THE HYPERCENTER OF A GROUP.

By

REINHOLD BAER

URBANA, ILLINOIS.

The hypercenter of a finite group may be characterized by various properties which, however, cease to be equivalent if applied to infinite groups. Of the possibilities thus arising we investigate here only one, the terminal member of the upper central chain; and our problem is the intrinsic characterization of the normal subgroups contained in this hypercenter. These hypercentral subgroups may be defined as exactly those subgroups N of G which satisfy the following simple condition: If M is a normal subgroup of G and if $M < N$, then N/M contains a center element, not 1, of G/M.

Of the fundamental properties of hypercentral subgroups N of G the following seem to be outstanding: (a) if the normal subgroup M of G is a proper part of N, if x is an element of order a power of p in N/M and if g is some element in G/M, then there exists an integer m such that $xg^{p^m} = g^{p^m}x$; (b) if T is a subgroup of G such that $T < NT$, then the normalizer of T in NT is different from T; (c) if the normal subgroup M of G is a proper part of N, then there exists a normal subgroup V of G such that $M < V \leq N$ and such that M is the intersection of all the normal subgroups X of G which satisfy: $M \leq X < V$ and V/X is a finite minimal normal subgroup of G/X. Actually it will be shown that each of the two combinations (a, c) and (b, c) is characteristic for hypercentrality.

One of the most interesting phenomena encountered in the course of this investigation is the fact that hypercentral subgroups are never "very infinite". To make this rather vague statement more precise we mention two results: The maximum condition is satisfied by the subgroups of every finitely generated subgroup of the hypercenter; and finitely many elements of finite order in the hypercenter generate a finite subgroup. The latter remark points to a fascinating undercurrent of complications arising from encounters with Burnside's celebrated conjecture which had to be either circumvented or, in rather special instances, proved. The preparatory discussions of section 1 are very much concerned with just this situation; and some concepts and results of independent interest may be found there.

Notations.

$C(S < G)$ = centralizer of the subset S in the group G
= totality of elements z in G which commute with every element s in S.
$Z(G)$ = center of the group G.
$[x, y] = x^{-1} y^{-1} xy$.
$[X, Y]$ = subgroup G which is generated by all the commutators $[x, y]$ for x in X and y in Y.
x^G = totality of elements $g^{-1} xg$ for g in G.
$\{S\}$ = subgroup generated by subset S.
$X < Y$ signifies that X is a proper part of Y and $X \leq Y$ signifies that X is contained in Y.
$X \cap Y$ = intersection of X and Y.
$G \simeq H$ signifies isomorphy of the groups G and H.
p-group = group all of whose elements have order a power of p.

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1. Finiteness, Minimality, Maximality.

In this section a number of general concepts and principles are collected which will prove useful in the course of our investigation. We begin by stating the following well known

Finiteness Principle: *If the group G is finitely generated, and if N is a normal subgroup of finite index in G, then N too is finitely generated.*

This one proves by straightforward application of the Reidemeister-Schreier method; see, for instance, Baer [1; p. 396, (1. 3)].

Lemma 1: *If the finitely generated normal subgroup N of G is infinite, then N contains a normal subgroup M of G such that N/M is infinite though N/U is finite whenever the normal subgroup U of G satisfies $M < U < N$.*

Proof: Denote by Φ the set of all the normal subgroups V of G such that $V < N$ and N/V is infinite. Since N is infinite, 1 belongs to Φ. Suppose that Θ is a not vacuous subset of Φ which is ordered by inclusion. Form the join J of all the subgroups in Θ. It is clear that J is a normal subgroup of G and that $J \leq N$. Assume by way of contradiction that N/J is finite. Since N is finitely generated and N/J is finite, we deduce from the Finiteness Principle that J is finitely generated. Denote by F a finite set of generators of J. Then it follows from the construction of J that there exists to every f in F a subgroup $S(f)$ in Θ which contains f. Since Θ is ordered by inclusion, there exists among the finitely many subgroups $S(f)$ a maximal one, say H. Then we have $F \leq H \leq J$ and this implies $J = H$, since J is generated by F. Consequently J belongs to Θ and hence to Φ; and this contradiction proves that N/J is infinite. Thus we have shown that J belongs to Φ. Now we may apply the Maximum Principle of Set Theory on the set Φ. Hence there exists a maximal subgroup M in Φ; and it is fairly clear that M has all the desired properties.

Lemma 2: *Suppose that the normal subgroup N of G satisfies the following condition.*

(F) *If M is a normal subgroup of G such that $M < N$, then there exists a normal subgroup K of G such that $M < K \leq N$ and K/M is finite.*

Then finite subsets of N generate finite subgroups of N.

Proof: Consider a finite subset F of N and denote by S the subgroup generated by F. Assume by way of contradiction that S is infinite. Then we deduce from Lemma 1 the existence of a normal subgroup W of S with the following properties: S/W is infinite; if V is a normal subgroup of S such that $W < V$, then S/V is finite.

Denote by Φ the set of all normal subgroups X of G such that $X \leq N$ and $X \cap S \leq W$. This set Φ contains $X = 1$. If Θ is a non vacuous subset of Φ which is ordered by inclusion, then denote by J the set theoretical join of all the subgroups X in Φ. It is fairly clear that J too belongs to Φ; and thus the Maximum Principle of Set Theory may be applied on Φ. Consequently there exists a maximal subgroup M in Φ. It is clear that M is a normal subgroup of G, that $M \leq N$ and that $M \cap S \leq W < S$. If M and N were equal, then we would have $S \leq W < S$ which is impossible. Hence $M < N$. Now we deduce from (F) the existence of a normal subgroup K of G such that $M < K \leq N$ and K/M is finite. Since $M < K$, K is not in Φ. Hence $K \cap S \not\leq W$. Consequently $W < (K \cap S) W \leq S$. Remembering the characteristic properties of W it follows now that $S/(K \cap S) W$ is finite. Next we note that $(K \cap S) M/M \leq K/M$. Thus the first of these groups is finite as a subgroup of the finite second group. From the isomorphism theorem we deduce that

$$(K \cap S) M/M \simeq (K \cap S)/(K \cap S \cap M) = (K \cap S)/(M \cap S).$$

But $M \cap S \leq W$ so that $M \cap S \leq K \cap W \leq K \cap S$ and $K \cap W$ is a normal subgroup of $K \cap S$. Thus $(K \cap S)/(K \cap W)$ is isomorphic to a quotient group of the finite group $(K \cap S) M/M$. It follows finally from the Isomorphism Theorem that $(K \cap S) W/W \simeq (K \cap S)/(K \cap W)$.

Consequently $(K \cap S)W/W$ is finite. Since $S/(K \cap S)W$ has been shown to be finite, it follows that S/W is finite contradicting our choice of W. Thus we have been led to a contradiction by assuming the infinity of S. Hence S is finite, as we wanted to prove.

Slightly extending a terminology that we adopted elsewhere we shall term the normal subgroup N of G locally finite, if every finite subset of N is contained in a finite normal subgroup of G which without loss in generality may be assumed to be part of N. It is clear that locally finite normal subgroups have the property (F) of Lemma 2 and that this property (F) is weaker than local finiteness. A still weaker property is the following one: (WF) *Every finite subset of N generates a finite subgroup of N.* It is the content of Lemma 2 that every normal subgroup with property (F) has likewise property (WF). The converse is false as there exist infinite groups with property (WF) without finite normal subgroups $\neq 1$; see, for instance, Baer [1; p. 412, Example 3.4]. All the elements in a normal subgroup with property (WF) are clearly of finite order; whether the converse is true is a question essentially equivalent with the strongest from of Burnside's celebrated conjecture.

It will be convenient to speak of WF-subgroups instead of subgroups with property (WF) and to denote by $W(G)$ *the product of all normal WF-subgroups of the group G.* Then we prove the following fact.

$W(G)$ is a normal WF-subgroup and $W[G/W(G)] = 1$.

Proof: We begin by verifying the following simple proposition.

(1) *If M and N are normal subgroups of G, if $M < N$ and if M is a normal WF-subgroup of G and N/M is a normal WF-subgroup of G/M, then N is a normal WF-subgroup of G.*

To prove this consider a finite subset F of N and denote by S the subgroup generated by F. Then SM/M is a finitely generated subgroup of N/M. Hence SM/M is finite. But $SM/M \simeq S/(S \cap M)$ so that the latter group is finite too. Since S is finitely generated, it follows from the Finiteness Principle that $S \cap M$ is finitely generated. But M is a WF-subgroup. Hence $S \cap M$ is finite. Since $S \cap M$ and $S/(S \cap M)$ are finite, S is finite. Consequently N has property (WF).

(2) *The product of two normal WF-subgroups is a normal WF-subgroup.*

Suppose that A and B are normal WF-subgroup. Then AB is a normal subgroup. Every finite subset of AB/A may be represented by a finite subset of B. Hence AB/A is a normal WF-subgroup of G/A. Now we deduce from (1) that AB is a normal WF-subgroup of G.

(3) *The product of a finite number of normal WF-subgroups is a normal WF-subgroup.*

This follows from (2) by an obvious inductive argument.

(4) *$W(G)$ is a normal WF-subgroup.*

If g is an element in $W(G)$, then there exist, by definition of $W(G)$, finitely many normal WF-subgroups $N(1), \ldots, N(k)$ such that g belongs to their product $N(1) \ldots N(k)$. Consequently every finite subset F of $W(G)$ is contained in a product P of finitely many normal

WF-subgroups of G. It follows from (3) that P is a normal WF-subgroup of G. Hence F generates a finite subgroup. Consequently $W(G)$ has property (WF).

(5) $W[G/W(G)] = 1$.

There exists one and only one normal subgroup T of G such that $W(G) \leq T$ and $T/W(G) = W[G/W(G)]$. It follows from (4) that $W(G)$ and $T/W(G)$ are normal WF-subgroups of G and $G/W(G)$ respectively. It follows from (1) that T is a normal WF-subgroup of G. Now it follows from the definition of $W(G)$ that $T \leq W(G) \leq T$ or $T = W(G)$. Hence $1 = T/W(G) = W[G/W(G)]$. This completes the proof.

Definition 1: *N is a finitely reducible subgroup of G, if N is a normal subgroup of G, if $N \neq 1$ and if 1 is the intersection of all normal subgroups X of G with the property:*

(M) *$X < N$ and N/X is a finite minimal normal subgroup of G/X.*

The exclusion of $N = 1$ is just a matter of technical convenience.

We note that every product of finite minimal normal subgroups of G is a finitely reducible subgroup of G; and that direct products of finitely reducible subgroups of G are finitely reducible subgroups of G. But it is not true that every product of finitely reducible subgroups is finitely reducible; if, for instance, G is the additive group of rational numbers, then every cyclic subgroup, not O of G is finitely reducible whereas their product G does not have this property. — Every free group $G \neq 1$ is a finitely reducible subgroup of itself; but their exist quotient groups of G which do not have this property [at least if the rank of G is greater than 1].

Definition 2: *The subgroup N of G is locally finitely reducible, if to every normal subgroup M of G satisfying $M < N$ there exists a finitely reducible subgroup of G/M which is part of N/M.*

It is almost obvious that locally finitely reducible subgroups need not be finitely reducible. Conversely consider the free group G possessing a normal subgroup N with the property: G/N is an infinite simple group. Then G is finitely reducible, but cannot be locally finitely reducible, since G/N does not contain finitely reducible subgroups.

Every locally finitely reducible subgroup is normal.

To verify this consider a locally finitely reducible subgroup N of G and form the product P of all the normal subgroups of G which are part of N. It is clear that P is a normal subgroup of G which is part of N; and an immediate application of Definition 2 shows the impossibility of $P < N$.

Lemma 3: *If N is a locally finitely reducible subgroup of G, and if M is a normal subgroup of G, then NM/M is a locally finitely reducible subgroup of G/M.*

Remark: This proposition becomes particularly interesting, if we remember that quotient groups of finitely reducible groups need not be finitely reducible.

Proof: Consider a normal subgroup U of G/M such that $U < MN/M$. Then there exists a uniquely determined normal subgroup H of G such that $M \le H < MN$ and $U = H/M$. It follows from Dedekind's Law that $H = M(H \cap N)$; and this implies $H \cap N < N$. Since N is a locally finitely reducible subgroup of G, there exists a finitely reducible subgroup R of $G/(H \cap N)$ which is part of $N/(H \cap N)$. There exists a uniquely determined normal subgroup K of G such that $H \cap N < K \le N$ and $R = K/(H \cap N)$.

Clearly KM is a normal subgroup of G such that

$$H = M(H \cap N) \le MK \le MN.$$

Next we deduce from Dedekind's Law that

$$N \cap H \le (N \cap H)(K \cap M) = K \cap M(N \cap H) = K \cap H \le N \cap H$$

or

$$N \cap H = (N \cap H)(K \cap M) = K \cap H;$$

and this implies in particular that $K \cap M \le N \cap H$.

If the normal subgroup W of G is situated between H and MK, then it follows from $M \le H$ and Dedekind's Law that $W = M(K \cap W)$. It is clear that $K \cap W$ is a normal subgroup of G satisfying

$$H \cap N = K \cap H \le K \cap W \le K,\ WK = M(K \cap W)K = MK,\ H(K \cap W) = W \cap HK = W,$$
$$K/(K \cap W) \simeq WK/W = MK/W.$$

If on the other hand the normal subgroup V of G is situated between $H \cap N$ and K, then HV is a normal subgroup between H and $HK = M(H \cap N)K = MK$ satisfying $HV \cap K = V(H \cap K) = V(H \cap N) = V$. Recalling that $R = K/(H \cap N)$ is a finitely reducible subgroup of $G/(H \cap N)$ it is now easily verified that KM/H is a finitely reducible subgroup of G/H which is part of NM/H. Consequently NM/M is a locally finitely reducible subgroup of G/M.

Every finite normal subgroup is locally finitely reducible.

This is practically obvious. Slightly deeper is the following criterion.

Lemma 4: *If the normal subgroup N of G is a finitely generated abelian group, then N is locally finitely reducible.*

Proof: Consider a normal subgroup M of G such that $M < N$. Then N/M is a finitely generated abelian group and the totality F of elements of finite order in N/M is a finite subgroup of N/M which is naturally a normal subgroup of G/M.

If $F \neq 1$, then F contains, as a finite normal subgroup of G/M, a finite minimal normal subgroup of G/M; and this proves the existence of a finitely reducible subgroup of G/M which is part of N/M.

If $F = 1$, then N/M and all its subgroups are free abelian groups of finite rank. Among the normal subgroups, not 1, of G/M which are part of N/M there exists one, W/M, of minimal rank. Denote by J the intersection of all the normal subgroups X of G/M with the property:

$X < W/M$ and $(W/M)/X$ is a finite minimal normal subgroup of $(G/M)/X$.

It is clear that J is a normal subgroup of G/M and that $J \leq W/M$.

Consider now a prime number p. Then $(W/M)^p < W/M$, $(W/M)^p$ is a normal subgroup of G/M [as a characteristic subgroup of a normal subgroup] and $(W/M)/(W/M)^p$ is finite, since W/M is a free abelian group of finite rank. But then there exists clearly at least one normal subgroup X of G/M such that $(W/M)^p \leq X < W/M$ and such that $(W/M)/X$ is a finite minimal normal subgroup of G/X. Clearly $J \leq X$ and the order of $(W/M)/X$ is divisible by p. Thus $(W/M)/J$ possesses quotient groups of order a multiple of p for every prime p. Hence $(W/M)/J$ is infinite. The rank of the free abelian group J is therefore smaller than the rank of W/M. Since the rank of W/M is minimal, it follows that $J = 1$. Application of Definition 1 shows that W/M is a finitely reducible subgroup of G/M which is part of N/M. This completes the proof of the fact that N is a locally finitely reducible subgroup of G.

Corollary 1: *If N is a normal subgroup of G such that the centralizer $C(N < G)$ has finite index in G, then N is a locally finitely reducible subgroup of G.*

Proof: Consider a normal subgroup M of G such that $M < N$ and let $G^* = G/M$ and $N^* = N/M$. Since $MC(N < G)/M \leq C(N^* < G^*)$, it follows from our hypothesis that $C(N^* < G^*)$ has finite index in G^*. Now we distinguish two cases.

Case 1: $N^* \cap C(N^* < G^*) = 1$.

Then N^* is isomorphic to the subgroup $N^*C(N^* < G^*)/C(N^* < G^*)$ of the finite group $G^*/C(N^* < G^*)$. Hence N^* is a finite normal subgroup, not 1, of G^*; and as such N^* contains a finitely reducible subgroup of G.

Case 2: $N^* \cap C(N^* < G^*) \neq 1$.

It is clear that $A = N^* \cap C(N^* < G^*)$ is an abelian normal subgroup of G^* which is part of N^*. From the hypothesis of our case we deduce the existence of an element $a \neq 1$ in A. Let B be the subgroup of G^* which is generated by the totality of elements conjugate to a in G^*. It is clear that $1 < B \leq A \leq N^*$. Since the index of $C(N^* < G^*)$ in G^* is finite, the number of elements conjugate to a is finite. Thus the abelian normal subgroup B of G^*

is finitely generated. It follows from Lemma 4 that B is a locally finitely reducible subgroup of G^*. Since $B \neq 1$, this implies the existence of a finitely reducible normal subgroup of G^* which is part of B and hence of N^*; and this completes the proof.

Remark: If N is a normal subgroup of G and $G/C(N < G)$ is finite, then it may be seen that $[G, N]$ is finite too; see Baer [2; Folgerung p. 167]. Our Corollary 1 is a simple consequence of this somewhat deeper fact.

Corollary 2: *If N is a normal subgroup of G, and if there exists to every normal subgroup M of G such that $M < N$ an element $t \neq 1$ in N/M which possesses only a finite number of conjugate elements in G/M, then N is a locally finitely reducible subgroup of G.*

This is an almost immediate consequence of Corollary 1.

Lemma 5: *If the locally finitely reducible subgroup L of G is finitely generated, and if M is a normal subgroup of G such that $M < L$ and L/M is a p-group, then L/M is finite.*

Remark: Whether or not the first hypothesis of this lemma can be omitted, is an open problem [Burnside's Conjecture].

Proof: Assume by way of contradiction that L/M is infinite. Since L is finitely generated, L/M is finitely generated too; and thus we may deduce from Lemma 1 the existence of a normal subgroup N of G with the following properties:

$M \leq N < L$, L/N is infinite,

if H is a normal subgroup of G such that $N < H < L$, then L/H is finite.

Since $N < L$, and since L is a locally finitely reducible subgroup of G, L/N contains a finitely reducible subgroup of G/N. Since finitely reducible subgroups are normal subgroups different from 1, there exists a normal subgroup K of G with the following properties:

$N < K \leq L$ and N is the intersection of all the normal subgroups X of G which satisfy:

$(+) N \leq X < K$ and K/X is a finite minimal normal subgroup of G/X.

Consider now some subgroup X with property $(+)$. Since L/M is a p-group, K/X is a p-group. But K/X is a minimal normal subgroup of G/X and therefore can not possess proper characteristic subgroups. Thus K/X is a finite p-group without proper characteristic subgroups. Hence K/X is abelian and this is equivalent to saying that $[K, K] \leq X$. Since N is the intersection of all these subgroups X, it follows that $[K, K] \leq N$. Hence K/N is abelian.

From $N < K \leq L$ and the choice of N it follows that L/K is finite. Since L is finitely generated, it follows from the Finiteness Principle that K too is finitely generated. Thus K/N is a finitely generated abelian group. Since L/M is a p-group, K/N is a p-group. Consequently K/N is a finitely generated abelian p-group; and this shows that K/N is finite. Since L/K is likewise finite, we see that L/N is finite. But this contradicts our choice of N. Thus we have been led to a contradiction by assuming that L/M is infinite, proving the desired finiteness of L/M.

2. Hypercentrality.

Of the various possible concepts of hypercentrality only two will be investigated. These we introduce in the present section which is devoted to a derivation of their basic properties.

Definition 1: *The normal subgroup N of G is a lower hypercentral subgroup of G, if it meets the following requirement.*

(L) *If the normal subgroup M of the subgroup S of G is finitely generated, and if $1 < M \leq N$, then $[M, S] < M$.*

If in particular G itself is a lower hypercentral subgroup of G, then we term *G a lower nilpotent group.* It is easily deduced from a Theorem of Magnus that every free group is lower nilpotent, though quotient groups of free groups are not always lower nilpotent. On the other hand it is quite obvious that $N \cap S$ is a lower hypercentral subgroup of the subgroup S of G whenever N is a lower hypercentral normal subgroup of G.

From the fact that free groups are lower nilpotent, it follows that lower hypercentrality will generally prove too weak a concept. This concept will accordingly only play a minor rôle in our discussion. The important concept for us is the following one.

Definition 2: *The subgroup N of G is an upper hypercentral subgroup of G, if N meets the following requirement.*

(U) *If M is a normal subgroup of G, and if $M < N$, then $(N/M) \cap Z(G/M) \neq 1$.*

If in particular G itself is an upper hypercentral subgroup of G, then we term *G an upper nilpotent group.* Note that the nonabelian free groups are lower nilpotent, but clearly not upper nilpotent.

Upper hypercentral subgroups are normal.

To prove this consider an upper hypercentral subgroup N of G and form the product P of all the normal subgroups of G which are part of N. It is clear that P is a normal subgroup of G and that $P \leq N$. Assume by way of contradiction that $P < N$. Then it follows from the upper hypercentrality of N that $1 \neq (N/P) \cap Z(G/P) = Q/P$ where Q is a uniquely determined subgroup of G such that $P < Q \leq N$. But subgroups of the center are normal so that Q itself is normal. Hence $Q \leq P < Q$, an impossibility. Consequently $N = P$ is a normal subgroup of G.

Lemma 1: *The following properties of the finitely reducible subgroup N of G are equivalent.*

(i) $N \leq Z(G)$.

(ii) *If the normal subgroup M of G is part of N, then N/M is a lower hypercentral subgroup of G/M.*

(iii) *N is an upper hypercentral subgroup of G.*

(iv) *If M is a normal subgroup of G, if $M < N$ and if N/M is a finite minimal normal subgroup of G, then $N/M \leq Z(G/M)$.*

Proof: It is fairly obvious that conditions (ii) and (iii) are consequences of condition (i).

Assume next the validity of one of the conditions (ii) and (iii). Consider a normal subgroup M of G such that $M < N$ and M/N is a finite minimal normal subgroup of G/M. If condition (ii) is satisfied by N, then N/M is a lower hypercentral subgroup of G/M. Since N/M is finite, N/M is a finitely generated normal subgroup. Thus we may apply (L) and find that $[N/M, G/M] < N/M$. But N/M is a minimal normal subgroup of G/M. Hence $[N/M, G/M] = 1$ or $N/M \leq Z(G/M)$. Thus (iv) is a consequence of (ii). — If (iii) is satisfied by N, then we deduce from $M < N$ and (U) that $(N/M) \cap Z(G/M) \neq 1$. But N/M is minimal. Hence $N/M = (N/M) \cap Z(G/M)$ or $N/M \leq Z(G/M)$. Thus (iv) is a consequence of (iii).

Assume finally the validity of (iv). If M is a normal subgroup of G such that $M < N$ and N/M is a finite minimal normal subgroup of G/M, then it follows from (iv) that $N/M \leq Z(G/M)$. This is equivalent to saying that $[G, N] \leq M$. Hence $[G, N]$ is part of the intersection J of all the normal subgroups X of G such that $X < N$ and N/X is a finite minimal normal subgroup of G. But N is a finitely reducible subgroup of G. Hence $J = 1$ and consequently $[G, N] = 1$. This last statement is equivalent to $N \leq Z(G)$; and this completes the proof.

It is clear that this lemma is our principal reason for introducing the concept of a finitely reducible subgroup. — Essential improvements upon this result will be found in section 4.

Proposition 1: *If N is an upper hypercentral subgroup of G, and if M is a normal subgroup of the subgroup S of G, then $(S \cap N)M/M$ is an upper hypercentral subgroup of S/M.*

Proof: It is clear that $S \cap N$ is a normal subgroup of S and that therefore $(S \cap N)M/M$ is a normal subgroup of S/M. If U is a normal subgroup of S/M such that $U < (S \cap N)M/M$, then there exists a normal subgroup V of S such that $M \leq V < (S \cap N)M$ and $V/M = U$. Consider the set Φ of all the normal subgroups X of G such that $X \leq N$

and $X \cap S \leq V$. This set Φ is not vacuous, since it contains $X = 1$. If the subset Θ of Φ is not vacuous and is ordered by inclusion, then the set theoretical join J of all the subgroups in Θ is clearly a normal subgroup of G which is part of N. If j belongs to $J \cap S$, then there exists a subgroup Y in Θ which contains j. Hence j is in $Y \cap S \leq V$ so that $J \cap S \leq V$. Thus J belongs to Φ; and we have shown that the Maximum Principle of Set Theory may be applied to Φ. Hence there exists a maximal subgroup W in Φ. We note that W is a normal subgroup of G, that $W \leq N$ and $W \cap S \leq V$. Assume by way of contradiction that $W = N$. Then.

$$(S \cap N)M = (S \cap W)M \leq VM = V < (S \cap N)M,$$

an impossibility. Thus $W < N$ and we may apply condition (U). Hence

$$D = (N/W) \cap Z(G/W) \neq 1.$$

Denote by T the uniquely determined normal subgroup of N such that $W < T \leq N$ and $T/W = D$. From the maximality of W it follows that $T \cap S \nleq V$. Thus there exists an element t in $T \cap S$ which does not belong to V. Since t is in T, Wt belongs to $Z(G/W)$. If s is an element in S, then Wt and Ws commute so that the set $[t, S]$ of commutators is part of W. But t is in S so that $[t, S] \leq S \cap W \leq V$ and then $Vt \neq 1$ belongs to $Z(S/V)$. On the other hand t is in $T \cap S$ and Vt therefore in $(T \cap S)V \leq (N \cap S)M$. This shows that

$$[(N \cap S)M/V] \cap Z(S/V) \neq 1$$

or

$$[(N \cap S)M/M]/[V/M] \cap Z[(S/M)/(V/M)] = [(N \cap S)M/M]/U \cap Z[(S/M)/U] \neq 1.$$

Hence condition (U) is satisfied by $(S \cap N)M/M$ and we have shown that $(S \cap N)M/M$ is an upper hypercentral subgroup of S/M.

Remark: We have pointed out before that no statement of this type can be true for lower hypercentral normal subgroups.

Corollary 1: *Every upper hypercentral subgroup is a lower hypercentral normal subgroup.*

Proof: Suppose that N is an upper hypercentral subgroup of G, that M is a normal subgroup of the subgroup S of G, that $1 < M \leq N$ and that M is finitely generated. It is a consequence of Proposition 1 that $S \cap N$ is an upper hypercentral subgroup of S. Form the set Φ of all the normal subgroups X of S such that $X < M$. This set Φ is not vacuous, since $X = 1$ belongs to Φ. If Θ is a non-vacuous subset of Φ and if Θ is ordered by inclusion, then we may form the set theoretical join J of all the subgroups in Θ. It is clear that J is a normal subgroup of S and that $J \leq M$. Assume by way of contradiction that $J = M$. There exists a finite set F of generators of M. To every element f in $F \leq M = J$ there exists a subgroup $X(f)$ in Θ which contains f. Since Θ is ordered by inclusion, there exists among the finitely many subgroups $X(f)$ in Θ a greatest one U. Then we have $F \leq U \leq M$ so that $U = M$ belongs to $\Theta \leq \Phi$, an impossibility. Hence $J < M$ so that J belongs to Φ. Thus we have shown that the Maximum Principle of Set Theory may be applied to Φ; and this shows the existence of a maximal subgroup V in Φ. Then V is a normal subgroup of S such that $V < M$ and M/V is a minimal normal subgroup of S/V. Since $M \leq N \cap S$, and since $N \cap S$ is an upper hypercentral normal subgroup of S, we may apply condition (U). Hence $(M/V) \cap Z(S/V) \neq 1$. But M/V is a minimal normal subgroup of S/V; and thus it follows that $(M/V) \cap Z(S/V) = M/V$ or $M/V \leq Z(S/V)$. This, however, is equivalent to saying that

$$[S, M] \leq V < M.$$

Hence condition (L) is satisfied by N and N is consequently a lower hypercentral normal subgroup of G.

Remark: The converse of this corollary is false, as has been pointed out before.

Proposition 2: *The following properties of the normal subgroup N of G are equivalent.*

(i) *N is an upper hypercentral subgroup of G.*

(ii) *N is a locally finitely reducible subgroup of G; and if H and K are normal subgroups of G such that $H < K \leq N$ and K/H is finite, then K/H is an upper hypercentral subgroup of G/H.*

(iii) *N is a locally finitely reducible subgroup of G; and if H and K are normal subgroups of G such that $H < K \leq N$ and K/H is finite, then K/H is a lower hypercentral subgroup of G/H.*

(iv) *N is a locally finitely reducible subgroup of G; and if H and K are normal subgroups of G such that $H < K \leq N$ and K/H is a finite minimal normal subgroup of G/H, then $K/H \leq Z(G/H)$.*

(v) *If M is a normal subgroup of G and $M < N$, then N/M contains a normal subgroup of G/M which is a finitely generated abelian group different from 1; and if H and K are normal subgroups of G such that $H < K \leq N$ and K/H is a finite minimal normal subgroup of G/H, then $K/H \leq Z(G/H)$.*

(vi) *If M is a normal subgroup of G and $M < N$, then N/M contains an element different from 1 which possesses only a finite number of conjugates in G/M; and if H and K are normal subgroups of G such that $H < K \leq N$ and K/H is a finite minimal normal subgroup of G/H, then $K/H \leq Z(G/H)$.*

(vii) *There exists an ascending central chain of G which terminates in N.*

(viii) *$N \leq Z_\sigma(G)$ for some finite or infinite ordinal σ.*

(ix) *N is part of the upper hypercenter $U(G)$ of G.*

Notational Remark: An ascending central chain of G which terminates in N is a well ordered set of normal subgroups N_σ such that

$$1 = N_0 \leq \ldots \leq N_\sigma \leq N_{\sigma+1} \leq \ldots \leq N_\alpha = N,\ [G, N_{\sigma+1}] \leq N_\sigma,$$

N_σ is the set theoretical join of all the N_ν with $\nu < \sigma$ in case σ is a limit ordinal.

The upper central chain $Z_\sigma = Z_\sigma(G)$ is defined inductively by the rules: $1 = Z_0$, $Z_{\sigma+1}/Z_\sigma = Z[G/Z_\sigma]$ and Z_ϱ is the set theoretical join of all the Z_ν with $\nu < \varrho$ in case ϱ is a limit ordinal. Clearly there exists a [first] ordinal τ such that $Z_\tau(G) = Z_{\tau+1}(G)$. This terminal member of the upper central chain is *the upper hypercenter* $U(G)$ *of* G.

Proof: Assume first that N is an upper hypercentral subgroup of G. If M is a normal subgroup of G such that $M < N$, then $(N/M) \cap Z(G/M)$ is different from 1 and contains therefore a cyclic subgroup different from 1. But subgroups of the center are normal; and cyclic normal subgroups, not 1, are finitely reducible. Thus we see that N is a locally finitely reducible subgroup of G; and the validity of the second part of condition (ii) is an immediate consequence of Proposition 1. Hence (i) implies (ii).

That (ii) implies (iii), may be deduced from Corollary 1; and that (iii) implies (iv), is a fairly immediate consequence of Lemma 1. Assume now the validity of (iv) and consider a normal subgroup M of G such that $M < N$. Then there exists a finitely reducible subgroup V of G/M such that $V \leq N/M$; and it follows from Lemma 1 and the second part of condition (iv) that $V \leq Z(G/M)$. Thus N is an upper hypercentral subgroup of G, proving the equivalence of the first four conditions.

Assume next the validity of the first four conditions. Then the second part of conditions (v) and (vi) is just a restatement of the second part of condition (iv). If M is a normal subgroup of G and if $M < N$, then we have $(N/M) \cap Z(G/M) \neq 1$ [because of (i)]. Thus $(N/M) \cap Z(G/M)$ contains a cyclic subgroup $V \neq 1$. Clearly V is a normal subgroup of G/M and the elements in V possess exactly one conjugate element in G/M. This shows that also the first parts of conditions (v) and (vi) are valid.

If conversely (v) or (vi) is true, then (iv) is valid too, as follows from § 1, Lemma 4 and § 1, Corollary 2. This completes the proof of the equivalence of the first six conditions.

Assume again the validity of (i). Suppose that we have constructed an ascending central chain N_σ of G all of whose terms are contained in N. If this chain has no last term, then its order type is a limit ordinal ν; and we let N_ν be the set theoretical join of all the N_σ [with $\sigma < \nu$]. If the chain has a last term N_ϱ, then $N_\varrho \leq N$. If $N = N_\varrho$, then we have completed our construction. If $N_\varrho < N$, then we deduce from condition (U) that $1 \neq (N/N_\varrho) \cap Z(G/N_\varrho)$. We denote by $N_{\varrho+1}$ the uniquely determined normal subgroup of G which contains N_ϱ and satisfies $N_{\varrho+1}/N_\varrho = (N/N_\varrho) \cap Z(G/N_\varrho)$. Thus we see that there exists an ascending central chain of G which terminates in N. Hence (vii) is a consequence of (i).

If the normal subgroups N_σ form an ascending central chain which terminates in $N[=N_\alpha]$, then one proves inductively that $N_\sigma \leq Z_\sigma(G)$ and now it is clear that (viii) is a consequence of (vii). That (viii) and (ix) are equivalent properties, is immediately clear, if one recalls the definition of the upper hypercenter $U(G)$ of G.

Assume finally the validity of (viii) and consider a normal subgroup M of G such that $M < N$. Then there exist ordinals σ, for instance the "terminal α", such that $Z_\sigma \cap N \not\leq M$. Consequently there exists a first ordinal β such that $Z_\beta \cap N \not\leq M$. From $Z_0 = 1$ we deduce $0 < \beta$. If β were a limit ordinal, then every element w in $Z_\beta \cap N$, but not in M, would belong to some $Z_\nu \cap N$ with $\nu < \beta$, contradicting the minimality of β. Hence $\beta = \gamma + 1$ and it follows from our minimal choice of β that $Z_\gamma \cap N \leq M$. Let $K = M(N \cap Z_\beta)$. Then K is a normal subgroup of G such that $M < K \leq N$. Furthermore

$$[K, G] = [M, G]\,[N \cap Z_\beta, G] \leq M(N \cap [Z_\beta, G]) \leq M(N \cap Z_\gamma) = M.$$

Hence $1 < K/M \leq (N/M) \cap Z(G/M)$. Thus condition (U) is satisfied by N and this completes the proof of the equivalence of our nine conditions.

Corollary 2: *If the normal subgroup N of G is part of an upper hypercentral subgroup of G, then N is an upper hypercentral subgroup of G.*

This is an immediate consequence of the equivalence of conditions (i) and (ix) of Proposition 2.

3. The Commutativity Relations.

We want to show in the present section that elements in hypercentral normal subgroups commute with "many" elements in the group.

Lemma 1: *If x is an element in the lower hypercentral normal subgroup N of G, if g is an element in G and if the orders of x and g are finite and relatively prime, then $xg = gx$.*

Proof: Denote by S the subgroup of G which is generated by x and g; and let $M = [S, S]$ be the commutator subgroup of S. Then S/M is an abelian group which is generated by two elements of finite order so that S/M is finite. Since S is finitely generated, it follows from § 1, Finiteness Principle that M is finitely generated.

Assume now by way of contradiction that $c = x^{-1}g^{-1}xg \neq 1$. Then $M \neq 1$. Since x belongs to N, c belongs to N. Since M is generated by c and elements conjugate to c, M is part of N. Thus we may apply condition (L). Hence $[M, S] < M$.

From $xc = g^{-1}xg$ one deduces easily that

$$g^{-i}xg^i \equiv xc^i \text{ modulo } [M, S],$$

since c is in M and g is in S. If h is the order of g, then it follows that

$$x = g^{-h} x g^h \equiv x c^h \text{ modulo } [M, S].$$

Consequently c^h belongs to $[M, S]$. If k is the order of x, then we see likewise that c^k belongs to $[M, S]$. Since h and k are relatively prime, it follows that c belongs to $[M, S]$. But $M/[M, S]$ is generated by $[M, S]\,c$. Hence $[M, S] = M$. This is the desired contradiction which proves that $c = 1$ or $xg = gx$.

Lemma 2: *If x is an element of finite order m in the upper hypercentral subgroup N of G, and if g is an element in G, then there exists a positive integer n all of whose prime divisors are divisors of m such that $xg^n = g^n x$.*

Proof: Assume first that g is an element of finite order. Then $g = g'\,g'' = g''\,g'$ where the orders of x and g' are relatively prime whereas every prime divisor of the order n of g'' is a divisor of the order m of x. It follows from § 2, Corollary 1 that N is a lower hypercentral normal subgroup of G. Since the orders of x and $g^n = g'^n$ are relatively prime, it follows now from Lemma 1 that $xg^n = g^n x$ as we wanted to show.

Assume next that g is of infinite order and suppose by way of contradiction that $xg^n \neq g^n x$ for every positive n. We form the subgroup S generated by x and g; and we deduce from § 2, Proposition 1 that $N \cap S$ is an upper hypercentral normal subgroup of S. It is clear that x belongs to $N \cap S$ and that therefore $S/(N \cap S)$ is a cyclic group generated by $(N \cap S)g$. Now we form the set Φ of all the normal subgroups X of S such that $X \leq N \cap S$ and such that $xg^n \not\equiv g^n x$ modulo X for every positive n. This set Φ is not vacuous, since it contains $X = 1$. If Θ is a non vacuous subset of Φ which is ordered by inclusion, then we form the set theoretical join J of all the subgroups in Θ. It is clear that J is a normal subgroup of S, and that $J \leq N \cap S$. If $xg^n \equiv g^n x$ modulo J for some positive n, then the commutator $[x, g^n]$ would belong to J and hence to some X in Θ. But then $xg^n \equiv g^n x$ modulo X which is impossible. Thus $xg^n \not\equiv g^n x$ modulo J for every positive n so that J belongs to Φ. Now we have shown that the Maximum Principle of Set Theory may be applied on the set Φ. Thus there exists a maximal subgroup W in Φ.

Since x is in $N \cap S$, we have $xg \equiv g \equiv gx$ modulo $N \cap S$. But $xg \not\equiv gx$ modulo W. Since W is part of $N \cap S$, we have shown that $W < N \cap S$. Since $N \cap S$ is an upper hypercentral normal subgroup of S, it follows now that

$$1 \neq [(N \cap S)/W] \cap Z(S/W) = V/W$$

where V is a uniquely determined normal subgroup of S. From $W < V \leq N \cap S$ and the maximality of W we deduce now the existence of a positive integer n such that $xg^n \equiv g^n x$ modulo V.

It will be convenient to let $s^* = Ws$ for every s in S. Then it follows from our construction of V that $1 < V^* = V/W \leq Z(S^*) = Z(S/W)$; and it follows from our choice of n that $x^*g^{*n} = g^{*n}x^*$ modulo V^*. The commutator $[x^*, g^{*n}]$ belongs therefore to $V^* \leq Z(S^*)$; and it is well known that this implies

$$[x^{*i}, g^{*n}] = [x^*, g^{*n}]^i = [x^*, g^{*ni}] \text{ for every positive } i.$$

If we apply this in particular on the order m of x, then we deduce from $x^m = 1$ successively that $x^{*m} = 1$ and that therefore

$$[x^*, g^{*nm}] = [x^{*m}, g^{*n}] = 1.$$

This is equivalent to saying that $[x, g^{mn}]$ belongs to W. Hence $xg^{mn} \equiv g^{mn}x$ modulo W which is impossible, since W belongs to Φ. Thus we have been led to a contradiction and consequently there exists a minimal positive integer n such that $xg^n = g^n x$.

Since g^n commutes with the two generators x and g of S, g^n belongs to $Z(S)$. Suppose now by way of contradiction that the prime divisor p of n is not a divisor of the order m of x. Let $y = g^{np^{-1}}$. Denote furthermore by C the cyclic subgroup of $Z(S)$ which is generated by g^n. Since g is an element of infinite order, Cy is an element of order p. Thus Cy and Cx are elements of finite relatively prime orders. It follows from § 2, Proposition 1 that $(N \cap S)C/C$ is an upper hypercentral normal subgroup of S/C; and it follows from §2, Corollary 1 that it is lower hypercentral too. Thus it follows from Lemma 1 that Cy and Cx commute. Hence $[x, y]$ belongs to the subgroup C of the center of S; and it follows from the customary arguments that $1 = [x^m, y] = [x, y]^m$. Since g and hence g^n is an element of infinite order, C is an infinite cyclic group; and we deduce $[x, y] = 1$ from $[x, y]^m = 1$. Consequently $xg^{np^{-1}} = g^{np^{-1}}x$ contradicting the minimal choice of n. Thus every prime divisor of n is a divisor of m; and this completes the proof.

Remark: It is impossible to substitute in Lemma 2 for the hypothesis of upper hypercentrality the weaker hypothesis of lower hypercentrality. This may be seen from the following interesting example. Denote by B a direct product of countably many cyclic groups of order p and denote by $b(i)$ for $i = 0, \pm 1, \pm 2, \ldots.$ a basis of B. Then there exists a well defined automorphism σ of B which maps $b(i)$ upon $b(i+1)$ for every i. The group G arises by adjoining to B the automorphism σ. Then G/B is an infinite cyclic group.

If S is a subgroup of G which is not part of B, and if $B \cap S \neq 1$, then $B \cap S$ has always an infinite basis. From this one sees that G is lower nilpotent. On the other hand it is clear that the result of Lemma 2 does not hold in G. It is furthermore worth noting that every proper quotient group of G is upper nilpotent.

Lemma 3: *Suppose that N is an upper hypercentral subgroup of G and that the element g in G induces in N an automorphism of finite order n. If the prime number p is a divisor of n, then N contains elements of order p.*

Proof: Clearly $n = n' p^m$ where $0 < m$ and n' is prime to p. Let $g' = g^{n'}$. Then g' induces in N an automorphism of exact order p^m. Denote by C the totality of elements in N which commute with g'. It is clear that C is a subgroup of N. Since g' induces in N an automorphism of order $p^m \neq 1$, g' does not commute with every element in N so that $C < N$.

Denote now by D the product of all the normal subgroups of G which are contained in C. It is clear that D is a normal subgroup of G and that $D \leq C < N$. We apply condition (U) and find that $1 \neq (N/D) \cap Z(G/D) = T/D$ where T is a uniquely determined normal subgroup of G such that $D < T \leq N$ and $[G, T] \leq D$. It follows from our construction of D that T is not part of C. Hence there exists an element t in T which does not belong to C. It is clear that $[t, g']$ belongs to $[G, T] \leq D \leq C$. Consequently g' commutes with $[t, g']$. Now we deduce from $g'^{-1} tg' = t[t, g']$ that $g'^{-i} tg'^{i} = t[t, g']^{i}$ for every positive i. Since t belongs to $T \leq N$, and since g' induces in N an automorphism of order p^m, we have

$$t = g'^{-p^m} tg'^{p^m} = t[t, g']^{p^m} \text{ or } [t, g']^{p^m} = 1.$$

Since t is not in C, $[t, g'] \neq 1$. Thus $[t, g']$ is an element in $D \leq N$ whose order is a multiple of p; and this shows the existence of elements of order p in N.

4. Subgroups of the Center.

In the light of § 2, Lemma 1 and § 2, Proposition 2 it is important to have criteria for a finite minimal normal subgroup or a finitely reducible subgroup to be part of the center. In this section such criteria will be obtained.

Proposition 1: *The following properties of the finite minimal normal subgroup M of G are equivalent.*

(i) $M \leq Z(G)$.

(ii) *If T is a maximal subgroup of the subgroup S of G, and if $M \cap S \nleq T$, then T is a normal subgroup of S.*

(iii) *If the element x in M is of order a power of p, and if g is an element in G, then there exists an integer $m = m(x, g)$ such that $x g^{p^m} = g^{p^m} x$.*

(iv) *There exists an element $t \neq 1$ in M whose order is a power of p such that to every g in G there exists an integer $m = m(g)$ satisfying $tg^{p^m} = g^{p^m} t$.*

Proof: Assume first that $M \leq Z(G)$. Consider the maximal subgroup T of the subgroup S of G which satisfies $M \cap S \nleq T$. Then $M \cap S \leq Z(G)$ and $T < T(M \cap S) \leq S$. It follows from the maximality of T that $S = (M \cap S)T$. To every element s in S there exist consequently elements u and v in $M \cap S$ and T respectively such that $s = uv$. Since u belongs to $Z(G)$, we find that

$$s^{-1}Ts = v^{-1}u^{-1}Tuv = v^{-1}Tv = T.$$

Hence T is a normal subgroup of S proving that (ii) is a consequence of (i).

Assume next the validity of (ii). If g is an element in G, then we form the subgroup $S = \{M, g\}$ of G. Clearly M is a normal subgroup of S and S/M is a cyclic group. The element g induces in M an automorphism of finite order, since M is a finite normal subgroup of S. If k is the order of this automorphism, then g^k commutes with every element in M. But g^k commutes with g too. Hence g^k belongs to $Z(S)$. This implies in particular that $S/MZ(S)$ is a finite cyclic group. Since M is finite, $MZ(S)/Z(S)$ is likewise finite; and thus we have shown that $S/Z(S)$ is finite.

Consider now a maximal subgoup of $S/Z(S)$. Such a maximal subgroup has the form $T/Z(S)$ where $Z(S) \leq T < S$ and T is a maximal subgroup of S. If T contains M, then T is a normal subgroup of S, since S/M is cyclic and since therefore every subgroup of S/M is normal. If T does not contain $M = M \cap S$, then we apply (ii) to see that T is a normal subgroup of S. Thus we have shown that every maximal subgroup of the finite group $S/Z(S)$ is normal. Now it follows from a Theorem of Wielandt that $S/Z(S)$ is a finite nilpotent group; see, for instance, Zassenhaus [1; p. 108, Satz 13]. But then it follows from § 2, Proposition 2 that S is upper nilpotent. If the element x in M is of order a power of p, then we may now deduce from § 3, Lemma 2 the existence of an integer m such that $xg^{p^m} = g^{p^m}x$; and thus we have shown that (iii) is a consequence of (ii).

It is almost obvious that (iv) is a consequence of (iii), if we remember only that M is finite.

Assume finally the validity of (iv). Then there exists an element $t \neq 1$ in M whose order is a power of p with the property:

(+) To every g in G there exists an integer $m = m(g)$ such that $tg^{p^m} = g^{p^m}t$.

Since M is a finite normal subgroup, every element conjugate to t in G belongs to M and their number is finite. Consequently there exists a finite set F in G such that the set of elements $f^{-1}tf$ with f in F is the totality of elements conjugate to t in G.

Consider now an element g in G and denote by $k(g)$ the maximum of the finitely many integers $m(fgf^{-1})$ for f in F. Since $(f^{-1}tf)g^{p^i} = g^{p^i}(f^{-1}tf)$ and $t(fgf^{-1})^{p^i} = (fgf^{-1})^{p^i}t$ are equivalent properties of the integer i, one sees easily the following fact.

(+ +) To every element g in G there exists an integer $k = k(g)$ such that $sg^{p^k} = g^{p^k}s$ for every element s conjugate to t in G.

Since $t \neq 1$ and since M is a minimal normal subgroup of G, M is generated by the elements conjugate to t in G. Thus it follows from (+ +) that

(*) to every element g in G there exists an integer $k = k(g)$ such that g^{p^k} commutes with every element in M.

Denote now by C the centralizer of M in G. Since M is a finite normal subgroup, C is a normal subgroup of finite index in G and G/C is essentially the same as the group of automorphisms of M which are induced by elements in G. It follows from (*) that to every g in G there exists an integer $k = k(g)$ such that g^{p^k} belongs to C. Consequently G/C is a finite p-group.

If $M \cap C = 1$, then M is isomorphic to the subgroup MC/C of the p-group G/C so that M is a p-group. If $M \cap C \neq 1$, then we infer $M \cap C = M$ from the minimality of M. Consequetly $M \leq C$. Hence M is abelian. Since M is a finite minimal normal subgroup of G, M does not contain proper characteristic subgroup. This implies that M is a primary abelian group. But M contains the element $t \neq 1$ of order a power of p; and thus it follows again that M is a p-group.

Since M is a finite p-group, not 1, and since G/C is essentially a finite p-group of automorphisms of M, it follows by the customary arguments that this p-group of automorphisms possesses fixed elements different from 1. Hence $M \cap Z(G) \neq 1$. It follows from the minimality of M that $M \cap Z(G) = M$ or $M \leq Z(G)$. Hence (i) is a consequence of (iv); and this completes the proof.

Remark: We used in the preceding proof Wielandt's Theorem asserting that a finite group is nilpotent if, and only if, all its maximal subgroups are normal. Thus one may wonder whether condition (ii) may be weakened correspondingly. That this is impossible, may be seen from the following simple *example:*

Consider an odd prime p, a divisor $i \neq 1$ of $p-1$, for instance $i = -1$. Denote by V a cyclic group of order p^2 and by σ the automorphism of V which maps every element in V upon its i-th power. Adjoin to V the automorphism σ. Then we obtain a finite group G. This group G contains V and V^p as normal subgroups and every maximal subgroup of G contains V^p. Hence V^p is a minimal normal subgroup M of G which satisfies [by default] the condition:

If the maximal subgroup S of G does not contain M, then S is a normal subgroup of G.

But $M = V^p$ is not part of the center $Z(G)$ of G, since σ does not leave invariant any element in V except 1.

Corollary: *The following properties of the finitely reducible subgroup M of G are equivalent.*

(i) $M \leq Z(G)$.

(ii) *If T is a maximal subgroup of the subgroup S of G and if $M \cap S \not\leq T$, then T is a normal subgroup of S.*

That (i) implies (ii), is shown by a verbal repetition of the argument in the first step of the proof of the preceding proposition where neither the finiteness nor the minimality of M has been used. — That conversely (i) is a consequence of (ii), is easily deduced from the preceding proposition and § 2, Lemma 1.

5. The Main Criteria for Hypercentrality.

The type of criterion for hypercentrality that we obtain will depend on the extent to which elements of infinite order are admitted.

Theorem 1: *The normal subgroup N of the group G without elements of infinite order is a lower hypercentral normal subgroup of G if, and only if, the following two conditions are satisfied by N and G.*

(a) *If x is an element in N and g an element in G, and if the orders of x and g are relatively prime, then $xg = gx$.*

(b) *If the normal subgroup M of the subgroup S of G is finitely generated, and if $1 < M \leq N$, then there exists a normal subgroup K of S such that $K < M$ and M/K is finite.*

Proof: If N is a lower hypercentral normal subgroup of G, then we deduce the validity of (a) from § 3, Lemma 1. If furthermore M is a normal subgroup of the subgroup S of G, if M is finitely generated and $1 < M \leq N$, then if follows from condition (L) that $[S, M] < M$. But $M/[S, M]$ is a finitely generated abelian group without elements of infinite order and such groups are finite. This proves the necessity of (b).

Assume conversely the validity of conditions (a) and (b). If M is a normal subgroup of the subgroup S of G, if M is finitely generated and $1 < M \leq N$, then we deduce from (b) the existence of a normal subgroup K of S such that $K < M$ and such that M/K is a finite minimal normal subgroup of S/K. Consider an element x^* of order a power of p in M/K and an element s^* in S/K. If s is an element in S such that $s^* = Ks$, then s is of finite order and there exists an integer m such that the order of s^{p^m} is prime to p. Since every element in G is of finite order, there exists an element x of order a power of p in M such that $x^* = Kx$. It follows from condition (a) that $x s^{p^m} = s^{p^m} x$ and this implies clearly that $x^* s^{*p^m} = s^{*p^m} x^*$. Thus condition (iii) of § 4, Proposition 1 is satisfied by the finite minimal normal subgroup M/K of S/K. Consequently $M/K \leq Z(S/K)$ and this is equivalent to saying that $[M, S] \leq K < M$. Thus condition (L) is satisfied so that N is a lower hypercentral normal subgroup of G.

Remark: If Burnside's celebrated conjecture were true, then finitely generated groups without elements of infinite order would be finite and condition (b) could certainly be omitted. Thus indispensability of condition (b) can only be proven by showing that Burnside's conjecture is false.

Theorem 2: *Suppose that the normal subgroup N of G has the following property.*

(C) *To every element x of order a power of p in N and to every element g in G there exists a non-negative integer $m = m(x, g)$ such that $xg^{p^m} = g^{p^m}x$.*

Then the following properties of N are equivalent.

(i) *N is an upper hypercentral subgroup without elements of infinite order.*

(ii) *If M is a normal subgroup of G and $M < N$, then N/M contains a finite normal subgroup, not 1, of G/M.*

(iii) *N is a locally finitely reducible subgroup of G without elements of infinite order.*

(iv) *N is a locally finitely reducible subgroup of G; and if R and S are normal subgroups of G such that $R < S \leq N$, then there exists an element of finite order in S which does not belong to R.*

Remark: It is a consequence of § 3, Lemma 2 that upper hypercentral subgroups of G have the property (C).

Proof: Assume first the validity of (i). Suppose that M is a normal subgroup of G such that $M < N$. Then it follows from condition (U) that $1 \neq Z(G/M) \cap (N/M)$. Every element in this subgroup is of finite order, since every element in N is of finite order; and every subgroup of this subgroup is a normal subgroup of G/M, since subgroups of the center are normal. It is clear now that $Z(G/M) \cap (N/M)$ contains a finite subgroup, not 1, which is a normal subgroup of G/M. Thus (ii) is a consequence of (i).

It is obvious that subgroups with property (ii) are locally finitely reducible; and it is a consequence of § 1, Lemma 2 that subgroups with property (ii) do not contain elements of infinite order. Thus (ii) implies (iii); and it is obvious that (iii) implies (iv).

Assume finally the validity of (iv). Suppose that R and S are normal subgroups of G with the properties:

$R < S \leq N$ and S/R is a finite minimal normal subgroup of G/R.

We deduce from (iv) the existence of elements of finite order in S which do not belong to R; and this implies the existence of an element s of order a power of p in S which does not belong to R. It is clear $s^* = Rs$ is an element different from 1 in S/R whose order is a power of p. If g is an element in G, then there exists by (C) an integer $m = m(g)$ such that s and g^{p^m} commute. Consequently s^* and $(Rg)^{p^m}$ commute too. Thus we have shown that the finite minimal normal subgroup S/R of G/R satisfies condition (iv) of § 4, Proposition 1; and this implies $S/R \leq Z(G/R)$. Now we have shown that condition (iv) of § 2, Proposition 2 is satisfied by N. Consequently N is an upper hypercentral subgroup of G.

There exists clearly a maximal normal subgroup H of G which is part of N and does not contain elements of infinite order. Assume by way of contradiction that $H < N$. Then we deduce from the upper hypercentrality of N that $1 \neq Z(G/H) \cap (N/H) = K/H$ where K is a uniquely determined normal subgroup of G such that $H < K \leq N$. We deduce from (iv) the existence of an element w of finite order in K which does not belong to H. From the normality of H it follows that $\{H, w\}/H$ is a finite group, not 1. Hence $H < \{H, w\}$ and every element in $\{H, w\}$ has finite order. Since every subgroup of the center is normal, $\{H, w\}$ is a normal subgroup of G. This contradicts the maximality of H. Our hypothesis that $H < N$ has led us to a contradiction. Hence $H = N$, proving that every element in N has finite order. Thus (i) is a consequence of (iv), completing the proof.

Lemma 1: *The following two properties of the normal subgroup N of G are equivalent.*

(i) *If T and S are subgroups of G such that $T < (S \cap N)T \leq S$, then the normalizer of T in S is different from T.*

(ii) *If T is a subgroup of G such that $T < NT$, then the normalizer of T in NT is different from T.*

Remark 1: If we let in particular $N = G$, then we obtain an earlier result of the author; see Baer [1; p. 423, Theorem 4.13].

Remark 2: The condition $T < (S \cap N)T$ is equivalent to $S \cap N \nleq T$.

Proof: Assume the validity of (i) and consider a subgroup T of G such that $T < NT$. Let $S = NT$. Then $T < NT = (S \cap N)T \leq S$; and it follows from (i) that the normalizer of T in $S = NT$ is different from T.

Assume next the validity of (ii) and suppose that S, T are subgroups of G satisfying $T < (S \cap N)T \leq S$. We define by transfinite induction an ascending chain of subgroups $R(\sigma)$ as follows: $R(0) = T$, $R(\sigma + 1)$ is the normalizer of $R(\sigma)$ in NT, $R(\nu)$ is the set theoretical join of all the $R(\sigma)$ with $\sigma < \nu$ whenever ν is a limit ordinal. Clearly every $R(\sigma)$ is part of NT and there exists a first ordinal α such that $R(\alpha) = R(\alpha + 1)$. From $T \leq R(\alpha)$ it follows that $NT = NR(\alpha)$; and from (ii) and $R(\alpha) = R(\alpha + 1)$ we infer the impossibility of $R(\alpha) < NT$. Thus $NT = R(\alpha)$.

Now we let $S(\sigma) = S \cap R(\sigma)$. It is clear that

$$S(0) = S \cap T = T \text{ and } T < (S \cap N)T = S \cap NT = S \cap R(\alpha) = S(\alpha).$$

Consequently there exists a first ordinal τ such that $T < S(\tau)$. It is clear that $0 < \tau$ and that τ is not a limit ordinal. Hence $\tau = \varrho + 1$; and it follows from our choice of τ that $S(\varrho) = T$. Since $R(\varrho)$ is a normal subgroup of $R(\varrho + 1)$, $T = S(\varrho)$ is likewise a normal subgroup of $S(\varrho + 1)$. Since $S(\varrho + 1)$ is part of the normalizer of T in $NT \cap S$, we have shown that the normalizer of T in S is different from T. Thus (i) is a consequence of (ii), completing the proof.

It is a consequence of § 2, Proposition 2 that every upper hypercentral subgroup is a locally finitely reducible subgroup. Thus it suffices to characterize the upper hypercentral subgroups among the locally finitely reducible subgroups; and this we are going to do next.

Theorem 3: *The following properties of the locally finitely reducible subgroup N of G are equivalent.*

(i) *N is an upper hypercentral subgroup of G.*

(ii) *If Q and R are normal subgroups of G, $Q < R \leq N$, and if R/Q is a finite minimal normal subgroup of G/Q, then $R/Q \leq Z(G/Q)$.*

(iii) *If T is a subgroup of G such that $T < NT$, then the normalizer of T in NT is different from T.*

(iv) *If T is a maximal subgroup of the subgroup S of G, and if $N \cap S \nleq T$, then T is a normal subgroup of S.*

(v) *If the normal subgroup M of G is part of N, if x is an element of order a power of p in N/M and g an element in G/M, then there exists an integer $m = m(x, g)$ such that $xg^{p^m} = g^{p^m}x$.*

(vi) *If x is in N and g is in G, then $\{x, g\}$ is an upper nilpotent group.*

(vii) *If x is in N and g is in G, then $N \cap \{x, g\}$ is an upper hypercentral subgroup of $\{x, g\}$.*

Proof: We note first that the equivalence of conditions (i) and (ii) is an immediate consequence of § 2, Proposition 2 — we have stated (ii) only, since this condition will be used several times during this proof.

Assume now that N is an upper hypercentral subgroup of G and consider a subgroup T of G such that $T < NT$. Denote by M the product of all the normal subgroups of G which are part of $N \cap T$. Then M is a normal subgroup of G and $M < N$, since otherwise $N \leq T$ contradicting $T < NT$. From the upper hypercentrality we infer now that $1 \neq Z(G/M) \cap (N/M) = H/M$ where H is a uniquely determined normal subgroup of G such that $M < H$. Because of the maximality of M the normal subgroup H which is part of N cannot be part of T. Thus there exists an element h in H which is not in T. It is clear that the set of commutators $[h, t]$ for t in T is part of $M \leq T$, since $[H, G] \leq M$. But then h belongs to the normalizer of T in NT, since h belongs to N. Thus (iii) is a consequence of (*i*).

Since our condition (iii) is identical with property (ii) of Lemma 1, the condition (i) of Lemma 1 may be deduced from our condition (iii). But our condition (iv) is a special case of Lemma 1, (i); and thus we see that (iii) implies (iv).

Assume next the validity of (iv) and consider normal subgroups Q and R of G with the following properties: $Q < R \leq N$ and R/Q is a finite minimal normal subgroup of G/Q. Consider furthermore a maximal subgroup T^* of the subgroup S^* of G/Q such that $S^* \cap R/Q \not\leq T^*$. There exist uniquely determined subgroups S and T of G such that $S^* = S/Q$ and $T^* = T/Q$. Then T is a maximal subgroup of S which does not contain $R \cap S \leq N \cap S$. Hence T does not contain $N \cap S$ and it follows from (iv) that T is a normal subgroup of S. Consequently T^* is a normal subgroup of S^*. Thus we see that the finite minimal normal subgroup R/Q of G/Q satisfies condition (ii) of § 4, Proposition 1. Hence $R/Q \leq Z(G/Q)$. Thus (ii) is a consequence of (iv) and we have verified the equivalence of the first four conditions.

If N is an upper hypercentral subgroup of G, and if the normal subgroup M of G is part of N, then it follows from § 2, Proposition 1 that N/M is an upper hypercentral subgroup of G/M. Now it follows from § 3, Lemma 2 that (v) is a consequence of (i). — If conversely (v) is satisfied by N, then we consider normal subgroups Q and R of G with the following properties: $Q < R \leq N$ and R/Q is a finite minimal normal subgroup of G/Q. The condition (iii) of § 4, Proposition 1 is satisfied by R/Q as a consequence of our present condition (v). Hence $R/Q \leq Z(G/Q)$. Thus (ii) is a consequence of (v); and this completes the proof of the equivalence of the first five conditions.

If x is an element in N and g is an element in G, then $\{x, g\}/[N \cap \{x, g\}]$ is a cyclic group. It follows therefore from § 2, Proposition 2 that $\{x, g\}$ is an upper nilpotent group if, and only if, $N \cap \{x, g\}$ is an upper hypercentral subgroup of $\{x, g\}$. This proves the equivalence of conditions (vi) and (vii).

It is a consequence of § 2, Proposition 1 that (vii) is a consequence of (i). Assume conversely the validity of (vii). If the normal subgroup M of G is part of N, if x^* is an element of order a power of p in N/M and g^* is an element in G/M, then we select elements x and g such that $x^* = Mx$ and $g^* = Mg$. Let $S = \{x, g\}$, $H = M \cap S$ and $K = N \cap S$. It follows from (vii) that K is an upper hypercentral normal subgroup of S. If p^k is the order of x^*, then x^{p^k} belongs to $M \cap S$. Hence the order of $Hx = x^{**}$ is likewise a power of p. Because of the upper hypercentrality of the subgroup K of S we may apply condition (v) on the element x^{**} in K/H and $g^{**} = Hg$ in S/H. Consequently there exists an integer i such that x^{**} and g^{**p^i} commute. This is equivalent to saying that the commutator $[x, g^{p^i}]$ belongs to $H \leq M$. But then $x^* = Mx$ and $g^{*p^i} = Mg^{p^i}$ commute too. Thus we have shown that our condition (v) is satisfied by the subgroup N of G. But (i) and (v) have been shown to be equivalent. Hence (i) is a consequence of (vii); and this completes the proof.

Remark: Quite a few of the implications of the preceding proof did not actually involve the hypothesis that N is a locally finitely reducible subgroup of G. But this condition is indipensable for the validity of this theorem, since there exist infinite p-groups whose center is 1 [though finitely many elements generate finite subgroups]; see, for instance, Baer [1; p. 412, Example 3.4]. This shows that (i) is not a consequence of conditions (v) to (vii) if we omit the hypothesis that N be locally finitely reducible. — It should be noted furthermore that this theorem generalizes well known characterizations of upper nilpotent groups, like those of Baer [1], Černikov [1, 2] and Schmidt [1] in two directions, since it not only characterizes the upper hypercentral subgroups instead of the upper nilpotent groups, but also substitutes for the customary solubility hypothesis the weaker hypothesis of local finite reducibility.

As a companionpiece to the preceding theorem we are now going to give a characterization of upper hypercentral subgroups which does not involve local finite reducibility and which may be related to chain conditions.

Theorem 4: *The normal subgroup N of G is an upper hypercentral subgroup of G if, and only if, the following conditions are satisfied.*

(a) *If S is a subgroup of G such that $S < NS$, then the normalizer of S in NS is different from S.*

(b) *If M is a normal subgroup of G such that $M < N$, then N/M contains an element, not 1, with maximal centralizer in G/M.*

(c) *If M is a normal subgroup of G such that $M < N$, and if S is a subgroup of G/M such that $(N/M)C(N/M < G/M) \leq S < G/M$ and $(N/M) \cap Z(S) \neq 1$, then there exists a subgroup $T \neq 1$ of $(N/M) \cap Z(S)$ such that the normalizer of T in G/M is different from S.*

Proof: Assume first that N is an upper hypercentral normal subgroup of G. Then we deduce the necessity of condition (a) from Theorem 3. Consider next a normal subgroup M of G such that $M < N$. Then N/M is an upper hypercentral subgroup of G/M [§ 2, Proposition 1]. This implies that $Z(G/M) \cap (N/M) \neq 1$. Thus there exists an element $z \neq 1$ in $Z(G/M) \cap (N/M)$. The centralizer of z in G/M is naturally G/M and is therefore maximal. This proves the necessity of (b). Next consider a subgroup S of G/M satisfying:

$$(N/M)\, C(N/M < G/M) \leq S < G/M \text{ and } (N/M) \cap Z(S) \neq 1.$$

Then we deduce from the upper hypercentrality of N that

$$1 \neq V = Z(G/M) \cap (N/M) \leq C(N/M < G/M) \leq S.$$

Clearly $V \leq (N/M) \cap Z(S)$ and the normalizer of V equals $G/M \neq S$. This proves the necessity of (c).

Assume conversely the validity of conditions (a) to (c) and consider a normal subgroup M of G such that $M < N$. It is clear that the normal subgroup $N^* = N/M$ of $G^* = G/M$ likewise satisfies conditions (a) to (c); and we note that, as a consequence of Lemma 1, condition (a) is equivalent with the following condition:

(a′) If T and S are subgroups of G^* such that $T < (S \cap N^*)T \leq S$, then the normalizer of T in S is different from T.

From $N^* \neq 1$ and (b) we deduce now the existence of an element $z \neq 1$ in N^* whose centralizer $S = C(z < G^*)$ is maximal. Assume by way of contradiction that $S \neq G^*$. Then we show the following fact.

(c′) There exists a subgroup $V \neq 1$ of $N^* \cap Z(S)$ such that the normalizer of V in G^* is different from S.

In the proof of (c′) we distinguish two cases.

Case 1: $N^* \leq S$.

Since z belongs to N^*, $C(N^* < G^*) \leq C(z < G^*)$. Thus S satisfies the conditions $N^* C(N^* < G^*) \leq S < G^*$. It is furthermore clear that $z \neq 1$ belongs to $N^* \cap Z(S)$. Hence $N^* \cap Z(S) \neq 1$. Thus we may apply condition (c), proving the validity of (c′) in this case.

Case 2: $N^* \nleq S$.

Then $S < N^*S$ and it follows from (a) that the normalizer T of S is different from S. Since $Z(S)$ is a characteristic subgroup of S, it is clear that $Z(S)$ is a normal subgroup of T. Hence T is part of the normalizer of $N^* \cap Z(S) = V$. Thus the normalizer of the subgroup V of $N^* \cap Z(S)$ is different from S. But $z \neq 1$ belongs to V so that $V \neq 1$; and thus we see again the validity of (c′).

Consider now some subgroup V such that $1 < V \leq N^* \cap Z(S)$ and such that the normalizer of V is different from S. Is clear that S is part of the normalizer of V. Hence there exists an element t in G^* which does not belong to S, but which does belong to the normalizer of V. We let $T = \{t\}$ and $R = \{V, t\}$. It is clear that V is a normal subgroup of R, since t belongs to the normalizer of V; and that consequently $R = VT$ and R/V is cyclic. We prove next that

(d) V contains an element $v \neq 1$ which commutes with t.

This is certainly true, if $V \leq T$. Hence we may assume that $V \nleq T$. Since $V \leq R \cap N^*$, it follows that $R \cap N^*$ is not part of T either; and this is equivalent to saying that $T < (R \cap N^*)T \leq R$. Thus we may apply condition (a'). Consequently the normalizer of T in R

is different from T. There exists therefore an element r in R which belongs to the normalizer of T, but which does not belong to T. From $R = VT$ we deduce the existence of elements v and v' in V and T respectively such that $r = vv'$. Since r does not belong to T, we have $v \neq 1$; and since r and v' both belong to normalizer of T, v belongs to the normalizer of T. Since v is an element in the normal subgroup V of R, and since v belongs to the normalizer of $T = \{t\}$, the commutator $c = [v, t]$ belongs to $V \cap T$. If $c = 1$, then $vt = tv$; and $v \neq 1$ is the desired element, not 1, in V which commutes with t. If $c \neq 1$, then c commutes with t as an element in $T = \{t\}$; and c is the desired element, not 1, in V which commutes with t.

Consider now any element $v \neq 1$ in V which commutes with t. Then t certainly belongs to $C(v < G^*)$. Since $V \leq N^* \cap Z(S)$, v belongs to N^* and S is part of $C(v < G^*)$. Hence v is an element, not 1, in N^* whose centralizer in G^* is greater than $S = C(z < G^*)$. But this contradicts our maximal choice of $C(z < G^*)$. Our assumption that $S < G^*$ has consequently led us to a contradiction. Hence G^* is the centralizer of the element $z \neq 1$ in N^*. Thus z is in the center of G^*; and we have shown that $N^* \cap Z(G^*) \neq 1$. This completes the proof of the fact that N is an upper hypercentral normal subgroup of G.

Corollary 1: *If the maximum condition or the minimum condition is satisfied by the subgroups of G, then condition (b) of Theorem 4 is satisfied by the normal subgroups of G.*

Proof: Consider normal subgroups M and N of G such that $M < N$. If the maximum condition is satisfied by the subgroups of G, then the maximum condition is satisfied by the subgroups of G/M. Consequently there exists among the elements $z \neq 1$ in N/M one with maximal centralizer $C(z < G/M)$. — If, however, the minimum condition is satisfied by the subgroups of G, then the minimum condition is satisfied by the subgroups of G/M. Consequently there exists among the elements $z \neq 1$ in N/M one with minimal $Z[C(z < G/M)]$. Suppose now that $z \neq 1$ is an element in N/M such that $Z[(C(z < G/M)]$ is minimal; and suppose that $w \neq 1$ is an element in N/M such that $C(z < G/M) \leq$ $\leq C(w < G/M)$. If t is an element in $Z[C(w < G/M)]$, then t commutes with z, since z belongs to $C(z < G/M)$ and therefore to $C(w < G/M)$. Hence t belongs to $C(z < G/M)$ and to the centralizer of $C(z < G/M)$. Thus t belongs to $Z[C(z < G/M)]$ so that $Z[C(w < G/M)] \leq Z[C(z < G/M)]$. It follows from the minimal choice of $Z[C(z < G/M)]$ that $Z[C(w < G/M)] = Z[C(z < G/M)]$. This implies in particular that z belongs to $Z[C(w < G/M)]$. Hence $C(w < G/M) \leq C(z < G/M) \leq C(w < G/M)$ or $C(z < G/M) =$ $= C(w < G/M)$, proving the desired maximality of $C(z < G/M)$.

Remark: If $N = G$, then condition (c) is satisfied by default; and thus it follows from Theorem 4 that *the group G is upper nilpotent if, and only if,*

(a) *every subgroup $S \neq G$ of G has a normalizer different from S and*

(b) *every quotient group, not 1, of G contains an element, not 1, with maximal centralizer.*

If the maximum or the minimum condition is satisfied by the subgroups of G, then it follows from Corollary 1 that condition (b) may be omitted. Thus we see that this result contains as special cases theorems due to Hirsch [1], Černikov [1] and O. Schmidt [1].

Corollary 2: *If N is a normal subgroup of G such that $G/NC(N<G)$ is upper nilpotent, then condition* (c) *of Theorem* 4 *is satisfied by N.*

Proof: Suppose that M is a normal subgroup of G and that $M<N$. Then $NC(N<G)/M \leq (N/M)C(N/M<G/M)$; and hence it follows from our hypothesis and § 2, Proposition 1 that $(G/M)/[(N/M)C(N/M<G/M)]$ is upper nilpotent. Suppose now that S is a subgroup of G/M which satisfies $(N/M)C(N/M<G/M) \leq S < G/M$. Then it follows from Theorem 4 that the subgroup $S/[(N/M)C(N/M<G/M)]$ of the upper nilpotent group $(G/M)/[(N/M)C(N/M<G/M)]$ is different from its normalizer. Hence the normalizer T of S in G/M is greater than S. But T is likewise part of the normalizer of the characteristic subgroup $Z(S)$ of S; and T is consequently part of the normalizer of $(N/M) \cap Z(S)$. Now it is clear that condition (c) of Theorem 4 is satisfied by N.

Remark: It is easily deduced from Theorem 3, (v) that $G/C(N<G)$ is a finite nilpotent group whenever N is a finite upper hypercentral normal subgroup of G; and in this case $G/NC(N<G)$ would be a finite nilpotent group too. But in general this is not the case as may be seen from the following example.

Denote by A a direct product of a countable infinity of cyclic groups of the same prime number order p; and denote by $a_1, \ldots, a_i, a_{i+1}, \ldots$ a basis of A Then a group is formed by the totality Φ of automorphisms σ of A which satisfy the following condition:

$$a_i^{\sigma} a_i^{-1} \text{ belongs to } \{a_1, \ldots, a_{i-1}\} \text{ for every } i.$$

The group G arises from A by adjunction of this group Φ of automorphisms of A.

It is clear that A is a normal subgroup of G and one sees easily that $A \cap Z_i(G) = \{a_1, \ldots, a_i\}$. This shows in particular that A is an upper hypercentral subgroup of G. One verifies that $A = C(A<G)$. Hence Φ is isomorphic to $G/AC(A<G)$. But $Z(\Phi) = 1$ proving that Φ is not upper nilpotent.

6. Hypercentral Subgroups without Elements of Infinite Order.

We begin by characterizing the finite upper hypercentral subgroups among the normal subgroups.

Theorem 1: *The following properties of the normal subgroup N of G are equivalent.*

(i) *N is a finite upper hypercentral subgroup of G.*

(ii) *N is a finite lower hypercentral subgroup of G.*

(iii) *N is a finitely generated lower hypercentral subgroup of G and the minimum condition is satisfied by the normal subgroups of G which are part of N.*

(iv) *If the maximal subgroup T of the subgroup S of G does not contain $N \cap S$, then T is a normal subgroup of S; N is finitely generated and the minimum condition is satisfied by the normal subgroups of G which are part of N.*

(v) *N is an upper hypercentral subgroup which is generated by a finite number of elements of finite order.*

(vi) *If x is an element of order a power of p in N and if g is an element in G, then there exists an integer $m = m(x, g)$ such that $xg^{p^m} = g^{p^m} x$; N is a locally finitely reducible subgroup of G which is generated by a finite number of elements of finite order.*

(vii) *If x is an element of order a power of p in N and if g is an element in G, then there exists an integer $m = m(x, g)$ such that $xg^{p^m} = g^{p^m} x$; N is generated by a finite number of elements of finite order; if M is a normal subgroup of G such that $M < N$ and N/M is a p-group, then N/M is finite.*

(viii) *If the order of the element x in N is a power of p, then there exists an integer $m = m(x)$ such that $xg^{p^m} = g^{p^m} x$ for every g in G; N is generated by a finite number of elements of finite order; N/N^n is finite for every prime power n.*

Remark: Note that the first of the three conditions (vii) is weaker than the first condition (viii) whereas the third condition (vii) is stronger than the third condition (viii).

Proof: It is a consequence of § 2, Proposition 1 that (i) implies (ii); and it is obvious that (ii) implies (iii).

Assume next the validity of (iii). Then there exists a minimal normal subgroup M of G such that $M \leq N$ and such that N/M is a finite upper hypercentral subgroup of G/M. Assume by way of contradiction that $M \neq 1$. Since N is finitely generated and N/M is finite, M is finitely generated [§ 1, Finiteness Principle]. Since N is a lower hypercentral subgroup of G, it follows now that $[M, G] < M$. Since M is finitely generated, $M/[M, G]$ is a finitely generated subgroup of $Z(G/[M, G])$. From $M/[M, G] \neq 1$ we deduce now the existence of a normal subgroup H of G such that $[M, G] \leq H < M$ and M/H is finite. N/H is finite, since N/M and M/H are finite. N/H is an upper hypercentral subgroup of G/H, since N/M is an upper hypercentral subgroup of G/M and $M/H \leq Z(G/H)$ [§ 2, Proposition 2]. Thus $H < M$ contradicts the minimal choice of M. Our assumption $M \neq 1$ has led us to a contradiction. Consequently $M = 1$ and $N = N/M$ is a finite upper hypercentral subgroup of $G = G/M$. Hence (i) is a consequence of (iii), completing the proof of the equivalence of the first three conditions.

It is a consequence of § 5, Theorem 3 that (i) implies (iv). Assume conversely the validity of (iv). Then there exists a minimal normal subgroup M of G such that $M \leq N$ and N/M is finite. Since N is finitely generated and N/M is finite, N/M is finitely generated [§ 1, Finiteness Principle]. Assume by way of contradiction that $M \neq 1$. Since M is a finitely generated group, not 1, we may deduce from the Maximum Principle of Set Theory the existence of a maximal subgroup T of M. Since T does not contain $N \cap M = M$, it follows from the first condition (iv) that T is a normal subgroup of M. But then M/T is a group without proper subgroups. Hence M/T is a cyclic group of order a prime. Consequently $[M, M] \leq T < M$. Since $M/[M, M]$ is a finitely generated abelian group, there exists a prime p such that $M/M^p[M, M]$ is a finite abelian group different from 1. Since $M^p[M, M]$ is a characteristic subgroup of the normal subgroup M of G, it is a normal subgroup of G. Since N/M and $M/M^p[M, M]$ are both finite, $N/M^p[M, M]$ is finite too. This contradicts the minimality of M, since $M^p[M, M] < M$. Hence $M = 1$ proving the finiteness of N. This implies in particular that N is a locally finitely reducible subgroup of G. Condition (iv) implies furthermore the validity of condition (iii) of § 5, Theorem 3. Consequently N is a finite upper hypercentral subgroup of G so that (i) is a consequence of (iv).

It is clear that (i) implies (v); and it is a consequence of § 2, Proposition 2 and § 5, Theorem 3 that (v) implies (vi). That (vi) implies (vii) may be deduced from § 1, Lemma 5.

Assume now the validity of (vii) and suppose by way of contradiction that N is infinite. Since N is finitely generated, we may deduce from § 1, Lemma 1 the existence of a normal subgroup M of G with the following properties:

$$\text{(a)} \begin{cases} M < N;\ N/M \text{ is infinite;} \\ \text{if } H \text{ is a normal subgroup of } G \text{ and } M < H < N, \text{ then } N/H \text{ is finite.} \end{cases}$$

Next we prove the following fact.

(b) $Z(N/M) = 1$.

If this were false, then we would deduce from (a) the finiteness of $(N/M)/Z(N/M)$, since the center of N/M is a normal subgroup of G/M as a characteristic subgroup of a normal subgroup of G/M. The finiteness of the central quotient group implies the finiteness of the commutator subgroup $[N/M, N/M]$; see, for instance, Baer [2; p. 163, Zusatz]. Since N is generated by a finite number of elements of finite order, the same is true of the abelian group $(N/M)/[N/M, N/M]$ and this abelian group is consequently finite. Thus $(N/M)/[N/M, N/M]$ and $[N/M, N/M]$ are both finite; and this implies the finiteness of N/M, contradicting (a). Hence we have been led to a contradiction by assuming that $Z(N/M) \neq 1$; and this proves (b).

The following notations will prove convenient for the purposes of this proof. The element x in N/M is a *proper* element, if $x \neq 1$ and if there exists an element y of prime power order in N such that $x = My$. It is clear that such a proper element has order a power of p; and this prime number p we shall call the *characteristic* of the proper element x.

(c) N/M is generated by its proper elements.

This is clear once we remember that N is generated by its elements of finite order and that N is therefore generated by its elements of prime power order.

(d) All the proper elements in N/M have the same characteristic p.

Since N/M is infinite [by (a)], there exist proper elements [by (*c*)]. Hence there exists some prime number p such that there exist proper elements of characteristic p. Assume now by way of contradiction the existence of proper elements of characteristic different from p. Then we denote by P the subgroup of N/M which is generated by the proper elements of characteristic p and denote by Q the subgroup of N/M which is generated by the proper elements of characteristic different from p. Both these subgroups P and Q are different from 1. Clearly $P = R/M$ where the uniquely determined subgroup R of N is obtained by adjoining to M all the elements of order a power of p. Since M and N are normal subgroups of G, R too is a normal subgroup of G and hence P is a normal subgroup of G/M. Similarly we see that Q is a normal subgroup of G/M; and it follows from (c) that $N/M = PQ$ and from (a) that $(N/M)/P$ and $(N/M)/Q$ are both finite.

If x and y are elements of prime power order in N, and if the orders of x and y are relatively prime, then it follows from the first condition (vii) that $xy = yx$. This implies that proper elements in N/M commute whenever their characteristics are different. Thus each of the proper elements generating P commutes with each of the proper elements generating Q: and this implies that every element in P commutes with every element in Q. From $N/M = PQ$ we deduce now that $P \cap Q \leq Z(N/M)$. But it follows from (b) that $Z(N/M) = 1$; and thus we have shown that N/M is the direct product of P and Q.

Thus Q is isomorphic to the finite group $(N/M)/P$ and P is isomorphic to the finite group $(N/M)/Q$. Hence P and Q and their product N/M are finite, contradicting (a). Our assumption of the existence of proper elements of characteristic different from p has led us to a contradiction. This completes the proof of (d).

(e) N/M is a p-group.

Since N is generated by a finite number of elements of finite order, N is also generated by a finite number of elements of prime power order. It follows from (d) that every element in N whose order is a power of a prime, not p, belongs to M. Consequently N/M is generated by a finite number of proper elements of characteristic p. Hence there exists a finite set S of elements of order a power of p in N such that $N = \{M, S\}$.

Every element in N/M has the form Mg with g in N. If s is an element in S, then we deduce from the first condition (vii) the existence of an integer $m(s, g)$ such that $sg^{p^{m(s,g)}} = g^{p^{m(s,g)}}s$. Let $m(g)$ be the maximum of the finitely many integers $m(s, g)$ for s in S. Then we have clearly

$$sg^{p^{m(g)}} = g^{p^{m(g)}}s \text{ for every } s \text{ in } S.$$

The element $[Mg]^{p^{m(g)}}$ commutes therefore with every Ms with s in S. Since N/M is generated by these elements Ms, it follows that $[Mg]^{p^{m(g)}}$ belongs to the center $Z(N/M)$ of N/M which is 1 by (b). Hence $[Mg]^{p^{m(g)}} = 1$ and we have shown that the order of every element in N/M is a power of p. This proves (e).

From (e) and the last condition (vii) we deduce the finiteness of N/M which contradicts (a). Thus we have been led to a contradiction by assuming that N is infinite. Hence N is finite.

The first condition (vii) assures us of the validity of condition (C) of § 5, Theorem 2 and the finiteness of N implies the validity of condition (iii) of § 5, Theorem 2. Hence N is a finite upper hypercentral group; and we have shown that (vii) implies (i).

Assume again the validity of (i). If C denotes the centralizer of N in G, then C is a normal subgroup of G and G/C is essentially the same as the group of automorphisms of N which are induced in N by elements in G. Since N is finite, every group of automorphisms of N is finite. Thus G/C is finite; and this implies the existence of a positive integer k such that g^k belongs to C for every g in G. If the order of the element x in N is a power of p, and if g is in G, then we deduce from (vii) [which is equivalent to (i)] the existence of a minimal integer m such that $xg^{p^m} = g^{p^m}x$. One verifies easily that p^m is a divisor of the order of the automorphism of N which g induces in N; and this implies that p^m is a divisor of k. If we denote by $p^{m(p)}$ the highest power of p which divides k, then it follows from what we have shown just now that $xg^{p^{m(p)}} = g^{p^{m(p)}}x$; and now it is clear that (viii) is a consequence of (i).

Assume conversely the validity of (viii). There exists a finite set F of elements of finite order in N which generates N; and we may assume without loss in generality that every element in F is of prime power order. We denote by P the finite set of primes which occur in the orders of the elements in F.

If p is a prime in P, then we denote by $F(p)$ the finite and not vacuous subset of those elements in F whose order is a power of p. If x belongs to $F(p)$, then there exists by (viii) an integer $m(x)$ such that $xg^{p^{m(x)}} = g^{p^{m(x)}} x$ for every g in G. Denote by $m(p)$ the maximum of the finitely many integers $m(x)$ for x in $F(p)$. Then

$(+)\, xg^{p^{m(p)}} = g^{p^{m(p)}} x$ for every x in $F(p)$ and every g in G.

We denote by J the intersection of the subgroups $N^{p^{m(p)}}$ for p in P. It is clear that $N^{p^{m(p)}}$ is a normal subgroup of G; and it follows from (viii) that $N/N^{p^{m(p)}}$ is finite. But the intersection of a finite number of normal subgroups of finite index is a normal subgroup of finite index. Hence J is a normal subgroup of G and G/J is finite.

Consider an element y in J. If x is an element in F, then x belongs to some $F(p)$. Since y belongs to $J \leq N^{p^{m(p)}}$, and since x commutes by $(+)$ with every element in $N^{p^{m(p)}}$, we have $xy = yx$. Thus y commutes with every element in F. Since N is generated by F, y belongs to $Z(N)$. Thus we have shown that $J \leq Z(N)$. Since N/J is finite, $N/Z(N)$ is finite too. But the finiteness of the central quotient group implies the finiteness of the commutator subgroup; see, for instance, Baer [2; p. 163, Zusatz]. Hence $[N, N]$ is finite. Since N is generated by a finite number of elements of finite order, the abelian group $N/[N, N]$ is generated by a finite number of elements of finite order. Such abelian groups are finite. Hence $N/[N,N]$ and $[N, N]$ are both finite. Thus N is finite. Now it is clear that (viii) implies (vii) which condition has been shown to be equivalent to (i). Hence all conditions (i) to (viii) are equivalent.

If S is a subset of the group G, then we denote by $F(S)$ *the totality of elements of finite order in the set* S. If G is a group, then $F(G)$ may or may not be a subgroup of G. But $F(G)$ will certainly be a characteristic subset and it will generate a characteristic subgroup.

Theorem 2: *The following properties of the normal subgroup N of G are equivalent.*

(i) *$F(N)$ is an upper hypercentral subgroup of G.*

(ii) *$F(N)$ is contained in an upper hypercentral subgroup of G.*

(iii) *If x is an element of order a power of p in N and g is an element in G, then there exists an integer $m = m(x, g)$ such that $xg^{p^m} = g^{p^m} x$; finite subsets of $F(N)$ generate finite subgroups and $\{F(N)\}$ is a locally finitely reducible subgroup of G.*

(iv) *If x is an element of order a power of p in N and g is an element in G, then there exists an integer $m = m(x, g)$ such that $xg^{p^m} = g^{p^m} x$; $\{F(N)\}$ is a locally finitely reducible subgroup of G and the subgroups generated by finitely many elements of finite order in N are locally finitely reducible* [*in themselves*].

Proof: It is clear that (i) implies (ii). If $F(N)$ is contained in an upper hypercentral subgroup of G, then $H = \{F(N)\}$ is an upper hypercentral subgroup of G[by § 2, Corollary 2]. We deduce the validity of the first and third condition (iii) from § 5, Theorem 3. It follows from § 2, Proposition 1 that every subgroup of H is upper nilpotent; and now it follows from Theorem 1 that finitely many elements in $F(N)$ generate a finite subgroup. Thus (iii) is a consequence of (ii) and it is obvious that (iii) implies (iv).

That (iv) implies (iii), may be deduced from Theorem 1. If (iii) is true, then it is clear that $F(N)$ is a subgroup and hence a normal subgroup of G. The normal subgroup $F(N)$ of G is a locally finitely reducible subgroup of G which satisfies condition (C) of § 5, Theorem 2. Hence $F(N)$ is an upper hypercentral subgroup of G; and this completes the proof.

Corollary 1: *G is an upper nilpotent group without elements of infinite order if, and only if, G is the direct product of primary locally finitely reducible subgroups.*

Remark: This is an improvement on Baer [1; p. 409, Theorem 3.2].

Proof: Assume first that G is an upper nilpotent group without elements of infinite order. Consider elements x and y of order a power of p in G; and denote by S the subgroup generated by x and y. It follows from § 2, Proposition 1 that S is upper nilpotent; and now it follows from Theorem 1, (v) that S is a finite nilpotent group. But a finite nilpotent group is the direct product of its primary components; see, for instance, Zassenhaus [1; p. 107, Satz 11]. Since S is generated by elements of order a power of p, S is a p-group. Hence xy^{-1} too has order a power of p; and we have shown that

the totality G_p of elements of order a power of p in G is a subgroup of G.

Since G does not contain elements of infinite order, it is evident that G is the direct product of its primary components G_p. Normal subgroups of upper nilpotent groups are upper hypercentral subgroups [§ 2, Proposition 2]. Hence every G_p is an upper hypercentral normal subgroup of G and as such it is locally finitely reducible [§ 2, Proposition 2].

Assume conversely that G is the direct product of primary locally finitely reducible subgroups G_σ. Then it is clear that G does not contain elements of infinite order. Consider now some definite G_σ. This is a p-group. If g is an element in G, then $g = g' g''$ where g' belongs to G_σ and g'' belongs to $\prod_{\tau \neq \sigma} G_\tau$. It is clear that g' and g'' commute and that g'' commutes with every element in G_σ. But g' has order p^n; and so $g^{p^n} = g''^{p^n}$ commutes with every element in G_σ. Thus we see that G_σ satisfies conditions (C) and (iii) of § 5, Theorem 2, proving that G_σ is an upper hypercentral subgroup of G. But then if follows from § 2, Proposition 2 that the product G of all the G_σ is an upper hypercentral subgroup of G. Hence G is an upper nilpotent group as we wanted to show.

Theorem 3: *If the order of the element t in G is a power of p, then the following properties of t [and G] are equivalent.*

(i) *t belongs to the upper hypercenter $U(G)$ of G.*

(ii) $\{t^G\}$ *is an upper hypercentral subgroup of* G.

(iii) *To every* g *in* G *there exists an integer* $m(g)$ *such that* $tg^{p^{m(g)}} = g^{p^{m(g)}}t$; *and to every normal subgroup* M *of* G *such that* $M < \{t^G\}$ *there exists a finite normal subgroup, not 1, of* G/M *which is part of* $\{t^G\}/M$.

(iv) *To every* g *in* G *there exists an integer* $m(g)$ *such that* $tg^{p^{m(g)}} = g^{p^{m(g)}}t$; *finitely many elements in* t^G *generate a locally finitely reducible group;* $\{t^G\}$ *is a locally finitely reducible subgroup of* G.

Here as always we denote by t^G the totality of elements in G which are conjugate to t in G.

Proof: It is an obvious consequence of § 2, Proposition 2 that (i) and (ii) are equivalent. Assume now the validity of (ii). Then it follows from Theorem 2 that every element in $\{t^G\}$ is of finite order; and it follows from Theorem 1, § 3, Lemma 2 and § 5, Theorem 2 that (iii) is a consequence of (ii). If (iii) is true, then it follows from § 1, Lemma 2 that finite subsets of $\{t^G\}$ generate finite subgroups; and now it is clear that (iii) implies (iv).

Assume finally the validity of (iv). If s is an element in t^G, then there exists an element x in G such that $s = x^{-1}tx$. If g is an element in G, then let $g' = xgx^{-1}$. Then it follows from (iv) that $tg'^{p^{m(g')}} = g'^{p^{m(g')}}t$. Hence

$$(+)\quad sg^{p^{m(g')}} = x^{-1}tg'^{p^{m(g')}}x = x^{-1}g'^{p^{m(g')}}tx = g^{p^{m(g')}}s.$$

Consider now a finite subset T of t^G and denote by S the subgroup generated by T. If w is an element in S and s is an element in T, then we infer from $(+)$ the existence of an integer $n(s)$ such that $sw^{p^{n(s)}} = w^{p^{n(s)}}s$. Denote by n the maximum of the finitely many integers $n(s)$ for s in T. Then $sw^{p^n} = w^{p^n}s$ for every s in T. Since w^{p^n} commutes with every element in the set T generating S, w^{p^n} belongs to the center $Z(S)$ of S. Hence $S/Z(S)$ is a finitely generated p-group. It follows from (iv) that S, and consequently $S/Z(S)$, is a locally finitely reducible group. Application of Theorem 1 shows that $S/Z(S)$ is a finite p-group. But then S is an upper nilpotent group which is generated by a finite number of elements of order a power of p; and it follows from Theorem 1 [and Corollary 1] that S is a finite p-group.

Having shown that finitely many elements in t^G generate a finite p-group we see that $\{t^G\}$ is a p-group. If x is an element in $\{t^G\}$, then there exists a finite subset T of t^G such that x belongs to $S = \{T\}$. If g is an element in G, then we deduce from $(+)$ the existence of integers $n(s)$ for s in T such that $sg^{p^{n(s)}} = g^{p^{n(s)}}s$. Denote by n the maximum of the finitely many integers $n(s)$ for s in T. Then $sg^{p^n} = g^{p^n}s$ for every s in T; and this implies that g^{p^n} commutes with every element in $S = \{T\}$. Hence we have in particular $xg^{p^n} = g^{p^n}x$.

Thus we have shown that the p-group $\{t^G\}$ satisfies conditions (C) and (iii) of § 5, Theorem 2. Consequently $\{t^G\}$ is an upper hypercentral subgroup of G so that (ii) is a consequence of (iv). This completes the proof.

7. Torsionfree Hypercentral Subgroups.

A group is called *torsionfree*, if it does not contain elements of finite order except 1. Their importance for our discussion stems from the following fact. If N is an upper hypercentral subgroup of G, then the totality $F(N)$ of elements of finite order in N is an upper hypercentral subgroup of G [§ 6, Theorem 2]. $N/F(N)$ is then a torsionfree upper hypercentral subgroup of $G/F(N)$ [§ 2, Proposition 1].

Proposition 1: *If C is the centralizer of the torsionfree upper hypercentral subgroup N of G, then G/C is torsionfree.*

Remark: This implies in particular that $F(G) \leq C$.

Proof: If G/C were not torsionfree, then G/C would contain elements of prime number order; and this is equivalent to saying that G contains an element w which induces in N an automorphism of order p. But this would imply, by § 3, Lemma 3, the existence of elements of order p in N, an impossibility. Hence G/C is torsionfree.

In order to obtain a simple and interesting generalization of this fact we introduce the iterated centralizers of a normal subgroup N inductively as follows:

$C_0(N) = 1$;

$C_{\sigma+1}(N)/C_\sigma(N)$ is the centralizer of $NC_\sigma(N)/C_\sigma(N)$ in $G/C_\sigma(N)$;

if σ is a limit ordinal, then $C_\sigma(N)$ is the set theoretical join of all the $C_\nu(N)$ with $\nu < \sigma$.

It is not difficult to verify that the $C_\sigma(N)$ form an ascending chain of normal subgroups of G; and that $Z_\sigma(G) \leq C_\sigma(N)$ for every σ. One may then deduce from § 2, Proposition 2 that $C_\sigma(N) = G$ for some σ, if N is an upper hypercentral subgroup of G [the converse is naturally not true, witness any abelian normal subgroup with abelian quotient group].

Corollary 1: *If N is a torsionfree upper hypercentral subgroup of G, then $G/C_\sigma(N)$ is torsionfree for every $\sigma > 0$.*

Proof: Our assertion is true for $\sigma = 1$, since $C_1(N)$ is just the centralizer of the torsionfree upper hypercentral subgroup N of G, and since we may apply therefore Proposition 1. Consequently we assume that $1 < \sigma$ and that $G/C_\nu(N)$ is torsionfree for every $\nu < \sigma$.

Case 1: $\sigma = \tau + 1$.

Then it follows from our inductive hypothesis that $G/C_\tau(N)$ is torsionfree; and it follows from § 2, Proposition 1 that $NC_\tau(N)/C_\tau(N)$ is a torsionfree upper hypercentral subgroup of $G/C_\tau(N)$. It follows from Proposition 1 that the centralizer quotient group is torsionfree; and this centralizer quotient group is, by definition, just $G/C_{\tau+1}(N) = G/C_\sigma(N)$.

Case 2: σ is a limit ordinal.

If g is an element in G, n a positive integer such that g^n belongs to $C_\sigma(N)$, then there exists an ordinal $\nu < \sigma$ such that g^n belongs to $C_\nu(N)$. It follows from our inductive hypothesis that $G/C_\nu(N)$ is torsionfree. Hence g itself belongs to $C_\nu(N) < C_\sigma(N)$; and this proves that $G/C_\sigma(N)$ is torsionfree. This completes the inductive proof.

Corollary 2: *If $x \neq 1$ is an element in the torsionfree upper hypercentral subgroup N of G, if g is an element in G, and if $x^n g^m = g^m x^n$ for positive m, n, then $xg = gx$.*

Proof: Let $S = \{x, g^m\}$, $M = N \cap S$ and denote by C the centralizer of M in S. It follows from § 2, Proposition 1 that M is a torsionfree upper hypercentral subgroup of S; and it follows therefore from Proposition 1 that S/C is torsionfree. It follows from our hypothesis that x^n belongs to $Z(S) \leq C$. Hence x itself belongs to C, since S/C is torsionfree. Thus x belongs to $M \cap C = Z(M)$. The torsionfree abelian group $Z(M)$ is a characteristic subgroup of the normal subgroup M of S. Hence $Z(M)$ is a normal subgroup of S. The element g^m induces therefore in $Z(M)$ an automorphism which leaves invariant x^n. But x is the one and only one solution in $Z(M)$ of the equation $t^n = x^n$ so that x itself is a fixed element of the automorphism which g^m induces in $Z(M)$. Thus we have shown that $xg^m = g^m x$.

Let $T = \{x, g\}$, $P = N \cap T$ and denote by Q the centralizer of P in T. We conclude as before that P is a torsionfree upper hypercentral subgroup of T and that T/Q is torsionfree. It follows from $xg^m = g^m x$ that g^m belongs to $Z(T) \leq Q$. Since T/Q is torsionfree, g itself belongs to Q. But x is in P. Hence $xg = gx$, as we wanted to show.

Proposition 2: *The torsionfree subgroup N of G is an upper hypercentral subgroup of G if, and only if, N [and G] satisfy the following conditions:*

(a) *N is a locally finitely reducible subgroup of G.*

(b) *If M is a normal subgroup of the subgroup S of G, if $M < N \cap S$ and $(N \cap S)/M$ is torsionfree, then M, N, S have the following properties:*

(b′) *If x belongs to $(N \cap S)/M$ and s to S/M and if $x^i s^j = s^j x^i$ for positive i and j, then $xs = sx$.*

(b″) *If s belongs to S/M, then $[(N \cap S)/M] \cap C(s < S/M)$ contains a normal subgroup, not 1, of S/M.*

Proof: We assume first that the torsionfree subgroup N of G is an upper hypercentral subgroup of G. Suppose that M is a normal subgroup of the subgroup S of G and that $M < N \cap S$ and $(N \cap S)/M$ is torsionfree. It follows from § 2, Proposition 1 that $(N \cap S)/M$

is an upper hypercentral subgroup of S/M. Now we deduce the validity of (b′) from Corollary 2 and the validity of (b″) is an immediate consequence of $[(N \cap S)/M] \cap Z(S/M) \neq 1$. The validity of (a) may be deduced from § 2, Proposition 2.

Assume conversely the validity of conditions (a) and (b). Consider an element a in N and an element g in G. Let $S = \{a, g\}$ and $T = N \cap S$. We begin by proving the following fact.

(1) *If M is a normal subgroup of S such that $M < T$ and T/M is torsionfree, then* $(T/M) \cap Z(S/M) \neq 1$.

It will be convenient to let $s^* = Ms$ for s in S; and if Y is a subset of S, then Y^* is the totality of elements y^* for y in Y. Denote by D^* the product of all the normal subgroups of S^* which are part of $T^* \cap C(a^* < S^*)$. This is naturally a normal subgroup of S^* which is part of $T^* \cap C(a^* < S^*)$; and there exists a uniquely determined normal subgroup D of S such that $M \leq D$ and $D/M = D^*$. It is a consequence of (b″) that $D^* \neq 1$. Hence $M < D \leq N \cap S$.

Next we form $E = \{D, g\}$. It is clear that D is a normal subgroup of E and that E/D is cyclic. Let $N \cap E = H$. Then $D \leq H$ so that H is a normal subgroup of E and E/H is the cyclic group generated by Hg. Note now that $M < D \leq H \leq E \leq S$. Since T/M is torsionfree, H/M is likewise torsionfree. Thus we may apply (b'') on the element $g^* = Mg$ in $E/M = E^*$; and it follows that $H^* \cap C(g^* < E^*)$ contains normal subgroups different from 1. It follows that the product P^* of all the normal subgroups of E^* which are part of $H^* \cap C(g^* < E^*)$ is different from 1.

Assume by way of contradiction that $D^* \cap C(g^* < E^*) = 1$. If we recall that g^* belongs to $C(g^* < E^*)$ and that $E^* = \{D^*, g^*\}$, then it follows that $C(g^* < E^*) = \{g^*\}$. From $1 < P^* \leq C(g^* < E^*) = \{g^*\}$ we deduce now the existence of a positive integer i such that $P^* = \{g^{*i}\}$. Since P^* and D^* are normal subgroups of E^* such that $P^* \cap D^* = 1$, g^{*i} commutes with every element in D^*. Since D^* is torsionfree, $D^* \leq (N \cap S)/M$, we may apply (b'). Consequently g^* itself commutes with every element in D^*. But then $1 < D^* \leq \leq C(g^* < E^*)$ contradicting our assumption that $D^* \cap C(g^* < E^*) = 1$. Thus we have shown that $D^* \cap C(g^* < E^*) \neq 1$.

If we remember that every element in D^* commutes with a^* and that $S^* = \{a^*, g^*\}$, then we find that

$$1 < D^* \cap C(g^* < E^*) \leq Z(S^*) \cap T^*;$$

and this completes the proof of (1).

(2) *If M is a normal subgroup of S such that $M < T$ and T/M is torsionfree, then* $(T/M)/[T/M \cap Z(S/M)]$ *is torsionfree.*

This is an almost obvious consequence of condition (b').

(3) *S is upper nilpotent.*

By a fairly obvious transfinite induction using (1) and (2) one proves the existence of an ascending central chain of S which connects 1 and T. But S/T is cyclic, since a is in T. Consequently there exists an ascending central chain of S which terminates with S. The upper nilpotency of S is now an immediate consequence of § 2, Proposition 2.

From (3) we deduce that condition (vi) of § 5, Theorem 3 is satisfied by the normal subgroup N of G. It follows from (a) that § 5, Theorem 3 is applicable. Thus N is an upper hypercentral subgroup of G. This completes the proof.

8. Finitely Generated Nilpotent Groups.

The importance of this class of groups for our discussion stems from § 5, Theorem 3, (vi).

Theorem: *The following properties of the group G are equivalent.*

(i) *G is a finitely generated upper nilpotent group.*

(ii) *G is a locally finitely reducible group whose maximal subgroups are normal; and the maximum condition is satisfied by the subgroups of G.*

(iii) *G is a finitely generated, locally finitely reducible group whose finite quotient groups are nilpotent; and the maximum condition is satisfied by the normal subgroups of G.*

Proof: We assume first that G is a finitely generated, upper nilpotent group. Then G is locally finitely reducible [by § 2, Proposition 2] and the maximal subgroups of G are normal [by § 5, Theorem 3].

In order to prove the validity of the maximum condition we have to analyze the descending central chain of G. It is defined inductively for integral n by the rules:

$$^{\circ}G = G, \quad ^{n+1}G = [G, {}^{n}G].$$

The transfinite terms of this series will not be needed.

Since G is finitely generated, there exists a finite set S of elements in G with the following two properties:

(1) $G = \{S\}$.

(2) If s is in S, then s^{-1} belongs to S.

Next we define inductively sets of elements $S(i)$ as follows:

$S(O) = S$; $S(i+1)$ is the totality of commutators $[t, s]$ with s in S and t in $S(i)$.

The set theoretical join of the sets $S(j)$ with $i \leq j$ will be denoted by $J(i)$.

(3) Every set $S(i)$ is finite.

If S contains n elements, then one verifies inductively that $S(i)$ contains at most n^{i+1} elements, proving the validity of (3).

(4) ${}^{i}G = \{J(i)\}$.

From (1) we deduce that ${}^{\circ}G = G = \{S\} = \{S(0)\} = \{J(0)\}$. Thus we may make the inductive hypothesis that ${}^{i}G = \{J(i)\}$. It is clear that $J(i+1) \leq {}^{i+1}G$. Letting $U = \{J(i+1)\}$ we have therefore $U \leq {}^{i+1}G$.

If t is in $J(i+1)$ and s is in S, then $s^{-1}ts = t[t, s]$ and t belongs to some $S(k)$ with $i+1 \leq k$ so that $[t, s]$ belongs to $S(k+1) \leq J(i+2) \leq J(i+1) \leq U$. Hence $s^{-1}ts$ belongs to U too; and thus we have shown that $s^{-1}J(i+1)s \leq U$. Since U is generated by $J(i+1)$, it follows that

$$s^{-1}Us \leq U \text{ for every } s \text{ in } S.$$

But it follows from (2) that s^{-1} belongs to S. Hence we have likewise $sUs^{-1} \leq U$ or $U \leq s^{-1}Us$; and thus we have shown $s^{-1}Us = U$ for every s in S. Since G is generated by S[by (1)], we have shown that

$$U \text{ is a normal subgroup of } G.$$

If t is in $J(i)$ and s is in S, then $[t, s]$ belongs to $J(i+1)$, as we have shown in the preceding paragraph of our proof. This implies

$$ts \equiv st \text{ modulo } U \text{ for every } s \text{ in } S \text{ and every } t \text{ in } J(i).$$

Since $G = \{S\}$ and ${}^{i}G = \{J(i)\}$ [by (1) and the inductive hypothesis], it follows that

$${}^{i}G/U \leq Z(G/U),$$

if we only note that $U = \{J(i+1)\} \leq \{J(i)\} = {}^{i}G$. Now we see that

$${}^{i+1}G = [G, {}^{i}G] \leq U \leq {}^{i+1}G \text{ or } {}^{i+1}G = U = \{J(i+1)\}.$$

This completes the inductive proof of (4).

(5) There exists an integer m such that $S(m)$ consists of the identity only.

If this were false, then the totality $S^*(i)$ of elements, not 1, in $S(i)$ would not be vacuous for any i. It follows from (3) that every $S^*(i)$ is finite.

If h is in $S^*(i)$, s in S and $[h, s] \neq 1$, then $[h, s]$ belongs to $S^*(i+1)$ and may be called, for the purposes of this proof, a direct descendent of h. Descendents are now defined inductively by the rules: direct descendents are descendents; if k is a descendent of h and k' is a direct descendent of k, then k' is a descendent of h.

The element u in $S^*(i)$ shall be termed a distinguished element if it possesses descendents in every $S^*(i+j)$ for $0 < j$. Since every element in $S^*(i+j)$ is a descendent of at least one element in $S^*(i)$ and since $S^*(i)$ is finite [by (3)], there exist distinguished elements in every $S^*(i)$.

Consequently there exists in particular a distinguished element $u(0)$ in $S^*(0)$. Assume now that we have already constructed distinguished elements $u(0), \ldots, u(i)$ such that

$u(j)$ belongs to $S^*(j)$ and such that $u(j+1)$ is a direct descendent of $u(j)$. Since $S^*(i+1)$ is finite [by (3)], $u(i)$ possesses only a finite number of direct descendents. If none of them were distinguished, then $u(i)$ could not be distinguished either. Consequently there exists a distinguished element $u(i+1)$ which is a direct descendent of $u(i)$.

Thus we have constructed a sequence of elements

$$u(0), \ldots, u(i), u(i+1), \ldots$$

such that $u(i)$ belongs to $S^*(i)$ and $u(i+1)=[u(i), s(i)]$ for some $s(i)$ in S. To prove the impossibility of the existence of such a sequence we recall that G is upper nilpotent. Consequently there exists, by § 2, Proposition 2, an ordinal σ such $G=Z_\sigma(G)$. There exists therefore to every integer i a first ordinal $\sigma(i)$ such that $u(i)$ belongs to $Z_{\sigma(i)}(G)$.

Since ordinal numbers are well ordered, there exists among the ordinals $\sigma(i)$ a first one $\sigma(m)$ [where m is a suitably selected integer]. Since $u(m)\neq 1$ belongs to $Z_{\sigma(m)}(G)$, $0<\sigma(m)$; and since $Z_{\sigma(m)}(G)$ is the first $Z_\nu(G)$ containing $u(m)$, $\sigma(m)$ is not a limit ordinal either. Hence $\sigma(m)=\tau+1$ for some ordinal τ. Since $u(m+1)=[u(m), s(m)]$ belongs to $[Z_{\sigma(m)}, G]=[Z_{\tau+1}, G]\leq Z_\tau$, it follows that

$$\sigma(m+1)\leq\tau<\tau+1=\sigma(m)\leq\sigma(m+1),$$

a contradiction which proves the validity of (5).

(6) ${}^mG=1$ for some m.

From $S(m)=1$ one deduces immediately $S(i)=1$ for every i exceeding m; and this implies $J(m)=1$. It follows from (4) that ${}^mG=\{J(m)\}=1$, as we claimed.

(7) Every ${}^iG$ is finitely generated.

It follows from (3) and (5) that every $J(i)$ is finite; and thus it follows from (4) that ${}^iG=\{J(i)\}$ is finitely generated.

Now it is easy to prove the validity of the maximum condition. It follows from (6) that the descending central chain terminates with 1. Hence

$$1={}^mG\leq\ \ldots\ \leq {}^{i+1}G\leq {}^iG\leq\ldots\leq {}^\circ G=G.$$

It follows from (7) that every ${}^iG/{}^{i+1}G$ is a finitely generated abelian group. Thus the maximum condition is satisfied by each of the factors ${}^iG/{}^{i+1}G$ of the finite descending central chain; and now it is a well known fact that the maximum condition is likewise satisfied by the subgroups of G.

This completes the proof of the fact that (ii) is a consequence of (i), a fact that we are going to use below.

Assume next the validity of (ii). Since the maximum condition is satisfied by all the subgroups of G, the maximum condition is likewise satisfied by the normal subgroups of G; and from the maximum condition we deduce furthermore that every subgroup of G is finitely generated. Hence G itself is finitely generated. If N is a normal subgroup of finite index of G, then G/N is a finite group all of whose maximal subgroups are normal. It follows from Wielandt's Theorem that G/N is nilpotent; see, for instance, Zassenhaus [1; p. 108, Satz 13]. Now it is clear that (iii) is a consequence of (ii).

Assume finally the validity of (iii) and assume by way of contradiction that G is not upper nilpotent. Then the class Φ of normal subgroups X such that G/X is not upper nilpotent is not vacuous, since it contains $X = 1$. Consequently there exists a maximal normal subgroup W in Φ [maximum condition]. Clearly W has the following properties:

(8) W is a normal subgroup of G.

(9) G/W is not upper nilpotent.

(10) If H is a normal subgroup of G such that $W < H < G$, then G/H is upper nilpotent.

We prove next the following property of W.

(11) $Z(G/W) = 1$.

If this were not true, then there would exist one and only one normal subgroup T of G such that $W < T \leq G$ and $T/W = Z(G/W)$. It follows from (10) that G/T is upper nilpotent. Hence $(G/W)/Z(G/W)$ would be upper nilpotent; and it would follow from § 2, Proposition 2 that G/W itself would be upper nilpotent which is impossible by (9). Thus (11) is true.

(12) G/W does not contain finite normal subgroups except 1.

If this were false, then there would exist a normal subgroup N of G such that $W < N$ and N/W is finite. It follows from (10) that G/N is upper nilpotent. But G is finitely generated. Thus G/N is a finitely generated, upper nilpotent group. Since we have shown in the first part of our proof that (i) implies (ii), it follows that the maximum condition is satisfied by the subgroups of G/N. Since G/N is upper nilpotent, it follows from § 6, Theorem 2 that the elements of finite order in G/N form a normal subgroup F. From the maximum condition it follows that F is finitely generated. Thus F is an upper hypercentral subgroup of G/N which is generated by finitely many elements of finite order. It follows from § 6, Theorem 1 that F is finite. There exists a uniquely determined normal subgroup M of G which contains N and satisfies $M/N = F$. Since M/N and N/W are finite, M/W is finite. It follows from our choice of F that $(G/N)/F \simeq G/M$ is a torsionfree group. If $G = M$, then G/W would be finite; and G/W would be nilpotent [by (iii)] contradicting (9). Thus G/M is an infinite torsionfree group which is upper nilpotent [by (10)], since $W < N \leq$

$\leq M$. It follows in particular that $Z(G/M) \neq 1$. Consequently there exists an element z in G such that $Mz \neq 1$ and Mz belongs to $Z(G/M)$.

We denote by n the order of the finite group M/W. It is clear that z induces in M/W an automorphism of finite order m. This is equivalent to saying that m is the minimal positive number such that $[z^m, M] \leq W$. If g is any element in G, then $[z^m, g]$ belongs to M, since Mz and Mz^m belong to the center $z(G/M)$. Hence

$$g^{-1}z^{mn}g = ((g^{-1}z^m g)^n = (z^m[z^m, g])^n \equiv z^{mn}[z^m, g]^n \equiv z^{mn} \text{ modulo } W,$$

since Wz^m commutes with every element in M/W and since $M^n \leq W$. Thus we see that Wz^{mn} belongs to $Z(G/W) = 1$ by (11). Hence $(Wz)^{mn} = 1$. But this is impossible, since $Mz \neq 1$ and G/M is torsionfree. We have arrived at a contradiction which proves (12).

So far we have not made any use of the fact that G is, by (iii), locally finitely reducible. Since $G/W \neq 1$ [by (9)], there exists a normal subgroup V of G such that $W < V$ and such that V/W is a finitely reducible subgroup of G/W. We denote by Θ the set of all the normal subgroups X of G with the following properties:

$W \leq X < V$ and V/X is a finite minimal normal subgroup of G/X.

Since V/W is a finitely reducible subgroup of G/W, W is the intersection of all the normal subgroups X in Θ.

Consider now some X in Θ. If X were W, then $V/X = V/W \neq 1$ would be a finite normal subgroup of G/W contradicting (12). Hence $W < X$; and it follows from (10) that G/X is upper nilpotent. But finite minimal normal subgroups of upper nilpotent groups are part of their center [§ 2, Lemma 1]. Hence $V/X \leq Z(G/X)$ and this implies $[G, V] \leq X$. Since this is true for every X in Θ, and since W is the intersection of all the X in Θ, it follows that $[G, V] \leq W$. Consequently $1 < V/W \leq Z(G/W)$ and this contradicts (11). Thus we have been led to a contradiction by assuming that G is not upper nilpotent; and this proves that (i) is a consequence of (iii), completing the proof of our theorem.

Corollary: *If G is a finitely generated upper nilpotent group, then $Z_m(G) = G$ and ${}^mG = 1$ for some finite m; and the elements of finite order form a finite subgroup of G.*

This corollary has been derived, more or less, in the course of the proof of our theorem; and a direct derivation from our theorem would be easy enough.

Remark 1: The implication of (i) by (ii) improves upon a result of Hirsch [1; p. 194, Theorem 3.3] who shows that the group G is upper nilpotent, if every maximal subgroup of G is normal, the maximum condition is satisfied by the subgroups of G and if $G^{(m)} = 1$ for some finite m. [Here the derived series is defined inductively by the rules $G^{(0)} = G$, $G^{(i+1)} = [G^{(i)}, G^{(i)}]$]. But the last two conditions imply, by § 1, Lemma 4, that G is locally finitely reducible; and thus our condition (ii) is a consequence of Hirsch's conditions.

Remark 2: Whether or not it is possible to omit from (ii) the hypothesis that G be locally finitely reducible, is an open question. It is, however, impossible to omit this hypothesis from (iii), since Higman [1] has constructed an example of an infinite, finitely generated, simple group.

Cp. also our Remark to § 5, Corollary 1.

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University of Illinois
Urbana, Illinois (USA).