma - \delta_n \tau$ in powers of X_n, as at (7.10), when we find that $\sigma'_{1+a} \perp g_{1+a}^{(n-1)}$. Applying the same argument to the coefficients of successively lower powers of X_n, we reduce σ modulo $\delta_n(g_{\mu+1})$ to an element

$$^*\sigma = \sum_0^\mu {}^*\sigma_\nu X_n^{\mu-\nu}/(\mu-\nu)!,$$

where $^*\sigma_\nu = \sigma_\nu$ for $\nu = 0, \ldots a$ and $^*\sigma_\nu \perp g_\nu^{(n-1)}$ for $\nu = a+1, \ldots, c$.

Define $^*g_\mu = \{ ^*\sigma \mid \sigma \in g_\mu \}$ for each $a+1 \leqslant \mu \leqslant c-1$. Then $\delta_n(^*g_{\mu+1}) \subset {}^*g_\mu$; and since by (7.11) $^*\sigma_0, \ldots, {}^*\sigma_a$ determine $^*\sigma$ uniquely, $\delta_n : {}^*g_{\mu+1} \to {}^*g_\mu$ is injective for $a+1 \leqslant \mu \leqslant c-2$. The number of maps here is equal to $c-a-2$ which is greater than dim $G_{a+1} \geqslant$ dim $^*g_{a+1}$. Thus at least one of them must be surjective. Therefore by Lemma 7.6 the maps $\delta_n : g_{\mu+1} \to g_\mu$ are surjective for all $\mu \geqslant c-2$. The proof is complete.

COROLLARY 7.7. *We can define the integer $\mu_1(n, m, \mu_0)$ by the following relations:*

(i) $\mu_1(0, m, 1) = 0$,

(ii) $\mu_1(n, m, 1) = m\binom{a+n}{n-1} + a + 1$ if $a = \mu_1(n-1, m, 1)$,

(iii) $\mu_1(n, m, \mu_0) = \mu_1(n, b, 1)$ if $b = \sum_0^{\mu_0} \binom{\nu+n-1}{n-1} m$.

Prop. 7.3 was first stated by D. C. Spencer [14] in the special case of the partial differential equations defining the infinitesimal transformations of a pseudogroup. Later S. Sternberg recognized the connection with E. Cartan's notion of involutiveness and introduced this concept into the theory. In [4] J. P. Serre proves that involutiveness is equivalent to the δ sequences being exact. For other proofs of the main theorem in this section see [3] and [12].

8. Exactness of the first Spencer sequences

We begin our discussion with the following proposition due to D. G. Quillen.

PROPOSITION 8.1. *Let $\mathcal{D}_0$ be a regular differential operator of order ν_0 from E_0 to E_1. Then there exists a sequence*

$$\underline{E}_0 \xrightarrow{\mathcal{D}_0} \underline{E}_1 \xrightarrow{\mathcal{D}_1} \underline{E}_2 \xrightarrow{\mathcal{D}_2} \underline{E}_3 \longrightarrow \cdots \tag{8.1}$$

of regular differential operators $\mathcal{D}_i$ *of order* ν_i *such that by passing to the jets we obtain exact sequences*

$$J_{\mu+\nu_i}(E_i) \rightarrow J_\mu(E_{i+1}) \rightarrow J_{\mu-\nu_{i+1}}(E_{i+2}) \tag{8.2}$$

for $i \geq 0$ *and* $\mu \geq \nu_{i+1}$.

Proof. It suffices to construct $\mathcal{D}_1$; the rest of (8.1) can then be obtained by iteration. We set $\nu_1 = n + \mu_1$ (n, fiber dim E_0, $\nu_0) - \nu_0$, and define E_2 to be the cokernel of $\varrho_{\nu_1+\nu_0}(j_{\nu_1}\mathcal{D}_0): J_{\nu_1+\nu_0}(E_0) \rightarrow J_{\nu_1}(E_1)$. Since $\mathcal{D}_0$ is regular, E_2 is a vector bundle. Composing j_{ν_1} with the natural map $J_{\nu_1}(E_1) \rightarrow E_2$, we obtain a differential operator $\mathcal{D}_1$ of order ν_1 from E_1 to E_2.

For $i = 0, 1, 2$ we write $G_{i,0} = E_i$ and $G_{i,\mu} = \ker(\pi: J_\mu(E_i) \rightarrow J_{\mu-1}(E_i))$ for $\mu \geq 1$. We prove by induction on μ that the sequences

$$G_{0,\mu+\nu_0} \rightarrow G_{1,\mu} \rightarrow G_{2,\mu-\nu_1} \tag{8.3}$$

are exact for $\mu \geq \nu_1$. (The maps here are $\varrho_{\mu+\nu_0}(j_\mu \mathcal{D}_0)$ and $\varrho_\mu(j_{\mu-\nu_1}\mathcal{D}_1)$.) Indeed, the case $\mu = \nu_1$ follows from the definition of E_2, and the inductive step is obtained by chasing the diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & g_{\mu+1+\nu_0} & \longrightarrow & G_{0,\mu+1+\nu_0} & \longrightarrow & G_{1,\mu+1} & \longrightarrow & G_{2,\mu+1-\nu_1} \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & g^1_{\mu+\nu_0} & \longrightarrow & G^1_{0,\mu+\nu_0} & \longrightarrow & G^1_{1,\mu} & \longrightarrow & G^1_{2,\mu-\nu_1} \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & g^2_{\mu+\nu_0-1} & \longrightarrow & G^2_{0,\mu+\nu_0-1} & \longrightarrow & G^2_{1,\mu-1} & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & g^3_{\mu-2+\nu_0} & \longrightarrow & G^3_{0,\mu+\nu_0-2}, & & & &
\end{array}$$

where the vertical maps are δ's and the columns are exact because of the choice of ν_1.

It now follows by induction on μ that the sequences (8.2) are exact for $i = 0$ and $\mu \geq \nu_1$. The case $\mu = \nu_1$ is trivial, and the inductive step is furnished by the diagram:

$$\begin{array}{ccccc}
0 & & 0 & & 0 \\
\downarrow & & \downarrow & & \downarrow \\
G_{0,\mu+1+\nu_0} & \longrightarrow & G_{1,\mu+1} & \longrightarrow & G_{2,\mu+1-\nu_1} \\
\downarrow & & \downarrow & & \downarrow \\
J_{\mu+1+\nu_0}(E) & \longrightarrow & J_{\mu+1}(E_1) & \longrightarrow & J_{\mu+1-\nu_1}(E_2) \\
\downarrow & & \downarrow & & \downarrow \\
J_{\mu+\nu_0}(E_0) & \longrightarrow & J_\mu(E_1) & \longrightarrow & J_{\mu-\nu_1}(E_2). \\
\downarrow & & \downarrow & & \downarrow \\
0 & & 0 & & 0
\end{array}$$

Since the regularity of $\mathcal{D}_1$ follows from (8.2), (8.3), and the diagram above, the proof is complete.

The following proposition shows that as far as exactness is concerned, the sequence (8.1) is "the best possible."

PROPOSITION 8.2. *Let $\mathcal{D}_0$ and $\mathcal{D}_1$ be as above, and suppose there exists an operator $\mathcal{D}$: $\underline{E} \to \underline{F}$ of order ν which makes $\underline{E}_0 \xrightarrow{\mathcal{D}_0} \underline{E}_1 \xrightarrow{\mathcal{D}} \underline{F}$ exact. Then $\underline{E}_0 \xrightarrow{\mathcal{D}_0} \underline{E}_1 \xrightarrow{\mathcal{D}_1} \underline{E}_2$ is exact.*

Proof. Since $j_{\mu-\nu}\mathcal{D}$ satisfies the same hypotheses as $\mathcal{D}$, we may assume that $\nu \geqslant \nu_1$. Note that $0 = \varrho_{\nu+\nu_0}(\mathcal{D}\mathcal{D}_0) = \varrho_\nu(\mathcal{D})\,\varrho_{\nu+\nu_0}(j_\nu \mathcal{D}_0)$, and recall that the sequence

$$J_{\nu+\nu_0}(E_0) \xrightarrow{\varrho_{\nu+\nu_0}(j_\nu \mathcal{D}_0)} J_\nu(E_1) \xrightarrow{\varrho_\nu(j_{\nu-\nu_1}\mathcal{D}_1)} J_{\nu-\nu_1}(E_2)$$

is exact. Thus we may factor $\varrho_\nu(\mathcal{D})$ through coker $\varrho_{\nu+\nu_0}(j_\nu \mathcal{D}_0)$, which is a bundle since $\mathcal{D}_0$ is regular, and extend the resulting map to a bundle map $h: J_{\nu-\nu_1}(E_2) \to F$. Thus $\varrho_\nu(\mathcal{D}) = h\varrho_\nu(j_{\nu-\nu_1}\mathcal{D}_1)$, and composing with j_ν we obtain $\mathcal{D} = hj_{\nu-\nu_1}\mathcal{D}_1$. Thus ker $\mathcal{D}_1 \subset$ ker $\mathcal{D} =$ im $\mathcal{D}_0$, and the proof is complete.

The relationship between (8.1) and the first Spencer sequence is expressed by the commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \theta & \longrightarrow & \underline{R_\mu} & \xrightarrow{D} & \underline{R^1_{\mu-1}} & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \underline{E}_0 & \longrightarrow & \underline{J_\mu(E_0)} & \xrightarrow{D} & \underline{J^1_{\mu-1}(E_0)} & \longrightarrow & \cdots \\
 & & \downarrow \mathcal{D}_0 & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \underline{E}_1 & \longrightarrow & \underline{J_{\mu-\nu_0}(E_1)} & \xrightarrow{D} & \underline{J^1_{\mu-\nu_0-1}(E_1)} & \longrightarrow & \cdots \\
 & & \downarrow \mathcal{D}_1 & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \underline{E}_2 & \longrightarrow & \underline{J_{\mu-\nu_0-\nu_1}(E_2)} & \xrightarrow{D} & \underline{J^1_{\mu-\nu_0-\nu_1-1}(E_2)} & \longrightarrow & \cdots \\
 & & \downarrow \mathcal{D}_2 & & \downarrow & & \downarrow & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array} \tag{8.4}$$

where θ is the solution sheaf for $\mathcal{D}_0 s = 0$ and μ is large. By Props. 6.2 and 8.1 the diagram is exact except possibly for the first row and first column. Thus by diagram chasing the cohomology of (8.1) is the same as the stable cohomolgy of the first Spencer sequences.

PROPOSITION 8.3. *Let $\mathcal{D}: \underline{E} \to \underline{F}$ be a regular differential operator of order μ_0. Then for $\mu \geqslant \mu_1(n, m, \mu_0)$ the Spencer sequence $(5.7)_\mu$ is exact*

(i) *if $M \subset \mathbf{R}^n$ and $\mathcal{D}$ has constant coefficients,*

(ii) *for real analytic germs if M, E, F, and $\varrho_{\mu_0}(\mathcal{D})$ are real analytic,*

(iii) *at $\underline{R}^1_{\mu+n-1}$ if and only if there exists an operator $\mathcal{D}'$ making $\underline{E} \xrightarrow{\mathcal{D}} \underline{F} \xrightarrow{\mathcal{D}'} \underline{F}'$ exact.*

Proof. (iii) follows from Prop. 8.2 and the remark following (8.3); (i) depends in addition on the theorem of Ehrenpreis and Malgrange. By the remarks following (8.4) statement (ii) is equivalent to the classical Cartan–Kähler theorem. Recently, L. Ehrenpreis, V. Guillemin, and S. Sternberg [3] have obtained estimates which, together with diagram chasing, yield a simple proof of this theorem. Using similar estimates obtained by the author [17], C. Buttin has given another proof of (ii) in [2].

Chapter II. The D-Neumann problem

1. The second Spencer sequence

At the end of Chapter I we saw how the exactness of the first Spencer sequences can be used to obtain existence theorems for the differential equation $\mathcal{D}s = t$. To this extent the theory generalizes the situation in complex analysis, where the Dolbeault sequence has proved to be an important tool for studying the Cauchy–Riemann equation. The problem of proving exactness is, of course, much more difficult in the general theory than in the case of the Dolbeault sequence; and, in fact, it is more difficult than it should be. To be specific, the sequence of symbol maps associated with $(\mathrm{I}.5.7)_\mu$ is exact only in the most trivial cases; thus the harmonic methods which establish exactness for the Dolbeault sequence cannot be applied to $(\mathrm{I}.5.7)_\mu$ even when "$\mathcal{D}s = 0$" is the Cauchy–Riemann equation. Thus we are led to consider the second Spencer sequence

$$0 \to \theta \to \underline{C}^0_\mu \to \underline{C}^1_\mu \to \dots \to \underline{C}^n_\mu \to 0, \tag{1.1}$$

which will be constructed from $(\mathrm{I}.5.7)_\mu$ by factoring out the degeneracy in the symbol sequence.

Let $\mathcal{D}: \underline{E} \to \underline{F}$ be a regular differential operator of order μ_0, and consider the corresponding data $\underline{g}^i_\mu$ and $\underline{R}^i_\mu$. For $\mu \geqslant \mu_1 = \mu_1$ (n, fiber dim E, μ_0) define $\alpha g^i_\mu = \{\zeta \in g^i_\mu \mid \delta\zeta = 0\}$. From the exactness of the δ-sequences for $\mu \geqslant \mu_1$ we conclude that αg^i_μ is a vector bundle; thus the cokernel of the inclusion map $\alpha g^i_{\mu+1} \to R^i_{\mu+1}$ is a vector bundle C^i_μ. For $\mu \geqslant \mu_1$ we have an exact sequence

$$0 \longrightarrow \alpha g^i_{\mu+1} \longrightarrow R^i_{\mu+1} \xrightarrow{\ p\ } C^i_\mu \longrightarrow 0 \tag{1.2}$$

and thus a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \underline{g}^i_{\mu+2} \arrow[r] \arrow[d, "-\delta"] & \underline{R}^i_{\mu+2} \arrow[r, "\pi"] \arrow[d, "D"] & \underline{R}^i_{\mu+1} \arrow[r] \arrow[d, "D'"] & 0 \\ 0 \arrow[r] & \underline{\alpha g}^{i+1}_{\mu+1} \arrow[r] & \underline{R}^{i+1}_{\mu+1} \arrow[r, "p"] & \underline{C}^{i+1}_{\mu} \arrow[r] & 0, \end{tikzcd} \tag{1.3}$$

where the rows are exact and D' is induced by D. We claim that D' factors through $\underline{C}^i_\mu$. Indeed, if $\sigma \in \underline{\alpha g}^i_{\mu+1}$ then $\sigma = -\delta\tau$ for some $\tau \in \underline{g}^{i-1}_{\mu+2}$ and hence $\sigma = -\delta\pi\tau_1$ for some $\tau_1 \in \underline{R}^{i-1}_{\mu+3}$; thus $D'\sigma = -D'\delta\pi\tau_1 = D'\pi D\tau_1 = pDD\tau_1 = 0$. Accordingly, we obtain a differential operator $D'' : \underline{C}^i_\mu \to \underline{C}^{i+1}_\mu$ such that $D' = D''p$. To show that D'' squares to zero we note that $p\pi^2 : \underline{R}^i_{\mu+3} \to \underline{C}^i_\mu$ is subjective and that $D''D''p\pi^2 = D''D'\pi^2 = D''pD\pi = D'D\pi = D'\pi D = pDD = 0$. Finally, we claim that the kernel of $D'' : \underline{C}^0_\mu \to \underline{C}^1_\mu$ is the solution sheaf θ for $\mathcal{D}s = 0$. In fact, by (1.2) we have $C^1_\mu = R_{\mu+1}$, and by chasing (1.3) we infer that the sequence

$$0 \longrightarrow \theta \xrightarrow{j_{\mu+1}} \underline{R}_{\mu+1} \xrightarrow{D_1} \underline{C}^1_\mu \tag{1.4}$$

is exact. Thus the construction of (1.1) is complete. It follows from (i) of Prop. I.5.1 that D'' is a derivation in the sense that

$$D''(\xi \wedge \sigma) = d\xi \wedge \sigma + (-1)^j \xi \wedge D'' \tag{1.5}$$

for each section σ of C^i_μ and each j-form ξ. In fact, writing $\sigma = p\pi\tau$, we find that

$$D''(\xi \wedge \sigma) = D''p\pi(\xi \wedge \tau) = pD(\xi \wedge \tau) = p(d\xi \wedge \pi\tau + (-1)^j \xi \wedge D\tau) = d\xi \wedge \sigma + (-1)^j \xi \wedge D''\sigma.$$

It will be convenient later to have a more explicit description of C^i_μ. For $\mu \geq \mu_0$ we choose maps $P : R_\mu \to R_{\mu+1}$ such that $\pi P = \mathbf{1}$; tensoring with $\mathbf{1}$ we thus obtain maps $P : R^i_\mu \to R^i_{\mu+1}$ with $\pi P = \mathbf{1}$. Setting $Q = \mathbf{1} - P\pi$, we consider the commutative program

$$\begin{tikzcd} & & 0 \arrow[d] & 0 \arrow[d] & \\ 0 \arrow[r] & \alpha g^i_{\mu+1} \arrow[r] \arrow[d, "\mathbf{1}"] & g^i_{\mu+1} \arrow[r, "\delta"] \arrow[d] & \alpha g^{i+1}_\mu \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \alpha g^i_{\mu+1} \arrow[r] & R^i_{\mu+1} \arrow[r, "p"] \arrow[d, "\pi"'] \arrow[u, bend right, "Q"'] & C^i_\mu \arrow[r] \arrow[d] & 0 \\ & & R^i_\mu \arrow[r, "\mathbf{1}"] \arrow[d] \arrow[u, bend right, "P"'] & R^i_\mu \arrow[d] & \\ & & 0 & 0 & \end{tikzcd} \tag{1.6}$$

where the last column is induced and is exact. We claim that the map $R^i_{\mu+1}$ to $R^i_\mu \oplus \alpha g^{i+1}_\mu$ defined by $\sigma \to (\pi\sigma, \delta Q\sigma)$ induces an isomorphism

$$C^i_\mu \to R^i_\mu \oplus \alpha g^{i+1}_\mu. \tag{1.7}$$

Since $\pi\sigma = \delta Q\sigma = 0$, when $\sigma \in \alpha g^i_{\mu+1}$, the map (1.7) is well defined. To prove injectivity we let $\sigma \in R^i_{\mu+1}$ satisfy $\pi\sigma = \delta Q\sigma = 0$. Then since $\sigma = P\pi\sigma + Q\sigma = Q\sigma$, we have $\pi\sigma = \delta\sigma = 0$. Thus $\sigma \in \alpha g^i_{\mu+1} = \ker p$. To prove surjectivity we let $\tau \in R^i_\mu$, $\zeta \in \alpha g^{i+1}_\mu$, choose $\xi \in g^i_{\mu+1}$ such that $\delta\xi = \zeta$, and note that (1.7) maps $p(P\tau + \xi)$ to (τ, ζ). A straightforward calculation shows that D'' is given by

$$(\sigma, \zeta) \to (DP\sigma - \zeta, (DP)^2\sigma - DP\zeta)$$

in terms of the isomorphisms (1.7); however, we shall not use this fact.

2. The symbol sequence

Let E and F be vector bundles over M, E having fiber dimension m. Let $\mathcal{D}$ be a regular differential operator of order μ_0 from E to F, let y be a point in M, and ξ be a non-zero cotangent vector at y. We choose a coordinate $x = (x^1, \ldots, x^n)$ on a small neighborhood U of y, and by trivializing $(R^i_\mu)_U$ and $(\alpha g^{i+1}_\mu)_U$ for $i = 0, \ldots, n$ we obtain trivializations for the bundles $(C^i_\mu)_U$, $i = 0, 1, \ldots, n$. Because of (1.5), the principal part of $D'': \underline{C}^i_\mu \to \underline{C}^{i+1}_\mu$ is given by $\sigma \to \sum_1^n dx^\nu \wedge \partial_\nu \sigma$ near y, where ∂_ν is defined by the coordinate x and the trivializations. In terms of the isomorphisms (1.7) the principal part is given by $(\sigma, \zeta) \to (\sum_1^n dx^\nu \wedge \partial_\nu \sigma,\ -\sum_1^n dx^\nu \wedge \partial_\nu \zeta)$. Thus the symbol sequence of (1.1) at ξ is the direct sum of the sequences

$$0 \longrightarrow (R_\mu)_y \xrightarrow{\xi\wedge} (R^1_\mu)_y \xrightarrow{\xi\wedge} \cdots \longrightarrow (R^n_\mu)_y \longrightarrow 0, \tag{2.1}$$

$$0 \longrightarrow (\alpha g^1_\mu)_y \xrightarrow{-\xi\wedge} (\alpha g^2_\mu)_y \xrightarrow{-\xi\wedge} \cdots \longrightarrow (\alpha g^n_\mu)_y \longrightarrow 0. \tag{2.2}$$

The first sequence is always exact; the second is the subject of the following proposition due to D. G. Quillen.

PROPOSITION 2.1. *The sequence* (2.2) *is exact for* $\mu \geqslant \mu_1(n, m, \mu_0)$ *if and only if* ξ *is not characteristic for* $\mathcal{D}$.

Proof. By the remark at the end of *I*, section 6 we may assume that $\mu_0 = 1$; we may also assume that $\xi = dx^n$. Throughout the proof we will suppress the subscript y.

To prove necessity assume that dx^n is characteristic for $\mathcal{D}$. Then by (I.4.4) there exists a non-zero element σ of g_μ such that $\sigma_\alpha = 0$ unless $\alpha_n = \mu$. Thus $\delta_\nu\sigma = 0$ for $\nu = 1, \ldots, n-1$, and $\xi \otimes \sigma \neq 0$ is in the kernel of the first map in (2.2).

To prove sufficiency assume that dx^n is not characteristic, and let $\sigma \in \alpha g^i_\mu$ satisfy $dx^n \wedge \sigma = 0$. Writing H for the subspace of T^* spanned by $dx^1, \ldots, dx^{n-1}$, we choose $\tau \in g_\mu \otimes \Lambda^{i-1}H$ such that $dx^n \wedge \tau = \sigma$. Then, setting $\delta' = \sum_1^{n-1} dx^\nu \wedge \delta_\nu$, we obtain $-\delta\sigma = dx^n \wedge \delta\tau = dx^n \wedge \delta'\tau$ and hence $\delta'\tau = 0$. Thus, if the sequence

$$g_{\mu+1}\otimes\Lambda^{i-2}H \xrightarrow{\delta'} g_\mu\otimes\Lambda^{i-1}H \xrightarrow{\delta'} g_{\mu-1}\otimes\Lambda^{i}H \tag{2.3}$$

is exact, we can write $\sigma = -\xi\wedge\delta'\eta = -\xi\wedge\delta\eta$, where $\delta\eta\in g_\mu^{i-1}$. Therefore it suffices to show that (2.3) is exact for $\mu_0 \geqslant \mu_1$.

For a polynomial $\sigma = \sum_{|\alpha|=\mu}\sigma_\alpha X^\alpha/\alpha!$ in g_μ we define

$$\varphi\sigma = \sum_{\substack{|\alpha|=\mu\\ \alpha_n=0}} \sigma_\alpha X^\alpha/\alpha!.$$

Thus φ is a mapping of g_μ into a space h_μ of homogeneous $\mathbf{R}^m$-valued polynomials in $X_1, \ldots, X_{n-1}$. We claim that φ is an isomorphism for each $\mu\geqslant 1$. Indeed, suppose $\sigma\in g_\mu$, $\sigma\neq 0$, satisfies $\varphi\sigma=0$. Among all multi-indices α with $|\alpha|=\mu$ and $\sigma_\alpha\neq 0$ choose a multi-index β for which β_n is minimal. Since $\varphi\sigma=0$ we have $\beta_n\geqslant 1$, and by minimality we have $\delta_\nu\delta^{\beta-1_n}\sigma=0$ for $\nu=1, \ldots, n-1$. Thus $\delta^{\beta-1_n}\sigma$ is an element of g_1 which depends only on X_n. By (I.4.3) this contradicts the assumption that dx^n is not characteristic. Since the isomorphisms φ are compatible with $\delta_1, \ldots, \delta_{n-1}$, it now suffices to show that

$$h_{\mu+1}\otimes\Lambda^{i-2}H \xrightarrow{\delta} h_\mu\otimes\Lambda^{i-1}H \xrightarrow{\delta} h_{\mu-1}\otimes\Lambda^{i-2}H \tag{2.4}$$

is exact for $\mu\geqslant\mu_1$.

Since the spaces h_μ do not come directly from a differential operator, we cannot immediately apply the proposition in I, section 7. Instead we introduce the inner product $\langle\sum X^\alpha\sigma_\alpha/\alpha!, \sum X^\alpha\tau_\alpha/\alpha!\rangle = \sum\langle\sigma_\alpha, \tau_\alpha\rangle/\alpha!$ on h_μ and note that the adjoint of δ_ν is multiplication by X_ν. Since $\delta_\nu(h_{\mu+1})\subset h_\mu$ for $\nu=1, \ldots, n-1$, the space $h^\perp = \sum_1^\infty h_\mu^\perp$ is a submodule of the $\mathbf{R}[X_1, \ldots, X_{n-1}]$-module of all $\mathbf{R}^m$-valued polynomials in $X_1, \ldots, X_{n-1}$. Thus by the Hilbert basis theorem there exists an integer μ_2 such that for $\mu\geqslant\mu_2$ we have $\sigma\in h_{\mu+1}^\perp$ if and only if $\sigma = X_1\sigma_1 + \ldots + X_n\sigma_n$ for some $\sigma_1, \ldots, \sigma_n\in h_\mu^\perp$. Thus if a homogeneous polynomial σ satisfies $\delta_\nu\sigma\in h_\mu$ for $\nu=1, \ldots, n-1$ then we can infer from $0=\sum_\nu\langle\delta_\nu\sigma, h_\mu^\perp\rangle = \sum_\nu\langle\sigma, X_\nu h_\mu^\perp\rangle = \langle\sigma, h_{\mu+1}^\perp\rangle$ that $\sigma\in h_\mu$ if and only if $\delta^\gamma\sigma\in h_{\mu_2}$ for each $|\gamma|=\mu-\mu_2$. It now follows from (I.3.6) that the spaces h_μ, $\mu\geqslant\mu_2$, are the g_μ-data for a differential operator of order μ_2. Thus by Prop. I.7.3 the sequence (2.4) and hence is exact for $\mu\geqslant\mu_2 + \mu_1(n, m, \mu_0)$. Using the diagram

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & 0\\
 & & \downarrow & & \downarrow & & \downarrow\\
 & & \alpha g_\mu^{i-1} & \longrightarrow & \alpha g_\mu^{i} & \longrightarrow & \alpha g_\mu^{i+1}\\
 & & \downarrow & & \downarrow & & \downarrow\\
g_\mu^{i-2} & \longrightarrow & g_\mu^{i-1} & \longrightarrow & g_\mu^{i} & \longrightarrow & g_\mu^{i+1}\\
\downarrow & & \downarrow & & \downarrow & & \\
\alpha g_{\mu-1}^{i-1} & \longrightarrow & \alpha g_{\mu-1}^{i} & \longrightarrow & \alpha g_{\mu-1}^{i+1} & & \\
\downarrow & & \downarrow & & \downarrow & & \\
0 & & 0 & & 0 & &
\end{array}$$

we conclude by downward induction that (2.2) is exact for all $\mu \geqslant \mu_1$. The proof is complete.

Prop. 2.1, together with the following proposition, removes the difficulty described at the beginning of section 1.

PROPOSITION 2.2. *For $\mu \geqslant \mu_1(n, m, \mu_0)$ the cohomology of the second Spencer sequence* (1.1) *is the same as the stable cohomology of the first Spencer sequences.*

Proof. Consider the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \underline{g^i_{\mu+2}} \arrow[r] \arrow[dd, "-\delta"'] & \underline{R^i_{\mu+2}} \arrow[r, "\pi"] \arrow[dd, "D"'] & \underline{R^i_{\mu+1}} \arrow[rr] \arrow[dd, "D'"'] \arrow[dr, "R"] & & 0 \\
& & & & \underline{C^i_\mu} \arrow[r] \arrow[dl, "D''"] & 0 \\
0 \arrow[r] & \underline{\alpha g^{i+1}_{\mu+1}} \arrow[r] \arrow[d] & \underline{R^{i+1}_{\mu+1}} \arrow[r, "p"] & \underline{C^{i+1}_\mu} \arrow[rr] & & 0 \\
& 0 & & & &
\end{tikzcd}$$

For each i we choose a map $q: C^i_\mu \to R^i_{\mu+1}$ such that $pq = 1$; we claim that p and q furnish the required isomorphism in homology. Since $pq = 1$ modulo coboundaries, we need only show that p maps cocycles to cocycles and coboundaries to coboundaries. The second assertion follows immediately from the diagram; to prove the first we let $\sigma \in \underline{R^i_{\mu+1}}$ satisfy $D\sigma = 0$. By chasing the diagram we can find $\tau \in \underline{R^i_{\mu+2}}$ such that $\pi\tau = \sigma$ and $D\tau = 0$. Thus $D''p\sigma = D''p\pi\tau = D'\pi\tau = pD\tau = 0$.

3. The *D*-Neumann problem

Our first task is to fix the notation which will be used in the rest of this chapter.

Let M be a smooth Riemannian n-manifold, and let $\Omega \subset M$ be a compact, smooth n-manifold-with-boundary which is imbedded smoothly in M. Thus from the Riemannian metric on M we obtain Riemannian metrics on Ω and its boundary ω. Let E and F be vector bundles over M, and let $\mathcal{D}$ be a regular elliptic differential operator from E to F. Whenever the data R_μ, g_μ, etc. occur in the future, it will be tacitly assumed that μ is greater than the integer μ_1 given in I, section 7. In situations where only one value of $\mu > \mu_1$ occurs, the subscript will be suppressed. We will denote the operator in the second Spencer sequence by D instead of D''.

We choose an inner product along the fibers of $J_\mu(E)$; using the Riemannian metric we thus obtain an inner product in each $J^j_\mu(E)$ and hence in each of the sub-bundles R^j_μ, g^j_μ, αg^j_μ. Using the isomorphism $C^j_\mu \approx R^j_\mu \oplus \alpha g^{j+1}_\mu$, we obtain an inner product in C^j_μ.

We denote by $\Gamma(\Omega, C^j)$ the space of all smooth sections of C^j over Ω. In accordance with the usual definition for manifolds-with-boundary, each section in $\Gamma(\Omega, C^j)$ can be extended smoothly across the boundary ω. We refer to Hörmander [5] for the definitions of the spaces $\mathcal{H}_{(s)}(\Omega, C^j)$. In the sequel we shall always consider $\mathcal{H}_{(0)}(\Omega, C^j)=L_2(\Omega, C^j)$ as a Hilbert space with inner product $\langle\sigma, \tau\rangle=\int\langle\sigma, \tau\rangle_x dv$, where dv is the Riemannian volume element and $\langle\cdot, \cdot\rangle_x$ is the inner product in C^j. The notation given here will also be applied to the bundles R^j, g^j, etc and to their restrictions to ω.

With the introduction of the Fourier transform spaces above, we assume that all bundles occurring have complex fiber. With $i=\sqrt{-1}$ we write $D_\nu=-i\partial_\nu$ in a local coordinate.

Now let D be the operator in the second Spencer sequence, and let D^* be its formal adjoint. Define the Neumann space $\mathbf{N}^j$ to be the space of all sections $u\in\Gamma(\Omega, C^j)$ satisfying the boundary conditions

$$\langle Dv, u\rangle=\langle v, D^*u\rangle \quad \text{for each} \quad v\in\Gamma(\Omega, C^{j-1}), \tag{3.1}$$

$$\langle Dv, Du\rangle=\langle v, D^*Du\rangle \quad \text{for each} \quad v\in\Gamma(\Omega, C). \tag{3.2}$$

Define the harmonic space $\mathbf{H}^j$ to be the set of all sections $u\in\mathbf{N}^j$ which are annihilated by the laplacian DD^*+D^*D. Since $\langle(DD^*+D^*D)u, u\rangle=\|Du\|^2+\|D^*u\|^2$ for $u\in\mathbf{N}^j$, we have $\mathbf{H}^j=\{u\in\mathbf{N}^j \mid Du=D^*u=0\}$.

Definition. We say that the D-Neumann problem is solvable for $\mathcal{D}$ on Ω if $\mathbf{H}^j$ is closed in $L_2(\Omega, C^j)$ for $j=1, ..., n$ and if there exist bounded operators $N: L_2(\Omega, C^j)\to L_2(\Omega, C^j)$, $j=1, ..., n$, mapping $\Gamma(\Omega, C^j)$ into $\mathbf{N}^j$, such that:

(i) $NH=HN=0$, where $H: L_2(\Omega, C^j)\to\mathbf{H}^j$ is the orthogonal projection;

(ii) each $u\in\Gamma(\Omega, C^j)$ can be written

$$u=DD^*Nu+D^*DNu+Hu, \tag{3.3}$$

where the terms are mutually orthogonal in view of (3.1) and (3.2);

(iii) $DN=ND$ holds.

Note that (iii) follows from (i) and (ii). Indeed, if $u\in\Gamma(\Omega, C^j)$, then

$$(D^*D+DD^*)(DN-ND)u=D(D^*D+DD^*)Nu-(D^*D+DD^*)NDu=Du-Du=0.$$

Also $(DN-ND)u\in\mathbf{N}^j$ and $H(DN-ND)u=0$. It follows that $(DN-ND)u=0$.

In view of (iii) the decomposition (3.3) gives a cochain homotopy $\mathbf{1}-H=D(D^*N)+(D^*N)D$; thus the cohomology of the sequence

$$0\to\Gamma(\Omega, C^0)\to\Gamma(\Omega, C^1)\to\dots\to\Gamma(\Omega, C^n)\to 0$$

is isomorphic to $\mathbf{H}=\sum_0^n\mathbf{H}^j$. Since the argument following (I.8.4) and the arguments leading to Prop. II.2.2 work for sections as well as for germs, it follows that if the D-Neumann problem is solvable for $\mathcal{D}$ on Ω, then there exists an operator $\mathcal{D}_1: \underline{F} \to \underline{F}_1$ such that in the sequence

$$\Gamma(\Omega, E) \xrightarrow{D} \Gamma(\Omega, F) \xrightarrow{\mathcal{D}_1} \Gamma(\Omega, F_1)$$

we have $\ker \mathcal{D}_1/\mathrm{im}\,\mathcal{D} \approx \mathbf{H}^1$. Moreover, this is the best possible result; for by the proof of Prop. I.8.2 if $\mathcal{D}': \underline{F} \to \underline{F}'$ satisfies $\mathcal{D}'\mathcal{D} = 0$, then $\ker \mathcal{D}' \supset \ker \mathcal{D}_1$.

4. General methods

In this section we recall some of the general Hilbert space methods which are relevant to the D-Neumann problem. Since much of the material presented here is classical, we will omit several proofs.

We first form the Friedrichs extension L of $DD^* + D^*D$ on $\mathbf{N}^j$ (see [13], p. 335). This is done as follows: Complete $\mathbf{N}^j$ abstractly to a Hilbert space $\mathbf{B}^j$ with the Dirichlet inner product $Q(\mu, v) = \langle Du, Dv\rangle + \langle D^*u, D^*v\rangle + \langle u, v\rangle$, and show that $\mathbf{B}^j$ can be considered as a subset of $L_2(\Omega, C^j)$. Define the domain of L to be the space of all $u \in \mathbf{B}^j$ such that $v \to Q(v, u)$ extends to a bounded functional on $L_2(\Omega, C^j)$; and for such u define $f = Lu$ by $Q(\cdot, u) = \langle \cdot, f\rangle + \langle \cdot, u\rangle$. The operator L is self adjoint, and $L + I$ has an inverse which is a bounded operator from $L_2(\Omega, C^j)$ to $\mathbf{B}^j$.

If $u \in \mathbf{N}^j$ and $v \in \Gamma(\Omega, C^j)$, then $|\langle Dv, u\rangle| = |\langle v, D^*u\rangle| \leqslant \|v\| Q(u, u)^{\frac{1}{2}}$. Thus for any $u \in \mathbf{B}^j$, $v \to \langle Dv, u\rangle$ extends to a bounded functional on $L_2(\Omega, C^j)$, and hence u satisfies the boundary condition (3.1). On the other hand, an element $u \in \mathbf{B}^j$ need not satisfy (3.2), even if u is smooth. However, if $u \in \Gamma(\Omega, C^j)$ is in the domain of L, then $|\langle Dv, Du\rangle| \leqslant |\langle D^*v, D^*u\rangle| + |Q(v, u)| + |\langle v, u\rangle| \leqslant \|v\| \{\|DD^*u\| + \|Lu\| + 2\|u\|\}$ for $v \in \mathbf{N}^j$, and hence u satisfies (3.2).

PROPOSITION 4.1. *Assume that* $\mathcal{D}$ *is elliptic and that the inclusions* $\mathbf{B}^j \to L_2(\Omega, C^j)$ *are completely continuous for* $j \geqslant 1$. *Then:*

(i) *we have* $u \in \Gamma(\Omega, C^j)$ *whenever* $Lu \in \Gamma(\Omega, C^j)$ *and* $j \geqslant 1$;

(ii) *for* $j \geqslant 1$, $\mathbf{H}^j$ *is finite dimensional, and the range of* L *on* $\Gamma(\Omega, C^j)$ *is closed*;

(iii) *the D-Neumann problem is solvable for* $\mathcal{D}$ *on* Ω.

Proof. (i) is a recent theorem due to J. J. Kohn and L. Nirenberg (see [9], Theorem 3). Our assumption that $\mathcal{D}$ is elliptic enters here via Prop. 2.1, which shows that $DD^* + D^*D$ is then elliptic. The Kohn–Nirenberg theorem requires that the boundary ω be non-characteristic for $DD^* + D^*D$.

Since $\mathbf{H}^j = \ker L$ by (i), statement (ii) is part of the standard theory of compact operators. To prove (iii) we first note that since L is self adjoint, $1 - H$ is the orthogonal projection on the range. Given $u \in L_2(\Omega, C^j)$, we define $v = Nu$ to be the unique element in the domain of L satisfying $Hv=0$, $Lv=u-Hu$. Then N is bounded because of the closed graph theorem, and it maps $\Gamma(\Omega, C^j)$ into $\mathbf{N}^j$ because of (i) and because of the remark preceeding this proposition.

5. Reduction to the boundary

As one easily verifies (see section 6 below), the boundary condition (3.1) on $u \in \Gamma(\Omega, C^j)$ is given by $Bu=0$ on ω for a certain bundle map $B: C^j \to C^{j-1}$. Thus in order to establish the complete continuity required in the hypothesis of Prop. 4.1, it suffices to show that for some $0 < r \leqslant 1$ the estimate

$$\|u\|_{(r)} \leqslant c\{Q(u) + {}^{\omega}\|Bu\|_{(\frac{1}{2})}\} \tag{5.1}$$

holds for all $u \in \Gamma(\Omega, C^j)$. Here $Q(u) = Q(u, u)^{\frac{1}{2}} = \{\|Du\|^2 + \|D^*u\|^2 + \|u\|^2\}^{\frac{1}{2}}$ is the Dirichlet norm for sections over Ω, and $\|\cdot\|_{(r)}$ and ${}^{w}\|\cdot\|_{(\frac{1}{2})}$ are any of the equivalent norms in $\mathcal{H}_{(r)}(\Omega, C^j)$ and $\mathcal{H}_{(\frac{1}{2})}(\omega, C^j)$ respectively. We begin our discussion of (5.1) with some technical material and a series of lemmata.

Recall that $\mathcal{H}_{(m,s)}(\mathbf{R}^n)$ is defined to be the space of all temperate distributions u on $\mathbf{R}^n$ such that with $\xi' = (\xi_1, ..., \xi_{n-1})$, the norm

$$\|u\|^2_{(m,s)} = (2\pi)^{-n} \int (1 + |\xi|^2)^m (1 + |\xi'|^2)^s |\hat{u}(\xi)|^2 d\xi$$

is finite. The space $\mathcal{H}_{(m,s)}(\overline{\mathbf{R}}^n_+)$ is defined as the space of all distributions on the open upper half space $\mathbf{R}^n_+ = \{x \in \mathbf{R}^n | x_n < 0\}$ which can be extended to elements of $\mathcal{H}_{(m,s)}(\mathbf{R}^n)$. Thus $\mathcal{H}_{(m,s)}(\overline{\mathbf{R}}^n_+)$ is the quotient of $\mathcal{H}_{(m,s)}(\mathbf{R}^n)$ by the closed subspace of elements with support in $\overline{\mathbf{R}}^n_- = \{x \in \mathbf{R}^n | x_n \leqslant 0\}$. The space $\mathcal{H}_{(m,s)}(\overline{\mathbf{R}}^n_-)$ is given the quotient inner product and norm. See Hörmander [5] for the basic properties.

LEMMA 5.1. *Let s be a real number, m a positive integer, and write $\mathbf{R}^n_0 = \{x \in \mathbf{R}^n | x_n = 0\}$. Then for each $u \in \mathcal{H}_{(m,s)}(\overline{\mathbf{R}}^n_+)$ the restriction $u | \mathbf{R}^n_0$ is a well-defined element of $\mathcal{H}_{(m+s-\frac{1}{2})}(\mathbf{R}^n_0)$ satisfying $\|(u|\mathbf{R}^n_0)\|_{(m+s-\frac{1}{2})} \leqslant \|u\|_{(m,s)}$. Moreover, for each $v \in \mathcal{H}_{(m+s-\frac{1}{2})}(\mathbf{R}^n_0)$ there exists $u \in \mathcal{H}_{(m,s)}(\overline{\mathbf{R}}^n_+)$ such that $u | \mathbf{R}^n_0 = v$.*

Proof. See Hörmander [5], Theorems 2.5.6 and 2.5.7.

LEMMA 5.2. *Let $G \subset \overline{\mathbf{R}}^n_+$ be open in $\overline{\mathbf{R}}^n_+$, and let P be an elliptic differential operator of order k from the trivial bundle $G \times R^a$ to $G \times R^b$. Let m, s, m_1, s_1, be real numbers satisfying*

$m+s=m_1+s_1$. *Then for each compact $U\subset G$ there exists a constant c such that*

$$\|u\|_{(m,s)}\leqslant c\{\|Pu\|_{(m-k,s)}+\|u\|_{(m_1,s_1)}\}$$

for sections u of $G\times R^a$ with supp $u\subset U$.

Proof. See Hörmander [7], Lemma 2.2.1.

LEMMA 5.3. *For each $0\leqslant s\leqslant 1$ the estimate*

$$\|u\|_{(s)}\leqslant c\{Q(u)+{}^{\omega}\|u\|_{(s-\frac{1}{2})}\}$$

holds for $u\in\Gamma(\Omega, C^j)$.

Proof. We first prove the lemma in the case $s=1$. By Lemma 5.1 the restriction map $u\to u|\omega$ is a bounded operator from $\mathcal{H}_{(1)}(\Omega, C^j)$ onto $\mathcal{H}_{(\frac{1}{2})}(\omega, C^j)$ whose kernel K is the closure in $\mathcal{H}_{(1)}(\Omega, C^j)$ of the space of smooth sections having support in the interior of Ω. For $u\in K$ the estimate follows from Gårding's inequality; if u is orthogonal to K, then $\|u\|_{(1)}\leqslant c^{\omega}\|u\|_{(\frac{1}{2})}$ by the closed graph theorem. To prove the estimate we write π for the orthogonal projection of $\mathcal{H}_{(1)}(\Omega, C^j)$ onto K and observe that $\|u\|_{(1)}=\|\pi u\|_{(1)}+\|(1-\pi)u\|_{(1)}\leqslant c\{Q(\pi u)+{}^{\omega}\|u\|_{(\frac{1}{2})}\}\leqslant c\{Q(u)+{}^{\omega}\|u\|_{(\frac{1}{2})}+Q((1-\pi)u)\}$. Since $Q((1-\pi)u)\leqslant\|(1-\pi)u\|_{(1)}\leqslant{}^{\omega}\|u\|_{(\frac{1}{2})}$, we have

$$\|u\|_{(1)}\leqslant c\{Q(u)+{}^{\omega}\|u\|_{(\frac{1}{2})}\},\quad\text{for}\quad u\in\Gamma(\Omega, C^j). \tag{5.2}$$

We now use (5.2) to prove the lemma in general. Observe that it suffices to prove the required estimate locally, for $u\in\Gamma(\Omega, C^j)$ having support in a coordinate disk U with coordinate $(x^1, ..., x^n)$. Since the interior case is simpler, we assume that U is a boundary disk where $\Omega\cap U$ and $\omega\cap U$ are given by $x^n\geqslant 0$ and $x^n=0$ respectively. By Prop. 2.1 and the ellipticity of $\mathcal{D}$, the map

$$s_{dx^n}(D)\oplus s_{dx^n}(D^*): C^j_U\to C^{j+1}_U\oplus C^{j-1}_U$$

is injective. Thus we may choose frames in C^j_U, C^{j+1}_U, and C^{j-1}_U in such a way that $s_{dx^n}(D)$ and $s_{dx^n}(D)$ and $s_{dx^n}(D^*)$ are represented by constant matrices; in these frames the coefficients of ∂_n in D and D^* will then be constant.

Now let $u\in\Gamma(\Omega, C^j)$ have support in a compact set U' which is contained in the interior of U. Denote by K the tangential pseudodifferential operator on $\overline{\mathbf{R}}^n_+$ with Fourier integral kernel $\varphi(\xi')|\xi'|^{-1+s}$, where $\xi'=(\xi_1, ..., \xi_{n-1})$ and $\varphi\in C^\infty(R^{n-1})$ is 0 on a neighborhood of the origin and 1 outside a slightly larger set. Let $\psi\in C^\infty_0(U)$ be 1 on a neighborhood of U'. Using (5.2) we obtain

$$\|\psi Ku\|_{(1)}\leqslant c\{Q(\psi Ku)+{}^{\omega}\|\psi Ku\|_{(\frac{1}{2})}\}. \tag{5.3}$$

Since K has tangential order $s-1$, we have ${}^{\omega}\|\psi Ku\|_{(\frac{1}{2})}\leqslant c^{\omega}\|u\|_{(s-\frac{1}{2})}$. Also, by the choice of frames, the operator $[\psi K, D]$ has tangential order $s-1$. Hence

$$\|D\psi Ku\| \leq \|\psi KDu\| + c\|u\|_{(0,s-1)} \leq c\{\|Du\| + \|u\|\},$$

since $s-1 \leq 0$. Treating the other terms in $Q(\psi Ku)$ similarly, we obtain

$$\|\psi Ku\|_{(1)} \leq c\{Q(u) + {}^{\omega}\|u\|_{(s-\frac{1}{2})}\}.$$

Now note that $\|u\|_{(0,s)} \leq c\{\|Ku\|_{(0,1)} + \|u\|\} = c\{\|K\psi u\|_{(0,1)} + \|u\|\} \leq c\{\|\psi Ku\|_{(1)} + \|u\|_{(0,s-2)} + \|u\|\} \leq c\{\|\psi Ku\|_{(1)} + \|u\|_{(1,-1)}\} \leq c\{\|\psi Ku\|_{(1)} + Q(u) + \|u\|\}$, where we have used Lemma 5.2 to obtain the last inequality. We now have that $\|u\|_{(0,s)} \leq c\{Q(u) + {}^{\omega}\|u\|_{(s-\frac{1}{2})}\}$. Another application of Lemma 5.2 now yields

$$\|u\|_{(s)} \leq c\{Q(u) + {}^{\omega}\|u\|_{(s-\frac{1}{2})}\},$$

as required.

LEMMA 5.4. *The estimate* (5.1) *holds, provided that the estimate*

$$ {}^{\omega}\|w\|_{(r-\frac{1}{2})} \leq c\{{}^{\omega}\|Dw\|_{(-\frac{1}{2})} + {}^{\omega}\|D^*w\|_{(-\frac{1}{2})} + {}^{\omega}\|w\|_{(-\frac{1}{2})} + {}^{\omega}\|Bw\|_{(\frac{1}{2})}\} \tag{5.4}$$

holds for all $w \in \Gamma(\Omega, C^j)$ *satisfying* $(DD^* + D^*D + I)w = 0$.

Proof. Assume that (5.4) holds, and let $u \in \Gamma(\Omega, C^j)$. Choose $w \in \Gamma(\Omega, C^j)$ such that $(DD^* + D^*D + I)w = 0$ and $w = u$ on ω. Using (5.4) and Lemma 5.3 we obtain

$$\|u\|_{(r)} \leq c\{Q(u) + {}^{\omega}\|w\|_{(r-\frac{1}{2})}\} \leq c\{Q(u) + {}^{\omega}\|Dw\|_{(-\frac{1}{2})} + {}^{\omega}\|D^*w\|_{(-\frac{1}{2})} + {}^{\omega}\|w\|_{(-\frac{1}{2})} + {}^{\omega}\|Bw\|_{(\frac{1}{2})}\}.$$

If $\psi \in C^\infty(M)$ has support in a boundary coordinate disk, then by Lemma 5.1 and Lemma 5.2 we have ${}^{\omega}\|\psi w\|_{(-\frac{1}{2})} \leq \|\psi w\|_{(1,-1)} \leq c\{\|(DD^* + D^*D + I)(\psi w)\|_{(-1,-1)} + \|\psi w\|\}$. If φ has support in the same disk and if $\varphi = 1$ in a neighborhood of supp ψ, then we obtain $\|(DD^* + D^*D + I)\psi w\|_{(-1,-1)} \leq \|\psi(DD^* + D^*D + I)\varphi w\|_{(-1,-1)} + c\|\varphi w\|_{(0,-1)} \leq 0 + c\|w\|$. Thus ${}^{\omega}\|\psi w\|_{(-\frac{1}{2})} \leq c\|w\|$, and since ω is compact we conclude that ${}^{\omega}\|w\|_{(-\frac{1}{2})} \leq c\|w\|$. In a similar fashion we find that ${}^{\omega}\|Dw\|_{(-\frac{1}{2})} \leq c\|Dw\|$ and ${}^{\omega}\|D^*w\|_{(-\frac{1}{2})} \leq c\|D^*w\|$. Substituting into the inequality above we have $\|u\|_{(r)} \leq c\{Q(u) + Q(w) + {}^{\omega}\|Bu\|_{(r-\frac{1}{2})}\}$. Since $Q(w) \leq Q(u)$ by Dirichlet's principle, we now have $\|u\|_{(r)} \leq c\{Q(u) + {}^{\omega}\|Bu\|_{(\frac{1}{2})}\}$ as required.

The estimate (5.4) is rather difficult to study directly because it presupposes the explicit solution of the Dirichlet problem. In his recent paper [7], L. Hörmander has approximated the operators in (5.4) with certain pseudo-differential operators on ω, whose symbols are, at least in principle, easy to compute. Before stating Hörmander's theorem, we must introduce additional data.

Define η to be the unit cotangent vector field $\eta:\omega \to T^*(M)$ which is orthogonal to $T^*(\omega)_y$ at every $y \in \omega$. Denote by $p(\xi)$ and $r(\xi)$ the principal symbols of D and $DD^* + D^*D$ respectively. In the notation of Chapter I we have, for example, $p(\xi) = is_\xi(D): C^j_y \to C^{j+1}_y$ for each $\xi \in T^*(M)_y$; thus p is a polynomial function of $\xi \in T^*(M)_y$ with values in Hom (C^j_y, C^{j+1}_y).

Note that the principal symbol of D^* is $p(\xi)^*$ and that $r(\xi) = p(\xi)p(\xi)^* + p(\xi)^*p(\xi)$, where the adjoints are defined by the inner product in the bundles C^j.

Now fix $y \in \omega$, $0 \neq \xi' \in T^*(\omega)_y$, and consider the initial value problem:

$$\begin{cases} r(\xi' + \eta D_t)\, V(t) = 0, \quad t \geqslant 0 \quad \left(D_t = -i\dfrac{d}{dt}\right) \\ V(0) = v \in C^j_y \end{cases} \tag{5.5}$$

for a C_y-valued function on the half line $t \geqslant 0$. Here $r(\xi' + \eta D_t)$ is defined by writing D_t for λ in the expansion

$$r(\xi' + \lambda\eta) = r_0(\xi', \eta) + r_1(\xi', \eta)\lambda + r_2(\eta)\lambda^2, \quad \lambda \in \mathbf{C}. \tag{5.6}$$

It is easy to verify that $r(\xi' + \eta D_t)$ is an elliptic self-adjoint operator in one variable; by a trivial application of the general theory we conclude that (5.5) has a unique exponentially decreasing solution. Thus for each $\xi' \in T^*(\omega)_y$ we have a linear map $h(\xi')$ from C^j_y to C^j_y defined by $h(\xi')v = D_t V(0)$, where $V(t)$ is the exponentially decreasing solution to (5.5). Now define

$$\begin{aligned} k(\xi') &= p_0(\xi', \eta) + p_1(\eta)\, h(\xi'), \\ k'(\xi') &= p_0(\xi', \eta)^* + p_1(\eta)^* h(\xi'), \end{aligned} \tag{5.7}$$

where $p_0(\xi', \eta)$ and $p_1(\eta)$ are defined by an expansion similar to (5.6). We note that $k(\xi')$ and $k'(\xi')$ are homogeneous of degree 1 in $\xi' \in T^*(\omega)_y$, and we choose pseudo-differential operators K and K' on ω with symbols $k(\xi')$ and $k'(\xi')$ respectively.

PROPOSITION 5.5. *The estimate* (5.1) *holds, provided that the estimate*

$$^{\omega}\|v\|_{(r-\frac{1}{2})} \leqslant c\{^{\omega}\|Kv\|_{(-\frac{1}{2})} + {}^{\omega}\|K'v\|_{(-\frac{1}{2})} + {}^{\omega}\|v\|_{(-\frac{1}{2})} + {}^{\omega}\|Bv\|_{(\frac{1}{2})}\} \tag{5.8}$$

holds for all $v \in \Gamma(\omega, C^j)$.

Proof. Let $w \in \Gamma(\Omega, C^j)$ satisfy $(DD^* + D^*D + I)w = 0$, and let $v = w|\omega$. It follows from (2.3.15) in Hörmander [7] that

$$^{\omega}\|Kv - Dw\|_{(-\frac{1}{2})} \leqslant c\{^{\omega}\|v\|_{(-\frac{1}{2})} + {}^{\omega}\|w_1\|_{(-\frac{3}{2})}\}$$

$$^{\omega}\|K'v - D^*w\|_{(-\frac{1}{2})} \leqslant c\{^{\omega}\|v\|_{(-\frac{1}{2})} + {}^{\omega}\|w_1\|_{(-\frac{3}{2})}\},$$

where $w_1 = -i(\partial/\partial n)w|\omega$ is the normal derivative of w restricted to the boundary. As in the proof of Lemma 5.4 we have $^{\omega}\|v\|_{(-\frac{1}{2})} \leqslant c\|w\|$ and a similar argument shows that $^{\omega}\|w_1\|_{(-\frac{3}{2})} \leqslant c\|w\|$. Hence, we have $^{\omega}\|w\|_{(r-\frac{1}{2})} \leqslant c\{^{\omega}\|Kv\|_{(-\frac{1}{2})} + {}^{w}\|K'v\|_{(-\frac{1}{2})} + {}^{\omega}\|w\|_{(-\frac{1}{2})} + {}^{\omega}\|Bw\|_{(\frac{1}{2})}\} \leqslant c\{^{\omega}\|Dw\|_{(-\frac{1}{2})} + {}^{\omega}\|D^*w\|_{(-\frac{1}{2})} + {}^{\omega}\|w\|_{(-\frac{1}{2})} + \|w\| + {}^{\omega}\|Bw\|_{(\frac{1}{2})}\}$. In the proof of Lemma 5.4 we showed that $^{\omega}\|Dw\|_{(-\frac{1}{2})} + {}^{\omega}\|D^*w\|_{(-\frac{1}{2})} + {}^{\omega}\|w\|_{(-\frac{1}{2})} \leqslant cQ(w)$. Thus we have

$$^{\omega}\|w\|_{(r-\frac{1}{2})} \leqslant c\{Q(w) + {}^{\omega}\|Bw\|_{(\frac{1}{2})}\}.$$

As in the proof of Lemma 5.4, this estimate, Dirichlet's principle, and Lemma 5.3 yield (5.1).

The following lemma will enable us to prove the converse of Prop. 5.5.

LEMMA 5.6. *Assume that* (5.1) *holds for all* $u \in \Gamma(\Omega, C^j)$. *Then the estimate*

$$\|u\|_{(r+2)} \leqslant c\{\|Du\|_{(2)} + \|D^*u\|_{(2)} + \|u\|_{(2)} + {}^{\omega}\|Bu\|_{(5/2)}\} \tag{5.9}$$

also holds for all $u \in \Gamma(\Omega, C^j)$.

Proof. Assume that (5.1) holds, and note that it suffices to prove (5.9) locally. Let U be a coordinate disk and let U' be a somewhat smaller disk. Since the interior case is simpler, we shall assume that U is a boundary disk. As in the proof of Lemma 5.3 we choose a tangential pseudodifferential operator $P: C_0^\infty(U) \to C_0^\infty(U)$ of tangential order 2 such that $[D, P]$ and $[D^*, P]$ have tangential order 2 and such that $\|u\|_{(0,r+2)} \leqslant c\{\|Pu\|_{(r)} + \|u\|_{(2)}\}$ holds for u with support in U'. Using Lemma 5.2 and (5.1) we obtain $\|u\|_{(r+2)} \leqslant c\{\|Du\|_{(r)} + \|D^*u\|_{(r)} + \|u\|_{(0,r+2)}\} \leqslant c\{\|Du\|_{(2)} + \|D^*u\|_{(2)} + \|u\|_{(2)} + \|DPu\| + \|D^*Pu\| + \|Pu\| + {}^{\omega}\|BPu\|_{(\frac{1}{2})}\} \leqslant c\{\|Du\|_{(2)} + \|D^*u\|_{(2)} + \|u\|_{(2)} + {}^{\omega}\|Bu\|_{(\frac{5}{2})}\}$, as required.

PROPOSITION 5.7. *Assume that* (5.1) *holds for all* $u \in \Gamma(\Omega, C^j)$. *Then the estimate*

$${}^{\omega}\|v\|_{(r+\frac{3}{2})} \leqslant c\{{}^{\omega}\|Kv\|_{(\frac{3}{2})} + {}^{\omega}\|K'v\|_{(\frac{3}{2})} + {}^{\omega}\|v\|_{(\frac{3}{2})} + {}^{\omega}\|Bv\|_{(\frac{5}{2})}\} \tag{5.10}$$

holds for all $v \in \Gamma(\omega, C^j)$.

Proof. Assume that (5.1) holds and let $v \in \Gamma(\omega, C^j)$. Let w be the unique element of $\Gamma(\Omega, C^j)$ such that $(DD^* + D^*D + I)w = 0$ and $w = v$ on ω. Then by Lemma 5.6 we have ${}^{\omega}\|v\|_{(r+\frac{3}{2})} \leqslant \|w\|_{(r+2)} \leqslant c\{\|Dw\|_{(2)} + \|D^*w\|_{(2)} + \|w\|_{(2)} + {}^{\omega}\|Bv\|_{(\frac{5}{2})}\}$. Using the well-known estimate

$$\|u\|_{(2)} \leqslant c\{\|(DD^* + D^*D + I)u\| + {}^{\omega}\|u\|_{(\frac{3}{2})}\} \tag{5.11}$$

for the Dirichlet problem, we see that

$$\|v\|_{(r+\frac{3}{2})} \leqslant c\{{}^{\omega}\|Dw\|_{(\frac{3}{2})} + {}^{\omega}\|D^*w\|_{(\frac{3}{2})} + {}^{\omega}\|w\|_{(\frac{3}{2})} + {}^{\omega}\|Bw\|_{(\frac{5}{2})}.$$

According to (2.3.15) in Hörmander [7] we have

$${}^{\omega}\|Dw - Kv\|_{(\frac{3}{2})} \leqslant c\{{}^{\omega}\|v\|_{(\frac{3}{2})} + {}^{\omega}\|w_1\|_{(\frac{1}{2})}\}$$
$${}^{\omega}\|D^*w - K'v\|_{(\frac{3}{2})} \leqslant c\{{}^{\omega}\|v\|_{(\frac{3}{2})} + {}^{\omega}\|w_1\|_{(\frac{1}{2})}\},$$

where w_1 has the same meaning as in the proof of Prop. 5.5. Since ${}^{\omega}\|v\|_{(\frac{3}{2})} + {}^{\omega}\|w_1\|_{(\frac{1}{2})} \leqslant c\|w\|_{(2)} \leqslant c\,{}^{\omega}\|w\|_{(\frac{3}{2})}$ by Lemma 5.1 and (5.11), the estimate (5.10) now follows.

PROPOSITION 5.8. *For each real number s the estimate*

$$^{\omega}\|v\|_{(r+s)} \leqslant c\{^{\omega}\|Kv\|_{(s)} + {}^{\omega}\|K'v\|_{(s)} + {}^{\omega}\|v\|_{(s)} + {}^{\omega}\|Bv\|_{(s+1)}\} \tag{5.12$_s$}$$

is equivalent to (5.1).

Proof. In view of Prop. 5.5 and Prop. 5.7 it is enough to prove that the estimates $(5.12)_s$ are all equivalent. However, this follows easily using pseudodifferential operators.

6. Local analysis

We first discuss the boundary condition (3.1). Let $U \subset M$ be a boundary coordinate disk for Ω with coordinate $x = (x^1, ..., x^n)$. Assume that $U \cap \Omega = \{x \in U \mid x^n \geqslant 0\}$ and that $U \cap \omega = \{x \in U \mid x^n = 0\}$; in fact, assume that x^n is the geodesic distance from $x \in U \cap \Omega$ to the boundary ω. Also choose trivializations for the bundles C^j_U. Now note that $u \in \Gamma(\Omega, C^j)$ satisfies (3.1) if and only if the mapping

$$\Gamma(\Omega, C^{j-1}) \ni v \to \langle Dv, u\rangle \tag{6.1}$$

extends to a bounded linear functional on $L_2(\Omega, C^{j-1})$. Thus the validity of (3.1) depends only on the principal part of D, which is given by $v \to \sum_1^n dx^\nu \wedge \partial_\nu v$ in the local coordinate x (see section 2). It also follows that (3.1) is a local condition on the boundary values of u. Thus let u have support in U and consider

$$\langle \textstyle\sum_1^n dx^\nu \wedge \partial_\nu v, u\rangle = \sum_1^n \displaystyle\int_{x_n \geqslant 0} \langle dx^\nu \wedge \partial_\nu v, u\rangle_x f(x)\, dx,$$

where f is the density for the volume measure. Integrating by parts, we find that (6.1) is equal to a bounded functional plus the mapping

$$v \to -\int_{x_n = 0} \langle v, Bu\rangle_{x'} f(x')\, dx',$$

where $x' = (x^1, \ldots, x^{n-1})$ and $B: C^j_{x'} \to C^{j-1}_{x'}$ is the adjoint of $dx^n \wedge$. If $Bu = 0$ on ω, then (3.1) holds. Conversely, if (3.1) holds, then for any $w \in \Gamma(\Omega, C^{j-1})$ with support in the interior of Ω we have

$$\left| \int_{x_n = 0} \langle Bu, Bu\rangle_{x'} f(x')\, dx' \right| \leqslant c\|Bu - w\|.$$

For suitable w the L_2 norm $\|Bu - w\|$ is arbitrarily small; thus $Bu = 0$ on ω. If we define $a(y)$ to be the geodesic distance from $y \in \Omega$ to the boundary ω, then $B: C^j_\omega \to C^{j-1}_\omega$ may be described globally as the adjoint of $da \wedge \cdot$.

Expressed in terms of the isomorphism (1.7), $da \wedge \cdot$ becomes the direct sum of the mappings $da \wedge \cdot : R^{j-1} \to R^j$ and $-da \wedge \cdot : \alpha g^j \to \alpha g^{j+1}$. Since $C^j \approx R^j \oplus \alpha g^{j+1}$ is an orthogonal decomposition, the adjoint B is a direct sum $B = B^1 \oplus B^2 : R^j \oplus \alpha g^{j+1} \to R^{j-1} \oplus \alpha g^j$. As we noted in section 2, the principal symbol $p(\xi) = is_\xi(D)$ has a similar decomposition $p(\xi) = p^1(\xi) + p^2(\xi)$. Thus we have decompositions $p(\xi)^* = p^1(\xi)^* \oplus p^2(\xi)^*$, $r(\xi) = r^1(\xi) \oplus r^2(\xi)$, $k(\xi) = k^1(\xi) \oplus k^2(\xi)$, etc., corresponding to the decompositions $C^j \approx R^j \oplus \alpha g^{j+1}$. Since we may choose the pseudo-differential operators K and K' as direct sums $K = K^1 + K^2$ and $K' = K^{1\prime} + K^{2\prime}$, we find that the estimate $(5.12)_s$ holds if and only if the estimates

$$^{\omega}\|\sigma\|_{(r+s)} \leq c\{^{\omega}\|K^1\sigma\|_{(s)} + {}^{\omega}\|K^{1\prime}\sigma\|_{(s)} + {}^{\omega}\|\sigma\|_{(s)} + {}^{\omega}\|B^1\sigma\|_{(s+1)}\} \tag{6.2$_s$}$$

$$^{\omega}\|\zeta\|_{(r+s)} \leq c\{^{\omega}\|K^2\zeta\|_{(s)} + {}^{\omega}\|K^{2\prime}\zeta\|_{(s)} + {}^{\omega}\|\zeta\|_{(s)} + {}^{\omega}\|B^2\zeta\|_{(s+1)}\} \tag{6.3$_s$}$$

hold for all $\sigma \in \Gamma(\omega, R^j)$ and $\zeta \in \Gamma(\omega, \alpha g^{j+1})$ respectively.

In order to study the estimates $(6.2)_s$ and $(6.3)_s$ we choose a boundary coordinate disk U, as above, with coordinate x. We also choose an orthonormal frame $\eta^1, ..., \eta^n$ in the real cotangent bundle over U; that is, we choose sections $\eta^1, ..., \eta^n$ of $T^*(N)_U$ such that at each $y \in U$, $\eta^1, ..., \eta^n$ form an orthonormal basis for $T^*(M)_y$. Assuming that x^n gives the geodesic distance from $x \in U \cap \Omega$ to ω, we may take $\eta^n = dx^n$ over $U \cap \Omega$. Taking exterior products of the forms η^ν, we obtain orthonormal frames in the bundles $\Lambda^j T^*(M)_U$; choosing orthonormal frames in R_U and g_U, we thus obtain orthonormal frames in each of the bundles g^j_U and R^j_U. We also choose orthonormal frames in the bundles αg^j_U.

As we noted in section 2, the principal symbol of D is given by

$$p(\xi)\, u = is_\xi(D)\, u = \textstyle\sum_1^n i\eta^\nu \wedge \xi_\nu u \quad \text{for} \quad \xi = \xi_1 \eta^1 + \ldots + \xi_n \eta^n \in T^*(M)_y$$

and $u \in C^j_y$. Therefore, we have

$$\begin{aligned} p^1(\xi)\,\sigma &= i \textstyle\sum_1^n \eta^\nu \wedge \xi_\nu \sigma, \\ p^2(\xi)\,\zeta &= -i \textstyle\sum_1^n \eta^\nu \wedge \xi_\nu \zeta, \\ p^1(\xi)^*\sigma &= -i \textstyle\sum_1^n \xi_\nu \sigma \,\overline{\wedge}\, \dfrac{\partial}{\partial \eta^\nu}, \\ p^2(\xi)^*\zeta &= i \textstyle\sum_1^n \xi_\nu \alpha \left(\zeta \,\overline{\wedge}\, \dfrac{\partial}{\partial \eta^\nu}\right), \end{aligned} \tag{6.4}$$

where $\partial/\partial\eta^\nu$ is the vector field dual to η^ν and α is the orthogonal projection of g^j onto αg^j. Using the identities

$$(\eta^\nu \wedge \sigma) \,\overline{\wedge}\, \frac{\partial}{\partial \eta^\mu} + \eta^\nu \wedge \left(\sigma \,\overline{\wedge}\, \frac{\partial}{\partial \eta^\mu}\right) = \begin{cases} \sigma & \text{if} \quad \nu = \mu \\ 0 & \text{if} \quad \nu \neq \mu \end{cases}$$

we find that $r^1(\xi)\sigma=\sum_1^n\xi_\nu^2\sigma=|\xi|^2\sigma$. Thus for $\xi'=\xi_1\eta^1+\dots\xi_{n-1}\eta^{n-1}\in T^*(\omega)_y$ the equation $r^1(\xi'+\eta^nD_t)\,V(t)=|\xi'|^2V(t)+D_t^2\,V(t)=0$ has exponentially decreasing solutions $V(t)=\exp(-|\xi'|t)\,V(0)$, and we find that $h^1(\xi')\sigma=i|\xi'|\sigma$. Therefore, the operators $(6.2)_s$ have symbols:

$$\begin{aligned} k^1(\xi')\sigma&=i\textstyle\sum_1^{n-1}\eta^\nu\wedge\xi_\nu\sigma-\eta^n\wedge|\xi'|\sigma,\\ k^{1'}(\xi')\sigma&=-i\textstyle\sum_1^{n-1}\xi_\nu\sigma\,\overline{\wedge}\,\dfrac{\partial}{\partial\eta^\nu}+|\xi'|\sigma\,\overline{\wedge}\,\dfrac{\partial}{\partial\eta^n},\\ B^1\sigma&=\sigma\,\overline{\wedge}\,\dfrac{\partial}{\partial\eta^n}.\end{aligned}\tag{6.5}$$

We claim that the map $\sigma\to(k^1(\xi')\sigma,\;k^{1'}(\xi')\sigma,\;B^1\sigma)$ is injective if $\xi\neq0$; indeed, if $k^1(\xi')\sigma=k^{1'}(\xi')\sigma=B^1\sigma=0$, then

$$|\xi'|\sigma=(\eta^n\wedge|\xi'|\sigma)\,\overline{\wedge}\,\frac{\partial}{\partial\eta^n}+\eta^n\wedge\left(|\xi'|\sigma\,\overline{\wedge}\,\frac{\partial}{\partial\eta^n}\right)=(\eta^n\wedge|\xi'|\sigma)\,\overline{\wedge}\,\frac{\partial}{\partial\eta^n}=-(k^1(\xi')\sigma)\,\overline{\wedge}\,\frac{\partial}{\partial\eta^n}=0.$$

Thus the pseudo-differential operator $K^1\oplus K^{1'}\oplus B^1$ is elliptic, and consequently the estimates $(6.2)_s$ hold with $r=1$.

The symbols of the operators occurring in $(6.3)_s$ are given by

$$\begin{aligned} k^2(\xi')\zeta&=i\textstyle\sum_1^{n-1}\eta^\nu\wedge\xi_\nu\zeta+i\eta^n\wedge h^2(\xi')\zeta,\\ k^{2'}(\xi')\zeta&=-i\textstyle\sum_1^{n-1}\alpha\left(\xi_\nu\zeta\,\overline{\wedge}\,\dfrac{\partial}{\partial\eta_\nu}\right)-i\alpha\left(h^2(\xi')\zeta\,\overline{\wedge}\,\dfrac{\partial}{\partial\eta^n}\right),\\ B^2\zeta&=\alpha\left(\zeta\,\overline{\wedge}\,\dfrac{\partial}{\partial\eta^n}\right).\end{aligned}\tag{6.6}$$

Suppose that $k^2(\xi')\zeta=k^{2'}(\xi')\zeta=B^2\zeta=0$. We apply $-i\alpha(\cdot\,\overline{\wedge}\,(\partial/\partial\eta^n))$ to the first line in (6.6), $i\eta^n\wedge\cdot$ to the second line, and add. We obtain

$$NM(\xi')\zeta=h^2(\xi')\zeta,\tag{6.7}$$

where
$$N^{-1}\tau=\eta^n\wedge\alpha\left(\tau\,\overline{\wedge}\,\frac{\partial}{\partial\eta^n}\right)+\alpha\left[(\eta^n\wedge\tau)\,\overline{\wedge}\,\frac{\partial}{\partial\eta^n}\right],$$

$$M(\xi')\zeta=\textstyle\sum_1^{n-1}\xi_\nu\left\{\eta^n\wedge\alpha\left(\zeta\,\overline{\wedge}\,\dfrac{\partial}{\partial\eta^\nu}\right)+\alpha\left[(\eta^\nu\wedge\zeta)\,\overline{\wedge}\,\dfrac{\partial}{\partial\eta^n}\right]\right\}.$$

It is readily verified that $V(t)=\exp(iNM(\xi')t)\zeta$ is a solution to $r^2(\xi'+\eta^nD_t)\,V(t)=0$, $t\geqslant0$. In view of (6.7) and the definition of $h^2(\xi')$ we see that $V(t)$ is exponentially decreasing.

Thus ζ belongs to the direct sum of those spaces in the Jordan decomposition which correspond to eigenvalues λ of $NM(\xi')$ satisfying Im $\lambda > 0$. Since this argument can be reversed, we obtain:

PROPOSITION 6.1. *A non-zero cotangent vector $\xi' \in T^*(\omega)_y$ is characteristic for the pseudo-differential operator $K^2 \oplus K^{2'} \oplus B^2$ if and only if the kernel of B^2 in $(\alpha g^{j+1})_y$ has a non-trivial intersection with the subspace of $(\alpha g^{j+1})_y$ corresponding, in the sense of the Jordan decomposition theorem, to the eigenvalues of $NM(\xi')$ with positive imaginary part.*

The estimates $(6.3)_s$ hold with $r=1$ if and only if $K^2 \oplus K^{2'} \oplus B^2$ has no characterictics; thus Prop. 6.1 gives a necessary and sufficient condition for $(6.3)_s$ to hold with $r=1$. Because of Prop. 5.7 we have:

COROLLARY 6.2. *The estimate* (5.1) *holds with $r=1$, if and only if for each $y \in \omega$ the kernel of B^2 in $(\alpha g^{j+1})_y$ has a trivial intersection with the subspace of $(\alpha g^{j+1})_y$ corresponding to the eigenvalues of $NM(\xi')$ with positive imaginary part.*

Thus Corollary 6.2 gives a necessary condition for the solvability of the D-Neumann problem. The condition is unrealistic, however, because it fails to hold even for the Cauchy–Riemann equation. In general one expects the solvability of the D-Neumann problem to depend on higher order properties of the boundary ω and of the symbols of the operators involved. Thus one is forced to consider the estimates $(6.3)_s$ for values $0 < r < 1$, and perhaps even for arbitrarily small positive values of r. In his paper [7], L. Hörmander gives a necessary and sufficient condition for the validity of estimates like $(6.3)_s$ with $r=\frac{1}{2}$. Since we are not able to simplify Hörmander's condition in this setting, we do not state it here. In Chapter III we will apply Hörmander's condition in a more tractible context.

Chapter III. The D_b-problem

1. Restriction to the boundary

In this chapter we construct a sequence of differential operators on ω by restricting the second Spencer sequence for $\mathcal{D}$. This construction generalizes the $\bar{\partial}_b$-sequence of Kohn–Rossi (see [8] and [11]) and provides some information about the convexity required for the solvability of the D-Neumann problem. We retain the notation of Chapter II and the assumption that $\mathcal{D}$ is elliptic.

Let $\iota : \omega \to M$ denote the inclusion, and choose a real function $a \in C^\infty(M)$ such that $a(y)=0$ and $da_y \neq 0$ whenever $y \in \omega$. Then for each $y \in \omega$ and each $j=0, 1, ..., n$ we obtain a surjective map $\iota^*: \Lambda^j T^*(M)_y \to \Lambda^j T^*(\omega)_y$, whose kernel consists of all elements $\xi \in \Lambda^j T^*(M)_y$ satisfying $da \wedge \xi = 0$. Accordingly, for each $y \in \omega$ we obtain exact sequences

$$0 \longrightarrow (R^j_\mu)^a_y \longrightarrow (R^j_\mu)_y \xrightarrow{1\otimes\iota^*} (R_\mu \otimes \Lambda^j T^*(\omega))_y \longrightarrow 0 \tag{1.1}$$

$$0 \longrightarrow (g^j_\mu)^a_y \longrightarrow (g^j_\mu)_y \xrightarrow{1\otimes\iota^*} (g_\mu \otimes \Lambda^j T^*(\omega))_y \longrightarrow 0, \tag{1.2}$$

where $(R^j_\mu)^a_y = \{\sigma \in (R^j_\mu)_y \mid da \wedge \sigma = 0\}$ and similarly for $(g^j_\mu)^a_y$. Since δ anti-commutes with $da \wedge \cdot$, we obtain the diagram

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow \delta & & \downarrow \delta & & \downarrow \delta_b & & \\
0 & \longrightarrow & (g^j_\mu)^a_y & \longrightarrow & (g^j_\mu)_y & \xrightarrow{\iota^*} & (g_\mu \otimes \Lambda^j T^*(\omega))_y & \longrightarrow & 0 \\
 & & \downarrow \delta & & \downarrow \delta & & \downarrow \delta_b & & \\
0 & \longrightarrow & (g^{j+1}_{\mu-1})^a_y & \longrightarrow & (g^{j+1}_{\mu-1})_y & \xrightarrow{\iota^*} & (g_{\mu-1} \otimes \Lambda^{j+1} T^*(\omega))_y & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}$$

where δ_b is induced by δ. Assuming that μ is sufficiently large, we know that the middle column is exact. It follows from the ellipticity of $\mathcal{D}$ that the first column is also exact. In fact, let $\zeta \in (g^{j+1}_{\mu-1})^a_y$ satisfy $\delta\zeta = 0$. Since $da \wedge \zeta = 0$, Prop. II.2.1 shows that $\zeta = da \wedge \zeta'$ for some $\zeta' \in (g^j_{\mu-1})_y$ satisfying $\delta\zeta' = 0$. Since $\zeta' = -\delta\zeta''$ for some $\zeta'' \in (g^{j-1}_\mu)_y$, we have $\zeta = \delta(da \wedge \zeta'')$ and $da \wedge \zeta'' \in (g^j_\mu)^a_y$. We may now conclude that the third column in the diagram is exact. Writing $\alpha_b g^j_\mu = \{\zeta \in g_\mu \otimes \Lambda^j T^*(\omega) \mid \delta_b \zeta = 0\}$ and $(\alpha g^j_\mu)^a_y = \{\zeta \in (g^j_\mu)^a_y \mid \delta\zeta = 0\}$, we thus obtain a diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & (\alpha g^j_\mu)^a_y & \longrightarrow & (\alpha g^j_\mu)_y & \xrightarrow{\iota^*} & (\alpha_b g^j_\mu)_y & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & (g^j_\mu)^a_y & \longrightarrow & (g^j_\mu)_y & \xrightarrow{\iota^*} & (g_\mu \otimes \Lambda^j T^*(\omega))_y & \longrightarrow & 0 \\
 & & \downarrow \delta & & \downarrow \delta & & \downarrow \delta_b & & \\
0 & \longrightarrow & (\alpha g^{j+1}_{\mu-1})^a_y & \longrightarrow & (\alpha g^{j+1}_{\mu-1})_y & \xrightarrow{\iota^*} & (\alpha_b g^{j+1}_{\mu-1})_y & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array} \tag{1.3}$$

with exact columns. The middle row is exact by (1.2); and if $j = n-1$, then the bottom row is exact. Thus by induction the top row is exact for all $j = 0, 1, \ldots, n$. Note that we can use the exactness on the right to show that $\alpha_b g^j_\mu$ is a vector bundle over ω.

From the top row in (1.3) we obtain an exact sequence

$$0 \longrightarrow \Gamma_a(\Omega, \alpha g^j_\mu) \longrightarrow \Gamma(\Omega, \alpha g^j_\mu) \xrightarrow{\varrho} \Gamma(\omega, \alpha_b g^j_\mu) \longrightarrow 0, \tag{1.4}$$

where $\Gamma_a(\Omega, \cdot)$ denotes the space of smooth sections ζ satisfying $da \wedge \zeta = 0$ on ω and where the restriction map ϱ is defined by $\varrho\zeta = \iota^*\zeta|\omega$. If we write $R^j_{b,\mu}$ for the bundle $R_\mu \otimes \Lambda^j T^*(\omega)$ over ω, then (1.1) yields a similar sequence:

$$0 \longrightarrow \Gamma_a(\Omega, R^j_\mu) \longrightarrow \Gamma(\Omega, R^j_\mu) \xrightarrow{\varrho} \Gamma(\omega, R^j_{b,\mu}) \longrightarrow 0. \tag{1.5}$$

Note that the restriction maps ϱ are compatible with exterior multiplication in the sense that $\varrho(\xi \wedge \sigma) = \varrho\xi \wedge \varrho\sigma$ if ξ is a ν-form on M and σ is a section of R^j_μ or αg^j_μ.

Now define the bundle $C^j_{b,\mu}$ over ω by the exact sequence

$$0 \to \alpha_b g^j_{\mu+1} \to R^j_{b,\mu+1} \to C^j_{b,\mu} \to 0$$

and consider the diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Gamma_a(\Omega, \alpha g^j_{\mu+1}) & \longrightarrow & \Gamma(\Omega, \alpha g^j_{\mu+1}) & \longrightarrow & \Gamma(\omega, \alpha_b g^j_{\mu+1}) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Gamma_a(\Omega, R^j_{\mu+1}) & \longrightarrow & \Gamma(\Omega, R^j_{\mu+1}) & \longrightarrow & \Gamma(\omega, R^j_{b,\mu+1}) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \Gamma_a(\Omega, C^j_\mu) & & \Gamma(\Omega, C^j_\mu) & & \Gamma(\omega, C^j_{b,\mu}). & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array} \tag{1.6}$$

By (1.4) and (1.5) the rows are exact, and the two columns on the right are exact because of the definitions of C^j_μ and $C^j_{b,\mu}$. We claim that the left column is also exact. Since exactness at the other places is easily verified, it is sufficient to show that $\Gamma_a(\Omega, R^j_{\mu+1}) \to \Gamma_a(\Omega, C^j_\mu)$ is surjective. Thus let σ be a section of C^j_μ such that $da \wedge \sigma = 0$ on ω. Write p for the mapping $R^j_{\mu+1} \to C^j_\mu$, and choose a section τ of $R^j_{\mu+1}$ such that $p\tau = \sigma$ on Ω. Since $da \wedge \sigma = 0$ on ω, we find that $(da \wedge \tau)|\omega$ is a section of $\alpha g^{j+1}_{\mu+1}$. By Prop. II.2.1 there exists $\tau' \in \Gamma(\omega, \alpha g^j_{\mu+1})$ such that $da \wedge \tau' = da \wedge \tau$ on ω. If $\tau'' \in \Gamma(\Omega, \alpha g^j_{\mu+1})$ is any extension of τ', then $p(\tau - \tau'') = \sigma$ on Ω and $da \wedge (\tau - \tau'') = 0$ on ω. Thus the required surjectivity has been established. Diagram (1.6) now yields an exact sequence

$$0 \longrightarrow \Gamma_a(\Omega, C^j_\mu) \longrightarrow \Gamma(\Omega, C^j_\mu) \xrightarrow{\varrho} \Gamma(\omega, C^j_{b,\mu}) \longrightarrow 0. \tag{1.7}$$

As before ϱ is compatible with exterior multiplication.

We claim that the operator D in the second Spencer sequence maps $\Gamma_a(\Omega, C^j_\mu)$ into $\Gamma_a(\Omega, C^{j+1}_\mu)$. Indeed, if $\sigma \in \Gamma_a(\Omega, C^j_\mu)$, then there exists $\tau \in \Gamma(\Omega, C^{j+1}_\mu)$ such that $da \wedge \sigma = a\tau$ in a neighborhood of ω. Thus

$$da \wedge D\sigma = -D(da \wedge \sigma) = -D(a\tau) = -da \wedge \tau - aD\tau = -aD\tau = 0$$

on ω. Using the sequences (1.7) we thus obtain an induced operator $D_b: C^j_{b,\mu} \to C^{j+1}_{b,\mu}$. Since $D^2 = 0$, it follows that $D^2_b = 0$; moreover, D_b is a derivation in the sense that

$$D_b(\xi \wedge \sigma) = d\xi \wedge \sigma + (-1)^\nu \xi \wedge D_b \sigma \tag{1.8}$$

for a ν-form ξ on ω and a section σ of $C^j_{b,\mu}$. To prove this choose $\tau \in \Gamma(\Omega, C^j_\mu)$ and $\eta \in \Gamma(\Omega, \Lambda^j T^*(M))$ such that $\varrho\tau = \sigma$ and $\varrho\eta = \xi$. Then

$$D_b(\xi \wedge \sigma) = \varrho(d\eta \wedge \tau + (-1)^\nu \eta \wedge D\tau) = \varrho d\eta \wedge \sigma + (-1)^\nu \xi \wedge D_b \sigma.$$

In a local coordinate it is easy to verify that $\varrho d\eta = d\varrho\eta = d\xi$; thus (1.8) holds.

2. The symbol sequence

Our purpose here is to investigate the sequence of symbol maps which is associated with the D_b-sequence

$$0 \to \underline{C^0_{b,\mu}} \to \underline{C^1_{b,\mu}} \to \dots \to \underline{C^{n-1}_{b,\mu}} \to 0 \tag{2.1}$$

introduced in section 1. In view of (1.8) we have that $s_\xi(D_b) = \xi \wedge \cdot$ for a non-zero cotangent vector $\xi \in T^*(\omega)_y$. If we introduce isomorphisms $C^j_{b,\mu} \approx R^j_{b,\mu} \oplus \alpha_b g^{j+1}_\mu$ as in II, section 1, then the symbol sequence of (2.1) at ξ is the direct sum of the sequences:

$$0 \longrightarrow (R^0_{b,\mu}) \xrightarrow{\xi\wedge\cdot} (R^1_{b,\mu})_y \xrightarrow{\xi\wedge\cdot} \dots \longrightarrow (R^{n-1}_{b,\mu})_y \longrightarrow 0, \tag{2.2}$$

$$0 \longrightarrow (\alpha_b g^1_\mu)_y \xrightarrow{-\xi\wedge\cdot} (\alpha_b g^2_\mu)_y \xrightarrow{-\xi\wedge\cdot} \dots \longrightarrow (\alpha_b g^{n-1}_\mu)_y \longrightarrow 0. \tag{2.3}$$

The first sequence is always exact; we begin our discussion of the second with the following:

Definition. Let ξ_1 and ξ_2 be linearly independent elements of $T^*(M)_y$, and let P be the subspace which they span. Then $\xi_1 \wedge \xi_2$ is said to be characteristic for $\mathcal{D}: \underline{E} \to \underline{F}$ if the composition

$$S^{\mu_0} P \otimes E_y \subset S^{\mu_0} T^*_y \otimes E_y \to J_{\mu_0}(E)_y \xrightarrow{\varrho_{\mu_0}(\mathcal{D})} F_y$$

has a non-trivial kernel.

If $x = (x^1, \dots, x^n)$ is a local coordinate at y and if $\xi_1 = dx^{n-1}$, $\xi_2 = dx^n$, then $\xi_1 \wedge \xi_2$ is characteristic for $\mathcal{D}$ if and only if $(g_{\mu_0})_y$ contains a non-zero element σ satisfying $\delta_1 \sigma = \dots = \delta_{n-2}\sigma = 0$. If $\xi_1 \wedge \xi_2$ is characteristic for $\mathcal{D}$, it does not follow that $\xi_1 \wedge \xi_2$ is characteristic for $j_\nu \mathcal{D}$, $\nu > 0$. For example, $\mu_0 = 1$, $n = 2$, and suppose that $(g_1)_y$ consists of all real multiples of eX^1+fX^2, where e and f are linearly independent elements of $\mathbf{R}^2$. For each $\sigma \in (g_2)_y$ there exist real numbers t_1 and t_2 such that $\delta_1\sigma = t_1(eX^1+fX^2)$ and $\delta_2\sigma = t_2(eX^1+fX^2)$. Then $t_1 f = \delta_2\delta_1\sigma = \delta_1\delta_2\sigma = t_2 e$; and by linear independence, $t_1 = t_2 = 0$. It follows that $\sigma = 0$ and hence $(g_2)_y = 0$. We find that $dx^1 \wedge dx^2$ is characteristic for $\mathcal{D}$, but not for $j_1\mathcal{D}$.

If $\mathcal{D}$ is elliptic and involutive, however, then $\xi_1 \wedge \xi_2$ is characteristic for $\mathcal{D}$ if and only if it is characteristic for $j_\nu\mathcal{D}$. Indeed, this will follow from the next proposition.

Proposition 2.1. *Let μ_1 be the integer defined in I, section 7, and let $da \in T^*(M)_y$ be orthogonal to $T^*(\omega)_y$. Then the sequence (2.3) is exact for $\mu \geqslant \mu_1$ if and only if $\xi \wedge da$ is not characteristic for $j_{\mu_1-\mu_0}\mathcal{D}$.*

Proof. For this proof we may assume that $\mathcal{D}$ is involutive and of order 1. Then δ_b sequences are exact for $\mu \geqslant 1$, and the cohomology of (2.3) is independent of $\mu \geqslant 1$. Indeed this follows from the argument used in concluding the proof of Prop. II.2.1.

We introduce a coordinate $x = (x^1, \ldots, x^n)$ on a neighborhood of y in M; we assume that $dx^1, \ldots, dx^{n-1} \in T^*(\omega)_y$ and that $dx^n = da$. We may also assume that $\xi = dx^{n-1}$. Using the definition of δ_b, we find that $\delta_b = \Sigma_1^{n-1} dx^\nu \wedge \delta_\nu$ in the fibers over y. Thus the kernel of $-dx^{n-1} \wedge \cdot$ in $(\alpha_b g_\mu^1)_y$ consists of all elements of the form $dx^{n-1} \otimes \sigma$, where $\sigma \in g_\mu$ satisfies $\delta_1\sigma = \ldots = \delta_{n-1}\sigma = 0$. Since the cohomology of (2.3) is independent of μ, we see that $dx^{n-1} \wedge dx^n$ is characteristic for $\mathcal{D}$. We also see that the condition in the proposition is necessary if (2.3) is to be exact.

The proof of sufficiency follows the proof of Prop. II.2.1 very closely. Using the arguments given there, we see that it suffices to show that the condition implies the exactness of

$$g_{\mu+1} \otimes \Lambda^{i-2} H \to g_\mu \otimes \Lambda^{i-1} H \to g_{\mu-1} \otimes \Lambda^i H \tag{2.4}$$

for large μ. Here $\delta' = \Sigma_1^{n-2} dx^\nu \wedge \delta_\nu$, and H is the subspace of $T^*(\omega)_y$ spanned by $dx^1, \ldots, dx^{n-2}$. Arguing as in the proof of Prop. II.2.1, we can identify the spaces g_μ with spaces of polynomials in $n-2$ variables and conclude, as before, that (2.4) is exact.

Note that there may be a large space of $\xi \in T^*(\omega)_y$ for which (2.3) is not exact. For example, let $\mathcal{D}$ be the determined operator given in a local coordinate by $\mathcal{D}s = \Sigma_1^n A_\nu \partial_\nu s$, where A_ν is a non-singular matrix. Assume that Ω is defined by $x^n \geqslant 0$ and that the metric is euclidean in the coordinate x. Then for any $f \in E_y$ we may set $e = -A_{n-1}^{-1} A_n f$ and obtain an element $eX^{n-1} + fX^n \in (g_1)_y$. Thus $dx^{n-1} \wedge dx^n$ is characteristic for $\mathcal{D}$; in fact, for any $0 \neq \xi \in T^*(\omega)_y$, $\xi \wedge dx^n$ is characteristic for $\mathcal{D}$. Since $\mathcal{D}$ is involutive, (2.3) is never exact.

In general, however, the space of characteristic ξ's will be a proper subspace of $T^*(\omega)_y$. If $\mathcal{D}s = 0$ gives the Cauchy–Riemann equation, then (2.3) fails to be exact for ξ lying in a 1-dimensional subspace. If $\mathcal{D}$ is the gradient operator, then (2.3) is always exact.

3. The D_b-problem

For $j=1, \ldots, n-1$ let $\mathbf{H}_b^j$ be the space of all $u \in \Gamma(\omega, C_b^j)$ satisfying $D_b u = D_b^* u = 0$.

Definition: We say that the D_b-problem is solvable for $\mathcal{D}$ on ω up to dimension m if $\mathbf{H}_b^j$ is a closed subspace of $L_2(\omega, C_b^j)$ for $j=1, \ldots, m$ and if there exist bounded operators $N_b: L_2(\omega, C_b^j) \to L_2(\omega, C_b^j)$, $j=1, \ldots, m$, mapping smooth sections to smooth sections, such that:

(i) $N_b H_b = H_b N_b = 0$, where H_b is the orthogonal projection onto $\mathbf{H}_b^j$;

(ii) each $u \in \Gamma(\omega, C_b^j)$ can be written

$$u = D_b D_b^* N_b u + D_b^* D_b N_b u + H_b u, \tag{3.1}$$

where the terms are mutually orthogonal;

(iii) N_b commutes with D_b and D_b^*.

Using the arguments described in II, section 5, we obtain:

PROPOSITION 3.1. *Assume that for some $0 < r \leqslant 1$ and for $j=1, \ldots, m$ the estimate*

$$\|u\|_{(r)} \leqslant c\{\|D_b u\| + \|D_b^* u\| + \|u\|\} \tag{3.2}$$

holds for all $u \in \Gamma(\omega, C_b^j)$. Then the D_b-problem is solvable for $\mathcal{D}$ on ω up to dimension m.

Now recall the exact sequence (1.7) and consider the commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & T_a & \longrightarrow & \Gamma_a(\Omega, C^0) & \xrightarrow{D} & \Gamma_a(\Omega, C^1) & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & T & \longrightarrow & \Gamma(\Omega, C^0) & \xrightarrow{D} & \Gamma(\Omega, C^1) & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & T_b & \longrightarrow & \Gamma(\omega, C_b^0) & \xrightarrow{D_b} & \Gamma(\omega, C_b^1) & \longrightarrow & \cdots \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array} \tag{3.3}$$

where T_a, T, and T_b are defined as the kernels in the corresponding rows. If $y \in \omega$ and if $\sigma \in C_y^0$ satisfies $dy \wedge \sigma = 0$, then $\sigma = 0$ because of Prop. II.2.1 and the ellipticity of $\mathcal{D}$. Thus $\Gamma_a(\Omega, C^0)$ consists of those sections in $\Gamma(\Omega, C^0)$ which vanish on ω. It also follows that C_b^0 is isomorphic to the restricted bundle C_ω^0, and thus we may consider T_b as a subspace of $\Gamma(\omega, C^0)$. If the top row is exact at $\Gamma_a(\Omega, C^1)$, then by diagram chasing we infer that the restriction map $T \to T_b$ is surjective. Thus $v \in \Gamma(\omega, C^0)$ is the boundary value of a section $u \in \Gamma(\Omega, C^0)$ satisfying $Du = 0$ if and only if $D_b v = 0$. Recall from II, section 1 that C_μ^0 is isomorphic to $R_{\mu+1}$ and that the canonical operator $j_{\mu+1}$ gives an isomorphism of T with the space of sections $u\in\Gamma(\Omega, E)$ satisfying $\mathcal{D}u=0$. Thus, under the exactness assumption above, the condition $D_b v=0$ is necessary and sufficient for $v\in\Gamma(\omega, R_{\mu+1})$ to be the Cauchy data for a solution to $j_{\mu-\mu_0+2}\mathcal{D}u=0$.

The exactness properties of the top row in (3.3) can be studied by a boundary problem very similar to the D-Neumann problem; in fact, the new problem is obtained by replacing (II.3.1) and (II.3.2) by the adjoint boundary conditions. Thus the fundamental estimate is obtained from (II.5.1) by replacing B with $B^*=da\wedge\cdot$.

The purpose of diagram (3.3) is to compare the exactness properties of the three rows. It follows by diagram chasing, for example, that if any two rows have finite dimensional cohomology, then the same is true for the third row. It is probably true that if the problems corresponding to any two rows are solvable, then the third problem is also solvable. For our purposes the following proposition is relevant.

PROPOSITION 3.1. *Recall the notation of II, section 5, and assume that $h(\xi)$ commutes with the boundary operator B. Assume that the estimate* (3.2) *holds for $u\in\Gamma(\omega, C_b^{j-1})$, where $j\geqslant 2$. Then the estimate* (II.5.1) *holds for $u\in\Gamma(\Omega, C^{j-1})$.*

Proof. The principal symbol of D_b splits into a direct sum $q(\xi)=is_\xi(D_b)=q^1(\xi)\oplus q^2(\xi)$ corresponding to the decomposition $C_b^j\approx R_b^j\oplus\alpha_b g^{j+1}$. For the purpose of this proof we may assume that D_b has a similar decomposition $D_b=D_b^1\oplus D_b^2$; this is because the estimate (3.2) depends only on $q(\xi)$. From the hypothesis of the proposition we now obtain the estimate

$$\|\zeta\|_{(r)}\leqslant c\{\|D_b^2\zeta\|+\|D_b^{2*}\zeta\|+\|\zeta\|\} \tag{3.4}$$

for $\zeta\in\Gamma(\omega, \alpha_b g^j)$.

We rewrite the exact sequence

$$0\longrightarrow(\alpha g^j)^a\longrightarrow\alpha g^j\xrightarrow{\ \iota^*\ }\alpha_b g^j\longrightarrow 0 \tag{3.5}$$

from the diagram (1.3). We assume that the inner product in $\alpha_b g^j$ is the quotient inner product defined by (3.5). Thus the adjoint t of ι^* is the splitting of (3.5) with least norm. It follows that $t\iota^*$ is the orthogonal projection of αg^j onto the orthogonal complement of $(\alpha g^j)^a$; that is, $t\iota^*$ is the orthogonal projection onto the kernel of B^2.

From (II.5.10) we have

$$k^2(\xi)\,\zeta=p_0^2(\xi)\,\zeta+da\wedge h^2(\xi)\,\zeta$$

$$k^{2\prime}(\xi)\,\zeta=p_0^2(\xi)^*\zeta+B^2h^2(\xi)\,\zeta,$$

where $p_0^2(\xi)\,\zeta = \xi \wedge \zeta$. Since ι^* is compatible with exterior multiplication, we have $q^2(\xi)\,\iota^* = \iota^* p_0^2(\xi)$. Taking adjoints, we obtain $tq^2(\xi)^* = p_0^2(\xi)^* t$. Note that $\iota^* p_0^2(\xi) = \iota^* k^2(\xi)$ and that $tq^2(\xi)^* = k^{2\prime}(\xi) t$ since $B^2 h^2(\xi)\, t = h^2(\xi)\, B^2 t = 0$. We thus obtain

$$q^2(\xi) = \iota^* k^2(\xi)\, t$$

$$q^2(\xi)^* = \iota^* k^{2\prime}(\xi)\, t$$

and hence

$$\begin{aligned} \|D_b^2\,\zeta\| &\leqslant c\{\|K^2 t\zeta\| + \|\zeta\|\} \\ \|D_b^{2*}\zeta\| &\leqslant c\{\|K^{2\prime} t\zeta\| + \|\zeta\|\} \end{aligned} \tag{3.6}$$

for $\zeta \in \Gamma(\omega, \alpha_b g^j)$.

For $\zeta \in \Gamma(\omega, \alpha g^j)$ we have

$$\|\zeta\|_{(r)} \leqslant \|t\iota^*\zeta\|_{(r)} + \|(\mathbf{1} - t\iota^*)\,\zeta\|_{(r)}. \tag{3.7}$$

Using (3.4) and (3.6) we see that

$$\|t\iota^*\zeta\|_{(r)} \leqslant c\|\iota^*\zeta\|_{(r)} \leqslant c\{\|D_b^2\,\iota^*\zeta\| + \|D_b^{2*}\iota^*\zeta\| + \|\iota^*\zeta\|\} \leqslant c\{\|K^2 t\iota^*\zeta\| + \|K^{2\prime} t\iota^*\zeta\| + \|\zeta\|\}.$$

Note that $\|K^2(\mathbf{1} - t\iota^*)\,\zeta\| \leqslant c\|(\mathbf{1} - t\iota^*)\,\zeta\|_{(1)} \leqslant c\{\|B^2\zeta\|_{(1)} + \|\zeta\|\}$. A similar estimate holds for $K^{2\prime}(\mathbf{1} - t\iota^*)\,\zeta$, and hence we obtain

$$\|t\iota^*\zeta\|_{(r)} \leqslant c\{\|K^2\zeta\| + \|K^{2\prime}\zeta\| + \|\zeta\| + \|B^2\zeta\|_{(1)}\}.$$

Combining this estimate with (3.7) and the estimate $\|(\mathbf{1} - t\iota^*)\zeta\|_{(r)} \leqslant c\{\|B^2\zeta\|_{(1)} + \|\zeta\|\}$, we obtain the estimate $(\text{II}.6.3)_0$ for $\zeta \in \Gamma(\omega, \alpha g_\mu^j)$. By the results of II, section 5 and II, section 6 we conclude that (II.5.1) holds for $u \in \Gamma(\Omega, C^{j-1})$. The proof is complete.

We close this section with a cautioning word about diagram (3.3). Namely, consider the end of that diagram:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
\cdots \arrow[r] & \Gamma_a(\Omega, C^{n-2}) \arrow[r] \arrow[d] & \Gamma_a(\Omega, C^{n-1}) \arrow[r] \arrow[d] & \Gamma_a(\Omega, C^n) \arrow[r] \arrow[d] & 0 \\
\cdots \arrow[r] & \Gamma(\Omega, C^{n-2}) \arrow[r] \arrow[d] & \Gamma(\Omega, C^{n-1}) \arrow[r] \arrow[d] & \Gamma(\Omega, C^n) \arrow[r] \arrow[d] & 0 \\
\cdots \arrow[r] & \Gamma(\omega, C_b^{n-2}) \arrow[r] \arrow[d] & \Gamma(\omega, C_b^{n-1}) \arrow[r] \arrow[d] & 0 & \\
 & 0 & 0 & &
\end{tikzcd}$$

On the summand αg^n of C^{n-1} the boundary condition defining $\Gamma_a(\Omega, C^{n-1})$ vanishes; thus the cohomology of the top row will often be infinite dimensional in dimension $j = n - 1$.

Consequently, we cannot expect the bottom row always to be well-behaved at $j=n-2$. It is also possible for this phenomenon to occur in dimension $j<n-2$; in section 5 below we discuss a case where the D_b-problem is solvable only up to dimension $\frac{1}{2}n-2$. In general we expect the D_b-problem to be well-behaved for $j=1, \ldots, m-2$, where m is the "cohomological dimension" of $\mathcal{D}: \underline{E} \to \underline{F}$; that is, where m is the length of the shortest sequence

$$0 \to \underline{E} \to \underline{F} \to \underline{F}_2 \to \ldots \to \underline{F}_m \to 0$$

of differential operators having the same cohomology as the Spencer sequence for $\mathcal{D}$.

4. Hörmander's condition

In this section we use a recent theorem due to L. Hörmander in order to study the estimate

$$\|u\|_{(\frac{1}{2})} \leqslant c\{\|D_b u\| + \|D_b^* u\| + \|u\|\}, \quad u \in \Gamma(\omega, C_b^j), \tag{4.1}$$

which is a special case of (3.2). Before stating Hörmander's theorem we must introduce new notation. Let $U \subset \omega$ be a coordinate disk with coordinate $y=(y^1, \ldots, y^{n-1})$; letting $(y^1, \ldots, y^{n-1}; \xi_1, \ldots, \xi_{n-1})$ correspond to the cotangent vector $\xi_1 dy^1 + \ldots + \xi_{n-1} dy^{n-1}$ in $T^*(\omega)y$, we obtain a coordinate (y, ξ) on $T^*(\omega)_U$. After choosing orthonormal frames in the bundles $(C_b^j)_U$, we may identify the principal symbol of D_b with a matrix-valued function $q(y, \xi)$. We define

$$q^{(\nu)}(y, \xi) = \frac{\partial}{\partial \xi_\nu} q(y, \xi) \quad \text{and} \quad q_{(\nu)}(y, \xi) = \frac{\partial}{\partial y^\nu} q(y, \xi).$$

For each $(y, \xi) \in T^*(\omega)$ we define a differential operator $Q(y, \xi)$ by setting

$$Q(y, \xi) v = \sum_\nu q^{(\nu)}(y, \xi) D_\nu v - \tfrac{1}{2} \sum_\nu q(y, \xi) \xi_\nu D_\nu v + \sum_\nu q_{(\nu)}(y, \xi) x^\nu v \tag{4.2}$$

for $(C_b^j)_y$-valued functions v of the variable $x=(x^1, \ldots, x^{n-1}) \in \mathbf{R}^{n-1}$. We write $Q(y, \xi)^*$ for the formal adjoint of $Q(y, \xi)$.

PROPOSITION 4.1. *The estimate* (4.1) *holds if and only if for every compact subset K of ω there exist constants c and N and a function $\varepsilon: \mathbf{R} \to \mathbf{R}$ vanishing at $+\infty$ such that the estimate*

$$\|v\|^2 \leqslant c\{\|q(y, \xi)\lambda v + Q(y, \xi) v\|^2 + \|q(y, \xi)^* \lambda v + Q(y, \xi)^* v\|^2 + \varepsilon(\lambda) \sum_{|\alpha|+|\beta| \leqslant N} \|x^\alpha D^\beta v\|^2\} \tag{4.3}$$

holds for all $|\xi|=1$, $y \in K$, and all $v \in C_0^\infty(\mathbf{R}^{n-1}, (C_b^j)_y)$.

Proof. See Hörmander [7], Theorem 1.1.4 and formula (1.1.18)′.

The estimate (4.3) is rather cumbersome; under additional assumptions Hörmander gives a simpler necessary and sufficient condition.

PROPOSITION 4.2. *Assume that the dimension of the kernel of $q(y,\xi)\oplus q(y,\xi)^*$ has locally constant dimension on the set where it is not 0. Also assume that for each compact subset K of ω there exists a constant c such that*

$$d(y,\xi)\|e\| \leqslant c\{\|q(y,\xi)e\| + \|q(y,\xi)^*e\|\}$$

for all $y \in K$, $|\xi|=1$, and $e \in (C_b^j)_y$, where $d(y,\xi)$ denotes the distance from (y,ξ) to the characteristic variety. Let $H(y,\xi)$ denote the orthogonal projection of $(C_b^j)_y$ onto the kernel of $q(y,\xi)\oplus q(y,\xi)^*$. Then (4.1) *holds if and only if for every compact subset K of ω there exists a constant c such that the estimate*

$$\|v\| \leqslant c\{\|H(y,\xi)Q(y,\xi)v\| + \|H(y,\xi)\,Q(y,\xi)^*v\|\} \tag{4.4}$$

holds for all $y \in K$, $|\xi|=1$, and all $v \in C^\infty(\mathbf{R}^{n-1}(C_b^j)_y)$ *satisfying* $H(y,\xi)v=v$.

Proof. See Hörmander [7], Theorem 1.1.7.

5. An example

Let $n=2m$ and let M be a complex manifold of complex dimension m; let $\Omega\subset M$ and $\omega\subset\Omega$ be as before. Let S be the bundle of holomorphic tangent vectors over M, and let $\bar{S}$ denote the conjugate bundle. Choose an inner product along the fibers of S; this induces an inner product on $\bar{S}$ and, hence, on the complexified tangent bundle $T_c(M)=S\oplus\bar{S}$. By construction, the conjugation map is a unitary map of $T_c(M)$ onto itself; and the restriction of the inner product to $T(M)=\{X\in T_c(M)\,|\,\bar{X}=X\}$ defines a Riemannian metric on M.

Let E be the trivial complex line bundle over M, and let $\bar{S}^*$ be the bundle dual to $\bar{S}$. Define the differential operator

$$\mathcal{D}:\underline{E}\to\underline{\bar{S}}^* \tag{5.1}$$

by setting

$$\mathcal{D}s=\sum_1^m \frac{\partial s}{\partial \bar{z}^\nu}\,d\bar{z}^\nu \tag{5.2}$$

in a local complex coordinate $z=(z^1,\dots,z^m)$. Formula (5.2) does not depend on the choice of coordinate, and thus (5.1) is well defined. Note that "$\mathcal{D}s=0$" is the Cauchy-Riemann equation in m complex variables. It is easily verified that $\mathcal{D}$ is elliptic and involutive.

LEMMA 5.1. *Let $y\in\omega$. Then there exist sections $Z^1,\dots,Z^m$ of S^* defined on a neighborhood U of y in M such that:* (i) *for each $x\in U\cap\Omega$ the covectors $Z^1,\dots,Z^m,\bar{Z}^1,\dots,\bar{Z}^m$ form an ortho-*

normal basis for $T_c^*(M)_x$; *and* (ii) *for each* $x \in U \cap \omega$ *the covectors* $Z^1, \ldots, Z^{m-1}, \bar{Z}^1, \ldots, \bar{Z}^{m-1}$, $\operatorname{Re} Z^m$ *form an orthonormal basis for* $T_c^*(\omega)$.

Proof. For each $x \in U \cap \omega$ define $S_{b,x}^*$ to be the space of all holomorphic cotangent vectors at x which belong to $T_c^*(\omega)$. Since "$Y \in T_c^*(\omega)$" gives at most one linear condition on the fiber S_x^*, the complex dimension of $S_{b,x}^*$ is either m or $m-1$. If the dimension were m, then $T_c^*(\omega)_x$ would contain every holomorphic cotangent vector; and since $T_c^*(\omega)$ is stable under conjugation, this would imply that $T_c^*(\omega)_x$ contains $T_c^*(M)_x$. The contradiction shows that $S_{b,x}^*$ has complex dimension $m-1$, and hence $S_b^* = \bigcup S_{b,x}^*$ is a subbundle of $T_c^*(\omega)$.

Now choose sections $Z^1, \ldots, Z^{m-1}$ of S_b^* in a neighborhood of y which give an orthonormal basis in each fiber $S_{b,x}^*$. Without loss of generality $Z^1, \ldots, Z^{m-1}$ can be extended to an orthonormal set of sections of S^* over a neighborhood of y in M.

Choose a section X of $T_c^*(M)$ over U such that $Z^1, \ldots, Z^{m-1}, \bar{Z}^1, \ldots, \bar{Z}^{m-1}, X$ form an orthonormal basis for $T_c^*(\omega)_x$ for each $x \in U$. Replacing X by $(X+\bar{X})/2$ if necessary, we may assume that $X = \bar{X}$. We may also assume that X has length 1 over each $x \in U \cap \Omega$.

Now let Z^m be a section of S^* over U such that $Z^1, \ldots, Z^m$ give an orthonormal basis in the fibers of S^* over U. Since X is orthogonal to Z^k and $\bar{Z}^k$ for $k=1, \ldots, m-1$, we have $X = aZ^m + b\bar{Z}^m$ for some $a, b \in \mathbf{C}$. Without loss of generality we may assume that a is real, and hence from $X = \bar{X}$ we conclude that $a = b$. We now have $X = 2a \operatorname{Re} Z^m$, and since $|X| = |\operatorname{Re} Z^m|$, we must have $2a = 1$. The proof is now complete.

Note that $Y^m = \operatorname{Im} Z^m$ is orthogonal to $T_c^*(\omega)_x$ at each $x \in \omega$. Also, if $X^m = \operatorname{Re} Z^m$, then $\xi = \pm X^m$ are the only cotangent vectors of unit length for which $\xi \wedge Y^m$ is characteristic for $\mathcal{D}$. Indeed, this is because the equation "$\mathcal{D}s = 0$" is equivalent to

$$\frac{\partial s}{\partial \bar{Z}^k} = 0 \quad (k = 1, \ldots, m),$$

where $\partial/\partial \bar{Z}^k$ is the vector field dual to $\bar{Z}^k$. Hence $\xi = \pm X^m$ are the only unit cotangent vectors for which the symbol of $D_b D_b^* + D_b^* D_b$ is not injective. Denoting the symbol of $D_b D_b^* + D_b^* D_b$ at ξ by $\Delta(\xi)$, we state:

LEMMA 5.2. *An element* ζ *of* $\alpha_b g_x^{j+1}$ *is in the kernel of* $\Delta(X^m)$ *if and only if it has the form*

$$\zeta = (X^m \wedge Z) \otimes \sigma, \tag{5.3}$$

where $X^m = \operatorname{Re} Z^m$, $Z \in \Lambda^j \bar{S}_b^*$, *and* $\sigma \in g_x$ *satisfies* $\delta_{Z^k}\sigma = \delta_{\bar{Z}^k}\sigma = 0$ *for* $k = 1, \ldots, m-1$. *Here* δ_{Z^k} *and* $\delta_{\bar{Z}^k}$ *are the bundle maps representing* $\partial/\partial Z^k$ *and* $\partial/\partial \bar{Z}^k$ *in the sense of jets.*

Proof. If α_b denotes the orthogonal projection of $g \otimes \Lambda^{j+1} T^*(\omega)$ onto $\alpha_b g^{j+1}$, then

$$\Delta(X^m)\,\zeta = X^m \wedge \alpha_b\left(\zeta \mathbin{\overline{\wedge}} \frac{\partial}{\partial X^m}\right) + \alpha_b\left[(X^m \wedge \zeta) \mathbin{\overline{\wedge}} \frac{\partial}{\partial X^m}\right]. \tag{5.4}$$

Also, for $\zeta \in g \otimes \Lambda^{j+1} T^*(\omega)$ the element $\delta_b \zeta$ is given by

$$\delta_b \zeta = \textstyle\sum_1^{m-1} Z^\nu \wedge \delta_{Z^\nu} \zeta + X^m \wedge \delta_{X^m} \zeta. \tag{5.5}$$

To prove sufficiency note that if ζ has the form (5.3), then $\delta_b \zeta = 0$ and $X^m \wedge \zeta = 0$. Also $\zeta \mathbin{\overline{\wedge}} \partial/\partial X^m$ is orthogonal to $\alpha_b g^j$. Indeed, (5.5) shows that $\alpha_b g^j$ is contained in the ideal generated by X^m and the Z^k's; by assumption $\zeta \mathbin{\overline{\wedge}} \partial/\partial X^m$ involves only $\bar{Z}^k$'s. Thus $\alpha_b(\zeta \mathbin{\overline{\wedge}} \partial/\partial X^m) = 0$ and hence $\Delta(X^m)\zeta = 0$.

To prove the necessity of (5.3) let $\zeta \in \alpha_b g^{j+1}$ satisfy $\Delta(X^m)\zeta = 0$. Since $X^m \wedge \zeta = 0$, we have $\zeta = X^m \wedge \tau$ for some $\tau \in g \otimes \wedge^j (S_b^* + \bar{S}_b^*)$. Since $\delta_b \zeta = 0$, we find that τ is annihilated by

$$\delta_b' = \textstyle\sum_1^{m-1} Z^\nu \wedge \delta_{Z^\nu}. \tag{5.6}$$

If we consider $g = g_\mu$ as a subset of $S^\mu T_c^*(M)$, then g consists of all homogeneous polynomials of degree μ in $Z^1, \ldots, Z^m$. It follows that $\delta_{Z^1}: g_{\mu+1} \to g_\mu$ is surjective, and thus we may choose $\eta \in g_{\mu+1} \otimes \Lambda^{j-1}(S_b^* + \bar{S}_b^*)$ such that $\delta_{Z^1}\eta = \tau \wedge \partial/\partial Z^1$. Arguing as in the proof of Prop. I.6.1, we see that $\eta' = \tau - \delta_b \eta$ satisfies $\delta_{Z^1}\eta' = 0$ and $\eta' \wedge \partial/\partial Z^1 = 0$. Repeating the argument several times, we obtain $\tau' \in g_{\mu+1} \otimes \Lambda^{j-1}(S_b^* + \bar{S}_b^*)$ such that $\tau'' = \tau - \delta_b' \tau'$ involves only $\bar{Z}^k$'s and satisfies $\delta_{Z^k} \tau'' = 0$ for $k = 1, \ldots, m-1$. Multiplying by X^m on the left, we now obtain

$$\zeta'' = \zeta - X^m \wedge \delta_b \tau', \tag{5.7}$$

where ζ'' has the form (5.3). By the first part of the proof ζ'' is annihilated by $\Delta(X^m)$; by hypothesis the same is true for ζ. But the last term on the right of (5.7) is in the range of $X^m \wedge \cdot : \alpha_b g^j \to \alpha_b g^{j+1}$ and is thus orthogonal to the kernel of $\Delta(X^m)$. We conclude that $\zeta = \zeta''$, and the proof is complete.

Let H denote the orthogonal projection of $C_b^j = R_b^j \oplus \alpha_b g^{j+1}$ onto the space of all $\zeta \in \alpha_b g^{j+1}$ having the form (5.3). We have:

LEMMA 5.3. *If $q(x, \xi)$ denotes the symbol of D_b at $\xi = \xi_1 Z^1 + \bar{\xi}_1 \bar{Z}^1 + \ldots + \xi_{m-1} Z^{m-1} + \bar{\xi}_{m-1} \bar{Z}^{m-1} + \xi_m X^m \in T^*(\omega)_x$, then*

$$\|q(x, \xi)\,\zeta\|^2 + \|q(x, \xi)\,\zeta\|^2 = 2\left(\sum_1^{m-1} |\xi_\nu|^2\right)\zeta \tag{5.8}$$

for ζ satisfying $H\zeta = \zeta$.

Proof. Let ζ satisfy $H\zeta=\zeta$, and let α_b denote the orthogonal projection of $g\otimes \wedge^j T^*(\omega)$ onto $\alpha_b g^j$. Then

$$\begin{aligned}\|q(x,\xi)\zeta\|^2+\|q(x,\xi)^*\zeta\|^2 &= \|\sum(\xi_\nu Z^\nu+\bar{\xi}_\nu \bar{Z}^\nu)\wedge\zeta\|^2+\|\alpha_b(\zeta \mathbin{\bar\wedge} \partial/\partial[\xi_\nu Z^\nu+\bar{\xi}_\nu\bar{Z}^\nu])\|^2\\ &= \|\sum(\xi_\nu Z^\nu+\bar{\xi}_\nu \bar{Z}^\nu(\wedge\zeta\|^2+\|\sum\zeta \mathbin{\bar\wedge} \partial/\partial(\xi_\nu Z^\nu+\bar{\xi}_\nu\bar{Z}^\nu)\|^2\\ &= \sum\langle Q_{\nu,\mu}\zeta,\zeta\rangle,\end{aligned}$$

where $Q_{\nu,\mu}\zeta=[(\xi_\nu Z^\nu+\bar{\xi}_\nu\bar{Z}^\nu)\wedge\zeta]\mathbin{\bar\wedge}\left(\xi_\nu\dfrac{\partial}{\partial Z^\nu}+\bar{\xi}_\nu\dfrac{\partial}{\partial\bar{Z}^\nu}\right)+(\xi_\nu Z^\nu+\bar{\xi}_\nu\bar{Z}^\nu)\wedge\left[\zeta\mathbin{\bar\wedge}\left(\xi_\nu\dfrac{\partial}{\partial Z^\nu}+\bar{\xi}_\nu\dfrac{\partial}{\partial\bar{Z}^\nu}\right)\right]$

$$=\sum 2\delta_{\nu,\mu}|\xi_\nu|^2=2\sum|\xi_\nu|^2.$$

Now let y be a coordinate defined on a coordinate disk $U\subset\omega$; let $q(y,\xi)$ and $Q(y,\xi)$ have the meaning described in section 4.

LEMMA 5.4. *If $\mathfrak{D}$ is given by* (5.1), *then the estimate* (4.1) *holds for all u with support in U if and only if for every compact $K\subset U$ there exists a constant c such that*

$$\|v\|^2\leqslant c\{\|HQ(y,\xi)v\|^2+\|HQ(y,\xi)^*v\|^2\}$$

holds for $\xi=\pm X^m$, for all $y\in K$, and for all $v\in C_0^\infty(\mathbf{R}^{n-1},(C_b^j)_y)$ satisfying $Hv=v$.

Proof. We claim that

$$\|(1-H)v\|^2+\|q(y,\xi)Hv\|^2+\|q(y,\xi)^*Hv\|^2\leqslant c\{\|q(y,\xi)v\|^2+\|q(y,\xi)^*v\|^2\} \tag{5.9}$$

holds with a constant c which is uniform in $|\xi|=1$ and for y in a compact subset of U. In fact, if (5.9) does not hold for any c, there exist sequences $\{y_\nu\}$, $\{\xi_\nu\}$, and $\{v_\nu\}$ such that

$$1=\|(1-H)v_\nu\|^2+\|q(y_\nu,\xi_\nu)Hv_\nu\|^2+\|q(y_\nu,\xi_\nu)^*Hv_\nu\|^2$$

but
$$\|q(y_\nu,\xi_\nu)v_\nu\|^2+\|q(y_\nu,\xi_\nu)v_\nu\|^2\leqslant 1/\nu.$$

Without loss of generality we may assume that $y_\nu\to y$, $\xi_\nu\to\xi$, and $v_\nu\to v$ as $\nu\to\infty$. By continuity we must have $q(y,\xi)v=q(y,\xi)v=0$. Hence $\xi=\pm X^m$ and $Hv=v$. But this contradicts

$$1=\|(1-H)v\|^2+\|q(y,\xi)Hv\|^2+\|q(y,\xi)^*Hv\|^2.$$

Therefore (5.9) holds for some constant c. If $d(y,\xi)$ denotes the distance from (y,ξ) to the characteristic variety, then by Lemma 5.3 $\|q(y,\xi)Hv\|^2+\|q(y,\xi)^*Hv\|^2=(d(y,\xi))^2\|Hv\|^2$. Hence if $|\xi|=1$ then (5.9) implies that

$$d(y,\xi)\|v\|\leqslant\sqrt{c}\{\|q(y,\xi)v\|+\|q(y,\xi)^*v\|\}.$$

The lemma now follows from Lemma 4.2.

We now compute the operators $HQ(y, \xi)$ and $HQ(y, \xi)^*$ occurring in (4.4). First note that if $Hv = v$, then

$$Hq(y, \xi)\,v = \sum_1^{m-1} \{ H(Z^k \wedge t_k(y, \xi)\,v) + H(\bar{Z}^k \wedge \bar{t}_k(y, \xi)\,v)\} = \sum_1^{m-1} \bar{Z}^k \wedge \bar{t}_k(y, \xi)\,v,$$

where $t_k(y, \xi)$ is the symbol of the vector field $\partial/\partial Z^k$. Also

$$Hq(y, \xi)^* v = \sum_1^{m-1} H\alpha_b(\bar{t}_k(y, \xi)\,v \mathbin{\bar{\wedge}} \partial/\partial Z^k + t_k(y, \xi)\,v \mathbin{\bar{\wedge}} \partial/\partial \bar{Z}^k) = \sum_1^{m-1} t_k(y, \xi)\,v \mathbin{\bar{\wedge}} \partial/\partial \bar{Z}^k.$$

If $T_k(y, \xi)$ denotes the operator $t_k(y, D) + \sum_1^{n-1} t_{k(\nu)}(y, \xi)\,x^\nu$ and if $T_k(y, \xi)$ denotes the operator obtained from $T_k(y, \xi)$ by putting a bar over occurrence of t, then

$$\begin{cases} HQ(y, \xi)\,v = \sum \bar{Z}^k \wedge \bar{T}_k(y, \xi)\,v \\ HQ(y, \xi)^* v = \sum T_k(y, \xi)\,v \mathbin{\bar{\wedge}} \partial/\partial \bar{Z}^k. \end{cases} \tag{5.10}$$

for $v \in C_0^\infty(\mathbf{R}^{n-1}, (C_b^j)_y)$ satisfying $Hv = v$.

We now set $\xi = sX^m$, where $s = \pm 1$, and evaluate the expression

$$\|HQ(y, \xi)\,v\|^2 + \|HQ(y, \xi)^* v\|^2 \tag{5.11}$$

for v satisfying $Hv = v$. The first term in (5.11) is equal to

$$\begin{aligned} &\sum_{\nu, \mu} \langle \bar{Z}^\nu \wedge \bar{T}_\nu(y, \xi)\,v,\ \bar{Z}^\mu \wedge \bar{T}_\mu(y, \xi)\,v\rangle \\ &\quad = \sum_{\nu, \mu} (\bar{Z}^\nu \wedge \bar{T}_\nu(y, \xi)\,v) \mathbin{\bar{\wedge}} \partial/\partial \bar{Z}^\mu, \bar{T}_\mu(y, \xi)\,v \\ &\quad = \sum_\nu \|\bar{T}_\nu(y, \xi)\,v\|^2 - \sum_{\nu, \mu} \langle \bar{T}_\nu(y, \xi)\,v \mathbin{\bar{\wedge}} \partial/\partial \bar{Z}^\nu,\ \bar{T}_\mu(y, \xi)\,v \mathbin{\bar{\wedge}} \partial/\partial \bar{Z}^\mu\rangle. \end{aligned}$$

The second term in (5.11) is equal to

$$\begin{aligned} &\sum_{\nu, \mu} \langle T_\nu(y, \xi)\,v \mathbin{\bar{\wedge}} \partial/\partial \bar{Z}^\nu,\ T_\mu(y, \xi)\,v \mathbin{\bar{\wedge}} \partial/\partial \bar{Z}^\mu\rangle \\ &\quad = \sum_{\nu, \mu} \langle \bar{T}_\nu(y, \xi)\,T_\mu(y, \xi)\,v \mathbin{\bar{\wedge}} \partial/\partial \bar{Z}^\mu,\ v \mathbin{\bar{\wedge}} \partial/\partial \bar{Z}^\nu\rangle \\ &\quad = \sum \langle \bar{T}_\nu(y, \xi)\,v \mathbin{\bar{\wedge}} \partial/\partial \bar{Z}^\mu,\ \bar{T}_\mu(y, \xi)\,v \mathbin{\bar{\wedge}} \partial/\partial \bar{Z}^\nu\rangle \\ &\qquad + \sum \langle [\bar{T}_\nu(y, \xi),\ T_\mu(y, \xi)]\,v \mathbin{\bar{\wedge}} \partial/\partial \bar{Z}^\mu,\ v \mathbin{\bar{\wedge}} \partial/\partial \bar{Z}^\nu\rangle, \end{aligned}$$

where $[\bar{T}_\nu(y, \xi),\ T_\mu(y, \xi)] = \bar{T}_\nu(y, \xi)\,T_\mu(y, \xi) - T_\mu(y, \xi)\,\bar{T}_\nu(y, \xi)$. We now find that (5.11) is equal to

$$\sum \|\bar{T}_\nu(y, \xi)\,v\|^2 + \sum \langle [\bar{T}_\nu(y, \xi),\ T_\mu(y, \xi)]\,v \mathbin{\bar{\wedge}} \partial/\partial \bar{Z}^\mu,\ v \mathbin{\bar{\wedge}} \partial/\partial \bar{Z}^\nu\rangle. \tag{5.12}$$

A straightforward computation shows that

$$[\bar{T}_\nu(y, \xi), T_\mu(y, \xi)] = -i \sum \{ \bar{t}_\nu^{(k)}(y, \xi) t_{\mu(k)}(y, \xi) - \bar{t}_{\nu(k)}(y, \xi) t_\mu^{(k)}(y, \xi) \},$$

which is the principal symbol of $[\partial/\partial \bar{Z}^\nu, \partial/\partial Z^\mu]$ at (y, ξ). It follows that $[\bar{T}_\nu(y, X^m), T_\mu(y, X^m)] = c_{\nu\mu}$ satisfies $c_{\nu\mu} = \bar{c}_{\mu\nu}$. Without loss of generality the Hermitian matrix $\{c_{\nu\mu}\}$ is in diagonal form so that $c_{\nu\mu} = \lambda_\nu$ if $\nu = \mu$ and $c_{\nu\mu} = 0$ otherwise. We now have $[\bar{T}_\nu(y, \xi), T_\mu(y, \xi)] = s c_{\nu\mu}$ if $\xi = sX^m$ so that (5.12) becomes

$$\sum \| \bar{T}_\nu(y, \xi) v \|^2 + \sum s\lambda_\nu \| v \bar{\wedge} \partial/\partial \bar{Z}^\nu \|^2. \tag{5.13}$$

Integrating by parts, we see that

$$\| \bar{T}_\nu(y, \xi) v \|^2 = \| T_\nu(y, \xi) v \|^2 - \langle [\bar{T}_\nu(y, \xi), T_\nu(y, \xi)] v, v \rangle = \| T_\nu(y, \xi) v \|^2 - s\lambda_\nu \| v \|^2.$$

Thus (5.13) is equal to

$$\sum \| T_\nu(y, \xi) v \|^2 - \sum s\lambda_\nu \| \bar{Z}^\nu \wedge v \|^2. \tag{5.14}$$

PROPOSITION 5.5. (See J. J. Kohn [8].) *Assume that for each $y \in \omega$ the eigenvalues of the matrix $c_{\nu\mu} = [\bar{T}_\nu(y, X^m), T_\mu(y, X^m)]$ are either all strictly positive or all strictly negative. Then the D_b-problem for the operator* (5.1) *on ω is solvable up to dimension $m-2$.*

Proof. If $1 \leqslant j \leqslant m-2$ and if $v \in C_0^\infty(\mathbf{R}^{n-1}, (C_b^j)_y)$ satisfies $Hv = v$, then the estimates

$$\| v \|^2 \leqslant c \sum_1^{m-1} \| Z^\nu \wedge v \|^2$$

$$\| v \|^2 \leqslant c \sum_1^{m-1} \| v \bar{\wedge} \partial/\partial \bar{Z}^\nu \|^2$$

hold. Using either (5.13) or (5.14) according as $s\lambda_\nu > 0$ or $s\lambda_\nu < 0$, we see that $\| v \|^2$ is majorized by a constant multiple of (5.11). The proposition now follows from Lemma 5.4 and Prop. 3.1.

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