

INEQUALITIES FOR STRONGLY SINGULAR CONVOLUTION OPERATORS

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Contents

I. Introduction	9
II. Air on the g -function	14
III. Weakly strongly singular integrals	21
IV. Results on the operators T_λ	28
References	35

I. Introduction

Suppose that f is an L^p function on the torus $T^n = S^1 \times S^1 \times \dots \times S^1$. Must the partial sums of the multiple Fourier series of f converge to f in the L^p norm? For the one-dimensional case, $T = S^1$, an affirmative answer has been known for many years. More specifically, suppose that $f \in L^p(S^1)$ has the Fourier expansion $f \sim \sum_{k=-\infty}^{\infty} a_k e^{ik\theta}$, and set $f_m(\theta) = \sum_{k=-m}^m a_k e^{ik\theta}$. Then f_m converges to f in $L^p(S^1)$, as $m \rightarrow \infty$ — provided $1 < p < +\infty$ (see [14]).

A whole slew of n -dimensional analogues of this theorem suggest themselves. Here are two natural conjectures.

(I) Let $f \in L^p(T^n)$ have the multiple Fourier expansion

$$f(\theta_1 \dots \theta_n) = \sum_{k_1 \dots k_n = -\infty}^{\infty} a_{k_1 \dots k_n} e^{i(k_1 \theta_1 + \dots + k_n \theta_n)}.$$

For each positive integer m , set

$$f_m(\theta_1 \dots \theta_n) = \sum_{|k_1| \leq m, |k_2| \leq m, \dots, |k_n| \leq m} a_{k_1 \dots k_n} e^{i(k_1 \theta_1 + \dots + k_n \theta_n)}.$$

Then $f_m \rightarrow f$ in $L^p(T^n)$, as $m \rightarrow \infty$.

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(II) Let f and its multiple Fourier series be given as above. For each positive real number R , define

$$f_R(\theta_1 \dots \theta_n) = \sum_{|k_1|^2 + |k_2|^2 + \dots + |k_n|^2 \leq R^2} a_{k_1 \dots k_n} e^{i(k_1 \theta_1 + \dots + k_n \theta_n)}$$

Then $f_R \rightarrow f$ in $L^p(T^n)$, as $R \rightarrow \infty$.

Conjectures (I) and (II) turn out to be enormously different, similar though they seem at first glance.

Elementary functional analysis reduces conjectures (I) and (II) to problems about "multiplier transformations". Every bounded real-valued function φ on R^n induces a bounded operator T_φ on $L^2(R^n)$, defined by the equation $(T_\varphi f)^\wedge(x) = \varphi(x) \hat{f}(x)$. (As always, $\hat{}$ denotes the Fourier transform on R^n). T_φ is called the multiplier transformation corresponding to φ .

Conjecture (I) is equivalent to the assertion that T_{φ_1} is a bounded operator on $L^p(R^n)$, where φ_1 denotes the characteristic function of the unit cube in R^n . Similarly, conjecture (II) is equivalent to the assertion that T_{φ_2} is a bounded operator on $L^p(R^n)$, where φ_2 denotes the characteristic function of the unit ball in R^n .

The operator T_{φ_1} can be handled, simply by using the one-dimensional result of M. Riesz; and it is well known that T_{φ_1} is a bounded operator on $L^p(R^n)$ for $1 < p < +\infty$. This proves conjecture (I).

On the other hand, the behavior of T_{φ_2} is far more subtle, and something stronger than Riesz's theory is needed to deal with it. To see what makes the problem of T_{φ_2} so thorny, let us examine it a little more closely. $T_{\varphi_2} f = \hat{\varphi}_2 * f$. Grubby computation shows that $\hat{\varphi}_2(x)$ is essentially $\sin|x|/|x|^{(n+1)/2}$ as $|x| \rightarrow +\infty$. Thus, the order of decrease of $|\hat{\varphi}_2(x)|$ at infinity is far from sufficient to put $\hat{\varphi}_2$ in the class $L^1(R^n)$. By way of contrast, $\int_{R < |y| < 2R} |\hat{\varphi}_1(y)| dy = O(R^e)$ as $R \rightarrow \infty$, so that $\hat{\varphi}_1$ is "almost" L^1 .

Let us apply T_{φ_2} to the simplest, most trivial kind of function—say, for instance, the function

$$f_0(x) = \begin{cases} 1 & \text{if } |x| < 1/10 \\ 0 & \text{if } |x| \geq 1/10. \end{cases}$$

If we merely had $\hat{\varphi}_2(x) = 1/|x|^{(n+1)/2}$, then we would find that $\hat{\varphi}_2 * f_0(x) \approx A/|x|^{(n+1)/2}$ as $|x| \rightarrow \infty$ for some constant A , so that $\hat{\varphi}_2 * f_0$ does not belong to L^p , unless $p > 2n/(n+1)$. But a moment's thought will convince the reader that the factor $\sin|x|$ in $\hat{\varphi}_2$ produces no significant cancellation in $\int_{R^n} \hat{\varphi}_2(x-y) f_0(y) dy$, so indeed, $T_{\varphi_2} f_0 \notin L^p(R^n)$ if $p \leq 2n/(n+1)$. Since f_0 belongs to all the L^p classes, it follows that T_{φ_2} cannot be a bounded operator on

$L^p(\mathbb{R}^n)$ for $1 \leq p \leq 2n/(n+1)$. Furthermore, an ‘‘adjoint’’ argument using the duality of L^p -spaces shows that if T_{φ_n} is not bounded on $L^p(\mathbb{R}^n)$, then neither is it bounded on $L^{p'}(\mathbb{R}^n)$, where p' is the exponent dual to p . Thus we have shown that T_{φ_n} cannot be a bounded operator on $L^p(\mathbb{R}^n)$ except for p in the range $2n/(n+1) < p < 2n/(n-1)$.

The natural conjecture is that T_{φ_n} is bounded on $L^p(\mathbb{R}^n)$ for $2n/(n+1) < p < 2n/(n-1)$. But how can we go about proving this conjecture? The standard methods for producing bounded operators on L^p , singular integrals and Littlewood–Paley theory, break down completely here, because they do not distinguish between different p . In other words, these techniques will only produce linear operators which are bounded on all the L^p spaces ($1 < p < +\infty$), and therefore they cannot be used to study an operator which is only bounded for some p in $(1, +\infty)$.

There is only one (previously) known method for handling operators which fail for some p —the method of interpolation. We shall illustrate this method by applying it to our conjecture on T_{φ_n} to produce a weak partial result. For $\lambda > 0$, consider the operator T_λ defined by $T_\lambda f = (\sin |x|/|x|^\lambda) * f$. Then, as we saw before, T_{φ_n} is essentially $T_{(n+1)/2}$.

Break up the operator T_λ as $T_\lambda = \sum_{k=1}^{\infty} T_{\lambda_k} + T_{\lambda_0}$ where

$$T_{\lambda_k} f(x) = \int_{2^k \leq |y| < 2^{k+1}} (\sin |y|/|y|^\lambda) f(x-y) dy \quad \text{and} \quad T_{\lambda_0} f(x) = \int_{2|y| \leq 1} (\sin |y|/|y|^\lambda) f(x-y) dy.$$

The operator T_{λ_0} is bounded on all L^p spaces, (if $\lambda < n$) so we needn't worry about it. Each operator T_{λ_k} is a convolution with an L^1 function of norm $2^{(n-\lambda)k}$. Hence

$$(A) \quad \|T_{\lambda_k} f\|_1 \leq 2^{(n-\lambda)k} \|f\|_1 \quad \text{for any } f \in L^1(\mathbb{R}^n).$$

On the other hand, an easy computation with the Plancherel formula shows that

$$(B) \quad \|T_{\lambda_k} f\|_2 \leq 2^{((n+1)/2-\lambda)k} \|f\|_2 \quad \text{for any } f \in L^2(\mathbb{R}^n).$$

Using the convexity theorem of M. Riesz, we can interpolate between the L^1 inequality (A), and the L^2 inequality (B), to obtain the inequality

$$(C) \quad \|T_{\lambda_k} f\|_p \leq 2^{(b(p)-\lambda)k} \|f\|_p, \quad \text{where } b(p) = \frac{n+1}{2} + (n-1) \left(\frac{1}{p} - \frac{1}{2} \right), \quad 1 < p < 2.$$

If $\lambda > b(p)$, then we can sum inequality (C) over all k , to obtain $\|T_\lambda f\|_p \leq A_{\lambda,p} \|f\|_p$ for any $f \in L^p(\mathbb{R}^n)$. In other words, T_λ is a bounded operator on $L^p(\mathbb{R}^n)$, if $\lambda > b(p)$.

This simple theorem is the best result previously known about the operators T_λ , and possibly represents the ultimate achievement obtainable by nothing more than some

clever decomposition $T_\lambda = \sum_{k=1}^{\infty} T_{\lambda k}$. It is far from optimal. For suppose that $T_{(n+1)/2}$ is indeed bounded on L^p for $2n/(n+1) < p < 2n/(n-1)$. Then by applying the same “interpolation” argument as before, we could deduce that T_λ is a bounded operator on L^p , whenever $\lambda > \tilde{b}(p) = n/p$. For every $\lambda < n$, this range of p is strictly larger than the range $\lambda > b(p)$. So it is plausible that for no $\lambda < n$, is the result “ $\|T_\lambda f\|_p \leq A_{\lambda p} \|f\|_p$ for $\lambda > b(p)$, $p \leq 2$ ” optimal. The optimal theorem should be “ $\|T_\lambda f\|_p \leq A_{\lambda p} \|f\|_p$ for $\lambda > n/p$, $p \leq 2$ ”. An argument like the one we gave for $T_{(n+1)/2}$ shows that T_λ cannot be bounded on $L^p(R^n)$ for $\lambda \leq n/p$.

Following a suggestion of E. M. Stein, we seek to understand the operators T_λ , by first studying some simpler operators, too singular to fall within the scope of the Calderón–Zygmund inequality of [1], but which can almost be handled by interpolation.

We begin by considering a sublinear operator g_λ^* on $L^p(R^n)$, one of the variants of the classical Littlewood–Paley g -function. $g_\lambda^*(f)$ is defined in terms of a certain “quadratic integral” involving the gradient of the Poisson integral of the function f . In [8], Stein proved by interpolation, that g_λ^* is a bounded operator on $L^p(R^n)$ for all p larger than a critical exponent, p_0 . As an application of that theorem, we mention the following result on the “smoothness” of fractional integrals:

The λ th fractional integral F , of a function $f \in L^p(R^n)$, is so well-behaved, that the function

$$D_\lambda(f)(x) \equiv \left(\int_{R^n} \frac{|F(x) - F(x-y)|^2}{|y|^{n+2\lambda}} dy \right)^{\frac{1}{2}}$$

is finite almost everywhere, and even belongs to $L^p(R^n)$ —provided $0 < \lambda < 1$ and $2n/(n+\lambda) < p$. See [9].

The purpose of section II of this paper is to show, without using interpolation, that if f belongs to the critical space $L^{p_0}(R^n)$, then $g_\lambda^*(f)$ is finite almost everywhere. This fact contains the L^p -boundedness of g_λ^* for $2 \geq p > p_0$, and might be used to show that $D_\lambda(f)$ is finite almost everywhere for $f \in L^{2n/(n+\lambda)}(R^n)$. But in fact, the technique of section II also shows how to estimate $D_\lambda(f)$, without ever mentioning g_λ^* .

In section III we study certain hypersingular integrals, of which the operator

$$T: f \rightarrow \int_{-\infty}^{\infty} \frac{e^{iy}}{y} f(x-y) dy$$

is a typical example. T is not a Calderón–Zygmund operator, since its convolution kernel oscillates far too violently near zero. An interpolation argument shows that T is a bounded operator on $L^p(R^1)$ for $1 < p < +\infty$, but is not precise enough to say anything about Tf for $f \in L^1(R^1)$. Our theorem 2 generalizes the Calderón–Zygmund inequality to cover T

and similar operators, and implies in particular that Tf is finite almost everywhere if $f \in L^1(R^1)$.

Finally, in section IV, we return to the question of the operators T_λ , and apply the techniques of sections II and III to prove a partial result more powerful than any now known from interpolation. For certain $\lambda < n$, we are actually able to prove the optimal estimate for T_λ , namely $\|T_\lambda f\|_p \leq A_p \|f\|_p$ for $n/\lambda < p \leq 2$.

II. Air on the g -function

The first operator which we study is the g_λ^* -function, a certain sublinear operation which arises in Littlewood–Paley theory (see [10]). For $f \in L^p(R^n)$, let $u(x, t)$ denote the Poisson integral of f , defined on $R_+^{n+1} = R^n \times (0, \infty)$. Then for any number $\lambda > 1$, the g_λ^* -function is a real-valued function on R^n defined by the equation

$$g_\lambda^*(f)(x) = \left(\int_{R_+^{n+1}} \left(\frac{t}{|x-y|+t} \right)^{n\lambda} t^{1-n} |\nabla u(y, t)|^2 dy dt \right)^{\frac{1}{2}}.$$

(∇u denotes the gradient of u).

A routine computation with the Plancherel formula shows that g_λ^* is (up to a constant factor) an isometry on $L^2(R^n)$. With much greater difficulty, it can be proved that for any p ($1 < p < +\infty$), $\|g_\lambda^*(f)\|_p$ and $\|f\|_p$ are equivalent norms. More precisely, suppose $1 < p < +\infty$, and $\lambda > 2/p$. Then for some constants A and A' , $A\|f\|_p \leq \|g_\lambda^*(f)\|_p \leq A'\|f\|_p$ (see [8]). In a moment, we shall see why the restriction $\lambda > 2/p$ is needed.

Littlewood and Paley introduced g_λ^* as a technical tool to prove the L^p -boundedness of various linear operators. In order to show that T is bounded on L^p , one need only prove that $\|g(Tf)\|_p \leq \|g_\lambda^*(f)\|_p$, ($g(f)$ is an auxiliary function, defined in much the same way as g_λ^*) which is often an easy task, even when the operator T is rather subtle and delicate (see [10] again).

At any rate, we have a family of operators $\{g_\lambda^*\}$. Each g_λ^* is bounded on some L^p -spaces, but not all. We seek to understand why. Two independent observations show that g_λ^* cannot be bounded on $L^p(R^n)$ if $p < 2/\lambda$.

(α) Let Q be the cylinder $\{(y, t) \in R_+^{n+1} \mid |y| < 1 \text{ and } 1 < t < 2\}$. Then

$$(g_\lambda^*(f)(x))^2 \geq \int_Q \left(\frac{t}{|x-y|+t} \right)^{n\lambda} t^{1-n} \left| \frac{\partial u}{\partial t}(y, t) \right|^2 dy dt.$$

But the right-hand side of this inequality simplifies enormously. For $t \approx 1$; and if $|x| > 10$, then $|x-y|+t \approx |x|$ when $(y, t) \in Q$. Therefore, we have

$$(g_\lambda^*(f)(x))^2 \geq \frac{1}{|x|^{n\lambda}} A \int_Q \left| \frac{\partial u}{\partial t}(y, t) \right|^2 dy dt \geq \frac{C}{|x|^{n\lambda}}$$

for all x of absolute value greater than 10. The constant

$$C = A \int_Q \left| \frac{\partial u}{\partial t}(y, t) \right|^2 dt$$

is non-zero unless $f=0$. Thus, in general, there is a constant $C > 0$ such that $g_\lambda^*(f)(x) \geq C/|x|^{n\lambda/2}$ for $|x| > 10$. If $\lambda \leq 2/p$, then $C/|x|^{n\lambda/2}$ decreases so slowly at infinity, that $g_\lambda^*(f)$ could never belong to $L^p(\mathbb{R}^n)$.

There is a deeper objection, which hints at the inner workings of g_λ^* .

(β) Let Q denote the cylinder $\{(y, t) \in \mathbb{R}_+^{n+1} \mid |y| < 1, |t| < 2\}$. Then

$$(g_\lambda^*(f)(x))^2 \geq \int_Q \left(\frac{t}{|x-y|+t} \right)^{n\lambda} t^{1-n} \left| \frac{\partial u(y, t)}{\partial t} \right|^2 dy dt \geq \frac{C}{|x|^{n\lambda}} \int_Q t^{n\lambda+1-n} \left| \frac{\partial u}{\partial t}(y, t) \right|^2 dy dt$$

if $|x| > 10$. It is not difficult to find functions $f \in L^p(\mathbb{R}^n)$, $p < 2/\lambda$, for which $\int_Q t^{n\lambda+1-n} |\partial u(y, t)/\partial t|^2 dy dt$ diverges. (In fact, we can take $f(x) = |x|^{-n\lambda/2}$ if $|x| < 1$, $f(x) = 0$ otherwise.) Thus $g_\lambda^*(f)(x) = +\infty$ for $|x| > 10$.

On the other hand, suppose $p \geq 2/\lambda$. Then

$$\int_Q t^{n\lambda+1-n} \left| \frac{\partial u}{\partial t}(y, t) \right|^2 dy dt \leq \int_{\mathbb{R}^n \times (0, 2)} t^{n\lambda+1-n} \left| \frac{\partial u}{\partial t}(y, t) \right|^2 dy dt = \int_{\mathbb{R}^n \times (0, 2)} t^{n\lambda+1-n} \left| \frac{\partial \hat{u}}{\partial t}(\xi, t) \right|^2 d\xi dt$$

by the Plancherel theorem (the Fourier transform $\hat{\cdot}$ is taken in the y variable),

$$= \int_{\mathbb{R}^n \times (0, 2)} t^{n\lambda+1-n} \left| |\xi| e^{-t|\xi|} \hat{f}(\xi) \right|^2 d\xi dt \approx \int_{\mathbb{R}^n} (|\xi| + 1)^{-n(\lambda-1)} |\hat{f}(\xi)|^2 d\xi$$

(this follows from doing the t -integration first) $= \|J^\alpha f\|_2^2$, again by the Plancherel theorem, where J^α denotes the Bessel potential of appropriate order. By the theory of fractional integrals (see [6], [11]) $\|J^\alpha f\|_2^2 \leq A \|f\|_p^2$ if $p \geq 2/\lambda$ and $p > 1$.

Thus, we have shown that if $p \geq 2/\lambda$, then

$$\int_Q t^{n\lambda+1-n} \left| \frac{\partial u}{\partial t}(y, t) \right|^2 dy dt \leq A \|f\|_p^2,$$

so that observation (β) poses no objection to the L^p -boundedness of g_λ^* if $p > 2/\lambda$.

We shall use the information and viewpoints provided by observations (α) and (β), to prove that g_λ^* is bounded on $L^p(\mathbb{R}^n)$ for $p > 2/\lambda$, without using interpolation or special

tricks. In fact, we shall prove a stronger theorem, valid for $p=2/\lambda$. That there should be a positive result for $p=2/\lambda$ seems reasonable, since objection (β) does not apply, and observation (α) suggests that although $g_\lambda^*(f)$ does not belong to $L^{2/\lambda}(R^n)$, it almost does.

THEOREM 1. *For $1 < p < 2$ and $\lambda = 2/p$, the operator g_λ^* has weak-type (p, p) . In other words, $|\{x \in R^n \mid g_\lambda^*(f)(x) > \alpha\}| \leq A/\alpha^p \|f\|_p^p$, for $f \in L^p(R^n)$.*

In Theorem 1, A is some positive “constant,” independent of f ; and $|E|$ denotes the Lebesgue measure of a set $E \subset R^n$.

The L^p -boundedness of g_λ^* for $\lambda > 2/p$, $p \leq 2$ follows from Theorem 1, by the Marcinkiewicz interpolation theorem.

One of the basic ideas of the proof of Theorem 1 is a carry-over from Calderón–Zygmund theory. The idea is basically that R^n can be divided into two parts—a set Ω of small measure, on which the function f is large; and the rest of the world, $R^n - \Omega$, on which f is small. Since Ω is a small set, we can suppose that Ω is written as a union of (essentially) disjoint cubes with small total volume. (A “cube” always means “a cube with sides parallel to the coordinate axes”, and two cubes are said to be “disjoint” if they have disjoint interiors.) The following lemma not only makes this idea precise, but also shows that the cubes can be picked to satisfy very strong conditions.

DECOMPOSITION LEMMA. *Let f be an L^p function on R^n , and let $\alpha > 0$ be given. There is a collection $\{I_j\}$ of pairwise disjoint cubes, with the following properties.*

$$\text{The } I_j\text{'s have small total volume, i.e. } \sum_j |I_j| \leq \frac{A}{\alpha^p} \|f\|_p^p. \quad (1)$$

$$|f(x)| \leq A\alpha \quad \text{for } x \notin \Omega = \bigcup_j I_j. \quad (2)$$

$$\frac{1}{|I_j|} \int_{I_j} |f(y)|^p dy \leq A\alpha^p, \text{ for any one of the cubes } \{I_j\}. \quad (3)$$

For any cube I_j of the collection, let \tilde{I}_j be a cube with the same center as I_j , but with twice as large a side. Then no point of R^n lies in more than N of the cubes \tilde{I}_j . We say that the \tilde{I}_j (4) have “bounded overlaps”.

The numbers N and A depend only on the dimension n , and not on f or p . Sketch of Proof of the Decomposition Lemma: The function $|f(x)|^p$ belongs to $L^1(R^n)$ and has norm $\|f\|_p^p$. Consider f^* , the Hardy–Littlewood maximal function of $|f(x)|^p$, given by

$$f^*(x) = \sup_{x \in I} \frac{1}{|I|} \int_I |f(y)|^p dy.$$

I some cube.

By the Hardy–Littlewood maximal theorem, the open set $\Omega = \{x \in R^n \mid f^*(x) > \alpha^p\}$ has measure at most $(A/\alpha^p) \|f\|_p^p$. (See [14], [10].)

The proof of the Whitney extension theorem (see [10]) includes a method which breaks down any open set U as a union of disjoint cubes, in such a way that the diameter of any cube is comparable to its distance from the complement of U . Applying this method to Ω , we obtain a decomposition $\Omega = \bigcup_j I_j$, where the I_j are pairwise disjoint cubes, satisfying $10 \cdot \text{diam}(I_j) \leq \text{distance}(I_j, R^n - \Omega) \leq 20 \cdot \text{diam}(I_j)$.

We shall prove that the collection $\{I_j\}$ satisfies conditions (1) through (4). Condition (1) is immediate, since $|\Omega| \leq (A/\alpha^p) \|f\|_p^p$. Condition (2) is no harder, since $x \in R^n - \Omega$ implies $f^*(x) \leq \alpha^p$, which implies that $|f(x)| \leq \alpha$ almost everywhere outside of Ω . To prove condition (3), take a cube I_j from the collection, and let I_j^* be the cube concentric with I_j , but with diameter $21n$ times as large. By construction, I_j^* contains a point $x \in R^n - \Omega$, i.e. a point x where

$$\sup_{x \in I} \frac{1}{|I|} \int_I |f(y)|^p dy \leq \alpha^p.$$

I any cube

It follows that $(1/|I_j^*|) \int_{I_j^*} |f(y)|^p dy \leq \alpha^p$, and since $|I_j^*| = (21n)^n |I_j|$, (3) follows, with $A = (21n)^n$.

Condition (4) follows from the geometry of the situation. For, if $x \in I_j$, then it follows that $9 \cdot \text{diam}(I_j) \leq \text{distance}(x, R^n - \Omega) \leq 21 \cdot \text{diam}(I_j)$. Therefore, the cube I_j has diameter at least $1/21$ distance $(x, R^n - \Omega) = d/21$ and is contained in a ball centered about x , of radius $\frac{1}{2}$ distance $(x, R^n - \Omega) = d/2$. Since at most N pairwise disjoint cubes of diameter $> d/21$ can be packed into a ball of radius $d/2$, condition (4) holds. Q.e.d.

Proof of Theorem 1. Let $f \in L^p(R^n)$ and $\alpha > 0$ be given. We have to show that

$$|\{x \in R^n \mid g_\lambda^*(f)(x) > A\alpha\}| \leq \frac{A}{\alpha^p} \|f\|_p^p$$

with A independent of f and α —for this is equivalent to the conclusion of Theorem 1.

Apply the decomposition lemma to f and α , to obtain a collection $\{I_j\}$ of cubes, satisfying conditions (1) through (4) above. Set $\Omega = \bigcup_j I_j$. We shall use the cubes I_j to decompose the function f into two parts, as follows. Define a function f' on R^n by saying that

$$f'(x) = \begin{cases} \frac{1}{|I_j|} \int_{I_j} f(y) dy & \text{if } x \in I_j, \\ f(x) & \text{if } x \notin \Omega. \end{cases}$$

Setting $f'' = f - f'$, we obtain a decomposition $f = f' + f''$ with the following properties.

$$|f'(x)| \leq A\alpha \text{ almost everywhere, and } \|f'\|_p \leq \|f\|_p. \tag{5}$$

$$f'' \text{ is supported on } \Omega. \tag{6}$$

$$\frac{1}{|I_j|} \int_{I_j} |f''(y)|^p dy \leq A\alpha^p \text{ for each cube } I_j \text{ from the collection.} \tag{7}$$

$$\int_{I_j} f''(y) dy = 0 \text{ for each cube } I_j \text{ from the collection.} \tag{8}$$

Property (5) clearly implies that $\|f'\|_2^2 \leq A\alpha^{2-p} \|f\|_p^p$. As we remarked earlier, g_λ^* is a bounded operator on $L^2(\mathbb{R}^n)$, so that by the Chebyshev inequality,

$$|\{x \in \mathbb{R}^n \mid g_\lambda^*(f')(x) > \alpha\}| \leq \frac{A}{\alpha^2} \|f'\|_2^2 \leq \frac{A}{\alpha^2} (A\alpha^{2-p} \|f\|_p^p) = \frac{A}{\alpha^p} \|f\|_p^p.$$

On the other hand, $g_\lambda^*(f) \leq g_\lambda^*(f') + g_\lambda^*(f'')$ which implies that $|\{x \in \mathbb{R}^n \mid g_\lambda^*(f)(x) > (A+1)\alpha\}| \leq |\{x \in \mathbb{R}^n \mid g_\lambda^*(f')(x) > \alpha\}| + |\{x \in \mathbb{R}^n \mid g_\lambda^*(f'')(x) > A\alpha\}| \leq (A/\alpha^p) \|f\|_p^p + |\{x \in \mathbb{R}^n \mid g_\lambda^*(f'')(x) > A\alpha\}|$, by what we have just proved. So in order to prove Theorem 1, it will be enough to prove that

$$|\{x \in \mathbb{R}^n \mid g_\lambda^*(f'')(x) > A\alpha\}| \leq \frac{A}{\alpha^p} \|f\|_p^p. \tag{9}$$

This inequality is much easier to get a hold on than Theorem 1 itself, because f'' lives on a small set, and has various other good properties.

In order to obscure things further, we introduce some notation. If $x \in \mathbb{R}^n$ and I_j is a cube from our collection, then $x \sim I_j$ means that x belongs to a cube I_l (also from the collection), which touches or coincides with I_j . Roughly speaking, $x \sim I_j$ means that x is not much further away from I_j than $\text{diam}(I_j)$. Note that for fixed x , $x \sim I_j$ holds for at most N Whitney cubes; and that if $x \notin \Omega$ then $x \sim I_j$ never holds. Finally, let $f_j = f'' \cdot \chi_{I_j}$ where χ_E always denotes the characteristic function of the set E , and let $h_j(x, t)$ denote the gradient of the Poisson integral of f_j .

Now we can give the basic decomposition of g_λ^* . By definition,

$$g_\lambda^*(f'')(x) = \left(\int_{\mathbb{R}^{n+1}_+} \left(\frac{t}{|x-y|+t} \right)^{n\lambda} t^{1-n} \left| \sum_j h_j(y, t) \right|^2 dy dt \right)^{\frac{1}{2}}.$$

Therefore $g_\lambda^*(f'')(x) \leq g_\lambda^1(f'')(x) + g_\lambda^2(f'')(x)$,

where $g_\lambda^1(f'')(x) = \left(\int_{\mathbb{R}_+^{n+1}} \left(\frac{t}{|x-y|+t} \right)^{n\lambda} t^{1-n} \left| \sum_{y \nearrow I_j} h_j(y, t) \right|^2 dy dt \right)^{\frac{1}{2}}$

and $g_\lambda^2(f'')(x) = \left(\int_{\mathbb{R}_+^{n+1}} \left(\frac{t}{|x-y|+t} \right)^{n\lambda} t^{1-n} \sum_{y \sim I_j} |h_j(y, t)|^2 dy dt \right)^{\frac{1}{2}}$.

So to prove inequality (9), and thus to prove Theorem 1, it will be enough to prove

$$|\{x \in \mathbb{R}^n \mid g_\lambda^1(f'')(x) > A\alpha\}| \leq \frac{A}{\alpha^p} \|f\|_p^p \quad (10)$$

and $|\{x \in \mathbb{R}^n \mid g_\lambda^2(f'')(x) > A\alpha\}| \leq \frac{A}{\alpha^p} \|f\|_p^p. \quad (11)$

Of these two inequalities, (10) is relatively easy, while (11) is deeper, and uses the relation $p = 2/\lambda$. In order not to spoil the plot, we prove (10) first.

To do so, we need a trivial inequality for $\sum_{y \nearrow I_j} h_j(y, t)$. Specifically, $|\sum_{y \nearrow I_j} h_j(y, t)| \leq A\alpha/t$. For if \mathbf{R} denotes the convolution kernel for the gradient of the Poisson integral, then

$$\begin{aligned} \left| \sum_{y \nearrow I_j} h_j(y, t) \right| &= \left| \sum_{y \nearrow I_j} \int_{I_j} \mathbf{R}(y-z, t) f''(z) dz \right| \leq \sum_{y \nearrow I_j} \int_{I_j} |\mathbf{R}(y-z, t)| |f''(z)| dz \\ &\leq \sum_{y \nearrow I_j} \sup_{z \in I_j} |\mathbf{R}(y-z, t)| \int_{I_j} |f''(z)| dz \leq \sum_{y \nearrow I_j} \sup_{z \in I_j} |\mathbf{R}(y-z, t)| A\alpha |I_j|, \end{aligned}$$

by inequality (7). On the other hand, anyone can verify that $\sup_{z \in I_j} |\mathbf{R}(y-z, t)| |I_j| \leq A \int_{I_j} |\mathbf{R}(y-z, t)| dz$ for any cube I_j satisfying $y \nearrow I_j$, and the "constant" A is independent of t . Therefore,

$$\left| \sum_{y \nearrow I_j} h_j(y, t) \right| \leq A \sum_{y \nearrow I_j} \alpha \int_{I_j} |\mathbf{R}(y-z, t)| dz \leq A\alpha \int_{\mathbb{R}^n} |\mathbf{R}(y-z, t)| dz \leq \frac{A\alpha}{t}$$

(since the cubes I_j are pairwise disjoint). Thus $|\sum_{y \nearrow I_j} h_j(y, t)| \leq A\alpha/t$.

Putting our estimate into the definition of g_λ^1 , we obtain

$$|g_\lambda^1(f'')(x)|^2 \leq A\alpha \int_{\mathbb{R}_+^{n+1}} \left(\frac{t}{|x-y|+t} \right)^{n\lambda} t^{1-n} \sum_{y \nearrow I_j} |h_j(y, t)| dy dt \equiv A\alpha \mathcal{J}(x).$$

So to prove (10), we need only show that $|\{x \in \mathbb{R}^n \mid \mathcal{J}(x) > \alpha\}| \leq (A/\alpha^p) \|f\|_p^p$ which in turn follows from the Chebyshev inequality and the estimate (as yet unproved)

$$\int_{\mathbb{R}^n} \mathcal{J}(x) dx \leq A \alpha^{1-p} \|f\|_p^p. \quad (12)$$

So (10) holds, provided (12) holds.

To prove (12), we compute $\int_{\mathbb{R}^n} \mathcal{J}(x) dx$ explicitly, using the definition of \mathcal{J} . In fact

$$\begin{aligned} \int_{\mathbb{R}^n} \mathcal{J}(x) dx &= \int_{\mathbb{R}^n} \int_{\mathbb{R}_+^{n+1}} \left(\frac{t}{|x-y|+t} \right)^{n\lambda} t^{-n} \left| \sum_{y \sim I_j} h_j(y, t) \right| dy dt dx \\ &= \int_{\mathbb{R}_+^{n+1}} \left[t^{-n} \int_{\mathbb{R}^n} \left(\frac{t}{|x-y|+t} \right)^{n\lambda} dx \right] \left| \sum_{y \sim I_j} h_j(y, t) \right| dy dt = A \int_{\mathbb{R}_+^{n+1}} \left| \sum_{y \sim I_j} h_j(y, t) \right| dy dt \\ &\leq A \int_{\mathbb{R}_+^{n+1}} \sum_{y \sim I_j} |h_j(y, t)| dy dt = A \sum_j \int_{\substack{(y, t) \in \mathbb{R}_+^{n+1} \\ y \sim I_j}} |h_j(y, t)| dy dt. \end{aligned} \quad (13)$$

Consider $\int_{\substack{(y, t) \in \mathbb{R}_+^{n+1} \\ y \sim I_j}} |h_j(y, t)| dy dt$, the j th summand in the right-hand side of (13). Written

out in full, the summand is $\int_{\substack{(y, t) \in \mathbb{R}_+^{n+1} \\ y \sim I_j}} \left| \int_{I_j} \mathbf{R}(y-z, t) f_j(z) dz \right| dy dt$. Since $\int_{I_j} f_j(z) dz = 0$ (see (8)).

$$\int_{\substack{(y, t) \in \mathbb{R}_+^{n+1} \\ y \sim I_j}} |h_j(y, t)| dy dt = \int_{\substack{(y, t) \in \mathbb{R}_+^{n+1} \\ y \sim I_j}} \left| \int_{I_j} (\mathbf{R}(y-z, t) - \mathbf{R}(y-z_j, t)) f_j(z) dz \right| dy dt$$

(where z_j denotes the center of the cube I_j)

$$\begin{aligned} &\leq \int_{\substack{(y, t) \in \mathbb{R}_+^{n+1} \\ y \sim I_j}} \int_{I_j} |\mathbf{R}(y-z, t) - \mathbf{R}(y-z_j, t)| |f_j(z)| dz dy dt \\ &= \int_{I_j} \left[\int_{\substack{(y, t) \in \mathbb{R}_+^{n+1} \\ y \sim I_j}} |\mathbf{R}(y-z, t) - \mathbf{R}(y-z_j, t)| dy dt \right] |f_j(z)| dz \leq A \int_{I_j} |f_j(z)| dz, \end{aligned}$$

for the term in brackets is bounded by a constant A which depends only on the dimension n . Thus, the j th summand in (13) is at most $A \int_{I_j} |f_j(z)| dz$, so that $\int_{\mathbb{R}^n} \mathcal{J}(x) dx \leq \sum_j A \int_{I_j} |f_j(z)| dz \leq A \sum_j \alpha |I_j|$ (by (7)), $= A \alpha |\Omega| \leq A \alpha^{1-p} \|f\|_p^p$, by (1). This completes the proof of (12). Since we have reduced inequality (10) to inequality (12), we have also proved (10).

Where do we stand? We began by reducing Theorem 1 to the proof of two inequalities, (10) and (11). By a laborious but conceptually simple argument we proved inequality (10), without resorting to the critical equation $p=2/\lambda$. To complete the proof of Theorem 1, it remains to prove inequality (11). Any proof of (11) will have to use $p=2/\lambda$. The argument below is neat, in that it not only proves (11), but also shows that the two objections (α) and (β) mentioned above are exactly the reasons why L^p -boundedness of g^* fails for $p < 2/\lambda$.

Recall that inequality (11) states that $|\{x \in R^n \mid g_\lambda^2(f'')(x) > A\alpha\}| \leq (A/\alpha^p) \|f\|_p^2$. Since $|\Omega| \leq A/\alpha^p \|f\|_p^2$, it will be enough to prove that

$$|\{x \in R^n - \Omega \mid g_\lambda^2(f'')(x) > A\alpha\}| \leq \frac{A}{\alpha^p} \|f\|_p^2. \quad (14)$$

Now if $x \in R^n - \Omega$, then $C_1|x - y_j| \leq |x - y| \leq C_2|x - y_j|$ where y_j denotes the center of I_j , and y is any point in I_j . Therefore, for $x \in R^n - \Omega$,

$$(g_\lambda^2(f'')(x))^2 = \int_{\Omega \times (0, \infty)} \left(\frac{t}{|x - y| + t} \right)^{n\lambda} t^{1-n} \left| \sum_{y \sim I_i} h_i(y, t) \right|^2 dy dt \quad (15)$$

(since $\sum_{y \sim I_i} h_i(y, t)$ is an empty sum if $y \notin \Omega$)

$$\begin{aligned} &= \sum_j \int_{I_j} \int_0^\infty \left(\frac{t}{|x - y| + t} \right)^{n\lambda} t^{1-n} \left| \sum_{y \sim I_i} h_i(y, t) \right|^2 dy dt \\ &\leq A \sum_j \frac{t}{|x - y_j|^{n\lambda}} \int_{I_j} \int_0^\infty t^{n\lambda+1-n} \left| \sum_{y \sim I_i} h_i(y, t) \right|^2 dy dt. \end{aligned}$$

We shall study the double integral in the far right-hand side of inequality (15). Since the relation $y \sim I_i$ depends only on which cube y is located in, we can rest assured that $|\sum_{y \sim I_i} h_i(y, t)|^2 = |\sum_{y_j \sim I_i} h_i(y, t)|^2$ for $y \in I_j$. On the other hand, $\sum_{y_j \sim I_i} h_i(y, t)$ is just the gradient of the Poisson integral of the function $f^j = \sum_{y_j \sim I_i} f_i$. So

$$\begin{aligned} \int_{I_j} \int_0^\infty t^{n\lambda+1-n} \left| \sum_{y \sim I_i} h_i(y, t) \right|^2 dy dt &= \int_{I_j} \int_0^\infty t^{n\lambda+1-n} |\mathbf{R} * f^j(y, t)|^2 dy dt \\ &\leq \int_{R_+^{n+1}} t^{n\lambda+1-n} |\mathbf{R} * f^j(y, t)|^2 dy dt \leq A \|f^j\|_p^2 \end{aligned}$$

if $\lambda = 2/p$ —we verified the last inequality of the chain, during the discussion of observation (β), above. But $\|f^j\|_p \leq \sum_{y_j \sim I_i} \|f_i\|_p \leq A\alpha \sum_{y_j \sim I_i} |I_i|^{1/p}$ (by (3)) $\leq A\alpha |I_j|^{1/p}$, because of the geometry of the cubes. Therefore,

$$\int_{I_j} \int_0^\infty t^{n\lambda+1-n} \left| \sum_{y \sim I_i} h_i(y, t) \right|^2 dy dt \leq A\alpha^2 |I_j|^{2/p}.$$

Putting this inequality into (15), we obtain

$$(g_\lambda^2(f'')(x))^2 \leq A\alpha^2 \sum_j \frac{1}{|x-y_j|^{n\lambda}} |I_j|^{2/p} \equiv A\alpha^2 \mathcal{F}(x). \quad (16)$$

So in order to complete the proof of inequality (14), and with it, that of Theorem 1, we have only to prove that

$$|\{x \in R^n - \Omega \mid \mathcal{F}(x) > 1\}| \leq \frac{A}{\alpha^p} \|f\|_p^p. \quad (17)$$

This last inequality is a standard lemma on the ‘‘Marcinkiewicz integral’’, and is proved by the following simple argument.

$$\begin{aligned} \int_{R^n - \Omega} \mathcal{F}(x) dx &= \sum_j |I_j|^{2/p} \int_{R^n - \Omega} \frac{dx}{|x-y_j|^{n\lambda}} \leq \sum_j |I_j|^{2/p} \int_{R^n - I_j} \frac{dx}{|x-y_j|^{n\lambda}} \\ &= A \sum_j |I_j|^{2/p+1-\lambda} = A \sum_j |I_j| \left(\text{if } p = \frac{2}{\lambda} \right) \leq \frac{A}{\alpha^p} \|f\|_p^p \end{aligned}$$

by inequality (1). (17) now follows from the Chebyshev inequality. The proof of Theorem 1 is complete. Q.e.d.

Note again, that the two applications we made of $p=2/\lambda$, reflect observations (α) and (β).

The method of proof of Theorem 1 also establishes a weaker result on the behavior of fractional integrals. In fact, suppose that $f \in L^p(R^n)$ ($1 < p < 2$), and that $0 < \lambda < 1$. A theorem of Stein [9] asserts that the fractional integral $I^\lambda(f)$ satisfies the ‘‘smoothness’’ condition that

$$D_\lambda(f)(x) = \left(\int_{R^n} \frac{|I^\lambda(f)(x) - I^\lambda(f)(x-y)|^2}{|y|^{n+2\lambda}} dy \right)^{\frac{1}{2}}$$

belongs to $L^p(R^n)$, provided that $2n/(n+2\lambda) < p$. This result is a consequence of the L^p inequalities for the g_β^* -function, which Stein proved in [8]. The connection between g_β^* and μ_λ is that if $2/\beta > 2n/(n+2\lambda)$, there is a pointwise inequality $D_\lambda(f)(x) \leq Cg_\beta^*(f)(x)$ for $x \in R^n$.

THEOREM 1': For $2n/(n+2\lambda) = p$, $0 < \lambda < 1$, $1 < p < 2$, the operator D_λ has weak-type (p, p) .

III. Weakly strongly singular integrals

We turn now to the study of *linear* operators which are bounded on some, but not all, of the L^p spaces. Our first example of such an operator is the ‘‘multiplier’’ transformation $T_{\alpha\beta}$, defined by the equation

$$(T_{\alpha\beta}f)^\wedge(x) = \frac{e^{i|x|^\alpha}}{|x|^\beta} \theta(x) \hat{f}(x), \quad \text{if } f \in C_0^\infty(R^n). \quad (18)$$

Here $0 < a < 1$, $\beta > 0$; and θ is a C^∞ function on R^n , which vanishes near zero, and equals 1 outside a bounded set. For a discussion of $T_{a\beta}$, see Hirschmann [4], Wainger [12], and Stein [7]. These papers demonstrate that the operator $T_{a\beta}$ is bounded on $L^p(R^n)$ when

$$\left| \frac{1}{2} - \frac{1}{p} \right| < \frac{\beta}{n} \left[\frac{n/2 + \lambda}{\beta + \lambda} \right], \quad \text{where } \lambda \equiv \frac{na/2 - \beta}{1 - a}.$$

The proof of this result is an ‘‘interpolation’’ argument not much different from the one sketched in Section I above—the interpolation is possible because $\left(\frac{e^{i|x|^a}}{|x|^\beta} \theta(x) \right)^\wedge$, the convolution kernel for $T_{a\beta}$, can be computed roughly. It turns out that essentially,

$$\left(\frac{e^{i|x|^a}}{|x|^\beta} \theta(x) \right)^\wedge (y) = \frac{e^{i|y|^{a'}}}{|\lambda|^{n+\lambda}},$$

where $a' = a/(a-1)$ and λ is as above. Wainger shows that $T_{a\beta}$ is unbounded on L^p if $\left| \frac{1}{2} - \frac{1}{p} \right| > \frac{\beta}{n} \left[\frac{n/2 + \lambda}{\beta + \lambda} \right]$. In [5], [7], and elsewhere, the question has been raised, whether $T_{a\beta}$ is bounded on the critical L^p space, $L^{p_0}(R^n)$. But nothing at all was known about the behavior of $T_{a\beta}$ on L^{p_0} .

THEOREM 2. *If $0 < a < 1$, $\beta > 0$, and $\frac{1}{p} - \frac{1}{2} = \frac{\beta}{n} \left[\frac{n/2 + \lambda}{\beta + \lambda} \right]$, then $T_{a\beta}$ extends to a bounded linear operator from $L^p(R^n)$ into the Lorentz space $L_{p,p'}(R^n)$, where p' is the exponent dual to p .*

For a discussion of Lorentz spaces, see [6].

Theorem 2 is stronger than a weak-type inequality, but not as strong as an inequality $\|T_{a\beta}f\|_p \leq A\|f\|_p$.

To prove Theorem 2, we interpolate between the two special cases $p=1$ and $p=2$. The simple-minded interpolation technique sketched in the introduction is inadequate, but we can use more sophisticated results related to the Riesz–Thorin convexity theorem. The exact results can be found in [2]. Here, we content ourselves with stating that Theorem 2 is essentially a consequence of the two special cases $p=1$ and $p=2$.

Of course, Theorem 2 is a triviality for $p=2$. We are thus left with the task of proving that for $\beta=na/2$, the operator $T_{a\beta}$ has weak type $(1, 1)$. More precisely, we have to prove that for $\beta=na/2$, the operator $T_{a\beta}$, defined on C^∞ functions of compact support, extends to an operator of weak type $(1, 1)$. This statement is a special case of the following generalization of the Calderón–Zygmund inequality.

THEOREM 2': Let K be a temperate distribution on R^n , with compact support; and let $0 < \theta < 1$ be given. Suppose that K is equal to a locally integrable function away from zero, that the Fourier transform \hat{K} is a function, and that

$$(i) \quad |\hat{K}(x)| \leq A(1 + |x|)^{-(n\theta/2)} \quad \text{for } x \in R^n$$

$$(ii) \quad \int_{|x| > 2|y|^{1-\theta}} |K(x) - K(x-y)| dx \leq A \quad \text{for all } y \in R^n (|y| \leq 1).$$

Then the convolution operator $T: f \rightarrow K * f$, defined for $f \in C_0^\infty(R^n)$, satisfies the a priori inequality $|\{x \in R^n \mid |Tf(x)| > \alpha\}| \leq (A'/\alpha) \|f\|_1$ for any $f \in C_0^\infty(R^n)$. Moreover, the "constant" A' depends only on A , n , θ , and the diameter of the support of K .

Obviously, then, T extends to an operator which has weak-type $(1, 1)$ and is therefore bounded on $L^p(R^n)$, $1 < p < \infty$. A typical concrete application of Theorem 2' is that the convolution operator $f \rightarrow f * (e^{i/x})$, defined for $f \in C_0^\infty(R^1)$, has weak-type $(1, 1)$.

Proof of Theorem 2'.

We shall prove the theorem for $K \in L^1(R^n)$, to avoid trivial technical problems. Since the constant A' in the conclusion of the theorem is independent of $\|K\|_1$, a routine limiting argument will allow us to conclude that Theorem 2' is valid for a general K .

(Indeed, we need only prove Theorem 2' with K replaced by $K * \varphi_\varepsilon$, where $\varphi_\varepsilon(x) = \varepsilon^{-n} \varphi(x/\varepsilon)$ and φ is a C_0^∞ function with integral 1. $(K * \varphi_\varepsilon)^\wedge(x) = \hat{K}(x) \hat{\varphi}_\varepsilon(x) = O(|x|^{-N})$ as $|x| \rightarrow \infty$, for any N ; so $K * \varphi_\varepsilon$ belongs to $C^\infty(R^n)$. We have only to check that conditions (i) and (ii) hold uniformly in ε . Condition (i) is obvious. To prove condition (ii), we consider two cases.

(a) Suppose $|y| > 10\varepsilon$. Then

$$\begin{aligned} \int_{|x| > 2|y|^{1-\theta}} |K * \varphi_\varepsilon(x) - K * \varphi_\varepsilon(x-y)| dx &\leq \int_{|x| > 2|y|^{1-\theta}} \int_{|z| < \varepsilon} |\varphi_\varepsilon(z)| |K(x-z) \\ &\quad - K(x-y-z)| dz dx \leq \int_{|z| < \varepsilon} |\varphi_\varepsilon(z)| \int_{|x| > 2|y|^{1-\theta}} |K(x) - K(x-z)| dx dz \\ &\quad + \int_{|z| < \varepsilon} |\varphi_\varepsilon(z)| \int_{|x| > 2|y|^{1-\theta}} |K(x) - K(x-y-z)| dx dz \leq A, \end{aligned}$$

by condition (ii) on K .

(b) Suppose $|y| \leq 10\varepsilon$. Then

$$\int_{|x| > \varepsilon^{1-\theta}} |K * \varphi_\varepsilon(x) - K * \varphi_\varepsilon(x-y)| dx \leq A$$

by the argument of (a), so we need only show that

$$\int_{|x| < \varepsilon^{1-\theta}} |K * \varphi_\varepsilon(x) - K * \varphi_\varepsilon(x-y)| dx \leq A'.$$

Let $F(x) \equiv K * \varphi_\varepsilon(x) - K * \varphi_\varepsilon(x-y)$. Easy computation shows that $\hat{F}(x) = \hat{\varphi}_\varepsilon(x) \hat{K}(x) (1 - e^{2\pi i x y})$, so that

$$\|F\|_2 \leq \|\hat{\varphi}_\varepsilon\|_2 \sup_{x \in \mathbb{R}^n} |\hat{K}(x) (1 - e^{2\pi i x y})| \leq A \|\varphi_\varepsilon\|_2 |y|^{n\theta/2} \text{ (by (i))} = A' \varepsilon^{-(n/2)} |y|^{n\theta/2}.$$

By Hölder's inequality,

$$\int_{|x| < \varepsilon^{1-\theta}} |F(x)| dx \leq \|F\|_2 (\varepsilon^{1-\theta})^{n/2} \leq A' \varepsilon^{-(n/2)} |y|^{n\theta/2} \varepsilon^{n/2 - n\theta/2} \leq A'$$

since $|y| \leq 10\varepsilon$. This completes the proof of condition (ii) for $K * \varphi_\varepsilon$. We can assume that $\text{diam}(\text{supp } K) < 1$.

Very well, let $f \in L^1(\mathbb{R}^n)$ and $\alpha > 0$ be given. We want to show that $|\{x \in \mathbb{R}^n \mid |K * f(x)| > A'\alpha\}| \leq (A'/\alpha) \|f\|_1$.

Apply the decomposition lemma with $p=1$, to f and α , to obtain cubes $\{I_j\}$ ($\bigcup_j I_j \equiv \Omega$), satisfying (1) through (4) above. Using the cubes, we can split f into two parts, $f = f' + f''$, simply by setting $f' = f \chi_{\mathbb{R}^n - \Omega}$ and $f'' = f \chi_\Omega \cdot f'' = \sum f_j$, where $f_j = f \chi_{I_j}$. Exactly as in the proof of Theorem 1, $f' \in L^2(\mathbb{R}^n)$, etc., so that $|\{x \in \mathbb{R}^n \mid |K * f'(x)| > \alpha\}| \leq (A'/\alpha) \|f'\|_1$. So to prove Theorem 2', we need only show that $|\{x \in \mathbb{R}^n \mid |K * f''(x)| > A'\alpha\}| \leq (A'/\alpha) \|f''\|_1$. Since $|\Omega| \leq (A'/\alpha) \|f''\|_1$, it is also enough to show that

$$|\{x \in \mathbb{R}^n - \Omega \mid |K * f''(x)| > A'\alpha\}| \leq \frac{A'}{\alpha} \|f''\|_1. \quad (19)$$

We shall return to (19) after a brief digression.

Let φ be a C^∞ function on \mathbb{R}^n , equal to zero outside the unit ball, and satisfying the conditions $\int_{\mathbb{R}^n} \varphi(y) dy = 1$, and $\varphi(y) \geq 0$ for all $y \in \mathbb{R}^n$. For any $\varepsilon > 0$, set $\varphi(y; \varepsilon) = \varepsilon^{-n} \varphi(y/\varepsilon)$; and set $\varphi_j(y) = \varphi(y; (\text{diam } I_j)^{1/\theta})$. Thus $\varphi_j(y)$ is a C^∞ function with integral 1 and "thickness" $(\text{diam } I_j)^{1/(1-\theta)}$.

Now define $\tilde{f}_j = f_j * \varphi_j$ and $\tilde{f} = \sum_{\text{diam } I_j < 1} \tilde{f}_j$. We are going to show that for $x \in \mathbb{R}^n - \Omega$, $K * f''(x)$ is approximately equal to $K * \tilde{f}$.

First of all, note that if $x \in \mathbb{R}^n - \Omega$, then $K * f_j(x) = 0$ when $\text{diam } I_j \geq 1$, for then $K * f_j$ will live inside a cube concentric with I_j , and with side twice that of I_j . So, for $x \in \mathbb{R}^n - \Omega$, we have $K * f''(x) = \sum_{j \in J} K * f_j(x)$, where for convenience we have set $J = \{j \mid \text{diam } I_j < 1\}$.

Now we can write

$$K * f''(x) - K * \tilde{f}(x) = \sum_{j \in J} (K * f_j(x) - K * \tilde{f}_j(x)) = \sum_{j \in J} (K * f_j(x) - K * \varphi_j * f_j(x)) \quad (20)$$

for $x \in \mathbb{R}^n - \Omega$. But for $j \in J$,

$$\begin{aligned} \int_{\mathbb{R}^n - \Omega} |K * f_j(x) - K * \varphi_j * f_j(x)| dx &\leq \int_{\mathbb{R}^n - \Omega} \int_{I_j} |K(x-y) - K * \varphi_j(x-y)| |f_j(y)| dy dx \\ &= \int_{I_j} \left[\int_{\mathbb{R}^n - \Omega} |K(x-y) - K * \varphi_j(x-y)| dx \right] |f_j(y)| dy \\ &\leq \left[\int_{|z| > \text{diam}(I_j)} |K(z) - K * \varphi_j(z)| dz \right] \int_{I_j} |f_j(y)| dy \end{aligned}$$

(since we can make the change of variable $z = x - y$, and then note that $|z| > \text{diam}(I_j)$ if $x \in \mathbb{R}^n - \Omega$ and $y \in I_j$) $\leq A' \int_{I_j} |f_j(y)| dy$, since

$$\begin{aligned} \int_{|z| > \text{diam}(I_j)} |K(z) - K * \varphi_j(z)| dz &= \int_{|z| > \text{diam}(I_j)} \left| K(z) - \int_{|y| < \text{diam}(I_j)^{1/(1-\theta)}} \varphi_j(y) K(z-y) dy \right| dz \\ &= \int_{|z| > \text{diam}(I_j)} \left| \int_{|y| < \text{diam}(I_j)^{1/(1-\theta)}} \varphi_j(y) [K(z) - K(z-y)] dy \right| dz \\ &\leq \int_{|y| < \text{diam}(I_j)^{1/(1-\theta)}} \varphi_j(y) \left| \int_{|z| > \text{diam}(I_j)} |K(z) - K(z-y)| dz \right| dy \\ &\leq A' \int_{|y| < \text{diam}(I_j)^{1/(1-\theta)}} \varphi_j(y) dy = A'. \end{aligned}$$

Summarizing the last sentence, we have $\int_{\mathbb{R}^n - \Omega} |K * f_j(x) - K * \varphi_j * f_j(x)| dx \leq A' \cdot \int_{I_j} |f_j(y)| dy$ for $j \in J$. If we sum this inequality over all $j \in J$, and look at equation (20), we see at once that

$$\int_{\mathbb{R}^n - \Omega} |K * f''(x) - K * \tilde{f}(x)| dx \leq A \sum_{j \in J} \int_{I_j} |f_j(y)| dy \leq A' \|f\|_1.$$

So $|\{x \in \mathbb{R}^n - \Omega \mid |K * f''(x) - K * \tilde{f}(x)| > \alpha\}| \leq (A'/\alpha) \|f\|_1$, by the Chebyshev inequality. Therefore, to prove (19), and with it Theorem 2', we have only to prove that

$$|\{x \in \mathbb{R}^n - \Omega \mid |K * \tilde{f}(x)| > A' \alpha\}| \leq (A'/\alpha) \|f\|_1. \quad (21)$$

The idea behind the proof of (21) is perfectly simple. We are going to show that $\|J^{n\theta/2} * \tilde{f}\|_2^2 \leq A' \alpha \|f\|_1$, where $J^{n\theta/2}$ denotes the Bessel potential of order $n\theta/2$. If this in-

equality can be proved, then by (i), $\|K * \tilde{f}\|_2^2 = \|(K * J^{-n\theta/2}) * (J^{n\theta/2} * \tilde{f})\|_2^2 \leq A^2 \|J^{n\theta/2} * \tilde{f}\|_2^2 \leq A' \alpha \|f\|_1$, and (21) follows, by the Chebyshev inequality.

So Theorem 2' reduces to the statement

$$\|J^{n\theta/2} * \tilde{f}\|_2^2 \leq A' \alpha \|f\|_1. \quad (22)$$

To prove (22), it will be convenient to use the notation " $x \sim I_j$ ", which means the same thing as it did in the proof of Theorem 1.

Now

$$\begin{aligned} J^{n\theta/2} * \tilde{f}(x) &= \sum_{j \in J} J^{n\theta/2} * \tilde{f}_j(x) = \sum_{j \in J} J^{n\theta/2} * \varphi_j * f_j(x) \\ &= \sum_{x \sim I_j, j \in J} J^{n\theta/2} * \varphi_j * f_j(x) + \sum_{x \not\sim I_j, j \in J} J^{n\theta/2} * \varphi_j * f_j(x) \equiv F_1(x) + F_2(x). \end{aligned}$$

First we shall show that $\|F_2\|_2^2 \leq A' \alpha \|f\|_1$. Obviously,

$$\begin{aligned} \|F_2\|_1 &= \int_{\mathbb{R}^n} \left| \sum_{x \not\sim I_j, j \in J} J^{n\theta/2} * \varphi_j * f_j(x) \right| dx \leq \int_{\mathbb{R}^n} \sum_{x \not\sim I_j, j \in J} |J^{n\theta/2} * \varphi_j * f_j(x)| dx \\ &\leq \int_{\mathbb{R}^n} \sum_{j \in J} |J^{n\theta/2} * \varphi_j * f_j(x)| dx = \sum_{j \in J} \|J^{n\theta/2} * \varphi_j * f_j\|_1 \leq A' \sum_{j \in J} \|f_j\|_1 \leq A' \|f\|_1 \end{aligned}$$

(since $\|J^{n\theta/2} * \varphi_j\|_1 \leq A'$ for any j). On the other hand, $\|F_2\|_\infty \leq A' \alpha$. To see this, note that if $x \not\sim I_j$, then

$$\begin{aligned} |(J^{n\theta/2} * \varphi_j) * f_j(x)| &\leq \int_{I_j} |(J^{n\theta/2} * \varphi_j)(x-y)| |f_j(y)| dy \leq \left[\sup_{y \in I_j} |J^{n\theta/2} * \varphi_j(x-y)| |I_j| \right] \\ &\quad \times \frac{1}{|I_j|} \int_{I_j} |f_j(y)| dy \leq \left[A' \int_{I_j} (J^{n\theta/2} * \varphi_j)(x-y) dy \right] \frac{1}{|I_j|} \int_{I_j} |f_j(y)| dy \end{aligned}$$

(since $x \not\sim I_j$ implies that $J^{n\theta/2} * \varphi_j(x-y)$ is roughly constant over the cube I_j)

$$= [A' (J^{n\theta/2} * \varphi_j) * \chi_{I_j}(x)] \frac{1}{|I_j|} \int_{I_j} |f_j(y)| dy = A' (J^{n\theta/2} * \varphi_j) * \left[\frac{1}{|I_j|} \int_{I_j} |f_j(y)| dy \right] \chi_{I_j}(x),$$

so that by (3), $|(J^{n\theta/2} * \varphi_j) * f_j(x)| \leq A' \alpha (J^{n\theta/2} * \varphi_j) * \chi_{I_j}(x)$.

$$\begin{aligned} \text{Hence, } |F_2(x)| &\leq \sum_{x \not\sim I_j, j \in J} |J^{n\theta/2} * \varphi_j * f_j(x)| \leq A' \alpha \sum_{x \not\sim I_j, j \in J} (J^{n\theta/2} * \varphi_j) * \chi_{I_j}(x) \\ &\leq A' \alpha \sum_{j \in J} J^{n\theta/2} * (\varphi_j * \chi_{I_j})(x) \leq A' \alpha \|J^{n\theta/2}\|_1 \left\| \sum_{j \in J} \varphi_j * \chi_{I_j} \right\|_\infty \leq A' \alpha, \end{aligned}$$

since the supports of the $\varphi_j * \chi_{I_j}$ have bounded overlaps.

So we have proved that $\|F_2\|_1 \leq A' \|f\|_1$ and that $\|F_2\|_\infty \leq A' \alpha$. (Note the strong resemblance between the proof that $\|F_2\|_\infty \leq A' \alpha$, and the proof that $|\sum_{y \sim I_j} h_j(y, t)| \leq A\alpha/t$, which occurred in the argument proving Theorem 1). It follows that

$$\|F_2\|_2^2 = \int_{\mathbb{R}^n} |F_2(x)|^2 dx \leq \|F_2\|_\infty \int_{\mathbb{R}^n} |F_2(x)| dx = \|F_2\|_\infty \|F_2\|_1 \leq A' \alpha \|f\|_1.$$

It remains only to prove that

$$\|F_1\|_2^2 \leq A' \alpha \|f\|_1, \quad (23)$$

for then equation (22) is proved, and with it, Theorem 2'. Set

$$F_1^j = \begin{cases} J^{n\theta/2} * \varphi_j * f_j(x), & \text{if } x \sim I_j, \\ 0 & \text{otherwise} \end{cases}$$

Then $F_1 = \sum_{j \in J} F_1^j$, and for each $x \in \mathbb{R}^n$ there are at most N values of j for which $F_1^j(x) \neq 0$ (by (4)). Hence $|F_1(x)|^2 \leq N \sum_{j \in J} |F_1^j(x)|^2$, so $\|F_1\|_2^2 \leq N \sum_{j \in J} \|F_1^j\|_2^2$. On the other hand,

$$\begin{aligned} \|F_1^j\|_2^2 &= \int_{x \sim I_j} |(J^{n\theta/2} * \varphi_j) * f_j(x)|^2 dx \leq \int_{\mathbb{R}^n} |(J^{n\theta/2} * \varphi_j) * f_j(x)|^2 dx \\ &\leq \|J^{n\theta/2} * \varphi_j\|_2^2 \|f_j\|_1^2 \leq \left(\frac{A'}{|I_j|}\right) \|f_j\|_1^2 \end{aligned}$$

(by an elementary computation)

$$\leq \left(\frac{A'}{|I_j|}\right) (A' \alpha^2 |I_j|^2) \text{ (by (3))} = A' \alpha^2 |I_j|.$$

So $\|F_1\|_2^2 \leq N \sum_{j \in J} A' \alpha^2 |I_j| \leq A' \alpha^2 |\Omega| \leq A\alpha \|f\|_1$ by (1). Thus, inequality (23), to which we reduced the proof of Theorem 2', holds. Q.e.d.

Under reasonable conditions on K , we can sharpen Theorem 2' by showing that the "maximal operator" $Mf(x) = \sup_{\varepsilon > 0} \left| \int_{|y| > \varepsilon} K(y) f(x-y) dy \right|$ has weak-type $(1, 1)$. This is the case when, say, $|K(x)| \leq 1/|x|$ and $(K * \varphi_\varepsilon - K_\varepsilon) * f$ is dominated uniformly by the maximal function of f . (Here, $\varphi_\varepsilon(x) = \varepsilon^{-n} \varphi(x/\varepsilon)$ as usual, and $K_\varepsilon(x) = K(x)$ if $|x| > \varepsilon^{1-\theta}$, $K_\varepsilon(x) = 0$ otherwise.) The sharper estimates prove, for instance, that the operator

$$f \rightarrow \sup_{0 < \varepsilon < R} \left| \int_{R > |y| > \varepsilon} \frac{e^{i|y|}}{y} f(x-y) dy \right|,$$

defined for $f \in L^1(\mathbb{R}^1)$, has weak-type $(1, 1)$, and is bounded on L^p ($1 < p < \infty$).

IV. Results on the operators T_λ

In this section, we apply the methods developed in sections II and III, to the study of the operators T_λ defined in section I. Our result is the following.

THEOREM 3. *Let $1 < p < 4n/(3n+1)$ be given. If $p > n/\lambda$, then T_λ is a bounded linear operator on $L^p(\mathbb{R}^n)$.*

In other words, the conjecture stated at the end of section I is true if $p < 4n/(3n+1)$.

As we have just said, the proof uses the same basic ideas as the arguments in sections II and III. This time, however, instead of the standard inequalities for fractional integrals, we make use of a remarkable observation by E. M. Stein, namely:

LEMMA. *Let $f \in C_0^\infty(\mathbb{R}^n)$ ($n > 1$), and let $1 \leq p < 4n/(3n+1)$ be given. Then we have an a priori inequality*

$$\left(\int_{S^{n-1}} |f(\theta)|^2 d\theta \right)^{\frac{1}{2}} \leq A_p \|f\|_p,$$

where $d\theta$ denotes hypersurface measure on the unit sphere S^{n-1} .

This lemma allows us to define the restriction $f|_{S^{n-1}}$ for $f \in L^p(\mathbb{R}^n)$, $1 \leq p < 4n/(3n+1)$, even though S^{n-1} has measure zero in \mathbb{R}^n .

Proof of the lemma. By the Fourier inversion formula, $\int_{S^{n-1}} |f(\theta)|^2 d\theta = \int_{\mathbb{R}^n} f \bar{f} d\theta = f * f * \widehat{d\theta}(0)$. But $\widehat{d\theta}$ is a function on \mathbb{R}^n , which belongs to $L^q(\mathbb{R}^n)$ for all $q > 2n/(n-1)$. (To see this, we write $\widehat{d\theta}(x) = \int_{S^{n-1}} e^{ix\theta} d\theta = \int_{-1}^1 e^{i|x|t} d\Omega(t)$, where $\Omega(t)$ denotes the hypersurface area of the set $\{\theta \in S^{n-1} \mid |x| \cdot \theta \leq t\}$. The integral can be evaluated explicitly in terms of Bessel functions by formula (3) p. 48 of [13], and the approximate size of the Bessel functions is given in formula (1) on p. 199 of [13]. Thus, $|\widehat{d\theta}(x)|$ can be computed approximately.) Therefore, $\int_{S^{n-1}} |f(\theta)|^2 d\theta = (f * f) * \widehat{d\theta}(0) \leq \|f * f\|_{q'} \|\widehat{d\theta}\|_q \leq A_q \|f * f\|_{q'}$, where q' is the exponent dual to q . In other words, $\int_{S^{n-1}} |f(\theta)|^2 d\theta \leq A_r \|f * f\|_r$ for any $r < 2n/(n+1)$. By Young's theorem on convolutions, $\|f * f\|_r \leq \|f\|_p^2$ (where $1/r = 2/p - 1$, so that if $p < 4n/(3n+1)$ then $r < 2n/(n+1)$). Hence $\int_{S^{n-1}} |f(\theta)|^2 d\theta \leq A_p \|f\|_p^2$ for $1 \leq p < 4n/(3n+1)$. Q.e.d.

This lemma is not the best possible such result, a point to which we shall return later.

Proof of Theorem 3. By the Marcinkiewicz interpolation theorem, it will be enough to prove that under the stated conditions on λ and p , T_λ has weak-type (p, p) . So let $f \in L^p(\mathbb{R}^n)$, and let $\alpha > 0$ be given. We want to show that $|\{x \in \mathbb{R}^n \mid |T_\lambda f(x)| > A\alpha\}| \leq (A/\alpha^p) \|f\|_p^p$. We can suppose that f is positive.

Because we are proving an inequality for $p > n/\lambda$ rather than a sharp result for $p = n/\lambda$, we shall encounter a few minor technical nuisances which did not occur before. To avoid trouble, it is convenient to arrange things so that when we apply the decomposition lemma, we will not have to worry about the small cubes. Therefore, we proceed as follows.

Let φ be a C^∞ function of rapid decrease on R^n . We are going to prove a weak-type inequality for $T_\lambda(\varphi * f)$ instead of for $T_\lambda f$. The advantage is that $\varphi * f$ is much smoother than f , so that local problems (which would arise from small cubes) disappear. We can deduce the inequality for $T_\lambda f$ from that for $T_\lambda(\varphi * f)$, since by using a suitable φ , we obtain

$$\|T_\lambda f - T_\lambda(\varphi * f)\|_p \leq A \|f\|_p. \quad (24)$$

To see this inequality, we write $(T_\lambda f - T_\lambda(\varphi * f))^\wedge(x) = m_\lambda(x)\hat{f}(x) - m_\lambda(x)\hat{\varphi}(x)\hat{f}(x) = [m_\lambda(x)(1 - \hat{\varphi}(x))]\hat{f}(x)$ where m_λ is the multiplier corresponding to T_λ . Since m_λ has no singularities except at the sphere $|x| = 1$, inequality (24) follows for all p ($1 \leq p \leq +\infty$) if $1 - \hat{\varphi}(x)$ vanishes to high enough order at $|x| = 1$.

So to prove our theorem, we need only show that

$$|\{x \in R^n \mid |T_\lambda(\varphi * f)(x)| > A\alpha\}| \leq \frac{A}{\alpha^p} \|f\|_p^p.$$

Now we have a pointwise inequality $\varphi * f \leq \psi * f$, where $\psi(x) \equiv A(1 + |x|)^{-2n}$, since φ is of rapid decrease. On the other hand, $\|\psi * f\|_p \leq A \|f\|_p$, since $\psi \in L^1(R^n)$. We shall apply the decomposition lemma to $\psi * f$, α , and p , to obtain a collection of cubes $\{I_j\}$ and an exceptional set $\Omega = \bigcup_j I_j$, with the properties

$$|\Omega| = \sum_j |I_j| \leq \frac{A}{\alpha^p} \|f\|_p^p. \quad (25)$$

$$|\psi * f(x)| \leq A\alpha \text{ if } x \in R^n - \Omega. \quad (26)$$

$$\frac{1}{|I_j|} \int_{I_j} |\psi * f(y)|^p dy \leq A\alpha^p \quad (27)$$

for each cube I_j from the collection, and satisfying all the various geometrical conditions which we have noted before.

Since $0 \leq \varphi * f \leq \psi * f$, (26) and (27) imply

$$|\varphi * f(x)| \leq A\alpha \text{ if } x \in R^n - \Omega, \quad (28)$$

and

$$\frac{1}{|I_j|} \int_{I_j} |\varphi * f(y)|^p dy \leq A\alpha^p \quad (29)$$

for each cube I_j from the collection.

Set $f_j = (\varphi * f) \chi_{I_j}$, and set $f' = (\varphi * f) \chi_{R^n - \Omega}$. Then of course $\varphi * f = f' + \sum_j f_j$. We are trying to show that $|\{x \in R^n \mid |T_\lambda(\varphi * f)(x)| > A\alpha\}| \leq (A/\alpha^p) \|f\|_p^p$. But by (28), $\|f'\|_2^2 \leq A\alpha^{2-p} \|f\|_p^p$, and thus, as in the proof of Theorems 1 and 2', $|\{x \in R^n \mid |T_\lambda f'(x)| > \alpha\}| \leq (A/\alpha^p) \|f\|_p^p$. Therefore, in order to prove Theorem 3, we need only show that

$$|\{x \in R^n \mid |\sum_j T_\lambda f_j(x)| > A\alpha\}| \leq \frac{A}{\alpha^p} \|f\|_p^p. \quad (30)$$

By the construction, each cube I_j has diameter 2^k for some (possibly negative) integer k . Let Q_k denote the collection of all Whitney cubes of diameter 2^k , and let $f^k = \sum_{I_j \in Q_k} f_j$. In our new notation, (30) becomes

$$|\{x \in R^n \mid |\sum_{k=-\infty}^0 T_\lambda f^k(x)| > A\alpha\}| \leq \frac{A}{\alpha^p} \|f\|_p^p. \quad (31)$$

We shall dispose of the terms $\sum_{k=-\infty}^0 T_\lambda f^k$ in one fell swoop. For since $f \geq 0$, it follows from the definition of φ that $A_1 \leq (\varphi * f(x))/(\varphi * f(y)) \leq A_2$ if $|x - y| \leq 1$, i.e. $\varphi * f$ is roughly constant over the cubes of Q_k , $k \leq 0$. From inequality (27) it follows that $(1/|I_j|) \int_{I_j} |\varphi * f(y)|^2 dy \leq A\alpha^2$ if $\text{diam}(I_j) \leq 1$. Adding up these inequalities, we obtain

$$\| \sum_{k \leq 0} f^k \|_2^2 \leq A\alpha^2 \sum_{\text{diam}(I_j) \leq 1} |I_j| \leq A\alpha^2 |\Omega| \leq A\alpha^{2-p} \|f\|_p^p.$$

Since T_λ is bounded on L^2 , we conclude that $\| \sum_{k=-\infty}^0 T_\lambda f^k \|_2^2 \leq A\alpha^{2-p} \|f\|_p^p$. So by the Chebyshev inequality, $|\{x \in R^n \mid |\sum_{k=-\infty}^0 T_\lambda f^k(x)| \geq A\alpha\}| \leq (A/\alpha^p) \|f\|_p^p$.

To prove (31), then, we have only to prove that

$$|\{x \in R^n \mid |\sum_{k=1}^{\infty} T_\lambda f^k(x)| > A\alpha\}| \leq \frac{A}{\alpha^p} \|f\|_p^p. \quad (32)$$

So far, we have really done nothing to the problem except remove some trivial error terms. As soon as we set up some notation, we shall give the decomposition that proves the theorem. Using this decomposition, we shall reduce (32) to more and more complicated inequalities, which finally become trivial.

Pick a small number $\delta > 0$ to be determined later. For each $k \geq 0$, let θ_k be a C^∞ function on R^1 , satisfying

- (i) $0 \leq \theta_k \leq 1$
- (ii) $\theta_k(x) = 1$ in a neighborhood $\{x \in R^1 \mid |x - 1| < c 2^{-k(1-\delta)}\}$ of $x = 1$.
- (iii) θ_k has "width" $2^{-k(1-\delta)}$. In other words,

$$\left| \frac{d^m}{dx^m} \theta_k(x) \right| \leq A_m 2^{km(1-\delta)} \quad \text{for all } x \in R^1 \text{ and } m > 0;$$

$$\text{and } \theta_k(x) = 0 \text{ if } |x - 1| \geq A 2^{-k(1-\delta)}.$$

Set $\varphi_k = 1 - \theta_k$.

Recall that the "multiplier" m_λ , defined by the equation $(T_\lambda f)^\wedge(x) = m_\lambda(x) \hat{f}(x)$ on R^n , is spherically symmetric and C^∞ away from the unit-sphere S^{n-1} and that near S^{n-1} , $|m_\lambda(x)| = O(|1 - |x||^\gamma)$, where $\gamma = \lambda - (n+1)/2$. For $f \in L^2(R^n)$ we can write $(T_\lambda f)^\wedge(x) = m_\lambda(x) \hat{f}(x) = m_\lambda(x) \theta_k(|x|) \hat{f}(x) + (m_\lambda(x) \varphi_k(|x|)) \hat{f}(x) \equiv m_\lambda(x) (S_k f)^\wedge(x) + (R_k f)^\wedge(x)$.

The operators S_k and R_k are given by convolution with L^1 kernels, which we call $s_k(x)$ and $r_k(x)$, respectively. If, finally, we define operators K_k by setting

$$(K_k f)^\wedge(x) = \begin{cases} m_\lambda(x) \hat{f}(x) & \text{if } ||x| - 1| \leq A 2^{-k(1-\delta)} \\ 0 & \text{otherwise} \end{cases}$$

then we obtain the equations

$$T_\lambda = T_\lambda S_k + R_k = K_k S_k + R_k. \quad (33)$$

Our basic decomposition is $T_\lambda f^k = K_k(S_k f^k) + R_k f^k$, which we shall use to prove the estimate (32). Of the two terms of the decomposition, the second is a trivial remainder term, and we shall rid ourselves of it right away, with a simple L^1 argument.

Note that $\int_{|y| > 2^k} |r_k(y)| dy \leq A$, for all $k \geq 0$. (Actually, we could do much better. In fact $\int_{|y| > 2^k} |r_k(y)| dy = O(2^{-Mk})$ for any $M > 0$. This is because r_k is a kernel with "thickness" only $2^{k(1-\delta)}$, which is far smaller than 2^k . More precisely, $|r_k(y)| \leq |y|^{-m} \cdot |[\nabla^m(m_\lambda(x) \varphi_\lambda(|x|))]^\wedge(y)| \leq A'_m |y|^{-m} 2^{km(1-\delta)}$, so that by taking m very large, we can deduce that $\int_{|y| > 2^k} |r_k(y)| dy = O(2^{-Mk})$. The same trick shows that $\int_{|y| > 2^k} |s_k(y)| dy = O(2^{-Mk})$, a fact which we shall soon use.) Therefore,

$$\left\| \sum_{k=1}^{\infty} R_k f^k \right\|_{L^1(R^n - \Omega)} \leq A \sum_{k=1}^{\infty} \|f^k\|_1 \leq A \sum_j \|f_j\|_1 \leq A \alpha \sum_j |I_j| \leq A \alpha^{1-p} \|f\|_p^p,$$

so that $|\{x \in R^n - \Omega \mid |\sum_{k=1}^{\infty} R_k f^k(x)| > A\alpha\}| \leq (A/\alpha^p) \|f\|_p^p$. Hence, also

$$|\{x \in R^n \mid |\sum_{k=1}^{\infty} R_k f^k(x)| > A\alpha\}| \leq (A/\alpha^p) \|f\|_p^p,$$

since Ω is a small set. In view of the basic decomposition of $T_\lambda f^k$, inequality (32) now reduces to

$$|\{x \in R^n \mid |\sum_{k=1}^{\infty} K_k(S_k f^k)(x)| > A\alpha\}| \leq \frac{A}{\alpha^p} \|f\|_p^p. \quad (34)$$

Look carefully at $S_k f^k - f^k = \sum_{I_j \in \mathcal{Q}_k} f_j$, so that $S_k f^k = \sum_{I_j \in \mathcal{Q}_k} S_k f_j$. For each j , let \tilde{I}_j denote the cube concentric with I_j , and having diameter twice as large. As we already noted, the \tilde{I}_j 's have bounded overlaps.

$$\text{Now} \quad S_k f^k = \sum_{I_j \in \mathcal{Q}_k} (s_k * f_j) \chi_{\tilde{I}_j} + \sum_{I_j \in \mathcal{Q}_k} (s_k * f_j) \chi_{R^n - \tilde{I}_j}. \quad (35)$$

We shall prove that the second term on the right is a trivial remainder term. Thus, (35) has the effect of “localizing” the problem to the individual cubes.

First of all,

$$\begin{aligned} \left\| \sum_{I_j \in Q_k} (s_k * f_j) \chi_{R^n - \tilde{I}_j} \right\|_1 &\leq \sum_{I_j \in Q_k} \left\| (s_k * f_j) \right\|_{L^1(R^n - \tilde{I}_j)} \\ &\leq A 2^{-k} \sum_{I_j \in Q_k} \|f_j\|_1 \leq A 2^{-k} \alpha \sum_{I_j \in Q_k} |I_j| \leq A 2^{-k} \alpha |\Omega| \\ &\leq A 2^{-k} \alpha^{1-p} \|f\|_p^p, \quad \text{since } \int_{|y| > 2^k} |s_k(y)| dy \leq A 2^{-k} \end{aligned}$$

(recall that $\int_{|y| > 2^k} |s_k(y)| dy = O(2^{-Mk})$ for any $M > 0$). On the other hand, for any $x \in R^n$,

$$\begin{aligned} \left| \sum_{I_j \in Q_k} (s_k * f_j) \chi_{R^n - \tilde{I}_j}(x) \right| &= \left| \sum_{I_j \in Q_k, x \notin \tilde{I}_j} (s_k * f_j)(x) \right| \leq \sum_{I_j \in Q_k, x \notin \tilde{I}_j} \|f_j\|_1 \sup_{y \in I_j} |s_k(x-y)| \\ &\leq A \alpha \sum_{I_j \in Q_k, x \notin \tilde{I}_j} |I_j| \sup_{y \in I_j} |s_k(x-y)| \leq A \alpha \int_{|x-y| \geq A 2^k} |s_k(x-y)| dy \leq A \alpha. \end{aligned}$$

In other words,

$$\left\| \sum_{I_j \in Q_k} (s_k * f_j) \chi_{R^n - \tilde{I}_j} \right\|_1 \leq 2^{-k} A \alpha^{1-p} \|f\|_p^p,$$

and

$$\left\| \sum_{I_j \in Q_k} (s_k * f_j) \chi_{R^n - \tilde{I}_j} \right\|_\infty \leq A \alpha.$$

So $\left\| K_k \left(\sum_{I_j \in Q_k} (s_k * f_j) \chi_{R^n - \tilde{I}_j} \right) \right\|_2 \leq A \left\| \sum_{I_j \in Q_k} (s_k * f_j) \chi_{R^n - \tilde{I}_j} \right\|_2 \leq A 2^{-k/2} \alpha^{(2-p)/2} \|f\|_p^{p/2}$.

By the triangle inequality,

$$\left\| \sum_{k=1}^{\infty} K_k \left(\sum_{I_j \in Q_k} (s_k * f_j) \chi_{R^n - \tilde{I}_j} \right) \right\|_2^2 \leq A \alpha^{2-p} \|f\|_p^p,$$

which proves that

$$\left| \{x \in R^n \mid \left| \sum_{k=1}^{\infty} K_k \left(\sum_{I_j \in Q_k} (s_k * f_j) \chi_{R^n - \tilde{I}_j} \right) (x) \right| > \alpha\} \right| \leq \frac{A}{\alpha^p} \|f\|_p^p.$$

This is exactly the inequality needed to estimate the last term in equation (35).

So to complete the proof of (34), which in turn proves Theorem 3, we have only to show that

$$\left| \{x \in R^n \mid \left| \sum_{k=1}^{\infty} K_k \left(\sum_{I_j \in Q_k} (s_k * f_j) \chi_{\tilde{I}_j} \right) (x) \right| > \alpha\} \right| \leq \frac{A}{\alpha^p} \|f\|_p^p.$$

This inequality follows from the Chebyshev inequality and the estimate

$$\left\| \sum_{k=1}^{\infty} K_k \left(\sum_{I_j \in Q_k} (s_k * f_j) \chi_{\tilde{I}_j} \right) \right\|_2^2 \leq A \alpha^{2-p} \|f\|_p^p. \quad (36)$$

We shall finish off the proof of Theorem 3 by proving (36).

Let us examine $\sum_{I_j \in Q_k} (s_k \times f_j) \chi_{I_j}$. Since the I_j 's have bounded overlaps, it follows that

$$\left\| \sum_{I_j \in Q_k} (s_k \times f_j) \chi_{I_j} \right\|_2^2 \leq N \sum_{I_j \in Q_k} \|s_k \times f_j\|_2^2 = N \sum_{I_j \in Q_k} \|\hat{s}_k \hat{f}_j\|_2^2. \tag{37}$$

Now
$$\begin{aligned} \|\hat{s}_k \hat{f}_j\|_2^2 &\equiv \int_{R^n} \theta_k(|x|)^2 |f_j(x)|^2 dx = \int_0^\infty r^{n-1} |\theta_k(r)|^2 \left(\int_{\omega \in S^{n-1}} |f_j(r\omega)|^2 d\omega \right) dr \\ &\leq A \|f_j\|_p^2 \int_0^\infty r^{n-1} |\theta_k(r)|^2 dr \leq A \|f_j\|_p^2 2^{-k(1-\delta)} \end{aligned}$$

(To prove this chain of inequalities, we have used properties (i)–(iii) of θ_k , and the lemma of Stein) $\leq A \|f_j\|_p^2 |I_j|^{-(1-\delta)/n}$ (since $I_j \in Q_k \leq A\alpha^2 |I_j|^{(2/p)-(1-\delta)/n}$). Substituting these inequalities in (37), we obtain

$$\left\| \sum_{I_j \in Q_k} (s_k \times f_j) \chi_{I_j} \right\|_2^2 \leq A\alpha^2 \sum_{I_j \in Q_k} |I_j|^{(2/p)-(1-\delta)/n}.$$

By definition, K_k is a bounded operator on $L^2(R^n)$, with norm roughly $2^{-k(1-\delta)(\lambda-(n+1)/2)}$. Hence,

$$\begin{aligned} \left\| K_k \left(\sum_{I_j \in Q_k} (s_k \times f_j) \chi_{I_j} \right) \right\|_2^2 &\leq 2^{-2k(1-\delta)(\lambda-(n+1)/2)} A\alpha^2 \sum_{I_j \in Q_k} |I_j|^{2/p-(1-\delta)/n} \\ &= A\alpha^2 \sum_{I_j \in Q_k} |I_j|^{-((1-\delta)/n)(2\lambda-n-1)+2/p-(1-\delta)/n} \tag{38} \end{aligned}$$

(since each I_j in the summation has $|I_j| = A 2^{kn}$)

$$= A\alpha^2 \sum_{I_j \in Q_k} |I_j|^{-((1-\delta)/n)(2\lambda-n)+2/p}.$$

Now remember—at the beginning of the proof we considered a small number $\delta > 0$, and all the estimates we have proved so far are valid no matter which δ we take. The time has come to pick a value for δ . If $\lambda > n/p$, then we can find a $\delta > 0$, so small that $-(1-\delta)(2\lambda-n)/n + 2/p < 1$, say $-(1-\delta)(2\lambda-n)/n + 2/p = 1 - \varepsilon$. With such a δ , (38) becomes

$$\left\| K_k \left(\sum_{I_j \in Q_k} (s_k \times f_j) \chi_{I_j} \right) \right\|_2^2 \leq A\alpha^2 \sum_{I_j \in Q_k} |I_j|^{1-\varepsilon} = A\alpha^2 2^{-n k \varepsilon} \sum_{I_j \in Q_k} |I_j| \leq A\alpha^2 2^{-n k \varepsilon} |\Omega| \leq 2^{-n k \varepsilon} A\alpha^{2-p} \|f\|_p^p.$$

Since $\varepsilon > 0$, the triangle inequality shows that $\left\| \sum_{k=1}^\infty K_k \left(\sum_{I_j \in Q_k} (s_k \times f_j) \chi_{I_j} \right) \right\|_2^2 \leq A\alpha^{2-p} \|f\|_p^p$, which is exactly inequality (36). This completes the proof of Theorem 3. Q.e.d.

As we mentioned just before the proof of Theorem 3, Stein's lemma is not the best possible inequality for the restriction of a Fourier transform to S^{n-1} . In particular, the author, in collaboration with Stein, has proved the following

LEMMA. *Let $f \in L^{4/3-\delta}(R^2)$. Then the Fourier transform \hat{f} restricts to an $L^{4/3}$ function on the circle S^1 , and satisfies the a priori inequality*

$$\|f\|_{L^{4/3}(S^1)} \leq A_\delta \|f\|_{L^{4/3-\delta}(R^2)}. \quad (39)$$

($\delta, \delta'',$ etc. denote small numbers).

Interpolation between (39) and $\|f\|_\infty \leq \|f\|_1$ yields, among other things, $\|f\|_{L^2(S^1)} \leq A_p \|f\|_{L^p(R^2)}$ for $1 \leq p < 6/5$. Using this improved L^2 -estimate, we can easily extend theorem 3 to cover the case $n=2, 1 \leq p < 6/5$. Stein's lemma covers only the case $1 \leq p < 8/7$. On the other hand, the present result deals only with R^2 . Presumably, both lemmas are approximations to an optimal n -dimensional restriction theorem.

Sketch of proof of the lemma. Boundedness of the operator $f \rightarrow \hat{f}|_{S^1}$ from $L^{4/3-\delta}$ to $L^{4/3}$ is obviously equivalent to the boundedness of the adjoint operator T^* from $L^4(S^1)$ into $L^{4+\delta'}(R^2)$. If $f \in L^4(S^1)$, then T^*f is simply the Fourier transform of the measure $f d\theta$, where $d\theta$ denotes Lebesgue measure on S^1 . To check that $(f d\theta)^\wedge \in L^{4+\delta'}(R^2)$, we shall prove that $(f d\theta)^\wedge \cdot (f d\theta)^\wedge \in L^{2+\delta'/2}(R^2)$. This follows from the assertion $(f d\theta) \times (f d\theta) \in L^{2-\delta''}(R^2)$, by the Hausdorff-Young inequality.

So to prove our restriction lemma, we merely have to show that $\|(f d\theta) \times (f d\theta)\|_{2-\delta''} \leq A_\delta \|f\|_4^2$. If we set $F = (f d\theta) \times (f d\theta)$, then obviously

$$\varepsilon^{-2} \int_{|y-x|<\varepsilon} F(y) dy = \varepsilon^{-2} \iint_B f(w_1) f(w_2) dw_1 dw_2, \quad (40)$$

where $B = \{(w_1, w_2) \in S^1 \times S^1 \mid |(w_1 + w_2) - x| < \varepsilon\}$. What does the set B look like? First of all, for $0 < |x| < 2$, there are precisely two pairs $(w_1, w_2) \in S^1 \times S^1$ such that $w_1 + w_2 = x$. Call these two pairs $(w_1(x), w_2(x))$ and $(\tilde{w}_1(x), \tilde{w}_2(x))$. Then $B = B_1 \cup B_2$, where B_1 consists entirely of pairs (w_1, w_2) which are close to $(w_1(x), w_2(x))$ in $S^1 \times S^1$, and similarly B_2 consists of pairs close to $(\tilde{w}_1(x), \tilde{w}_2(x))$. Equation (40) now shows that, approximately,

$$\begin{aligned} \varepsilon^{-2} \int_{|x-y|<\varepsilon} F(y) dy &\approx f(w_1(x)) f(w_2(x)) \left[\varepsilon^{-2} \iint_{B_1} dw_1 dw_2 \right] \\ &\quad + f(\tilde{w}_1(x)) f(\tilde{w}_2(x)) \left[\varepsilon^{-2} \iint_{B_2} dw_1 dw_2 \right]. \end{aligned}$$

Letting ε tend to zero, we obtain $F(x) = (f(w_1(x))f(w_2(x)) + f(\tilde{w}_1(x))f(\tilde{w}_2(x)))\varphi(x)$, where $\varphi(x)$ is defined as the common limit of the two quantities in square brackets. In view of its elementary definition, φ can be computed explicitly. We spare the reader the details of our computation, but it turns out that $\varphi(x) = O(|x|^{-1})$ near $x=0$, $\varphi(x) = O((2-|x|)^{\frac{1}{2}})$ near $|x|=2$, and φ is bounded elsewhere. Of course, φ is a radial function of x .

Now, using polar coordinates for x , we write

$$\|F\|_{\frac{2}{2-\delta''}}^{2-\delta''} \leq A \int_0^2 r |\varphi(r)|^{2-\delta''} \left[\int_0^{2\pi} |f(w_1(r, \theta)) f(w_2(r, \theta))|^{2-\delta''} d\theta + \int_0^{2\pi} |f(\tilde{w}_1(r, \theta)) f(\tilde{w}_2(r, \theta))|^{2-\delta''} d\theta \right] dr.$$

Hölder's inequality shows easily that the first integral in brackets is smaller than

$$(\|f(w_1(r, \cdot))\|_4 \|f(w_2(r, \cdot))\|_4)^{2-\delta''} \leq \|f\|_4^{2(2-\delta'')}.$$

Similarly, the second integral in brackets is at most $\|f\|_4^{2(2-\delta'')}$. Thus

$$\|F\|_{\frac{2}{2-\delta''}}^{2-\delta''} \leq A \|f\|_4^{2(2-\delta'')} \left(\int_0^2 r |\varphi(r)|^{2-\delta''} dr \right).$$

The final integral converges for $\delta'' > 0$, which proves the a priori inequality. Q.e.d.

The reader may note the systematic completeness with which every single step in the above argument breaks down in n dimensions ($n > 2$).

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