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Torus actions on manifolds of positive sectional curvature

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1. Introduction

We present several new results on isometric torus actions on positively curved manifolds. The symmetry rank was introduced by Grove and Searle as one possible way to measure the amount of symmetry of a Riemannian manifold (M, g). It is defined as the rank of the isometry group,

$$\operatorname{symrank}((M,g)) = \operatorname{rank}(\operatorname{Iso}(M,g)),$$

or equivalently as the largest number d such that a d-dimensional torus acts effectively and isometrically on M.

Grove and Searle [13] showed that $\operatorname{symrank}((M,g)) \leqslant \left[\frac{1}{2}(n+1)\right]$ provided that M is a compact manifold of positive sectional curvature. They also studied the case of equality and showed that this can only occur if the underlying manifold is diffeomorphic to $\mathbf{CP}^n$, $\mathbf{S}^n$, or to a lens space.

Our main new tool is the following basic result.

THEOREM 1. *Let M^n be a compact Riemannian manifold with positive sectional curvature. Suppose that $N^{n-k} \subset M^n$ is a compact totally geodesic embedded submanifold of codimension k. Then the inclusion map $N^{n-k} \to M^n$ is $(n-2k+1)$-connected.*

Recall that a map $f\colon N \to M$ between two manifolds is called h-connected, if the induced map $\pi_i(f)\colon \pi_i(N) \to \pi_i(M)$ is an isomorphism for $i<h$ and an epimorphism for $i=h$. If f is an embedding, this is equivalent to saying that up to homotopy M can be obtained from $f(N)$ by attaching cells of dimension $\geqslant h+1$. It is easy to find various examples where the conclusion of Theorem 1 is optimal. For example the 24-dimensional

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positively curved Wallach flag [24] $M^{24}:=\mathsf{F}_4/\mathsf{Spin}(8)$ contains a totally geodesic 15-dimensional submanifold $N^{15}=\mathsf{Spin}(9)/\mathsf{Spin}(7)$ which is diffeomorphic to $\mathbf{S}^{15}$. Since M^{24} is 7-connected but not 8-connected, the same holds for the inclusion map $N^{15}\to M^{24}$. One can refine Theorem 1 if there are two totally geodesic submanifolds, or if there is an isometric group action fixing the submanifold, see Theorem 2.1.

Theorem 1 can be viewed as a generalization of a theorem of Frankel [10] stating that the inclusion map of a compact immersed minimal hypersurface N in a manifold of positive Ricci curvature is 1-connected. In the same paper he also showed that the inclusion map of a totally geodesic submanifold N in a positively curved manifold M is 1-connected provided that $2\dim(N)\geqslant\dim(M)$. Frankel obtained his results by showing that the inverse image of N in the universal cover is connected. Our proof relies on a very simple Morse theory argument in the space of all paths in M starting and ending in N. The setup has been used in the literature before, and like in the proof of Frankel's theorem [9] the sectional curvature condition comes in via a Synge-type argument. Theorem 1 can be generalized to positive k-Ricci curvature, see Remark 2.4. Upon composing the Poincaré duality of M with the one of N we will show that the cohomology ring of M exhibits a certain periodicity, provided that the codimension of N is small, see Lemma 2.2.

Recall that the fixed-point components of an isometry are totally geodesic submanifolds. That is why Theorem 1 turns out to be a very powerful tool in the analysis of group actions on positively curved manifolds. As a first application we prove

THEOREM 2. *Let M^n be a simply-connected n-dimensional manifold of positive sectional curvature, $n\geqslant 10$, and let*

$$d\geqslant \tfrac{1}{4}n+1.$$

Suppose that there is an effective isometric action of a d-dimensional torus T^d on M^n. Then M^n is homeomorphic to $\mathbf{HP}^{n/4}$ or to $\mathbf{S}^n$, or M is homotopy equivalent to $\mathbf{CP}^{n/2}$.

The theorem remains valid for $n\leqslant 9$ except for $n=7$. For $n\leqslant 6$ this is just a consequence of the above-mentioned result of Grove and Searle [13]. Fang and Rong [8] showed that the theorem remains valid for $n=8,9$. Their proof relies on Theorem 2.1 combined with a thorough study of the orbit space. For $n=7$ the theorem is incorrect since the Aloff–Wallach examples [1] as well as the Eschenburg examples [6] have symmetry rank 3.

Notice that F_4, the isometry group of $\mathrm{Ca}\mathbf{P}^2$, has rank 4. Thus in dimension 16 the result is optimal. Similarly the isometry group of the 12-dimensional Wallach flag has rank 3. In dimension 13 the Berger space $\mathrm{SU}(5)/\mathsf{S}^1\cdot\mathsf{Sp}(2)$ [3] as well as the Bazaĭkin examples [2] are positively curved manifolds with symmetry rank 4.

The fact alone that in Theorem 2 a large-dimensional torus acts on the manifold could potentially allow to strengthen its conclusion. A conjecture of Mann [18] asserts that an exotic sphere Σ^n can not support an effective smooth action of a d-dimensional torus with $d \geqslant \frac{1}{4}n+1$. Let M^{2m} be a compact manifold that is homotopy equivalent to $\mathbf{CP}^m$, and suppose that M^{2m} supports a nontrivial smooth S^1-action. The Petrie conjecture [20] asks whether the total Pontrjagin class of M^{2m} is standard, i.e., whether it is given by $(1+x^2)^{m+1}$, where $x \in H^2(M, \mathbf{Z})$ is a generator. Dessai and Wilking [5] show that the conclusion of the Petrie conjecture holds if M^{2m} supports an effective smooth action by a d-dimensional torus with $d > \frac{1}{4}(m+1)$. More precisely, it is shown that such an action is linear in the sense of Petrie [19].

In the proof of Theorem 2 we will only recover the cohomology ring of the manifold. The recovery of the homeomorphism type of $\mathbf{HP}^k$ is a consequence of the following topological rigidity result which we prove in §10.

THEOREM 3. *Let M^{4k} be a simply-connected compact manifold with the integral cohomology ring of $\mathbf{HP}^k$. Suppose that a d-dimensional torus T^d acts smoothly and effectively on M. Then $d \leqslant k+1$ and if $d=k+1$, then M is homeomorphic to $\mathbf{HP}^k$.*

In view of the above-quoted theorem of Grove and Searle, it is natural to ask whether one can recover the fundamental group of a positively curved manifold with a large symmetry rank. This actually has been answered by Rong [21].

THEOREM 4 (Rong). *If M^n is a positively curved compact manifold supporting an effective isometric action of a d-dimensional torus with $d \geqslant \frac{1}{4}n+1$, then $\pi_1(M)$ is cyclic.*

Actually Rong only proved this under the slightly stronger assumption $d \geqslant \frac{1}{4}(n+6)$. As a very simple application of Theorem 2.1 we improve this result to the present version in §9.

It is easy to see that for $n \equiv 3 \bmod 4$ the (improved) lower bound for d is optimal. In fact let F be a noncyclic finite subgroup of S^3. Then there is a free action of F on $\mathbf{S}^{4m+3}$ such that $\mathsf{Sp}(m+1)$ acts by isometries on the quotient $\mathbf{S}^{4m+3}/\mathsf{F}$.

Grove (1991) proposed to classify manifolds of positive sectional curvature with a large isometry group. The charm of this proposal is that everyone who starts to work on this problem is himself in charge of what 'large' means and what classify means. One can relax the assumption if one gets new ideas. One potential hope could be that if one understands the obstructions for positively curved manifolds with a 'large' amount of symmetry, one may get an idea for a general obstruction. However, the main hope of Grove's program is that the process of relaxing the assumptions should lead toward constructing new examples. Of course, Theorem 2 fits very well in this context. Following Grove's proposal for finding examples one should try and relax the condition further until one arrives at structure results for potential new examples. The following theorem is a modest attempt in that direction.

THEOREM 5. *Let M^n be a simply-connected n-dimensional manifold of positive sectional curvature, $n \geqslant 6000$, and let*

$$d \geqslant \tfrac{1}{6}n+1.$$

Suppose that there is an effective isometric action of a d-dimensional torus T^d *on M^n. Then one of the following statements holds*:

(a) *M is homotopy equivalent to $\mathbf{S}^n$ or $\mathbf{CP}^{n/2}$.*

(b) *M has the same integral cohomology ring as $\mathbf{HP}^{n/4}$.*

(c) *$n \equiv 2 \mod 4$ and with respect to any field $\mathbf{F}$ the cohomology ring of M is isomorphic to $H^*(\mathbf{CP}^{n/2}, \mathbf{F})$ or $H^*(\mathbf{S}^2 \times \mathbf{HP}^{(n-2)/4}, \mathbf{F})$.*

(d) *$n \equiv 3 \mod 4$ and with respect to any field $\mathbf{F}$ the cohomology ring of M is either given by $H^*(\mathbf{S}^n, \mathbf{F})$ or by $H^*(\mathbf{S}^3 \times \mathbf{HP}^{(n-3)/4}, \mathbf{F})$.*

It is interesting to notice that any $\mathbf{S}^3$-bundle over $\mathbf{HP}^{(n-3)/4}$ has a cohomology ring satisfying (d), and any $\mathbf{S}^2$-bundle over $\mathbf{HP}^{(n-2)/4}$ has a cohomology ring satisfying (c). However, it is the author's belief that it is a little premature to look among $\mathbf{S}^2$-bundles over $\mathbf{HP}^n$ for positively curved metrics with a symmetry rank as large as in Theorem 5. In fact it seems more likely that a better understanding of the relations between the topology of the manifold and the topology of fixed-point sets could actually help to rule out possibilities (c) and (d).

The fact that we are only able to prove Theorem 5 in dimensions above 6000 is related to a problem in error-correcting code theory, see §3. The statement needed from the theory of error-correcting codes does not hold in low dimensions, it is however not clear whether 6000 is a good estimate, see also Remark 3.3. As a consequence in low dimensions the analogous assumption on the symmetry rank does not ensure the existence of totally geodesic submanifolds of sufficiently small codimension. This in turn allows potentially different phenomena to occur. A typical problem is for example that one can only recover the first third of the cohomology ring of a manifold, and that one can not say much more.

The rough idea for the proofs of Theorem 2 and Theorem 5 is as follows. Consider an involution in T^d fixing a submanifold $N \subset M^n$ of the smallest possible codimension k. Suppose that $k \leqslant \frac{1}{4}(n+3)$. Then by Theorem 1 the cohomology ring of N determines the cohomology ring of M. Since the torus acts on N, one may hope that the induction hypothesis is satisfied for N, in which case one would be done.

There are basically three major problems that can occur in this line of argument. The first problem is the induction start. In the proof of Theorem 2 we first establish it in the first four dimensions, see §6.1. In the case of Theorem 5 we actually prove a slightly more general theorem for all dimensions involving an additional hypothesis, see §8. The additional hypothesis needed becomes redundant in large dimensions.

The second problem is that the submanifold N may not satisfy the induction hypothesis. Under the hypothesis of Theorem 2 the second problem can essentially only occur if N has codimension 2 and is fixed by an S^1-subaction. In this case we can actually use a result of Grove and Searle [14] to conclude that M is diffeomorphic to $\mathbf{S}^n$ or $\mathbf{CP}^{n/2}$. However, under the hypothesis of Theorem 5 this could for example happen if N has codimension 4 and is fixed by an S^1-subaction. This is the situation we analyze in §7 using §5. In §5, which has a more topological flavor, we study $(\mathbf{Z}/p\mathbf{Z})$-actions with connected fixed-point sets.

The third problem is that the hypothesis on the symmetry rank of M may not ensure a fixed-point set of codimension $\leqslant\frac{1}{4}(n+3)$. We resolve it by establishing first the existence of a fixed-point set of codimension $\leqslant\frac{1}{4}(n+d+2)$, see §3. We then use the induction hypothesis and Theorem 1 to recover the cohomology of M up to dimension $\left[\frac{1}{2}n\right]-l$ with $l\leqslant\frac{1}{2}d$. Namely we show that M is either $\left(\left[\frac{1}{2}n\right]-l\right)$-connected, or the cohomology ring up to dimension $\left[\frac{1}{2}n\right]-l$ is given by the cohomology ring of $\mathbf{CP}^{n/2}$. It is not so hard to reduce the problem to the case of a $\left(\left[\frac{1}{2}n\right]-l\right)$-connected manifold. Fitting precisely this situation we prove in §4 a topological result that allows to recover the underlying manifold as a sphere, provided one has sufficiently good control over fixed-point sets. More precisely Theorem 4.1 says that if in this situation the fixed-point set of any element of finite prime order p in T^d is given by a $(\mathbf{Z}/p\mathbf{Z})$-homology sphere then the underlying manifold is a homology sphere.

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2. Proof of the connectedness lemma

In this section we prove Theorem 1. As mentioned before we actually prove the following refinement.

THEOREM 2.1. (Connectedness lemma.) *Let M^n be a compact Riemannian manifold with positive sectional curvature.*

(a) *Suppose that $N^{n-k}\subset M^n$ is a compact totally geodesic embedded submanifold of codimension k. Then the inclusion map $N^{n-k}\to M^n$ is $(n-2k+1)$-connected. If there is a Lie group G acting isometrically on M^n and fixing N^{n-k} pointwise, then the inclusion map is $(n-2k+1+\delta(\mathsf{G}))$-connected, where $\delta(\mathsf{G})$ is the dimension of the principal orbit.*

(b) *Suppose that $N_1^{n-k_1}, N_2^{n-k_2}\subset M^n$ are two compact totally geodesic embedded submanifolds, $k_1\leqslant k_2$, $k_1+k_2\leqslant n$. Then the intersection $N_1^{n-k_1}\cap N_2^{n-k_2}$ is a totally geodesic* *embedded submanifold as well, and the inclusion*

$$N_1^{n-k_1} \cap N_2^{n-k_2} \to N_2^{n-k_2}$$

is $(n-k_1-k_2)$*-connected.*

Part (b) of Theorem 2.1 says in particular that $N_1^{n-k_1} \cap N_2^{n-k_2}$ is not empty, which is exactly the content of Frankel's theorem.

Theorem 2.1 turns out to be a very powerful tool in the analysis of group actions on positively curved manifolds. In fact by combining the theorem with the following lemma, one sees that a totally geodesic submanifold of low codimension in a positively curved manifold has immediate consequences for the cohomology ring.

LEMMA 2.2. *Let* M^n *be a closed differentiable oriented manifold, and let* N^{n-k} *be an embedded compact oriented submanifold without boundary. Suppose that the inclusion* $\iota: N^{n-k} \to M^n$ *is* $(n-k-l)$*-connected and* $n-k-2l>0$*. Let* $[N] \in H_{n-k}(M, \mathbf{Z})$ *be the image of the fundamental class of* N *in* $H_*(M, \mathbf{Z})$*, and let* $e \in H^k(M, \mathbf{Z})$ *be its Poincaré dual. Then the homomorphism*

$$\cup e: H^i(M, \mathbf{Z}) \to H^{i+k}(M, \mathbf{Z})$$

is surjective for $l \leqslant i < n-k-l$ *and injective for* $l < i \leqslant n-k-l$*.*

Recall that the pullback of e to $H^k(N, \mathbf{Z})$ is the Euler class of the normal bundle of N in M. Notice that in the case of a simply-connected manifold M the submanifold N is simply-connected as well, and hence orientable.

Remark 2.3. (a) Fang, Mendonça and Rong [7] observed an analogy between Theorem 2.1 and similar theorems in algebraic geometry. They show that a totally geodesic immersed submanifold of a simply-connected positively curved n-manifold of codimension less than $\frac{1}{2}n$ is automatically embedded. All other theorems for totally geodesic submanifolds in that paper could have been deduced from Theorem 2.1 by elementary topological means. However, in order to exhibit the analogy better they give proofs using the general framework set up in [11].

(b) It is interesting to notice that Theorem 2.1 fails completely for manifolds with positive sectional curvature almost everywhere. In fact in [25] it was shown that the projective tangent bundles $P_{\mathbf{R}}T\mathbf{RP}^n$, $P_{\mathbf{C}}T\mathbf{CP}^n$ and $P_{\mathbf{H}}T\mathbf{HP}^n$ of $\mathbf{RP}^n$, $\mathbf{CP}^n$ and $\mathbf{HP}^n$ admit metrics with positive sectional curvature on open and dense sets of points. Furthermore with respect to these metrics all natural inclusions remain totally geodesic embeddings. However, the inclusion map $P_{\mathbf{R}}T\mathbf{RP}^n \to P_{\mathbf{R}}T\mathbf{RP}^{n+1}$ is only $(n-1)$-connected. Turning the problem around this feature partly explains the size of the set of points at which zero-curvature planes occur.

Proof of Theorem 2.1. (a) We essentially use the same setup as in [22]. Consider the space $\Omega_N M$ of piecewise smooth paths in M starting and ending in N^{n-k} which are parameterized on the unit interval. As usual we define the energy of a path c by

$$E(c)=\frac{1}{2}\int_0^1 \|\dot{c}(t)\|^2\,dt.$$

The set of paths having energy $\leqslant C$ may be approximated by a finite-dimensional manifold of broken geodesics. Hence we can explore the topology of that space using Morse theory.

Notice that N^{n-k} embeds naturally into $\Omega_N M$ as the set of point paths. Clearly $N^{n-k}=E^{-1}(0)$. We claim that the inclusion

$$N^{n-k}\to\Omega_N M$$

is $(n-2k)$-connected.

The critical points of the energy function are geodesics that start and end perpendicularly to N. We can estimate the index of such a geodesic c from below as follows: Notice that there are at least $n-2k+1$ linearly independent normal parallel vector fields along c that start and end tangentially to N. Hence we deduce that the index of c is at least $n-2k+1$. Notice that E is a Morse–Bott function in a neighborhood of $E^{-1}(0)$. It is then well known that one can find a Morse–Bott function E' on $\Omega_N M$ (or more precisely on a finite-dimensional approximation of $E^{-1}([0,C])$) that is C^∞-close to E such that $E'=E$ in a neighborhood of $E^{-1}(0)$ and any critical point $p\in\Omega_N M\setminus N$ of E' is nondegenerate and has index $\geqslant n-2k+1$.

By the usual Morse theory argument it follows that up to homotopy $\Omega_N M$ can be obtained from N^{n-k} by attaching cells of dimension $\geqslant n-2k+1$.

It follows that the inclusion $N^{n-k}\to\Omega_N M$ is $(n-2k)$-connected. Let D^i be the closed disk with boundary ∂D^i. Any map

$$(D^i,\partial D^i)\to(M,N)$$

induces naturally a map

$$(D^{i-1},\partial D^{i-1})\to(\Omega_N M,N),$$

and vice versa. Hence the relative homotopy group $\pi_i(M,N)$ satisfies $\pi_i(M,N)\cong\pi_{i-1}(\Omega_N M,N)$. Consequently, $\pi_i(M,N)=0$, $i=1,...,n-2k+1$. By the exact homotopy sequence for a pair it follows that the inclusion map $N\to M$ is $(n-2k+1)$-connected.

Suppose now that a Lie group G acts isometrically on M leaving N pointwise fixed. In that case we can improve the lower bound for the index of a critical point c to $n-2k+1+\delta(\mathsf{G})$ as follows. Roughly speaking we just lift the corresponding index estimate from the Alexandrov space M/G to M. For practical reasons it is convenient to stick with manifolds. Choose a fixed biinvariant metric $\langle\cdot,\cdot\rangle$ on G. Consider the diagonal free action of G on $M\times\mathsf{G}$, and for $\lambda>0$ put $(M,g_\lambda):=(M,g)\times(\mathsf{G},\lambda\langle\cdot,\cdot\rangle)/\mathsf{G}$ where g_λ denotes the induced orbit metric. Clearly, the sectional curvature of (M,g_λ) is positive as well. It is elementary to check that the geodesic $c\colon[0,1]\to(M,g)$ is a geodesic of length $\|\dot c(0)\|_g$ with respect to the metric g_λ as well. In fact $(c(t),e)\in M\times\mathsf{G}$ is a horizontal geodesic. Furthermore the index of $c\in\Omega_N M$ with respect to the metric g is equal to the index of c with respect to the metric g_λ.

We consider first the case that for some t, the isotropy group of $c(t)$ is trivial. Let

$$W:=\left\{X\;\middle|\;\begin{array}{l}X \text{ is a piecewise smooth vector field along } c \text{ with } X(0)\in T_{c(0)}N,\\ X(1)\in T_{c(1)}N,\ X(t)\perp\mathsf{G}*c(t),\ X'(t)\in T_{c(t)}(\mathsf{G}*c(t)) \text{ for } t\in[0,1]\end{array}\right\}.$$

Clearly $\dim(W)\geqslant n-2k+1+\delta(\mathsf{G})$. Another characterization of W can be given as follows. Let $0=t_0<\ldots<t_k=1$ be precisely those times at which the point $c(t)$ has a nontrivial isotropy group. A continuous vector field X along c with $X(0)\in T_{c(0)}N$ and $X(1)\in T_{c(0)}N$ is in W if and only if the following holds: the vector field $X|_{(t_i,t_{i+1})}$ is horizontal with respect to the submersion $\mathrm{pr}\colon M\backslash\mathrm{Sing}\to(M\backslash\mathrm{Sing})/\mathsf{G}$, and its projection $\mathrm{pr}_*\circ X|_{(t_i,t_{i+1})}$ is a parallel vector field along the geodesic $\mathrm{pr}\circ c|_{(t_i,t_{i+1})}$. Here $\mathrm{Sing}\subset M$ denotes the set of all points which have a nontrivial isotropy group. Since the metric g_λ has the same horizontal distribution as the metric g, this characterization implies that W does not depend on the choice of λ for the underlying metric g_λ.

Furthermore, it is easy to see that $\|X'(t)\|_{g_\lambda}$ converges monotonically to 0 as $\lambda\to0$, $X\in W$. The sectional curvature of the plane spanned by $X(t)$ and $\dot c(t)$ with respect to the metric g_λ increases as $\lambda\to0$. Clearly this implies that the index form of c corresponding to the metric g_λ is negative definite on W for λ sufficiently small. Therefore $\mathrm{index}(c)\geqslant\dim(W)\geqslant n-2k+1+\delta(\mathsf{G})$.

If $c(t)$ has a nontrivial isotropy group for all t, then there is a subgroup $\mathsf{H}\subset\mathsf{G}$ such that c lies in a component F of the fixed-point set $\mathrm{Fix}(\mathsf{H})$. Let k_2 be the codimension of F in M. The codimension of N in F is then given by $k-k_2$, and by induction on the dimension n we can assume that the index of c is $\geqslant n+k_2-2k+1+\delta(\mathsf{K})$, where K denotes the normalizer of H in G and $\delta(\mathsf{K})$ is the dimension of a principal K-orbit in F. Evidently $\delta(\mathsf{K})\geqslant\delta(\mathsf{G})-k_2$. Hence the result follows.

Next we want to prove statement (b). First notice that $N_1\cap N_2$ is indeed a totally geodesic submanifold. It is however not clear that $N_1\cap N_2$ is connected and that all connected components have the same dimension. Consider the space $\Omega_{N_1,N_2}M$ of all paths starting in N_1 and ending in N_2. Again it is not hard to check that the energy function E is a Morse–Bott function in a neighborhood of $E^{-1}(0)=N_1\cap N_2$. Similarly as above we can estimate the index of any critical point from below by $n-k_1-k_2+1$.

As above we use Morse theory to conclude that the inclusion map

$$N_1 \cap N_2 \to \Omega_{N_1,N_2} M \text{ is } (n-k_1-k_2)\text{-connected.}$$

Notice that the space $\Omega_{N_1,N_2}M$ fibers over $N_1 \times N_2$ with homotopy fiber being the based loop space of M.

We also consider the path space $\Omega_{M,N_2}M$ of all paths starting in M and ending in N_2. Since the inclusion $N_1 \to M$ is $(n-2k_1+1)$-connected, it follows that the inclusion $\Omega_{N_1,N_2}M \to \Omega_{M,N_2}M$ is $(n-2k_1+1)$-connected.

Combining the two statements we find that the inclusion map $N_1 \cap N_2 \to \Omega_{M,N_2}M$ is $(n-k_1-k_2)$-connected. Clearly $\Omega_{M,N_2}M$ is canonically homotopy equivalent to N_2, and thus the result follows. $\square$

Remark 2.4. Notice that the result carries over in parts to manifolds with positive l-Ricci curvature. In fact we only have to change the conclusion as follows: (a) the map is then $(n-2k+2-l)$-connected, and (b) the map is then $(n-k_1-k_2+1-l)$-connected. This is straightforward to check, as the only time where the lower curvature bound enters the proof is in the estimates of the indices of geodesics.

2.1. Proof of Lemma 2.2

The lemma follows from the fact that the map $\cup e\colon H^i(M,\mathbf{Z}) \to H^{i+k}(M,\mathbf{Z})$ is the composition of the four maps

$$\begin{array}{ccccccccc} H^i(M,\mathbf{Z}) & \to & H^i(N,\mathbf{Z}) & \to & H_{n-k-i}(N,\mathbf{Z}) & \to & H_{n-k-i}(M,\mathbf{Z}) & \to & H^{i+k}(M,\mathbf{Z}), \\ & w \mapsto \iota^* w, & & x \mapsto x \cap [N], & & y \mapsto \iota_* y, & & z \cap [M] \mapsto z, & \end{array}$$

where the first map is the pullback, the second map is the Poincaré duality of N, the third map is the push forward, and the last map is the Poincaré duality of M. The second map and the last map are isomorphisms for all i. Since the inclusion $N \to M$ is $(n-k-l)$-connected, the first map is an isomorphism for $i<n-k-l$, and it is injective for $i=n-k-l$. The third map is an isomorphism for $l<i$, and surjective for $i=l$.

3. Upper bounds on error-correcting codes

Put $\mathbf{Z}_2 := \mathbf{Z}/2\mathbf{Z}$. Consider the group $\mathbf{Z}_2^n$ with the Hamming distance

$$\operatorname{dist}(a,b) = \operatorname{ord}(\{i \mid a_i \neq b_i\}) \quad \text{for } a, b \in \mathbf{Z}_2^n.$$

A (binary) error-correcting code C of length n and distance b is a subset of $\mathbf{Z}_2^n$ such that $\operatorname{dist}(x,y)\geqslant b$ for all $x,y\in C$ with $x\neq y$. The code is called linear if C is a subgroup of $\mathbf{Z}_2^n$. The number $A(n,b)$ is defined as the optimal bound of the order of error-correcting codes of length n and distance b.

In our applications later on C will be a linear code. In fact C will be obtained as follows. Consider a manifold of positive sectional curvature M^n with an isometric action of an l-dimensional torus T^l. It is well known that if n is even, then T^l has a fixed point, and if n is odd-dimensional, then T^l has an isotropy group of codimension 1, see Lemma 6.1. Assume for simplicity that T^l itself has a fixed point p, and consider the isotropy representation

$$\varrho: \mathsf{T}^l \to \mathrm{O}(T_pM^n).$$

If n is odd, then there is a 1-dimensional trivial subrepresentation, and the orthogonal complement of this 1-dimensional subspace can be identified with $\mathbf{C}^{[n/2]}$ such that ϱ induces a representation

$$\hat{\varrho}: \mathsf{T}^l \to \mathsf{D} \subset \mathsf{U}\big(\big[\tfrac{1}{2}n\big]\big) \subset \mathrm{SO}(n-1),$$

where $\mathsf{D}\subset\mathsf{U}\big(\big[\tfrac{1}{2}n\big]\big)$ is the maximal torus of diagonal matrices in $\mathsf{U}\big(\big[\tfrac{1}{2}n\big]\big)$. If n is even we can identify T_pM itself with $\mathbf{C}^{[n/2]}$ such that ϱ induces a representation $\hat{\varrho}$ as above.

In order to find fixed-point sets of low codimension, one proceeds as follows. Consider the group of involutions $\mathsf{L}\subset\mathsf{T}^l$ and $\mathsf{H}\subset\mathsf{D}$ in T^d and D, respectively. Notice that H can be canonically identified with $\mathbf{Z}_2^{[n/2]}$ written multiplicatively. Furthermore $\mathsf{L}\cong\mathbf{Z}_2^l$. Of course ϱ induces an embedding $\varrho: \mathsf{L}\to\mathbf{Z}_2^{[n/2]}$. If for any involution $\iota\in\mathsf{L}$ the fixed-point set $\operatorname{Fix}(\iota)$ has codimension $\geqslant 2k$, then $C:=\varrho(\mathsf{L})$ is an error-correcting code of length $\big[\tfrac{1}{2}n\big]$ and distance k. Therefore $A\big(\big[\tfrac{1}{2}n\big],k\big)\geqslant 2^l$. Vice versa if we can establish $A\big(\big[\tfrac{1}{2}n\big],k+1\big)<2^l$, then there must be a fixed-point set of codimension $\leqslant 2k$.

Given an error-correcting code C of length n and distance b one can estimate its size as follows. The metric balls of distance less than $\tfrac{1}{2}b$ around each point in C are pairwise disjoint. On the other hand such a ball has

$$\sum_{i=0}^{[(b-1)/2]} \binom{n}{i}$$

elements. This gives the well-known estimate

$$A(n,b)\leqslant \frac{2^n}{\sum_{i=0}^{[(b-1)/2]}\binom{n}{i}},$$

see [17]. Often it is useful to consider a refinement of this estimate. A code $C_w \subset \mathbf{Z}_2^n$ of length n and distance b is said to be of constant weight w if $\operatorname{dist}(x,0)=w$ for all $x \in C_w$. In other words, C_w is a subset of the w-distance sphere around 0. The optimal bound on the order of C_w is denoted by $A(n,b,w)$. One can use weighted codes to establish different estimates on $A(n,b)$. Let C be again a code of length n and distance b. Consider around each point in C the distance sphere of radius w in $\mathbf{Z}_2^n$. It is straightforward to check that an element $x \in \mathbf{Z}_2^n$ lies in at most $A(n,b,w)$ of these distance spheres. This gives another well-known result,

$$A(n,b) \leqslant \frac{2^n \cdot A(n,b,w)}{\binom{n}{w}}.$$

The following proposition is based on this inequality combined with the estimate $A(n,b,w) \leqslant n+1$ for $w < \frac{1}{2}\left(n-\sqrt{n^2-2b}\right)$. The author was not able to find the latter inequality in the literature, which is probably due to the fact that he is not very familiar with it.

PROPOSITION 3.1. *Suppose that n and b are positive integers with $b<\frac{1}{2}n$. Let w be the smallest integer less than $\frac{1}{2}\left(n-\sqrt{n^2-2nb}\right)+1$. Then*

$$\begin{aligned}\log_2(A(n,b)) \leqslant n\Big(1+\frac{w}{n}\log_2\Big(\frac{w}{n}\Big)+\frac{n-w}{n}\log_2\Big(\frac{n-w}{n}\Big)\Big)\\ +\frac{3}{2}\log_2(n)+\frac{1}{2}\log_2\Big(\frac{n-w}{w}\Big)+\frac{3}{2}.\end{aligned}$$

Proof. Put $w_0=w-1$. We plan to prove $A(n,b,w_0) \leqslant n+1$. Let $S \subset \mathbf{Z}_2^n$ be a subset of elements with $\operatorname{dist}(x,0)=w_0$ for all $x \in S$ and $\operatorname{dist}(x,y) \geqslant b$ for $x \neq y$, $x,y \in S$. Therefore each element $x=(x_1,...,x_n) \in S$ has precisely w entries which are 1, and the other elements are 0. Unlike before we use here the additive notation of $\mathbf{Z}_2$. Consider $\alpha,\beta>0$ with $w\alpha^2+(n-w_0)\beta^2=1$ and the map $f: S \to \mathbf{S}^{n-1}$, $x \mapsto (a_1,...,a_n)$, where $a_i=\alpha$ if $x_i=1$, and $a_i=-\beta$ if $x_i=0$. Notice that the inner product between two elements $f(x), f(y)$ in $\mathbf{R}^n$ satisfies

$$\langle f(x), f(y)\rangle \leqslant \left(n-w_0-\tfrac{1}{2}b\right)\beta^2-b\alpha\beta+\left(w_0-\tfrac{1}{2}b\right)\alpha^2 \tag{1}$$

for all $x,y \in S$ with $x \neq y$. If the right-hand side of this inequality is negative, then the angle between any two elements in $f(S)$ is larger than $\frac{1}{2}\pi$. That implies that the order of S is bounded by $n+1$. Notice that we can choose α,β such that the right-hand side of (1) is negative if and only if the discriminant is positive,

$$0<b^2-4\left(n-w_0-\tfrac{1}{2}b\right)\left(w_0-\tfrac{1}{2}b\right)=2nb-4nw_0+4w_0^2,$$

which is equivalent to

$$w_0 < \tfrac{1}{2}\big(n-\sqrt{n^2-2nb}\,\big).$$

Thus

$$A(n,b) \leqslant \frac{2^n A(n,b,w_0)}{\binom{n}{w_0}} \leqslant \frac{2^n(n+1)(n-w+1)}{w\binom{n}{w}}.$$

Applying the Stirling formula for $w\geqslant 2$, $n\geqslant 9$ gives

$$\begin{aligned}
A(n,b) &\leqslant 2^n\cdot\frac{n^2}{w}\,\frac{\sqrt{2\pi w}\cdot w^w\cdot\sqrt{2\pi(n-w)}\cdot(n-w)^{n-w}}{\sqrt{2\pi n}\,n^n} \\
&\qquad \times e^{1/12(n-1)+1/12(w-1)+1/12(n-w-1)} \\
&\leqslant n^{3/2}\cdot 2^n\cdot\Big(\frac{w}{n}\Big)^{w-1/2}\cdot\Big(\frac{n-w}{n}\Big)^{n-w+1/2}\cdot\sqrt{2\pi}\cdot e^{1/9} \\
&\leqslant n^{3/2}\cdot 2^n\cdot\Big(\frac{n-w}{w}\Big)^{1/2}\cdot\Big(\frac{w}{n}\Big)^{w}\cdot\Big(\frac{n-w}{n}\Big)^{n-w}\cdot 2^{3/2}.
\end{aligned}$$

If $w=1$ or $n<9$ one can check the inequality by more direct means, but in this range the inequality is useless anyway.

The following corollary is all we need from the theory of error-correcting codes for the remainder of the paper.

COROLLARY 3.2. *Let $\varrho\colon \mathsf{T}^d\to\mathrm{O}(n)$ be a representation of a d-dimensional torus.*

(a) *If $d\geqslant\frac{1}{6}n$ and $n\geqslant 6000$, then there is an involution $\iota\in\mathsf{T}^d$ such that the multiplicity of the eigenvalue -1 in $\varrho(\iota)$ is at most $\frac{7}{24}n$.*

(b) *If $n=n_0-2k$ with $2k\leqslant\frac{7}{24}n_0$ and $d\geqslant\max\{\frac{1}{8}n_0+12,\frac{1}{6}n_0-1\}$, then there is an involution $\iota\in\mathsf{T}^d$ such that the multiplicity of the eigenvalue 1 in $\varrho(\iota)$ is at least $\frac{1}{2}n_0$.*

(c) *If $d\geqslant\max\{\frac{1}{8}n+12,\frac{1}{6}n\}$, then there is an involution $\iota\in\mathsf{T}^d$ such that the multiplicity of the eigenvalue -1 in $\varrho(\iota)$ is at most $\leqslant\frac{5}{12}n$.*

(d) *Suppose that $n\geqslant 11$, $n\neq 15$ and $d\geqslant\frac{1}{4}n+1-(n-2[\frac{1}{2}n])$. Then there is an involution $\iota\in\mathsf{T}^d$ such that the multiplicity of the eigenvalue -1 in $\varrho(\iota)$ is at most $\frac{1}{16}(5n+13)$.*

Proof. Of course we may assume that there is no involution in the kernel of ϱ. Therefore $n\geqslant 2d$.

(a) Put $m:=[\frac{1}{2}n]$. Let b be the smallest integer $\leqslant\frac{7}{24}m+1$. In particular $b>\frac{7}{24}m$. As explained at the beginning of this section it suffices to show $A(m,b)<2^{m/3}\leqslant 2^{n/6}\leqslant 2^d$. Let w be the smallest integer less than $\frac{1}{2}(m-\sqrt{m^2-2mb}\,)+1$. Thus $w\geqslant\frac{1}{4}m(2-\sqrt{5/3}\,)$. Since the right-hand side in the inequality of Proposition 3.1 is decreasing in w, we obtain

$$\log_2(A(m,b)) < 0.325973\,m+\tfrac{3}{2}\log_2(m)+2.6074.$$

The right-hand side is less than $\frac{1}{3}m \leqslant \frac{1}{6}n$ for $m \geqslant 2700$.

(b) Put $k_0 := \left[\frac{7}{48}n_0\right]$. So $k \leqslant k_0$. It suffices to consider the case of $k = k_0$. Otherwise just consider an $(n_0 - 2k_0)$-dimensional invariant subspace and apply the special statement $k = k_0$ to the induced representation on this subspace.

Notice that $n = n_0 - 2\left[\frac{7}{48}n_0\right]$ implies $n \leqslant n_0 - \frac{7}{24}n_0 + 2$ and $n_0 \geqslant \frac{24}{17}(n-2)$. Therefore

$$d \geqslant \max\left\{\tfrac{1}{8}n_0 + 12, \tfrac{1}{6}n_0 - 1\right\} \geqslant \max\left\{\tfrac{3}{17}(n-2) + 12, \tfrac{4}{17}(n-2) - 1\right\}.$$

Furthermore it suffices to find an involution $\iota \in \mathsf{T}^d$ such that the multiplicity of the eigenvalue -1 is at most $n - \frac{1}{2}n_0 \geqslant \frac{5}{17}n$. Put $m = \left[\frac{1}{2}n\right]$. Let b be the smallest integer $\leqslant \frac{5}{17}m + 1$. As explained at the beginning of this section it suffices to show $A(m, b) < 2^d$.

Let w be the smallest integer less than $\frac{1}{2}\left(m - \sqrt{m^2 - 2mb}\right) + 1$. Thus we have $w \geqslant \frac{1}{2}m\left(1 - \sqrt{7/17}\right)$. Since the right-hand side in the inequality of Proposition 3.1 is decreasing in w, we obtain

$$\log_2(A(m, b)) < 0.321774m + \tfrac{3}{2}\log_2(m) + 2.6.$$

The right-hand side is less than $\frac{4}{17}(n-2) - 1$ for $m \geqslant 91$, while for $m \leqslant 90$ it is less than $\frac{3}{17}(n-2) + 12$.

(c) is proved similarly.

(d) Put $m := \left[\frac{1}{2}n\right]$. Notice that $d \geqslant \frac{1}{2}(m+1)$. Let b be the smallest integer $\leqslant \frac{5}{16}m + \frac{45}{32}$. In particular $b \geqslant \frac{1}{16}(5m+7)$. As before it suffices to show $A(m, b) < 2^d$.

Put $h := \left[\frac{1}{2}(b-1)\right] \geqslant \frac{1}{32}(5m - 25)$. Notice that $h \geqslant \frac{2}{15}m$ for $m \geqslant 25$. Furthermore

$$A(m, b) \leqslant \frac{2^m}{\sum_{i=0}^{h}\binom{m}{i}} \leqslant \frac{15 \cdot 2^m}{17\binom{m}{h}}.$$

By the Stirling formula,

$$\begin{aligned}
\log_2(A(m, b)) \leqslant\; & m\left(1 + \frac{h}{m}\log_2\left(\frac{h}{m}\right) + \frac{m-h}{m}\log_2\left(\frac{m-h}{m}\right)\right) \\
& + \log_2(e)\frac{1}{12}\left(\frac{1}{h-1} + \frac{1}{m-1} + \frac{1}{m-h-1}\right) \\
& + \log_2\left(\frac{15}{17}\right) + \frac{1}{2}\log_2\left(\frac{h}{m}\right) + \frac{1}{2}\log_2\left(\frac{m-h}{m}\right) + \frac{1}{2}\log_2(2\pi)\frac{1}{2}\log_2(m) \\
\leqslant\; & 0.4335m - 0.64 + \tfrac{1}{2}\log_2(m),
\end{aligned}$$

which is $\leqslant \frac{1}{2}(m+1) \leqslant d$ for $m \geqslant 25$. For $m \leqslant 24$ one can use tables on bounds of error-correcting codes to get the desired result, see [17]. In some instances one has to use the fact that the codes are linear. □

Remark 3.3. In the literature estimates of the quantity

$$R(\delta) := \limsup_{n\to\infty} \frac{1}{n} \log_2(A(n,[\delta n]))$$

for $\delta \in \left(0, \frac{1}{2}\right)$ are called asymptotic bounds of $A(n,d)$. The best known bound for $R(\delta)$ is due to McEliece, Rodemich, Rumsey and Welch [16]. Their estimate relies on linear programming and on some inequalities involving Krantchouk and Hahn polynomials. One can use their method in order to get estimates for $A(n,d)$. In fact this only requires to do the bookkeeping in a few rather elementary estimates. It is conceivable that one could lower the bound 6000 in Theorem 5 with this approach.

4. A recognition theorem for the sphere

Let p be a prime number and put as before $\mathbf{Z}_p := \mathbf{Z}/p\mathbf{Z}$. A well-known theorem of Smith says that for a $\mathbf{Z}_p$-homology sphere any diffeomorphism of order p has either an empty fixed-point set or the fixed-point set is also a $\mathbf{Z}_p$-homology sphere, see [4, Chapter III, Theorem 5.1]. The following theorem says that under very special circumstances one can conclude the other way around: if the topology of the singular set of a torus action is sufficiently simple, then the manifold has to be a sphere.

THEOREM 4.1. *Let M^n be a compact $\left(\left[\frac{1}{2}n\right]-k\right)$-connected manifold. Suppose that a d-dimensional torus T^d acts smoothly and effectively on M. Furthermore we assume that either $d \geqslant 2k+2$ or T^d has a fixed point and $d \geqslant 2k-1+n-2\left[\frac{1}{2}n\right]$. Finally, we assume for any element $\sigma \in \mathsf{T}^d$ of prime order p that the fixed-point set* Fix(σ) *is either empty or* Fix(σ) *is a $\mathbf{Z}_p$-homology sphere.*

Then M is a homology sphere.

It should be understood that Fix(σ) being a homology sphere implies that Fix(σ) is connected unless it consists of two isolated points. It is crucial for the proof that the range of the unknown Betti numbers of M is smaller than the dimension of the torus.

Proof. It is convenient to fix a Riemannian metric on M invariant under the action of T^d. We first reduce the problem to the case where T^d has a fixed point. So assume $d \geqslant 2k+2$. Notice first that the torus can not act freely on M^n. Otherwise $N := M^n/\mathsf{T}^d$ would be a compact $(n-d)$-dimensional manifold. Since M is highly connected, it follows that the Betti numbers of N satisfy

$$\begin{aligned} b_{2i}(N,\mathbf{Q}) &= b_{2i}(BT^d,\mathbf{Q}) = \binom{i+d-1}{i} && \text{for } 2i \leqslant \left[\tfrac{1}{2}n\right]-k, \\ b_{2i+1}(N,\mathbf{Q}) &= b_{2i+1}(BT^d,\mathbf{Q}) = 0 && \text{for } 2i+1 \leqslant \left[\tfrac{1}{2}n\right]-k. \end{aligned}$$

Finally if $h=\left[\frac{1}{2}n\right]-k+1$ is even, then

$$b_h(N,\mathbf{Q})\geqslant\binom{\frac{1}{2}h+d-1}{d-1}.$$

Since $\left[\frac{1}{2}n\right]-k>\frac{1}{2}(n-d)$, this is a contradiction to the Poincaré duality of N.

Hence T^d acts not freely. Let $\sigma\in\mathsf{T}^d$ be an element of prime order for which its fixed-point set $\mathrm{Fix}(\sigma)$ is not empty. Since $\mathrm{Fix}(\sigma)$ is a homology sphere, it follows that T^d has a fixed point if n is even or that T^d has a circle orbit if n is odd. In the latter case we pass from T^d to a subtorus of codimension 1 with a nonempty fixed-point set. Subsequently we can and do assume that $d\geqslant 2k-1+n-2\left[\frac{1}{2}n\right]$ and T^d has a fixed point.

For a connected subgroup $\mathsf{H}\subset\mathsf{T}^d$ we consider the space $M\times_{\mathsf{H}}E\mathsf{H}$. Notice that this space fibers over the classifying space $B\mathsf{H}$. Furthermore since the action of H on M has a fixed point, the fibration has a section. Hence the induced homomorphism

$$\iota\colon H^i(B\mathsf{H},\mathbf{Z}_p)\to H^i(M\times_{\mathsf{H}}E\mathsf{H},\mathbf{Z}_p)\tag{2}$$

is injective. Since M is $\left(\left[\frac{1}{2}n\right]-k\right)$-connected, ι is an isomorphism for $i\leqslant\left[\frac{1}{2}n\right]-k$.

Next we plan to show that ι is an isomorphism for $i=n-2,n-1$. Consider all elements $h_1,...,h_l\in\mathsf{H}$ of order p. By hypothesis the fixed-point set F_i of h_i is a $\mathbf{Z}_p$-homology sphere. Put

$$F:=\bigcup_{i=1}^{l}F_i.$$

Notice that the intersection $F_{i_1}\cap...\cap F_{i_q}$ is given by the fixed-point set of the group $\langle h_2,...,h_{i_q}\rangle$ with respect to the induced action of H on F_{i_1}. By a theorem of Smith $F_{i_1}\cap...\cap F_{i_q}$ is a $\mathbf{Z}_p$-homology sphere as well.

Let $x_0\in F$ be a fixed point of H, and let $B_r(x_0)$ be a ball of radius r around x_0, where r is less than the injectivity radius of the compact manifold M. Notice that the induced action of H on $B_r(x_0)$ is linear.

Put $F':=F\setminus(B_r(x_0)\cap F)$. Notice that F' is a finite union of $\mathbf{Z}_p$-homology disks, and that any subfamily of those disks intersect in a homology disk as well. From the Mayer–Vietoris sequence it follows that $H^i(F',\mathbf{Z}_p)=0$ for $i>0$.

Let A be a closed neighborhood of F' in $M\setminus B_r(x_0)$ such that the inclusion $F'\to A$ is a homotopy equivalence and such that the boundary ∂A is a topological embedded submanifold. By construction $H^*(\partial A,\mathbf{Z}_p)\cong H^*(\mathbf{S}^{n-1},\mathbf{Z}_p)$.

Using Mayer–Vietoris we find that the natural map

$$H^i(M\times_{\mathsf{H}}E\mathsf{H},\mathbf{Z}_p)\to H^i((M\setminus A)\times_{\mathsf{H}}E\mathsf{H},\mathbf{Z}_p)$$

is an isomorphism for $i\leqslant n-1$.

For any point x in $M\setminus(A\cup B_r(x_0))$ the isotropy group of x has no p-torsion.

This implies for any H-invariant submanifold $B\subseteq M\setminus(A\cup B_r(x_0))$ that $B\times_{\mathsf{H}}E\mathsf{H}$ has finite $\mathbf{Z}_p$-homology. In fact $H^i(B\times_{\mathsf{H}}E\mathsf{H},\mathbf{Z}_p)=0$ for $i>n-\dim(\mathsf{H})-1$.

Using Mayer–Vietoris we deduce that

$$H^i((M\setminus A)\times_{\mathsf{H}}E\mathsf{H},\mathbf{Z}_p))\to H^i(B_r(x_0)\times_{\mathsf{H}}E\mathsf{H},\mathbf{Z}_p)\cong H^i(B\mathsf{H},\mathbf{Z}_p) \tag{3}$$

is an isomorphism for $i\geqslant n$. Notice that A has the $\mathbf{Z}_p$-homology of a point. Using Mayer–Vietoris it follows that $H^i(M\setminus A,\mathbf{Z}_p)=0$ for $i\geqslant\left[\frac{1}{2}(n+1)\right]+k$. From this it follows easily that $H^*((M\setminus A)\times_{\mathsf{H}}E\mathsf{H},\mathbf{Z}_p)$ is injective as a $B\mathsf{H}$-module in dimensions above $\left[\frac{1}{2}(n+1)\right]+k$, i.e., for $i\geqslant\left[\frac{1}{2}(n+1)\right]+k$, $x\in H^i((M\setminus A)\times_{\mathsf{H}}E\mathsf{H},\mathbf{Z}_p)$, $y\in H^*(B\mathsf{H},\mathbf{Z}_p)$ the quantity $\iota(y)\cup x$ only vanishes if $y=0$ or $x=0$. Therefore the map from (3) has to be an isomorphism for $i\geqslant\left[\frac{1}{2}(n+1)\right]+k$. Hence $b_i(B\mathsf{H},\mathbf{Z}_p)=b_i(M\times_{\mathsf{H}}E\mathsf{H},\mathbf{Z}_p)$ for $i=n-1,n-2$. Consequently the monomorphism ι in equation (2) is an isomorphism for $i=n-2,n-1$.

Next we choose a chain of connected subgroups $\{e\}=\mathsf{H}_0\subset\mathsf{H}_1\subset\ldots\subset\mathsf{H}_d=\mathsf{T}^d$ such that $\mathsf{H}_i/\mathsf{H}_{i-1}\cong\mathsf{S}^1$. We claim that the map

$$\iota_j\colon H^i(B\mathsf{H}_j,\mathbf{Z}_p)\to H^i(M\times_{\mathsf{H}_j}E\mathsf{H}_j,\mathbf{Z}_p) \tag{4}$$

is an isomorphism for $\left[\frac{1}{2}(n-1)\right]+k-j\leqslant i\leqslant n-1$.

By hypothesis the statement is true for $j=0$, and we have just seen that it is true for $i=n-2,n-1$ and arbitrary j. Suppose, on the contrary, that the statement is false for some (j,i). We may assume that $j>0$ is as small as possible and that $i<n-2$ is as large as possible. Using the Gysin sequence of the fibration $\mathsf{S}^1\to M\times_{\mathsf{H}_{j-1}}E\mathsf{H}_{j-1}\to M\times_{\mathsf{H}_j}E\mathsf{H}_j$ we deduce that the statement is false for $(j,i+2)$ as well—a contradiction.

Applying the above claim for $\mathsf{H}=\mathsf{T}^d$, we see that

$$\iota\colon H^i(B\mathsf{T}^d,\mathbf{Z}_p)\to H^i(M\times_{\mathsf{H}_j}E\mathsf{T}^d,\mathbf{Z}_p)$$

is an isomorphism for $\left[\frac{1}{2}(n-1)\right]+k-d\leqslant i\leqslant n-1$. Since M is $\left(\left[\frac{1}{2}(n-1)\right]-k\right)$-connected, it follows that ι is an isomorphism for $i\leqslant\left[\frac{1}{2}n\right]-k$. In summary ι is an isomorphism for $i\leqslant n-1$, but that proves that M is $(n-1)$-connected. □

5. Cyclic group actions with connected fixed-point sets

THEOREM 5.1. *Let (M^n,g) be a simply-connected compact Riemannian manifold of positive sectional curvature, and let p be a prime. Suppose that the cyclic group $\mathbf{Z}_p$ acts on M^n by isometries, and assume that the fixed-point set N is a connected submanifold of codimension k. Then*

$$H^i(M,\mathbf{Z}_p)=0 \quad \textit{for } k\leqslant i\leqslant n-2k+1.$$

For the proof we need the following lemma.

LEMMA 5.2. *Let $\mathbf{Z}_p$ and M^n be as in Theorem 5.1. Consider the Borel construction $M\times_{\mathbf{Z}_p}E\mathbf{Z}_p$. The natural map $H^i(M\times_{\mathbf{Z}_p}E\mathbf{Z}_p,\mathbf{Z}_p)\to H^i(M,\mathbf{Z}_p)$ is surjective for $i\leqslant n-2k$ and for $i\geqslant 2k-1$.*

Proof. Throughout the proof we use cohomology with $\mathbf{Z}_p$-coefficients. The inclusion map $N\to M$ is $(n-2k+1)$-connected, see Theorem 2.1. The same holds for the map $N\times_{\mathbf{Z}_p}E\mathbf{Z}_p\to M\times_{\mathbf{Z}_p}E\mathbf{Z}_p$. Thus $H^i(M\times_{\mathbf{Z}_p}E\mathbf{Z}_p)\to H^i(N\times_{\mathbf{Z}_p}E\mathbf{Z}_p)$ is an isomorphism for $i\leqslant n-2k$. Clearly $N\times_{\mathbf{Z}_p}E\mathbf{Z}_p=N\times B\mathbf{Z}_p$. Combining these statements we see that the map $H^i(M\times_{\mathbf{Z}_p}E\mathbf{Z}_p)\to H^i(N)$ is surjective for $i\leqslant n-2k$. Finally this implies the lemma for $i\leqslant n-2k$.

Consider a tubular neighborhood U of N, and put $M':=M\setminus U$. Notice that $H^i(M')=0$ for $i\geqslant 2k-1$. Using the exact cohomology sequence of the pair (M,M') we see that the natural map $H^i(M,M')\to H^i(M)$ is surjective for $i\geqslant 2k-1$. The result follows now from the fact that the natural map $H^i(M\times_{\mathbf{Z}_p}E\mathbf{Z}_p, M'\times_{\mathbf{Z}_p}E\mathbf{Z}_p)\to H^i(M,M')$ is surjective for all i, which can be easily deduced from the excision theorem. □

Proof of Theorem 5.1. Throughout the proof we use cohomology with $\mathbf{Z}_p$-coefficients. Notice that Lemma 2.2 remains valid for $\mathbf{Z}_p$-coefficients We may assume that $k\leqslant\frac{1}{3}(n+1)$. Consider the Borel construction $M\times_{\mathbf{Z}_p}E\mathbf{Z}_p$ and the fibration

$$M\to M\times_{\mathbf{Z}_p}E\mathbf{Z}_p\to B\mathbf{Z}_p.$$

Recall that all Betti numbers of $B\mathbf{Z}_p$ are equal to one. Combining Lemma 5.2 with the spectral sequence of this fibration we see that the Betti numbers of $M\times_{\mathbf{Z}_p}E\mathbf{Z}_p$ satisfy

$$b_j(M\times_{\mathbf{Z}_p}E\mathbf{Z}_p)\geqslant\sum_{i\leqslant n-2k}b_i(M)+\sum_{i>\max\{n-2k,2k-2\}}b_i(M)$$

for $j>n$. On the other hand it is well known that this number is equal to the total Betti number of $\mathrm{Fix}(\mathbf{Z}_p)=N$, see [4]. If $k\leqslant\frac{1}{4}(n+2)$, then this shows $b(M)\leqslant b(N)$. It is apparent from Lemma 2.2 that this can only occur if $e=0$, and this in turn implies the theorem. If $\frac{1}{4}(n+2)<k\leqslant\frac{1}{3}(n+1)$, then

$$\sum_{i\leqslant n-2k}b_i(M)+\sum_{i\geqslant 2k-1}b_i(M)\geqslant 2\sum_{i\leqslant(n-k)/2}b_i(M)=2\sum_{i\leqslant(n-k)/2}b_i(N)\geqslant b(N).$$

Since equality must hold, we find $b_i(M)=0$ for $\frac{1}{2}(n-k)<i\leqslant n-2k+1$. In particular we find $b_k(M)=0$. As before, the theorem is now a consequence of Lemma 2.2. □

COROLLARY 5.3. *Let M^n be a compact manifold of positive sectional curvature. Suppose that S^1 acts effectively on M by isometries. Suppose that the fixed-point set F*

of S^1 has codimension $k \leqslant \frac{1}{3}(n+1)$. Moreover we assume that F is connected and that G acts freely on $M \backslash F$. Then $H^i(M^n, \mathbf{Z})=0$ for $i=k-1, ..., n-2k+2$.

If we assume in addition that $k \leqslant \frac{1}{4}(n+5)$ and that there is a totally geodesic submanifold N_2 of codimension $\leqslant \frac{1}{2}(n+3-k)$ intersecting N transversely, then M is a homology sphere.

Proof. Note that every cyclic subgroup of prime order p in S^1 has F as its fixed-point set. Applying Theorem 5.1 gives $H^k(M, \mathbf{Z})=0$. By Theorem 2.1 the inclusion map $F \to M$ is $(n-2k+2)$-connected, and the result follows from Lemma 2.2.

In order to prove the addendum put $n-k_2=\dim(N_2)$. We consider first the case of $k_2 \geqslant k$. Note that $N_2 \cap N \to N_2$ is $(n-k_2-k)$-connected and $\dim(N_2 \cap N)=n-k_2-k$. By Lemma 2.2 there is a cohomology class $e \in H^k(N_2, \mathbf{Z})$ that gives a period in the cohomology ring of N_2. The pullback $\iota^* e$ of e to $H^k(N_2 \cap N, \mathbf{Z})$ is the Euler class of the normal bundle of $N_2 \cap N$ in N. The normal bundle of $N_2 \cap N$ in N is the pullback bundle of the normal bundle of N in M. The latter has a vanishing Euler class, as $H^k(N, \mathbf{Z})=0$. Thus $\iota^* e=0$. Since the inclusion map $N_2 \cap N \to N_2$ is $(n-k_2-k)$-connected, this implies $e=0$. Hence N_2 is a homology sphere. Since the inclusion map $N_2 \to M$ is $(k-2)$-connected, it follows that $H^i(M, \mathbf{Z})=0$ for $i \leqslant n-2k+2$.

If $k_2<k$, the inclusion map $N_2 \cap N \to N$ is $(n-k_2-k)$-connected and $\dim(N_2 \cap N)= n-k_2-k$. Similarly as above it follows from Lemma 2.2 that N has a k_2-periodic cohomology ring. On the other hand we know that $H^*(N, \mathbf{Z})=0$ for $i=k-1, ..., n-2k+1$. Thus N is a homology sphere, and so is M. □

6. Proof of Theorem 2

Our proof is by induction on the dimension $n \geqslant 9$, but as mentioned in the introduction the case of dimension 9 is due to Fang and Rong. For convenience we include a different proof for the case of dimension 9, since it is important for the induction start. It is also a first nice application of Theorem 2.1 and Theorem 4.1.

Before treating the special cases we will say a few words about the setup. Based on a lemma of Berger [3], Grove and Searle [13] proved the following lemma.

LEMMA 6.1. *Suppose that a torus T^d acts effectively on a positively curved manifold M^n. If the dimension n is even, then T^d has a fixed point. If the dimension is odd, then there is a circle orbit.*

Notice that at a circle orbit the isotropy group is $(d-1)$-dimensional. Therefore in odd dimensions a suitable subtorus of codimension 1 has a fixed point. Corollary 3.2 (d) guarantees involutions with fixed-point sets of codimension $\leqslant\frac{1}{16}(5n+13)$ for $n\geqslant 11$ and $n\neq 15$.

Throughout this section let (M,g) and T^d be as in Theorem 2. We may assume that $d=\mathrm{symrank}(M,g)$. We fix an involution $\iota_0\in\mathsf{T}^d$ such that the fixed-point set $\mathrm{Fix}(\iota_0)$ has a component N_0 of minimal codimension.

The torus T^d acts on N_0. The minimality of the codimension implies that the kernel of that action is at most 1-dimensional. Therefore

$$\mathrm{symrank}(N_0)\geqslant d-1$$

with equality only if N_0 is fixed by a circle subaction $\mathsf{S}^1\subset\mathsf{T}^d$.

We claim that we can always assume $\mathrm{codim}(N_0)\geqslant 4$. Suppose $\mathrm{codim}(N_0)=2$. If N_0 is fixed by a subaction of a circle $\mathsf{S}^1\subset\mathsf{T}^d$, then M is fixed-point homogeneous, i.e., the circle is acting on the unit normal sphere of its fixed-point set. By Grove and Searle [13] that implies that the simply-connected manifold M is diffeomorphic to $\mathbf{S}^n$ or $\mathbf{CP}^{n/2}$.

If N_0 is not fixed by a circle, then $\mathrm{symrank}(N_0)\geqslant d\geqslant\frac{1}{4}n+1$. If $n=9,10$, then N_0 has maximal symmetry rank and is diffeomorphic to a sphere or a complex projective space. If $n\geqslant 11$, then it follows that N_0 satisfies the induction hypothesis, and we conclude that N_0 has the cohomology ring of $\mathbf{CP}^{(n-2)/2}$ or $\mathbf{S}^{n-2}$. Since the inclusion $N_0\to M$ is $(n-3)$-connected, M has the corresponding cohomology ring.

Therefore we have $\mathrm{codim}(N_0)\geqslant 4$.

6.1. Proof of Theorem 2 in dimensions 9, 10, 11 and 12

T^4 *on* M^9. Because $\mathrm{symrank}(N_0)\geqslant 3$, it follows that $\dim(N_0)=5$ and N_0 is fixed by a circle subaction. Thus the inclusion map $N_0\to M^9$ is 3-connected. Therefore N_0 is a positively curved simply-connected manifold of maximal symmetry rank. By Grove and Searle N_0 is diffeomorphic to $\mathbf{S}^5$. Consequently M^9 is 3-connected.

By Lemma 6.1 there is a subtorus of codimension 1, $\mathsf{T}^3\subset\mathsf{T}^4$, that has a fixed point q_0, and we may assume $q_0\in N_0^5$. In order to recognize M^9 as a sphere we plan to apply Theorem 4.1 to M^9 equipped with the T^3-subaction.

It suffices to prove that for any element $\iota\in\mathsf{T}^3$ of prime order p the fixed-point set $\mathrm{Fix}(\iota)$ is a homotopy sphere. We first want to show that $\mathrm{Fix}(\iota)$ is connected. Suppose that there is a component $F\subset\mathrm{Fix}(\iota)$ with $q_0\notin F$. Choose a point $q_1\in F$ that lies on a circle orbit. The isotropy group of the T^3-action at q_1 is 2-dimensional. In particular it contains three involutions. It is now easy to see that one of these involutions ι_1 has a 5-dimensional fixed-point component N_1^5 with $q_1\in N_1$. Notice that N_1 is diffeomorphic to a sphere, too.

By a theorem of Smith it follows that $\mathrm{Fix}(\iota|_{N_1})$ is a $\mathbf{Z}_p$-homology sphere. In particular, $N_1 \cap \mathrm{Fix}(\iota)$ is connected, and hence $q_0 \notin N_1$. In other words the fixed-point set of ι_1 is not connected either, and we might as well assume that $\iota=\iota_1$ and $F=N_1$. By Frankel N_1 and N_0 intersect. That implies that $\mathrm{Fix}(\iota|_{N_0})$ is disconnected—a contradiction.

It remains to check that the connected set $F=\mathrm{Fix}(\iota)$ is a homotopy sphere. If $\dim(F)=5$, then this follows as for N_0. If $\dim(F)=3$, then the action of T^4 on F has 2-dimensional kernel. Thus there is an involution with a 5-dimensional fixed-point set N_2^5 such that $F \subset N_2^5$. Since N_2^5 is a homotopy sphere and $F \to N_2^5$ is 2-connected, it follows that F is a homotopy sphere, too.

T^4 *on* M^{10}. Since $\mathrm{symrank}(N_0) \geqslant 3$, it follows that N_0 is 6-dimensional and fixed by a circle subaction $\mathsf{S}^1 \subset \mathsf{T}^4$. Furthermore it is easy to see that the nonexistence of fixed-point sets of codimension 2 implies that $\mathrm{Fix}(\mathsf{T}^4)$ consists of isolated points.

The inclusion map $N_0^6 \to M^{10}$ is 4-connected, and, in particular, N_0^6 is simply-connected. Since N_0 has maximal symmetry rank, it is either diffeomorphic to $\mathbf{CP}^3$ or $\mathbf{S}^6$.

If N_0^6 is a sphere, then it follows that M^{10} is 4-connected. Notice that $b_5(M^{10}, \mathbf{Q})$ is even. Thus $b_5(M^{10}, \mathbf{Q}) \neq 0$ would imply that the Euler characteristic $\chi(M^{10})$ is non-positive. On the other hand by Lefschetz $\chi(M)=\chi(\mathrm{Fix}(\mathsf{T}^4))$—a contradiction.

In the case of $N_0^6 \cong \mathbf{CP}^3$ we can argue as follows: If the even number $b_5(M^{10}, \mathbf{Q})$ is not zero, then $\chi(M) \leqslant 4 = \chi(N_0^6)$. It is easy to find another manifold N_2^6 fixed by a different S^1-action. By Theorem 2.1 the intersection $N_2^6 \cap N_0^6$ is connected. In particular, $\chi(N_2^6 \cap N_0^6) < 4$. But that implies that T^4 has at least one fixed point outside of N_0^6, and thus $\chi(M)=\chi(\mathrm{Fix}(\mathsf{T}^4)) > 4 = \chi(N^6)$—a contradiction.

Notice that the natural map $H^*(M^{10}, \mathbf{Z}) \to H^*(N^6, \mathbf{Z})$ is surjective, as its image contains the generator of $H^*(N^6, \mathbf{Z})$. So the even cohomology groups of M^{10} coincide with the cohomology groups of $\mathbf{CP}^5$. Since the Euler characteristics are equal, we conclude that the odd-dimensional cohomology groups of M^{10} coincide with the cohomology groups of $\mathbf{CP}^5$ as well. But this shows that the natural map $H^i(M^{10}, \mathbf{Z}) \to H^i(N^6, \mathbf{Z})$ is an isomorphism for $i \leqslant 6$. This determines the ring structure of $H^*(M^{10}, \mathbf{Z})$.

T^4 *on* M^{11}. The first step is to show that M is 3-connected. Suppose, on the contrary, that M is not 3-connected.

Consider a point q_0 whose isotropy group H_{q_0} has dimension $\geqslant 3$. It is straightforward to check that there are at least two totally geodesic submanifolds N_1, N_2 of codimension 4 fixed by involutions $\iota_1, \iota_2 \in \mathsf{H}_{q_0}$.

Since the inclusion map $N_i^7 \to M$ is 4-connected, N_i^7 is not a sphere. By Grove and Searle that implies $\mathrm{symrank}(N_i^7) \leqslant 3$. Hence we can find a 1-dimensional subgroup of T^4

that fixes N_i. Therefore the inclusion map $N_i \to M$ is 5-connected.

Furthermore it is clear that N_i^7 does not have totally geodesic submanifolds of codimension 2 either. In particular it follows that N_1 and N_2 intersect transversely. Thus $N_1^7 \cap N_2^7 \to N_i^7$ is 3-connected. In particular, $N_1^7 \cap N_2^7$ is a simply-connected 3-manifold and hence a homotopy sphere. It follows that N_1^7 is 2-connected. Let $\mathbf{F}$ be a field. Since the inclusion $N_1^7 \cap N_2^7 \to N_1^7$ is 3-connected, it follows that $b_3(N_1^7, \mathbf{F}) \leqslant 1$. By Poincaré duality $b_4(N_1^7, \mathbf{F}) = b_3(N_1^7, \mathbf{F}) \leqslant 1$ and $b_5(N_1^7, \mathbf{F}) = 0$. Since the inclusion $N_1^7 \to M^{11}$ is 5-connected, this leaves only two possibilities for the total Betti number $b(M^{11}, \mathbf{F})$, namely $b(M, \mathbf{F})$ is either 2 or 6.

Since N_1 and N_2 are fixed by circles and do not have totally geodesic submanifolds of codimension 2, we deduce that H_{q_0} is connected, $\dim(\mathsf{H}_{q_0}) = 3$ and the fixed-point set H_{q_0} through q_0 is the circle $\mathsf{T}^4 * q_0$. The isotropy representation of H_{q_0} induces a complex structure on the normal space $\nu_p(\mathsf{T}^4 * q_0)$ of the circle $\mathsf{T}^4 * q_0$. We can identify $\nu_p(\mathsf{T}^4 * q_0)$ with $\mathbf{C}^5$ such that the image of the isotropy representation is generated by

$$a(z) = (z, z, 1, 1, 1), \quad b(z) = (1, 1, z, z, 1), \quad c(z) = (z, 1, z, 1, z),$$

$z \in \mathsf{S}^1$, where $a(z)$, $b(z)$, $c(z)$ represent diagonal matrices in $\mathsf{U}(5)$.

We identify H_{q_0} with the image in $\mathsf{U}(5)$, and we may assume that $a(\mathsf{S}^1)$ fixes N_1^7 and $b(\mathsf{S}^1)$ fixes N_2^7. Let N_3^5 be the component of the fixed-point set of $c(\mathsf{S}^1)$ through q_0.

Notice that $\mathrm{symrank}(N_3^5) = 3$. By Grove and Searle N_3^5 is equivariantly diffeomorphic to a lens space or a sphere.

Next we claim that N_3^5 is simply-connected. Suppose not. Let p be a prime number that divides the order of the fundamental group of N_3^5. The total Betti number of N_3^5 is 6 with respect to $\mathbf{Z}_p$-coefficients.

Notice that the fixed-point set of $c(\mathbf{Z}_p)$ has N_3^5 as a component as well. By a theorem of Floyd the total Betti number of the fixed-point set of $c(\mathbf{Z}_p)$ is bounded by $b(M, \mathbf{Z}_p)$, see [4, Chapter III, Theorem 4.1]. Hence $b(M, \mathbf{Z}_p) = 6$ and the fixed-point set of $c(\mathbf{Z}_p)$ is equal to N_3^5.

Let N_4^5 be the component of the fixed-point set of $d(z) = b(\bar{z})c(z) = (z, 1, 1, \bar{z}, z)$. The intersection $N_4^5 \cap N_3^5$ has a 3-dimensional component B, and the inclusion $B \to N_i^5$ is 2-connected, $i = 3, 4$. In particular, the fundamental groups of N_4^5 and N_3^5 are isomorphic.

The manifold N_4^5 is diffeomorphic to a lens space, too. The group $c(\mathbf{Z}_p)$ fixes the 3-manifold B. Since the action is linear and p divides the order of the fundamental group of N_4^5, it follows that the fixed-point set of $c(\mathbf{Z}_p)|_{N_4^5}$ has precisely one more component, a circle N^1. The circle N^1 is not fixed by $c(\mathsf{S}^1)$, and therefore N^1 is not contained in N_3^5. Hence $\mathrm{Fix}(c(\mathbf{Z}_p))$ has at least two components—a contradiction.

Therefore N_3^5 is simply-connected and diffeomorphic to a sphere. We plan to prove that the fixed-point set of $a(\mathsf{S}^1)$ has no components other than N_1^7 and that $a(\mathsf{S}^1)$ acts freely on $M^{11}\backslash N_1^7$.

Suppose, on the contrary, that the fixed-point set of the cyclic group $a(\mathbf{Z}_p)$ has a second component B for some prime number p. Choose a point $q_2 \in B$ such that the isotropy group H_{q_2} has dimension 3. Since all arguments from above apply equally with q_0 replaced by q_2, there is an element $\iota \in \mathsf{H}_{q_2}$ such that the component F of the fixed-point set of ι through q_2 is diffeomorphic to $\mathbf{S}^5$. By Frankel F and N_1^7 intersect. Hence the fixed-point set of $a(\mathbf{Z}_p)|_F$ has at least two odd-dimensional components. On the other hand F is diffeomorphic to $\mathbf{S}^5$—a contradiction.

Therefore the fixed-point set of $a(\mathsf{S}^1)$ is N_1^7, and $a(\mathsf{S}^1)$ acts freely on $M^{11}\backslash N_1^7$. By Corollary 5.3 we get $H^i(M, \mathbf{Z})=0$ for $i=3,4$—a contradiction.

Thus we have proved that M^{11} is 3-connected. Of course we can still prove the existence of a totally geodesic submanifold N_1^7. Since the inclusion map $N_1^7 \to M^{11}$ is 4-connected, it follows that N_1^7 is a homotopy sphere, and hence M^{11} is 4-connected. If we can choose N_1^7 such that it is fixed by a circle subaction, then the inclusion map $N^7 \to M^{11}$ is 5-connected, and we are done. Thus we may may assume that $\operatorname{symrank}(N_1^7)=4$ for any possible choice of N_1^7.

Next we claim that for any element $\iota \in \mathsf{T}^4$ of prime order p the fixed-point set $\operatorname{Fix}(\iota)$ is either empty or given by a homotopy sphere. Because of Theorem 4.1 this finishes the proof.

Let us first check that each component N of $\operatorname{Fix}(\iota)$ is a homotopy sphere. Suppose first that $\dim(N)=5$. If N is contained in a totally geodesic 7-manifold, then clearly N is a sphere. Thus we may assume that the kernel of the action of T^4 on N^5 is isomorphic to S^1. Thus N^5 is diffeomorphic to a lens space by Grove and Searle. It is easy to see that there is totally geodesic 7-manifold N_2^7 (fixed by an involution) that intersects N in a 3-dimensional manifold. Clearly, the intersection $N_2^7 \cap N$ has the same fundamental group as N. On the other hand N_2^7 is equivariantly diffeomorphic to $\mathbf{S}^7$, and hence $N_2^7 \cap N$ is diffeomorphic to $\mathbf{S}^3$.

If $\dim(N)=3$, then N is contained in a totally geodesic submanifold of dimension 5 or 7. In either case N is a homotopy sphere.

In order to prove that $\operatorname{Fix}(\iota)$ is connected we can argue as follows. Otherwise we can choose two points q_1, q_2 in different components of $\operatorname{Fix}(\iota)$ such that the corresponding isotropy groups H_1 and H_2 have dimension at least 3. Choose an involution $\sigma \in \mathsf{H}_1 \cap \mathsf{H}_2$ such that the component F of $\operatorname{Fix}(\sigma)$ through q_1 has dimension at least 5. Since F is a homotopy sphere the fixed-point set of $\iota|_F$ is connected. Hence $q_2 \notin F$. In other words we may assume $\iota=\sigma$. By Frankel $F \cap N_1^7$ is not empty, and clearly the intersection contains a circle orbit. Therefore $q_1 \in N_1^7$ without loss of generality. Switching the roles of q_1 and q_2 shows that we also may assume $q_2 \in N_1^7$. But this is a contradiction as $N_1^7 \cap \mathrm{Fix}(\iota)$ is connected.

T^4 *on* M^{12}. By Corollary 3.2 we have $\mathrm{codim}(N_0)=4$. Consider a fixed point $q_0 \in N_0$. There is a 2-dimensional subtorus $\mathsf{T}^2 \subset \mathsf{T}^4$ that acts trivially on the normal bundle of N_0^8 at q_0. It is now easy to find an involution $\iota_1 \in \mathsf{T}^2$ such that the q_0-component N_1 of $\mathrm{Fix}(\iota_1)$ has codimension 4.

By construction the intersection $N_2 := N_0^8 \cap N_1^8$ is 4-dimensional.

Clearly T^4 can not act trivially on N_2^4. Thus N_2^4 has a Killing field, and by Hsiang and Kleiner [15] N_2^4 is homeomorphic to $\mathbf{S}^4$ or $\mathbf{CP}^2$. Since the inclusion map $N_2^4 \to N_0^8$ is 4-connected, it follows that N_0^8 has the cohomology ring of a compact symmetric space of rank 1.

If we can find an N_0^8 that is fixed pointwise by a subgroup $\mathsf{S}^1 \subset \mathsf{T}^4$, then the inclusion $N_0^8 \to M^{12}$ is 6-connected. Clearly this implies that M^{12} has the cohomology ring of a compact symmetric space of rank 1, too.

If we can find no such N_0^8, then T^4 acts with finite kernel on N_0^8, and by Grove and Searle [13] N_0^8 is either diffeomorphic to $\mathbf{S}^8$ or to $\mathbf{CP}^4$. Furthermore it follows that T^4 has only isolated fixed points.

The inclusion $N^8 \to M^{12}$ is 5-connected.

Thus the odd-dimensional homology and cohomology groups of M^{12} are 0. By the universal coefficient theorem the cohomology groups are torsion-free. Therefore it suffices to show that $b_6(M, \mathbf{Q}) = b_2(M, \mathbf{Q})$.

We argue by contradiction: If $b_6(M, \mathbf{Q}) > b_2(M, \mathbf{Q})$, then clearly $\chi(M) \geqslant 3$. Thus there are at least three fixed points of T^4. We claim that for any three fixed points of T^4 we can find a fixed-point component N^8 containing the three fixed points.

Let q_1, q_2, q_3 be three different fixed points of T^4. We consider the subgroup $(\mathbf{Z}_2)^4 \subset \mathsf{T}^4$ and the direct sum of the three isotropy representations in $T_{q_i}M$. This is a 36-dimensional vector space. Since $(\mathbf{Z}_2)^4$ is contained in a torus, the irreducible subrepresentations come in equivalent pairs. Thus we may think of this representation as an 18-dimensional complex representation.

$\mathbf{Z}_2^4$ has only fifteen nontrivial representations. So it follows that at least two of the eighteen subrepresentations are equivalent. We choose such a subrepresentation and consider its kernel $\mathbf{Z}_2^3$. $\mathbf{Z}_2^3$ has now $18-l$ nontrivial subrepresentations and l trivial subrepresentations ($l \geqslant 2$). If $l \leqslant 3$ we can find at least three nontrivial subrepresentations that are pairwise equivalent. If $l \leqslant 10$ we can find at least two such subrepresentations.

We consider the kernel $\mathbf{Z}_2^2$ of the pairwise equivalent subrepresentations. The representation of $\mathbf{Z}_2^2$ consists of k trivial and $18-k$ nontrivial subrepresentations, $k \geqslant 5$. If $k=5$, then at least five of the nontrivial subrepresentations are equivalent. If $k \leqslant 8$, then at least four of the nontrivial subrepresentations are equivalent. If $k \leqslant 10$, then at least three of the nontrivial subrepresentations are equivalent. We consider the kernel of these pairwise equivalent subrepresentations. It has at least ten trivial subrepresentations. Consider now the fixed-point set of the nontrivial element in the kernel. Let N_{q_i} be the connected component containing q_i. Clearly $\dim(N_{q_1})+\dim(N_{q_2})+\dim(N_{q_3}) \geqslant 20$. Furthermore $\dim(N_{q_i}) \leqslant 8$. Thus after reordering we have $\dim(N_{q_1})=8$ and $\dim(N_{q_2}), \dim(N_{q_3}) \geqslant 4$. By Frankel's theorem N_{q_1} and N_{q_2} intersect. Thus $N_{q_1}=N_{q_2}$, and similarly $N_{q_1}=N_{q_3}$.

Since N_{q_1} contains three fixed points of T^4, it can not be a sphere. Thus as explained before it has to be diffeomorphic to $\mathbf{CP}^4$. Since T^4 acts almost effectively on N_{q_1}, we can find a totally geodesic 4-manifold $N^4_{123} \cong \mathbf{CP}^2$ containing all three fixed points.

Notice that N^4_{123} contains precisely three fixed points.

We now give two different estimates for the number of these submanifolds containing a given point q_1. First this number is at least $\frac{1}{2}(\chi(M)-1)(\chi(M)-2)$.

On the other hand such a manifold is determined by its tangent space at q_1, and there are only $\frac{1}{2}\cdot 6\cdot 5$ possibilities.

Thus we have proved $\chi(M) \leqslant 7$ and hence $b_6(M)=b_2(M)=1$.

6.2. Proof of Theorem 2 in dimensions above 12

There are two different arguments for the induction step. The first is

LEMMA 6.2. *Suppose that we have proved Theorem 2 in all dimensions less than n. Let M^n, T^d be as in Theorem 2. If there is an element $\iota \in \mathsf{T}^d \backslash e$ such that the fixed-point set has codimension $\leqslant \frac{1}{4}(n+3)$, then the conclusion of Theorem 2 holds for M as well.*

Proof. This is almost a direct consequence of Theorem 2.1. Let F be the component of $\mathrm{Fix}(\iota)$ of codimension $\leqslant \frac{1}{4}(n+3)$. We may assume that $\dim(F)$ is maximal. If F has codimension 2 and is fixed by a circle, then we are done by Grove and Searle [14]. If not, then $\mathrm{symrank}(F) \geqslant \frac{1}{4}\dim(F)+1$. Thus the cohomology ring of F is isomorphic to the cohomology of a compact symmetric space of rank 1. Since the inclusion map $F \to M$ is $\left[\frac{1}{2}n\right]$-connected, the same holds for M. □

Using the theory of error-correcting codes it is not hard to see that the hypothesis on the existence of a fixed-point set of low codimension can actually be removed from Lemma 6.2 for almost all dimensions n, since in those dimensions it follows from the assumption on the symmetry rank. However, for the other dimensions one needs additional arguments, and here it is where Theorem 4.1 is crucial. In fact this approach works uniformly in all dimensions.

LEMMA 6.3. *Suppose that we have proved Theorem 2 in all dimensions less than n. Let M^n, T^d be as in Theorem 2. If there is an element $\iota \in \mathsf{T}^d \backslash e$ such that the fixed-point set has codimension $\leqslant \frac{1}{16}(5n+13)$, then the conclusion of Theorem 2 holds for M as well.*

By Corollary 3.2, the condition on the existence of a fixed-point set of codimension $\leqslant \frac{1}{16}(5n+13)$ is implied by the other assumptions, unless possibly $n=15$. For $n=15$ the existence is also implied by the other conditions, but unlike in the other dimensions the argument is of global nature. We will treat the case of $n=15$ at the end of this section in a separate lemma. This finishes the proof of Theorem 2.

Proof of Lemma 6.3. By the previous lemma we may assume that there are no elements in T^d fixing a totally geodesic submanifold of codimension $\leqslant \frac{1}{4}(n+3)$. Furthermore $n \geqslant 13$ and thus there are no fixed-point sets of codimension 4. Thus the statement of the lemma is only nontrivial if $n \geqslant 17$. Let N_0 be the fixed-point component of ι of codimension $\leqslant \frac{1}{16}(5n+13)$. We may assume that N_0 has minimal codimension among all fixed-point sets of involutions.

Consider a component N of $\mathrm{Fix}(\sigma)$ for some $\sigma \in \mathsf{T}^d$. Suppose that N is maximal, i.e., N is not contained in a component $N' \subset \mathrm{Fix}(\sigma')$ with $\dim(N') > N$. Then the symmetry rank of N is at least $d-1 \geqslant \frac{1}{4}n$. Furthermore the intersection $N \cap N_0$ has symmetry rank $\geqslant \frac{1}{4}n-1$. Therefore $\dim(N \cap N_0) \geqslant \frac{1}{2}n-3$ with equality only if $N \cap N_0$ is fixed by a 2-torus. Because of $\mathrm{codim}(N), \mathrm{codim}(N_0) \geqslant 6$, Theorem 2.1 implies that the inclusion maps $N \cap N_0 \to N$ and $N \cap N_0 \to N_0$ are 2-connected. Since M and N_0 are simply-connected, the same holds for N.

Since the codimension of N is larger than 4, we have

$$\mathrm{symrank}(N) > \tfrac{1}{4}\dim(N)+1. \tag{5}$$

Thus N has the homotopy type of $\mathbf{S}^k$ or $\mathbf{CP}^{k/2}$—N can not be homeomorphic to $\mathbf{HP}^{k/4}$, since we have the strict inequality in (5).

Notice that the previous discussion applies in particular to $N=N_0$. If N_0 is a homotopy sphere, then we can argue as follows. Notice first that M is l-connected with $l \geqslant n-2k+1 \geqslant \frac{1}{8}(3n-5)$.

By Theorem 4.1 it suffices to prove that for any element $\sigma \in \mathsf{T}^d$ of prime order p, the fixed-point set $\mathrm{Fix}(\sigma)$ is either empty or a $\mathbf{Z}_p$-homology sphere. In the odd-dimensional case one has to pass from T^d to a subtorus of codimension 1 with a nonempty fixed-point set in order to satisfy the hypothesis of Theorem 4.1.

We first prove that any component N of $\mathrm{Fix}(\sigma)$ is a $\mathbf{Z}_p$-homology sphere. Suppose, on the contrary, that N is a counterexample of minimal codimension. If there is a group

$\mathbf{Z}_p^2 \subset \mathsf{T}^d$ that fixes N pointwise, then N is strictly contained in a component F of a fixed-point set of some other element in $\mathbf{Z}_p^2 - \{0\}$. Since $\dim(F) > \dim(N)$, it follows that F is a $\mathbf{Z}_p$-homology sphere. By a theorem of Smith $\mathrm{Fix}(\sigma|_F)$ is a $\mathbf{Z}_p$-homology sphere as well. Since N is a connected component of $\mathrm{Fix}(\iota|_F)$, it has to be a $\mathbf{Z}_p$-homology sphere as well.

Thus we may assume that N is not fixed by a group $\mathbf{Z}_p^2$. That implies $\mathrm{symrank}(N) \geqslant d-1$. As before it follows that N is homotopy equivalent to a sphere or a complex projective space. In order to rule out the latter, notice that the inclusion $N \cap N_0 \to N$ is h-connected with $h = n - \mathrm{codim}(N_0) - \mathrm{codim}(N) \geqslant 4$.

The manifold $N \cap N_0$ is a component of the fixed-point set of $\mathrm{Fix}(\sigma|_{N_0})$. Since N_0 is a homotopy sphere, $N \cap N_0$ is a $\mathbf{Z}_p$-homology sphere. Hence N can not be a complex projective space.

Next we prove that $\mathrm{Fix}(\sigma)$ is connected, unless $\mathrm{Fix}(\sigma)$ consists of two isolated points. We first argue for n odd. Suppose that $\mathrm{Fix}(\sigma)$ has at least two connected components. Then we can find two points q_1 and q_2 in the different components N_1 and N_2 of $\mathrm{Fix}(\sigma)$ such that their isotropy groups H_{q_1} and H_{q_2} have dimensions at least $d-1$.

Clearly we can choose an element $\sigma' \in \mathsf{H}_{q_1} \cap \mathsf{H}_{q_2}$ of order p such that $\mathrm{Fix}(\sigma')$ contains N_1, and the component F_1 of $\mathrm{Fix}(\sigma')$ with $q_1 \in F_1$ has dimension at least

$$2(\dim(\mathsf{H}_{q_1} \cap \mathsf{H}_{q_2}) - 1) + 1 \geqslant \tfrac{1}{2}(n+1) - 3.$$

Since F_1 is a $\mathbf{Z}_p$-homology sphere, the set $\mathrm{Fix}(\sigma|_{F_1})$ is a homology sphere as well, and hence $\mathrm{Fix}(\sigma|_{F_1}) = N_1$. In particular $q_2 \notin F_1$. Hence $\mathrm{Fix}(\sigma')$ is not connected either, and we might as well assume $\sigma' = \sigma$.

By Frankel the manifolds F_1 and N_0 intersect. Clearly the intersection $F_1 \cap N_0$ contains a circle orbit of T^d as well. Thus we may assume $q_1 \in N_0$. Switching the roles of q_1 and q_2 in the above argument shows that we also may assume $q_2 \in N_0$. But that proves that $\sigma|_{N_0}$ has two odd-dimensional components—a contradiction since N_0 is a homotopy sphere.

In even dimensions we can argue as follows: Since all connected components of $\mathrm{Fix}(\sigma)$ are homology spheres and $\chi(\mathrm{Fix}(\sigma)) = \chi(M)$, it suffices to prove $\chi(M) = 2$. For that it suffices to show that $\mathrm{Fix}(\iota) = N_0$. Suppose that $q \in \mathrm{Fix}(\iota) \setminus N_0$ is a fixed point of T^d.

It is easy to find an involution ι' such that the connected component F of $\mathrm{Fix}(N')$ through q has dimension greater than $\frac{1}{2}n$. We have seen that F is a $\mathbf{Z}_2$-homology sphere. On the other hand $F \cap N_0$ has positive dimension by Frankel. That shows that $\mathrm{Fix}(\iota|_F)$ is not a $\mathbf{Z}_2$-homology sphere, a contradiction.

Finally we have to consider the case of N_0 being homotopy equivalent to $\mathbf{CP}^{(n-k)/2}$.

Then $H^2(M,\mathbf{Z})=\mathbf{Z}$. There is a principal S^1-bundle

$$\mathsf{S}^1 \to B^{n+1} \to M^n$$

whose Euler class is the generator of $H^2(M,\mathbf{Z})$, where B^{n+1} is a compact manifold. There is a torus action T^{d+1} on B^{n+1} and a homomorphism $\mathsf{T}^{d+1}\to\mathsf{T}^d$ such that the projection $\mathrm{pr}: B^{n+1}\to M^n$ becomes equivariant.

Clearly fixed-point sets in B^{n+1} are preimages of fixed-point sets in M^n. Hence inclusion maps of fixed-point sets of low codimension are highly connected; more precisely, for a submanifold of codimension k the inclusion map is $(n+1-2k)$-connected. Furthermore, Frankel's theorem holds in B^{n+1} for fixed-point sets, as it holds for their images in M^n. Using all this it is easy to see that the proof carries over to B^{n+1}. Thus B^{n+1} is a homotopy sphere, and M^n is a homotopy $\mathbf{CP}^{n/2}$. □

T^5 *on* M^{15}.

LEMMA 6.4. *Let M^{15} be a 15-manifold of positive sectional curvature. Suppose that a 5-dimensional torus T^5 acts isometrically and effectively on M^{15}. Then there is an involution T^5 with a fixed-point set of codimension $\leqslant 4$.*

Proof. If the torus action has a fixed point or a nonisolated circle orbit, then the statement is trivial. If there is an involution $\iota\in\mathsf{T}^5$ such that $\mathrm{Fix}(\iota)$ has a 5-dimensional component N^5, then we can argue as follows. The action of T^5 on N has kernel of dimension at least 2. Hence we find another involution $\sigma\in\mathsf{T}^5\setminus\{\iota\}$ fixing N pointwise. It is now clear that either σ or $\sigma\cdot\iota$ has a fixed-point component of codimension $\leqslant 4$.

Suppose, on the contrary, that there is no involution with a fixed-point set of codimension $\leqslant 4$. It is easy to see that any circle orbit must be contained in a totally geodesic submanifold N_1^9 fixed by an involution ι_1. The manifold N_1^9 has symmetry rank $\geqslant 4$, and thus N_1^9 is homeomorphic to a sphere.

It is also clear that there must be another involution ι_2, fixing a 7-manifold N_2^7 that intersects N_1^9 transversely. Furthermore there is a third manifold N_3^9 intersecting N_2^7 in a 5-manifold B^5. It follows that $\pi_1(N_2^7)=\pi_1(B^5)=\pi_1(N_3^9)=\pi_1(M)=0$. Thus N_2^7 is simply-connected, and it is easy to see that $\mathrm{symrank}(N_2^7)\geqslant 4$. Hence N_2^7 is a sphere as well.

Notice that the intersection $N_1^9\cap N_2^7$ is connected by Theorem 2.1. Furthermore $S_0^1:=N_1^9\cap N_2^7$ is 1-dimensional, and thus it is necessarily given by a circle orbit. Since N_1 and N_2 are spheres, there must be four more circle orbits $S_1^1,...,S_4^1$ in N_1^9 apart from S_0^1. Furthermore there are another three circle orbits in $S_5^1,...,S_7^1$ in N_2^7.

The action of T^d on S_i^1 can be naturally identified with a 1-dimensional complex representation in $\mathbf{C}$. The sum of all those representations is a representation ϱ of T^5

in $\mathbf{C}^8$. By Corollary 3.2 there must be an involution ι such that $\varrho(\iota)$ has 1 as a (complex) eigenvalue with multiplicity at least 6. Translating back gives that ι fixes at least six of the circle orbits pointwise.

Suppose for a moment that all these circle orbits are contained in the same connected component F of $\mathrm{Fix}(\iota)$. Clearly we may assume $\mathrm{symrank}(F)\geqslant 4$, otherwise we can find a larger fixed-point set containing F. Since $\dim(F)\leqslant 9$, Theorem 2 implies that F is a sphere, and hence it has at most $\frac{1}{2}(\dim(F)+1)$ circle orbits.

It follows that there are two different connected components F_1 and F_2 of $\mathrm{Fix}(\iota)$ intersecting N_1 and N_2, respectively.

The fixed-point sets $\mathrm{Fix}(\iota|_{N_i})=N_i\cap\mathrm{Fix}(\iota)$ are necessarily homology spheres, and in particular they are connected. Hence $\mathrm{Fix}(\iota)\cap N_i=F_i\cap N_i$ and ι does not fix the circle orbit $S_0^1=N_1^9\cap N_2^7$. Since $F_2\cap N_1^9$ is empty, Frankel's theorem implies $\dim(F_2)\leqslant 5$. As shown above, the case of $\dim(F_2)=5$ can not occur, and thus $\dim(F_2)\leqslant 3$. Therefore F_2 contains at most two circle orbits.

This shows that $F_1\cap N_1^9$ contains each of the circle orbits $S_1^1, ..., S_4^1$. Hence $\dim(N_1^9\cap F_1)=7$. Since $F_1\cap N_2^7$ is empty, Frankel's theorem implies that $\dim(F_1)\leqslant 7$. Consequently $F_1\subset N_1^9$. But then $\iota\circ\iota_1$ has a fixed-point set of codimension 2—a contradiction. □

7. Totally geodesic submanifolds

LEMMA 7.1. *Let M^n be a simply-connected n-manifold, and let N^{n-k} be a submanifold such that the inclusion map $N^{n-k}\to M$ is $(n-k)$-connected. If k divides n, then M^n has the integral cohomology ring of a compact symmetric space of rank 1. If k divides $n-1$, then N is a homology sphere.*

The proof is a simple cohomology computation. In fact one just uses Lemma 2.2 to show that the integral cohomology ring of the manifold is generated by one element. By Adams that implies the desired result.

The following lemma is a simple application of Theorem 2.1 and Lemma 2.2.

LEMMA 7.2. *Suppose that the positively curved manifold M^n has two totally geodesic submanifolds $N_1^{n-k_1}$ and $N_2^{n-k_2}$. Assume that $k_1\leqslant\frac{1}{4}(n+3)$ and that N_2 is homotopy equivalent to a sphere or a complex projective space. If the inclusion map of $N_2\to M$ is (k_1+1)-connected, then M is homotopy equivalent to a sphere or a complex projective space as well.*

Now we are ready to show

PROPOSITION 7.3. *Suppose that M^n is a compact simply-connected n-manifold of positive sectional curvature.*

(a) *If n is odd and M^n contains one totally geodesic complete submanifold N of codimension 2, then M^n is homeomorphic to a sphere.*

(b) *If n is even and M^n contains a totally geodesic complete submanifold N of codimension 2 and one totally geodesic submanifold N' of codimension less than $\frac{1}{2}n$ with $N' \not\subset N$, then M^n is homotopy equivalent to a sphere or to $\mathbf{CP}^{n/2}$.*

(c) *Suppose that $n\equiv 0 \bmod 4$ or $n\equiv 1 \bmod 4$, $n\geqslant 13$. If M contains one totally geodesic submanifold N_1 of codimension 4 and one totally geodesic submanifold N_2 of codimension $\leqslant \frac{1}{2}n-3$, such that N_1 and N_2 intersect transversely, then M has the cohomology ring of a symmetric space of rank 1.*

Proof. (a) By Theorem 2.1 the inclusion map $N^{n-2}\to M^n$ is $(n-3)$-connected. Combining Lemma 2.2 with $H^1(M,\mathbf{Z})\cong H^{n-1}(M,\mathbf{Z})\cong 0$ we see that M is a homology sphere.

(b) Whether or not N and N' intersect transversely Theorem 2.1 (b) or (a) implies that $N\cap N'\to N'$ is $\dim(N\cap N')$-connected. By Lemma 7.1 this shows that N' is homotopy equivalent to a sphere or a complex projective space. Theorem 2.1 implies that M has the corresponding homotopy type.

(c) We only treat the harder case $n\equiv 0 \bmod 4$. Notice that the inclusion map $N_2\to M$ is 7-connected. By Theorem 2.1 the inclusion $N_2\cap N_1\to N_2$ is $\dim(N_2\cap N_1)$-connected. Let $e\in H^4(M,\mathbf{Z})$ be the Poincaré dual of the image of the fundamental class of N_1 in $H_{n-4}(M,\mathbf{Z})$. Since e pulls back to the Euler class of the normal bundle of N_1 in M, the pullback e to $H^4(N_2\cap N_1,\mathbf{Z})$ is the Euler class of the normal bundle of $N_2\cap N_1$ in N_2. This consideration shows that the pullback of e to N_2 gives a period in the cohomology ring of N_2 in sense of Lemma 2.2.

Since the cohomology rings of M and N_2 coincide up to dimension 7, we deduce that $\cup e: H^i(M,\mathbf{Z})\to H^{i+4}(M,\mathbf{Z})$ is an isomorphism for $0<i\leqslant 7$ and surjective for $i=0$.

Since $N_1\to M$ is $(n-7)$-connected, the above map is actually an isomorphism for $0<i<n-7$. It is easy to deduce from this that all odd cohomology groups of M vanish. Furthermore M is a homology sphere unless $H^4(M,\mathbf{Z})\cong\mathbf{Z}$.

If the torsion-free group $H^2(M,\mathbf{Z})$ vanishes, then it is easy to see that $H^*(M,\mathbf{Z})\cong H^*(\mathbf{HP}^{n/4},\mathbf{Z})$.

Otherwise one can use Poincaré duality to show that $e=x\cup y$ for $x,y\in H^2(M,\mathbf{Z})$. It is easy to see that then $\cup x: H^i(M,\mathbf{Z})\to H^{i+2}(M,\mathbf{Z})$ is an isomorphism, $i=0,\dots,n-2$. Thus $H^*(M,\mathbf{Z})\cong H^*(\mathbf{CP}^{n/2},\mathbf{Z})$. □

PROPOSITION 7.4. *Suppose that $n\equiv 2,3 \bmod 4$. Let M^n be a simply-connected n-manifold of positive sectional curvature. Let N^{n-4} be the fixed-point component of an* *isometry ι. Finally assume that $N^{n-4}\to M$ is $(n-4)$-connected. Then there are only the following possibilities for the cohomology ring of M.*

(a) *$n=4m+3$ and with respect to any field $\mathbf{F}$ we either have $H^*(M,\mathbf{F})\cong H^*(\mathbf{S}^n,\mathbf{F})$ or $H^*(M,\mathbf{F})\cong H^*(\mathbf{HP}^m\times\mathbf{S}^3,\mathbf{F})$.*

(b) *$n=4m+2$ and with respect to any field $\mathbf{F}$ either $H^*(M,\mathbf{F})\cong H^*(\mathbf{CP}^{m+1},\mathbf{F})$ or $H^*(M,\mathbf{F})\cong H^*(\mathbf{HP}^m\times\mathbf{S}^2,\mathbf{F})$.*

(c) *$n=4m+2$ and M is a homology sphere.*

(d) *$n=4m+2$ and the isometry ι has finite order l. If the characteristic of a field $\mathbf{F}$ is 0 or if it divides l, then $H^*(M,\mathbf{F})\cong H^*(\mathbf{S}^n,\mathbf{F})$. For a general field we either have $H^*(M,\mathbf{F})\cong H^*(\mathbf{S}^n,\mathbf{F})$ or the Betti numbers with respect to $\mathbf{F}$ are given as $b_i(M,\mathbf{F})=1$ if $i\equiv 2,0 \bmod 4$ and $b_i(M,\mathbf{F})=2$ if $i\equiv 3 \bmod 4$.*

Notice that the possibility (d) can not occur if N^{n-4} is fixed pointwise by an isometric circle action.

Proof. Notice that Lemma 2.2 remains true if we replace the coefficient ring $\mathbf{Z}$ by the field $\mathbf{F}$.

If $n=4m+3$, then it follows that $0=H^i(M,\mathbf{F})$ for $i=1,5,\dots,4m+1$ and for $i=2,\dots,4m+2$. Furthermore we get that the map $\cup e\colon H^0(M,\mathbf{F})\to H^4(M,\mathbf{F})$ is surjective. If $H^4(M,\mathbf{F})=0$, then we conclude that M is a homology sphere with respect to $\mathbf{F}$. Otherwise we find that $\mathbf{F}\cong H^i(M,\mathbf{F})$ for $i=0,4,\dots,4m$ and for $i=3,\dots,4m+3$. More precisely we get $H^*(M,\mathbf{F})\cong H^*(\mathbf{HP}^m\times\mathbf{S}^3,\mathbf{F})$.

It remains to consider the case of $n=4m+2$. Since ι acts trivially on the cohomology of N and thereby trivially on the cohomology of M, the Lefschetz theorem implies that the Euler characteristic of M is given by $\chi(\mathrm{Fix}(\iota))$. By Frankel $\mathrm{Fix}(\iota)$ can apart from N only have components of dimension 0 and 2. Thus all other components, if any, have positive Euler characteristic.

Therefore we get $\chi(N)\leqslant\chi(M)$. On the other hand we know that M and N have a 4-periodic cohomology ring, and hence

$$\chi(M)-\chi(N)=-b_1(M,\mathbf{F})+b_2(M,\mathbf{F})-b_3(M,\mathbf{F})+b_4(M,\mathbf{F}).$$

As before, the map $\cup e\colon H^0(M,\mathbf{F})\to H^4(M,\mathbf{F})$ is surjective. If $e=0$, then Lemma 2.2 implies that M is an $\mathbf{F}$-homology sphere. Thus we may assume $b_4(M,\mathbf{F})=b_2(M,\mathbf{F})=1$. Furthermore we know that $b_1(M,\mathbf{F})=0$. Finally Poincaré duality implies that $b_3(M,\mathbf{F})$ is even unless possibly if $\mathbf{F}$ has characteristic 2. Since the parity of $0\leqslant\chi(M)-\chi(N)=b_2(M,\mathbf{F})-b_3(M,\mathbf{F})+b_4(M,\mathbf{F})$ is independent of the field, it follows that $b_3(M,\mathbf{F})$ is even with respect to any field. Hence $b_3(M,\mathbf{F})\in\{0,2\}$. If $b_3(M,\mathbf{F})=0$ with respect to any field, then it is easy to see that (b) is satisfied.

So we may assume that $b_3(M,\mathbf{F})=2$ for some field $\mathbf{F}$. Then $\chi(M)=2$. Thus if $b_3(M,\mathbf{F}')=0$ with respect to a different field $\mathbf{F}'$, then we must necessarily have $b_2(M,\mathbf{F}')=b_4(M,\mathbf{F}')=0$. Hence M is an $\mathbf{F}'$-homology sphere. If ι has infinite order, then we obtain an isometric S^1-action on M fixing N pointwise. It is straightforward to check that the S^1-action on $M\setminus N$ is free. Thus Corollary 5.3 implies that M is an integral homology sphere—a contradiction.

Thus ι has finite order l. Similarly Theorem 5.1 implies that the characteristic of $\mathbf{F}$ does not divide l. □

PROPOSITION 7.5. *Let M^n be a simply-connected compact manifold of positive sectional curvature. Suppose that $N_1^{n-k_1}$ and $N_2^{n-k_2}$ are two totally geodesic submanifolds intersecting transversely. Finally assume that $k_1\leqslant\frac{1}{4}(n+3)$, $2k_2+k_1\leqslant n$ and that k_1 is odd. Then for all $i\in\{1,...,n-1\}$ and $x\in H^i(M^n,\mathbf{Z})$ we have $2x=0$.*

Proof. The inclusion map $N_1\to M$ is $(n-2k_1+1)$-connected, and by Lemma 2.2 there is a class $e\in H^{k_1}(M,\mathbf{Z})\cong H^{k_1}(N_1,\mathbf{Z})$ such that $\cup e\colon H^i(M,\mathbf{Z})\to H^{i+k}(M,\mathbf{Z})$ is an isomorphism for $i=k,...,n-2k$. The pullback of e to $H^{k_1}(N_1,\mathbf{Z})$ is the Euler class of the normal bundle of N_1 in M. Since the codimension is odd, $2e=0$. That proves the statement for $i=k,...,n-k$.

Next consider $N_3=N_1\cap N_2$ and put $n_3:=\dim(N_3)=n-k_1-k_2$. If $k_1\leqslant k_2$, then the inclusion map $N_3\to N_2$ is n_3-connected. Similarly as above we get $2\cdot H^i(N_2,\mathbf{Z})=0$ for $i=1,...,n-k_2-1$. Since the inclusion map $N_2\to M$ is k_1-connected, that finishes the argument.

If $k_1>k_2$, then the inclusion map $N_3\to N_1$ is n_3-connected. By Lemma 2.2, $H^*(N_1,\mathbf{Z})$ has k_2 as a period. But this finishes the proof since we already established the desired result for $H^i(N_1,\mathbf{Z})\cong H^i(M,\mathbf{Z})$ with $i=k_1,...,n-2k_1$. □

8. Proof of Theorem 5

Theorem 5 follows from Corollary 3.2 combined with the following proposition.

PROPOSITION 8.1. *Let M^n be a simply-connected compact manifold of positive sectional curvature, and let T^d be a d-dimensional torus acting effectively and isometrically on M^n with $d\geqslant\max\{\frac{1}{8}n+14,\frac{1}{6}n+1\}$. Suppose that there is one involution $\iota\in\mathsf{T}^d$ fixing a submanifold N of codimension $k\leqslant\frac{7}{24}n$. Then the cohomology ring of M is given by one of the possibilities described in Theorem 5.*

Proof. We argue by induction on the dimension. First notice that the proposition is an immediate consequence of Theorem 2 for $n\leqslant 108$. Therefore assume $n\geqslant 109$ without loss of generality. We may assume that ι is chosen such that it maximizes the dimension of N. In particular $\operatorname{symrank}(N) \geqslant d-1$.

We consider first the case of $\operatorname{symrank}(N) < \max\{\frac{1}{8}\dim(N)+14, \frac{1}{6}\dim(N)+1\}$. Then N is fixed by a circle, and $\operatorname{codim}(N) \in \{2,4,6\}$. If $\operatorname{codim}(N)=2$, then M is fixed-point homogeneous, and the result follows from Grove and Searle [13]. In the case of $\operatorname{codim}(N)=4$, it is easy to find an involution ι_2 such that $\operatorname{Fix}(\iota)$ has a component N_2 of dimension $n_2 \geqslant \frac{1}{2}n+4$ intersecting N transversely. The inclusion map $N_2 \cap N \to N_2$ is (n_2-4)-connected. By Proposition 7.4 and Lemma 7.1, N_2 has one of the cohomology rings described in Theorem 5. Up to dimension 8 the cohomology rings of N_2 and M are equal. Since the inclusion map $N_1^{n-4} \to M^n$ is $(n-7)$-connected, the result now follows from Lemma 2.2.

In the case of $\operatorname{codim}(N)=6$, we can find similarly an involution fixing a submanifold N_2 of dimension $n_2 \geqslant \frac{1}{2}n+6$ intersecting N transversely. The inclusion map $N_2 \cap N \to N_2$ is (n_2-6)-connected. Let $e \in H^6(M, \mathbf{Z})$ be the Poincaré dual of $\operatorname{in}_*([N]) \in H_{n-6}(M, \mathbf{Z})$. We claim that the map $\cup e\colon H^i(M,\mathbf{Z}) \to H^{i+6}(M,\mathbf{Z})$ is an isomorphism for $0<i<n-7$, an epimorphism for $i=0$, and a monomorphism for $i=n-6$. In fact, for $5<i<n-12$ this is a consequence of Lemma 2.2 as $N^{n-6} \to M$ is $(n-11)$-connected. For $0<i<7$ we can make use of the fact that the cohomology groups of N_2 and M coincide up to dimension 12, and once again the statement follows from Lemma 2.2 and the fact that $N_2 \cap N \to N_2$ is $\dim(N_2 \cap N)$-connected. For $n-7<i \leqslant n-6$, the statement then follows from Poincaré duality and the fact that one can prove in dimension less than $\frac{1}{2}n$ the analogous statement for the cap product $\cap e\colon H_i(M,\mathbf{Z}) \to H_{i-6}(M,\mathbf{Z})$.

If $n \equiv 0 \bmod 6$, then Poincaré duality implies that $H^*(M,\mathbf{Z})$ is generated by one element, and it follows that M is homotopy equivalent to a sphere or a complex projective space. If $n \equiv 1 \bmod 6$, it follows that $H^6(M,\mathbf{Z}) \cong H^{n-1}(M,\mathbf{Z})=0$, and thus M is a homology sphere.

Otherwise the argument is a little more subtle: Choose a maximal collection of involutions $\iota=\sigma_1, \dots, \sigma_l$ such that $\operatorname{Fix}(\sigma_i)$ has a component N_i of codimension 6 and $N_1, \dots, N_l$ intersect pairwise transversely.

Put $B=N_1 \cap \dots \cap N_l$. Clearly $\dim(B)=n-6l$. In particular $l<\frac{1}{6}n \leqslant d-1$. We choose a point $p \in B$ such that the isotropy group H_p has dimension $\geqslant d-1$.

We can choose an involution $\iota_{l+1} \in \mathsf{H}_p \backslash \langle \iota_1, \dots, \iota_l \rangle$ such that the following holds:

The multiplicity $m_{\iota_{l+1}}$ of the eigenvalue -1 in $\iota_{l+1*}|_{T_p(B)}$ is as small as possible. It is easy to check that $k \leqslant \frac{1}{2}(\dim(B)+3)$, and that $k \leqslant \frac{1}{2}(\dim(B)-4)$ if $\dim(B)>80$. Notice that we can replace ι_{l+1} by any element in $\iota_{l+1} \cdot \langle \iota_1, \dots, \iota_l \rangle$ without changing $m_{\iota_{l+1}}$. Hence we may choose ι_{l+1} such that the multiplicity of the eigenvalue -1 in $\iota_{l+1*}|_{\nu_p(B)}$ is at most $\frac{1}{3}\dim(\nu_p(B))$.

By assumption $l\geqslant 1$, and thus the component N_{l+1} of $\mathrm{Fix}(\iota_l)$ with $p\in N_{l+1}$ has dimension $n_{l+1}\geqslant\frac{1}{2}(n+6)$. The inclusion map $N_{l+1}\to M$ is 7-connected. Suppose for a moment that N_{l+1} does not intersect one N_i transversely for $i\leqslant l$ suitable. Then $N_{l+1}\cap N_i$ has codimension 4 in N_{l+1}. It follows that $N_{l+1}\cap N_i\to M$ is 7-connected, too. Consider next the product $\iota_{l+1}\cdot\iota_i$ and the component N'_{l+1} of $\mathrm{Fix}(\iota_{l+1}\cdot\iota_i)$ with $p\in N_{l+1}$. Clearly N'_{l+1} contains $N_{l+1}\cap N_i$ as a submanifold of codimension 2. In odd dimensions it follows that N'_{l+1} and $N_{l+1}\cap N_i$ are homology spheres. That implies that M is 6-connected, and hence M is a homology sphere, too. In even dimensions, we can make use of the additional information that $H^6(M,\mathbf{Z})\cong H^6(N_{l+1}\cap N_i,\mathbf{Z})$ is cyclic or 0, to see that $N_{l+1}\cap N_i$ is homotopy equivalent to a sphere or a complex projective space. Clearly it follows that M has the corresponding homotopy type.

Thus we may assume that N_{l+1} intersects all submanifolds $N_1,...,N_l$ transversely. By assumption this implies $\mathrm{codim}(N_{l+1})>6$. Furthermore, $\mathrm{symrank}(N_{l+1})\geqslant d-1$. Thus $\mathrm{symrank}(N_{l+1})\geqslant\max\bigl\{\frac{1}{8}\dim(N_l+1)+14, \frac{1}{6}\dim(N_l+1)+1\bigr\}$. Furthermore the fixed-point set of $\iota|_{N_{l+1}}$ has codimension 6. From the induction hypothesis it follows that N_{l+1} has one of the cohomology rings described in Theorem 5. In odd dimensions it follows that $H^6(M,\mathbf{Z})=0$, and hence we are done. In even dimensions, it follows that the generator $e\in H^6(M,\mathbf{Z})$ can be expressed as $e=x\cup y$ where $x\in H^2(M,\mathbf{Z})$ and $y\in H^4(M,\mathbf{Z})$. It is easy to check that this implies that M has the cohomology ring of a sphere or a complex projective space.

Thus we may assume that $\mathrm{symrank}(N)\geqslant\max\bigl\{\frac{1}{8}\dim(N)+14, \frac{1}{6}\dim(N)+1\bigr\}$. Next we consider the case of $\mathrm{codim}(N)\leqslant\frac{1}{4}(n+3)$. The cohomology ring of N determines the cohomology ring of M, and hence it suffices to prove that N has one of the cohomology rings described in Theorem 5. If there is a fixed-point set in N of codimension $\leqslant\frac{7}{24}(n-k)$, this follows from the induction hypothesis. Thus we may assume that such a fixed-point set does not exist. By Corollary 3.2 we can find an involution $\sigma\in\mathsf{T}^d$ such that a component $N_2\subset N$ of $\mathrm{Fix}(\sigma|_N)$ has dimension $\geqslant\frac{1}{2}n$. We choose the involution σ such that the dimension of N_2 is as large as possible, and put $k_2:=n-k-\dim(N_2)$.

The inclusion map $N_2\to M$ is $(k+1)$-connected. By Lemma 7.2 it suffices to prove that N_2 has the cohomology ring of $\mathbf{S}^{n_2}$ or $\mathbf{CP}^{n_2/2}$. By Theorem 2, we may assume $\mathrm{symrank}(N_2)\leqslant\frac{1}{4}\dim(N_2)+1$. Since $\dim(N)-\dim(N_2)=k_2>\frac{1}{4}(n-k)$ and

$$\mathrm{symrank}(N_2)\geqslant\max\bigl\{\tfrac{1}{8}\dim(N)+13, \tfrac{1}{6}\dim(N)\bigr\},$$

it follows that $n-k\geqslant 96$. Because of

$$\mathrm{symrank}(N_2)\geqslant\tfrac{1}{6}n-1$$

we also may assume that $k+k_2\leqslant\frac{1}{3}n-1$ and $k_2>2k$.

By Corollary 3.2 there is an involution σ' such that $\mathrm{Fix}(\sigma')$ has a component N_3 of dimension $\geqslant \frac{7}{12}\dim(N_2)$. The inclusion map $N_2 \to N_3$ is h-connected with $h \geqslant \frac{1}{6}\dim(N_2)+1 \geqslant k+1$. By Lemma 7.2 it suffices to prove that N_3 is homotopy equivalent to a sphere or a complex projective space. Without loss of generality σ' is chosen such that the dimension of N_3 is as large as possible. Then $\mathrm{symrank}(N_3) \geqslant \mathrm{symrank}(N)-2$. By construction the fixed-point sets of the involutions $\sigma'|_N$ and $\sigma\cdot\sigma'|_N$ have dimension at most $\dim(N_2)$. Hence

$$\dim(N_2)-\dim(N_3) \geqslant \tfrac{1}{2}(\dim(N)-\dim(N_2)).$$

In summary we can say that $\mathrm{symrank}(N_3) \geqslant \frac{1}{6}(n-k)-1$ and $\dim(N_2) < \frac{7}{16}(n-k)$. Because of $n>64$ we obtain $\mathrm{symrank}(N_3) > \frac{1}{4}\dim(N_3)+1$. By Theorem 2, N_3 is homotopy equivalent to a sphere or a complex projective space.

It remains to consider the case of $\mathrm{codim}(N)=k>\frac{1}{4}(n+3)$. A first step is to show that N is homotopy equivalent to $\mathbf{S}^{n-k}$ or $\mathbf{CP}^{(n-k)/2}$. By Corollary 3.2, there is an involution ι_2 such that $\mathrm{Fix}(\iota_2|_{N_2})$ has a component N_2 of dimension $\geqslant \frac{1}{2}n$. The inclusion map $N_2 \to M$ is $(k+1)$-connected. Since the dimensions $\mathrm{Fix}(\iota_2)$ and $\mathrm{Fix}(\iota\cdot\iota_2)$ are at most $\dim(N)$, it follows that $\dim(N)-\dim(N_2) \geqslant \frac{1}{2}k$. Again without loss of generality $\mathrm{symrank}(N_2) \geqslant \mathrm{symrank}(M)-1$, and hence $\mathrm{symrank}(N_2) > \frac{1}{4}\dim(N_2)+1$. Thus N_2 is homotopy equivalent to a sphere or a complex projective space. If N_2 is a homotopy sphere, then M is $(k+1)$-connected. Since the inclusion map $N^{n-k} \to M$ is $(n-2k+1)$-connected, we can use Lemma 2.2 to see that M is actually $(n-2k+1)$-connected. This implies that N is $(n-2k)$-connected, and hence N is a homotopy sphere. If N_2 is a complex projective space, one can consider the S^1-bundle $\mathsf{S}^1 \to P^{n+1} \to M$, whose Euler class is the generator of $H^2(M, \mathbf{Z})$. Repeating the argument for P^{n+1} shows that P^{n+1} is $(n-2k+1)$-connected. That in turn shows that N is homotopy equivalent to a complex projective space.

In order to show that M has the corresponding homotopy type we distinguish again between two cases. Suppose first that N is a homotopy sphere.

We claim that for any element $a \in \mathsf{T}^d$ of prime order p the fixed-point set of a is either empty or given by a $\mathbf{Z}_p$-homology sphere. We first want to prove that each component F of $\mathrm{Fix}(a)$ is a $\mathbf{Z}_p$-homology sphere. We argue by induction on $\mathrm{codim}(F)$. If $\mathrm{codim}(F) < \frac{1}{4}n$, then the inclusion map $F \to M$ is $(n-2\,\mathrm{codim}(F)+1)$-connected. From Lemma 2.2 we get additional information on the cohomology ring of M. Combining with the fact that M is $\left(\left[\frac{5}{12}n\right]+1\right)$-connected, this implies that M is an integral homology sphere, and hence F is a homotopy sphere, too.

If $\frac{1}{4}n \leqslant \mathrm{codim}(F) \leqslant \frac{1}{3}n$, the fact that the inclusion map $F \to M$ is $(n-2\,\mathrm{codim}(F)+1)$-connected implies that F is h-connected with $h \geqslant \frac{1}{2}\dim(F)$. This also implies that F is a homotopy sphere.

If $\frac{1}{3}n \leqslant \operatorname{codim}(F) < \frac{5}{12}n$, then the inclusion map $F \cap N \to F$ is h-connected, with $h = \dim(F) - k > \frac{1}{2}\dim(F)$. The intersection $F \cap N$ is a component of the fixed-point set of $\operatorname{Fix}(\sigma|_N)$. Since N is a homotopy sphere and σ is of order p, it follows that $\operatorname{Fix}(\sigma|_N) = F \cap N$ is a $\mathbf{Z}_p$-homology sphere. That implies that F is a $\mathbf{Z}_p$-homology sphere as well.

If F is fixed by a group $\mathbf{Z}_p^2 \subset \mathsf{T}^d$, then F is contained in a $\mathbf{Z}_p$-homology sphere $\bar{F}$. Thus $F = \operatorname{Fix}(\sigma|_{\bar{F}})$ is a $\mathbf{Z}_p$-homology sphere as well.

If F is not fixed by a group $\mathbf{Z}_p^2 \subset \mathsf{T}^d$ and $\operatorname{codim}(F) \geqslant \frac{5}{12}n$, then $\operatorname{symrank}(F) \geqslant \frac{1}{6}n > \frac{1}{4}\dim(F) + 2$ and $\dim(F) \geqslant \frac{1}{3}n$. By Theorem 2 the universal cover of F is a homotopy sphere, and by Theorem 4 the fundamental group of M is cyclic. Since the inclusion map $F \cap N \to F$ is $\left[\frac{1}{24}(n+23)\right]$-connected, and $F \cap N$ is a $\mathbf{Z}_p$-homology sphere, it follows that the first $\left[\frac{1}{24}(n-1)\right]$-homology groups of F are zero with respect to $\mathbf{Z}_p$. That implies that the cyclic fundamental group of F has order prime to F, and hence F is a $\mathbf{Z}_p$-homology sphere as well.

In order to show that the fixed-point set $\operatorname{Fix}(\sigma)$ is connected unless it consists of two isolated points, one can argue as in the proof of Lemma 6.3.

By Theorem 4.1 it follows that M is a homology sphere.

In the case that N is homotopy equivalent to a complex projective space one lifts the discussion as before to the total space of an S^1-bundle over M whose Euler class is a generator of $H^2(M, \mathbf{Z})$. □

9. Proof of Theorem 4

We argue by induction on n. As mentioned in the introduction it suffices to treat the case of $n \equiv 3 \mod 4$. For $n=3$ the theorem is a consequence of Grove and Searle [13]. Suppose that $n = 4m+3$ with $m \geqslant 1$. As before we consider a point $q_0 \in M$ sitting on a circle orbit of the isometric action of the torus $\mathsf{T}^d \subset \operatorname{Iso}(M, g)$. In the $(d-1)$-dimensional isotropy group at q_0 we choose an involution ι such that the q_0-component N of $\operatorname{Fix}(\iota)$ has the largest possible dimension. Then $\operatorname{symrank}(N) \geqslant d-1 \geqslant m+1$. From Grove and Searle it follows that $\dim(N) \geqslant \frac{1}{2}(n-1)$, and equality can only occur if N is fixed pointwise by an isometric circle action. By Theorem 2.1 (a) the inclusion map $N \to M$ is 1-connected, and hence it suffices to prove that the fundamental group of N is cyclic.

If $\operatorname{symrank}(N) \geqslant \frac{1}{2}\dim(N) + 1$, then this follows from the induction hypothesis. Otherwise we have $\operatorname{codim}(N) = 2$, and N is fixed by an S^1-subaction. Thus M is fixed-point homogeneous, and by Grove and Searle [13] M is diffeomorphic to a lens space.

10. Proof of Theorem 3

LEMMA 10.1. *Let M^{4l} be a compact manifold whose $\mathbf{Z}_2$-cohomology ring is isomorphic to the $\mathbf{Z}_2$-cohomology ring of $\mathbf{HP}^l$. Suppose that a d-dimensional torus T^d acts effectively and smoothly on M^{4l}. Then $d \leqslant l+1$, and if equality holds, then there is an S^1-subaction fixing a submanifold N of codimension 4. Furthermore N is a $\mathbf{Z}_2$-cohomology $\mathbf{HP}^{l-1}$.*

Proof. We argue by induction on k. Suppose that we have proved the statement for $k' \leqslant l-1$. Since M has nonzero Euler characteristic, it follows that T^d has a fixed point. If $l=1$, then the estimate $d \leqslant 2$ follows from the fact that the isotropy representation at p is faithful. Furthermore if $d=2$ we can clearly find an S^1-subaction with an isolated fixed point. For $l \geqslant 3$ we can find an involution fixing a connected submanifold N of codimension less than $2l$. We may assume that the involution is chosen such that N has minimal codimension. It follows from [4, Chapter VII, Theorem 3.1] that the fixed-point set of that involution has the $\mathbf{Z}_2$-cohomology ring of a quaternionic space. Since the codimension of N is minimal, it follows that the induced action of T^d on N has at most a 1-dimensional kernel. The induction hypothesis implies that $d \leqslant k+1$ and that equality can only hold if N is fixed by an S^1-subaction.

If $l=2$ we can argue as follows. By [4, Chapter VII, Theorem 3.1] it is not possible to find an involution whose fixed-point set has codimension 2 or 6. Using this it is easy to see that there is an involution with an isolated fixed point. By Bredon the fixed-point set of such an involution has precisely one more component N, and N has the $\mathbf{Z}_2$-homology of $\mathbf{S}^4$. The result now follows as above. □

Theorem 3 now is an immediate consequence of Lemma 10.1 and the following lemma.

LEMMA 10.2. *Let M^{4l} be a simply-connected manifold whose integral cohomology ring is isomorphic to the cohomology ring of $\mathbf{HP}^l$. Suppose that there is an effective smooth S^1-action on M fixing a submanifold N of codimension 4. Assume furthermore that N has a $\mathbf{Z}_2$-cohomology of $\mathbf{HP}^{l-1}$. Then M is homeomorphic to $\mathbf{HP}^l$.*

Proof. In the presence of an invariant positively curved metric one can actually give a slightly simpler proof since in that case it is known that N is simply-connected. We argue again by induction on l. There is nothing to prove for $l=1$. Since M has the integral cohomology ring of $\mathbf{HP}^l$, we can use [4, Chapter VII, Theorem 5.1] to see that a generator $H^4(M,\mathbf{Z})$ restricts to a generator of $H^4(N,\mathbf{Z})$ and $H^*(N,\mathbf{Z}) \cong H^*(\mathbf{HP}^{l-1},\mathbf{Z})$.

Let $B_r(N)$ be a tubular neighborhood of N and put $M' = M \setminus B_r(N)$. Using the Mayer–Vietoris sequence it is easy to see that M' is acyclic, i.e., $H^i(M',\mathbf{Z})=0$ for $i>0$. Using Bredon it easy to see that the action of S^1 on M' is semifree. In fact for any element $a \in \mathsf{S}^1$ of prime order p the fixed-point set $\mathrm{Fix}(\iota)$ has no component of codimension less than 4. Thus N is a component of $\mathrm{Fix}(\iota)$. Furthermore any other component of $\mathrm{Fix}(\iota)$ has the $\mathbf{Z}_p$-cohomology of a quaternionic or complex projective space, and $\chi(\mathrm{Fix}(\iota)\setminus N)=1$. This shows that there is precisely one fixed point of S^1 in M', and S^1 acts freely away from that fixed point. Notice that the S^1-action induces a complex structure on the normal bundle of N. Therefore the structure group reduces to $\mathsf{U}(2)$. Consider the induced $(\mathsf{U}(2)/\mathrm{SU}(2))$-bundle. Since N is homologically 2-connected, this circle bundle is trivial. Hence the structure group reduces further to $\mathrm{SU}(2)\cong\mathsf{S}^3$. In other words the unit normal bundle $\nu^1(N)$ is a principal S^3-bundle. Furthermore the S^3-action on $\nu^1(N)$ may be viewed as an extension of the given S^1-action. Consider the classifying map $f\colon N\to\mathbf{HP}^{n-1}\subset\mathbf{HP}^\infty$ of the principal S^3-bundle. Since the Euler class in $H^4(N,\mathbf{Z})\cong H^4(M,\mathbf{Z})$ is the Poincaré dual of the image of the fundamental class of N in $H_{4l-4}(M,\mathbf{Z})$, it follows that the Euler class of the normal bundle represents a generator of $H^4(N,\mathbf{Z})$. Hence f induces an isomorphism on cohomology.

Next we claim that f pulls the Pontrjagin classes of $\mathbf{HP}^{n-1}$ back to the Pontrjagin classes of N. Let $p\in M'$ be the unique fixed point of S^1, and let $B_r(p)$ be a small ball around p. Notice that S^1 acts freely on the integral homology sphere $S{:=}M'\setminus B_r(p)$. The inclusion maps of each of the two boundary components $\partial M'\cong\nu^1(N)$ and $\partial B_r(p)\cong T^1_pM$ induce isomorphisms on cohomology. The same holds for the boundary components of S/S^1. Combining with the fact that S^1 acts on $\partial B_r(p)\cong T^1_pM$ by the natural linear Hopf action we conclude that the total Pontrjagin class of $\nu^1(N)/\mathsf{S}^1$ is given by $(1+x^2)^{2l}$ where $x\in H^2(\nu^1(N)/\mathsf{S}^1)$ is a generator. Notice that $\nu^1(N)/\mathsf{S}^1$ is an $\mathbf{S}^2$-bundle over N. In particular, the natural projection induces an isomorphism on cohomology in dimensions divisible by 4. Furthermore it follows that the Pontrjagin classes of N pull back to the Pontrjagin classes of the horizontal distribution of the projection $\mathrm{pr}\colon\nu^1(N)/\mathsf{S}^1\to N$. It is easy to see that the Euler class of the vertical distribution is twice a generator of $H^2(\nu^1(N)/\mathsf{S}^1)$. That implies that the total Pontrjagin class of the vertical distribution is given by $1+4x^2$. Consequently the total Pontrjagin class of the horizontal distribution is given by

$$(1+4x^2)^{-1}(1+x^2)^{2l}\in H^*(\nu^1(N)/\mathsf{S}^1,\mathbf{Z})\cong H^*(\mathbf{CP}^{2l-1},\mathbf{Z}).$$

This determines the total Pontrjagin class of N. Since we can do the same computation for $\mathbf{HP}^{n-1}$, it follows that f pulls the Pontrjagin classes of $\mathbf{HP}^{n-1}$ back to the Pontrjagin classes of N.

The pullback of the normal bundle of $\mathbf{HP}^{n-1}\subset\mathbf{HP}^n$ is the normal bundle of $N\subset M$. Thus we can extend f to a map

$$(B_r(N),\partial B_r(N))\to(B_r(\mathbf{HP}^{n-1}),\partial B_r(\mathbf{HP}^{n-1})).$$

Since $\mathbf{HP}^n \backslash B_r(\mathbf{HP}^{n-1})$ is contractible it follows that we can extend f further to a map $\bar{f}: M \to \mathbf{HP}^n$. Clearly $\bar{f}$ induces an isomorphism on cohomology, and since M is simply-connected, $\bar{f}$ is a homotopy equivalence by Whitehead. Furthermore, it is easy to see that $\bar{f}$ maps the first $l-1$ Pontrjagin classes of $\mathbf{HP}^l$ onto the first $l-1$ Pontrjagin classes of M. Because of Hirzebruch's signature formula the same holds for the lth class. By Sullivan's classification [23] $\bar{f}$ is homotopic to a homeomorphism. □

References

[1] ALOFF, S. & WALLACH, N. R., An infinite family of 7-manifolds admitting positively curved Riemannian structures. *Bull. Amer. Math. Soc.*, 81 (1975), 93–97.

[2] BAZAĬKIN, YA. V., On a family of 13-dimensional closed Riemannian manifolds of positive curvature. *Sibirsk. Mat. Zh.*, 37 (1996), 1219–1237; English translation in *Siberian Math. J.*, 37 (1996), 1068–1085.

[3] BERGER, M., Les variétés riemanniennes homogènes normales simplement connexes à courbure strictement positive. *Ann. Scuola Norm. Sup. Pisa (3)*, 15 (1961), 179–246.

[4] BREDON, G. E., *Introduction to Compact Transformation Groups.* Pure Appl. Math., 46. Academic Press, New York–London, 1972.

[5] DESSAI, A. & WILKING, B., Torus actions on homotopy complex projective spaces. To appear in *Math. Z.*

[6] ESCHENBURG, J.-H., New examples of manifolds with strictly positive curvature. *Invent. Math.*, 66 (1982), 469–480.

[7] FANG, F., MENDONÇA, S. & RONG, X., A connectedness principle in the geometry of positive curvature. Preprint, 2002.

[8] FANG, F. & RONG, X., Homeomorphism classification of positively curved manifolds with almost maximal symmetry rank. Preprint, 2002.

[9] FRANKEL, T., Manifolds with positive curvature. *Pacific J. Math.*, 11 (1961), 165–174.

[10] — On the fundamental group of a compact minimal submanifold. *Ann. of Math. (2)*, 83 (1966), 68–73.

[11] GROVE, K., Geodesics satisfying general boundary conditions. *Comment. Math. Helv.*, 48 (1973), 376–381.

[12] — Geometry of, and via, symmetries, in *Conformal, Riemannian and Lagrangian Geometry* (Knoxville, TN, 2000), pp. 31–53. Univ. Lecture Ser., 27. Amer. Math. Soc., Providence, RI, 2002.

[13] GROVE, K. & SEARLE, C., Positively curved manifolds with maximal symmetry-rank. *J. Pure Appl. Algebra*, 91 (1994), 137–142.

[14] — Differential topological restrictions by curvature and symmetry. *J. Differential Geom.*, 47 (1997), 530–559.

[15] HSIANG, W.-Y. & KLEINER, B., On the topology of positively curved 4-manifolds with symmetry. *J. Differential Geom.*, 30 (1989), 615–621.

[16] MCELIECE, R. J., RODEMICH, E. R., RUMSEY, H., JR. & WELCH, L. R., New upper bounds on the rate of a code via the Delsarte–MacWilliams inequalities. *IEEE Trans. Information Theory*, IT-23 (1977), 157–166.

[17] MACWILLIAMS, F. J. & SLOANE, N. J. A., *The Theory of Error-Correcting Codes*, Vol. I. North-Holland Math. Library, 16. North-Holland, Amsterdam–New York–Oxford, 1977.

[18] MANN, L. N., Differentiable actions of compact abelian Lie groups on S^n. *Proc. Amer. Math. Soc.*, 16 (1965), 480–484.

[19] PETRIE, T., Smooth S^1 actions on homotopy complex projective spaces and related topics. *Bull. Amer. Math. Soc.*, 78 (1972), 105–153.

[20] — Torus actions on homotopy complex projective spaces. *Invent. Math.*, 20 (1973), 139–146.

[21] RONG, X., Positively curved manifolds with almost maximal symmetry rank. *Geom. Dedicata*, 95 (2002), 157–182.

[22] SCHOEN, R. & WOLFSON, J., Theorems of Barth–Lefschetz type and Morse theory on the space of paths. *Math. Z.*, 229 (1998), 77–89.

[23] SULLIVAN, D. P., Triangulating and smoothing homotopy equivalences and homeomorphisms. Geometric topology seminar notes, in *The Hauptvermutung Book*, pp. 69–103. K-Monogr. Math., 1. Kluwer, Dordrecht, 1996.

[24] WALLACH, N. R., Compact homogeneous Riemannian manifolds with strictly positive curvature. *Ann. of Math. (2)*, 96 (1972), 277–295.

[25] WILKING, B., Manifolds with positive sectional curvature almost everywhere. *Invent. Math.*, 148 (2002), 117–141.

[26] — *Group Actions on Manifolds of Positive Sectional Curvature.* XII Escola de Geometria Diferencial. Universidade Federal de Goiás, Goiânia, 2002.

[27] — Positively curved manifolds with symmetry. Preprint, 2003.

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