Acta Math., 168 (1992), 273–318

Julia–Fatou–Sullivan theory for real one-dimensional dynamics

by

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Introduction

Our aim is to show that the Julia–Fatou–Sullivan structure theory for the dynamics of rational maps is also valid for smooth endomorphisms of the circle (and of the interval) under extremely mild smoothness and non-flatness conditions.

In order to stress the similarity between real and complex one-dimensional dynamics let us recall the main results from the Fatou–Julia–Sullivan theory. If f is a rational map then there is a dynamical decomposition of the Riemann sphere into the disjoint union of two totally invariant (i.e., both forward and backward invariant) sets $J(f), F(f)$. Here, $F(f)$ is the domain of normality of the family of iterates of f, and is called the Fatou set. Its complement, which is called the Julia set of f, is a compact set, which contains all the complications of the dynamics of f. The connected components of the open set $F(f)$ are mapped onto each other by f. Hence the orbit of a component of $F(f)$ is the union of some components of $F(f)$. Julia proved in the beginning of the century that if a component of $F(f)$ is periodic and contains an attracting periodic point then the orbit of this component must contain a critical point. Sullivan, in the remarkable paper [Su], showed via quasi-conformal deformations that the components are eventually periodic and fall into finitely many orbits.

Let N be either the circle S^1 or a compact interval of the real line and $f\colon N\to N$ be a smooth endomorphism. A *critical point* of f is a point where the derivative vanishes. A critical point is *non-flat* if some (higher) derivative is non-zero. A critical point is an *inflection point* if it has a neighbourhood where f is monotone. Otherwise it is called a *turning point*. Assume f is not a homeomorphism (if f is a homeomorphism then these maps correspond to degree ± 1 rational maps of the Riemann sphere and the situation is

(1) Supported by N.W.O.

simple). If f has turning points then we define the singular set of f, Sing(f), as the union of the set of turning points of f and the boundary points of N. If f has no turning points then f is a circle map of degree ≥ 2 (or ≤ -2) and we define Sing(f) to be the set of fixed points of f. Finally we define the *Julia set* of f to be the α-limit set of Sing(f), i.e., $x \in \alpha(\mathrm{Sing}(f))$ iff there exists $y_i \to x$ and a sequence $n_i \to \infty$ such that $f^{n_i}(y_i) \in \mathrm{Sing}(f)$. Denote this set by $J(f)$. It is forward invariant. Thus its complement, called the *Fatou set* $F(f)$, is backward invariant: $f^{-1}(F(f)) \subset F(f)$. In general the Fatou set is not forward invariant but if U is a connected component of $F(f)$ which does not contain a turning point then $f(U)$ is also a component of $F(f)$.

MAIN THEOREM. *Let $f: N \to N$ be a smooth map such that all its critical points are non-flat. Then:*

(1) *All the connected components of $F(f)$ are eventually periodic (i.e., eventually mapped into a periodic component of $F(f)$);*

(2) *The number of periodic components of $F(f)$ is finite.*

Part (1) of the above theorem can be reformulated by stating that there are no *wandering intervals* for such maps. By a wandering interval we mean an interval J such that all forward iterates of J are disjoint and such that the ω-limit set of J is not a periodic orbit. The first result in this direction was obtained by Denjoy in 1932, [D]. He proved that a C^2 diffeomorphism of the circle does not have wandering intervals (more precisely, his proof applies to all C^1 diffeomorphisms $f: S^1 \to S$ such that $\log Df$ has bounded variation). His proof relies on a detailed understanding of the dynamics of rotations and on the control of the distortion of iterates of the map on intervals whose iterates are all disjoint. Later, in 1963, A. Schwartz [Sc] gave a different proof of Denjoy's result. His proof does not rely on precise dynamical properties but requires that $\log Df$ is Lipschitz. For maps with critical points the techniques of Denjoy and Schwartz cannot be used. In 1979, J. Guckenheimer was able to deal with critical points in some special cases. He proved in [Gu1] the non-existence of wandering intervals for unimodal maps of the interval with negative Schwarzian derivative and no inflection points, see also [Mi]. J. C. Yoccoz, [Y], proved the non-existence of wandering intervals for C^∞ homeomorphisms of the circle having only non-flat critical points. He combines techniques of Denjoy away from the critical points with some analytical estimates near the critical points which are related to the Schwarzian derivative. In [MS1] the same result was proven for smooth unimodal maps (not necessarily having negative Schwarzian derivative) with a non-flat critical point and also for maps satisfying the so-called Misiurewicz condition. In [MS1] the main tool is the control of the distortion of the cross-ratio under iterates. This control implies that under some disjointness assumptions the diffeomorphic inverse branches of iterates of a smooth map behaves very much like univalent holomorphic maps. This similarity is clear from the minimum principle and the Koebe distortion principle for real maps, see § 2 and § 4 of this paper. In 1988, Blokh and Lyubich proved in [L], [BL] the non-existence of wandering intervals for smooth maps whose only critical points are turning points. They introduced some very nice and powerful new topological tools generalizing those of [Gu1] and used the analytical tools developed in [MS1]. Our proof of the first part of the Main Theorem combines the analytical tools developed in [MS1] and [MMMS] (which allow for inflection points) with an extension of the topological ingredients of [L] and [BL]. In fact many of the ideas of § 6–§ 9 in this paper are simplifications and modifications of results contained in [L] and [BL]. We should note however that for the proof in [L] and [BL] it is necessary that the maps are C^2 and that our proof of part (1) of the Main Theorem also works for piecewise linear maps (see § 7). To be more specific our proof applies to all diffeomorphisms of the circle for which Denjoy's results hold. So Theorem A is a natural extension of Denjoy's original ideas to maps with critical points.

The first contribution to an analogue of part (2) of the Main Theorem for rational maps on the Riemann sphere is due to Julia. He proved in [J] (see also [Fa]) that if the orbit of a periodic domain contains an attracting point then it must also contain a critical point. Hence the number of orbits of these periodic domains must be bounded by the number of critical points. In [Si], Singer introduced for the first time the Schwarzian derivative in one dimensional dynamics and proved the same result of Julia for maps with negative Schwarzian derivative. Mañé, using estimates related to Denjoy and Schwartz proved that (2) holds for maps of the circle without critical points. Instead of (2) we will prove a still stronger result. We show that for each smooth map satisfying the hypothesis of the Main Theorem there exists $\varrho>0$ such that if p is a periodic points of sufficiently high period n then $|Df^n(p)|\geqslant 1+\varrho$. This last estimate is new even for maps from the family $x\to ax(1-x)$.

Let us finish this introduction by stating a corollary of the Main Theorem.

COROLLARY. (For the proof see [Me].) *In the space $\mathcal{U}^r$ of C^r unimodal maps of the interval* $[-1, 1]$ *endowed with the C^r topology, $r\geqslant 3$ the set of structurally stable maps is open and dense.*

The above corollary is the analogue of Mañé–Sad–Sullivan Theorem on the density of structural stable rational maps on the Riemann sphere. Note however that it does not follow that structurally stable maps satisfy the Axiom A condition.

COROLLARY. *Let $f: [-1,1] \to [-1,1]$ be a C^2 unimodal map which is infinitely renormalisable and whose turning point is non-flat. Then f has an attracting Cantor set and its basin contains intervals.*

Proof. By Theorem B the turning point of f cannot be approximated by periodic attractors. Therefore after renormalizing a finite number of times, the new renormalized map has no periodic attractors. So for this new map all points, except those which are eventually mapped into periodic orbits, tend to the closure of the forward orbit of the turning point. Q.E.D.

Many of D. Sullivan's ideas were an important source of inspiration for our work. We are grateful to him for inviting the last two authors to C.U.N.Y. and his interest in this work. Also we would like to thank the mathematics department of the University of Warwick at which this paper was finished.

§ 1. Definitions and statement of the theorems

Let N be a smooth compact 1-dimensional manifold (i.e., a finite union of closed intervals or circles). We say that $p \in N$ is a *periodic point* of f of period n if $f^n(p)=p$ and $f^i(p) \neq p$ for $0<i<n$. $J \subset N$ is a *periodic interval* of period n if $f^n(J) \subset J$ and $f^i(J) \cap J = \emptyset$ for $0<i<n$. An interval $J \subset N$ is a *wandering interval* for f if (i) $f^i(J) \cap f^j(J) = \emptyset$ for all $0 \leq i < j$ and (ii) the ω-limit of J, $\omega(J)=\{x; \exists n_i \to \infty$ and $y \in J$ such that $f^{n_i}(y) \to x\}$, is not a periodic orbit. (In fact, if it is a periodic orbit then it necessarily has to be a (possibly one-sided) attracting periodic orbit.) If f is not a homeomorphism and has a finite number of turning points then a component of J of the Fatou set of f is either eventually periodic (i.e., eventually mapped into a periodic component of $F(f)$) or a wandering interval. If U is a periodic component of period $n>0$ of the Fatou set of f (i.e. $f^n(U) \subset U$), then either the orbit of U contains a critical point of f or $f^n|U$ is monotone and f^n has a non-repelling fixed point in the closure of U. Therefore our Main Theorem follows from the following two theorems.

THEOREM A. *Let $f: N \to N$ be a C^∞ map whose critical points are non-flat. Then f has no wandering interval.*

THEOREM B. *Let $f: N \to N$ be a C^∞ map whose critical points are non-flat. Then* *there exists n_0 and $\varrho>0$ such that for every periodic points of f of period $n\geqslant n_0$ one has*

$$|Df^n(p)|\geqslant 1+\varrho.$$

The assumptions in these theorems are satisfied for analytic maps $f\colon N\to N$. In fact Theorem B implies that analytic maps can have at most a finite number of non-repelling periodic orbits (unless f or f^2 is the identity map).

The non-flatness conditions in Theorems A and B cannot be dropped. Indeed if one considers C^∞ maps with a flat critical point then Theorems A and B are false. Such maps may have wandering intervals and an infinite number of attractors. The first example of such a map was given by Hall, [Ha], but see also [SI], [Me]. Moreover, Theorem A and B are actually proved under much weaker smoothness assumptions. Theorem A is for example also proved for general continuous piecewise linear maps (in fact it is proved under the same smoothness conditions as the result of [D] for circle diffeomorphism).

More precisely we will prove

THEOREM A′. *Let $f\colon N\to N$ be in the clase $\mathscr{A}$ (defined presently). Then f has no wandering interval.*

Here $\mathscr{A}$ is the class of absolutely continuous maps $f\colon N\to N$ such that the following two conditions are satisfied:

(A1) There exists a finite set K_f such that for each neighbourhood of U of the set K_f, the map $N\setminus U\ni x\mapsto \log|Df(x)|$ (which exists almost everywhere because f is absolutely continuous) extends to a map of bounded variation on $N\setminus U$;

(A2) For each $x_0\in K_f$ there exists $\alpha\geqslant 1$, a neighbourhood $U(x_0)$ of x_0 and a homeomorphism $\phi\colon U(x_0)\to(-1,1)$ such that $\phi(x_0)=0$, ϕ, ϕ^{-1} are absolutely continuous, $(-1,1)\ni x\mapsto \log D\phi(x)$ (which exists almost everywhere) can be extended to a map of bounded variation, and such that

$$f(x)=\pm|\phi(x)|^\alpha+f(x_0),\quad \forall x\in U(x_0).$$

This number α is called the *order* of $x_0\in K_f$.

Notice that condition (A2) gives a non-flatness condition at the set of critical points. (If x_0 is an inflection point we can also allow that $f(x)=\pm|\phi(x)|^{\alpha_1}+f(x_0)$ on one side of x_0 and $f(x)=\pm|\phi(x)|^{\alpha_2}+f(x_0)$ on the other side, where $\alpha_1,\alpha_2\geqslant 1$ need not be the same. On the other hand, if x_0 is a turning point then we need the order or flatness to be the same on both sides; indeed, otherwise the involution τ from Section 4 below would not be Lipschitz.) This class $\mathscr{A}$ contains the class of C^∞ maps with non-flat critical points (and therefore Theorem A′ implies Theorem A) and also for example the class of continuous piecewise linear maps. For these piecewise linear maps even the non-existence of wandering intervals whose orbits stays away from turning points was previously unknown. The classical proof based on Denjoy–Schwartz requires $\log|Df|$ to be Lipschitz, see [Sc], and also [CE], [Str2]. Our proof relies on a disjointness result for backward iterates of certain intervals, see Theorem 6.4. In this sense our proof is closer to the original ideas of Denjoy than the later developments of Schwartz.

Instead of Theorem B we will prove the following stronger result:

THEOREM B′. *Let $f: N \to N$ be in the clase $\mathscr{B}$ (defined presently). Then there exists n_0 and $\varrho>0$ such that for every periodic point of f of period $n \geqslant n_0$ one has*

$$|Df^n(p)| \geqslant 1+\varrho.$$

The class $\mathscr{B}$ is somewhat smaller than the class $\mathscr{A}$. Indeed, $f \in \mathscr{B}$ if and only if:

(B1) f is C^2;

(B2) Let K_f be the set of critical points of f. Then for every $x_0 \in K_f$, there exist $\alpha>1$ and a C^2 coordinate system ϕ: $U(x_0) \to (-1, 1)$ on a neighbourhood $U(x_0)$ of x_0 such that $\phi(x_0)=0$ and

$$f(x) = \pm|\phi(x)|^\alpha + f(x_0), \quad \forall x \in U(x_0).$$

Clearly $\mathscr{B}$ includes the class of C^∞ maps with non-flat critical points (and therefore Theorem B′ implies Theorem B). Moreover, it is not difficult to prove that condition (B2) is satisfied if for each critical point c there exists $k \geqslant 2$ such that f is C^{k+1} at c and $Df^k(c) \neq 0$. It is not clear whether we can weaken the condition that f is C^2 in the proof of Theorem B′.

The numbers n_0 and ϱ from Theorem B obviously depend in an essential way on the maps f. Indeed, there exists a sequence of analytic maps f_n converging to an analytic map f such that f_n has an attracting periodic orbit of period $>n$. (Take for example f to be a quadratic map with an eventually periodic critical point.) Nevertheless, our proof of Theorem B gives more uniform estimates. More precisely, if $\mathscr{K}$ is a compact family of maps in $\mathscr{B}$ then there exists $\varrho>0$ and $n_0 \in \mathbf{N}$ such that if $f \in \mathscr{K}$ and p is a periodic point of period $n \geqslant n_0$ of f then one of the following possibilities hold.

(i) p is an attracting periodic point whose immediate basin of attraction contains a critical point;

(ii) p is in the boundary of the immediate basin of a periodic attractor which attracts a critical point;

(iii) $|Df^n(p)|\geq 1+\varrho$.

In particular the number of periodic orbits of maps in $\mathcal{K}$ of type (i) and (ii) is bounded by the number of critical points.

Notation. We will use the following notation. J_n will denote $f^n(J)$. If I, J are intervals in the same component of N, let $[I, J]$ be the (smallest) convex hull of I and J. (Even if this component of N is equal to S^1 it will always be clear which interval we mean. If I and J are in different components then we define $[I, J]=N$.) $(I, J]$ denotes the set $[I, J]\setminus I$. Similarly define $[I, J)$ and (I, J). The Lebesgue measure of a measurable set $I\subset N$ is denoted by $|I|$.

2. Analytical estimates on diffeomorphic branches of f^n when $f\in\mathcal{B}$: the Koebe and the minimum principle

In Denjoy's theory for circle diffeomorphisms the main technical tool is the control of the distortion of iterates of the map restricted to some interval under some disjointness assumptions on the iterates of this interval. Here the distortion of a differentiable map f on an interval T is defined as the maximal ratio of the absolute values of the derivative in two different points. This number measures the non-linearity of the map. Another way to present the same concept is to consider pairs of intervals $L, R\subset T$, intersecting at a common boundary point, and the distortion of the ratio $D(L, R)=|L|/|R|$ by the map f, i.e., the number $D(f, L, R)=D(f(L), f(R))/D(L, R)$. It is easy to see that the distortion of a differentiable map f in the interval N is bounded if and only if there is an upperbound for $D(f, L, R)$ for any pair of intervals $L, R\subset T$.

If a map f has critical points we cannot hope to get a bound for its non-linearity. Hence, instead of the distortion of a pair of consecutive intervals, or three consecutive points, we have to analyze the distortion of a more complicate configuration called the cross-ratio of four points.

2.1. *Definition*. Let $J\subset T$ be open and bounded intervals in N such that $T\setminus J$ consists of intervals L and R. Define the cross ratio of these intervals as

$$D(T, J)=\frac{|J|\,|T|}{|L|\,|R|},$$

(where $|I|$ denotes the length of an interval I). If $g\colon T\to N$ is continuous and monotone define

$$B(g, T, J) = \frac{D(g(T), g(J))}{D(T, J)}.$$

We notice that if $f^n|T$ is monotone and continuous then

$$B(f^n, T, J) = \prod_{i=0}^{n-1} B(f, f^i(T), f^i(J)).$$

Of course this cross-ratio is related to the hyperbolic metric. Indeed, let T be an open and bounded interval on N. For $x, y \in T$ let

$$\varrho_T(x, y) = \frac{1}{2}\mathrm{Log}\frac{|L \cup J||J \cup R|}{|L||R|} = \frac{1}{2}\mathrm{Log}(1+D(T, J))$$

where J is the interval bounded by the points x, y. Then ϱ_T is a metric in T and the group of isometries of this metric is exactly the group $\mathcal{M}_T$ of all Möbius transformations that map T onto T. Furthermore, the group $\mathcal{M}_T$ acts transitively on T, namely, given $x, y \in T$, there exists an isometry $\phi \in \mathcal{M}_t$ such that $\phi(x)=y$. If we take $T=(-1, 1)$ and $x, y \in T$ then $\varrho_T(x, y)$ is exactly the hyperbolic distance between the two points $(x, 0)$, $(y, 0)$ in the unit disc. Therefore we shall call ϱ_T the *hyperbolic metric* of the interval T.

As is well known, holomorphic maps of the unit disc contract the hyperbolic metric. Moreover holomorphic maps of the disc have many very powerful properties: for example there is a universal bound for the non-linearity of these maps on a smaller disc (this follows from the Koebe Lemma). We shall show that diffeomorphisms of the interval T which contract the hyperbolic metric ϱ_T satisfy similar properties.

One way to check that a map contracts the hyperbolic metric on an interval is through the Schwarzian derivative.

2.2. *Definition.* Let $g: T \to N$ be a C^3-map on the interval $T \subset N$. Then

$$Sg(x) = \frac{g'''(x)}{g'(x)} - \frac{3}{2}\left(\frac{g''(x)}{g'(x)}\right)^2$$

is called the Schwarzian derivative of g.

2.3. PROPOSITION. *Let $g: T \to N$ be a C^3 diffeomorphism on the open interval $T \subset N$ such that for every $x \in T$, $Sg(x) < 0$. Let $\mathrm{cl}(I) \subset \mathrm{int}(T)$. Then*

$$B(g, T, I) > 1.$$

Furthermore if $Sg(x) \leqslant 0$, $\forall x \in T$, then $B(g, T, I) \geqslant 1$.

Proof. The proof is well known and can be found in for example [MS].

In particular a diffeomorphism $g: T\to T'$ having negative Schwarzian derivative expands the hyperbolic metric: $\varrho_{T'}(g(x), g(y))>\varrho_T(x, y)$ for all $x, y\in T$ and $x\neq y$. Therefore the inverse of g contracts these metrics.

So our aim is to show that inverses of diffeomorphisms $g: T\to T'$ with negative Schwarzian derivative behave in many ways as holomorphic maps on a disc in the complex plane.

Unfortunately the condition that a map has negative Schwarzian derivative is not very natural: it is not preserved under coordinate changes, has no dynamic interpretation and it excludes a large class of maps.

Therefore we will not only consider maps with negative Schwarzian derivative but maps with some smoothness properties. By studying the distortion of the cross-ratio under iteration, we will first show in this section that the iterates of a smooth map restricted to an interval where it is a diffeomorphism, does not contract the metric too much provided some disjointness assumption on the iterates of the interval under consideration. Then we will show that in this case one obtains analogues of the Koebe and Maximum Principle for conformal maps (the analogy is with respect to the inverse of the maps). These results will hold for maps $f\in\mathcal{B}$. In § 4 the case is considered that f^n is a homeomorphism which may have critical points of inflection type and when $f\in\mathcal{A}$.

In the following theorem we give a lower bound for $B(f^n, T, J)$ if $f\in\mathcal{B}$, see also in [MS1] or in [Str3], see also [Str2] and [MS2] (where also a different cross-ratio is considered).

2.4. Theorem. *Let $f\in\mathcal{B}$. Then there exists a bounded continuous function $\sigma: [0,\infty)\to\mathbf{R}_+$ such that $\sigma(t)\to 0$ as $t\to 0$ with the following property. If T is an interval such that $f^m|T$ is diffeomorphism then for any interval $J\subset T$ with* $\mathrm{cl}(J)\subset\mathrm{int}(T)$:

$$B(f^m, T, J)\geqslant \exp\{-\sigma(\tau)\cdot\sum_{i=0}^{m-1}|f^i(T)|\}. \tag{2.1}$$

Here $\tau=\max_{i=0,\dots,m-1}|f^i(T)|$.

Proof. Since $B(f^n, T, J)=\prod_{i=0}^{n-1}B(f, f^i(T), f^i(J))$, it suffices to show that there exists a constant $C_0\in(0,\infty)$ and an increasing continuous function $\sigma: [0,\infty)\to[0, C_0)$ with $\lim_{t\to 0}\sigma(t)=0$ such that

$$B(f, T, J)\geqslant \exp\{-\sigma(|T|)\cdot|T|\}$$

for all intervals $J \subset T \subset N$ such that $Df(x) \neq 0$ for all $x \in T$. It is enough to prove that

$$B(f, T, J) - 1 \geq -\sigma(|T|) \cdot |T| \tag{2.2}$$

for some (other) function σ as above. Let $V \subset \mathrm{cl}(V) \subset U$ be a neighbourhood of K_f such that each component U' of U contains a unique critical point and such that f is of the form $f(x) = \pm|\phi(x)|^\alpha + f(x_0)$, for all $x \in U'$ where $\alpha \geq 1$ and $\phi: U' \to (-1, 1)$ is a diffeomorphism.

Case 1. $T \subset N \setminus V$. In this case write $T = [a, d]$ and $M = [b, c]$. Then

$$B(f, T, J) - 1 = \frac{\frac{f(c)-f(b)}{c-b} \cdot \frac{f(d)-f(a)}{d-a} - \frac{f(b)-f(a)}{b-a} \cdot \frac{f(d)-f(c)}{d-c}}{\frac{f(b)-f(a)}{b-a} \cdot \frac{f(d)-f(c)}{d-c}}$$

$$\geq \frac{-1}{K^2} \cdot \left| \frac{f(c)-f(b)}{c-b} \cdot \frac{f(d)-f(a)}{d-a} - \frac{f(b)-f(a)}{b-a} \cdot \frac{f(d)-f(c)}{d-c} \right|,$$

where $K = \inf\{|Df(x)|; x \in N \setminus V\}$. Writing

$$f(a+x) = f(a) + \mu(a, x) \cdot x$$

one gets

$$\frac{f(d)-f(a)}{d-a} = \mu(a, d-a),$$

$$\frac{f(b)-f(a)}{b-a} = \mu(a, b-a).$$

Also

$$\begin{aligned} f(c)-f(b) &= \mu(a, c-a) \cdot (c-a) - \mu(a, b-a) \cdot (-c+b-a+c) \\ &= [\mu(a, c-a) - \mu(a, b-a)](c-a) + \mu(a, b-a) \cdot (c-b) \end{aligned}$$

and therefore

$$\frac{f(c)-f(b)}{c-b} = \mu(a, b-a) + \frac{\mu(a, c-a) - \mu(a, b-a)}{c-b} \cdot (c-a)$$

and similarly

$$\frac{f(d)-f(c)}{d-c} = \mu(a, d-a) + \frac{\mu(a, d-a) - \mu(a, c-a)}{d-c} \cdot (c-a).$$

Hence

$$
\begin{aligned}
B(f,T,J)-1 \geq \frac{-1}{K^2}\cdot \Bigg| \mu(a,d-a)\left(\mu(a,b-a)+\frac{\mu(a,c-a)-\mu(a,b-a)}{c-b}\cdot(c-a)\right) \\
-\mu(a,b-a)\cdot\left(\mu(a,d-a)+\frac{\mu(a,d-a)-\mu(a,c-a)}{d-c}\cdot(c-a)\right)\Bigg| \\
=\frac{-1}{K^2}\cdot|c-a|\cdot\Bigg|\mu(a,d-a)\cdot\frac{\mu(a,c-a)-\mu(a,b-a)}{c-b} \\
-\mu(a,b-a)\cdot\frac{\mu(a,d-a)-\mu(a,c-a)}{d-c}\Bigg|.
\end{aligned}
$$

Hence

$$
\begin{aligned}
\frac{B(f,T,J)-1}{|T|} \geq \frac{-1}{K^2}\cdot\Bigg|\mu(a,d-a)\cdot\frac{\mu(a,c-a)-\mu(a,b-a)}{c-b} \\
-\mu(a,b-a)\cdot\frac{\mu(a,d-a)-\mu(a,c-a)}{d-c}\Bigg|.
\end{aligned}
$$

Since f is C^2

$$
\sigma(t)=\sup_{|T|\leq t}\left|\mu(a,d-a)\cdot\frac{\mu(a,c-a)-\mu(a,b-a)}{c-b}-\mu(a,b-a)\cdot\frac{\mu(a,d-a)-\mu(a,c-a)}{d-c}\right|
$$

is a bounded increasing function with $\sigma(t)\to 0$ as $t\to 0$. From this the result follows.

Case 2. $T\subset U$. Since $\alpha\geq 1$, the map $\phi_\alpha(x)=x^\alpha$ has Schwarzian derivative ≤ 0. Then Proposition 2.3 implies $B(\phi_\alpha,\phi(T),\phi(U))\geq 1$. Therefore

$$
B(f,T,U)=B(\phi_\alpha,\phi(T),\phi(U))\cdot B(\phi,T,U)\geq B(\phi,T,U).
$$

Since ϕ is C^2 and $\inf\{|D\phi(x)|; x\in T\}$ is bounded from below, $B(\phi,T,U)\geq 1-\sigma(|T|)\cdot|T|$ follows as in Case 1. Combining this gives (2.2).

Case 3. If T contains a component of $U\setminus V$ then $(B(f,T,M)-1)/|T|$ is bounded from below since $|T|$ is bounded from below. Combining Cases 1, 2 and 3 this finishes the proof of Theorem 2.4. Q.E.D.

Remark. We should emphasise that it is really essential in this theorem that f is C^2. It is *not* sufficient that f' is Lipschitz. This is in contrast with the usual bounded non-linearity results in the theory of A. Denjoy and of A. Schwartz. If in addition f'' is

Lipschitz then it is proved in [MS2] that there exists $C_0>0$ such that

$$B(f^m, T, J) \geqslant \exp\left\{-C_0 \cdot \sum_{i=0}^{m-1} |f^i(T)|^2\right\} \geqslant \exp\left\{-C_0 \cdot \tau \cdot \sum_{i=0}^{m-1} |f^i(T)|\right\}.$$

The previous theorem shows how to find lower bounds for $B(f^m, T, J)$. From the next results the usefulness of these lower bounds will become clear. The first of these are similar to well known properties of (quasi)-conformal mappings.

2.5. "MINIMUM PRINCIPLE." *Let* $T \subset N$ *and* $g: T \to g(T) \subset N$ *be a* C^1 *diffeomorphism with* $T=[a, b] \subset N$. *Let* $x \in (a, b)$. *If for any* $J^* \subset T^* \subset T$.

$$B(g, T^*, J^*) \geqslant C > 0$$

then

$$|Dg(x)| \geqslant C^3 \min\{|Dg(a)|, |Dg(b)|\}. \tag{2.3}$$

Proof. The proof of this principle is given already in [MS1]. In order to be self-contained let us give it here too. Let us consider the following two operators:

$$B_0(g, T^*) = \frac{|g(T^*)|^2}{|T^*|^2} \frac{1}{|Dg(a^*)|\,|Dg(b^*)|}$$

$$B_1(g, T, x) = \frac{|Dg(x)| \dfrac{|g(T)|}{|T|}}{\dfrac{|g(L)|}{|L|} \dfrac{|g(R)|}{|R|}},$$

where $T^*=[a^*, b^*] \subset T$ and L and R are the connected components of $T-\{x\}$. Observe that

$$B_0(g, T^*) = \lim_{J \to T^*} B(g, T^*, J), \quad B_1(g, T, x) = \lim_{J \to x} B(g, T, J).$$

Hence $B_0(g, L), B_0(g, R), B_1(g, T, x) \geqslant C > 0$. Since

$$B_0(g, L), B_0(g, R) \geqslant C$$

we have:

$$\left(\frac{|g(L)|}{|L|}\right)^2 \geqslant C|Dg(a)|\,|Dg(x)|, \quad \left(\frac{|g(R)|}{|R|}\right)^2 \geqslant C|Dg(x)|\,|Dg(b)|.$$

Since $B_1(g, T, x) \geq C > 0$ we have

$$|Dg(x)| \frac{|g(T)|}{|T|} \geq C \frac{|g(L)|}{|L|} \frac{|g(R)|}{|R|}.$$

Since $g|T$ is a diffeomorphism,

$$\min\left\{\frac{|g(L)|}{|L|}, \frac{|g(R)|}{|R|}\right\} \leq \frac{|g(T)|}{|T|} \leq \max\left\{\frac{|g(L)|}{|L|}, \frac{|g(R)|}{|R|}\right\}.$$

Then

$$|Dg(x)|^2 \geq C^2 \left(\frac{|g(L)|/|L| \cdot |g(R)|/|R|}{|g(T)|/|T|}\right)^2 \geq C^2 \min\left\{\left(\frac{|g(L)|}{|L|}\right)^2, \left(\frac{|g(R)|}{|R|}\right)^2\right\}$$
$$\geq C^3 \min\{|Dg(a)|\,|Dg(x)|, |Dg(b)|\,|Dg(x)|\}.$$

Hence $|Dg(x)| \geq C^3 \min\{|Dg(a)|\,|Dg(b)|\}$. Q.E.D.

For maps with negative Schwarzian derivative one version of the next principle was first used and proved in [Str1] and later rediscovered by S. Johnson and J. Guckenheimer, see [Gu2]. See also [Str2] and [Str3].

In order to state this principle it is convenient to introduce the following terminology. Let $U \subset V$ be two intervals. We say that V is a *δ-scaled neighbourhood of U* if each component of $V \setminus U$ has length $\delta|U|$. Similarly we define when V *contains a δ-scaled neighbourhood of U.*

2.6. "Koebe Principle." *For each $C, \tau > 0$ there exists $K(C, \tau) < \infty$ with the following property. Let $g: T \to g(T) \subset N$ be a C^1 diffeomorphism where T is a subinterval of N. Assume that for any intervals J^* and T^* with $J^* \subset T^* \subset T$ one has*

$$B(g, T^*, J^*) \geq C > 0.$$

Let $M \subset T$ and assume that $g(T)$ contains a τ-scaled neighbourhood of $g(M)$ then

$$\frac{1}{K(C, \tau)} \leq \frac{|g'(x)|}{|g'(y)|} \leq K(C, \tau), \quad \forall x, y \in M. \tag{2.4}$$

Proof. See [Str3] for a more general statement of this principle. In order to be complete we will include a proof of (2.4) here. By rescaling we may assume that $M = g(M) = [0, 1]$ and that g is increasing. Let $a, b \in T$ be such that $a < 0 < 1 < b$ and that

$g(a)=-\tau$ and $g(b)=1+\tau, L=[a,0]$, $M=[0,1]$ and $R=[1,b]$. As in the proof of (2.3) one has

(2.5) $$|Dg(0)|\,|Dg(1)|\leq\frac{1}{C}\left(\frac{|g(M)|}{|M|}\right)^2.$$

Similarly

$$|Dg(0)|\geq C\cdot\frac{|g(L)|/|L|\cdot|g(M)|/|M|}{|g(L\cup M)|/|L\cup M|}.$$

Using $|g(M)|=|M|=1$ and

$$\frac{|g(L)|}{|L|}\frac{|L\cup M|}{|g(L\cup M)|}=\frac{\tau}{|L|}\frac{|L|+|M|}{1+\tau}\geq\frac{\tau}{1+\tau},$$

this gives

(2.6) $$|Dg(0)|\geq\frac{C\tau}{1+\tau}.$$

Similarly

(2.7) $$|Dg(1)|\geq\frac{C\tau}{1+\tau}.$$

Combining (2.5)–(2.7) gives that there exists $K'<\infty$ such that

(2.8) $$\frac{1}{K'}\leq|Dg(0)|,\ |Dg(1)|\leq K'.$$

Using the Minimum Principle one obtains that for each $x\in[0,1]$,

(2.9) $$|Dg(x)|\geq\frac{C^3}{K'}.$$

Let $U=[0,x]$ and $V=[x,1]$. Since g is a diffeomorphism one has either $|g(U)|/|U|\leq |g(M)|/|M|=1$ or $|g(V)|/|V|\leq|g(M)|/|M|=1$. If the former holds then

$$\frac{[|g(U)|/|U|]^2}{|Dg(0)|\,|Dg(x)|}\geq C$$

gives

$$|Dg(x)|\,|Dg(0)|\leq\frac{1}{C}\cdot 1.$$

Using (2.6) this gives

$$|Dg(x)|\leq\frac{1}{C}\frac{1+\tau}{C\tau}.$$

From this and (2.9) it follows that there exists $K''<\infty$ such that

$$\frac{1}{K''}\leqslant Dg(x)|\leqslant K''. \tag{2.10}$$

Therefore

$$\frac{1}{(K'')^2}\leqslant\frac{|Dg(x)|}{|Dg(y)|}\leqslant(K'')^2,\quad \forall x,y\in M.$$

Q.E.D.

The next two results will play an important role in proving that the periodic points of high period of a map in $\mathscr{B}$ are repelling and are concerned with the situation that $B(g,T,M)-1$ is positive and bounded from below. The next result states that whenever an interval is mapped monotonically over itself with expansion of the cross-ratios then the map is really 'bending' and therefore at some point expanding.

2.7. "Expansion Principle." *Let T be an interval in N and $g\colon T\to g(T)\subset N$ a C^1 diffeomorphism. For every $\varepsilon>0$ and $\delta>0$ there exists $\varrho>0$ such that the following holds. Let T,M be intervals such that both components of $T\setminus M$ have at least length $\delta\cdot|T|$. If $B(g,T,M)\geq 1+\varepsilon$ and $g(T)\supset T$ then there exists $\theta\in T$ with $Dg(\theta)\geq 1+\varrho$.*

Proof. Let $\xi>0$ be so small that $(1+\varepsilon)(1-\xi)^2\geq 1+\frac{1}{2}\varepsilon$.

Case 1. First suppose that $|g(L)|/|L|\geq 1-\xi$ and $|g(R)|/|R|\geq 1-\xi$. Then, using $B(g,T,M)\geq 1+\varepsilon$, we get

$$\frac{|g(T)|}{|T|}\cdot\frac{|g(M)|}{|M|}\geqslant(1+\varepsilon)(1-\xi)^2\geqslant 1+\frac{1}{2}\varepsilon.$$

So at least one of the terms $|g(T)|/|T|, |g(M)|/|M|$ is greater or equal than $\sqrt{1+\varepsilon/2}$. Using the mean value theorem we obtain in either case $\theta\in T$ so that

$$|Dg(\theta)|\geqslant\sqrt{1+\varepsilon/2}. \tag{2.11}$$

Case 2. Suppose that $|g(L)|/|L|<1-\xi$. The case that $|g(R)|/|R|<1-\xi$ is proved similarly. Then, using $|L|,|R|\geq\delta|T|$, we get

$$\begin{aligned}\frac{|g(M\cup R)|}{|M\cup R|}&=\frac{|g(T)|-|g(L)|}{|M|+|R|}\geqslant\frac{|T|-|g(L)|}{|M|+|R|}\\&>\frac{|T|-(1-\xi)|L|}{|M|+|R|}\geqslant 1+\delta\xi.\end{aligned}$$

The mean value theorem then gives $\theta \in M \cup R$ such that

$$|Dg(\theta)| \geqslant 1+\delta\varepsilon. \tag{2.12}$$

Together, Case 1 and Case 2 prove the result. Q.E.D.

We will need to use the previous lemma in the case where g is an iterate f^n of f. Hence we will need that $B(f^n, T, J)$ is strictly bigger than one. In order to get such a lower bound for $B(f^n, T, J)$ we will use the following result which is based on the non-flatness of the critical point. (This lemma does not hold for piecewise linear maps.)

2.8. LEMMA. *Let $g \in \mathcal{B}$ and c a critical point of g. Then for every $\tau>0$ there exists a neighbourhood U of c and a number $\xi>0$ such that if T is an interval in $U \setminus \{c\}$ with $|T| \geqslant \tau |[c, T)|$ and M the middle third interval in T then*

$$B(g, T, M) \geqslant 1+\xi. \tag{2.13}$$

Proof. We may assume that $N=[-1, 1]$. Let $\hat{B}(g, T)=B(g, T, M)$ where M is the middle third interval in T. Let $b \neq c$, k the order of the critical point and define

$$\phi_b: [c, b] \to [0, 1] \quad \text{by} \quad \phi_b(x) = \frac{|x-c|}{|b-c|},$$

$$\psi_b: [g(c), g(b)] \to [0, 1] \quad \text{by} \quad \psi_b(y) = \frac{|y-g(c)|}{|g(b)-g(c)|},$$

$$g_b: [0, 1] \to [0, 1] \quad \text{by} \quad g_b(x) = \psi_b \circ g \circ \phi_b^{-1}(x)$$

and $g_c(x)=x^k$. Since ϕ_b and ψ_b are affine transformations and the crossratio are invariant under affine transformations, one has for any interval $T=[a, b] \subset [c, b]$

$$\hat{B}(g, T) = \hat{B}\left(g_b, \left[\frac{|a-c|}{|b-c|}, 1\right]\right). \tag{2.14}$$

Let $\Phi: [c, b] \times [0, 1/(1+\tau)] \to \mathbf{R}$ be defined by

$$\Phi(x, y) = \hat{B}(g_x, [y, 1]).$$

Since $g(x)=\pm|\phi(x)|^k+g(x_0)$ where ϕ is a C^2 diffeomorphism and $k>1$ one has that g_b depends continuously on b and g_b tends in the C^0 topology to g_c as $b \to c$ where g_c is the function $g_c(x)=x^k$. Therefore Φ is continuous.

Notice that since $k \geqslant 2$, the map g_c has negative Schwarzian derivative and therefore $B(g_c, V, U)>1$ for any intervals $U\subset V\subset[0,1]$ with $\mathrm{cl}(U)\subset\mathrm{int}(V)$. Therefore the map $[0,1/(1+\tau)]\ni y\to\hat{B}(g_c,[y,1])-1$ has a positive minimum, say $2\xi>0$. So $\Phi(0,y)\geq 1+2\xi$ for all $y\in[0,1/(1+\tau)]$. Let y_0 be such that

$$\Phi(x,y)\geq 1+\xi,\quad \forall(x,y)\in[c,y_0]\times\left[0,\frac{1}{1+\tau}\right]. \tag{2.15}$$

Now let U be a neighbourhood of c with diameter $\leq y_0$. If $I=(a,b)$ is an interval in $U\setminus\{c\}$ and $|I|/|[c,I)|\geq\tau$, then $|b-a|/|c-a|\geq\tau$. Therefore $|a-c|/|b-c|\in[0,1/(1+\tau)]$ and from (2.15),

$$\hat{B}(g,I)=\hat{B}\left(g_b,\left[\frac{|a-c|}{|b-c|},1\right]\right)=\Phi\left(b,\frac{|a-c|}{|b-c|}\right)\geq 1+\xi.$$

This finishes the proof of the lemma. Q.E.D.

3. Analytical estimates on monotone branches of f^n when $f\in\mathscr{A}$: the Macroscopic Koebe Principle

In this section we will analyze the distortion of the cross-ratio for iterates of a map $f\in\mathscr{A}$. We will face two new difficulties. First we have less differentiability as in the previous section and, therefore, we will have to rely on more disjointness assumptions in order to get good bounds for the distortion of the cross ratio of iterates. The second type of problems is that we will need later some estimates on the cross-ratios when the iterates of the interval may contain critical points of inflection type. So we will prove here a version of the Koebe Principle for branches of f^n which are only monotone, and not necessarily diffeomorphic.

The basic strategy is the same as before: we need to estimate the contraction of the cross ratio under high iterates of f on an interval T. Because in this section $f\in\mathscr{A}$, we only have that $\log Df$ has bounded variation. Therefore we need to split up the iterates of T into collections of disjoint intervals. So we start by discussing some disjointness properties of families of intervals.

3.1. *Definition.* The *intersection multiplicity* of a finite collection of intervals in N is the maximal number of intervals in this collection whose interior has a non-empty intersection.

3.2. PROPOSITION. *Let $\mathscr{W}$ be a finite collection of intervals in N with intersection multiplicity at most p then there exists a partition of the collection*

$$\mathscr{W} = A_1 \cup A_2 \cup \ldots \cup A_{2p},$$

such that A_k consists of mutually disjoint intervals for $k=1,2,\ldots,2p$.

Proof. Clearly we may assume that N is connected. So we only need to consider the cases that N is equal to an interval or a circle.

Proof if $N=[-1,1]$. Of course we may assume that all the intervals in $\mathscr{W}$ are open. We claim that if $N=[-1,1]$ then there are p classes

$$A_1, A_2, \ldots, A_p,$$

which form the desired partition of $\mathscr{W}$. Indeed let $\mathscr{I}$ be some collection of intervals in $[-1,1]$. For $I \in \mathscr{I}$ let $\operatorname{next}\{I,\mathscr{I}\}$ be some interval $I' \in \mathscr{I}$ such that (i) I' does not intersect I and is to the right of I and (ii) there is no interval $J \in \mathscr{I}$ satisfying (i) which is closer to I (if there is no interval to the right of I take $\operatorname{next}\{I,\mathscr{I}\}$ to be the empty set). Similarly $\mathscr{R}(\mathscr{I})$ is some interval in $\mathscr{I}$ such that there is no interval in $\mathscr{I}$ which has points to the right of $\mathscr{R}(\mathscr{I})$. For $k=1,2,\ldots,p$ we define inductively A_k as follows. Let $A_0=\emptyset$ and suppose that we have defined by induction A_{k-1}. If $k \le p$ and $\mathscr{B}_k=\mathscr{W} \setminus \bigcup_{i=1,\ldots,k-1} A_i=\emptyset$ then let $A_k=\emptyset$. If $\mathscr{B}_k$ is non-empty then take an interval $I_k \in \mathscr{B}_k$ so that there is no interval in $\mathscr{B}_k$ containing points to the left of I_k. Then define

$$A_{k,1} = \{I_k\},$$

$$A_{k,n+1} = A_{k,n} \cup \operatorname{next}\{\mathscr{R}(A_{k,n}), (\mathscr{W} \setminus (A_0 \cup \ldots \cup A_{k-1} \cup A_{k,n}))\},$$

and

$$A_k = \bigcup_{n \ge 1} A_{k,n}.$$

Then A_k is a collection of intervals with disjoint interiors and the collections $A_1,\ldots,A_p$ are mutually disjoint. Now we will show that $\mathscr{W}=A_1 \cup \ldots \cup A_p$. Suppose this is not the case and there exists an interval I in the collection $\mathscr{W} \setminus (A_1 \cup \ldots \cup A_p)$. Then

$$I \in \mathscr{W} \setminus (A_0 \cup \ldots \cup A_{k-1} \cup A_k)$$

for each $k=1,\ldots,p$ and since $I \notin A_k$ it follows from the inductive definition above that there exist for $i=1,2,\ldots,p$, $T_i \in A_i$ such that the left boundary point of T_i is to the left of (or equal to) the left boundary point x of I. But then since all these intervals are open, points just to the right of this boundary point x are contained in I as well as in T_k,

$k=1,2,\dots,p$. This contradicts the assumption that the intersection multiplicity of $\mathcal{W}$ is at most p.

Proof if $N=S^1$. If $N=S^1$ then choose some $x\in S^1$ and let $I_1,\dots,I_r$ be intervals in $\mathcal{W}$ which contain x. Then $r\leq p$. Since $S^1\setminus\{x\}\simeq(-1,1)\subset[-1,1]$ it follows from the previous case that $\mathcal{W}\setminus\{I_1,\dots,I_r\}$ can be disjointly decomposed into collections $A_1,\dots,A_p$ of disjoint intervals. Then $\mathcal{W}=A_1\cup\dots\cup A_p\cup\bigcup_{k=1,\dots,r}\{I_k\}$ has the desired properties. Q.E.D.

The main result of this section is a version of the Koebe Principle for maps f^n restricted to an interval T such that $f^n|T$ is monotone (but may have critical points of inflection type) and such that the collection $T,f(T),\dots,f^{n-1}(T)$ has low intersection multiplicity.

3.3. Macroscopic Koebe Principle. *Let $f\in\mathcal{A}$. Then there exists a strictly positive function $B_0\colon(0,1)\to(0,1)$ such that for any pair of intervals $M\subset T$, any $n\geq0$ and any $0<\varepsilon<1$, satisfying the following conditions:*

(a) *$f^n(T)$ contains an ε-scaled neighbourhood of $f^n(M)$;*

(b) *the intersection multiplicity of $\{T,f(T),\dots,f^{n-1}(T)\}$ is at most* 17;

(c) *$f^i(M)\cap K_f=\varnothing$, $i\in\{0,1,\dots,n-1\}$*

one has:

T is a $B_0(\varepsilon)$-scaled neighbourhood of M.

For the proof of this theorem we will need a number of lemmas. Let $f\in\mathcal{A}$. Let $\mathcal{V}\subset\mathrm{cl}(\mathcal{V})\subset\mathrm{int}(\mathcal{U})$ be interval neighbourhoods of K_f such that the number of components of $\mathcal{V}$ and $\mathcal{U}$ are both the same as $\#K_f$.

3.4. Lemma. *Let f be a map in $\mathcal{A}$ and $\Xi=\{T_1,T_2,\dots,T_n\}$ a collection of intervals in N and $M_i\subset T_i$. Assume that the intersection multiplicity of Ξ is at most* 17 *and that none of the intervals T_i contains a component of $\mathcal{U}\setminus\mathcal{V}$ or contains points of K_f. Then there exists $V<\infty$ such that*

$$\sum_{i=1}^{n}\log B(f_i,T_i,M_i)\geq -V. \tag{3.1}$$

Proof. Since $f\in\mathcal{A}$, the restriction of $\log|Df|$ to $N\setminus\mathcal{V}$ exists almost everywhere and $\log|Df|$ has bounded variation on this set. Let

$$V_f=\mathrm{Var}(\log[|Df|(N\setminus\mathcal{V})])<\infty.$$

For each $x_i \in K_f$ let U_i be the component of $\mathcal{U}$ which contains x_i. Since $f \in \mathcal{A}$, $f(x) = \phi_{\alpha_i} \circ \phi_i(x) + f(x_i)$, where $\phi_{\alpha_i}(x) = \pm|x|^{\alpha_i}$ and $\alpha_i \geq 1$. Here $\phi_i \colon U_i \to (-1, 1)$ is a homeomorphism and $\log|D\phi_i|$ exists almost everywhere and has bounded variation. Hence $V' = \Sigma_i \operatorname{Var}(\log|D\phi_i|)$ is finite. Let $V = 34(V_f + V')$.

Let $I_1 = \{i; T_i \cap \mathcal{V} = \emptyset\}$ and $I_2 = \{i; T_i \subset \mathcal{U}\}$. Since the intervals T_i never contain a component of $\mathcal{U} \setminus \mathcal{V}$, $I_1 \cup I_2 = \{1, 2, \ldots, n\}$.

First assume that $i \in I_1$. Let L_i and R_i be the components of $T_i \setminus M_i$. For $u_i, v_i \in T_i$ let (u_i, v_i) be the open interval connecting u_i and v_i. Because Df exists almost everywhere there exist $m_i \in M_i$, $l_i \in L_i$, $r_i \in R_i$ and $\tau_i \in T_i$ such that Df exists in these points and such that $|f(M_i)|/|M_i| \geq |Df(m_i)|$, $|f(T_i)|/|T_i| \geq |Df(\tau_i)|$, $|f(L_i)|/|L_i| \leq |Df(l_i)|$ and $|f(R_i)|/|R_i| \leq |Df(r_i)|$. Using this in the definition of $B(f, T_i, M_i)$ we find that there exist $l_i \in L_i$, $r_i \in R_i$, $m_i \in M_i$ and $\tau_i \in T_i$ such that

$$\log B(f, T_i, M_i) \geq \log\left(\frac{|Df(m_i)|\,|Df(\tau_i)|}{|Df(l_i)|\,|Df(r_i)|}\right), \tag{3.2}$$

and

$$m_i \in (l_i, r_i). \tag{3.3}$$

From (3.2) and the choice of the points l_i, m_i, r_i, τ_i one has

$$\log B(f, T_i, M_i) \geq -\{|\log|Df(m_i)| - \log|Df(l_i)|| + |\log|Df(\tau_i)| - \log|Df(r_i)||\} \tag{3.4}$$

and also

$$\log B(f, T_i, M_i) \geq -\{|\log|Df(m_i)| - \log|Df(r_i)|| + |\log|Df(\tau_i)| - \log|Df(l_i)||\}. \tag{3.5}$$

Rename the points l_i, m_i, r_i, τ_i in increasing order a_i, b_i, c_i, d_i. From (3.3) one gets that either $(l_i, m_i) \cap (\tau_i, r_i) = \emptyset$ or $(\tau_i, l_i) \cap (m_i, r_i) = \emptyset$, and so we can use either (3.4) or (3.5) and get

$$\begin{aligned} \log B(f, T_i, M_i) &\geq -\{|\log|Df(b_i)| - \log|Df(a_i)||\} + |\log|Df(d_i)| - \log|Df(c_i)||\} \\ &\geq -\operatorname{Var}(\log|Df|T_i|) \end{aligned} \tag{3.6}$$

and therefore

$$\log B(f, T_i, M_i) \geq -\operatorname{Var}(\log|Df|T_i|). \tag{3.7}$$

Now consider $i \in I_2$. Then T_i is contained in some component U_i of $\mathcal{U}$ (and does not intersect K_f) and so f has the form $f(x)=\phi_{\alpha_i}\circ\phi_i$. Hence

$$B(f, T_i, M_i) = B(\phi_{\alpha_i}, T_i', M_i')\times B(\phi_i, T_i, M_i).$$

Here $T_i'=\phi_i(T_i)$ and $M_i'=\phi_i(M_i)$. Since the Schwarzian derivative of ϕ_{α_i} is less or equal to 0 (because $\alpha_i\geqslant 1$) one gets $B(\phi_{\alpha_i}, T_i', M_i')\geqslant 1$. Hence, as above,

$$\begin{aligned}\log B(f, T_i, M_i) &\geqslant 0+\log B(\phi_i, T_i, M_i)\\ &\geqslant -\mathrm{Var}(\log|D\phi_i|T_i|).\end{aligned}\tag{3.8}$$

Since the intersection multiplicity of Ξ is at most 17, using Proposition 3.2, one can write $\Xi=A_1\cup A_2\ldots\cup A_{34}$ where A_j consists of a collection of mutually disjoint intervals. Hence from (3.7) and (3.8) one gets

$$\sum_{i=1}^{n}\log B(f, T_i, M_i)\geqslant -34\cdot(V_f+V').\tag{3.9}$$

The lemma follows. Q.E.D.

3.5. Lemma. *Let $f\in\mathscr{A}$. Then there exists $A_0>0$ such that if $I\subset W$ are intervals in N such that*

(a) *$W\setminus I$ consists of one component H;*

(b) $|H|\leqslant|I|$;

(c) $I\cap K_f=\varnothing$,

then

$$\frac{|f(I)|}{|I|}\geqslant A_0\cdot\frac{|f(H)|}{|H|}.\tag{3.10}$$

Proof. The proof of this lemma is elementary and can be found in [MMMS]. Q.E.D.

3.6. Lemma. *Let $f\in\mathscr{A}$ and I, T be intervals with* $\mathrm{cl}(I)\subset\mathrm{int}(T)$ *and L and R be the components of $T\setminus I$. Let $\gamma\in(0,1)$. If*

$$\frac{|f(I)|}{|f(R)|}\geqslant\gamma\cdot\frac{|I|}{|R|}\quad\textit{and}\quad\frac{|f(I)|}{|f(L)|}\geqslant\gamma\cdot\frac{|I|}{|L|},$$

then

$$B(f, T, I)\geqslant\gamma^2.$$

Proof. Let

$$d=\left(\frac{|I||I|}{|R||L|}\right)\Big/\left(\frac{|I|}{|R|}+\frac{|I|}{|L|}\right).$$

Then

$$B(f,T,I)=\frac{\dfrac{|f(I)|}{|f(R)|}+\dfrac{|f(I)|}{|f(R)|}\dfrac{|f(I)|}{|f(L)|}+\dfrac{|f(I)|}{|f(L)|}}{\dfrac{|I|}{|R|}+\dfrac{|I|}{|R|}\dfrac{|I|}{|L|}+\dfrac{|I|}{|L|}}\geqslant\frac{\gamma+\gamma^2\cdot d}{1+d}=\gamma\cdot\frac{1+\gamma\cdot d}{1+d}\geqslant\gamma^2.$$

The last inequality follows since $\gamma\in(0,1)$ and therefore $(1+\gamma x)/(1+x)$ is always greater than γ for $x>0$. The result follows. Q.E.D.

3.7. LEMMA. *Let $f\in\mathscr{A}$. Then there exists $A_1>0$ such that if I, T are intervals with* $\mathrm{cl}(I)\subset\mathrm{int}(T)$ *and L and R the components of $T\setminus I$ such that*

(a) $|L|\leqslant|I|$ *or* $|R|\leqslant|I|$;

(b) $I\cap K_f=\varnothing$;

then

$$B(f,T,I)\geqslant A_1.$$

Proof. Let A_0 be the number from Lemma 3.5. We may assume that $A_0\in(0,1)$. We will prove the lemma for $A_1=\frac{1}{3}\cdot(A_0)^2$. By possibly renaming L and R, we may consider the case that $|R|\leqslant|I|$. Then from Lemma 3.5 we get

$$\frac{|f(I)|/|I|}{|f(R)|/|R|}\geqslant A_0, \tag{3.11}$$

and hence

$$B(f,T,I)\geqslant\frac{|f(T)|}{|f(L)|}\cdot\frac{|L|}{|T|}\cdot A_0\geqslant A_0\cdot\frac{|L|}{|T|}.$$

If $|L|\geqslant|I|$ then it follows from this and $|R|\leqslant|I|$ that

$$B(f,T,I)\geqslant A_0\cdot\frac{|L|}{|L|+|I|+|R|}\geqslant A_0\cdot\frac{|L|}{|L|+|L|+|L|}=A_0\cdot\frac{1}{3}\geqslant A_1$$

and the lemma is proved. So assume that $|L|\leqslant|I|$. Then applying Lemma 3.5 again we get

$$\frac{|f(I)|/|I|}{|f(L)|/|L|} \geq A_0, \tag{3.12}$$

and it follows from (3.11), (3.12) and Lemma 3.6 that $B(f,T,I) \geq A_0 \cdot A_0 > A_1$. Q.E.D.

Proof of Theorem 3.3. Let $\varepsilon \in (0,1)$. Let $m \leq n$ be the smallest number such that $|f^m(L)| \geq \varepsilon |f^m(M)|$ and $|f^m(R)| \geq \varepsilon |f^m(M)|$. Let A_1 be the number from Lemma 3.7 and assume $A_1 < 1$. Let $V > 0$ be the number from Lemma 3.4. Let

$$B_1 = [A_1]^{51\#K_f+1} \cdot e^{-V}.$$

We claim that

$$B(f^m, T, M) \geq B_1.$$

Indeed let $t(1) < t(2) < \ldots < t(s) < m$ be the integers $t < m$ such that $f^t(T)$ contains a component of $\mathcal{U} \setminus \mathcal{V}$ or such that $f^t(T) \cap K_f \neq \emptyset$. Since the intersection multiplicity of $\{T, f(T), \ldots, f^{n-1}(T)\}$ is at most 17 one gets $s < 51\#K_f$. From the choice of m either $|f^{t(i)}(L)| \leq \varepsilon |f^{t(i)}(M)| \leq |f^{t(i)}(M)|$ or $|f^{t(i)}(R)| \leq \varepsilon |f^{t(i)}(M)| \leq |f^{t(i)}(M)|$. So from Lemma 3.7

$$B(f, f^{t(i)}(T), f^{t(i)}(M)) \geq A_1.$$

From Lemma 3.4

$$\sum_{j \leq m-1,\, j \notin \{t(1), \ldots, t(s)\}} \log B(f, f^j(T), f^j(M)) \geq -V.$$

Hence the claim follows. From the claim and the definition of the operator B the theorem easily follows. Q.E.D.

4. Some simplifications and the induction assumption

The proof of Theorem A will go by induction on the number of turning points of f. So let $\mathcal{A}^d$ be the collection of all endomorphisms of N in $\mathcal{A}$ with $f(\partial N) \subset \partial N$ and with precisely d turning points. So we will prove inductively that

$$\text{maps in } \bigcup_{i=0}^{d} \mathcal{A}^i \text{ have no wandering intervals.} \tag{Ind$_d$}$$

Because the proof of Theorem A goes by induction on the number of turning points we have to consider a more general situation: the manifold N is not necessarily connected (but does consist of a finite number of components). This does not give, however, a more general result. In fact if a map $f: N \to N$ has a wandering interval then one of the connected components N' of N is periodic of period s and the map $f^s: N' \to N'$ has a wandering interval. Therefore, if we prove the theorem for connected manifolds we get also the same theorem for non-connected manifolds. However, if f has d turning points, the map f^s may have more than d turning points. This is the main reason to start with a disconnected manifold. So if Ind_{d-1} holds then we may use the following fact: *if there exists a finite disjoint union of intervals which is invariant by f and contains a wandering interval then the union of these intervals contains at least d turning points.*

By extending f to a slightly bigger interval we may assume that

$$f(\partial N) \subset \partial N.$$

It suffices to prove the theorem for maps f such that none of the turning points is contained in (the closure) of a wandering interval. Indeed, otherwise f also has a wandering interval W containing a turning point c in its interior. Now modify the map f in a small neighbourhood $V \subset W$ of c to a map $g \in \mathcal{A}$ such that $f = g$ on $N - V$ for which $g|V$ has a unique turning point in c and $g(c)$ is not contained in a wandering interval. Then $f(W)$ is still a wandering interval for g (the forward iterates of $f(W)$ under f and g are the same because these iterates never enter W) and $g(c)$ is not contained in a wandering interval of g. Repeating this procedure for every wandering interval which contains a turning point in its closure we get a map g which still has wandering intervals but such that none of its turning points is contained in the closure of a wandering interval. So it suffices to prove the theorem for g.

Furthermore, for each turning point c of f there exists a neighbourhood S_c of c and a continuous involution $\tau: S_c \to S_c$ (which is not the identity) such that $f(\tau(x)) = f(x)$ for all $x \in S_c$. From the non-flatness conditions for maps in $\mathcal{A}$ the map τ is Lipschitz.

5. The pullback of space: the Koebe/Contraction Principle

In this section we will start the proof of the non-existence of wandering intervals for maps in $\mathcal{A}$. This will be proved by contradiction. More precisely, suppose that J is a wandering interval and that J is not contained in a larger wandering interval. The strategy of the proof is to show that there exists necessarily an interval I which strictly contains J such that $\inf_{n \geq 0} |f^n(I)| = 0$. Using the next principle, which also can be found in [L], it follows that I is a wandering interval (since $f^n(J)$ does not converge to a periodic orbit the same holds for $f^n(I)$), contradicting the maximality of J.

5.1. CONTRACTION PRINCIPLE. *Suppose I is an interval such that*

$$\inf_{n\geq 0} |f^n(I)| = 0$$

then I is a wandering interval or there exists a periodic orbit $\mathcal{O}$ such that $f^k(I)\to\mathcal{O}$ as $k\to\infty$. In particular, if I contains a wandering interval J then it is also a wandering interval.

Proof. Let $\mathcal{I}=\bigcup_{n\geq 0} f^n(\operatorname{int}(I))$. Then $\mathcal{I}$ is forward invariant.

First suppose that there exists a component U of $\mathcal{I}$ and $n>0$ such that $f^n(U)\cap U\neq\varnothing$. Since $\mathcal{I}$ is forward invariant this implies $f^n(U)\subset U$. There are three cases.

(1) U is an interval which contains a fixed point p of $f^n\colon U\to U$ in its interior. In this case some iterate of I contains this fixed point of f^n in its closure and since $\inf_{k\geq 0}|f^k(I)|=0$ this fixed point of f^n must attract I, $f^k(I)\to O(p)$ as $k\to\infty$. So we are finished in this case.

(2) U is an interval and there exists no fixed point as in (1). Then $\operatorname{cl}(U)$ contains in its boundary an attracting fixed point p of $f^n\colon \operatorname{cl}(U)\to\operatorname{cl}(U)$. If $f^n(U)\neq U$ then every point in $\operatorname{cl}(U)$ is asymptotic to $O(p)$, $\omega(I)=O(p)$. If $f^n(U)=U$ then the boundary point $\{q\}=\partial U\setminus\{p\}$ is a repelling periodic point and $\inf_{k\geq 0}|f^k(I)|=0$ implies that no iterate of I contains q in its closure. Since every point in $\operatorname{int}(U)$ is asymptotic to $O(p)$ this implies that $f^k(I)\to O(p)$. Again the result follows.

(3) U is a circle. Then there exists a finite collection $n_1<\ldots<n_r$ of positive integers such that $\bigcup_{i=1}^r f^{n_i}(\operatorname{int}(I))$ covers S^1. But since $\inf_{n\geq 0}|f^n(I)|=0$ this implies that there exists an integer $n>n_r$ such that $f^n(I)$ is strictly contained in $f^{n_i}(\operatorname{int}(I))$ for some $i=1,\ldots,r$. But then f^{n-n_i} has an attracting fixed point in $f^{n_i}(\operatorname{int}(I))$ which attracts I. So again the result follows.

Now assume that for every component U of $\mathcal{I}$ one has $f^n(U)\cap U=\varnothing$ for all $n\geq 1$. Since $\mathcal{I}$ is forward invariant and this holds for each component, this implies that $f^n(U)\cap f^m(U)=\varnothing$ for all $n>m\geq 0$. It follows that U and therefore I is a wandering interval (or asymptotic to a periodic orbit). Q.E.D.

5.2. *Definition.* The *pullback* of $P_n\supset J_n$ is the sequence of intervals

$$\{P_i \mid i=0,1,\ldots,n\}$$

where P_{i-1} is the maximal interval containing J_{i-1} such that

$$f(P_{i-1})\subset P_i$$

for each $i=1, \ldots, n$. The integers i for which P_i contains a turning point in its closure are called the *cutting times*. It will turn out to be useful to call n also a cutting time. This pullback is *monotone* if $f^n|P_0$ is monotone and $f^n(P_0)=P$. Furthermore it is said to be *diffeomorphic* if $f^n|P_0$ is a diffeomorphism and $f^n(P_0)=P$ and *unimodal* if P_i contains at most one turning point for each i.

All the pullbacks we will consider are unimodal because of the following lemma:

5.3. LEMMA. *There exists $\eta>0$ such that if $P_n \supset J_n$ and $|P_n| \leq \eta$ then the pullback of P_n is unimodal.*

Proof. Follows from the Contraction Principle and since by assumption no wandering interval contains two turning points. Q.E.D.

5.4. CONTRACTION/KOEBE PRINCIPLE. *For each $\varepsilon>0$ and each $p \in \mathbf{N}$ there exists $N_0(\varepsilon, p)$ with the following properties. Let $P_n \supset J_n$ contain an ε-scaled neighbourhood of J_n. If the pullback of P_n has intersection multiplicity $\leq p$ then $n \leq N_0(\varepsilon, p)$.*

Proof. Let $\varrho=\min(\varepsilon, 1/2)$ and $\hat{P}_n \subset P_n$ be a ϱ-scaled neighbourhood of J_n. Let $m(0)<m(1)<\ldots<m(l)=n$ be the cutting times of the pullback of P_n. Since the intersection multiplicity of the pullback is $\leq p$, $l \leq p \cdot d$. Let $\hat{P}_i^{\pm}$ be the components of $\hat{P}_i \setminus J_i$. For $t=0, \ldots, l-1$, let $P_{m(t)}^{+}$ be the component which contains the turning point. Now the map

$$f^{n-m(l-1)+1}: \hat{P}_{m(l-1)+1} \rightarrow \hat{P}_n$$

is monotone. Since $\hat{P}_n$ is a ϱ-scaled neighbourhood of J_n it follows from the Macroscopic Koebe Principle that both components of $\hat{P}_{m(l-1)+1} \setminus J_{m(l-1)+1}$ have length at least $B_0(\varrho)$. From this and the non-flatness of the turning points we get that there exists a universal constant $C \in (0,1)$ such that

$$|\hat{P}_{m(l-1)}^{-}| \geq C \cdot B_0(\varrho) \cdot |J_{m(l-1)}|.$$

Because $\hat{P}_{m(l-1)}$ is symmetric around a turning point, we also have that $|\hat{P}_{m(l-1)}^{+}| \geq |J_{m(l-1)}|$. Therefore

$$|\hat{P}_{m(l-1)}^{\pm}| \geq C \cdot B_0(\varrho) \cdot |J_{m(l-1)}|.$$

Repeating this $l \leq p \cdot d$ times we get, letting $g(x)=C \cdot B_0(x)$,

$$(*) \qquad |\hat{P}_0^{\pm}| \geq g^l(\varrho) \cdot |J|.$$

Hence there exists an interval P which does not depend on n and which *strictly* contains the wandering interval J such that $|f^n(P)| \le (1+2\varrho)|J_n| \le 2|J_n|$ for each n as above. Now if there exists no upperbound $N_0(\varepsilon, p)$ for n then we would get $|f^{n_i}(P)| \le 2|J_{n_i}| \to 0$ for some sequence $n_i \to \infty$ and since P contains J this would imply from the Contraction Principle that P is also a wandering interval. But this contradicts the maximality of J. Q.E.D.

6. Disjointness of orbits of intervals

In this section we will give some upperbounds on the intersection multiplicity of orbits of intervals. These bounds are needed in order to apply the Macroscopic Koebe Principle. Let $J_n = f^n(J)$. We will define a natural neighbourhood T_n of J_n such that its monotone pullback has good disjointness properties. (For those familiar with the circle homeomorphisms these neighbourhoods will coincide with the neighbourhood $[f^{q_{k-2}}(J), f^{q_{k-2}+(a_k-1)q_{k-1}}(J)]$ of $f^{q_k}(J)$ when $n = q_k$ and with

$$[f^{q_{k-2}+(i-1)q_{k-1}}(J), f^{q_{k-2}+(i+1)q_{k-1}}(J)]$$

of $f^{q_{k-2}+iq_{k-1}}(J)$ when $n = q_{k-2}+iq_{k-1}$ and $1 \le i \le a_k - 1$.)

6.1. *Definition*. We say that J_{n_1} and J_{n_2} have the *same orientation* if the forward iterate of f which sends one of these intervals in the other is orientation preserving. If $n \in \mathbb{N}$, we say that J_k is a *predecessor* of J_n if $0 \le k < n$, if J_k and J_n have the same orientation and if $J_s \subset (J_k, J_n)$ and $0 \le s < n$ implies that J_s and J_n have different orientations. If J_n has a predecessor to its left (right) then we denote the corresponding iterate by $L(n)$ (respectively $R(n)$). J_n has a *successor* J_{n+a} if

(1) J_{n-a} is predecessor of J_n (with $0 < a \le n$);

(2) $f^a|[J_{n-a}, J_{n+a}]$ is monotone, orientation preserving, and its image contains no predecessor of J_n (if $L(n)$ and $R(n)$ both exist and if for example $n-a = L(n)$ then this implies that $f^t[J_{n-a}, J_{n+a}] \subset [J_n, J_{R(n)})$;

(3) if $J_k \subset (J_n, J_{n+a})$ and $k = 0, 1, \ldots, n+a-1$ then the intervals J_k and J_{n+a} have different orientations.

Next we define the *natural neighbourhood* of J_n to be the biggest open interval containing J_n which contains no neighbourhood of a predecessor or successor.

Remark. Of course J_n can have at most one predecessor on each side and it has a predecessor to, say, its right if there exists an interval J_s with $s < n$ to the right of J_n with the *same orientation* as J_n. Moreover, as we will see in the lemma below an interval J_n

has at most one successor; so denote this successor by $s(n)$. Therefore, if J_n has two predecessors and no successor then the natural neighbourhood T_n of J_n is equal to $T_n=[J_{L(n)}, J_{R(n)}]$ and if it has a successor then it is equal to

$$T_n=[J_{L(n)}, J_{s(n)}] \quad \text{or} \quad T_n=[J_{s(n)}, J_{R(n)}].$$

6.2. LEMMA. *For every $n\in\mathbf{N}$, J_n can have at most one successor.*

Proof. Suppose that J_n had two successors. Then it also has two predecessors, $J_{L(n)}$ and $J_{R(n)}$. By definition $f^{n-L(n)}\colon [J_{L(n)}, J_n]\to[J_n, J_{R(n)}]$ and $f^{n-R(n)}\colon [J_n, J_{R(n)}]\to[J_{L(n)}, J_n]$ are both monotone. Hence J is attracted to a periodic point, a contradiction. Q.E.D.

6.3. LEMMA. *Assume that the interval J_n has two predecessors $J_{L(n)}, J_{R(n)}$, and a successor $J_{s(n)}$. If this successor is to the right of J_n then the predecessors of $J_{s(n)}$ are J_n and $J_{R(n)}$ and if $J_{s(n)}$ has a successor then this successor is between $J_{s(n)}$ and $J_{R(n)}$.*

Proof. The left predecessor of $s(n)$ is n by the definition of $s(n)$. The right predecessor of $s(n)$ is certainly defined because $J_{R(n)}$ and $J_{s(n)}$ have the same orientation and $R(n)<n<s(n)$. Let us show that $R(n)$ is this predecessor. If this was not the case then there exists $k<s(n)$ so that $J_k\subset(J_{s(n)}, J_{R(n)})$ and so that J_k and $J_{s(n)}$ have the same orientation. Because of the definition of $R(n)$ this implies certainly that $k>n$. Because $k<s(n)$ this implies that $0\leq k-n<s(n)-n=n-L(n)$ and $L(n)+k-n<n$. From the first of these two inequalities and the definition of $s(n)$ it follows that f^{k-n} is monotone and orientation preserving on $H=[J_{L(n)}, J_n]$. In particular $J_{L(n)+k-n}$ has the same orientation as J_n. From this, the definition of $L(n)$ and $R(n)$ and the second of these inequalities it follows $J_{L(n)+k-n}$ cannot be between $J_{L(n)}$ and $J_{R(n)}$. It follows that

$$f^{k-n}(H)\supset[J_{L(n)}, J_k]\supset[J_n, J_{s(n)}]=f^{n-L(n)}(H).$$

In particular, $f^{(n-L(n))-(k-n)}$ maps $f^{k-n}(H)$ monotonically into itself, and hence J would be attracted to a periodic attractor, a contradiction.

Let us finally show that $s(n)$ cannot have a successor to its left. Indeed if it did, then by definition $f^{s(n)-R(n)}$ would map $[J_{s(n)}, J_{R(n)}]$ monotonically into $[J_n, J_{s(n)}]$. From the definition of $s(n)$, $f^{s(n)-n}$ maps $[J_n, J_{s(n)}]$ monotonically into $[J_{s(n)}, J_{R(n)}]$. Combining this gives that J is attracted to a periodic attractor. With this contradiction the proof of this lemma is completed. Q.E.D.

Remark. The previous lemma implies that if J_n has a successor $J_{s(n)}$ and $J_{s(n)}$ also has a successor $J_{s(s(n))}$ then $s(n)-n=a=s(s(n))-s(n)$ and $J_{s(n)}$ is between J_n and $J_{s(s(n))}$. So continuing this there exists a maximal integer k such that $J_{s^{i+1}(n)}$ is a successor of $J_{s^i(n)}$ for $i=0,1,\dots,k-1$. In this case the intervals

$$J_{s(n)},\ J_{s(s(n))},\ \dots,\ J_{s^k(n)}(n)$$

lie ordered and $f^a|[J_n, J_{s^k(n)}]$ is monotone. So f^a acts as a translation on these intervals.

6.4. THEOREM. *Let $n\in\mathbf{N}$ and assume that J_n has two predecessors $J_{R(n)}$ and $J_{L(n)}$. Let M_n be an open interval contained either in $[J_n, J_{R(n)}]$ or in $[J_{L(n)}, J_n]$. Assume that $\{M_{t_0}, M_{t_0+1}, \dots, M_n\}$ is a monotone pullback of M_n. If the intersection multiplicity of this piece is at least $2p$ and $p\geq 2$ then there exists $t\in\{t_0,\dots,n\}$ such that*

(1) $J_{s(t)}, J_{s^2(t)}, \dots, J_{s^{2p-2}}(t)$ are defined;

(2) $n=s^p(t)$ and $J_{s^j(t)}$ is contained in M_n for $j=p,\dots,2p-2$.

COROLLARY. *If the pullback of an interval $T\supset J_n$ with T contained in the natural neighbourhood T_n is monotone then the intersection multiplicity of this pullback is at most* 11. *Similarly, if $T\supset J_n$ and $s^k(n)$ does not exist then the monotone pullback of T has at most intersection multiplicity $2k+4$.*

Proof of Corollary. Consider the pullback of $T\cap[J_n, J_{R(n)}]$ and $T\cap[J_{L(n)}, J_n]$ separately. If the intersection multiplicity of T is at least 12 then the pullback of either $T\cap[J_n, J_{R(n)}]$ or $T\cap[J_{L(n)}, J_n]$ has intersection multiplicity ≥ 6. So take $p=3$ and the previous theorem implies that $p+1\leq 2p-2$ and $J_{s(n)}$ is contained in $T\cap[J_n, J_{R(n)}]$, which is impossible since $T\subset T_n$. The second statement follows also immediately. (Note that $2k+4\geq 6$ for $k\geq 1$.) Q.E.D.

Proof of Theorem 6.4. In order to be definite assume that $M_n\subset[J_n, J_{R(n)}]$. By assumption there exists a point y which is contained in $2p$ of the intervals $M_{t_0},\dots,M_n$. Of course this implies that for at least p of these intervals M_i the corresponding intervals J_i all lie on the same side of y. So let $N\geq p$ be the maximal number of distinct integers $t\leq i(1),\dots,i(N)\leq n$ such that each of the intervals $M_{i(1)},\dots,M_{i(N)}$ contains this point y, $J_{i(j)}$ all lie on one side of y and that these are labeled so that

$$[J_{i(1)}, y]\supset[J_{i(2)}, y]\supset\dots\supset[J_{i(N)}, y].$$

Since M_i has a common endpoint with J_i this implies that the intervals $J_{i(1)},\dots,J_{i(N)}$ all have the same orientation.

Claim 1. $i(1)<i(2)<\dots<i(N)$ and we may assume that $i(N)=n$. Furthermore $J_{i(1)}$ cannot be contained in $f^t(M_{i(1)})$ for $t=1,\dots,i(N)-i(1)$ if f^t is orientation preserving on

$M_{i(1)}$. In particular if we take $a=i(N)-i(N-1)$ then f^a maps $[J_{i(1)}, J_{i(N-1)}]$ (which is contained in $M_{i(1)}$) monotonically into $(J_{i(1)}, J_{i(N)}]$.

Proof of Claim 1. For $j \in \{2,3,\dots,N\}$ the interval $J_{i(j)}$ is contained in $[J_{i(j-1)}, y] \subset M_{i(j-1)}$ and $J_{i(j)}$ and $J_{i(j-1)}$ have the same orientation. Since $f^{n-i(j-1)}: M_{i(j-1)} \to M_n \subset [J_n, J_{R(n)}]$ is monotone this implies that $J_{i(j)+n-i(j-1)}$ has the same orientation as J_n and is between J_n and $J_{R(n)}$. Since $J_{R(n)}$ is a predecessor of J_n this implies that $i(j)+n-i(j-1)>n$ and therefore $i(j)>i(j-1)$. Let us show that we may assume that $i(N)=n$. Indeed from what we have shown $f^{n-i(N)}$ is monotone on $M_{i(j)}$ for $j=1,\dots,N$. In particular letting $i'(j)=i(j)+(n-i(N))$, and taking the images of these intervals under this map we get that $M_{i'(1)},\dots,M_{i'(N)}$ all contain one point y', $J_{i'(j)}$ all lie on one side of y' and that these are labeled so that $[J_{i'(1)}, y] \supset [J_{i'(2)}, y] \supset \dots \supset [J_{i'(N)}, y]$. Since $i'(N)=n$ we may as well assume that $i(N)=n$. The first statement of the claim follows and it follows that f^a is monotone and orientation preserving on $M_{i(1)} \supset [J_{i(1)}, J_{i(N-1)}]$. Now $J_{i(1)}$ cannot be contained in $f^t(M_{i(1)})=M_{i(1)+t}$ for $t=1,\dots,i(N)-i(1)$ because otherwise $J_{i(N)-t}$ would be contained in $M_{i(N)}=M_n$ and would have the same orientation as J_n. But this is impossible because $M_n \subset [J_n, J_{R(n)}]$ and $J_{R(n)}$ is a predecessor of J_n. Q.E.D.

Claim 2. If $k<n$, J_k and J_n have the same orientation and $J_k \subset [J_{i(1)}, J_n]$ then $k \in \{i(1), i(2),\dots, i(N)\}$. Furthermore if we let $a=i(N)-i(N-1)$ then $i(j+1)-i(j)=a$ for $j=1,2,\dots,N-1$.

Proof. Let $j<n$ be maximal such that $J_k \subset [J_{i(j)}, y] \subset M_{i(j)}$. Therefore $f^{n-i(j)}$ maps J_k into M_n. Because J_k, $J_{i(j)}$ and J_n all have the same orientation, it follows that $J_{k+n-i(j)}$ also has the same orientation. This implies that $k \geqslant i(j)$. Suppose $k>i(j)$. If $M_k \not\supset J_n$ then $M_k \subset M_{i(j)}$ and $f^{k-i(j)}$ maps $M_{i(j)}$ monotonically into $M_k \subset M_{i(j)}$. This implies that J is attracted by a periodic orbit, contradiction. Hence, $M_k \not\supset J_n$ and by the maximality of N, $k \in \{i(0),\dots, i(N)\}$. This proves the first statement of the claim. From Claim 1, f^a maps $[J_{i(1)}, J_{i(N-1)}]$ monotonically and orientation preserving into $(J_{i(1)}, J_{i(N)}]$. It follows from this and the first part of this claim that $i(j)+a=i(j+1)$ for $j=1,\dots,N-2$. Thus Claim 2 is proved.

Define $i(N+j)=n+j\cdot a$ for $j=1,\dots,N$. As we have shown in the previous claim this formula holds for j negative. So we get

$$i(N+j) = n+j\cdot a \quad \text{for} \quad j=-N+1, -N+2, \dots, N-1, N.$$

Now consider the interval

$$H = [J_{i(1)}, J_{i(N)}].$$

The map $f^{n-i(1)}=f^{(N-1)a}$ maps $H\subset M_{i(1)}$ monotonically and orientation preserving into $M_n\subset[J_n, J_{R(n)}]$. In particular f^a maps $[J_{i(1)}, J_{i(2N-2)}]$ monotonically and orientation onto $[J_{i(2)}, J_{i(2N-1)}]$. Therefore $J_{i(j)}$ lies between $J_{i(j-1)}$ and $J_{i(j+1)}$ for $|j|\leqslant N-2$. Furthermore if $f^t(H)\cap[J_{i(1)}, J_{R(n)}]\neq\emptyset$ for some $t=1,\ldots,(N-1)a$ then $f^t|H$ is either orientation reversing or t is a multiple of a. Indeed, assume that $f^t|H$ is orientation preserving and that this intersection is non-empty. By Claim 1, $f^t(H)\subset f^t(M_{i(1)})$ does not contain $J_{i(1)}$. So $f^t(J_{i(1)})$ must be contained in $[J_{i(1)}, J_{R(n)}]$. But $f^t(J_{i(1)})\subset f^t(H)$ cannot be contained in $[J_n, J_{R(n)}]$ since $J_{R(n)}$ is a predecessor of J_n. So $f^t(J_{i(1)})\subset[J_{i(1)}, J_n]$ and it follows from the previous claim that t must be a multiple of a. Furthermore consider J_k with $k<n$ and with the same orientation as J_n. This interval cannot be contained in $[J_{i(1)}, J_n]$ (see Claim 2) and neither in $[J_n, J_{R(n)}]$ from the definition of $J_{R(n)}$. Combining all this shows that the only intervals J_k with $k\leqslant i(2N-1)$ inside $[J_{i(1)}, J_{R(n)}]$ with the same orientation as J_n are intervals of the form $k=i(1)+j\cdot a$. It follows that $J_{i(j+1)}$ is a successor of $J_{i(j)}$ for $j=1,\ldots,2N-2$. Q.E.D.

7. Wandering intervals accumulate on turning points

In this section we are going to prove that the ω-limit set of a wandering interval contains at least one turning point and thus prove that Ind_0 holds. From this is follows in particular that maps without turning points, e.g. circle homeomorphisms in $\mathscr{A}$, cannot have wandering intervals. So the proof in this section includes the classical proof of Denjoy.

Suppose we have a map $f\colon N\to N$ which has a wandering interval J which stays away from the turning points of f. But then we can modify the map f near these turning points without affecting the orbit of J. So change f so that each maximal interval on which f is monotone is mapped by f onto a component of N. Once we have done this we may assume that every pullback is monotone.

7.1. Proposition. *There exists n_0 such that if J_n, $n\geqslant n_0$, has two predecessors $J_{L(n)}$ and $J_{R(n)}$ and $|J_{R(n)}|>|J_n|$ and $|J_{L(n)}|>|J_n|$ then J_n has a successor and $|J_{s(n)}|<|J_n|$.*

Remark. Suppose for example that $J_{s(n)}$ is to the right of J_n. Then Lemma 6.3 implies that the interval $J_{s(n)}$ has two predecessors, namely J_n and $J_{R(n)}$ and that it has no left successor. From the conclusion of this proposition it follows that the assumptions of this proposition are again satisfied for $J_{n'}$ where $n'=s(n)$. So one can apply the proposition infinitely often!

Proof. Let $n_0=N_0(1;11)$ be as in the Koebe/Contraction Principle and let $n\geq n_0$. Suppose $s(n)$ is not defined and let $T_n=[J_{L(n)}, J_{R(n)}]$ be the natural neighbourhood of J_n. By the corollary to Theorem 6.4 the monotone pullback of T_n has intersection multiplicity bounded by 11. Because the intervals $|J_{L(n)}|, J_{R(n)}|\geq|J_n|$ this contradicts the Koebe/Contraction Principle. Hence $s(n)$ is defined. From Lemma 6.3 $L(s(n))=n$ and $R(s(n))=R(n)$. Furthermore $s(n)$ has no left successor. Now the natural neighbourhood of J_n is $T_n=[J_{L(n)}, J_{s(n)}]$ and again by the corollary to Theorem 6.4 the monotone pullback of T_n has intersection multiplicity bounded by 11. Since $|J_{L(n)}|>|J_n|$ it follows from the Koebe/Contraction Principle that $|J_{s(n)}|<|(J_n, J_{s(n)}]|<|J_n|$. Q.E.D.

7.2. THEOREM. *If $f\in\mathcal{A}$ and f has a wandering interval the ω-limit set of J contains a turning point.*

Proof. Let us first show that $\omega(J)$ cannot be finite. Indeed otherwise $\omega(J)$ contains a periodic point p of, say, period k and there exists a neighbourhood U of p such that $U\cap\omega(J)=\{p\}$. Furthermore there exist a neighbourhood $V\subset U$ of p such that $f^k(V)\subset U$ and an integer n' for which $J_{n'}\subset V$ and such that $J_n\subset V$ whenever $n\geq n'$ and $J_n\cap U\neq\emptyset$. Since $f^k(V)\subset U$ this implies by induction that $J_n\subset V\cup f(V)\cup\ldots\cup f^{k-1}(V)$ for all $n\geq n'$ and therefore that $\omega(J)=O(p)$. Hence $\omega(J)$ is attracted to a periodic orbit, a contradiction.

Therefore, and since all intervals J_i are disjoint, there exists arbitrarily large integers $l, r<n$ such that J_l, J_r, J_n are in he same component of N, have the same orientation, $J_n\subset(J_l, J_r)$, such that

$$|J_n|\leq\min(|J_l|, |J_r|)$$

and, for $i=0, 1, \ldots, n-1$,

$$J_i\cap(J_l, J_r) \text{ implies that } J_i \text{ has a different orientation.}$$

Assume that l, r, n are bigger than the number n_0 from above. It follows that J_l and J_r are the predecessors of J_n. So we can apply Proposition 7.1 and hence J_n has infinitely many successors $J_{s^k(n)}$, $k=1,2,\ldots$. From the description in Lemma 6.3, all these successors are contained in $[J_l, J_r]$, they either all lie to the right of the previous one or all to the left. Moreover $s^k(n)-s^{k-1}(n)$ is independent of k. It follows that as k tends to infinity these intervals $J_{s^k(n)}$ converge to a fixed point of f^a where $a=s(n)-n$. Hence this fixed point is an attracting fixed point with J in its basin, a contradiction. Q.E.D.

Now we know that iterates of wandering intervals tend to some turning point. In order to analyze the metric properties of iterates of a wandering interval we will pay special attention to the moments where the iterates get closest to some turning point. This is formalised in the following definition.

7.3. *Definition*. If c is a turning point in $\omega(J)$, $J_n\subset S_c$ and $m\geqslant n$, then we say that J_n is the *m-closest approach* to c if $(J_n, \tau(J_n))\cap(\bigcup_{i\leqslant m}J_i)=\emptyset$. Similarly J_n is the *closest approach* to c if it is the n-closest approach to c. Now fix a turning point c in $\omega(J)$ and let $N(c)$ be the collection of integers $i\in\mathbf{N}$ with

$$J_i\subset S_c \quad\text{and}\quad [J_i, \tau(J_i)]\cap(\bigcup_{0\leqslant j<i} J_j)=\emptyset.$$

From now on let

$$N(c)=\{n(1), n(2), \dots\}$$

where $n(1)<n(2)<\dots$ We call $J_{n(1)}, J_{n(2)}, \dots$ the *sequence of closest approach to c*.

7.4. LEMMA. *If $J_{n(k)}$ has a successor $J_{s(n(k))}$ then $s(n(k))=n(k+1)$ and $J_{n(k+1)}$ is between $J_{n(k)}$ and c. Furthermore, if there exists an integer j such that $s(j)$ and $s^2(j))$ are both defined and such that $j<n(k)<s(j)$ then*

$$n(k+1)=s(n(k)).$$

Proof. In order to be definite assume that $J_{n(k)}$ is to the left of c. Since $J_{n(k)}$ is a closest interval to c, any predecessor of $J_{n(k)}$ to its right must also be to the right of c. Therefore $c\in[J_{n(k)}, J_{R(n(k))}]$ and there can be no successor of $J_{n(k)}$ to its left. Hence if $J_{n(k)}$ has a successor then it must be to its right and because $f^{s(n(k))-n(k)}|[J_{L(n)}, J_{s(n)}]$ is monotone it even must be between $J_{n(k)}$ and c. So if $s(n(k))\neq n(k+1)$ then $n(k+1)<s(n(k))$. Consider $H=[J_{L(n(k))}, J_{n(k)}]$ and let $a=n(k)-L(n(k))=s(n(k))-n(k)$. Then f^{2a} is monotone on H and $a>n(k+1)-n(k)$. Since $L(n(k))+n(k+1)-n(k)<n(k)$ the interval $J_{L(n(k))+n(k+1)-n(k)}\subset f^{n(k+1)-n(k)}(H)$ is not contained in $[J_{n(k)}, \tau(J_{n(k)})]$ whereas by assumption $J_{n(k+1)}\subset f^{n(k+1)-n(k)}(H)$ is contained in this interval $[J_{n(k)}, \tau(J_{n(k)})]$. It follows that either $J_{n(k)}$ or $\tau(J_{n(k)})$ is contained in $f^{n(k+1)-n(k)}(H)$. Hence for $t=a-(n(k+1)-n(k))$ one has $0<t<a$ and

$$(*)\qquad f^t(J_{n(k)})\subset f^a(H)=[J_{n(k)}, J_{s(n(k))}]\subset[J_{n(k)}, c].$$

Since $J_{s(n(k))}$ is the successor of $J_{n(k)}$ this implies that f^t is orientation reversing on H. So $J_{L(n(k))+t}$ is to the right of $J_{n(k)}$. But since $L(n(k))+t<n(k)$ this interval cannot be contained in $[J_{n(k)}, \tau(J_{n(k)})]$. Therefore, and because of $(*)$, $f^t(H)$ contains c; this contradicts the monotonicity of $f^a|H$. This proves the first statement of this lemma.

Let us now prove the second statement. According to the first part it is enough to show that $s(n(k))$ is defined. Now let $a=s(j)-j$ then $L(j)=j-a$, $s(j)=j+a$ and $s(s(j))=j+2a$. Since f^a is monotone on $[L_{j-a}, L_{j+2a}]$ and $n(k)-j<s(j)-j=a$ it follows that $J_{n(k)}$ is contained in $T=[J_{n(k)-a}, J_{n(k)+a}]$ and that f^a is monotone on T. Furthermore there is no predecessor of $J_{n(k)}$ in $f^a(T)$ because otherwise there would be a predecessor of J_{j+a} in $f^a([L_j, L_{j+2a}])$, contradicting that $s^2(j)$ exists. So property (1) and (2) of the definition of successor of $J_{n(k)}$ hold. Finally there is also no interval J_t in $[J_{n(k)}, J_{n(k)+a}]$ with $t<n(k)+a$ and with the same orientation as $J_{n(k)}$ because $J_{t+(s(j)-n(k))}$ would have the same orientation as $J_{s(j)}$ and be contained in $[J_{s(j)}, J_{s(j)+a}]$, contradicting the definition of $s^2(j)$. Q.E.D.

8. Topological properties of unimodal pullback's

From now on we will assume

$$(\mathrm{Ind}_{d-1}) \qquad \text{maps in } \bigcup_{i=0}^{d} \mathcal{A}^i \text{ have no wandering intervals}$$

and try to show that this implies Ind_d. Throughout this section we will consider properties of pullbacks of two intervals. The first of these intervals is $Q_{n(k)} \supset J_{n(k)}$: this is the interval in $M \setminus \{c\}$ such that

$$f(Q_{n(k)}) = [f(J_{n(k-1)}), f(J_{n(k+1)})].$$

Similarly let

$$\hat{Q}_{n(k)} = Q_{n(k)} \cup [J_{n(k+1)}, \tau(J_{n(k+1)})].$$

In this section we will describe the structure of the unimodal pullback $P_0, \ldots, P_{n(k)}$ of intervals $P_{n(k)}$ which are contained in $Q_{n(k)}$ or in $\hat{Q}_{n(k)}$.

In the next result it is shown that the intervals from the pullback of $\hat{Q}_{n(k)}$ meet the turning points in a periodic way.

8.1. Structure Theorem. *Let $P_{n(k)} \supset J_{n(k)}$ be an interval which is contained in $\hat{Q}_{n(k)}$. Let $m(0)<m(1)\ldots<m(l)=n(k)$ be the cutting times of its unimodal pullback $P_0, \ldots, P_{n(k)}$ and let c_j denote the turning point in $P_{m(j)}$. Then we have the following properties.*

(1) *If $i \in \{0, \ldots, l-d\}$ then $J_{m(i)}$ is a $m(i+d)-1$ closest approach to c_i;*

(2) *If $i \in \{l-d+1, \ldots, l\}$ then $J_{m(i)}$ is a $n(k)$ closest approach to c_i;*

(3) $c_i = c_{i+d}$ *for* $i=0, \ldots, l-d$ *and* $\{c_{l-d+1}, \ldots, c_l\}$ *are distinct*;

(4) *If* $J_{m(i)+j} \subset (J_{m(i)}, \tau(J_{m(i)}))$, $i \in \{0, \ldots, l-1\}$ *and* $m(i) < m(i)+j \leq n(k)$ *then*

$$P_{m(i)+j} \subset [J_{m(i)}, \tau(J_{m(i)})];$$

(5) $P_{m(i+d)} \subset [J_{m(i)}, \tau(J_{m(i)})] \subset P_{m(i)}$ *for* $i=0,1,\ldots,l-d$ *and therefore* $f^{m(i+d)-m(i)}$ *maps* $P_{m(i)}$ *into itself.*

Proof of (4). If property (4) does not hold then the closure of $J_{m(i)}$ is contained in the interior of $P_{m(i)+j}$. So the closure of $J_{m(i)+n(k)-(m(i)+j)}$ is contained in the interior of $P_{n(k)}$. From the definition of $P_{n(k)}$ this implies $m(i)+n(k)-m(i)-j \geq n(k)$ and therefore $j \leq 0$, a contradiction.

Proof of (1) *and* (2). Let us just prove (1). Statement (2) is proved in exactly the same way. Suppose by contradiction that there exists $l \in \{0, \ldots, m(i+d)-1\}$ such that $J_l \subset (J_{m(i)}, \tau(J_{m(i)}))$. Then $J_{l+n(k)-m(i)} = f^{n(k)-m(i)}(J_l) \subset f^{n(k)-m(i)}(P_{m(i)}) \subset P_{n(k)}$ and $l \neq m(i)$. Hence $l > m(i)$. From statement (4) we know that $P_l \subset P_{m(i)}$. So $f^{l-m(i)}$ maps $P_{m(i)}$ into $P_l \subset P_{m(i)}$. Because $l < m(i+d)$ the map $f\colon \bigcup_{t=0}^{l-m(i)-1} f^t(P_{m(i)}) \to \bigcup_{t=0}^{l-m(i)-1} f^t(P_{m(i)})$ has at most $d-1$ turning points. Since $J_{m(i)}$ is a wandering interval of this map, we get a contradiction with the induction hypothesis.

Proof of (3). Suppose there are $i-d < j \leq i \leq s$ with $c_i = c_j$. From (1) we get that $J_{m(i)}$ and $J_{m(j)}$ are both $m(j+d)$ closest approaches to $c_i = c_j$. Hence because $m(i), m(j) < m(j+d)$ this implies $i=j$. Since f has precisely d turning points one gets that $c_{l-d+1}, \ldots, c_l$ are distinct and that $c_i = c_{i+d}$ for $i \in \{0, \ldots, l-d\}$.

Proof of (5). The proof of statement (5) follows immediately from the other statements. Q.E.D.

The following theorem shows that we can even take monotone pullbacks of intervals which contain topologically rather large sets. A unimodal version of this theorem was already used by J. Guckenheimer for his proof of the non-existence of wandering intervals for unimodal maps with negative Schwarzian derivative.

8.2. Monotone Extension Theorem. *Let* $P_{n(k)} = \hat{Q}_{n(k)}$ *and* $\{P_0, P_1, \ldots, P_{n(k)}\}$ *be its unimodal pullback. Let* $m(0) < m(1) < \ldots < m(l) = n(k)$ *be the cutting times and* c_i *the turning point in* $P_{m(i)}$. *Let* $H_{m(i)} \supset J$ *be the maximal interval such that* $f^{m(i)}$ *is monotone on* $H_{m(i)}$. *Let* $R_{m(i)}$ *be the component of* $P_{m(i)} \setminus J_{m(i)}$ *which contains* c_i *and* $L_{m(i)}$ *the other component. If the number* l *of cutting times of the pullback is at lest* $d+1$ *then*

$$f^{m(i)}(H_{m(i)}) \supset [L_{m(i)}, c_i],$$

$$f^{m(i-1)}(H_{m(i)}) \supset [L_{m(i-1)}, c_{i-1})$$

and

$$f^{m(i)-m(i-1)}(L_{m(i-1)}) \subset L_{m(i)}$$

for $i=0, 1, \ldots, l-d$.

Proof. Let us first show that

$$(*) \qquad f^{m(i+1)-m(i)}(P_{m(i)}) \supset [J_{m(i+1)}, c_{i+1}]$$

for $i=0, 1, 2, \ldots, l-d$. Suppose by contradiction that there exists $i \in \{0, 1, \ldots, l-d\}$ with $P_{m(i+1)} \supset f^{m(i+1)-m(i)}(P_{m(i)}) \not\ni c_{i+1}$. By statement (5) of the previous theorem $f^{m(i+d)-m(i)}$ maps $P_{m(i)}$ into itself. Now $f^{m(i+1)-m(i)}(P_{m(i)}) \not\ni c_{i+1}$ implies that

$$T = \bigcup_{t=0}^{m(i+d)-m(i)-1} f^t(P_{m(i)})$$

does not contain c_{i+1}. Hence f maps T into itself and has at most $d-1$ turning points. Since $J \subset P_{m(i)}$ it follows from the induction hypothesis that J is not a wandering interval, a contradiction. This proves (*). Furthermore

$$(**) \qquad f^{m(i+1)-m(i)}\colon L_{m(i)} \to L_{m(i+1)} \text{ is monotone and onto}$$

for $i=0, 1, 2, \ldots, l-d$ because otherwise there exists such an integer i with

$$f^{m(i+1)-m(i)}(R_{m(i)}) = L_{m(i+1)}$$

and then as before

$$T = \bigcup_{t=0}^{m(i+d)-m(i)-1} f^t(R_{m(i)} \cup J_{m(i)})$$

contains at most $d-1$ turning points and f maps this interval into itself. Since J is contained in T this contradicts the induction hypothesis. It follows from (*) and (**) that $f^{m(i+1)-m(i)}$ maps $[L_{m(i)}, c_i]$ monotonically over $[L_{m(i+1)}, c_{i+1}]$. The theorem clearly follows. Q.E.D.

Next we give two results about the disjointness of unimodal pullbacks of intervals in $Q_{n(k)}$ and in $\hat{Q}_{n(k)}$. The first result deals with the unimodal pullback of $Q_{n(k)}$.

8.3. THEOREM. *Let $m(0)<m(1)<\ldots<m(l)=n(k)$ be the cutting times of the unimodal pullback of $Q_{n(k)}$. Then:*

(1) $l\leqslant d-1$.

(2) *For every $0\leqslant j\leqslant l$ the second successor $s^2(m(j))$ is not defined.*

(3) *The intersection multiplicity of the unimodal pullback of $Q_{n(k)}$ is universally bounded (in fact by $12d$).*

Proof. Suppose by contradiction that $l\geqslant d$. By statement (5) of Theorem 8.1 it follows that $Q_{m(l)}=Q_{m(k)}$ is contained in $Q_{m(l-d)}$. Hence f maps

$$\bigcup_{t=0}^{m(l)-m(l-d)-1} f^t(Q_{m(l-d)})$$

into itself and since $Q_{n(k)}$ contains no turning point this map has at most $d-1$ turning points. This contradicts Ind_{d-1}. So let us prove statement 2 by assuming by contradiction that there exists $j\in\{0,1,\ldots,l\}$ for which $s(m(j))$ and $s(s(m(j)))$ are defined. Because $m(j)$ is $n(k)$-closest we get $s(m(j))>n(k)$. Hence from Lemma 7.4 we get that $n(k+1)=s(n(k))$ and $s(n(k))-n(k)=s(j)-j$. Because the closure of $J_{s(m(j))}$ is contained in $Q_{m(j)}$ we get that the closure of $J_{n(k+1)}=f^{n(k)-m(j)}(J_{s(m(j))})$ is contained in $f^{n(k)-m(j)}(Q_{m(j)})\subset Q_{n(k)}$ which contradicts the definition of $Q_{n(k)}$. So let us prove statement (3). From statement (2) and the corollary of Theorem 6.4 it follows that the intersection multiplicity of $\{P_{m(j)},\ldots,P_{m(j+1)}\}$ for $j=-1,0,1,\ldots,l-1$ (where we let $m(-1)=0$) is bounded by 11. Since $l\leqslant d-1$ the theorem follows. Q.E.D.

8.4. THEOREM. *Assume $n(k)>n(k-1)+(n(k-1)-n(k-2))$. Let $m(0)<m(1)<\ldots<m(l)=n(k)$ be the cutting times of the unimodal pullback of $\hat{Q}_{n(k)}$ and $l\geqslant 2d$. Then $s(n(k-1))=s(m(l-d))$ and $s^l(m(j))$ are both not defined if $j\in\{l-ld,\ldots,l-ld+d-1\}$ and $l\geqslant 2$. Furthermore, if $j\in\{l-ld,\ldots,l-ld+d-1\}$, the intersection multiplicity of*

$$\{\hat{Q}_{m(j)},\ldots,\hat{Q}_{m(j+1)}\}$$

is bounded by $4+2l$. Similarly the intersection multiplicity of $\{\hat{Q}_0,\hat{Q}_1,\ldots,\hat{Q}_{m(0)}\}$ is bounded by $5+2[l/d]$. (Here $[l/d]$ is the smallest integer smaller than l/d.)

Proof. If $s(n(k-1))$ were defined then $J_{n(k-2)}\subset[J_{L(n(k-1))},J_{n(k-1)})$ and $n(k)=s(n(k-1))$. Therefore, for $a=n(k)-n(k-1)=s(n(k-1))-n(k-1)$,

$$J_{n(k-2)+a}\subset[J_{n(k-1)},J_{n(k)}).$$

But since $n(k-2)+a<n(k-1)+a=n(k)$ this implies that $n(k-2)+(n(k)-n(k-1))=n(k-2)+a$ must be equal to $n(k-1)$ and this contradicts the assumption of the theorem.

Let us show that if $l\geq 2$ and $j\in\{l-ld,\dots,l-ld+d-1\}$ then $s^l(m(j))$ is not defined. So suppose by contradiction $s^l(m(j))$ exists. We claim that then $s^{i+1}(m(j))>n(k-1)>s^i(m(j))$ for some $i=0,1,\dots,l-1$. Indeed, since $m(0)<n(k-1)$ we may otherwise assume that $s^{l-1}(m(j))<n(k-1)$. But then $J_{m(j+(l-1)d)}$ is the $(l-1)$th successor of $J_{m(j)}$ and, again by Lemma 7.4, the successor of $J_{m(j+(l-1)d)}$ must be between $J_{m(j+(l-1)d)}$ and c_j. By statement (2) of Theorem 8.1 this implies that $s^l(m(j))>n(k-1)$. This proves the claim. Hence Lemma 7.4 and $s^{i+1}(m(j))>n(k-1)>s^i(m(j))$ imply that $s(n(k-1))$ exists and so we get a contradiction.

The disjointness statements immediately follow from the Corollary of Theorem 6.4. Q.E.D.

9. The non-existence of wandering intervals

In Section 7 we have proved Ind_0 and so Theorem A follows from

$$\mathrm{Ind}_{d-1} \Rightarrow \mathrm{Ind}_d.$$

So let us assume that Ind_{d-1} holds and that there exists a map $f\in\mathcal{A}_d$ which has a wandering interval J. Asume that J is maximal in the sense that J is not contained in any strictly larger wandering intervals. From the Contraction Principle this implies that

$$\frac{|H_n|}{|J|}\to 1 \quad \text{if} \quad n\to\infty$$

where H_n is the maximal interval containing J on which f^n is monotone.

From Section 7 we know that J accumulates at a turning point, say c. Consider the sequence of closest approach to c, $\{J_{n(k)}\}_{k\geq 0}$.

9.1. THEOREM. *There exists k_0 such that for all $k\geq k_0$*

$$n(k)-n(k-1)\leq n(k-1)-n(k-2).$$

COROLLARY. *J is not a wandering interval.*

Proof of Corollary. Since $n(k)-n(k-1)\leq n(k-1)-n(k-2)$ it follows that $n(k)-n(k-1)$ is eventually equal to some integer a for all k sufficiently large. In particular since $J_{n(k-1)}$ and $J_{n(k)}$ tends to c it follows that c is an attractive fixed point of f^a which attracts J. Hence J is not a wandering interval. Q.E.D.

Proof of Theorem 9.1. Since the intervals $J_{n(k)}$ are disjoint there exist arbitrarily large k such that $|J_{n(k-1)}|>|J_{n(k)}|$. Let $Q_{n(k)}$ be as before. If $|J_{n(k)}|<|J_{n(k+1)}|$ then $|J_{n(k\pm 1)}|>|J_{n(k)}|$. Because f is non-flat in the critical point c this implies that $Q_{n(k)}$ is a 1/2-scaled neighbourhood of $J_{n(k)}$ (if k is large). By Theorem 8.3 the intersection multiplicity of the unimodal pullback of $Q_{n(k)}$ is at most $12d$ and therefore the Koebe/Contraction Principle gives a contradiction when $n(k)$ is large. So we have shown that $k\mapsto|J_{n(k)}|$ is monotone decreasing for k large.

Let us now show that $n(k)\leqslant n(k-1)+(n(k-1)-n(k-2))$ for k large. So assume by contradiction $n(k)>n(k-1)+(n(k-1)-n(k-2))$. Consider the unimodal pullback of $\hat{Q}_{n(k)}$. If $l\leqslant 2d$ then from Theorem 8.4 the intersection multiplicity of $\{\hat{Q}_0, \ldots, \hat{Q}_{n(k)}\}$ is uniformly bounded. Because $\hat{Q}_{n(k)}$ contains a 1/2-scaled neighbourhood of $J_{n(k)}$ this gives a contradiction with the Koebe/Contraction Principle.

This implies that for k large the number of cutting times l is at least $2d+1$. Now let L_i and R_i be as in Theorem 8.2. From Theorem 8.2 and for $j=1,2,\ldots l-d$, there exists an interval $H\supset J$ which is mapped by $f^{m(j)}$ monotonically onto $[L_{m(j)}, c_{j-d}]$. We claim that $f^{m(j)}(H)$ does not contain $\tau(J_{m(j)})$. Indeed, since Theorem 8.2 gives

$$f^{m(j)}(H)\supset[L_{m(j)}, c_{m(j)}]$$

one would otherwise have

$$f^{m(j)}(H)\supset[L_{m(j)}, \tau(J_{m(j)})].$$

If we take $j\in\{l-2d, \ldots, l-d\}$ then $s^2(m(j))$ does not exist by Theorem 8.4, and since $f^{m(j)}$ is monotone on H we can apply the Corollary of Theorem 6.4 and the intersection multiplicity of $H, f(H), \ldots, f^{m(j)}(H)$ is at most 8. Furthermore, from the definition $\hat{Q}_{n(k)}$, $[L_{m(j)}, c_{j-d}]$ contains $[J_{m(j-d)}, c_j]$ or it contains $[\tau(J_{m(j-d)}), c_j]$. Since the length of the closest approach intervals decreases one has $|J_{m(j-d)}|>|J_{m(j)}|$ and therefore $f^{m(j)}(H)$ contains a 1/2-scaled neighbourhood of $f^{m(j)}(J)$. Therefore we get a contradiction with the Koebe/Contraction Principle and this proves the claim.

Now let $F_{m(j)}=[J_{m(j)}, \tau(J_{m(j)})]$. By definition $f^{m(j)}|H$ is monotone and by Theorem 8.2 $f^{m(j-1)}(H)\supset[J_{m(j-1)}, c_{j-1})$ and $f^{m(j)-m(j-1)}$ maps $[J_{m(j-1)}, c_{j-1})$ monotonically over $[J_{m(j)}, c_j]$. By the previous claim, the image of $[J_{m(j-1)}, c_{j-1})$ under this map is contained in $F_{m(j)}$ and therefore we get

$$f^{m(j)-m(j-1)}(F_{m(j-1)})\subset F_{m(j)}$$

for all $j \in \{l-2d, \dots, l-d\}$. In particular

$$f^{n(k-1)-n(k-2)}(F_{n(k-2)}) \subset F_{n(k-1)}.$$

Therefore

$$J_{n(k-1)+(n(k-1)-n(k-2))} \subset [J_{n(k-1)}, \tau(J_{n(k-1)})].$$

Since $n(k-1)+(n(k-1)-n(k-2))>n(k-1)$ and $J_{n(k)}$ is a closest approach this gives $n(k-1)+(n(k-1)-n(k-2)) \geqslant n(k)$, a contradiction. Q.E.D.

10. The proof of Theorem B: finiteness of attractors

In this section we will prove Theorem B. If $f: N \to N$ is a diffeomorphism then the period of periodic orbits of f is bounded. So Theorem B holds trivially. So from now on assume that f is not a diffeomorphism and that $f \in \mathcal{B}$. Of course some points in $\mathcal{K}_f$ may be attracted by periodic orbits so let $\hat{n}$ be an upper bound for the period of this attracting orbits.

In this section we have to show that one has some expansion near periodic orbits. For this it will be convenient to consider the orientation preserving period of a periodic orbit. More precisely let $\mathcal{O}$ be a periodic orbit of period $k > \max(\hat{n}, 300)$. Then $p \in \mathcal{O}$ implies $Df^k(p) \neq 0$. Let $n=2k$ if $Df^k(p)<0$ and $n=k$ otherwise.

The main idea of the proof of Theorem B is to choose $p \in \mathcal{O}$ and get on both sides of p points θ^1 and θ^2, very close to p, with $Df^n(\theta^i) \geqslant 1+2\varrho$. Using the Minimum Principle we will get $Df^n(p) \geqslant 1+\varrho$ for large n and we are done.

For $p \in \mathcal{O}$ define T_p to be the maximal open interval such that both components of $T_p \setminus \{p\}$ contain at most one point of $\mathcal{O}$. (So the closure of T_p contains at most 5 points of $\mathcal{O}$.) The interval $T_q, q \in \mathcal{O}$, is a *direct neighbours* of T_p if $T_q \cap T_p = \emptyset$ and $\mathrm{cl}(T_p)$ and $\mathrm{cl}(T_q)$ have one point in common.

10.1. Lemma. *There exists a number $\tau>0$ such that for each periodic orbit $\mathcal{O}$ of period $\geqslant \max(\hat{n}, 300)$ there exists $p \in \mathcal{O}$ such that:*

(1) *T_p has direct neighbours on both sides;*

(2) *$|T_{q_1}| \geqslant 2\tau |T_p|$ and $|T_{q_2}| \geqslant 2\tau |T_p|$ where T_{q_1} and T_{q_2} are the two direct neighbours of T_p.*

Proof. Let $s \in \mathcal{O}$ be such that (i) $\#(\mathrm{cl}(T_s) \cap \mathcal{O})=5$ and (ii) $|T_s| \leqslant |T_q|$ for all q with $\#(\mathrm{cl}(T_q) \cap \mathcal{O})=5$. If T_s has neighbours on both sides then we take $p=s$ and we are done.

Otherwise $N=[-1,1]$, and then let T^l and T^r be the smallest open intervals containing points of respectively $\{-1\}$ and $\{1\}$ such that $\#(\mathrm{cl}(T^l)\cap\mathcal{O})=\#(\mathrm{cl}(T^r)\cap\mathcal{O})=5$. Because T_s has no two neighbours $T_s\subset T^l$ or $T_s\subset T^r$. Since the interval $\mathrm{cl}(T_s)$ contains five points of $\mathcal{O}$, and n is at most twice the period of p, there are at most $5\times 2\times(5+5)=100$ integers t, $0\leqslant t\leqslant n$ such that f^t maps a point in $\mathrm{cl}(T_s)\cap\mathcal{O}$ into a point of $\mathrm{cl}(T^r)\cup\mathrm{cl}(T^l)\cap\mathcal{O}$. So there exists a $0\leqslant t\leqslant 101$ such that $f^t(\mathrm{cl}(T_s)\cap\mathcal{O})$ is between T^l and T^r. Let p be the middle point of $f^t(\mathrm{cl}(T_s)\cap\mathcal{O})$. Let $S=\max\{1,\sup_{x\in N}|Df(x)|\}$. Since $T_p\subset f^t(T_s)$, we get

$$|T_p|\leqslant|f^t(T_s)|\leqslant\left(\sup_{x\in N}|Df(x)|\right)^t|T_s|\leqslant S^t|T_s|.$$

As $t\leqslant 101$ and $|T_s|\leqslant|T_{q_i}|$, $i=1,2$ the lemma is proved. Q.E.D.

Let $p\in\mathcal{O}$ be the point from Lemma 10.1 and as before let n be the period or twice the period of p. Let U_n be the interval around T_p such that

(10.1) $$U_n \text{ is a } 2\tau\text{-scaled neighbourhood of } T_p.$$

So $U_n\subset T_{q_1}\cup T_p\cup T_{q_2}$. Let J be the maximal interval around p satisfying:

(10.2) $$f^n(J)\subset U_n$$

and

(10.3) $$f^n|J \text{ is an orientation preserving diffeomorphism.}$$

Let J^l and J^r be the components of $J\setminus\{p\}$ and let U_n^i be the component of $U_n\setminus\{p\}$ which contains J^i for $i\in\{l,r\}$. Let $\hat{n}$ be as in the beginning of this section.

10.2. Lemma. *If $n>\hat{n}$ we have*

(10.4) $$J\subset T_p,$$

(10.5) $$f^n(J^i)\supset J^i,\quad i=l,r.$$

Proof. We claim that $J\cap P$ consists of at most 2 points. Indeed, take I to be the maximal interval containing p such that $\partial I\subset P$ and such that $f^n|I$ is a diffeomorphism. Since f is not a diffeomorphism, I is an interval (i.e. not equal to S^1). Then $f^k|I$ is a diffeomorphism for all $k\geqslant 0$ and since I is maximal, $f^i(I)\cap I\neq\emptyset$ implies that $f^i(I)=I$. In particular the boundary points of I (which are in P) cannot be mapped into $\mathrm{int}(I)$ and so I contains at most two points of P. (This argument also shows that if I contains two points of P then f^n interchanges these two points.) This proves statement (10.4). If

(10.5) does not hold then $f^n(J^i) \subset J^i$ and a critical point of f is attracted by a periodic orbit of period n. This implies $n \leqslant \hat{n}$. Q.E.D.

The main step in the proof of Theorem B is the proof of the following:

10.3. PROPOSITION. *There exist a number $\varrho>0$ and an integer n_0 such that if the number n corresponding to p is greater than n_0 then there exist $\theta^i \in J^i$, $i=l, r$, such that*

$$Df^n(\theta^i) \geqslant 1+2\varrho.$$

Before proving Proposition 10.3 we will make a few remarks. Let τ be the number from Lemma 10.1. If there exists $\theta^i \in J^i$ with $Df^n(\theta^i) \geqslant 1+\tau$ then we are done with J^i. So from now on we may assume that

$$0 < Df^n(x) < 1+\tau, \quad \forall x \in J^i. \tag{10.6}$$

By the previous lemma, for $n>\hat{n}$ and $i=l, r$, $J^i \subset f^n(J^i) \subset U^i$. Then $f^n(J^i)=U^i$ is impossible because otherwise

$$\frac{|f^n(J^i)|}{|J^i|} \geqslant \frac{2\tau|T_p|+|J^i|}{|J^i|} \geqslant (1+2\tau),$$

a contradiction with (10.6). Hence in this case

$$f^n(J^i) \subsetneqq U^i.$$

Let $\{U_0, \ldots, U_n\}$ be the diffeomorphic pullback of $U_n \supset f^n(J)$: U_i is the maximal interval containing $f^i(J)$ which is mapped by f diffeomorphically into U_{i+1}. Since $f^n|J$ is a diffeomorphism this is well defined and $U_k \supset f^k(J)$. Furthermore from the maximality of J one has $U_0=J$.

10.4. LEMMA. (i) *There is a universal upperbound for the intersection multiplicity of $\{U_0, U_1, \ldots, U_n\}$ (in fact it is at most* 74);

(ii) *For every $\varepsilon>0$ there exists n_0 such that if $n>n_0$ then $|U_k| \leqslant \varepsilon$ for all $k=0, 1, \ldots, n$.*

Proof. Let $U_{k(1)} \cap \ldots \cap U_{k(r)} \ni x$ with $k(1)<\ldots<k(r) \leqslant n$. Because $f^{n-k}(U_k) \subset \operatorname{int}[T_{q_1}, T_{q_2}]$ and $f(U_k) \subset U_{k+1}$ we get that

$$f^{n-k(r)}(x) \in U_{k(1)+n-k(r)} \cap \ldots \cap U_n.$$

Therefore $\#\{k; U_k \cap \mathrm{int}[T_{q_1}, T_{q_2}] \neq \emptyset\} \geq r$. Hence statement (i) follows from

$$\#\{k | U_k \cap \mathrm{int}[T_{q_1}, T_{q_2}] \neq \emptyset\} \leq 74. \tag{10.7}$$

Let us prove (10.7). Notice that $\mathrm{int}[T_{q_1}, T_{q_2}]$ contains 11 points of $\mathcal{O}$. So there are at most 22 integers $0 \leq k < n$ such that $f^k(p) \in \mathrm{int}[T_{q_1}, T_{q_2}]$. Hence there are at most 22 integers $0 \leq k < n$ such that $U_k \subset \mathrm{int}[T_{q_1}, T_{q_2}]$. Now let a and b be the boundary points of $[T_{q_1}, T_{q_2}]$, so $a, b \in \mathcal{O}$. If U_k is not contained in $\mathrm{int}[T_{q_1}, T_{q_2}]$ but has a non-empty intersection with this set then $f^{n-k}(a)$ or $f^{n-k}(b) \in \mathrm{int}[T_{q_1}, T_{q_2}]$. Because $\mathrm{cl}([T_{q_1}, T_{q_2}])$ contains only 13 points of $\mathcal{O}$, there exist at most $2 \times 2 \times 13 = 52$ integers $0 \leq k < n$ with this property. This implies inequality (10.7) and finishes the proof of (i).

Statement (ii) follows from the Contraction Principle and the fact that U_k contains at most 5 points of $\mathcal{O}$. Q.E.D.

Proof of Proposition 10.3. Let $C \in (0, 1)$ be so that if $T \supset I$ are intervals, $f|T$ is an diffeomorphism and $f(T)$ is an ε-scaled neighbourhood of $f(I)$ then T is a $C\varepsilon$-scaled neighbourhood of I. Since f is non-flat at its critical points such a constant exists.

Let $m(1) < m(2) < \ldots < m(l)$ be the 'cutting' times of the pullback, i.e., the integers for which U_j contains a turning point in its closure. Let L_j and R_j be the components of $U_j \setminus f^j(J^i)$ and $J^i_j = f^j(J^i)$ and let $R_{m(k)}$ be the component which contains c_k in its boundary. Now U_n contains a τ scaled neighbourhood of $f^n(J^i)$. Since $f^{n-m(l)+1}\colon U_{m(l)+1} \to U_n$ is a diffeomorphism it follows from the Macroscopic Koebe Principle that there exists a positive function B_0 (which only depends on f and the intersection multiplicity 74 from Lemma 10.6) such that $U_{m(l)+1}$ contains a $B_0(\tau)$-scaled neighbourhood of $J^i_{m(l)+1}$. Now let $g(x) = C \cdot B_0(x)$. If $U_{m(l)}$ contains a $C \cdot B_0(\tau) = g(\tau)$-scaled neighbourhood of $J^i_{m(l)}$ we repeat this procedure and we get from the Macroscopic Koebe principle again that $U_{m(l-1)+1}$ contains a $B_0(g(\tau)))$-scaled neighbourhood of $J^i_{m(l-1)+1}$. If $U_{m(l-1)}$ contains a $g^2(\tau) = C \cdot B_0(g(\tau)))$-scaled neighbourhood of $J^i_{m(l-1)}$ then we repeat this procedure again. Since $U = J$ this procedure must stop however. Say it stops at $m(r)$ where $r \leq l$ and then (by the definition of C above)

$$|R_{m(r)}| < g^r(\tau) |J^i_{m(r)}|,$$

where, since intersection multiplicity is at most 74, one has $r \leq 74 \cdot \# K_f$.

Now let M' be the middle third interval of $J^i_{m(r)}$ and $M \subset J^i$ be such that $f^{m(r)}(M) = M'$. From the Macroscopic Koebe principle J^i is δ-scaled neighbourhood of M. From Lemma 10.4, $|U_k|$ is small if n is large. So we can apply Lemma 2.8 and get some universal constant $\xi>0$ such that

$$B(f, f^{m(r)}(J^i), M') \geqslant 1+\xi.$$

Let λ be such that $(1-\lambda)^2(1+\xi) \geqslant 1+\frac{1}{2}\xi$. From the disjointness property of the orbit of J^i we get

$$B(f^{m(r)}, J^i, M) \geqslant 1-\lambda,$$

$$B(f^{n-m(r)-1}, f^{m(r)+1}(J^i), f(M')) \geqslant 1-\lambda,$$

for n large enough. Hence

$$B(f^n, J^i, M) \geqslant 1+\frac{1}{2}\xi,$$

for n large enough. Because of the First Expansion Principle, see Theorem 1.3, it suffices to show that the length of both components of $J^i \setminus M$ is at least $\delta \cdot |J^i|$. But since $f^{n-m(r)-1}$ has bounded distortion on $f(J^i_{m(r)})$, since f is non-flat at critical points and since the components of $J^i_{m(r)} \setminus M'$ have the same length as M', there exists a universal constant β such that the length of both components of $f^n(J^i \setminus M) = f^{n-m(r)}(J^i_{m(r)} \setminus M')$ is at least $\beta \cdot |f^n(J^i)|$. However, by Lemma 10.2, $f^n(J^i) \supset J^i$ and by assumption $|Df^n| \leqslant 1+\tau$ on J^i. It follows that both components of $J^i \setminus M$ have at least size $(\beta/(1+\tau)) \cdot |J^i|$.

Q.E.D.

Proof of Theorem B. Let $\mathcal{O}$ and as before let n bet he period or twice the period of $\mathcal{O}$. Assume that $n>n_0$ where n_0 is as in Proposition 10.3. So there exist two points θ^1, θ^2 such that $p \in T = [\theta^1, \theta^2]$,

$$Df^n(\theta^i) \geqslant 1+2\varrho, \quad i=l, r.$$

For n large, $|f^i(T)|$ is small for all $i \in \{0, 1, \ldots, n\}$ and the orbits has intersection multiplicity $\leqslant 74$. Therefore we get

$$[B(f^n, T^*, J^*)]^3 \geqslant \frac{1+\varrho}{1+2\varrho}$$

for all intervals $J^* \subset T^* \subset T$, provided n is large. Now we apply the Minimum Principle. Then

$$|Df^n(x)| \geqslant [\inf B(f^n, T^*, J^*)]^3 (1+2\varrho) \geqslant 1+\varrho$$

for all $x \in T$. So $Df^n(p) \geqslant 1+\varrho$. This proves Theorem B. Q.E.D.

References

[BL] BLOKH, A. M. & LYUBICH, M. JU., Non-existence of wandering intervals and structure of topological attractors of one dimensional dynamical systems 2. The smooth case. *Ergodic Theory Dynamical Systems*, 9 (1989), 751–758.

[CE] COLLET, P. & ECKMANN, J., *Iterated Maps of the Interval as Dynamical Systems*. Birkhäuser, 1980.

[D] DENJOY, A., Sur les courbes definie par les equations differentielles a la surface du tore. *J. Math. Pures Appl. 9*, 11 (1932), 333–375.

[Fa] FATOU, P., Sur les equations fonctionnelles. *Bull. Soc. Math. France*, 47 (1919), 161–271; 48 (1920), 33–94; 48 (1920), 208–314.

[Gu1] GUCKENHEIMER, J., Sensitive dependence on initial conditions for one dimensional maps. *Comm. Math. Phys.*, 70 (1979), 133–160.

[Gu2] — Limit sets of S-unimodal maps with zero entropy. *Comm. Math. Phys.*, 110 (1987), 655–659.

[Ha] HALL, C. R., A C^∞ Denjoy counterexample. *Ergodic Theory Dynamical Systems*, 1 (1981), 261–272.

[J] JULIA, G., Memoire sur l'iteration des fonctions rationnelles. *J. de Math.*, 8 (1918), 47–245.

[L] LYUBICH, M. JU., Non-existence of wandering intervals and structure of topological attractors of one dimensional dynamical systems 1. The case of negative Schwarzian derivative. *Ergodic Theory Dynamical Systems*, 9 (1989), 737–750.

[Ma] MAÑÉ, R., Hyperbolicity, sinks and measure in one dimensional dynamics. *Comm. Math. Phys.*, 100 (1985), 495–524; Erratum. *Comm. Math. Phys.*, 112 (1987), 721–724.

[Me] DE MELO, W., A finiteness problem for one dimensional maps. *Proc. Amer. Math. Soc.*, 101 (1987), 721–727.

[Mi] MISIUREWICZ, M., Absolutely continuous measure for certain maps of an interval. *Publ. Math. I.H.E.S.*, 53 (1981), 17–51.

[MS1] DE MELO, W. & VAN STRIEN, S. J., A structure theorem in one-dimensional dynamics. *Ann. of Math.*, 129 (1989), 519–546.

[MS2] — Schwarzian derivative and beyond. *Bull. Amer. Math. Soc.*, 18 (1988), 159–162.

[MMMS] MARTENS, M., DE MELO, W., MENDES, P. & VAN STRIEN, S., Cherry flows on the torus: towards a classification. *Ergodic Theory Dynamical Systems*, 10 (1990), 531–554.

[MT] MILNOR, J. & THURSTON, W., On iterated maps of the interval: I, II. Preprint Princeton, 1977.

[Si] SINGER, D., Stable orbits and bifurcations of maps of the interval. *SIAM J. Appl. Math.*, 35 (1978), 260–267.

[Sc] SCHWARTZ, A., A generalization of a Poincaré–Bendixon theorem to closed two dimensional manifolds. *Amer. J. Math.*, 85 (1963), 453–458.

[SI] SHARKOVSKII, A. N. & IVANOV, A. F., C^∞-mappings of an interval with attracting cycles of arbitrary large periods. *Ukrain. Mat. Zh.*, 4 (1983), 537–539.

[Str1] VAN STRIEN, S., On the bifurcations creating horseshoes, in *Dynamical Systems and Turbulence*, Warwick, 1980. Lecture Notes in Mathematics, 898 (1981), 316–351. Springer-Verlag, Berlin.

[Str2] — Smooth dynamics on the interval, in *New Direction in Chaos*. Cambridge University Press, Cambridge, 1987, pp. 57–119.

[Str3] — Hyperbolicity and invariant measures for general C^2 interval maps satisfying the Misiurewicz condition. *Comm. Math. Phys.*, 128 (1990), 437–496.

[Su] SULLIVAN, D., Quasiconformal homeomorphisms and dynamics I: a solution of

Fatou–Julia problem on wandering domains. *Ann. of Math.*, 122 (1985), 401–418; II: Structural stability implies hyperbolicity for Kleinian groups. *Acta Math.*, 155 (1985), 243–260.

[Y] Yoccoz, J. C., Il n'y a pas de contre-examples de Denjoy analytique. *C. R. Acad. Sci. Paris Sér. I Math.*, 298 (1984), 141–144.

Received January 19, 1989
Received in revised form January 7, 1991