TRANSFORMATION OF BOUNDARY PROBLEMS

BY

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Introduction

In this paper the calculus of pseudodifferential and Fourier integral operators introduced in [13] is examined in more detail. There are several alterations which have been made to extend and simplify the theory. In particular a natural vector bundle $\tilde{T}^*M$, the compressed cotangent bundle, is defined for any manifold with boundary. This is the appropriate space for microlocalization with respect to the pseudodifferential operators in the space $L_b^\infty(M)$. In local coordinates these operators are of the form

$$Au(x,y) = (2\pi)^{-n-1}\int e^{ix\xi+iy\cdot\eta}a(x,y,x\xi,\eta)\,\hat{u}(\xi,\eta)\,d\xi\,d\eta, \tag{0.1}$$

where (x,y) are the coordinates in the standard manifold with boundary $Z=\overline{\mathbf{R}_x^+}\times\mathbf{R}_y^n$. The importance of the space $\tilde{T}^*M$ is that it carries, invariantly, functions of the type of the symbol $a(x,y,x\xi,\eta)$ in (0.1) with its special dependence on the variable, ξ, dual to x.

Under certain mild lacunary conditions on the symbol, A in (0.1) preserves the property that the distribution u has restrictions, or traces, of all orders to the boundary and

$$Au\,|\,(x=0) = A_0(u\,|\,(x=0)) \tag{0.2}$$

with A_0 a pseudodifferential operator on the boundary. This is the fundamental property of these pseudodifferential operators, and also the Fourier integral operators discussed here, because it allows the operators to act on distributions satisfying differential or pseudodifferential equations and boundary conditions. Since no applications are presented here the interested reader is referred to the lectures in [13], [14] and [15] for an indication of this approach to the examination of singularities. The details will appear elsewhere.

One of the technical difficulties in a systematic approach to boundary problems is the abundance of spaces of distributions which enter. The central position here is given to the space $\dot{\mathcal{D}}'(M)$ of distributions supported on M, a manifold with boundary. In Chapter I the more formal properties of boundary problems, posed in this way, are examined. In the first section the various standard spaces are introduced as is the space

$$\dot{\mathcal{A}}(M)\subset\dot{\mathcal{D}}'(M)$$

of almost regular distributions. These are determined by the property that they have fixed regularity after the arbitrary action of vector fields tangent to the boundary. They are characterized as the Lagrangian distributions, supported on M, associated to the conormal bundle $N^*\partial M$. The elements of $\dot{\mathcal{A}}(M)$ are regarded as, essentially, negligible distributions as far as singularities are concerned, although not quite as negligible as the space

$$\dot{C}^\infty(M)\subset\dot{\mathcal{D}}'(M)$$

of C^∞ functions vanishing to all orders at the boundary. The standard notion of wavefront set, taken with respect to an open extension of M, determines $\dot{C}^\infty(M)$ in $\dot{\mathcal{A}}(M)$:

$$\text{If}\quad u\in\dot{\mathcal{A}}(M)\quad\text{then}\quad u\in\dot{C}^\infty(M)\Leftrightarrow \mathrm{WF}\,(u)=\varnothing. \tag{0.3}$$

The compressed tangent and cotangent bundles are introduced in Section two as geometric constructs closely related to the definition of $\dot{\mathcal{A}}$. Thus, $\tilde{T}M$ is the natural bundle of which the vector fields tangent to the boundary are sections. Then $\tilde{T}^*M$ is the dual of $\tilde{T}M$. The natural vector bundle map

$$T^*M \to \tilde{T}^*M$$

has image, of corank one over the boundary, canonically identifiable with

$$T^*\partial M \cup T^*\mathring{M} \subset \tilde{T}^*M. \tag{0.4}$$

This space has been used (see for example [5], [11], [16]) as the carrier of the set $\mathrm{WF}_b\,(u)$, for special distributions.

The third section contains a rather formal treatment of noncharacteristic boundary problems. The important item introduced here is the subspace

$$\mathcal{A}'(M) \subset \dot{\mathcal{D}}'(M) \tag{0.5}$$

given as the dual of $\dot{\mathcal{A}}_c(M, \Omega)$, the space of compactly supported almost regular densities. The usual trace or restriction map R: $C^\infty(M) \to C^\infty(\partial M)$ extends to $\mathcal{A}'(M)$. This allows the weak formulation of boundary problems (I.3.20), due essentially to Schwartz, to be recovered in a strong sense with minimal regularity assumptions on the data. It should be noted that $\mathcal{A}'(M)$ contains the usual spaces of distributions for which all traces are defined and can be used in place of the awkward spaces of distribution regular in a normal variable used previously, for instance in [1].

Chapter II treats the calculus of pseudodifferential operators $L_b^\infty(M)$ of totally characteristic type. Some standard properties of symbol spaces are briefly recalled in the first section and the lacunary condition imposed on the symbol in (0.1) is examined, and shown to impose conditions only on the residual part, i.e., to be trivial modulo $S^{-\infty}$. The second section consists of a short exposition of the theory of pseudodifferential operators on open sets for purposes of comparison. In Section three the operators (0.1) are defined on Z, by analysis of the formal adjoint, and the simplest mapping properties are deduced.

The next two sections, four and five, treat the more significant properties of the operators (0.1). First the precise nature of the kernels near the corner $\partial Z \times \partial Z$ is investigated. This is done in terms of the 'stretched product' $N \,\hat{\times}\, M$ of two manifolds with boundary; in fact the kernels lift to standard Lagrangian sections of an appropriate line bundle on $Z \,\hat{\times}\, Z$. This leads to the proof of coordinate invariance and hence the global definition of $L_b^\infty(M)$, for any manifold with boundary M. In Section five it is shown that A acts on spaces of distributions with restriction properties to the boundary, even maps $\mathcal{A}_c'(Z)$ to

$\mathcal{A}'(Z)$ and that (0.2) holds. In Section six the symbolic properties of the operators are shown to closely parallel the open case, with the principal symbol defined on $\tilde{T}^*M$:

$$\sigma_m: L_b^m(M)/L_b^{m-1}(M) \cong S^m(\tilde{T}^*M)/S^{m-1}(\tilde{T}^*M).$$

This, combined with (0.4), gives a symbolic version of (0.2)

$$\sigma_m(A_0) = \sigma(A) \,|\, T^*\partial M.$$

Moreover the composition and symbolic properties of the residual operators, $L_b^{-\infty}(M)$, are examined. This space is filtered by a sequence of residual symbol maps defined on the corner $\partial M \times \partial M$ of the product:

$$\sigma_{-\infty,-k}: L^{-\infty,-k}(M)/L^{-\infty,-k-1}(M) \to \mathcal{S}(\bar{S}^+; B_k) \tag{0.6}$$

where S is a line bundle on $\partial M \times \partial M$ and B_k a bundle over S. In fact the product formula (II.6.16) involves convolution in the $\mathbf{R}^+$ structure of the positive side of S.

After the usual formula for the product of pseudodifferential operators is proved in Section seven it is applied to the construction of parametrices, modulo $L_b^{-\infty}$, of elliptic operators. This so closely resembles the familiar case that, in Section eight, the definition of wavefront set used by Hörmander can be applied almost verbatim to fix

$$\mathrm{WF}_b\,(u) \subset \tilde{T}^*M \diagdown 0, \quad \forall u \in \dot{\mathcal{D}}'(M). \tag{0.7}$$

The only real departure from the usual properties of wavefront set is the almost regularity property:

$$\mathrm{WF}_b\,(u) = \varnothing \iff u \in \dot{\mathcal{A}}(M).$$

Combined with (0.3) this still gives a rather complete range of indicators of singularity, with an obvious sheaf-theoretic interpretation on $\tilde{T}^*M$.

Using the symbol calculus the space of normally regular distributions, $\mathcal{N}(M) \subset \mathcal{A}'(M)$, is discussed in Section nine and a slightly strengthened form of Peetre's theorem on hypoellipticity up to the boundary is proved. Again using the calculus in a standard way the L^2 boundedness of operators in $L_b^0(M)$ is shown in Section ten.

The analogous spaces of Fourier integral operators on a manifold with boundary are the subject of Chapter III. First the properties of canonical transformations, on the usual cotangent space, preserving the boundary, are considered. A boundary-canonical transformation is then taken to be a C^∞ diffeomorphism

$$\chi: \tilde{T}^*M \to \tilde{T}^*N \tag{0.8}$$

which is homogeneous and canonical for the singular symplectic structure on the compressed cotangent bundle. In the second section the existence of suitable local parametrizations of these boundary-canonical transformations is discussed. After the appropriate lacunary conditions on symbols are shown to present no problems this leads to the local definition of Fourier integral operators in terms of oscillatory integrals, in Section three. The kernels of the totally characteristic Fourier integral operators can be identified with generalized sections, of an appropriate bundle over $N \hat{\times} M$, with only Lagrangian singularities at a Lagrangian submanifold

$$\hat{\Lambda} \subset T^*(N \hat{\times} M) \setminus 0. \tag{0.9}$$

For a boundary-canonical transformation this Lagrangian can be identified with the twisted graph. The calculus of these operators can then be routinely developed by combining the methods of Chapter II with the original work of Hörmander [10]. In particular the symbol is a section of the Maslov bundle, with an appropriate density factor, over $\hat{\Lambda}$. The action of Fourier integral operators as transformations of the pseudodifferential operator ring, i.e. Egorov's theorem, follows from the calculus. A suitable formal definition of non-characteristic boundary problems is given in Section five, to extend these transformation properties.

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Chapter I: Manifolds with boundary

I.1. Spaces of distributions

Let M be a C^∞ manifold with boundary. If E is any vector bundle over M we shall denote by $C^\infty(M, E) = C^\infty(E)$ the space of sections of E which are C^∞ up to the boundary, ∂M, of M and by $C_c^\infty(E)$ the subspace of compactly supported sections. Both of these spaces are equipped with the usual topologies of uniform convergence of all derivatives on compact subsets of M. By $\dot{C}^\infty(E) \subset C^\infty(E)$ and $\dot{C}_c^\infty(E) \subset C_c^\infty(E)$ we denote the closed subspaces consisting of those sections vanishing to all orders on ∂M.

The standard spaces of distributional sections of E over M are then defined as the duals of these spaces of C^∞ sections of $E^* \otimes \Omega$, where Ω is the density bundle of M. Thus,

$$\mathcal{D}'(M, E) = \mathcal{D}'(E) = [\dot{C}_c^\infty(M, E^* \otimes \Omega)]'$$

is the space of extendible distributions whereas

$$\dot{\mathcal{D}}'(M, E) = \dot{\mathcal{D}}'(E) = [C_c^\infty(M, E^* \otimes \Omega)]'$$

is the space of distributions supported by M. If $M \hookrightarrow N$ is realized as a submanifold with boundary of a manifold N, $\partial N = \emptyset$, then $\dot{\mathcal{D}}'(E)$ is naturally identified with the subspace of $\mathcal{D}'(N, E)$, for any extension of E to N, of distributions supported in M. Similarly the space $\mathcal{D}'(E)$ can be regarded as the set of restrictions to $\mathring{M}$, the interior of M, of elements of $\mathcal{D}'(N, E)$, that is as the quotient

$$\mathcal{D}'(M, E) \simeq \mathcal{D}'(N, E)/\dot{\mathcal{D}}'(N\setminus\mathring{M}, E).$$

Following Schwartz we shall denote by

$$\mathcal{E}'(E) \subset \mathcal{D}'(E), \quad \dot{\mathcal{E}}'(E) \subset \dot{\mathcal{D}}'(E)$$

the subspaces of compactly supported distributions. For $u \in \mathcal{D}'(E)$ the support is the closure in M of the support of $u|\mathring{M} \in \mathcal{D}'(\mathring{M}, E)$.

Consider the relationship between extendable and supported distributions. Let $\dot{\mathcal{D}}'(M, \partial M; E) \subset \dot{\mathcal{D}}'(M; E)$ be the subspace of distributions supported in the closed set $\partial M \subset M$. Then, using the restriction map to $\mathring{M}$, we have a sequence:

$$0 \hookrightarrow \dot{\mathcal{D}}'(M, \partial M; E) \hookrightarrow \dot{\mathcal{D}}'(M, E) \xrightarrow{|\mathring{M}} \mathcal{D}'(M; E) \to 0. \tag{1.1}$$

LEMMA 1.2. *The sequence* (1.1) *is exact.*

Proof. The only point not immediately clear is the surjectivity at $\mathcal{D}'$. Using a partition of unity, it suffices to show surjectivity locally. However, the structure theorem of Schwartz shows that any extension $u' \in \mathcal{D}'(N, E)$ of $u \in \mathcal{D}'(M, E)$ is locally of the form Pv with P a differential operator with C^∞ coefficients and v a continuous section of E. Replacing v by

$$v' = \begin{cases} v & \text{in } M \\ 0 & \text{in } N\setminus M \end{cases}$$

gives an element $Pv' \in \dot{\mathcal{D}}'(M, E)$ which restricts to u locally in $\mathring{M}$. This proves the lemma.

Schwartz in [19] gives a complete description of $\dot{\mathcal{D}}'(M; \partial M)$ each element being locally a finite sum of Dirac distributions on the hypersurface ∂M. If $x \in C^\infty(M)$ is a function which vanishes to precisely first order on ∂M, for each integer $m \geq 0$, consider the kernel of the map given by multiplication by x^{m+1}:

$$\dot{\mathcal{D}}'_m(M, \partial M; E) = \{u \in \dot{\mathcal{D}}'(M, \partial M; E);\, x^{m+1}u = 0\}, \quad m \geq 0. \tag{1.3}$$

PROPOSITION 1.4. *For each $m \geq 0$ there is a vector bundle $E_{(m)}$ over ∂M and a natural isomorphism*

$$\dot{\mathcal{D}}'_m(M, \partial M; E) \leftrightarrow \mathcal{D}'(\partial M; E_{(m)}) = \mathcal{D}'(E_{(m)}). \tag{1.5}$$

There are natural injections $E_{(m)} \hookrightarrow E_{(m+1)}$ *such that*

$$E_{(m)}/E_{(m-1)} \cong E|_{\partial M} \otimes (N^*\partial M)^{-m}, \quad m \geqslant 1.$$

Proof. The theorem of Schwartz shows that in any local coordinates $x, y_1, \ldots, y_n$ at a point in the boundary of M, $\dim M = n+1$, $u \in \dot{\mathcal{D}}'_m(M, \partial M; E)$ is of the form:

$$u = \sum_{0 \leqslant j \leqslant m} \sum_{r=1}^{\dim E} u_{r,j}(y) \otimes \frac{d^j}{dx^j} \delta(x) \cdot e_r \tag{1.6}$$

with respect to a local basis e_r of E. The coefficients $u_{r,j}$ are distributions on $\mathbf{R}^n$. This clearly provides local trivializations for $E_{(m)}$. The remainder of the proposition is straightforward.

Recall the standard continuous inclusions

$$\dot{C}^\infty(E) \subset C^\infty(E) \begin{matrix} \nearrow \mathcal{D}'(E) \\ \searrow \dot{\mathcal{D}}'(E) \end{matrix} \tag{1.7}$$

where the interpretation of an element of $C^\infty(E)$ as a distribution supported in M is through 'cutting off at the boundary'. There are similar inclusions for compactly supported distributions. $C^\infty(E)$ is a natural space of smooth extendible distributions, ignorable within the context of singularities. It is important, in certain cases, to enlarge the class of smooth elements of $\dot{\mathcal{D}}'(E)$ to include as well the elements

$$C^\infty(\partial M; E_{(m)}) \hookrightarrow \dot{\mathcal{D}}'(E). \tag{1.8}$$

In fact it is convenient to further enlarge the space of 'ignorable' distributions. We shall proceed to define

$$\dot{\mathcal{A}}(M, E) = \dot{\mathcal{A}}(E) \subset \dot{\mathcal{D}}'(E)$$

the subspace of almost regular distributional sections of E.

Let

$$\mathcal{V} \subset C^\infty(TM)$$

be the space of C^∞ vector fields on M everywhere tangent to ∂M:

$$V \in \mathcal{V} \quad \Leftrightarrow \quad V_m \in T_m \partial M \subset T_m M, \quad \forall m \in \partial M.$$

The linear space $\mathrm{Diff}^k(M)$ of differential operators of order at most k on M is locally finitely generated as a $C^\infty(M)$-module by j-fold products, for $0 \leqslant j \leqslant k$, of vector fields acting by Lie derivation. This allows us to define

$$\mathrm{Diff}_b^k(M) \subset \mathrm{Diff}^k(M),$$

the submodule of totally characteristic operators of order at most k, as the span of the

$$\mathcal{V}^j = \mathcal{V} \circ \ldots \circ \mathcal{V} \quad (j \text{ factors})$$

for $0 \leqslant j \leqslant k$. The description of these operators as totally characteristic is only slightly contrary to standard notation. Thus, multiplication by a C^∞ function is here regarded as a totally characteristic differential operator of order zero. In general if E_1, E_2 are C^∞ vector bundles over M the space $\mathrm{Diff}^k(M; E_1, E_2)$, of differential operators of order at most k from sections of E_1 to sections of E_2, consists precisely of the linear operators

$$P: C^\infty(M, E_1) \to C^\infty(M, E_2)$$

such that $\varrho_2 \cdot P \cdot \varrho_1 \in \mathrm{Diff}^k(M)$ whenever ϱ_1 is a section of E_1 and ϱ_2 is a section of E_2^*. The formal adjoint of $P \in \mathrm{Diff}^k(M; E_1, E_2)$,

$$P^* \in \mathrm{Diff}^k(M; E_2^* \otimes \Omega, E_1^* \otimes \Omega)$$

is uniquely specified by:

$$\int_M (P\varphi, \psi) = \int_M (\varphi, P^*\psi) \tag{1.9}$$

for all $\varphi \in \dot{C}_c^\infty(M, E_1)$, $\psi \in \dot{C}_c^\infty(M; E_2^* \otimes \Omega)$, in terms of the sesquilinear pairing between vector bundle and dual. The totally characteristic differential operators, $\mathrm{Diff}_b^k(M; E_1, E_2)$, are then precisely those for which (1.9) holds for all $\varphi \in C_c^\infty(M, E_1)$, $\psi \in C_c^\infty(M, E_2^* \otimes \Omega)$; those for which no boundary terms arise in using the adjoint equation.

Let $K \subset\subset M$ be a compact set and put

$$\dot{\mathcal{E}}'(K, E) = \{u \in \dot{\mathcal{E}}'(M, E); \operatorname{supp} u \subset K\}.$$

The topology of $\dot{\mathcal{E}}'(K, E)$ is the inductive limit topology

$$\dot{\mathcal{E}}'(K, E) = \lim_{s \to -\infty} \dot{H}^s(K, E)$$

over the Hilbertable topologies of the Sobolev spaces $\dot{H}^s(K, E)$. For each s set

$$\dot{\mathcal{A}}^{(s)}(K, E) = \{u \in \dot{\mathcal{E}}'(K, E): Pu \in \dot{H}^s(K, E), \ \forall P \in \mathrm{Diff}_b^\infty(M; E, \mathbf{C})\}. \tag{1.10}$$

$\dot{\mathcal{A}}^{(s)}(K, E)$ is to be topologized as the projective limit of the spaces

$$\{u \in \dot{\mathcal{E}}'(K, E); Pu \in \dot{H}^s(K, E), \ \forall P \in \mathrm{Diff}_b^k(M; E, \mathbf{C})\},$$

which are clearly Hilbertable for each k. Thus each $\dot{\mathcal{A}}^{(s)}(K, E)$ is a Fréchet space with the inclusions

$$\dot{\mathcal{A}}^{(s)}(K, E) \hookrightarrow \dot{\mathcal{A}}^{(s')}(K, E),$$

for $s \geq s'$, continuous and, as a result of the analysis below, dense. In consequence the inductive limit

$$\dot{\mathcal{A}}(K, E) = \bigcup_s \dot{\mathcal{A}}^{(s)}(K, E)$$

is a Mackey space (see [18]). A map from $\dot{\mathcal{A}}(K, E)$ into a locally convex topological space is continuous if and only if it is continuous on each $\dot{\mathcal{A}}^{(s)}(K, E)$. For convenience we also give

$$\dot{\mathcal{A}}_c(M, E) = \bigcup_K \dot{\mathcal{A}}(K, E)$$

the strict inductive limit topology over an exhaustive sequence of compacta $K_j \to M$. Since we can always work locally on M, as $\varrho u \in \dot{\mathcal{A}}(K, E)$ if $\varrho \in C_c^\infty(M)$ has support in K and $u \in \dot{\mathcal{A}}_c(M, E)$, this topology is never really used. Similarly, the space $\dot{\mathcal{A}}(M, E) \subset \dot{\mathcal{D}}'(M, E)$ is defined as consisting of those distributional sections, u, such that $\varrho u \in \dot{\mathcal{A}}_c(M, E)$ whenever $\varrho \in C_c^\infty(M)$ and is then topologized in the usual way.

LEMMA 1.11. $\quad C^\infty(E) \oplus C^\infty(E_{(m)}) \subset \dot{\mathcal{A}}(M, E), \quad \forall m.$

Proof. This follows immediately from the fact that

$$\mathrm{Diff}_b^k(M; E, \mathbf{C})(C^\infty(E) \oplus C^\infty(E_{(m)})) \subset C^\infty(E) \oplus C^\infty(E_{(m)})$$

for every m, k. In view of the definition of Diff_b this in turn is a consequence of

$$\mathcal{V}(C^\infty(E) \oplus C^\infty(E_{(m)})) \subset C^\infty(E) \oplus C^\infty(E_{(m)}), \tag{1.13}$$

which we prove in local coordinates. Any element $V \in \mathcal{V}$ is of the form

$$V = \sum_j c_j(z)\, \partial_{y_j} + x b(z)\, \partial_x,$$

$z = (x, y)$ so the lemma follows from the identity:

$$x \frac{d^j}{dx^j} \delta(x) = -j \frac{d^{j-1}}{dx^{j-1}} \delta(x), \quad j \geq 1.$$

Conversely the elements of $\dot{\mathcal{A}}(M, E)$ are clearly C^∞ in the interior of M, they are also singular only in the normal direction at the boundary.

PROPOSITION 1.14. *$\dot{\mathcal{A}}(M, E)$ is the space of E-valued Lagrangian distributions of type* $(1, 0)$ *supported in M and associated to the conormal bundle $N^*\partial M \subset T^*M$.*

Proof. The result is clearly local in nature and coordinate independent so it suffices to consider the case $M=Z=\overline{\mathbf{R}_x^+}\times\mathbf{R}_y^n$, and to assume that E is trivial and so suppose $u\in\dot{\mathcal{A}}_c(Z)$. Then we can show that

$$u(z)=(2\pi)^{-1}\int e^{ix\xi}\hat{a}(\xi,y)\,d\xi \tag{1.15}$$

where the partial Fourier transform $\hat{a}=\mathbf{C}\times\mathbf{R}_y^n\to\mathbf{C}$ is C^∞ and entire in the first variable with

$$|\partial_y^\alpha\partial_\xi^k\hat{a}(\xi,y)|\leqslant C_{\alpha,k}(1+|\xi|)^{m-k},\quad \forall k,\alpha \tag{1.16}$$

in $\operatorname{Im}\xi\leqslant 0$ for some fixed m. In fact the defining condition (1.10) is just

$$(x\partial_x)^p\partial_y^\alpha a(x,y)\in\dot{H}^s(Z)$$

for some fixed s. Now, $\hat{a}$ is defined by (1.15) and the Sobolev embedding theorem shows that for some fixed m,

$$|(\partial_\xi\xi)^p\partial_y^\alpha\hat{a}|\leqslant C_{\alpha,\varrho}(1+|\xi|)^m,\quad\forall\alpha,p \tag{1.17}$$

in $\operatorname{Im}\xi\leqslant 0$. A simple inductive argument reduces these estimates to (1.16). This proves the proposition.

As an immediate consequence of Proposition 1.14

$$\operatorname{WF}(u)\subset N^*(\partial M)\subset T^*M,\quad\forall u\in\dot{\mathcal{A}}(M) \tag{1.18}$$

where the wavefront set is calculated with respect to any extension of M to an open manifold.

I.2. Compressed contangent bundle

Consider first the geometry of the subspace $\mathcal{V}\subset C^\infty(TM)$ of totally characteristic vector fields on M. As a locally free $C^\infty(M)$-module of finite rank $\mathcal{V}$ is the space of sections of a C^∞ vector bundle which we denote $\tilde{T}M$. For $m\in M$ consider the equivalence relation on $\mathcal{V}$:

$$V\underset{\tilde{m}}{\sim}V'\Leftrightarrow\begin{cases}(V'-V)f(m)=0, & \forall f\in C^\infty(M);\\ \text{if } m\in\partial M \text{ then } d((V'-V)g)(m)=0, & \forall g\in C^\infty(M) \text{ with } g=0 \text{ on } \partial M.\end{cases} \tag{2.1}$$

LEMMA 2.2. *We can identify $\tilde{T}_mM=\mathcal{V}/\tilde{m}$ so that $\mathcal{V}=C^\infty(\tilde{T}M)$ and there is a natural C^∞ vector bundle map*

$$\tilde{T}M\to TM \tag{2.3}$$

with range $T\mathring{M}\cup T\partial M$.

Proof. Near any point $m \in \mathring{M} \setminus \partial M$ the second condition on the right in (2.1) is vacuous so defines the fibre of the tangent bundle, clearly

$$\tilde{T}_m M = T_m M, \quad m \notin \partial M.$$

Suppose $m \in \partial M$ then we introduce local coordinates (x, y) in M, with $x=0$ on ∂M, $x>0$ in $\mathring{M}$ and $y=(y_1, ..., y_n)$, $n=\dim \partial M$. The elements of $\mathcal{V}$ are locally of the form:

$$V=\sum_{j=1}^{n} a_j(x,y)\,\partial_{y_j}+xb(x,y)\,\partial_x$$

$a_j, b \in C^\infty$. The equivalence relation (2.1) shows that $[V]$ is exactly determined by the vector $(b(m), a_1(m), ..., a_n(m))$. This agrees with the abstract definition of $\tilde{T}M$. That this structure is coordinate invariant can be seen explicitly; under a change of coordinates, to (x', y'),

$$\partial_x = e_{00}\,\partial_{x'}+\sum_{j=1}^{n} e_{0j}\,\partial_{y'_j}$$

$$\partial_{y_k} = x\,e_{k0}\,\partial_{x'}+\sum_{j=1}^{n} e_{kj}\,\partial_{y'_j}$$

with $e_{00}>0$, $\det(e_{kj}) \neq 0$. In such local coordinates the vector bundle map (2.3) takes $(b, a_1, ..., a_n)$ to $(xb, a_1, ..., a_n)$ and is clearly C^∞ with the indicated range.

Let $\tilde{T}^*M$ be the dual to $\tilde{T}M$. The dual to the map (2.3) gives a C^∞ vector bundle mapping

$$\pi\colon T^*M \to \tilde{T}^*M. \tag{2.4}$$

In local coordinates (x, y) a section of $\tilde{T}^*M$ is of the form

$$\alpha=\lambda(x,y)\,x^{-1}\,dx+\sum_{j=1}^{n}\eta_j(x,y)\,dy_j$$

where dy_j, $x^{-1}dx$ is the dual basis to ∂_{y_j}, $x\partial_x$ and λ, $\eta_j \in C^\infty$. Then, (2.4) is the map $\xi dx + \eta\cdot dy \mapsto (x\xi)x^{-1}dx+\eta\cdot dy$ in local coordinates i.e.

$$\lambda = x\xi, \quad \eta=\eta, \tag{2.5}$$

justifying the notation.

The inclusion map $\iota\colon \partial M \hookrightarrow M$ defines by duality a C^∞ map

$$\iota^*\colon \partial T^*M \to T^*\partial M,$$

where $\partial T^*M = T^*M \,|\, \partial M$ is the boundary of the cotangent bundle of M; ι^* is the boundary projection of M.

LEMMA 2.6. *ι^*: $\partial T^*M \to T^*\partial M$ is an affine line bundle with fibres the leaves of the Hamilton foliation of ∂T^*M in T^*M. The maps ι^* and $\pi|\partial T^*M$ can be canonically identified so that the range of π in (2.4) is*

$$T^*\partial M \cup T^*\mathring{M} \subset \tilde{T}^*M. \tag{2.7}$$

Proof. We work in the canonically dual coordinates (x, y, ξ, η) to some local coordinates in M at $m \in \partial M$, so ∂T^*M is the surface $x=0$. Since $(0, y, \xi, \eta)$ are the coordinates of the 1-form $\xi dx + \eta \cdot dy$ at $(0, y)$ and (y, η) are the coordinates of the 1-form $\eta \cdot dy = \iota^*(\xi dx + \eta \cdot dy)$,

$$\iota^*(0, y, \xi, \eta) = (y, \eta)$$

in these coordinates. Thus ι^* has the structure of an affine line bundle with fibres the ξ-lines. The Hamilton vector field of x is $-\partial_\xi$ so these lines are also the leaves of the Hamilton foliation of ∂T^*M. A change to new canonically dual coordinates, corresponding to a change of coordinates in M, is linear in (ξ, η) so certainly affine-linear along the ξ-lines. Thus, the affine structure of ι^* is coordinate-free. Finally, from (2.5), it follows that the image, $\lambda=0$, $x=0$ of ∂T^*M under π can be naturally identified with $T^*\partial M$, proving the lemma.

This lemma is fundamental to the philosophy of this paper, in that the space $\tilde{T}^*M$ is regarded as the natural manifold for the micro-localization of boundary problems. Thus, (2.7) shows that $\tilde{T}^*M$ contains the subspace $T^*\partial M \cup T^*\mathring{M}$ in which most previous micro-localization has been carried out (see [16], [11]).

If $\varrho \in \partial T^*M$, let $C^\infty(\varrho)$ be the ring of germs at ϱ of C^∞ functions on T^*M. Thus if $U \subset T^*M$ is an open neighbourhood of ϱ the natural projection $C^\infty(U) \to C^\infty(\varrho)$ is surjective.

Definition 2.8. A germ $f \in C^\infty(\varrho)$ is said to have polynomial traces of order $t(\in \mathbf{Z})$ if in any canonically dual coordinates (x, y, ξ, η) at ϱ,

$$\partial_x^k f(0, y; \xi, \eta) \text{ is a polynomial in } \xi \text{ of degree at most } k+t, \ \forall k \geq 0. \tag{2.9}$$

The space of such germs will be denoted $C_t^\infty(\varrho)$. The space $C_t^\infty(U)$ of C^∞ functions in the open set $U \subset T^*M$ having polynomial traces of order t consists of all $f \in C^\infty(U)$ having polynomial traces of order t at each $\varrho \in U \cap \partial T^*M$.

We remark that (2.9) need only be verified in any one canonically dual coordinate system. Indeed, the choice of new coordinates $(x', y'; \xi', \eta')$ corresponds to the choice of new coordinates

$$x = X(x', y'), \quad y = Y(x', y')$$

so

$$\xi' = (\partial_{x'} X)\xi + (\partial_{x'} Y)\eta, \quad \eta' = (\partial_{y'} X)\xi + (\partial_{y'} Y)\eta,$$

which shows that x', y', ξ', η' are respectively functions of polynomial traces of orders $-1, 0, 1, 0$, in the original coordinates. From this it follows easily that f has polynomial traces of a given order in one canonically dual coordinate system if it satisfies the same conditions in any other such system. More abstractly we can see this invariance as follows.

PROPOSITION 2.10. *Let $x \in C^\infty(M)$ be positive in $\mathring{M}$ and vanish simply on ∂M. If $U \subset T^*M$ is open, $t \in \mathbf{Z}$ and $f \in C^\infty(U)$ then $f \in C_t^\infty(U)$ if and only if there exists $g \in C^\infty(V)$, V some neighbourhood of $\pi(U) \subset \tilde{T}^*M$, such that*

$$\begin{cases} f = x^{-t}\pi^*g & (t \leqslant 0) \\ x^t f = \pi^* g & (t \geqslant 0) \end{cases} \tag{2.11}$$

where we write x for the pull-back of x to T^*M.

Proof. The result is clearly local in nature, so introduce (x, y) as coordinates. By definition of $f \in C_t^\infty(U)$ the Taylor series of $x^t f$ has as coefficient of x^p a polynomial in ξ of degree at most p. In view of (2.4) we can write (2.11), taking $t \geqslant 0$ for definiteness,

$$x^t f(x, y; \xi, \eta) = g(x, y, x\xi, \eta). \tag{2.12}$$

So if g is to satisfy (2.11) the Taylor series of the right-hand side must be of the same form and indeed

$$\partial_x^k g(x, y; x\xi, \eta)\,|\,(x=0) = \sum_{0 \leqslant j \leqslant k} \binom{k}{j} \partial_x^j \partial_\lambda^{k-j} g(0, y, 0, \eta)\, \xi^{k-j}.$$

Thus, the Taylor series of g at $x = \lambda = 0$ is uniquely determined by (2.12), with the coefficient of $x^k \lambda^p$ being fixed by the coefficient of ξ^p in the term x^{p+k} of the Taylor series of $x^t f$ at $x = 0$. By Borel's theorem we can choose a germ g_1 with this Taylor series at $x = \lambda = 0$.

Then,

$$E(x, y; \xi, \eta) = x^t f(x, y, \xi, \eta) - g_1(x, y, x\xi, \eta) \tag{2.13}$$

is defined in some neighbourhood of ϱ and vanishes to all orders at $x = 0$. Define

$$g_2(x, y, \lambda, \eta) = E(x, y, \lambda/x, \eta) \tag{2.14}$$

in some region

$$A_\varepsilon = \{(x, y, \lambda, \eta);\, 0 < x < \varepsilon,\, 0 < |y| < \varepsilon,\, |\lambda| < \varepsilon x\}.$$

For $\varepsilon > 0$ small g_2 is clearly C^∞ on the open set A_ε. Since E is always of the form $x^k E_k$, $k \in \mathbf{N}$, where E_k is C^∞, it follows that all derivatives of g_2 are bounded uniformly in A_ε for ε small, since λ/x is bounded above. Thus g_2 extends to a C^∞ function on $\overline{A_\varepsilon}$ and then obviously g_2 can be extended to a germ at $\pi(\varrho)$. This extension does not affect (2.13) or (2.14) so $g = g_1 + g_2$ satisfies (2.12). The converse is trivial so the proposition is proved.

Remark 2.15. If f is homogeneous in (2.11) it is clear that g can be chosen homogeneous of the same degree.

I.3. Non-characteristic operators

If P is a linear differential operator, from a vector bundle E to a vector bundle F, the basic objective of the study of boundary problems is the relation of properties of $u\in\dot{\mathcal{D}}'(M,E)$, specifically regularity properties, to properties of $Pu\in\dot{\mathcal{D}}'(M,F)$ under various conditions on P itself. One such condition, that of being totally characteristic at ∂M, is introduced above. At the opposite extreme is the more familiar condition that the boundary be non-characteristic for P. This is normal ellipticity:

Definition 3.1. $P\in \mathrm{Diff}^m(M;E,F)$ is non-characteristic at $m\in\partial M$ if $\sigma_m(P)\colon \pi^*E\to\pi^*F$ is an isomorphism over $N^*_m(\partial M)\setminus 0$.

Here we use the usual notions of principal symbol and $\pi\colon T^*M\to M$ is the projection.

LEMMA 3.2. *If P is non-characteristic at $m\in\partial M$ then for every $f\in\dot{\mathcal{A}}(M,F)$ there exists $u\in\dot{\mathcal{A}}(M,E)$ such that $Pu-f\in\dot{C}^\infty(M,F)$ near m; $u\in\dot{\mathcal{A}}(M,E)$ is uniquely specified near m modulo $\dot{C}^\infty(M,E)$.*

Proof. As a consequence of Proposition 1.14 this is a standard computation with Lagrangrian distributions. In local coordinates (x,y) based at m we can take $E=F=M\times\mathbf{C}^n$ trivial, in view of the existence of an isomorphism $\sigma_m(P)$. The condition of Definition 3.1 then states that in the formula

$$P=\sum_{k+|\alpha|\leqslant m} p_{k,\alpha}(x,y)\,D_x^kD_y^\alpha \tag{3.3}$$

the coefficient of D_x^m is invertible:

$$\det\left(p_{m,0}(0,0)\right)\neq 0. \tag{3.4}$$

We can take f of the form (1.15) with vector-valued symbol $\hat{f}$ and look for u in the same form with symbol $\hat{u}$. Then

$$Pu=(2\pi)^{-1}\int e^{ix\xi}\sum p_{k,\alpha}\,\xi^kD_y^\alpha\hat{u}\,d\xi. \tag{3.5}$$

We choose successive symbols $\hat{u}_r$, $r=0,1,\ldots$ by requiring

$$\hat{u}_0=[p_{m,0}(0,y)]^{-1}\xi^{-m}\hat{f} \tag{3.6}$$

and the recursion formula

$$\xi^m p_{m,0}(0,y)\,\hat{u}_r=-\sum\frac{1}{j!}\,p_{l,\alpha}^{(j)}(0,y)\,(-D_\xi)^j(\xi^lD_y^\alpha\hat{u}_s) \tag{3.7}$$

with the sum extended over $l-j-s=m-r$, $l<m$. Here $p_{l,\alpha}^{(j)}=\partial_x^jp_{l,\alpha}(0,y)$ and (3.7) is obtained from (3.5) by expanding in Taylor series at $x=0$ and integrating by parts. Clearly

the $\hat{u}_r$ are holomorphic in $\operatorname{Im} \xi < 0$ if $\hat{f}$ is. A suitable asymptotic summation of the $\hat{u}_r$ completes the proof of existence. Uniqueness follows from the necessity of the formula (3.6), modulo lower order terms, and a simple inductive argument. This proves the lemma.

This result shows the degree to which the elements of $\dot{\mathcal{A}}(M)$ are ignorable in the study of non-characteristic boundary problems; it can be strengthened slightly.

LEMMA 3.8. *If P is of order k, non-characteristic at $m \in \partial M$ and $x \in C^\infty(M)$ vanishes simply on ∂M near m then for each $f \in \dot{\mathcal{A}}(M, F)$ there exists $u \in \dot{\mathcal{A}}(M, E)$ such that $x^k Pu - f \in \dot{C}^\infty$ near m. Similarly there exists $u' \in \dot{\mathcal{A}}(M, E)$ such that $Px^k u' - f \in \dot{C}^\infty$ near m.*

Proof. A similar argument to that of Lemma 3.2 applies when P is replaced by $x^k P$ or Px^k. Thus, in using (3.5), the initial equation (3.6) becomes

$$(-D_\xi)^k \xi^k p_{k,0}(0, y)\hat{u}_0 = \hat{f} \tag{3.9}$$

where $\hat{f}$ is entire in ξ and clearly satisfies symbol estimates

$$|D_\xi^r D_y^\alpha \hat{f}(\xi, y)| \leqslant C_{\alpha,r} |\xi|^{M-r}, \quad \operatorname{Im} \xi < 0, \ |\xi| > 1 \tag{3.10}$$

for all r, α. If $\hat{f}$ is a function holomorphic in $|\xi| > 1$, $\operatorname{Im} \xi < 0$ and satisfying (3.10), integration of $e^{ix\xi}\hat{f}(\xi)$ along a suitable contour on which $\operatorname{Im} \xi$ is bounded near infinity shows $\hat{f}$ to be the Fourier–Laplace transform of an element of $\dot{\mathcal{A}}(Z)$ of finite exponential growth at infinity.

Introducing the variable $s = \log(\xi)$, where s lies in the half-strip

$$G = \{0 < \operatorname{Im}(s) < \pi, \operatorname{Re}(s) > 0\}$$

converts the estimates (3.10) on a holomorphic function to

$$|D_s^r D_y^\alpha g(y, s)| \leqslant C_{r,\alpha} e^{M \operatorname{Re}(s)} \quad \text{in } G, \quad g(y, s) = \hat{f}(y, \xi), \tag{3.11}$$

and the equation (3.9) becomes

$$(-e^{-s} D_s)^k (e^{ks} v) = p_{k,0}^{-1} g, \quad v(y, s) = \hat{u}_0(y, \xi).$$

Now, this can be rewritten

$$(D_s - i) \dots (D_s - ik) v = p_{k,0}^{-1} g.$$

If g satisfies (3.11) such a linear differential equation with constant coefficients always has a solution satisfying the same type of estimates with M replaced by $M + \varepsilon$ for any $\varepsilon > 0$. To see this it suffices to consider each linear factor separately. In solving

$$(D_s - ip) v(s) = g(s)$$

set $v(\frac{1}{2}+i\frac{1}{2}\pi)=0$ if $M\geq p$, $v(\infty+i\frac{1}{2}\pi)=0$ if $M<p$. Clearly this allows (3.9) to be solved with the holomorphic symbol $\hat{u}_0$ of order ε greater than f.

Proceeding by induction allows the equation (3.5) to be solved modulo $\dot{C}^\infty(Z)$ as in (3.7).

Note that the solution of $x^kPu=f$ is not in general unique modulo $\dot{C}^\infty$ because x^k annihilates terms supported on the boundary up to order k.

Now we have a continuous injection, with dense range

$$C_c^\infty(M, E) \hookrightarrow \dot{\mathcal{A}}_c(M, E).$$

By duality consider $\mathcal{A}'(M, E)=(\dot{\mathcal{A}}_c(M, E^*\otimes\Omega))'$ as a subspace of $\dot{\mathcal{D}}'(M, E)$,

$$\mathcal{A}'(M, E) \hookrightarrow \dot{\mathcal{D}}'(M, E).$$

We shall always take the weak topology on $\mathcal{A}'(M, E)$.

Proposition 3.12. *If $u\in\dot{\mathcal{D}}'(M, E)$ and $P\in\mathrm{Diff}^k(M; E, F)$ is non-characteristic for ∂M then*

$$x^kPu\in\mathcal{A}'(M, F) \quad\Rightarrow\quad u\in\mathcal{A}'(M, E).$$

Proof. The construction of Lemma 3.8 actually gives a continuous parametrix for x^kP on bounded subsets. Thus, if $B\subset\dot{\mathcal{A}}(M, F)$ is bounded there exists

$$Q\colon B\to\dot{\mathcal{A}}(M, E)$$

continuous and such that

$$x^kPQ-\mathrm{Id}\colon B\to\dot{C}^\infty(M, E).$$

Since the formal dual of P,

$$P^*\colon C^\infty(M, F^*\otimes\Omega)\to C^\infty(M, E^*\otimes\Omega)$$

is a differential operator non-characteristic with P, Lemma 3.8 applies equally to P^*x^k. Observe that

$$\langle u, g\rangle = \langle u, P^*x^kQ'g\rangle+\langle u, R'g\rangle = \langle x^kPu, Q'g\rangle+\langle u, R'g\rangle$$

where $Q'\colon B'\to\dot{\mathcal{A}}_c(M, F^*\otimes\Omega)$ for a bounded set $B'\subset\dot{\mathcal{A}}_c(M, E^*\otimes\Omega)$ and $R'\colon B'\to\dot{C}_c^\infty(M, E)$. Thus u extends by continuity to $\dot{\mathcal{A}}_c(M, E^*\otimes\Omega)$, so is an element of $\mathcal{A}'(M, E)$ as claimed.

The importance of the space $\mathcal{A}'(M, E)$ is that the elements have well-defined restriction properties. That is, there is a continuous linear map

$$R\colon\mathcal{A}'(M, E)\to\mathcal{D}'(\partial M, E) \tag{3.13}$$

which extends the trace map on $C^\infty(M, E)$. To define (3.13) simply observe that the map

$$T\colon C_c^\infty(\partial M, E\otimes\Omega_{\partial M})\ni\varphi\mapsto\delta(x)\otimes\varphi\in\dot{\mathcal{A}}(M, E\otimes\Omega_M) \tag{3.14}$$

is continuous, independent of the choice of function x vanishing simply on ∂M. Then, if $u \in \mathcal{A}'(M, E)$ the map

$$C_c^\infty(\partial M, E \otimes \Omega_{\partial M}) \ni \varphi \mapsto \langle u, T\varphi \rangle$$

is linear and continuous, hence a distribution. This defines the map (3.13).

The existence of such a restriction, or trace, map allows a strong definition of boundary problems for non-characteristic operators. We first give the weak definition, due to Schwartz and then show the usual 'weak equals strong' theorem.

A boundary problem for an operator $P \in \operatorname{Diff}^m(M; E, F)$ is provided by a C^∞ differential operator $B \in \operatorname{Diff}(M; E, G)$. The classical problem is the search for a solution $u \in C^\infty(M, E)$ to

$$\begin{cases} Pu = f \in C^\infty(M, F) & \text{in } \mathring{M} \\ Bu = g \in C^\infty(M, G) & \text{at } \partial M, \end{cases} \tag{3.15}$$

possibly with additional constraints in the form of support or other conditions. We also assume that B is of lower order than P.

Assuming $u \in C^\infty(M; E)$ we write $(u)_c$ for the image of u in $\dot{\mathcal{D}}'(M; E)$. The first equation in (3.15) becomes

$$h = P(u_c) - (f)_c \in \mathcal{D}'(\partial M; F_{(m-1)}). \tag{3.16}$$

The boundary conditions then specify at least part of h. To give these in weak form consider the map

$$\dot{P}: C^\infty(M, E) \to C^\infty(\partial M, F_{(m-1)}) \tag{3.17}$$

defined by $\dot{P}u = P(u_c) - (Pu)_c$.

PROPOSITION 3.18. *If P is non-characteristic then* (3.17) *is surjective. Moreover if $B \in C^\infty(M; E, G)$ there is a uniquely defined differential operator*

$$\dot{B}_p: C^\infty(\partial M; F_{(m-1)}) \to C^\infty(\partial M; G)$$

such that

$$\begin{tikzcd} & C^\infty(\partial M; F_{(m-1)}) \arrow[dd, "\dot{B}_p"] \\ C^\infty(M; E) \arrow[ur, "\dot{P}"] \arrow[dr, "R \circ B"'] & \\ & C^\infty(\partial M; G) \end{tikzcd}$$

commutes.

Proof. Working in local coordinates one can easily see the surjectivity of $\dot{P}$ in (3.17) (see e.g. [2]). Moreover $\dot{P}$ has a right inverse Q

$$\dot{P}Q = \mathrm{Id}, \quad Q\dot{P} - \mathrm{Id} \subset \ker(\dot{P}), \tag{3.19}$$

and $Q: C^\infty(M, F_{(m-1)}) \to C^\infty(M, E)$ is well-defined modulo a map into the kernel of $\dot{P}$. Then, $\dot{B}_p$ is well-defined as

$$\dot{B}_p = R \circ B \circ Q$$

since $R \circ B(u) = 0$ if $u \in \ker(\dot{P})$.

With these preliminaries we can recast (3.15) into the form

$$\begin{cases} Pv - f \in \mathcal{D}'(\partial M, F_{(m-1)}) \\ \dot{B}(Pv - f) = g \in \mathcal{D}'(\partial M, G) \end{cases} \tag{3.20}$$

where $v = (u_c)$, $f = (f)_c$. In this weak form the boundary problem makes sense for arbitrary data

$$f \in \dot{\mathcal{D}}'(M, F), \quad g \in \mathcal{D}'(\partial M, G),$$

with the solution sought in $\dot{\mathcal{D}}'(M, E)$.

Now, the space $\dot{\mathcal{A}}_c(M)$ is closed under differentiation but it is important to realize that this does not carry over by formal duality to $\mathcal{A}'(M)$. Indeed, by definition, $u \in \mathcal{A}'(M)$ if $u \in \dot{\mathcal{D}}'(M)$ extends by continuity from $C_c^\infty(M)$ to $\dot{\mathcal{A}}_c(M)$. However, $C_c^\infty(M)$ as a subspace of $\dot{\mathcal{A}}_c(M)$ is not closed under differentiation because of the appearance of boundary terms. We shall let $\dot{\mathcal{B}}(M)$ be the minimal extension of $\mathcal{A}'(M)$ which is closed under differentiation. Elements of $\dot{\mathcal{B}}(M)$ are just, locally, finite sums of terms each a differential operator applied to an element of $\mathcal{A}'(Z)$. Clearly,

$$\dot{\mathcal{D}}'(M, \partial M) \subset \dot{\mathcal{B}}(M). \tag{3.21}$$

In fact much more is true since $\dot{\mathcal{B}}(M)$ splits:

$$\dot{\mathcal{B}}(M) = \dot{\mathcal{D}}'(M, \partial M) \oplus \mathcal{A}'(M). \tag{3.22}$$

R is extended to $\dot{\mathcal{B}}(M)$ by defining it to vanish on the first factor in (3.22). To see this observe that $\dot{C}_c^\infty(M) \subset \dot{\mathcal{A}}_c(M)$ is a dense subspace closed under differentiation. If P is a differential operator and $\{\varphi_n\}$ is a sequence in $\dot{C}_c^\infty(M)$ converging in $\dot{\mathcal{A}}_c(M)$ then $\{P\varphi_n\}$ converges in $\dot{\mathcal{A}}_c(M)$ too. Thus the restriction map, to the interior,

$$\dot{\mathcal{B}}(M) \to \mathcal{D}'(M) \tag{3.23}$$

actually defines a projection:

$$\dot{\mathcal{B}}(M) \to \mathcal{A}'(M). \tag{3.24}$$

Clearly the kernel of (3.24) is $\dot{\mathcal{D}}'(M, \partial M)$ and it is certainly surjective. In particular it follows from (3.22) that

$$\dot{\mathcal{D}}'(M, \partial M) \cap \mathcal{A}'(M) = \{0\}. \tag{3.25}$$

Of course, for sections of a vector bundle over M, (3.22) becomes

$$\dot{\mathcal{B}}(M; E) = \mathcal{A}'(M; E) \oplus \lim_{m} \mathcal{D}'(\partial M; E_{(m)}). \tag{3.26}$$

Now if $u \in \mathcal{A}'(M, E)$ the map (3.17) extends, by continuity, to

$$\dot{P}\colon \mathcal{A}'(M, E) \to \mathcal{D}'(\partial M; F_{(m-1)}) \tag{3.27}$$

where $\dot{P}u = Pu - r(Pu)$, r the map (3.24), on sections. Proposition 3.18 holds with the maps extended to these more singular spaces and this allows us to extend and strengthen Proposition 3.12.

PROPOSITION 3.28. *If P is non-characteristic and $f \in \mathcal{A}'(M, F)$, then any solution v to (3.20) lies in $\mathcal{A}'(M, E)$ and*

$$RBv = g. \tag{3.29}$$

Proof. Directly from the definition of $\dot{B}$ and R it follows that $R \circ Bv = \dot{B}_p \cdot \dot{P}v = g$.

This result shows that, for non-characteristic boundary problems, the very weak formulation (3.20) is equivalent to the natural formulation with boundary values from $\dot{\mathcal{B}}(M, E)$. We shall also use the notation

$$\dot{\mathcal{B}}^{(k)}(M) = \mathcal{A}'(M) \oplus \dot{\mathcal{D}}_k(M, \partial M),$$

when there are boundary terms only up to order k, see (1.5).

Chapter II: Pseudodifferential operators

II.1. Symbols

Recall some standard results on symbols and symbol spaces. A symbol of order m on the product manifold $\mathbf{R}^n \times \mathbf{R}^N$ is a C^∞ function which satisfies bounds

$$\|a\|^{(m)}_{\gamma,\alpha,\gamma} = \sup_{(z,\theta)\in\gamma} |D_z^\alpha D_\theta^\beta a(z,\theta)|(1+|\theta|)^{-m+|\beta|} < \infty, \tag{1.1}$$

for each pair α, β of n-, N-multiindices and set $\gamma = K \times \mathbf{R}^N$ with $K \subset\subset \mathbf{R}^n$. The space of all such symbols, $S^m(\mathbf{R}^n \times \mathbf{R}^N)$ is a Fréchet space with the seminorms (1.1). Moreover, if $m' > m$ then $S^m(\mathbf{R}^n \times \mathbf{R}^N) \hookrightarrow S^{m'}(\mathbf{R}^n \times \mathbf{R}^N)$ is locally compactly included and S^m is dense in $S^{m'}$ in the topology of $S^{m''}$ whenever $m'' > m' > m$. In fact

$$C_c^\infty(\mathbf{R}^n \times \mathbf{R}^N) \subset S^{-\infty}(\mathbf{R}^n \times \mathbf{R}^N) = \bigcap_{m \in \mathbf{R}} S^m(\mathbf{R}^n \times \mathbf{R}^N)$$

is dense in S^m in the topology of $S^{m'}$. If $\Gamma \subset \mathbf{R}^n \times \mathbf{R}^N$ is an open cone, so $(z, t\theta) \in \Gamma$ if $t > 0$ and $(z, \theta) \in \Gamma$, denote by $S^m(\Gamma)$ the Fréchet space of those functions $a \in C^\infty(\Gamma)$ satisfying estimates (1.1) for each closed cone $\gamma \subset \Gamma$ with compact base (projection into $\mathbf{R}^n$); the seminorms of $S^m(\Gamma)$ are given by (1.1). If $\gamma \subset \Gamma$ is a closed subset we denote by $S^m(\Gamma, \gamma)$ the closed subspace of those symbols which have support in γ.

Let $\chi\colon \Gamma_1 \to \Gamma_2$ be a C^∞ diffeomorphism of open cones, which is homogeneous. Then

$$\chi^*\colon S^m(\Gamma_2) \to S^m(\Gamma_1)$$

is an isomorphism. Thus, if $\pi\colon V \to M$ is a vector bundle over a manifold we can define $S^m(V)$, $S^m(\Gamma)$ for $\Gamma \subset V$ an open conic set, and $S^m(V, \gamma)$ for $\gamma \subset V$ a closed set, by reference to local coordinates and trivializations (see [10]).

When $a_j \in S^{m_j}(\Gamma)$ for $j = 0, 1, \ldots$ and $m_j \to -\infty$ as $j \to \infty$, $a \in S^m(\Gamma)$ is said to be an asymptotic sum of the a_j if

$$a - \sum_{j<N} a_j \in S^{d_N}(\Gamma)$$

where $d_N \to -\infty$ as $N \to \infty$. This relationship is written

$$a \sim \Sigma\, a_j$$

and determines a modulo $S^{-\infty}(\Gamma)$; one can take $m = \sup m$; and $d_N = \sup_{j \geq N} m_j$.

The fundamental property of symbols is that inverse Fourier transformation

$$u(z, t) = (2\pi)^{-N} \int e^{it\cdot\theta} a(z, \theta)\, d\theta \tag{1.2}$$

gives a distribution $u \in C^\infty(\mathbf{R}^n_z; S'(\mathbf{R}^N_t))$ which is C^∞ outside $\{t = 0\}$ in $\mathbf{R}^{n+N}$ and rapidly decreasing with all its derivatives as $|t| \to \infty$, uniformly on compact subsets of $\mathbf{R}^n$. The singularity type of such distributions is coordinate free and this leads to the space

$$I^\infty(\mathbf{R}^{n+N}; N^*\{t = 0\}) \subset \mathfrak{D}'(\mathbf{R}^{n+N})$$

of distributions everywhere locally the sum of a distribution (1.2), for some m, and a term in C^∞.

Now, suppose $a \in S^m(\mathbf{R}^n \times \mathbf{R}^{N+1})$. Consider the splitting $\mathbf{R}^{N+1} = \mathbf{R} \times \mathbf{R}^N$, with variables $\theta = (\theta_1, \theta')$. Clearly, $a \in S^m((\mathbf{R}^n \times \mathbf{R}^N) \times \mathbf{R})$, so the translated partial Fourier transform

$$Ma(z, \theta'; t) = \int e^{i\theta_1(1-t)} a(z, \theta_1, \theta')\, d\theta_1 \tag{1.3}$$

is C^∞ away from $t = 1$. We shall say that $a \in S^m(\mathbf{R}^n \times \mathbf{R}^{1+N})$ is lacunary, or satisfies the lacunary condition if

$$Ma(z, \theta'; t) = 0 \quad \text{in } t < 0. \tag{1.4}$$

The subspace of lacunary symbols will be denoted $S^m_{\mathrm{la}}(\mathbf{R}^n \times \mathbf{R}^{1+N})$.

LEMMA 1.5. *There is a continuous linear map*

$$T\colon S^\infty(\mathbf{R}^n \times \mathbf{R}^{1+N}) \to S^{-\infty}(\mathbf{R}^n \times \mathbf{R}^{1+N})$$

such that $(\mathrm{Id} + T)\colon S^m(\mathbf{R}^n \times \mathbf{R}^{1+N}) \to S^m_{\mathrm{la}}(\mathbf{R}^n \times \mathbf{R}^{1+N})$ *for every* m.

Proof. If $\varrho\in C^\infty(\mathbf{R})$ has $\operatorname{supp}(\varrho)\subset(-\infty,\frac{1}{2}]$ with $\varrho(t)=1$ if $t<1/4$ then the map $Ta=-M^{-1}(\varrho(t)Ma)$ has the desired properties. Certainly $(\mathrm{Id}+T)a$ satisfies the lacunary condition (1.4) since $Ma+M\cdot Ta=(1-\varrho)(t)Ma=0$ in $t<0$. When $a\in S^{-\infty}(\mathbf{R}^n\times\mathbf{R}^{1+N})$ the integral defining Ma is absolutely convergent and on the support of ϱ integration by parts using the identity

$$e^{i(1-t)\theta_1}=(1-t)^{-1}D_{\theta_1}e^{i(1-t)\theta_1}$$

allows a to be replaced by $(1-t)^{-k}D^k_{\theta_1}a$ for any $k\in\mathbf{N}$. Since, for $a\in S^m$, $D^k_{\theta_1}a\in S^{m-k}$ the integral with a so replaced is absolutely convergent when $k>m+1$ and, by the density of $S^{-\infty}$ in S^∞, valid. Clearly then, T is continuous into $S^{-\infty}$.

$S^m_{\mathrm{la}}(\mathbf{R}^n\times\mathbf{R}^{1+N})$ is a closed subspace of $S^m(\mathbf{R}^n\times\mathbf{R}^{1+N})$ and in view of Lemma 1.5

$$S^{-\infty}_{\mathrm{la}}(\mathbf{R}^n\times\mathbf{R}^{1+N})\hookrightarrow S^m_{\mathrm{la}}(\mathbf{R}^n\times\mathbf{R}^{1+N})$$

is dense in the topology of $S^{m'}(\mathbf{R}^n\times\mathbf{R}^{1+N})$ for any $m'>m$. Indeed, if $a\in S^m_{\mathrm{la}}$ and $a_n\in S^{-\infty}$ converges to a in the topology of $S^{m'}$ then

$$a_n+Ta_n\to a+Ta \quad \text{in } S^{m'}_{\mathrm{la}},$$

by the continuity of T. Thus, $b_n=a_n+Ta_n-Ta\in S^{-\infty}_{\mathrm{la}}$ converges to a in $S^{m'}_{\mathrm{la}}$. Lemma 1.5 also shows that the inclusion of S^m_{la} in S^m defines an isomorphism

$$S^m_{\mathrm{la}}(Z\times\mathbf{R}^{1+N})/S^{-\infty}_{\mathrm{la}}(Z\times\mathbf{R}^{1+N})\cong S^m(Z\times\mathbf{R}^{1+N})/S^{-\infty}(Z\times\mathbf{R}^{1+N}), \tag{1.6}$$

since the variables in the base appear as parameters throughout.

In defining totally characteristic pseudodifferential operators in Section 3 below, we need to examine the properties of Ma more closely. If $\dot{S}'([0,\infty))\subset S'(\mathbf{R})$ is the subspace of temperate distributions with support in $[0,\infty)$ then clearly

$$M\colon S^m_{\mathrm{la}}(Z\times\mathbf{R}^{1+N})\to C^\infty(Z\times\mathbf{R}^N;\dot{S}'([0,\infty))).$$

As noted above, Ma is certainly C^∞ away from $t=1$. In fact the proof of Lemma 1.5 shows that if $\varrho\in C^\infty(\mathbf{R})$ is of slow growth in the sense of Schwartz and $1\notin\operatorname{supp}(\varrho)$ then

$$\varrho(t)Ma(z,\theta';t)\in S^{-\infty}(Z\times\mathbf{R}^{N+1}). \tag{1.7}$$

It is important to note that, even near the singularity at $t=1$, Ma inherits some symbolic properties from a, in the remaining θ variables. If $\varrho\in C^\infty_c(\mathbf{R})$,

$$\langle\varrho(t),Ma(z,\theta';t)\rangle\in S^m(Z\times\mathbf{R}^N). \tag{1.8}$$

This can be put in a stronger form in which the singularity in the t variable and symbol order in the θ' variables are related. Consider the identity for $\psi = (1-t)\theta_1$:

$$e^{i\psi} = (1+|\theta|^2)^{-1}(-\theta_1 D_t + 1 + |\theta'|^2)e^{i\psi}.$$

Inserting this into the definition (1.3) of Ma gives

$$Ma(z, \theta'; t) = D_t M(-\theta_1(1+|\theta|^2)^{-1}a) + M((1+|\theta'|^2)(1+|\theta|^2)^{-1}a).$$

Iteration of this identity gives

$$Ma = \sum_{0\leqslant j\leqslant k} D_t^j M((1+|\theta|^2)^{-k}P_{k,j}(\theta)\,a) \tag{1.9}$$

where $P_{k,j}$ is a polynomial in θ of degree $2k$, but of degree at most k in θ_1. For $k > m+1+r$,

$$M((1+|\theta|^2)^{-k}P_{k,j}(\theta)a) \in C^r(\mathbf{R}_t; S^{k+2m}(Z\times\mathbf{R}^N)).$$

Thus, (1.9) proves:

LEMMA 1.10. *If $a\in S^m(Z\times\mathbf{R}^{1+N})$ then for each $r\in\mathbf{N}$ there exists $R\,(>2(m+r+1))$ such that*

$$Ma = \sum_{0\leqslant j\leqslant R} D_t^j \beta_{j,r}(z, \theta'; t)$$

where $\beta_{j,r}\in C^r(\mathbf{R}_t; S^{m+R}(Z\times\mathbf{R}^N))$.

II.2. Operators on open sets

We briefly review the calculus of pseudodifferential operators on open subsets of manifolds. If $\Omega\subset\mathbf{R}^n$ is an open set and $a\in S^m(\Omega\times\mathbf{R}^n)$ the map

$$C_c^\infty(\mathbf{R}^n)\ni u\mapsto a(z, D_z)u = \int e^{iz\cdot\zeta}a(z,\zeta)\,\hat{u}(\zeta)\,d\zeta(2\pi)^{-n}\in C^\infty(\Omega), \tag{2.1}$$

where $\hat{u}(\zeta)$ is the Fourier transform of u, is a pseudodifferential operator. In fact, the bilinear map

$$S^{-\infty}(\Omega\times\Omega\times\mathbf{R}^n)\times C_c^\infty(\Omega)\ni(a, u)\mapsto(2\pi)^{-n}\int e^{i(z-z')\cdot\zeta}a(z, z', \zeta)\,u(z')\,dz'\,d\zeta$$

which reduces to (2.1) when a is independent of z', extends to a separately continuous bilinear map

$$S^\infty(\Omega\times\Omega\times\mathbf{R}^n)\times\mathcal{E}'(\Omega)\to\mathcal{D}'(\Omega). \tag{2.2}$$

A pseudodifferential operator A: $\mathcal{E}'(\Omega)\to\mathcal{D}'(\Omega)$ is a linear map which is defined from a symbol, $a\in S^m(\Omega\times\Omega\times\mathbf{R}^n)$, through (2.2). The space of such operators will be denoted by $L^m(\Omega)$, with $L^\infty(\Omega)=\bigcup_{m\in\mathbf{R}}L^m(\Omega)$ and $L^{-\infty}(\Omega)=\bigcap_{m\in\mathbf{R}}L^m(\Omega)$ the space of smoothing operators. The subspace $L_p^m(\Omega)\subset L^m(\Omega)$ of properly supported pseudodifferential operators consists of those $A\in L^m(\Omega)$ such that for each $K\subset\subset\Omega$ there exists $K'\subset\subset\Omega$ with $\operatorname{supp}(Au)\subset K'$ and $\operatorname{supp}(A^*u)\subset K'$ whenever $\operatorname{supp}(u)\subset K$, $u\in C_c^\infty(\Omega)$. $L_p^\infty(\Omega)$ is a ring under composition, $L_p^m(\Omega)\circ L_p^{m'}(\Omega)\subset L_p^{m+m'}(\Omega)$. Moreover, if $A\in L^m(\Omega)$ then the full symbol of $A, \sigma_f(A)\in S^m(\Omega\times\mathbf{R}^n)/S^{-\infty}(\Omega\times\mathbf{R}^n)$, is well-defined and the sequence

$$0\hookrightarrow L_p^{-\infty}(\Omega)\hookrightarrow L_p^m(\Omega)\xrightarrow{\sigma_f} S^m(\Omega\times\mathbf{R}^n)/S^{-\infty}(\Omega\times\mathbf{R}^n)\to 0, \tag{2.3}$$

is exact, as is the corresponding sequence without support condition. If $A\in L_p^m(\Omega)$, $B\in L_p^{m'}(\Omega)$ then

$$\sigma_f(A\circ B)\sim\sum_\alpha \partial_\zeta^\alpha\sigma_f(A)(z,\zeta)\cdot D_z^\alpha\sigma_f(B)(z,\zeta)/\alpha!. \tag{2.4}$$

Any operator $A\in L^m(\Omega)$ can be written as the sum $A'+S$ of an operator $A'\in L_p^m(\Omega)$ and a smoothing operator, that is $S\in L^{-\infty}(\Omega)$.

Now, if $\chi\colon\Omega\to\Omega'$ is a diffeomorphism then the pull-back operators $\chi^*\colon \mathcal{E}'(\Omega')\to\mathcal{E}'(\Omega)$, $\chi^*\colon \mathcal{D}'(\Omega')\to\mathcal{D}'(\Omega)$ are isomorphisms and the operator $(\chi^*)^{-1}\circ A\circ\chi^*\colon \mathcal{E}'(\Omega')\to\mathcal{D}'(\Omega')$, obtained by conjugation from a pseudodifferential operator on Ω, is a pseudodifferential operator on Ω'; in fact the resulting maps $\chi^*\colon L^m(\Omega)\to L^m(\Omega')$, $L_p^m(\Omega)\to L_p^m(\Omega')$ are isomorphisms. If one makes the identification $T^*\Omega=\Omega\times\mathbf{R}^n$ in each coordinate system, the principal symbol map

$$L_p^m(\Omega)\ni A\mapsto[\sigma_f(A)]\in S^m(T^*\Omega)/S^{m-1}(T^*\Omega)$$

is coordinate-free and the sequence

$$0\hookrightarrow L_p^{m-1}(\Omega)\hookrightarrow L_p^m(\Omega)\xrightarrow{\sigma_m} S^m(T^*\Omega)/S^{m-1}(T^*\Omega)\to 0$$

is exact. This coordinate invariance allows one to define the space $L^m(M)$ of pseudodifferential operators acting on generalized functions (or distributional sections of other line bundles) on a given manifold, M, by reference to local coordinates and with principal symbol map

$$\sigma_m\colon L^m(M)\to S^m(T^*M)/S^{m-1}(T^*M).$$

If $A\in L^m(M)$ and $(p,\nu)\in T^*M\setminus 0$ then A is elliptic at (p,ν) if the principal symbol, $\sigma_m(A)$, has a representative $a\in S^m(T^*M)$ which is elliptic at (p,ν) in the sense that there is an open conic neighbourhood of (p,ν), $\Gamma\subset T^*M\setminus 0$, with $1/a\in S^{-m}(\Gamma)$. The set of non-elliptic points $\Sigma(A)\subset T^*M\setminus 0$ is the characteristic set of A. For $u\in\mathcal{D}'(M)$,

$$\operatorname{WF}(u)=\bigcap\,\{\Sigma(A);\,A\in L_p^0(M),\,Au\in C^\infty(M)\},$$

is the wavefront set of u. If $u \in \mathcal{D}'(\Omega)$, $\Omega \subset \mathbf{R}^n$, the wavefront set of u is the closed cone of those points $(\bar{z}, \bar{\zeta})$ such that for every $\varrho \in C_c^\infty(\Omega)$ with $\varrho(\bar{z}) \neq 0$ and every neighbourhood B of $\bar{\zeta} \in \mathbf{R}^n$

$$\sup_{\zeta \in B, t \geq 1} \left| t^N \widehat{\varrho u}(t\zeta) \right| = \infty \quad \text{for some } N.$$

For general $u \in \mathcal{D}'(M)$, WF (u) is just the union of the images of the wavefront sets of its local coordinate representatives. Under the natural projection π: $T^*M \to M$, WF (u) projects to the closed set sing supp (u) of points at which u is not locally equal to a C^∞ function.

Pseudodifferential operators are *microlocal*: WF $(Au) \subset$ WF (u) for all $u \in \mathcal{E}'(M)$, $A \in L^m(M)$ $(u \in \mathcal{D}'(M)$, $A \in L_p^m(M))$ and microlocally invertible at elliptic points. Thus, if $A \in L^m(M)$ and $(p, \nu) \notin \Sigma(A)$ there exists $B \in L_p^{-m}(M)$ such that $(p, \nu) \notin$ WF $((A \cdot B - \mathrm{Id})u)$, WF $((B \cdot A - \mathrm{Id})u)$ for all $u \in \mathcal{E}'(M)$.

The main obstacle to the free use of pseudodifferential operators in the treatment of boundary problems is connected with the non-locality of their operation (indeed if $A \in L^\infty(M)$ and supp $(Au) \subset$ supp (u) for all $u \in \mathcal{E}'(M)$ then A is a differential operator, see Peetre [17]). To define pseudodifferential operators on a manifold with boundary, M, one can embed M in an open extension $\tilde{M}$ and allow the elements $A \in L^\infty(\tilde{M})$ to act on the space $\dot{\mathcal{E}}'(M) \cong \{u \in \mathcal{E}'(\tilde{M});\ \text{supp}\ (u) \subset M\}$, of (compactly) supported distributions on M, by

$$u \mapsto Au|_{\mathring{M}} \in \mathcal{D}'(M), \quad \mathring{M} = M \setminus \partial M$$

where $\mathcal{D}'(M) \subset \mathcal{D}'(\mathring{M})$ is the space of extendible distributions on M. Such a definition is intrinsic, i.e., independent of the choice of extension, modulo $C^\infty(M)$, but in general the operators so defined do not preserve regularity up to the boundary. Thus, the space $C_c^\infty(M)$ of compactly supported functions smooth on M (i.e. smoothly extendible into elements of $C_c^\infty(\tilde{M})$) is naturally included in $\dot{\mathcal{E}}'(M)$ by cutting off at the boundary (i.e. taking the unique extension to a locally integrable function on $\tilde{M}$ vanishing in $\tilde{M} \setminus M$) and $A(C_c^\infty(M)) \subset C^\infty(\mathring{M})$ but in general Au is not smooth up to the boundary, $Au \notin C^\infty(M)$. In [3] Boutet de Monvel introduced the *transmission condition* on (classical) symbols under which the corresponding operators A do map $C_c^\infty(M)$ into $C^\infty(M)$. Suppose we introduce local coordinates $z = (x, y) \in Z = \overline{\mathbf{R}}_+ \times \mathbf{R}^n$ in M, so that the boundary is locally defined by $x = 0$ and we can take $M = Z$. Then the space $C_c^\infty(\overline{\mathbf{R}}_+;\ \mathcal{E}'(\mathbf{R}^n))$ of compactly supported distributions smooth in the normal variable x is naturally included in $\dot{\mathcal{E}}'(Z)$, by cutting off at the boundary, and if $A \in L^\infty(\mathbf{R}^{n+1})$ then $Au|_{\mathring{Z}} \in C^\infty(\mathbf{R}_+;\ \mathcal{D}'(\mathbf{R}^n))$. When A satisfies the transmission condition,

$$A\colon C_c^\infty(\overline{\mathbf{R}}_+;\ \mathcal{E}'(\mathbf{R}^n)) \to C^\infty(\overline{\mathbf{R}}_+;\ \mathcal{D}'(\mathbf{R}^n)),$$

but in general $Au|_{x=0}$ is not closely related to $u|_{x=0}$; it is not even determined modulo $C^\infty(\mathbf{R}^n)$ by a finite truncation of the Taylor series of u at $x=0$. This is a grave problem when u is the solution of a boundary problem, since the boundary condition on u can not be easily reinterpreted as a boundary condition on Au.

In view of these difficulties we proceed to a different notion of pseudodifferential operators on a manifold with boundary.

II.3. Definition on Z

The natural coordinates in the standard manifold with boundary

$$Z=\overline{\mathbf{R}^+}\times\mathbf{R}^n$$

will be denoted $z=(x,y)$. If $P\in\mathrm{Diff}_b(Z)$ is totally characteristic then its action can be written in pseudodifferential form:

$$Pu(z)=(2\pi)^{-n-1}\int e^{i(z-z')\cdot\zeta}\,\tilde{p}(z,\zeta)\,u(z')\,dz'\,d\zeta \tag{3.1}$$

where $\tilde{p}(z,\zeta)=\tilde{p}(x,y,\xi,\eta)=p(x,y,x\xi,\eta)$ is a polynomial in ξ,η defined by putting P in the form

$$\begin{cases} P=\sum\limits_{j+|\alpha|\leqslant m} p_{j,\alpha}(x,y)\,x^jD_x^jD_y^\alpha \\ p(x,y,\lambda,\eta)=\sum\limits_{j+|\alpha|\leqslant m} p_{j,\alpha}(x,y)\,\lambda^j\eta^\alpha. \end{cases} \tag{3.2}$$

To define operators written formally as oscillatory integrals (0.1),

$$Au(z)=(2\pi)^{-n-1}\int e^{i(z-z')\cdot\zeta}\,\tilde{a}(z,\zeta)\,u(z')\,dz'\,d\zeta \tag{3.3}$$

where $\tilde{a}(z,\zeta)=a(x,y,x\xi,\eta)$ is a more general amplitude, it is useful to rewrite the formal adjoint of A by making a singular coordinate change. Thus, one expects

$$A^*f(z')=(2\pi)^{-n-1}\int e^{i(z'-z)\cdot\zeta}\,\bar{\tilde{a}}(z,\zeta)\,f(z)\,dz\,d\zeta. \tag{3.4}$$

Suppose that $f\in C_c^\infty(Z)$. In (3.4) introduce $\lambda=x\xi$, $s=x/x'$ as variables of integration.

$$A^*f(z')=(2\pi)^{-n-1}\int e^{i(-1+1/s)\lambda+i(y'-y)\cdot\eta}\,\bar{a}(x's,y,\lambda,\eta)\,f(x's,y)\,d\lambda\frac{ds}{s}\,dy\,d\eta. \tag{3.5}$$

Proposition 3.6. *If $a\in S_{\mathrm{la}}^{-\infty}(Z\times\mathbf{R}^{n+1})$ satisfies the lacunary conditions,* (1.4), *and $f\in C_c^\infty(Z)$ then the successive integrals in* (3.5) *converge absolutely and uniformly and so define a continuous bilinear form*

$$S_{\mathrm{la}}^{-\infty}(Z\times\mathbf{R}^{n+1})\times C_c^\infty(Z)\to C^\infty(Z). \tag{3.7}$$

Proof. The first integral in (3.5) converges since, by assumption, $\tilde{a}$ is rapidly decreasing as $|\lambda| \to \infty$. In fact this integral is just Fourier transformation in λ. The resulting function

$$M^{(*)} a(x', y, \eta; s) = \int e^{i(-1+1/s)\lambda} \tilde{a}(x's, y, \lambda, \eta) \frac{d\lambda}{s} \tag{3.8}$$

is discussed following (1.3), in view of (3.17) below. Since

$$f(x's, y) \in C^\infty(Z_{(x',y)} \times \overline{\mathbf{R}_s^+})$$

is polynomially bounded:

$$|D_y^\alpha D_{x'}^j D_s^k f(x's, y)| \leqslant C_{\alpha,j,k}(1+|s|)^j,$$

the product $M^{(*)}a \cdot f(x's, y)$ is rapidly decreasing as $|(s, \eta)| \to \infty$, and compactly supported in y. Thus, not only do the remaining integrals converge but they remain convergent after arbitrary differentiation. This defines the bilinear form (3.7) and its joint continuity follows directly from the estimates discussed above.

PROPOSITION 3.9. *The bilinear form* (3.7) *extends to a separately continuous form*

$$S_{\mathrm{la}}^\infty(Z \times \mathbf{R}^{n+1}) \times C_c^\infty(Z) \to C^\infty(Z). \tag{3.10}$$

Proof. Fixing $f \in C_c^\infty(Z)$ it is necessary to show that (3.5) extends by continuity to all lecunary symbols. Since $S_{\mathrm{la}}^{-\infty}(Z \times \mathbf{R}^{n+1}) \subset S_{\mathrm{la}}^\infty(Z \times \mathbf{R}^{n+1})$ is dense it suffices to observe that if $\varrho \in C_c^\infty(\mathbf{R}^+)$ is identically equal to 1 near 1 then the map

$$S_{\mathrm{la}}^\infty(Z \times \mathbf{R}^{n+1}) \ni a \mapsto (1-\varrho(s)) M^{(*)} a \tag{3.11}$$

defined using (3.8), is continuous into the space $C^\infty(Z; \mathcal{S}(\mathbf{R}^{n+1}_{(s,\eta)}))$. That is, $(1-\varrho(s))M^{(*)}a$ is C^∞ and rapidly decreasing with all derivatives as $|(s, \eta)| \to \infty$, uniformly for (x', y) in a compact subset of Z.

Inserting the cutoff ϱ into (3.5) gives

$$\begin{aligned} A^* f(z') = (2\pi)^{-n} \int e^{i(y'-y)\cdot\eta} (1-\varrho(s)) M^{(*)} a f(x's, y)\, ds\, dy\, d\eta \\ + (2\pi)^{-n} \int e^{i(y'-y)\cdot\eta} \varrho(s) M^{(*)} a f(x's, y)\, ds\, dy\, d\eta, \end{aligned} \tag{3.12}$$

where the first term is absolutely convergent as before and in the second the s-integral, really the pairing of a compactly supported distribution with 1, gives a symbol with support in a compactly based cone, in view of (1.8), so the remaining integrals have an oscillatory sense. The continuity of (3.10) is again easily verified.

These propositions provide the desired meaning for the oscillatory integral operator (0.1).

Definition 3.13. If $a \in S^{\infty}_{\mathrm{la}}(Z \times \mathbf{R}^{n+1})$ the operator

$$A: \dot{\mathcal{E}}'(Z) \to \dot{\mathfrak{D}}'(Z),$$

written formally (3.3), is the adjoint of A^* defined by (3.5), (3.7), (3.10).

Of course this definition by duality actually gives a meaning to the oscillatory integral (3.3) in the sense that it can be identified with a separately continuous bilinear mapping:

$$\mathrm{Os}: S^{\infty}_{\mathrm{la}}(Z \times \mathbf{R}^{n+1}) \times \dot{\mathcal{E}}'(Z) \to \dot{\mathfrak{D}}'(Z). \tag{3.14}$$

Both to justify (3.3) and to deduce some properties of the integral operators defined this way it is useful to show that (3.14) can be obtained in a fashion quite similar to (3.10). Consider the formula

$$Ag(z) = (2\pi)^{-n-1} \int e^{i(1-t)\lambda + i(y-y')\cdot\eta} a(x, y, \lambda, \eta)\, g(xt, y')\, d\lambda\, dt\, dy'\, d\eta. \tag{3.15}$$

PROPOSITION 3.16. *If $a \in S^{-\infty}_{\mathrm{la}}(Z \times \mathbf{R}^{n+1})$, $g \in C^{\infty}_{c}(Z)$ the successive integrals in* (3.15) *converge and define a continuous bilinear form*

$$\mathrm{Os}: S^{-\infty}_{\mathrm{la}}(Z \times \mathbf{R}^{n+1}) \times C^{\infty}_{c}(Z) \to C^{\infty}(Z)$$

which is the restriction of (3.14).

Proof. The convergence of (3.15) can be shown by following the proof of Proposition 3.6. Thus,

$$Ma(x, y, \eta; t) = \int e^{i(1-t)\lambda} a(x, y, \lambda, \eta)\, d\lambda$$

is C^{∞} and rapidly decreasing as $|(t, \eta)| \to \infty$, and vanishes with all derivatives at $t=0$. Now,

$$M^{(*)}a(x', y, \eta; s) = \overline{Ma}\left(x's, y, \eta; \frac{1}{s}\right) \cdot s^{-1}. \tag{3.17}$$

Since Ma can be approximated by C^{∞} functions with support in $Z \times \mathbf{R}^{n} \times (0, \infty)$, the change of variables $x = x't$, $s = 1/t$ relating (3.15) to (3.5), (3.8) is justified, so (3.15) does define the restriction of (3.14).

COROLLARY 3.18. *With A as in Definition* 3.13

$$A: C^{\infty}_{c}(Z) \to C^{\infty}(Z).$$

Indeed, the proof of Proposition 3.9 applies to show that the bilinear form of Proposition 3.16 extends to be separately continuous as in (3.10), which shows that Os in (3.14) has this restriction property.

To obtain the totally characteristic pseudodifferential operators on Z from the operators of Definition 3.13 it is only necessary to admit certain extra 'smoothing operators'. These arise naturally from the fact that the existence of an oscillatory integral representation (3.3) is not a strictly local condition on the kernel of an operator.

Definition 3.19. The space $L_b^m(Z)$ of totally characteristic pseudodifferential operators on Z, of order m, consists of those continuous linear maps

$$A: C_c^\infty(Z) \to C^\infty(Z) \tag{3.20}$$

such that $\varrho' A \varrho$ is of the form (3.3) whenever $\varrho, \varrho' \in C_c^\infty(Z)$.

From the discussion above, if $A \in L_b^m(Z)$ then

$$A: \dot{\mathcal{E}}'(Z) \to \dot{\mathcal{D}}'(Z). \tag{3.21}$$

In consequence the adjoints of the operators in $L_b^m(Z)$ also have the properties (3.20), (3.21).

II.4. Kernels and adjoints

The Schwartz kernel theorem shows that a continuous linear operator

$$A': C_c^\infty(\mathbf{R}^{n+1}) \to \mathcal{D}'(\mathbf{R}^{n+1})$$

is represented uniquely by its kernel $k(z, z') \in \mathcal{D}'(\mathbf{R}^{2n+2})$. Any $A \in L_b^\infty(Z)$ defines such an operator by restriction

$$A': C_c^\infty(\mathbf{R}^{n+1}) \xrightarrow{|_Z} C_c^\infty(Z) \xrightarrow{A} \dot{\mathcal{D}}'(Z) \hookrightarrow \mathcal{D}'(\mathbf{R}^{n+1}),$$

so has a kernel in $\mathcal{D}'(\mathbf{R}^{2n+2})$ with support in $Z \times Z$. These pseudodifferential operators will be characterized by their kernels. Such a characterization leads directly to the proof of coordinate invariance and the fact that $A^* \in L_b^\infty(Z)$. First consider a technical result which helps to simplify the discussion.

LEMMA 4.1. *A continuous linear map*

$$A: \dot{\mathcal{E}}'(Z) \to \dot{\mathcal{D}}'(Z)$$

such that $A(C_c^\infty(\mathring{Z})) \subset C^\infty(Z)$ *is determined by the restriction of its Schwartz kernel to the open quarter space* $\mathring{Z} \times \mathring{Z}$.

Proof. Since $C_c^\infty(\mathring{Z}) \hookrightarrow \dot{\mathcal{E}}'(Z)$ is dense the mapping A is determined by its restriction

$$A\colon C_c^\infty(\mathring{Z}) \to \dot{\mathcal{D}}'(Z).$$

By assumption $Au \in C^\infty(Z)$ if $u \in C_c^\infty(\mathring{Z})$ so this map is in turn determined by its restriction to the interior

$$\mathring{A}\colon C_c^\infty(\mathring{Z}) \to \mathcal{D}'(\mathring{Z}).$$

By the Schwartz kernel theorem on $\mathring{Z}$ this is determined by, and determines, the restriction to $\mathring{Z} \times \mathring{Z}$ of the Schwartz kernel of A.

The continuity properties of the bilinear map of Proposition 3.16 show that in $x, x' > 0$ the oscillatory integral formula (3.3) is valid for the Schwartz kernel of an operator A, determined by $a \in S_{1a}^m(Z \times \mathbf{R}^{n+1})$

$$k(z, z') = (2\pi)^{-n-1} \int e^{i(z-z')\cdot\zeta} \tilde{a}(x, y, \zeta)\, d\zeta \tag{4.2}$$

where $z = (x, y)$, $z' = (x', y')$. From (3.15) one obtains, with $s = x'/x$

$$k(z, z') = \frac{1}{x}\, \alpha\!\left(x, y, \frac{x'}{x}, y - y'\right) \tag{4.3}$$

where

$$\alpha(z, y, s, \mu) = (2\pi)^{-n-1} \int e^{i(1-s)\lambda + i\mu\cdot\eta} a(x, y, \lambda, \eta)\, d\lambda\, d\eta. \tag{4.4}$$

Since a is a symbol in (λ, η) this shows that α is a C^∞ function of $z \in Z$, with values in the space of Lagrangian distributions associated to the conormal bundle to $s = 1$, $\mu = 0$. To formalize the use of these singular coordinates, which serve to simplify the form of the distributions occurring as the kernels, we shall define the *stretch* of a manifold with corner.

Let Q be a manifold with corner. That is, each point has a coordinate neighbourhood and coordinate mapping with image which can be taken to be Euclidean space $\mathbf{R}^d$, the half space Z^d or else the quarter space

$$Q^d = \{x \in \mathbf{R}^d;\, x_1 \geqslant 0,\, x_2 \geqslant 0\}.$$

The corresponding subsets of Q are denoted $\mathring{Q}$, the topological interior, $\partial_1 Q$, the part of the boundary of codimension one and $\partial_2 Q$, the corner. We shall show the existence of a new manifold with corner, $\widehat{Q}$, 'Q stretched', which corresponds invariantly to the introduction of polar coordinates near $\partial_2 Q$. In terms of the disjoint union:

$$Q = \mathring{Q} \cup \partial_1 Q \cup \partial_2 Q$$

the underlying set for the stretch of Q is

$$\hat{Q} = \mathring{Q} \cup \partial_1 Q \cup (\partial_2 Q \times [-1, 1]) = \mathring{\hat{Q}} \cup (\partial_1' \hat{Q} \cup \partial_1'' \hat{Q}) \cup \partial_2 \hat{Q} \tag{4.5}$$

where $\partial_1' \hat{Q} = \partial_1 Q$, $\partial_1'' \hat{Q} = \partial_2 Q \times (-1, 1)$ and

$$\partial_2 \hat{Q} = \partial_2 Q \times (\{-1\} \cup \{1\}). \tag{4.6}$$

To make $\hat{Q}$ into a manifold with corner so that (4.5) is the decomposition into open submanifolds of varying dimensions, the natural projection

$$\hat{\pi}: \hat{Q} \to Q$$

can be used to induce coordinates in $\hat{Q}$. Here, $\hat{\pi}$ is the identity on $\mathring{Q} \cup \partial_1 Q$ and

$$\hat{\pi}: \partial_1'' \hat{Q} \cup \partial_2 \hat{Q} \to \partial_2 Q$$

is just projection onto the first factor in $\partial_2 Q \times [-1, 1]$. Now, $\hat{\pi}$ is a bijection onto $\mathring{\hat{Q}} \cup \partial_1' \hat{Q}$ and can be used to induce the corresponding C^∞ structure on that part of $\hat{Q}$. Suppose that

$$x: A \to Q^d \tag{4.7}$$

is a C^∞ coordinate system near some point of the corner of Q. In $\hat{A} = \hat{\pi}^{-1}(A)$ consider the map

$$\hat{x}(p') = \begin{cases} \left(\tfrac{1}{2}(x_1(p) + x_2(p)), \dfrac{x_1(p) - x_2(p)}{x_1(p) + x_2(p)}, x_3(p), \ldots, x_d(p)\right) & \text{if } p' = p \in \mathring{Q} \cup \partial_1 Q \\ (0, r, x_3(p), \ldots, x_d(p)) \quad \text{if } \partial_1'' \hat{Q} \cup \partial_2 \hat{Q} \ni p' = (p, r). \end{cases} \tag{4.8}$$

PROPOSITION 4.9. *If x in (4.7) is a coordinate system then $\hat{x}$ in (4.8) is a bijection*

$$\hat{x}: \hat{A} \leftrightarrow \{(w, r, x_3, \ldots, x_d);\ w \geq 0,\ |r| \leq 1,\ (x_3, \ldots, x_d) \in \mathbf{R}^{d-2}\}.$$

These maps, together with the C^∞ structure on $\mathring{\hat{Q}} \cup \partial_1' \hat{Q}$, make $\hat{Q}$ a C^∞ manifold with corner such that $\hat{\pi}: \hat{Q} \to Q$ is C^∞.

Proof. Clearly $\hat{x}$ is a bijection onto the manifold with corner given by the half strip $w \geq 0$, $|r| \leq 1$. Thus, to prove the proposition, it is only necessary to show the consistency of the various coordinate systems on $\hat{Q}$. Suppose that $y: A' \to Q^d$ is another coordinate system with $A' \cap A \cap \partial_2 Q \neq \varnothing$. Write

$$\hat{y} = (w', r', y_3, \ldots, y_d)$$

for the corresponding map on $\hat{A}'$. In the first instance we can suppose that

$$y_1 = x_1 a(x), \quad y_2 = x_2 b(x) \tag{4.10}$$

with $a, b > 0$ on $A \cap A'$. Then, on $\mathring{Q} \cup \partial_1 Q$,

$$w' = \tfrac{1}{2}(y_1 + y_2) = \tfrac{1}{2}(x_1 a(x) + x_2 b(x)) \tag{4.11}$$

and since $x_1, x_2, \ldots, x_d$ are C^∞ functions on $\hat{A}$, this shows that w' is C^∞ in terms of the coordinates $\hat{x}$. Similarly the functions $y_3, \ldots, y_d$ are C^∞ and

$$r' = \frac{y_1 - y_2}{y_1 + y_2} = \frac{(1+r)\,a - (1-r)\,b}{(1+r)\,a + (1-r)\,b} \tag{4.12}$$

that is,

$$\frac{1+r'}{1-r'} = \frac{a}{b} \cdot \frac{1+r}{1-r} \tag{4.13}$$

so r' is also C^∞ in terms of the coordinates $\hat{x}$.

If the assumption (4.10) does not hold it can be ensured by first exchanging the variables x_1, x_2. This latter transformation has a C^∞ lift to the coordinates $\hat{x}$, namely $r \to -r$. Compatibility between the coordinates $\hat{x}$ and those in $\mathring{Q} \cup \partial_1 Q$ is obvious so the proposition is proved.

Now, if M, N are C^∞ manifolds with boundary then $N \times M$ is a manifold with corner and the *stretched product* $N \mathbin{\hat{\times}} M$ is defined as

$$N \mathbin{\hat{\times}} M = \widehat{N \times M}.$$

Clearly $N \mathbin{\hat{\times}} M$ and $M \mathbin{\hat{\times}} N$ are isomorphic in a natural way.

LEMMA 4.14. *The part $\partial_1'' \hat{Q} \subset \hat{Q}$ of the boundary of Q stretched is a fibre bundle over $\partial_2 Q$ with fibre $(-1, 1)$ and structure group the real fractional linear transformations preserving it. The natural subbundle $F \subset T^*(\partial_1'' \hat{Q})$, consisting of the conormals to the fibres is itself a fibre bundle*

$$\begin{array}{c} F \\ \downarrow \\ T^* \partial_2 Q \end{array} \tag{4.15}$$

with the same fibre and structure group.

Proof. That $\partial_1'' \hat{Q} = \partial_2 Q \times (-1, 1)$ is a fibre bundle follows from the change of coordinate formulae (4.12), (4.13) since $x_1 = x_2 = w = 0$ on $\partial_1'' \hat{Q}$ and by (4.12) the variable r in $(-1, 1)$ undergoes a fractional linear transformation. Note that the transformations (4.12) together with $r \to -r$, which comes from exchanging the roles of x_1 and x_2, give the group of all fractional linear transformations preserving $(-1, 1)$. The rest of the lemma is obvious.

Note that the transformation $r \to -r$ on the fibres of $\partial_1'' \hat{Q}$ comes only from the exchange of the two local boundary components of Q near $\partial_2 Q$. Without this transformation the map $u = (1+r)/(1-r)$ reduces $(-1, 1)$ to $\mathbf{R}^+$ with its self action. Thus we obtain:

COROLLARY 4.16. *If M, N are manifolds with boundary then the part $\partial_1''(N \hat{\times} M) \subset N \hat{\times} M$ of the boundary of the stretched product is a principal $\mathbf{R}^+$ bundle over $\partial N \times \partial M$; the bundle of conormals, F, is a principal $\mathbf{R}^+$ bundle over $T^*(\partial N \times \partial M)$.*

Further specializing we consider the stretched product $Z \hat{\times} Z$. This has the natural coordinates $w = \frac{1}{2}(x + x')$, $r = (x - x')/(x + x')$, y, y' identifying it with the half strip

$$Z \hat{\times} Z = \{(w, r, y, y') \in \mathbf{R}^{2n+2};\ w \geq 0,\ |r| \leq 1\}.$$

Observe that distributions of the form (4.4) can be transferred to $Z \hat{\times} Z$ by setting

$$s = \frac{1-r}{1+r}, \quad x = (1+r)\,w. \tag{4.17}$$

Let $\mathcal{K}^m(Z \hat{\times} Z) \subset \mathcal{D}'(Z \hat{\times} Z)$ be the space of those distributions α with the following properties:

$$\alpha \text{ is singular only at } r = 0,\ y = y'. \tag{4.18}$$

$$\begin{gathered}\alpha \text{ is } C^\infty \text{ up to the boundary (and corner) of } Z \hat{\times} Z \text{ away from } r = 0,\ y = y' \\ \text{and vanishes to all orders at the part } \partial_1'(Z \hat{\times} Z).\end{gathered} \tag{4.19}$$

$$\begin{gathered}\text{Near } r = 0,\ \alpha \text{ is the restriction to } w \geq 0 \text{ of a Lagrangian distribution} \\ \text{of order } m \text{ associated to the submanifold } r = 0,\ y = y'.\end{gathered} \tag{4.20}$$

Since these distributions are regular in the normal variable there is a natural inclusion $\mathcal{K}^m(Z \hat{\times} Z) \subset \dot{\mathcal{D}}'(Z \hat{\times} Z)$ as well.

PROPOSITION 4.21. *If $a \in S^m_{1a}(Z \times \mathbf{R}^{n+1})$ the distribution given by (4.4) is in $\mathcal{K}^m(Z \hat{\times} Z)$. Conversely, each $\alpha \in \mathcal{K}^m(Z \hat{\times} Z)$ can be represented in the form (4.4), with $a \in S^m_{1a}$ in any region $(w, y, y') \in K \subset\subset \overline{\mathbf{R}^+} \times \mathbf{R}^{2n}$ of $Z \hat{\times} Z$.*

Proof. For a distribution of the form (4.4) the condition (4.18) and its refinement (4.20) are immediate, essentially from the definition of a Lagrangian distribution. Similarly the fact that α is the Fourier transform in the s variable of a symbol shows that it is rapidly decreasing with all derivatives as $s \to \infty$; this gives the part of (4.19) corresponding to the boundary component $r=-1$, in view of (4.17). The other part of (4.19) is precisely the lacunary condition on the symbol a, namely that α vanishes identically in $s<0$, i.e. $r<-1$.

The converse part of the proposition is similar. Given $\alpha\in\mathcal{K}^m(Z\hat{\times}Z)$ and $\varrho\in C_c^\infty(\overline{\mathbf{R}^+}\times\mathbf{R}^{2n})$ set

$$a(z,\lambda,\eta)=\int e^{i(s-1)\lambda-i\mu\cdot\eta}\varrho(x(1-r)^{-1},r,y,y-\mu)\,\alpha(x(1-r)^{-1},r,y,y-\mu)\,ds\,d\mu, \quad (4.21)$$

where $r=(1-s)/(1+s)$. Note that x is bounded from above on the support of ϱ in (4.21) so the integrand is a C^∞ function of r, away from $r=0$, vanishing to infinite order at $r=-1, 1$. Thus, after the substitution for s the integrand is rapidly decreasing as $s\to\infty$, vanishes to all orders at $s=0$ and is Lagrangian at $s=1$, $\mu=0$ with compact support in y,μ. The Fourier transform (4.21) is therefore a symbol satisfying the lacunary condition. Clearly (4.4) now gives a representation of α in any region where $\varrho=1$, proving the proposition.

The fact that the space of kernels $\mathcal{K}^m(Z\hat{\times}Z)$ is invariant under all C^∞ diffeomorphisms of the manifold with corner $Z\hat{\times}Z$ allows us to use the following direct definition of a pseudodifferential on any manifold with boundary M.

Definition 4.22. Let E, F be vector bundles over M. A continuous linear map

$$A\colon C_c^\infty(M,E)\to C^\infty(M,F) \quad (4.23)$$

is an element of the space $L_b^m(M;E,F)$ of totally characteristic pseudodifferential operators if $\varrho' A\varrho$ is a matrix of elements of $L_b^m(Z)$ whenever $\varrho',\varrho\in C_c^\infty(M)$ have supports in coordinate neighbourhoods over which E, F are trivial.

The invariant symbol calculus for these operators will be examined in section six. First we examine in more detail the space $L_b^m(Z)$. Definition 3.19 is consistent with Definition 4.22 above and we note that the elements of $L_b^m(Z)$ are in one to one correspondence with those of $\mathcal{K}^m(Z\hat{\times}Z)$ via (4.3), (4.4), (4.17). Using this isomorphism it follows that

$$L_b^{-\infty}(Z)=\bigcap_m L_b^m(Z) \quad (4.24)$$

corresponds via (4.3) to the space $\mathcal{K}^{-\infty}(Z\hat{\times}Z)\subset C^\infty(Z\hat{\times}Z)$ consisting of those functions vanishing to all orders at $r=\pm 1$.

PROPOSITION 4.25. *Each $A\in L_b^m(Z)$ can be written in the form $A=A_1+A_2$ where $A_2\in L_b^{-\infty}(Z)$ and A_1 is of the form (3.3) and properly supported. Associating to A_1 the symbol $a\in S_{\mathrm{la}}^m(Z\times\mathbf{R}^{n+1})$ defines the full symbol mapping*

$$\sigma_f\colon L_b^m(Z)\to S^m(Z\times\mathbf{R}^{n+1})/S^{-\infty}(Z\times\mathbf{R}^{n+1}), \quad (4.26)$$

which has the property

$$\ker(\sigma_f)=L_b^{-\infty}(Z). \quad (4.27)$$

Proof. Suppose $\varrho \in C_c^\infty(\mathbf{R}^{n+1})$ is identically equal to 1 near the origin. Then if A has kernel $k(z, z')$ the distribution

$$k_1(z, z') = \varrho(x-x', y-y')\, k(z, z') \tag{4.28}$$

can be taken as the kernel of A_1. That $A_1 \in L_b^m(Z)$ follows from the fact that the corresponding distribution on $Z \hat{\times} Z$, is

$$\alpha_1(w, r, y, y') = \varrho(2wr, y-y')\, \alpha(w, r, y, y') \in \mathcal{K}^m(Z \hat{\times} Z).$$

If $\varrho = 1$ in $|x-x'| \leqslant \varepsilon$, $|y-y'| \leqslant \varepsilon$ then $\alpha_1 = \alpha$ in $|y-y'| \leqslant \varepsilon$, $|2wr| \leqslant \varepsilon$ which is a neighbourhood of $r=0$, $y=y'$. Thus $\alpha_2 = \alpha - \alpha_1$ is C^∞, i.e. $A_2 \in L_b^{-\infty}(Z)$. Similarly, the fact that (4.26) is well-defined is just the statement that this operator $A_1 \in L_b^{-\infty}(Z)$ precisely when α_1 is C^∞, i.e. $a \in S_{\mathrm{la}}^{-\infty}$. Note that in defining (4.26) the isomorphism (1.6) has been used.

PROPOSITION 4.29. *If* $A \in L_b^m(Z)$ *then* $A^* \in L_b^m(Z)$.

Proof. If A has Schwartz kernel $k(z, z')$ then the kernel of A^* is

$$k^*(z, z') = \bar{k}(z', z). \tag{4.30}$$

The distribution α^* defined on $Z \hat{\times} Z$ by k^* is therefore

$$\alpha^*(w, r, y, y') = s^{-1} \bar{\alpha}(w, -r, y', y) \tag{4.31}$$

where (4.7) holds. Since $\bar{\alpha}$ vanishes rapidly at $r = \pm 1$ this is again an element of $\mathcal{K}^m(Z \hat{\times} Z)$, providing the proposition.

Now, Proposition 4.29 shows that $\sigma_f(A^*)$ is well-defined for all $A \in L_b^m(Z)$ and is determined by $\sigma_f(A)$. To obtain an explicit formula it is enough to write α^*, modulo C^∞, in the form (4.4), starting from the corresponding formula for α. This is just an exercise using the lemma of stationary phase and a short calculation gives:

$$\sigma_f(A^*) \sim \sum_{\alpha, k} \frac{(-i)^{|\alpha|+k}}{\alpha!\, k!} \prod_{j=0}^{k} (x\partial_x + \lambda \partial_\lambda + j)\, \partial_\lambda^k \partial_\eta^\alpha \partial_y^\alpha \sigma_f(A). \tag{4.32}$$

This is easily seen to be the usual formula for the full symbol of the adjoint of a pseudodifferential in the open region if it is used to compute

$$\tilde{\sigma}(A^*) = \sigma_f(A^*)(x, y, x\xi, \eta)$$

in terms of $\tilde{\sigma}_f(A)$. Note that all terms in (4.32) with $k + |\alpha| = N$ are of order $m - N$.

II.5. Boundary values

In order to consider in more detail the mapping properties of the space $L_b^\infty(Z)$ of totally characteristic pseudodifferential operators it is useful to extend the oscillatory integral representation (3.15). To do this consider the transformed distribution

$$u_t(z) = u\left(\frac{x}{t}, y\right), \quad t > 0. \tag{5.1}$$

LEMMA 5.2. *If $u \in \dot{\mathcal{E}}'(Z)$ then* (5.1) *defines a C^∞ map u_t: $\mathbf{R}^+ \to \dot{\mathcal{E}}'(Z)$. If $u \in \dot{H}_c^s(Z)$ then $u_t \in \dot{H}_c^s(Z)$ for each $t>0$ and*

$$\|u_t\|_{H^s} \leqslant C(t^{\frac{1}{2}-s} + t^{\frac{1}{2}})\|u\|_{H^s}. \tag{5.3}$$

Proof. For $u \in \dot{H}^s(Z)$ the Sobolev norm is

$$\|u\|_{H^s}^2 = \int (1 + |\zeta|^2)^s |\hat{u}(\zeta)|^2 d\zeta.$$

Since $\hat{u}_t(\zeta) = \hat{u}_t(\xi, \eta) = t\hat{u}(t\xi, \eta)$,

$$\|u_t\|_{H^s}^2 = t^2 \int (1 + |\xi|^2 + |\eta|^2)^s |\hat{u}(t\xi, \eta)|^2 d\xi\, d\eta = t \int \left(1 + \frac{r^2}{t^2} + |\eta|^2\right)^s |\hat{u}(r, \eta)|^2 dr\, d\eta. \tag{5.4}$$

Using the bounds

$$\left(1 + \left(\frac{r}{t}\right)^2 + |\eta|^2\right)^s \leqslant \begin{cases} (1 + r^2 + |\eta|^2)^s, & s \geqslant 0, t \geqslant 1 \quad \text{or } s \leqslant 0, t \leqslant 1 \\ t^{-2s}(1 + r^2 + |\eta|^2)^s, & s \geqslant 0, t \leqslant 1 \quad \text{or } s \leqslant 0, t \geqslant 1 \end{cases}$$

(5.3) follows easily from (5.4).

Since $D_t u_t = (ix/t^2)(D_x u)(x/t, y)$ it follows inductively from (5.3) that for any $j \in \mathbf{N}$,

$$\|D_t^j u_t\|_{H^{s-j}} \leqslant C_j(t^{\frac{1}{2}-s} + t^{\frac{1}{2}-j}) \|(xD_x)^j u\|_{H^{s-j}}.$$

Summing these inequalities over j we conclude that for any $k \in \mathbf{N}$, $u \in \dot{H}_c^s(Z)$,

$$\sum_{0 \leqslant j \leqslant k} \|D_t^j u_t\|_{H^{s-k}} \leqslant C_k(t^{\frac{1}{2}-s} + t^{\frac{1}{2}-k} + t^{\frac{1}{2}})\|u\|_{H^s}. \tag{5.5}$$

Now, let us examine the representation (3.15), carrying out the λ-integral as Fourier transform:

$$Ag(z) = (2\pi)^{-n-1} \int e^{i(y-y')\cdot\eta} Ma(z, \eta; t)\, g_{1/t}(x, y')\, dy'\, d\eta\, dt, \tag{5.6}$$

where Ma is defined in (1.3). Choose $\varrho \in C_c^\infty(\mathbf{R}^+)$ with $\varrho(t) = 0$ in $|t-1| > \frac{1}{2}$, $\varrho(t) = 1$ in $|t-1| < \frac{1}{4}$. Then, for $a \in S_{1a}^{-\infty}$, $g \in C_c^\infty(Z)$

$$Ag = A_1 g + A_2 g \tag{5.7}$$

where, using the expansion of Lemma 1.10 and integration by parts,

$$A_1 g = (2\pi)^{-n-1} \sum_{j\leq R} \int e^{i(y-y')\cdot\eta} \beta_{j,r}(z,\eta;t)\,(-D_t)^j\,(\varrho(t)\,g_{1/t}(x,y'))\,dy'\,d\eta\,dt, \tag{5.8}$$

and

$$A_2 g = (2\pi)^{-n-1} \int e^{i(y-y')\cdot\eta} Ma(z,\eta;t)\,g_{1/t}(x,y')\,(1-\varrho(t))\,dy'\,d\eta\,dt. \tag{5.9}$$

PROPOSITION 5.10. *If $a\in S^m_{\mathrm{la}}(Z\times\mathbf{R}^{1+n})$ and $g\in\dot{\mathcal{E}}'(Z)$ then (5.7) holds if (5.9) is interpreted as Fourier transform in y' followed by integration in η, t and (5.8), with r large enough, as an oscillatory integral in (y',η) and distributional pairing in t.*

Proof. In view of (1.7), (5.5) and Lemma 1.10 it is clear that both terms (5.8), (5.9) make sense as stated and are separately continuous in a and g provided that r is large enough. Thus the validity of (5.7), (5.8) and (5.9) follows by continuity.

There are several direct consequences of this representation. First the uniform continuity of the operators in L^m_b; these results are by no means optimal but are very useful in establishing finer continuity properties.

COROLLARY 5.11. *For each pair s, $m\in\mathbf{R}$ there exists $p=p(s,m)$ such that every $A\in L^m_b(Z)$ is a continuous linear map $\dot{H}^s_c(Z)\to H^p_{\mathrm{loc}}(Z)$.*

Since the operators in L^∞_b are modelled on the totally characteristic differential operators, $\mathrm{Diff}^m_b(Z)$, it is to be expected that they have similar mapping properties. To see this we need an elementary composition result.

LEMMA 5.12. *For any $m\in\mathbf{R}$, $k\in\mathbf{N}$,*

$$\mathrm{Diff}^k_b(Z)\cdot L^m_b(Z)\subset \mathrm{span}\,(L^m_b(Z)\cdot\mathrm{Diff}^k_b(Z)),\quad \mathrm{Diff}^k_b(Z)\cdot L^{-\infty}_b(Z)\subset L^{-\infty}_b(Z). \tag{5.13}$$

Proof. To obtain (5.13) it suffices to prove the individual decompositions: If $A\in L^m_b(Z)$ there exist A'_j, $B'_j\,L^m_b(Z)$, $j=0,\ldots,n$ such that

$$\begin{cases} xD_x\cdot A = A'_0\cdot xD_x + B'_0 \\ D_{y_j}\cdot A = A'_j\cdot D_{y_j}+B'_j. \end{cases} \tag{5.14}$$

The continuity properties of the operators in $L^m_b(Z)$ show that it is enough to demonstrate (5.14) on the domain $\dot{C}^\infty_c(Z)$ where (3.3) converges absolutely with all derivatives. Then one obtains (5.14) immediately with $A'_j=A$, $j=0,\ldots,n$ and B'_j given by an integral of the form (3.3) with symbol

$$b'_0 = xD_x a(x,y,y',\lambda,\eta),\quad b'_j=(D_{y_j}-D_{y'_j})a(x,y,y',\lambda,\eta)$$

when A is of this form with symbol a. Since (5.13) is local on the kernels this proves the first part of it; the other part is similar.

Since the remainder term in the first identity (5.14) has a factor x the same argument establishes

LEMMA 5.15. *For any* $m \in \mathbf{R}$, $k \in \mathbf{N}$, $\operatorname{Diff}^k(Z) \cdot L_b^m(Z) \subset \operatorname{span}(L_b^m(Z) \cdot \operatorname{Diff}^k(Z))$.

The analogue of the second part of (5.13) is not valid.

PROPOSITION 5.16. *If* $A \in L_b^m(Z)$ *then*

$$A\colon \dot{\mathcal{A}}_c(Z) \to \dot{\mathcal{A}}(Z). \tag{5.17}$$

If $A \in L_b^{-\infty}(Z)$ *then*

$$A\colon \dot{\mathcal{E}}'(Z) \to \dot{\mathcal{A}}(Z). \tag{5.18}$$

Proof. From the definition (I.1.10) of $\dot{\mathcal{A}}$ it suffices to show that if $u \in \dot{\mathcal{A}}_c(Z)$ and $\varrho \in C_c^\infty(Z)$ then there exists $p \in \mathbf{R}$ such that

$$P(\varrho A u) \in \dot{H}^p(Z), \quad \forall P \in \operatorname{Diff}_b(Z). \tag{5.19}$$

According to Lemma 5.12 we can always write

$$P(\varrho A u) = \sum_{\text{finite}} \varrho' A_k \cdot P_k u \tag{5.20}$$

with $P_k \in \operatorname{Diff}_b(Z)$, $A_k \in L_b^m(Z)$ and $\varrho' \in C_c^\infty(Z)$ fixed (with $\varrho' = 1$ on $\operatorname{supp} \varrho$). By assumption $u \in \dot{\mathcal{A}}_c(Z)$ so there exists $s \in \mathbf{R}$ such that $P_k u \in \dot{H}_c^s(Z)$, for all k. Using Corollary 5.11, (5.20) implies (5.19) with $p = N(s, m)$. This proves (5.17). Similarly, one obtains (5.18) by using the second part of (5.13) and Corollary 5.11 to prove (5.19) for every $u \in \dot{\mathcal{E}}'(Z)$.

For special distributions the representations (5.7), (5.8), (5.9) give simple results. Of particular importance is the action of the pseudodifferential operators on distributions supported at the boundary. If

$$f(x, y) = \delta^{(k)}(x) \otimes f_k(y), \quad f_k \in \mathcal{E}'(\mathbf{R}^n)$$

then $f_t(x, y) = t^{-1-k} f(x, y)$. Thus,

$$Af = (2\pi)^{-n} \sum_{j \leqslant k} \int e^{i(y-y')\cdot\eta} \alpha_{k,j}(y, \eta) f_k(y')\, dy'\, d\eta\, \delta^{(j)}(x) \tag{5.21}$$

where, if A is of the form (3.3) then the symbols $\alpha_{k,j}$ are given by

$$\frac{1}{2\pi} \int Ma(z, \eta; t)\, t^{-k-1} \delta^{(k)}(x)\, dt = \sum_{j \leqslant k} \alpha_{k,j}(y, \eta)\, \delta^{(j)}(x). \tag{5.22}$$

PROPOSITION 5.23. *If* $A \in L_b^m(Z)$ *then*

$$A\colon \mathcal{A}_c'(Z) \to \mathcal{A}'(Z). \tag{5.24}$$

There is a well-defined map

$$(\ )_\partial\colon L_b^m(Z) \to L^m(\mathbf{R}^n) \tag{5.25}$$

such that for any $u \in \mathcal{A}_c'(Z)$, $A \in L_b^m(Z)$,

$$Au|_{x=0} = A_\partial(u|_{x=0}). \tag{5.26}$$

Proof. Since $\mathcal{A}'(Z) \subset \dot{\mathfrak{D}}'(Z)$ consists of those distributions which extend by continuity from $C_c^\infty(Z)$ to $\dot{\mathcal{A}}_c(Z)$ the first part of the proposition is simply the dual statement to (5.17). Then

$$\langle u|_{x=0}, \varphi\rangle = \langle u, \delta(x)\otimes\varphi(y)\rangle, \quad \varphi \in C_c^\infty(\mathbf{R}^n)$$

and we conclude directly form (5.21) that

$$\langle Au|_{x=0}, \varphi\rangle = \langle Au, \delta\otimes\varphi\rangle = \langle u, A^*(\delta\otimes\varphi)\rangle = \langle A_\partial(u|_{x=0}), \varphi\rangle.$$

This proves (5.25), (5.26) with the explicit formula

$$A_\partial u = (2\pi)^{-n-1}\int e^{i(y-y')\cdot\eta}\left(\int M(0, y, \eta; t)\,dt\right)u(y')\,dy'\,d\eta \tag{5.27}$$

given as an oscillatory integral. It should be noted that the terms in the kernel coming from the difference between (3.3) and the Definition 3.19 can be ignored.

Notice that, directly from (1.3), if A is given by (3.3) then

$$A_\partial u = (2\pi)^{-n}\int e^{i(y-y')\cdot\eta}a(0, y, 0, \eta)\,u(y')\,dy'\,d\eta. \tag{5.28}$$

From (5.21) it follows that

$$A\colon \dot{\mathcal{E}}'(Z, \partial Z) \to \dot{\mathcal{D}}'(Z, \partial Z). \tag{5.29}$$

In particular this means that the totally characteristic pseudodifferential operators act in a natural way on extendible distributions.

PROPOSITION 5.30. *If* $A \in L_b^m(Z)$ *then* $A\colon \mathcal{E}'(Z) \to \mathcal{D}'(Z)$.

Proof. If $v \in \mathcal{E}'(Z)$ there exists $u \in \dot{\mathcal{E}}'(Z)$ with $v = u|_{\mathring{Z}}$. According to (I.1.1) u is well-defined up to an element of $\dot{\mathcal{E}}'(Z, \partial Z)$. Then one can set $Av = Au|_{\mathring{Z}}$, unambiguously because of (5.29).

II.6. Symbols and residual operators

Let $L_b^m(M)$ be the space of linear operators introduced in Definition 3.19:

$$A: \dot{\mathcal{E}}'(M) \to \dot{\mathcal{D}}'(M)$$

with kernels in $\mathcal{K}^m(M)$. Following (4.26) we define

$$\tilde{S}^m(T^*M) \subset C^\infty(T^*M)$$

to be the space of C^∞ functions on T^*M which are of the form $\pi^*a = \tilde{a}$ with π: $T^*M \to \tilde{T}^*M$ the map of (I.2.4) and $a \in S^m(\tilde{T}^*M)$. The results of Section 4 now give:

PROPOSITION 6.1. *The principal symbol map*

$$\tilde{\sigma}_m: L_b^m(M)/L_b^{m-1}(M) \to \tilde{S}^m(T^*M)/\tilde{S}^{m-1}(T^*M) \simeq S^m(\tilde{T}^*M)/S^{m-1}(\tilde{T}^*M),$$

is an isomorphism well-defined by projection from (4.26) *in any local coordinates. If* $A \in L_b^m(M)$ *we denote by* $\sigma_m(A)$ *the image of the principal symbol in* $S^m(\tilde{T}^*M)/S^{m-1}(\tilde{T}^*M)$.

Similarly the results of Section 5 can be restated invariantly. If $A \in L_b^m(M)$ then

$$A: \dot{\mathcal{A}}_c(M) \to \dot{\mathcal{A}}(M) \tag{6.2}$$

and as $L_b^m(M)$ is closed under the adjoint operation

$$A: \mathcal{A}_c'(M) \to \mathcal{A}'(M). \tag{6.3}$$

Using the trace map of Section I.3 and Proposition 5.23

$$R(Au) = A_0(Ru) \tag{6.4}$$

for every $u \in \mathcal{A}_c'(M)$, with $A_0 \in L^m(\partial M)$,

$$\sigma_m(A_0) = \sigma_m(A) \,|\, \tilde{T}^*\partial M. \tag{6.5}$$

Now, we consider the residual operators, in $L_b^{-\infty}(M)$. From (5.18) it is clear that

$$A: \dot{\mathcal{E}}'(M) \to \dot{\mathcal{A}}(M) \quad \text{if } A \in L_b^{-\infty}(M). \tag{6.6}$$

It is possible to define symbol maps which allow one to examine the degree to which such an operator differs from a smoothing operator. Returning to local coordinates we see from (4.3) that the function

$$\alpha_0(y, y', s) = \alpha(0, y, s, y-y') \in C^\infty(\mathbf{R}^{2n}; \mathcal{S}(\mathbf{R})) \tag{6.6}$$

can be associated with the kernel k. Here we extend α as zero into $s \leqslant 0$, using the fact that α vanishes to infinite order at $s = 0$.

PROPOSITION 6.7. *If $A \in L_b^{-\infty}(Z)$ the function α_0 is well-defined and the residual symbol mapping*

$$\sigma_{-\infty,0}\colon L_b^{-\infty}(Z) \to C^\infty(\mathbf{R}^{2n}; \dot{S}([0,\infty)), \quad A \mapsto \alpha_0,$$

is surjective.

Proof. Since, in terms of the kernel $k(z, z')$,

$$\alpha_0(y, y', t) = \lim_{x\downarrow 0} x \cdot k(x, y, xt, y') \tag{6.8}$$

it is clear that α_0 is well-defined. Moreover, the distribution k defined by (4.3) with α replaced by α_0, which is independent of x, is the kernel of some element of $L_b^{-\infty}(Z)$, so $\sigma_{-\infty,0}$ is surjective.

Consider the behaviour of the kernels under coordinate changes. Let $N^* = N^*\partial Z$ be the conormal bundle to the boundary of Z and on the product $\partial Z \times \partial Z$ consider the line bundle

$$S = (\pi_1^* N)^{-1} \otimes (\pi_2^* N)$$

where N is the dual to N^*; thus $N = T_{\partial Z} Z / T\partial Z$ is the normal bundle.

LEMMA 6.9. *The $\mathbf{R}^+$-line bundle S^+ is canonically identified with the line bundle $\partial_1''(Z \hat{\times} Z)$ of Corollary 4.16 by $(dx)^{-1} \otimes dx' \cong s$.*

Proof. This is just the identification of a vector space and its cotangent space, $x \cong dx$ and is coordinate independent because of (4.13).

If we wish to define this first residual symbol map on $L_b^{-\infty}(M)$ we need only take account of the density terms in the kernel. Thus, $k \in \mathcal{K}^{-\infty}(M)$ transforms as a density on the second M factor. We therefore have:

PROPOSITION 6.10. *The map*

$$\sigma_{-\infty,0}\colon L_b^{-\infty}(M) \to \dot{S}(\overline{S^+; B}),$$

where $S = \partial_1''(M \hat{\times} M)$ over $\partial M \times \partial M$ and $B = T_f(S) \otimes (\pi_2 \circ p)^*(\Omega_{\partial M})$, is well defined and surjective, with $T_f(S)$ the cotangent bundle to the fibres.

Suppose we are given two sections $\gamma_1, \gamma_2 \in S(\overline{S^+; B})$ one at least properly supported in $\partial M \times \partial M$. Identifying S^+ with the associated R^+ bundle we define composition as follows, in local coordinates. If

$$\gamma_i = r_i(y, y', s)\, ds\, |dy'| \tag{6.11}$$

we set

$$\gamma_1 \# \gamma_2 = \left(\int r_1(y, y', s)\, r_2\left(y, y'', \frac{t}{s}\right) \frac{ds}{s}\, dy' \right) dt\, |dy''|. \tag{6.12}$$

It is straightforward to verify that this is a section of B over $\overline{S^+}$ and is rapidly decreasing at 0 and ∞. In fact this also follows from

PROPOSITION 6.13. *If $A, B \in L_b^{-\infty}(M)$ and one at least is properly supported then*

$$\sigma_{-\infty,0}(A \circ B) = \sigma_{-\infty,0}(A) \# \sigma_{-\infty,0}(B).$$

Proof. If $k(x, y, x', y') = (1/x)\alpha(x, y, x'/x, y')$ is the kernel of A and $l(x', y', x'', y'') = (1/x')\beta(x', y', x''/x', y'')$ is the kernel of B then the kernel in (x, y, x'', y'') of $A \circ B$ is of the form $(1/x)\gamma(x, y, x''/x, y'')$ with

$$\gamma(x, y, t, y'') = \int \alpha(x, y, s, y') \frac{1}{x'} \beta\left(x', y', \frac{x''}{x'}, y''\right) dx'\, dy'$$

where $s = x'/x$, $t = x''/x$. Thus,

$$\gamma(x, y, t, y'') = \int \alpha(x, y, s, y') \beta\left(xs, y', \frac{t}{s}, y''\right) \frac{ds}{s} dy'. \tag{6.14}$$

Taking the limit as $x \downarrow 0$ we obtain (6.12), proving the proposition.

Consider the subspace of $L_b^{-\infty}(M)$ defined by the vanishing of this first residual symbol. In local coordinates the kernel, in $L_b^{-\infty}(Z)$ is of the form

$$k(z, z') = \frac{1}{x} \alpha\left(x, y, \frac{x'}{x}, y'\right)$$

with $\alpha(0, y, s, y') \equiv 0$. We can then consider higher terms in the Taylor series:

$$\alpha_{-j}(y, y', s) = (j!)^{-1}(\partial_x^j \alpha)(0, y, s, y').$$

If $\alpha_{-j} \equiv 0$ for $j < p$ then

$$\alpha_{-p}(y, y', s) = \lim_{x \downarrow 0} x^{1-p} k(x, y, xs, y'), \quad p \geqslant 0. \tag{6.15}$$

For each $p \geqslant 0$ define the bundles

$$B_p = B_{p-1} \otimes ((\pi_1 \circ p)^* N^* \partial M)^{-1}, \quad p \geqslant 1.$$

Then, if $\alpha_{-j} \equiv 0$ for all $j < p$ (6.15) defines a coordinate-free map into $\mathcal{S}(\overline{S^+}; B_p)$. Differentiating (6.14) with respect to x leads to the following natural definition:

$$\gamma_1 \# \gamma_2 = \int r_1(y, y', s)\, r_2\left(y', y'', \frac{t}{s}\right) s^{k_2} \frac{ds}{s} dy' \tag{6.16}$$

when $\gamma_i = r_i(y, y', s)(dx)^{-p} ds\, |dy'| \in \mathcal{S}(\overline{S^+}; B_{k_i})$.

THEOREM 6.17. *The spaces $L_b^{-\infty,p}(M)$, fixed by the condition that $\alpha_{-j}\equiv 0$ for $j<p$ in any (one) local coordinate system, are invariantly defined as is the map*

$$\sigma_{-\infty,-p}\colon L_b^{-\infty,-p}(M)\to S(\overline{S^+};\, B_p),\quad p\geqslant 0.$$

If $A\in L_b^{-\infty,-p_1}(M)$, $B\in L_b^{-\infty,-p_2}(M)$ and one is properly supported then

$$A\circ B\in L_b^{-\infty,-p_1-p_2}(M),\quad \sigma_{-\infty,-p_1-p_2}(A\circ B)=\sigma_{-\infty,-p_1}(A)\#\sigma_{-\infty,-p_2}(B)$$

using the product (6.16).

Proof. This follows directly from (6.14).

PROPOSITION 6.18. *If $A\in L_b^{-\infty}(M)$ then A has a kernel $k\in\dot{C}^\infty(Z\times Z)$ if, and only if,*

$$A\in L_b^{-\infty,-\infty}(M)=\bigcap_{p\geqslant 0} L_b^{-\infty,-p}(M),$$

i.e. if and only if all the successive residual symbols $\sigma_{-\infty,-p}$ vanish identically.

Proof. Of course, this result is really local near points of $\partial M\times\partial M$. In local coordinates the vanishing of all the $\sigma_{-\infty,-p}(A)$ means that

$$k(z,z')=\frac{1}{x}\,\alpha\left(x,y,\frac{x'}{x},y'\right)$$

with α vanishing to all orders at $x=0$. Clearly then $k\in\dot{C}^\infty(Z\times Z)$. Conversely if $\sigma_{-\infty,p}\neq 0$ at some point $(\bar{y},\bar{y}',\bar{s})$ for some p, where $\sigma_{-\infty,-j}\equiv 0$ for $j<p$ consider

$$xA(\delta(x'-r)\otimes\delta(y'-\bar{y}'))=\alpha\left(x,y,\frac{r}{x},\bar{y}'\right).\tag{6.19}$$

If this is C^∞ in $x\geqslant 0$ we can set $y=\bar{y}$, $x=r/\bar{s}$, since certainly $\bar{s}\neq 0$, after differentiating (6.19) p times. Then we have

$$\lim_{r\to 0}\partial_x^p\alpha\left(x,\bar{y},\frac{r}{x},\bar{y}'\right)\bigg|_{x=r/\bar{s}}=\lim_{r\to 0}\,(\partial_x^p\alpha)\,(x,\bar{y},\bar{s},\bar{y}')\neq 0.$$

However, if $A:\dot{\mathcal{E}}'(Z)\to\dot{C}^\infty(Z)$ then this limit must be zero from (6.19).

This result can be refined to give a useful description of the operators in $L_b^{-\infty,-p}$. For each $m\in\mathbf{R}$ we can consider inside $\dot{\mathcal{A}}(Z)$ the subspace of Lagrangian distributions of order m:

$$\dot{\mathcal{A}}^m(Z)=\left\{u\in\dot{\mathcal{A}}(Z);\ \text{near } x=0\ u \text{ is of the form (I.1.15) with symbol of order } m+\frac{n-1}{4}\right\}.$$

PROPOSITION 6.20. *Each* $A \in L_b^{-\infty}(Z)$ *maps* $\dot{\mathcal{A}}_c^m(Z)$ *into* $\dot{\mathcal{A}}^m(Z)$ *for every* $m \in \mathbf{R}$. $A\colon \dot{\mathcal{A}}_c^m(Z) \to \dot{\mathcal{A}}^{m-p}(Z)$ *for some fixed* $p>0$ *and every* m *if, and only if,* $A \in L_b^{-\infty,-p}(Z)$.

Proof. Since A is in $L_b^{-\infty}(Z)$, and can be assumed of the form (3.3), we can carry out the η-integral to give

$$Au = (2\pi)^{-1} \int e^{i(x-x')\xi} K(x, y, y', x\xi)\, u(x', y')\, dx'\, d\xi$$

where $K(x, y, y', \lambda)$ is smooth in all variables and in $S_{\mathrm{la}}^{-\infty}$ with respect to the last; in fact we can take the support to be compact in z, y'. Assuming $u \in \dot{C}_c^\infty(Z)$ and writing it in Lagrangian form

$$u = (2\pi)^{-1} \int e^{ix\xi} b(y, \xi)\, d\xi$$

we can substitute directly into the formula for Au and find

$$Au = (2\pi)^{-1} \int e^{ix\xi} c(y, \xi)$$

where we have used the Fourier inversion formula in one variable to write

$$c(y, \xi') = (2\pi)^{-1} \int e^{i(\xi-\xi')x} a(x, y, y', x\xi)\, b(y', \xi)\, dy'\, dx\, d\xi.$$

This can be rearranged to give

$$c(y, \xi) = (2\pi)^{-1} \int_0^\infty \int e^{i(\lambda-\mu)} a\left(\frac{\mu}{\xi}, y, y', \lambda\right) b\left(y', \frac{\lambda\xi}{\mu}\right) dy'\, d\lambda \frac{d\mu}{\mu} \tag{6.21}$$

If $b \in S^N$ the fact that K is rapidly decreasing means that (6.21) can be interpreted as an oscillatory integral, defining $c \in S^N$. Suppose

$$\partial_x^j K(0, y, y', \lambda) \equiv 0, \quad \text{for } j < p, \tag{6.22}$$

then the leading term in (6.21) is

$$c(y, \xi) \equiv \int \left(\frac{\mu}{\xi}\right)^p e^{i(\lambda-\mu)} \partial_x^p K(0, y, y', \lambda)\, b\left(y', \frac{\lambda}{\mu}\xi\right) dy' \frac{d\mu}{\mu} d\lambda (p!)^{-1}, \tag{6.23}$$

modulo S^{N-p-1}. The mapping properties of $A \in L_b^{-\infty,-p}(Z)$ follow directly from (6.23) since (6.22) is just the succession of conditions $\alpha_{-j} \equiv 0$, $j<p$, i.e. the definition of $A_b^{-\infty,-p}(Z)$. The validity of (6.21) and (6.23) for general $u \in \dot{\mathcal{A}}_c^m(Z)$ follows by the continuity of the formula (6.21) and the density of $\dot{C}_c^\infty(Z)$ in $\dot{\mathcal{A}}_c(Z)$.

To prove the remainder of the proposition we need to show that if $A \in L_b^{-\infty,-p}(Z)$ and

$$\sigma_{m-p}(Au) = 0, \quad \forall u \in \dot{\mathcal{A}}_c^m(Z) \tag{6.24}$$

then $\sigma_{-\infty,-p}(A) = 0$. To do this we will rewrite (6.23). First note that the symbol map

$$\dot{\mathcal{A}}^m(Z)/\dot{\mathcal{A}}^{m-1}(Z) \to S^{m+(n-1)/4}(\mathbf{R}^n, \mathbf{R})/S^{m+((n-1)/4)-1}(\mathbf{R}^n, \mathbf{R})$$

can be considered as an isomorphism:

$$\dot{\mathcal{A}}^m(Z)/\dot{\mathcal{A}}^{m-1}(Z) \to \frac{\dot{\mathcal{A}}^{m+(n-1)/4}(N\,\partial Z)}{\dot{\mathcal{A}}^{m+((n-1)/4)-1}(N\,\partial Z)},$$

the image being the space of Lagrangian distributions on the fibres of the normal bundle to the boundary, $N\partial Z$, supported on the positive side. If we apply the Plancherel formula to (6.23) then, since

$$\alpha_{-p}(y, y', s) = \int e^{i(1-s)\lambda}\, \partial_x^p K(0, y, y', \lambda)\, d\lambda,$$

$$c(y, \eta) = \int \alpha_{-p}(y, y', s) \cdot \left(\frac{\mu}{\eta}\right)^p \cdot \beta\left(y', s\frac{u}{\eta}\right) e^{-i\mu} \frac{ds}{\eta}\, d\mu\, dy' \tag{6.25}$$

where

$$\beta(y', t) = \int e^{it\lambda'} b(y', \lambda')\, d\lambda'.$$

Now, changing variable in (6.25), $\mu = t\eta$, gives

$$c(y, \eta) = \int e^{-it\eta} t^p \int \alpha_{-p}(y, y', s)\, \beta(y', st)\, ds\, dy'\, dt. \tag{6.26}$$

With the interpretation of the symbol above we see that if $\beta \in \dot{\mathcal{A}}^{m+(n-1)/4}(N\,\partial Z)$ represents $u \in \dot{\mathcal{A}}^m(Z)$ then Au is represented by

$$t^p \int a_{-p}(y, y', s)\, \beta(y', st)\, ds\, dy'. \tag{6.27}$$

Here β is a Lagrangian distribution in the second variable with support in $st \geqslant 0$.

Taking product distributions $\beta(y, x) = p(y')h(x)$ with h Lagrangian on $\mathbf{R}$, associated to $T_0^*\mathbf{R}$, and with support in $x \geqslant 0$, we can now prove the remainder of Proposition 6.20. We take $h(x) = x_+^z$, $z \in \mathbf{R}$ not a negative integer. Then from (6.27) we must have

$$t^p \int \alpha_{-p}(y, y', s)\, \varrho(y')\, (st)_+^z\, ds\, dy' = t_+^{z+p} \int a_{-p}(y, y', s)\, \varrho(y')\, s^z\, ds\, dy' = 0 \tag{6.28}$$

since as a homogeneous distribution, it vanishes with its symbol. Thus we must have

$$\int \alpha_{-p}(y, y', s)\, \varrho(y')\, s^z\, ds\, dy' = 0 \tag{6.29}$$

for all $\varrho \in C_c^\infty(\mathbf{R}^n)$ and $z \in \mathbf{R} \setminus (-\mathbf{N})$. By continuity, (6.29) gives the vanishing of the Mellin transform of α_{-p} in s. Thus, $\alpha_{-p}=0$, proving the proposition.

COROLLARY 6.30. *Operators in $L_b^{-\infty}(Z)$ map the classical distributions in $\dot{\mathcal{A}}_c(Z)$ to classical distributions.*

Proof. This is evident from (6.23).

Remark 6.31. The proofs above, of Propositions 6.10, 6.13 and 6.20 do not really depend on the assumption that $A \in L_b^{-\infty}(M)$. Thus, for each m the subspaces

$$\ldots \subset L_b^{m,-k}(M) \subset L_b^{m,-k+1}(M) \subset \ldots \subset L_b^{m,0}(M) = L_b^m(M)$$

can be defined successively by the conditions

$$\partial_x^j \alpha(0, y, y', s) = 0, \quad \text{for } j < k$$

in any local coordinates. Then the symbol maps and product formula extend with

$$\sigma_{(m),k}\colon L_b^{m,-k}(M) \to \dot{S}'(\overline{S^+}; B_k), \quad k \geqslant 0$$

having values in the space of Lagrangian distributions, supported in $\overline{S^+}$, of order $m-\frac{1}{2}$ associated to the surface $s=1$, which is invariantly defined in S, and decreasing rapidly at ∞ on the fibres.

In particular note that if $A \in L_b^m(M)$ then $A \in L_b^{m,-k}(M)$ if and only if

$$A\colon \dot{\mathcal{A}}_c^r(M) \to \dot{\mathcal{A}}^{r-k}(M), \quad \forall r. \tag{6.32}$$

The formula (6.27) also carries over, when the symbol isomorphism is viewed as

$$\sigma_r\colon \dot{\mathcal{A}}_c^r(M)/\dot{\mathcal{A}}_c^{r-1}(M) \cong \dot{\mathcal{A}}_c^{r+\frac{1}{2}(n-1)}(N^*\partial M)/\dot{\mathcal{A}}_c^{r+\frac{1}{2}(n-1)-1}(N^*\partial M). \tag{6.33}$$

So, for $A \in L_b^{m,-k}(M)$, $u \in \dot{\mathcal{A}}_c^r(M)$,

$$\sigma_{r-k}(Au)\,(t) = t^k \int \sigma_{(m),-k}(A)\,(y, y', s)\, \sigma_r(u)\,(st, y')\, dy'\, ds. \tag{6.34}$$

II.7. Composition and ellipticity

The composite of two operators is not defined unless there is a suitable restriction on the support or growth of the kernels. Recall that an operator

$$A\colon \dot{\mathcal{E}}'(M) \to \dot{\mathcal{D}}'(M)$$

is properly supported if the support of its Schwartz kernel

$$\operatorname{supp} a \subset M \times M$$

is a proper relation, with proper inverse:

$$\begin{cases} \operatorname{supp}(a)(K) \subset\subset M, & \forall K \subset\subset M \\ \operatorname{supp}(a)^{-1}(K') \subset\subset M, & \forall K' \subset\subset M. \end{cases} \tag{7.1}$$

LEMMA 7.2. *If $A \in L_b^m(M)$ there is a properly supported operator $A' \in L_b^m(M)$ such that*

$$A - A' \in L_b^{-\infty}(M).$$

Proof. Cutting off the kernel of A away from the diagonal affects it only by an element in $K^{-\infty}(M)$, proving the lemma.

Note that it is not in general possible to modify A by a smoothing operator and make it proper.

THEOREM 7.3. *If $A \in L_b^m(M)$, $B \in L_b^{m'}(M)$ and one at least is properly supported then $C = B \cdot A \in L_b^{m+m'}(M)$ and*

$$\sigma_{m+m'}(C) = \sigma_m(A) \cdot \sigma_{m'}(B). \tag{7.4}$$

The main element in the proof of this theorem is the corresponding local result.

PROPOSITION 7.5. *If $a \in S_{\mathrm{la}}^m(Z \times \mathbf{R}^{n+1})$ has support in a compactly based cone and $b \in S_{\mathrm{la}}^{m'}(Z \times \mathbf{R}^{n+1})$ then there exists $c \in S_{\mathrm{la}}^{m+m'}(Z \times \mathbf{R}^{n+1})$ such that*

$$\tilde{b}(z, D_z) \circ \tilde{a}(z, D_z) = \tilde{c}(z, D_z).$$

Proof. Following Corollary 3.18 it suffices to show that

$$\tilde{b}(z, D_z)(e^{iz\cdot\zeta}\tilde{a}(z, \zeta)) = e^{iz\cdot\zeta}\tilde{c}(z, \zeta). \tag{7.6}$$

Now,

$$\tilde{b}(z, D_z)(e^{iz\cdot\zeta}\tilde{a}(z, \zeta)) \in C^\infty(Z \times \mathbf{R}^{m+1})$$

so, provided $c \in S^{m+m'}(Z \times \mathbf{R}^{n+1})$, we only need verify (7.6) in $x > 0$, and only for $b \in S_{\mathrm{la}}^{-\infty}$, if we show that c depends continuously on b in symbol spaces.

For $x>0$, we make successive changes of variable from ξ' to $\lambda'=x\xi'$ and from x' to $s=x'/x$ to show that

$$\begin{aligned}
&e^{-iz\cdot\zeta}\tilde{b}(z, D_z)\,(e^{iz\cdot\zeta}\tilde{a}(z,\zeta))\\
&\quad=(2\pi)^{-n-1}\int\int_0^\infty e^{i(x-x')(\xi'-\xi)+i(y-y')\cdot(\eta'-\eta)}\,b(z,x\xi',\eta')\,a(z',x'\xi,\eta)\,dx'\,dy'\,d\xi'\,d\eta'\\
&\quad=(2\pi)^{-n-1}\int\int_0^\infty e^{i(1-x'/x)(\lambda'-x\xi)+i(y-y')(\eta'-\eta)}\,b(z,\lambda',\eta')\,a(z',x'\xi,\eta)\,dx'/x\,dy'\,d\lambda'\,d\eta'\\
&\quad=(2\pi)^{-n-1}\int\int_0^\infty e^{i(1-s)(\lambda'-x\xi)+i(y-y')\cdot(\eta'-\eta)}\,b(z,\lambda',\eta')\,a(sx,y',sx\xi,\eta)\,ds\,dy'\,d\lambda'\,d\eta'.
\end{aligned}$$

Thus, if we take (7.6) as the definition of c we have

$$c(z,\lambda,\eta)=(2\pi)^{-n-1}\int\int_0^\infty e^{i(1-s)(\lambda'-\lambda)+i(y-y')\cdot(\eta'-\eta)}\,b(z,\lambda',\eta')\,a(sx,y',s\lambda,\eta)\,ds\,dy'\,d\lambda'\,d\eta'. \tag{7.7}$$

To see that this is a symbol we first note that the λ',η' integrals in (7.7) can be carried out as the inverse Fourier transform of the symbol b. Setting

$$\check{b}(z,t,r)=(2\pi)^{-n-1}\int e^{it\lambda'+ir\cdot\eta'}\,b(z,\lambda',\eta')\,d\lambda'\,d\eta'$$

we find

$$c(z,\lambda,\eta)=\int\int_0^\infty e^{i(s-1)\lambda-i(y-y')\cdot\eta}\,\check{b}(z,1-s,y-y')\,a(sx,y',s\lambda,\eta)\,ds\,dy'. \tag{7.8}$$

Here, $\check{b}(z,1-s,y-y')$ is a Lagrangian distribution singular only at $s=1$, $y=y'$ (and C^∞ for $b\in S^{-\infty}$). The lacunary condition (1.4) means that $\check{b}(z,1-s,y-y')$ vanishes identically in $s<0$. Thus, we can write (7.8) as

$$c(z,\lambda,\eta)=\int\int H(s)\,\check{b}(z,1-s,y-y')\,a(sx,y',s\lambda,\eta)\,e^{i(s-1)\lambda-i(y-y')\cdot\eta}\,ds\,dy',$$

with the integrand still C^∞ away from $s=1$, $y=y'$. In fact, the integrand is rapidly decreasing as $s\to\infty$ or $|y'|\to\infty$, so if $\varrho\in C_c^\infty(\mathbf{R})$ has $\varrho(s)=1$ in $|s-1|<1/4$ and $\operatorname{supp} p\subset[1/2,3/2]$ then

$$\begin{aligned}
c(z,\lambda,\eta)&\equiv\int\varrho(s)\,\check{b}(z,1-s,y-y')\,a(sx,y',s\lambda,\eta)\,e^{i(s-1)\lambda-i(y-y')\cdot\eta}\,ds\,dy'\\
&=(2\pi)^{-n-1}\int\varrho(s)\,e^{i(1-s)(\lambda'-\lambda)+i(y-y')\cdot(\eta'-\eta)}\,b(z,\lambda',\eta')\,a(sx,y',s\lambda,\eta)\,ds\,dy'\,d\lambda'\,d\eta'
\end{aligned} \tag{7.9}$$

modulo $S^{-\infty}$; the amplitude is a symbol since s is bounded away from zero.

The standard stationary phase argument, as used to prove the composition law for pseudodifferential operators (see Hörmander [9]), now applies to the last integral in (7.9) and shows that c is indeed a symbol depending continuously on a and b. Indeed,

$$c(z,\lambda,\eta)\sim\sum_{k,\alpha}(-i)^{k+|\alpha|}\partial_\lambda^k\partial_\eta^\alpha b(z,\lambda,\eta)\,\partial_s^k\partial_y^\alpha a(sx,y,s\lambda,\eta)\big|_{s=1}\Big/k!\,\alpha! \tag{7.10}$$

Returning to (7.7), it remains to show that c satisfies the lacunary conditions (3.9). Calculating directly,

$$\begin{aligned} Mc(z,\eta,t) &= (2\pi)^{-n-1}\int e^{i(s-t)\lambda}a(sx,y',s\lambda,\eta)\,e^{i(1-s)\lambda'+(y-y')\cdot(\eta'-\eta)}\varrho(s)\,b(z,\lambda',\eta')\,ds\,d\lambda'\,dy'\,d\eta'\,d\lambda \\ &= (2\pi)^{-n-1}\int e^{i(1-ts^{-1})\mu}a(sx,y',\mu,\eta)\,e^{i(y-y')\cdot(\eta'-\eta)}Mb(z,\eta',s)\,\varrho(s)\,s^{-1}\,ds\,d\mu\,d\eta'\,dy' \\ &= (2\pi)^{-n-1}\int Ma(sx,y',\eta;ts^{-1})\,Mb(z,\eta';s)\,e^{i(y-y')\cdot(\eta'-\eta)}\varrho\frac{ds}{s}\,dy'\,d\eta'. \end{aligned} \tag{7.11}$$

The lacunary conditions on a and b show that this vanishes identically if $t<0$, so c satisfies (1.4) too and the proof of the theorem is complete.

The formula (7.10) specifying c modulo $S^{-\infty}$ is actually just the usual asymptotic series for the symbol of the product of two pseudodifferential operators, in that it implies

$$\tilde{c}(z,\zeta)\sim\sum_\beta(-i)^{|\beta|}\partial_\zeta^\beta\tilde{b}(z,\zeta)\cdot\partial_z^\beta\tilde{a}(z,\zeta)/\beta!. \tag{7.12}$$

Proof of Theorem 7.3. Decomposing the kernels of A and B by a locally finite partition of unity we can apply Proposition 7.5 to all terms except those with one factor in $L_b^{-\infty}$. Since the latter contributions are shown to be in $L_b^{-\infty}$ below it follows that $C\in L_b^{m+m'}(M)$ and the formula (7.4) follows directly from (7.12).

PROPOSITION 7.13. *Suppose B: $C_c^\infty(Z)\to C^\infty(Z)$ is defined by an integral*

$$B\varphi(z)=\int_Z\beta(z,y',s)\,\varphi(xs,y')\,dy'\,ds \tag{7.14}$$

where $Z\ni z\mapsto\beta(z,y',s)\in\dot{\mathcal{D}}'(Z)$ is a C^∞ map such that β is C^∞ in all variables in $s<\varepsilon$, $s>1/\varepsilon$ for some $\varepsilon>0$ and vanishes rapidly as $s\to0$, $s\to\infty$ then for any $A\in L_{b,P}^{-\infty}(Z)\,A\cdot B\in L_b^{-\infty}(Z)$.

Proof. Clearly it is enough to suppose that A is also of the form (7.14) with kernel $\alpha\in C^\infty$, compactly supported in the y' variable and vanishing rapidly as $s\to0,\infty$. Similarly one can suppose that β is supported in $x \leqslant 1$. Then the composite operator can be written in the same form (7.14) with kernel

$$\gamma(z, y', t) = \int_Z \alpha(z, y'', t)\,\beta(xt, y'', y', s/t)\,dy''\,\frac{ds}{t}, \tag{7.15}$$

which is clearly in $\mathcal{K}^{-\infty}(Z)$.

For $A \in L_b^m(M)$ we define the characteristic variety

$$\Sigma_{\cdot}(M) = \{\varrho \in \tilde{T}^*M \setminus 0;\ \sigma_m(A) \text{ is not elliptic in a cone around } \varrho\}.$$

The standard microlocal invertibility of pseudodifferential operators at elliptic points comes over to this setting.

PROPOSITION 7.13. *If $A \in L_b^m(M)$ is elliptic at ϱ then there exists $B \in L_b^{-m}(M)$ with proper support such that*

$$R_1 = \mathrm{Id} - AB, \quad R_2 = \mathrm{Id} - BA$$

have full symbols (in any coordinates) of order $-\infty$ in a conic neighbourhood of ϱ.

Proof. Since the construction just depends on the formal properties of the symbol maps we refer the reader to [4].

II.8. Wavefront set

The notion of wavefront set of a distribution on an open set, as introduced by Hörmander [10] and by Sato in the analytic category, is the basic starting point of microlocal analysis. It is intimately related to the invertibility of pseudodifferential operators, discussed in Proposition 7.13.

Definition 8.1. If $u \in \dot{\mathcal{D}}'(M)$

$$\mathrm{WF}_b\,(u) = \bigcap\,\{\Sigma_b(A);\ A \in L_b^0(M) \text{ is properly supported and } Au \in \dot{\mathcal{A}}(M)\}.$$

The notation used here is not quite the same as that of [11], [16]. We shall show in Section II.9 below that the various definitions agree in their common domain. The most straightforward properties of WF_b can be proved as in the standard case of WF.

LEMMA 8.2. $$\mathrm{WF}_b\,(u) = \emptyset \Leftrightarrow u \in \dot{\mathcal{A}}(M).$$

Proof. This result follows from the locality of the symbol product (7.10). Thus, if $A_j \in L_b^0(M)$ are proper such that the $\tilde{T}^*M \setminus (0 \cup \Sigma_b(A_j))$ give a locally finite open covering of $\tilde{T}^*M \setminus 0$ and $A_j u \in \dot{\mathcal{A}}(M)$ for every j then one can find $B_j \in L_b^0(M)$ proper such that

$$\mathrm{Id} \equiv \sum B_j \cdot A_j' \cdot \operatorname{mod} L_b^{-\infty}, \tag{8.3}$$

where $A_j - A'_j \in L_b^{-\infty}$ are such that the sum in (8.3) is locally finite. Then $u \in \dot{\mathcal{A}}(M)$ because of (6.6). The converse is trivial.

Also as a direct consequence of the definition we have the microlocality of pseudodifferential operators. If $A \in L_b^m(M)$ we define the essential support of A as

$$\mathrm{WF}_b\,(A) = \{\varrho \in \tilde{T}^*M\setminus 0;\ \sigma_f(A) \text{ is not of order } -\infty \text{ in any conic neighbourhood of } \varrho\}. \tag{8.4}$$

This is clearly independent of the choice of coordinates. Then,

$$\mathrm{WF}_b\,(Au) \subset \mathrm{WF}_b\,(A) \cap \mathrm{WF}_b\,(u). \tag{8.5}$$

For distributions with additional regularity we can strengthen (8.1). We need a preliminary result:

PROPOSITION 8.6. *If $a \in S^m(\mathbf{R}^n \times \mathbf{R}^N)$ has support in a compactly based cone and $b \mapsto (b, a)$ extends by continuity from $S^{-\infty}(\mathbf{R}^n \times \mathbf{R}^N)$ to $S^\infty(\mathbf{R}^n \times \mathbf{R}^N)$ then $a \in S^{-\infty}(\mathbf{R}^n \times \mathbf{R}^N)$.*

Proof. Choose $\sigma \in C_c^\infty(\mathbf{R}^N)$ with $\sigma \geqslant 0$ and $\sigma(\theta) = 1$ for θ near 0. Since $a \in S^m(\mathbf{R}^n \times \mathbf{R}^N)$ for any choice of multiindices α, β, γ there is some $M \in \mathbf{R}$ such that

$$c_n = D_\theta^\alpha D_x^\beta \theta^\gamma \sigma(\theta/n)\,\theta^\gamma D_x^\beta D_\theta^\alpha a \quad \text{is bounded in } S^M(\mathbf{R}^n \times \mathbf{R}^N)$$

for all $n \in \mathbf{N}$. The continuity assumption implies that

$$(c_n, a) = \int \varrho(\theta/n)\,|\theta^\gamma D_x^\beta D_\theta^\alpha a|^2 \quad \text{is bounded as } n \to \infty.$$

Thus certainly, $a \in S^{-\infty}(\mathbf{R}^n \times \mathbf{R}^N)$.

LEMMA 8.7. *If $Z_+ = \overline{\mathbf{R}_x^+} \times \mathbf{R}_y^n$, $Z_- = \overline{\mathbf{R}_x^-} \times \mathbf{R}_y^n$ then the space of Lagrangian distributions splits:*

$$I^\infty(\mathbf{R}^{1+n};\ N^*\{x = 0\}) = \dot{\mathcal{A}}(Z_+) + \dot{\mathcal{A}}(Z_-).$$

Proof. Clearly we can assume that $w \in I^\infty(\mathbf{R}^{1+n};\ N^*\{x=0\}) \cap \dot{\mathcal{E}}'(\mathbf{R}^{1+n})$. Thus, for some k,

$$(xD_x)^p D_y^\alpha w \in H^{-k}(\mathbf{R}_x;\ L^2(\mathbf{R}^n)).$$

Define $v \in L^2_{\mathrm{loc}}(\mathbf{R}^{1+n})$ by $D_x^k v = w$, $v = 0$ for $x \gg 0$. Now, for some constants $c_{p,j}$

$$D_x^k(x^p D_x^p D_y^\alpha v) = \sum_{j \leqslant p} c_{p,j} D_y^\alpha x^{p-j} D_x^{p-j} D_x^k v = \sum_{j \leqslant p} c_{p,j} D_y^\alpha x^{p-j} D_x^{p-j} w.$$

This shows that $x^p D_x^p D_y^\alpha v \in L^2_{\mathrm{loc}}(\mathbf{R}^{1+n})$ for all p, α.

Consider $u = H(x)v(x, y)$, with H the Heaviside function. For $\varphi \in C_c^\infty(\mathbf{R}^{1+n})$,

$$\begin{aligned}\langle x^p D_x^p D_y^\alpha u, \varphi\rangle &= \langle u, D_y D_x^p x^p \varphi\rangle \\ &= \int_0^\infty v(x, y) \sum_{r \leq p} \overline{c_{p,r} x^{p-r} D_x^{p-r} D_y \varphi}\, dx\, dy \\ &= \int_0^\infty \sum_{r \leq p} \bar{c}_{p,r} D_x^{p-r} x^{p-r} D_y^\alpha v \bar{\varphi}\, dx\, dy,\end{aligned}$$

where the last integration by parts is justified by the fact that $x^{p-r}v$ is in the Sobolev space $H_{\mathrm{loc}}^{p-r}(\mathbf{R}^{1+n})$ and vanishes with its first $p-r-1$ derivatives at $x=0$. Thus, $x^p D_x^p D_y^\alpha u \in L^2_{\mathrm{loc}}(\mathbf{R}^{1+n})$ for all p, α so $u \in \dot{\mathcal{A}}(Z_+)$. In consequence, $D_x^k u \in \dot{\mathcal{A}}(Z_+)$ and $w - D_x^k u \in \dot{\mathcal{A}}(Z_-)$ proving the lemma.

PROPOSITION 8.8. *On any manifold with boundary*

$$\mathcal{A}'(M) \cap \dot{\mathcal{A}}(M) = C^\infty(M).$$

Proof. The result is clearly local in nature so we can assume $u \in \mathcal{A}_c'(Z) \cap \dot{\mathcal{A}}_c(Z)$. We first observe that the trace Ru of u is C^∞. For $\varphi \in C_c^\infty(\mathbf{R}^n)$

$$\langle Ru, \varphi\rangle = \langle u, \delta(x) \otimes \varphi\rangle$$

where $\delta(x) \otimes \varphi \in \dot{\mathcal{A}}_c(Z)$. Now,

$$\delta(x) \otimes \varphi = \frac{d}{dx}(H(x) \otimes \varphi)$$

and clearly $H(x) \otimes \varphi \in C^\infty(Z)$ is in $\mathcal{A}'(Z)$. Moreover we can choose a sequence $\psi_n \to H(x) \otimes \varphi$ in $\mathcal{A}'(Z)$ with $\psi_n \in C_c^\infty(Z)$ and $\partial_x \psi_n$ converging in $\dot{\mathcal{A}}_c(Z)$. Since $u \in \mathcal{A}_c'(Z)$ and $\partial_x u \in \dot{\mathcal{A}}_c(Z)$ this implies

$$(Ru, \varphi) = -(\partial_x u, H(x) \otimes \varphi).$$

Now the map $\mathcal{E}'(\mathbf{R}^n) \ni \varphi \mapsto H(x) \otimes \varphi \in \mathcal{A}'(Z)$ is (weakly) continuous, so shows that Ru extends by continuity to $\mathcal{E}'(\mathbf{R}^n)$. Thus $Ru \in C_c^\infty(\mathbf{R}^n)$. The same argument applies to all boundary values of u, so using Borel's theorem we can choose $u' \in C_c^\infty(Z) \subset \mathcal{A}_c'(Z) \cap \dot{\mathcal{A}}_c(Z)$ such that

$$v = u - u' \in \mathcal{A}_c'(Z) \cap \dot{\mathcal{A}}_c(Z)$$

has the property

$$\left(v, \frac{d^k}{dx^k} \delta(x) \otimes \varphi\right) = 0, \quad \forall k, \varphi \in C_c^\infty(\mathbf{R}^n). \tag{8.9}$$

Using this we can extend $v \in \dot{\mathcal{E}}'(Z) \subset \mathcal{E}'(\mathbf{R}^{n+1})$ to a linear form on the space of Lagrangian distributions, on $\mathbf{R}^{n+1}$, associated to $N^*\{x=0\}$. By Lemma 8.8, if $w \in I^\infty(\mathbf{R}^{n+1}; N^*\{x=0\})$

there exists $w' \in \dot{\mathcal{A}}_c(Z)$, the subspace supported in $x \geqslant 0$, such that $w = w'$ in $x > 0$. Moreover w' is unique up to a term

$$\sum_{\text{finite}} \frac{d^k}{dx^k} \delta(x) \otimes \varphi_k(y) \quad \varphi_k \in C_c^\infty(\mathbf{R}^n). \tag{8.10}$$

Now, if we set

$$(v, w) = (v, w') \tag{8.11}$$

the fact that v vanishes on all distributions of the form (8.10), i.e. (8.9) holds, means that this extension is well-defined and continuous.

Taking the partial Fourier transform of v with respect to x we have

$$\hat{v}(\xi, y) \in C_c^\infty(\mathbf{R}_y^n; S^m(\mathbf{R})),$$

since $v \in \dot{\mathcal{A}}_c(Z)$. The topology in $I^m(\mathbf{R}^{n+1}; N^*\{x=0\})$ is defined by the C^∞ and symbol norms, so the continuity of (8.11) means that $\hat{v}$ extends to $S^\infty(\mathbf{R}^n \times \mathbf{R})$. Applying Proposition 8.6 we conclude that

$$\hat{v} \in S^{-\infty}(\mathbf{R}^n \times \mathbf{R}).$$

Thus, v is C^∞. This completes the proof of the proposition.

PROPOSITION 8.12. *If* $u \in \mathcal{A}'(M)$ *then*

$$\mathrm{WF}_b(u) = \bigcap \{\Sigma_b(A);\ A \in L_b^0(M) \textit{ is properly supported and } Au \in C^\infty(M)\}.$$

Proof. If $u \in \mathcal{A}'(M)$ and A is properly supported then $Au \in \mathcal{A}'(M)$. So, for all A in Definition 8.1 $Au \in \mathcal{A}'(M) \cap \dot{\mathcal{A}}(M) = C^\infty(M)$ from Proposition 8.8 and the proposition follows.

THEOREM 8.13. *If* $u \in \mathcal{A}'(M)$ *then*

$$\mathrm{WF}(Ru) \subset \mathrm{WF}_b(u) \cap T^*\partial M.$$

Proof. If $\varrho \in T^*\partial M \hookrightarrow \tilde{T}^*M$ is not in $\mathrm{WF}_b(u)$ then we can find $A_0 \in L^m(\partial M)$ elliptic at ϱ and such that there is an element $A \in L_b^m(M)$ with

$$R(Au) \equiv A_0(Ru) \tag{8.14}$$

and $\mathrm{WF}_b(A) \cap \mathrm{WF}_b(u) = \emptyset$. Proposition 8.12 shows that the left side of (8.14) is C^∞ so $\varrho \notin \mathrm{WF}(Ru)$ and the theorem is proved.

Using the notion of wavefront set we can introduce the usual formal structures of microlocal analysis. Let

$$\tilde{S}^*M = (\tilde{T}^*M \setminus 0)/\mathbf{R}^+$$

be the compressed cosphere bundle of M. The ring of pseudodifferential operators on M induces a sheaf of rings on $\tilde{S}^*M$, the sheaf of totally characteristic microdifferential operators. Thus, suppose $\Omega \subset \tilde{S}^*M$ is an open set. Let

$$X^m(\Omega) = \{A \in L_b^m(M);\ \mathrm{WF}_b\,(A) \subset \Omega^c\}$$

be the subspace of operators with essential support in the complement of Ω. We define

$$\mathcal{L}_b^m(\Omega) = L_b^m(M)/X^m(\Omega).$$

THEOREM 8.15. *$\mathcal{L}_b^\infty$ is a sheaf of filtered rings on $\tilde{S}^*M$.*

Proof. This is essentially immediate from the results proved above concerning $L_b^m(M)$. The definition and properties of the restriction map, from Ω to $\Omega' \subset \Omega$, are clear. The presheaf property, that a section over Ω vanishes if it vanishes in a covering of Ω follows from the definition of essential support. The intrinsically global nature of the definition then implies the sheaf property.

PROPOSITION 8.16. *$\mathcal{L}_b^\infty$ is a fine sheaf.*

Proof. Clearly microlocal partitions of unity can be constructed as in the theory of pseudodifferential operators on open manifolds.

We remark that the support of a section of $\mathcal{L}$ is just the essential support of a defining pseudodifferential operator. Similarly we define

$$\mathcal{M}(\Omega) = \dot{\mathcal{D}}'(M)/Y(\Omega) \tag{8.17}$$

where

$$Y(\Omega) = \{u \in \dot{\mathcal{D}}'(M);\ \mathrm{WF}_b\,(u) \subset \Omega^c\}. \tag{8.18}$$

$\mathcal{M}(\Omega)$ is the space of microfunctions on $\Omega \subset \tilde{S}^*M$.

THEOREM 8.19. *$\mathcal{M}$ is a sheaf of $\mathcal{L}$-modules over $\tilde{S}^*M$.*

Next we consider some simple functorial properties of distributions on manifolds with boundary related to this notion of wavefront set. Let $W \hookrightarrow M$ be a submanifold in the sense that W is an embedded submanifold with boundary defined by the transversal intersection.

$$\partial W = \partial M \mathrel{\overline{\pitchfork}} W.$$

Near a point $m \in \partial W$ local coordinates x, y can be chosen so that $x \geq 0$ on M and

$$W = \{y_j = 0,\quad 0 \leq j \leq \mu\} \quad \text{near } m.$$

In particular from the standard theory of Lagrangian distributions we have:

PROPOSITION 8.20. *The restriction map*

$$\dot{\mathcal{A}}(M) \ni u \mapsto u \,|\, W \in \dot{\mathcal{A}}(N)$$

is well-defined by continuous extension from the dense subspace $\dot{C}^\infty(M)$ and is surjective.

The inclusion of W in M induces a projection

$$\iota_W^*\colon \tilde{T}^*_W M \to \tilde{T}^* W.$$

The inverse image in $\tilde{T}_W M$ of the zero section is the compressed conormal bundle of W in M

$$\tilde{N}^* W \subset \tilde{T}^*_W M.$$

LEMMA 8.21. *$\tilde{N}^*W$ is a vector bundle of rank equal to the codimension of W in M and can be canonically identified with N^*W so that the diagram*

$$\begin{array}{ccc} T^*M & \longrightarrow & \tilde{T}^*M \\ \uparrow & & \uparrow \\ N^*W & \dashleftarrow\dashrightarrow & \tilde{N}^*W \end{array}$$

commutes.

Proof. The map $\pi\colon \tilde{T}^*_W M \to \tilde{T}^* W$ is the dual of the inclusion $\tilde{T}W \hookrightarrow \tilde{T}_W M$ of tangent vectors tangent to the boundary. Thus, in local coordinates of the type introduced above we have

$$\eta \cdot dy + \frac{\lambda}{x}\, dx \in \tilde{N}^* W$$

if, and only if, $\lambda = 0$, $\eta_j = 0$ for $j > \mu$. Clearly $\sum_{j=1}^{m-n} \eta_j dy_j$ can be canonically identified with the corresponding covectors in N^*W in a coordinate independent way. Observe that in (8.22) the image of $\tilde{N}^*W$ always lies in $T^*\partial M$ over ∂W so the lemma is proved.

If $\Gamma \subset \tilde{T}^*M$ is a closed cone, let

$$\dot{\mathcal{D}}'(M, \Gamma) = \{u \in \dot{\mathcal{D}}'(M);\ \mathrm{WF}_b\,(u) \subset \Gamma\}$$

be topologized by the seminorms of $\dot{\mathcal{D}}'(M)$ and by the seminorms $|Au|$ where $|\ \ |$ is a continuous seminorm on $\dot{\mathcal{A}}(M)$ and $A \in L^0_b(M)$ is properly supported with essential support in Γ^c.

THEOREM 8.23. *If $W \hookrightarrow M$ is an embedded submanifold with boundary $\partial W = \partial M \pitchfork W$ and $\Gamma \subset \tilde{T}^*M$ is a closed cone with*

$$\Gamma \cap \tilde{N}^* W = \varnothing$$

then the restriction map

$$|\, W\colon \dot{\mathcal{D}}'(M, \Gamma) \to \dot{\mathcal{D}}'(W)$$

is well-defined by continuity from the dense subspace $\dot{C}^\infty(M)$ *and*

$$\mathrm{WF}_b\,(u\,|\,W) \subset \iota_W^*\,(\mathrm{WF}_b\,(u) \cap \tilde{T}_W^* M).$$

The proof is omitted since this result will not be used below.

II.9. Normal regularity

Although the ring of operators $L_b^\infty(M)$ behaves well on the space of distributions $\mathcal{A}'(M)$, for which boundary values are well defined, the solutions of non-characteristic boundary value problems have more special regularity properties. We introduce below a coordinate independent space of distributions in which these lie, in particular the boundary values are then taken in the strong sense. This approach gives a new proof of Peetre's theorem (see [8]) on hypoellipticity at the boundary, which implies that the solutions to non-characteristic problems satisfy

$$u \in C^\infty([0, \infty);\ \mathcal{D}'(\mathbf{R}^n)) \hookrightarrow \dot{\mathcal{D}}'(Z), \tag{9.1}$$

in terms of local coordinates on M.

In any local coordinates the conormal direction at the boundary is always the span of dx, $x=0$ on ∂M, and this is the kernel of the map

$$\pi\colon T^*M \to \tilde{T}^*M.$$

However, the choice of local coordinates does define a compressed conormal direction

$$\tilde{n}_x = \frac{1}{x}\,dx \in \tilde{T}^*M. \tag{9.2}$$

PROPOSITION 9.3. *If* $m \in \partial M$ *and* $u \in \mathcal{A}'(M)$ *has*

$$\tilde{n}_x \notin \mathrm{WF}_b\,(u)$$

for some local coordinates (x, y) *then near* m, u *is regular up to the boundary in the sense of* (9.1).

Proof. Clearly we can assume that u has compact support near m. By assumption there is an operator $A \in L_b^0(M)$ which is properly supported and elliptic at $\tilde{n}_x$ and for which $Au \in \dot{\mathcal{A}}_c(M)$. Propositions 8.7 and 8.12 show that $Au \in C_c^\infty(M)$. Thus, in the given local coordinates there is a symbol $a(\lambda, \eta) \in S_{1a}^0$ with $a \equiv 0 \pmod{S^{-\infty}}$ in $|\lambda| \geqslant C|\eta|$, and

$$u = \tilde{a}(D_z)\,u + v, \quad v \in C^\infty(Z).$$

To complete the proof we use:

LEMMA 9.4. *If* $a(x, y, \lambda, \eta) \in S^m_{1a}$ *has* $a \equiv 0 \pmod{S^{-\infty}}$ *in* $|\lambda| \geq C|\eta|$ *for some* C *then*

$$\tilde{a}(z, D_z)u \in C^\infty([0, \infty); \mathcal{D}'(\mathbf{R}^n)) \hookrightarrow \dot{\mathcal{D}}'(Z)$$

for each $u \in \mathcal{A}'_c(Z)$.

Proof. Let $Q \in L_p^{-\infty}(\mathbf{R}^n)$ be a properly supported smoothing operator. Then,

$$Q(e^{iy\cdot\eta}a(x, y, \lambda, \eta)) = e^{iy\cdot\eta}b(x, y, \lambda, \eta)$$

with $b \in S^{-\infty}_{1a}$. Applying this to the representation (3.15) of Au we conclude that

$$Q\tilde{a}(z, D_z)u = \tilde{b}(z, D_z)u \in C^\infty(Z)$$

since $\tilde{b}(z, D_z) \in L_b^{-\infty}(Z)$ and $u \in \mathcal{A}'_c(Z)$. Thus, u lies in the space (9.1) and the lemma is proved.

To exploit this result we introduce the following space of distributions.

Definition 9.5. Let $\mathcal{N}(M) \subset \mathcal{A}'(M)$ be the space of *normally regular* distributions, $u \in \mathcal{N}(M)$ if

$$\mathrm{WF}_b\,(u) \cap \partial\tilde{T}^*M \subset T^*\partial M. \tag{9.6}$$

COROLLARY 9.7. *If* $u \in \mathcal{N}(M)$ *then in every local coordinate system* (x, y) *near the boundary,*

$$u \in C^\infty([0, \varepsilon); \mathcal{D}'(\mathbf{R}^n)) \quad \textit{for some } \varepsilon > 0.$$

Proof. For any coordinates, $\tilde{n}_x \in \partial T^*M \setminus T^*\partial M$.

Recall, from section I.1, that $\dot{\mathcal{D}}'_m(M, \partial M)$ is the space of distributions supported in ∂M which are annihilated by x^{m+1}.

THEOREM 9.8. *Let* $P \in \mathrm{Diff}^m(M)$ *be a* C^∞ *differential operator for which* ∂M *is non-characteristic. If* $u \in \dot{\mathcal{D}}'(M)$, $f \in \mathcal{N}(M)$ *and* $g \in \dot{\mathcal{D}}'_{m-1}(M, \partial M)$ *with*

$$Pu = f + g \tag{9.9}$$

then $u \in \mathcal{N}(M)$.

Proof. Proposition I.3.12 shows that $u \in \mathcal{A}'(M)$. Moreover the operator $A = x^m P \in L_b^m(M)$ is clearly elliptic in $\partial\tilde{T}^*M$ away from $T^*\partial M$ since its symbol is a non-vanishing multiple of λ^m over $x = 0$. Thus, the fact that

$$\mathrm{WF}_b\,(x^m(f+g)) \subset \mathrm{WF}_b\,(f) \subset T^*\partial M \cup \tilde{T}^*\mathring{M}$$

implies the same for u and the proof is complete.

COROLLARY 9.10. *If* P *is non-characteristic,* $u \in \mathcal{D}'(M)$ *and* $Pu \in C^\infty(M)$ *then near each boundary point and in any local coordinates* u *is in the space* (9.1).

Proof. We can easily choose an extension of u to $v \in \dot{\mathcal{D}}'(M)$ so that (9.9) holds with $g \in \dot{\mathcal{D}}'_{m-1}(M, \partial M)$.

It is a consequence of Theorem 9.8 that $\mathrm{WF}_b(u)$, as defined in Section 8, agrees with the definition given in [16] for distributions satisfying non-characteristic differential equations. This latter definition states that $(y, \eta) \in T^*\mathbf{R}^n \setminus 0$ is not in $\mathrm{WF}_b(u)$, in any local coordinates, if there is a properly supported pseudo-differential operator Q in the tangential variables such that for some $\varepsilon > 0$

$$Q(y, D_z)\, u(x, y) \in C^\infty([0, \varepsilon) \times \mathbf{R}^n), \tag{9.11}$$

and Q is elliptic at (y, η).

By Theorem 9.8 if $A \in L_b^0(Z)$ has $A \equiv \mathrm{Id}$ in a conic neighbourhood of $T^*\mathbf{R}^n \subset T^*Z$ then

$$Au - u \in C^\infty([0, \varepsilon) \times \mathbf{R}^n).$$

We can further suppose that $A \equiv 0$ in a conic neighbourhood of the compressed conormal direction in these coordinates. The composition formula for pseudo-differential operators, in this case with parameters, shows easily that $QA \in L_b^\infty([0, \varepsilon) \times \mathbf{R}^n)$ is elliptic or not at any point $(y, \eta) \in T^*\mathbf{R}^n$ with Q. This shows the equivalence of (9.11) and the characterization of WF_b for elements of $\mathcal{A}'$, hence $\mathcal{N}$, in Proposition 8.12.

The occurrence of boundary terms in (9.9) makes it natural to consider

$$\dot{\mathcal{N}}_k(M) = \mathcal{N}(M) \oplus \dot{\mathcal{D}}'_k(M, \partial M)$$

and the inductive limit, $\dot{\mathcal{N}}(M)$, over k. These spaces are easily characterized as in Definition 9.5:

$$u \in \dot{\mathcal{N}}(M) \quad \Leftrightarrow \quad u \in \dot{\mathcal{B}}(M) \quad \text{and} \quad \mathrm{WF}_b(u) \subset T^*\partial M \cup T^*\mathring{M} = BM.$$

There is, apart from Proposition 8.7, another intrinsic relation between certain spaces of distributions on a manifold with boundary which can be obtained using the calculus. In [12] the set of distributions $\mathcal{D}'_\partial(M) \subset \mathcal{D}'(M)$ was defined, as a general class admitting boundary values. Explicitly, $u \in \mathcal{D}'_\partial(M)$ if near each point $m \in \partial M$ there is a coordinate system (x, y) and $\varepsilon > 0$ such that

$$u(x, y)\big|_{x<\varepsilon} \in C^\infty([0, \varepsilon);\, \mathcal{D}'(\mathbf{R}^n)), \tag{9.12}$$

in the coordinate patch. Since the coordinate system in which (9.12) holds may vary from point to point $\mathcal{D}'_\partial(M)$ is not even a linear space. The span

$$\mathrm{sp}\,(\mathcal{D}'_\partial(M)) \subset \mathcal{D}'(M)$$

is not, at least a priori, an easy space to deal with nor is it of obvious general significance. However, this is precisely the space introduced above.

THEOREM 9.13. *For any manifold with boundary, M,*

$$\mathcal{D}'(M)\supset\mathcal{A}'(M)=\mathcal{D}'_\partial(M)+\mathcal{D}'_\partial(M)=\operatorname{sp}(\mathcal{D}'_\partial(M)).$$

Proof. In view of the definition (9.12) it is clear that

$$\mathcal{D}'_\partial(M)\subset\mathcal{A}'(M).$$

To prove the result it is therefore only necessary to show that any $u\in\mathcal{A}'(M)$ can be written as the sum of two elements $u_1, u_2\in\mathcal{D}'_\partial(M)$. As vector bundles $\tilde{T}^*M\cong T^*M$ since near the boundary both bundles are isomorphic to $T^*\partial M\oplus N^*\partial M$. For T^*M this follows from the usual isomorphism

$$T^*_{\partial M}M/N^*\partial M\cong T^*\partial M.$$

For the compressed cotangent bundle $T^*\partial M\hookrightarrow\tilde{T}^*_{\partial M}M$ and the isomorphism

$$\tilde{T}^*_{\partial M}M/T^*\partial M\cong N^*\partial M$$

is dual to

$$\tilde{T}_{\partial M}M/\tilde{N}\partial M=T\partial M$$

where $\tilde{N}\partial M\hookrightarrow\tilde{T}_{\partial M}M$ is the kernel of the natural restriction from $\tilde{T}_{\partial M}M$ to $T\partial M$. It is therefore possible to choose a global compressed conormal subbundle, spanned by $\tilde{n}_m\in\tilde{T}^*_mM\setminus T^*_m\partial M$ for all $m\in\partial M$.

Now, choose a symbol $b\in S^0(\tilde{T}^*M)$ with $b\in S^{-\infty}$ in a conic neighbourhood of $\pm\tilde{n}_m$ near ∂M. Clearly using the symbol calculus there exists $B\in L^0_b(M)$ with $\sigma_0(B)=b$. More particularly, given an open conic neighbourhood Γ of $T^*\partial M$, b can be chosen so $b=1$ and $B=\mathrm{Id}$ on Γ, provided $\tilde{n}_m\notin\Gamma_m$ for all $m\in\partial M$. Furthermore B can be chosen properly supported.

With such a choice of B set

$$u_1=Bu,\quad u_2=(\mathrm{Id}-B)u.$$

By construction $\tilde{n}_m\notin\mathrm{WF}_b(u_1)$, $T^*\partial M\cap\mathrm{WF}_b(u_2)=\emptyset$. Near each $m\in\partial M$ Proposition 9.3 can be applied to u_1 in local coordinates with $dx/x=\tilde{n}_m$; similarly Proposition 9.3 applies to u_2 in any local coordinates with $dx'/x'\in\Gamma$. This proves the theorem.

II.10. L^2 estimates

It is relatively straightforward to show that operators in $L^0_b(Z)$ are locally L^2 bounded modulo a continuous seminorm on $C^\infty(\overline{\mathbf{R}^+};L^2(\mathbf{R}^n))$. The main step in showing that these operators are actually locally L^2 bounded is the boundedness of the residual operators in $L_b^{-\infty}$.

LEMMA 10.1. *If* $a \in S_{1a}^{-\infty}(Z \times \mathbf{R}^n \times \mathbf{R}^{n+1})$ *then the operator* $\tilde{a}(z, D_z)$ *defined by* (3.16) *maps* $L_c^2(Z)$ *to* $L_{\mathrm{loc}}^2(Z)$.

Proof. If $u \in C_c^\infty(Z)$ then

$$\tilde{a}(z, D_z)\, u = (2\pi)^{-n-1} \int e^{i(1-s)\lambda + i(y-y')\cdot\eta}\, a(z, y', \lambda, \eta)\, u(sx, y')\, ds\, dy'\, d\lambda\, d\eta$$

$$= (2\pi)^{-n-1} \int \alpha(z, y', s)\, u(sx, y')\, dy'\, ds, \tag{10.2}$$

where $\alpha(z, y', s) \in C^\infty(Z \times \mathbf{R}^n; \dot{S}([0, \infty))$. Thus, if $\varrho \in C_c^\infty(Z)$ then

$$\begin{aligned}\|\varrho a(z, D_z)\, u\|_{L^2}^2 &= \int_Z |\varrho(z)|^2 \left| \int \alpha(z, y', s)\, u(sx, y')\, ds\, dy' \right|^2 dz \\ &\leqslant \int_{[0,R]\times K_1} dz \left\{ \left(\int_{\mathbf{R}^+ \times K_1} |\alpha(z, y', s)|^2 (1+s^2)\, s^{-1}\, ds\, dy' \right) \right. \\ &\quad \times \left. \left(\int_{\mathbf{R}^+ \times K_1} |u(sx, y')|^2\, s(1+s^2)^{-1}\, ds\, dy' \right) \right\} \\ &\leqslant C \int_0^R \int_{K_1} \int_0^\infty |u(sx, y')|^2 (1+s^2)^{-1}\, dsx\, dy'\, ds \\ &\leqslant C' \|u\|_{L^2}^2.\end{aligned}$$

where, supp $(\varrho) \cup$ supp $(u) \subset [0, R] \times K_1$, $K_1 \subset\subset \mathbf{R}^n$. The lemma follows from the density of $C_c^\infty(Z)$ in $L_c^2(Z)$.

THEOREM 10.3. *For any manifold with boundary,* M, *if* $A \in L_b^0(M)$ *then*

$$A\colon L_c^2(M) \to L_{\mathrm{loc}}^2(M). \tag{10.4}$$

Proof. The result is local and follows form the construction of an approximate square root for an elliptic self-adjoint operator in $L_b^0(M)$. Thus, in (10.4) it can be assumed that A is properly supported, the error term being handled by Lemma 10.1. Then, it suffices to construct a square root, modulo $L_b^{-\infty}(M)$, for

$$C - A^*A \in L_b^0(M), \quad C > 0 \text{ large.}$$

Since the proof follows as in the standard calculus (see [4]) it is omitted.

The local boundedness of operators in L_b^0, on L^2, together with certain commutation results easily leads to boundedness on Sobolev spaces of integral order. Recall from Lemma

5.15 that if P is a differential operator of order k and $A \in L_b^m(Z)$ then there exist finitely many differential operators P_s, $s \in S$, of order at most k and operators $A_s \in L_b^m(Z)$ such that

$$P \cdot A = \sum_{s \in S} A_s \cdot P_s. \tag{10.5}$$

Now, for any $k \in \mathbf{Z}$ define

$$\dot{H}^k(Z) = \{u \in H^k(\mathbf{R}^{n+1}); \operatorname{supp}(u) \subset Z\}.$$

The standard properties of Sobolev spaces show that if $k \leqslant 0$ then $u \in H^k(\mathbf{R}^{n+1})$ if, and only if, it can be written in the form

$$u = \sum_{|\beta| \leqslant -k} D_z^\beta u_\beta(z), \quad u_\beta \in L^2(Z) = \dot{H}^0(Z). \tag{10.6}$$

Applying (10.5) repeatedly to the adjoint of $A \in L_b^m(Z)$ shows that

$$A \cdot D_z^\beta = \sum_{|\gamma| \leqslant |\beta|} D_z^\gamma \cdot A_\gamma \tag{10.7}$$

with $A_\gamma \in L_b^m(Z)$.

THEOREM 10.8. *If $k \in \mathbf{Z}$ then any $A \in L_b^0(M)$ maps $\dot{H}_c^k(M)$ into $\dot{H}_{\mathrm{loc}}^k(M)$.*

Proof. Of course this is purely local, and for $k \leqslant 0$ follows directly from (10.6), (10.7) and Theorem 10.3. For $k > 0$ it follows similarly, since $u \in \dot{H}^k(Z)$ if, and only if,

$$D_z^\beta u \in L^2(Z) \quad \forall\, |\beta| \leqslant k. \tag{10.9}$$

Thus, using (10.5) and assuming $u \in \dot{H}_c^k(Z)$

$$D_z^\beta A u = \sum_{|\gamma| \leqslant |\beta|} A_\gamma \cdot D_z^\gamma u \in L_{\mathrm{loc}}^2(Z)$$

for all $|\beta| \leqslant k$.

The spaces $H_{\mathrm{loc}}^k(M) \subset \mathcal{D}'(M)$ of distributions extendible to elements of $H_{\mathrm{loc}}^k(\tilde{M})$, whenever $\tilde{M}$ is an extension of M across ∂M, can be characterized as the duals of the corresponding $\dot{H}_c^{-k}(M)$, $k \in \mathbf{Z}$. It follows that any $A \in L_b^0(M)$ defines maps

$$A: H_c^k(M) \to H_{\mathrm{loc}}^k(M) \quad \forall k \in \mathbf{Z}, \tag{10.10}$$

the details of the proof being left to the reader.

Chapter III: Fourier integral operators

III.1. Boundary canonical transformations

A canonical transformation from a boundaryless manifold X to a manifold Y is a C^∞ map $\chi: \Omega \to T^*Y \setminus 0$ defined on an open conic subset $\Omega \subset T^*X \setminus 0$ which is symplectic

$$\chi^* \omega_Y = \omega_X \tag{1.1}$$

and homogeneous

$$\chi \cdot M_s = M_s \cdot \chi \quad s > 0, \tag{1.2}$$

where M_s is multiplication by s in the fibres. In view of (1.2) the condition (1.1) that χ pulls the fundamental 2-form of Y back to the corresponding form on T^*X implies that

$$\chi^* \alpha_Y = \alpha_X \tag{1.3}$$

with α_X, α_Y the fundamental 1-forms.

If X and Y are both manifolds with boundary it is natural to demand that, in addition to (1.1), (1.2) χ preserve the boundary $\partial\Omega = \Omega \cap \partial T^*X$:

$$\chi(\partial\Omega) \subset \partial T^*Y. \tag{1.4}$$

PROPOSITION 1.5. *If X and Y are manifolds with boundary and $\chi: \Omega \to T^*Y \setminus 0$ is a canonical transformation satisfying* (1.4) *then, provided $(\iota_X^*)^{-1}(\iota_X^* \varrho) \cap \partial\Omega$ is connected for each $\varrho \in \partial\Omega$, there is a unique canonical transformation $\partial\chi: \iota^*(\partial\Omega \setminus N^*\partial X) \to T^*\partial Y \setminus 0$, from ∂X to ∂Y such that the following diagram commutes:*

$$\begin{array}{ccc} \partial T^*X \supset (\partial\Omega \setminus N^*\partial X) & \xrightarrow{\chi} & \partial T^*Y \\ \downarrow \iota^* & & \downarrow \\ T^*\partial X \supset \Omega_\partial & \xrightarrow{\partial\chi} & T^*\partial Y \end{array} \tag{1.6}$$

Proof. Recall from Lemma I.2.6 that $\partial T^*X \subset T^*X$ has Hamilton foliation defined by the projection ι^*. Since this is a symplectic invariant χ must map the fibres of ι_X^* in $\partial\Omega$ into the fibres of ι_Y^*. As the fibres are connected χ projects to a map as indicated in (1.6) and it remains only to show that $\partial\chi$ is canonical. Since ι^* is homogeneous, so is $\partial\chi$ and

$$(\iota_X^*)^* (\partial\chi)^* \alpha_{\partial Y} = \chi^* (\iota_Y^*) \alpha_{\partial Y} = \chi^* (\alpha_Y|_{\partial T^*Y}) = \alpha_X|_{\partial\Omega}$$

shows that $(\partial\chi)^* \alpha_{\partial Y} = \alpha_{\partial X}$, concluding the proof.

We have defined $\Omega_\partial = \iota^*(\partial\Omega \setminus N^*\partial X)$, excluding the conormal directions from $\partial\Omega$, since $\iota^*(N^*\partial X)$ is the zero section of $T^*\partial X$. If χ is a canonical transformation satisfying (1.4) and $\partial\Omega \cap N^*\partial X \neq \emptyset$ then the projection $\partial\chi$ is C^∞ on all of $\iota^*(\partial\Omega)$, which meets the zero section $0_{\partial X}$. As $\partial\chi$ is canonical it must be given near $0_{\partial X} \cap \iota^*(\partial\Omega)$ by the lift of a coordinate transformation, and is then essentially trivial from the point of view of canonical transformations.

To remove the awkwardness associated with the assumption of connectedness of the fibres of ι^* in $\partial\Omega$ we shall work at the germ level. Recall that $C^\infty_{\partial,k}(\Omega) \subset C^\infty(\Omega)$, the subspace of C^∞ functions on Ω satisfying the trace conditions (of order k) is invariantly defined for $\Omega \subset T^*X$ open. Let

$$C^\infty_{\partial,k}(\varrho) = C^\infty_{\partial,k}(\varrho, X)$$

be the space of germs at $\varrho \in \partial T^*X$ of such functions. Since the symbols of elements of L^∞_b are functions of this type, it is natural to examine the canonical transformations preserving these spaces.

Definition 1.7. If $\varrho' \in \partial T^*X \setminus N^*\partial X$, $\varrho \in \partial T^*Y$, *a germ of boundary canonical transformation* is a germ $\chi: T^*X, \varrho' \to T^*Y, \varrho$ of canonical transformation satisfying (1.4) which in addition satisfies

$$\chi^*: C^\infty_{\partial,k}(\varrho, Y) \to C^\infty_{\partial,k}(\varrho', X), \quad \forall k \in \mathbf{Z}. \tag{1.8}$$

The fact that χ is a canonical transformation implies that the preservation of the spaces of functions having polynomial traces, (1.8), follows from a particular case.

PROPOSITION 1.9. *If $\chi: T^*X, \varrho' \to T^*Y, \varrho$ is a germ of canonical transformation satisfying* (1.4) *then χ is a germ of boundary canonical transformation provided $\chi^*\mu$ has polynomial traces of order* 1 *for some germ $\mu \in C^\infty_{\partial,1}(\varrho, Y)$ which is not constant along the fibre $(\iota^*)^{-1}(\iota^*\varrho)$ through ϱ.*

Proof. Certainly the condition given is necessary, since it follows from (1.8). To prove sufficiency we introduce local coordinates $z' = (x', y')$ near $\pi\varrho'$ in X and $z = (x, y)$ near $\pi\varrho$ in Y, with corresponding dual coordinates (x', y', ξ', η') in T^*X and (x, y, ξ, η) in T^*Y. Now, the assumption on μ is that $\mu(x, y, \xi, \eta)$ is not constant along the fibres of ι^* through $\varrho = (0, 0, \bar\xi, \bar\eta)$. As μ has polynomial traces of order 1 this means

$$\mu(0, 0, \xi, \bar\eta,) = a + b\xi \quad \text{with } b \neq 0.$$

We introduce new Darboux coordinates (i.e. coordinates in which the 1-form is still $\xi\, dx + \eta \cdot dy$) (X, Y, Ξ, H) in T^*Y near ϱ by setting $\Xi = \mu$ and

$$H_\Xi Y = H_\Xi H = 0, \quad H_\Xi X = 1, \quad H = \eta, \quad Y = y, \quad X = 0 \quad \text{on } x = 0. \tag{1.10}$$

Since $b \neq 0$ the Hamilton field of Ξ is transversal to $x = 0$ so (1.10) has a unique local solution. The new coordinates are Darboux since from the initial conditions in (1.10), $\Xi\, dX + H \cdot dY = \xi\, dx + \eta \cdot dy = \alpha$ when restricted to $x = 0$ and $-\langle \alpha, H_\Xi \rangle = \Xi$ so

$$L_{H_\Xi} \alpha = H_\Xi \lrcorner\, \omega + d\Xi = 0.$$

Next we observe directly from (1.10) that $X \in C^{\infty}_{\partial,-1}(\varrho')$, $Y_j, H_j \in C^{\infty}_{\partial,0}(\varrho')$. For example the differential equation for the Y_j's is

$$\partial_\xi \Xi\, \partial_x Y - \partial_x \Xi\, \partial_\xi Y + \sum_{j=1}^{n-1} (\partial_{\eta_j} \Xi\, \partial_{y_j} Y - \partial_{y_j} \Xi\, \partial_{\eta_j} Y) = 0. \tag{1.11}$$

The initial conditions for Y show $Y|_{x=0}$ to be independent of ξ so

$$\partial_\xi \Xi\, \partial_x Y = - \sum_{j=1}^{n-1} (\partial_{\eta_j} \Xi\, \partial_{y_j} Y - \partial_{y_j} \Xi\, \partial_{\eta_j} Y)$$

and $\partial_x Y$ is therefore a polynomial of degree one in ξ. Proceeding by induction we can assume that $\partial_x^k Y|_{x=0}$ is a polynomial of degree at most k in ξ for $k \leqslant j$. Applying ∂_x^j to (1.11)

$$\partial_\xi \Xi\, \partial_x^{j+1} Y = - \sum_{j>p} \binom{j}{p} \partial_x^{j-p} \Xi\, \partial_x^{p+1} Y + \partial_x^j \left(\partial_x \Xi\, \partial_\xi Y - \sum_{j=1}^{n-1} \partial_{\eta_j} \Xi\, \partial_{y_j} Y - \partial_{y_j} \Xi\, \partial_{\eta_j} Y \right)$$

is easily seen to be a polynomial of degree at most $j+1$ in ξ at $x=0$.

Let us write Φ for the canonical transformation $(x, y, \xi, \eta) \mapsto (X, Y, \Xi, H)$ in $T^*\mathbf{R}^n$. The pull-back of a function f is

$$\Phi^* f(x, y, \xi, \eta) = f(X, Y, \Xi, H)$$

so, carrying out the differentiation

$$\partial_x^k \Phi^* f|_{x=0} = \sum Q_{lj\alpha}\, \partial_X^l \partial_\Xi^j \partial_{(Y,H)}^\alpha f(0, Y, \Xi, H)$$

where Y, Ξ, H are evaluated at $x=0$ and for each pair l, j of integers and each $(2n-2)$-multiindex α, $Q_{lj\alpha}$ is a polynomial in the derivatives of the X, Y, Ξ, H. Now, $Y, H|_{x=0}$ are independent of ξ and $\Xi|_{x=0}$ is linear so $\partial_x^l \partial_\xi^j \partial_{(y,\eta)}^\alpha f$ is a polynomial of degree at most $l+t-j$ in ξ if f has polynomial traces of order t. Moreover, $Q_{lj\alpha}$ is a sum of terms

$$\prod_{s=1}^{l} \partial_x^{r_s} T_{t_s}$$

where for each s, $t_s \in \{1, \ldots, 2n+2\}$, $T_1 = X$, $T_q = Y_{q-1}$, $2 \leqslant q < n+2$, $T_{n+2} = \Xi$, $T_q = H_{q-n-1} n+2 < q \leqslant 2n+2$ the total order $\sum_s r_s = k$ and the numbers of appearances of X and Ξ are $\#\{t_s = 1\} = l$, $\#\{t_s = n+2\} = j$. Thus, $\partial_x^k \Phi^* f|_{x=0}$ is a polynomial in ξ of degree at most $k - l + j + (l + t - j) = k + t$ and we have shown that

$$\Phi^*: C^{\infty}_{\partial,t}((0, 0, \bar{\xi}, \bar{\eta})) \to C^{\infty}_{\partial,t}((0, 0, \bar{\mu}, \bar{\eta})), \quad \forall t,$$

In the local coordinates z', z introduced in X and Y the canonical transformation $\partial\chi$ from ∂X to ∂Y has a natural 'flat' extension to a boundary canonical transformation χ_0 from X to Y, where $\chi_0^*(x)=x'$, $\chi_0^*(\xi)=\xi'$, $\chi_0^*(y,\eta)=(\partial\chi)^*(y,\eta)$. Thus, replacing χ by $\chi_0^{-1}\cdot\chi$ it suffices to consider the special case $\partial\chi=\mathrm{Id}$. Let Φ be the boundary canonical transformation from Y to $\mathbf{R}^n$ defined above by demanding that $\partial\Phi$ be the lift of the coordinate map $y\colon \partial Y\to\mathbf{R}^{n-1}$ and $\Phi^*(\xi)=\mu$. If Ψ is the boundary canonical transformation from X to $\mathbf{R}^n$ with $\partial\Psi$ the lift of the coordinate map y' and $\Psi^*(\xi)=\chi^*(\mu)$, we have $\chi=\Phi^{-1}\cdot\Psi$, so χ is a boundary canonical transformation. This completes the proof of the proposition.

We can easily restate this proposition in terms of the compressed cotangent bundle:

PROPOSITION 1.12. *A germ of canonical transformation $\chi\colon T^*X,\varrho'\to T^*Y,\varrho$ satisfying* (1.4) *is a germ of boundary canonical transformation if, and only if, it lifts to a local diffeomorphism.*

$$\begin{array}{ccc} \tilde\chi: \tilde T^*X, \tilde\varrho' & \longrightarrow & \tilde T^*Y, \tilde\varrho \\ \uparrow & & \uparrow \\ \chi: T^*X, \varrho' & \longrightarrow & T^*Y, \varrho, \end{array} \tag{1.13}$$

for $\tilde\varrho'=\pi(\varrho')\in\tilde T^*X$, $\tilde\varrho=\pi(\varrho)\in\tilde T^*Y$.

Proof. The condition that the lift $\tilde\chi$ be C^∞ is simply that χ preserve the rings of C^∞ germs, $C^\infty_{\partial,k}$ on $\tilde T^*X$ at $\tilde\varrho$. For $k<0$ $C^\infty_{\partial,k}(X)$ is just the space of functions vanishing to order k at $x=0$ and for $k>0$ $C^\infty_{\partial,k}$ is just the space of functions f such that $x^kf\in C^\infty_{\partial,0}$. Thus if χ preserves $C^\infty_{\partial,0}$ it preserves all the $C^\infty_{\partial,k}$.

The region of definition of the lift $\tilde\chi$ of a realization of the germ χ is not a neighbourhood of $\tilde\varrho'$; it can be taken to be of the form

$$B=\{(x,y,\lambda,\eta);\ (y,\eta)\in B',\ 0\leqslant x\leqslant\varepsilon,\ |\lambda|\leqslant C|x|\,|\eta|\} \tag{1.14}$$

where B' is a smoothly bounded conic region. Using Proposition I.2.10 it follows that $\tilde\chi$ is a diffeomorphism on its domain, so it can be extended to a local diffeomorphism as claimed.

To extend Definition 1.7 to be global, and also to include some transformations more general than symplectic diffeomorphism on T^*X we simply choose (1.13) as the fundamental concept.

Definition 1.15. A boundary canonical transformation from X to Y is a diffeomorphism defined in an open cone $\Gamma\subset\tilde T^*X$:

$$\tilde\chi\colon\Gamma\to\tilde T^*Y$$

such that $\tilde{\chi}(\Gamma \cap \partial\tilde{T}^*X) \subset \partial\tilde{T}^*Y$ and the transformation

$$\begin{tikzcd} \chi: T^*\mathring{X} \arrow[r] \arrow[d, leftrightarrow] & T^*\mathring{Y} \arrow[d, leftrightarrow] \\ \tilde{T}^*X \setminus \partial\tilde{T}^*X \arrow[r] & \tilde{T}^*Y \setminus \partial\tilde{T}^*Y \end{tikzcd} \tag{1.16}$$

is canonical.

Observe immediately that near each point $\varrho' \in T^*\partial X \cap \Gamma$, $\tilde{\chi}$ projects to a germ of boundary canonical transformation, in the sense of Definition 1.7.

To properly reconcile Definition 1.14 and Definition 1.7 it is necessary to improve Proposition 1.12 slightly so that the extension of χ, provided by $\tilde{\chi}$, is everywhere canonical in its domain of definition. The last condition (1.16) can be replaced by an intrinsic condition on the compressed cotangent bundles, namely that χ relates the invariantly defined singular symplectic forms:

$$\tilde{\omega}_x = \sum_{j=i}^{n} d\eta_j \wedge dy_j + d\,\log x \wedge d\lambda \tag{1.17}$$

and is homogeneous. Thus a C^∞ map from $\tilde{T}^*X$ to $\tilde{T}^*Y$ is a boundary canonical transformation if, in canonical coordinates (x, y, λ, η), (X, Y, Λ, H) in $\tilde{T}^*X$, $\tilde{T}^*Y$, $\chi(x, y, \lambda, \eta) = (X, Y, \Lambda, H)$ it satisfies the Poisson bracket conditions:

$$\begin{gathered} \tilde{H}_X Y_j = \tilde{H}_X H_j = 0, \quad \tilde{H}_X \Lambda = -1, \quad \tilde{H}_\Lambda X = 1, \quad \tilde{H}_\Lambda Y_j = \tilde{H}_\Lambda H_j = 0, \\ \tilde{H}_{Y_k} X = \tilde{H}_{Y_k} \Lambda = \tilde{H}_{Y_k} Y_j = 0, \quad \tilde{H}_{Y_k} H_j = -\delta_{kj}, \quad \tilde{H}_{H_k} X = \tilde{H}_{H_k} \Lambda = \tilde{H}_{H_k} H_j = 0, \quad \tilde{H}_{H_k} Y_j = \delta_{kj}. \end{gathered} \tag{1.18}$$

Here, for $f = f(x, y, \lambda, \eta)$

$$\tilde{H}_f = \frac{\partial f}{\partial \lambda} x \partial_x + \frac{\partial f}{\partial \eta} \cdot \partial_y - x \frac{\partial f}{\partial x} \partial_\lambda - \frac{\partial f}{\partial y} \cdot \partial_\eta \tag{1.19}$$

is the Hamilton vector field of f with respect to (1.17).

The system of equations (1.18) holds for the lift $\tilde{\chi}$ of χ in the region B in (1.14). Using Proposition I.2.10 $\Xi = \Lambda/X$ can be extended to a C^∞ function near the base point, $\Xi = a(x, y, \lambda, \eta) + (\lambda/x) b(y, \lambda, \eta)$, $b \neq 0$. Since the system (1.18) is in involution it is only necessary to solve

$$\tilde{H}_\Xi X = 1, \quad \tilde{H}_\Xi Y_j = \tilde{H}_\Xi H_j = 0, \tag{1.20}$$

using the given initial data on $x = \lambda = 0$. The solution is unique and smooth since

$$x\tilde{H}_\Xi = b(x\partial_x + \lambda\partial_\lambda) + xV$$

where V is tangent to $x = 0$. This is essentially radial in x, λ and since the C^∞ extension of $\tilde{\chi}$ gives solutions to infinite order at $x = \lambda = 0$ smoothness of the solution follows. In view of this identification the same notation, χ, will be used for the action of a boundary canonical transformation either on T^*X or $\tilde{T}^*X$.

Since the volume form $\tilde{\omega}^{n+1}$ of the 2-form $\tilde{\omega}$ in (1.17) is $x^{-1}\omega$, with ω a non-vanishing C^∞ volume form it is clear that a diffeomorphism (1.16) which is singular-canonical in the sense that it is homogeneous and

$$\chi^* \tilde{\omega}_y = \tilde{\omega}_x \tag{1.21}$$

is automatically a boundary canonical transformation. Indeed χ must map the two singular surfaces $x=0$, $x'=0$ to each other. Now, since χ is homogeneous the symplectic condition (1.21) is equivalent to the canonical condition:

$$\chi^* \tilde{\alpha}_y = \tilde{\alpha}_x \tag{1.22}$$

where $\tilde{\alpha}_x$, $\tilde{\alpha}_y$ are the singular-canonical 1-forms:

$$\tilde{\alpha} = \sum_{j=1}^{n} \eta_j dy_j + \lambda d \log x \tag{1.23}$$

LEMMA 1.24. *If (x, y, λ, η), $(x', y', \lambda', \eta')$ are canonical coordinates in $\tilde{T}^*X$, $\tilde{T}^*Y$ and χ is a boundary canonical transformation from X to Y then*

$$\chi^*(\lambda') \,|\, (x=0) = \lambda.$$

Proof. Since $\chi^*(x') = xa$ with $a > 0$ and C^∞

$$\chi^* (\log x') \equiv \log x \mod C^\infty.$$

Thus, condition (1.22) implies

$$\lambda dx = \chi^*(\lambda') dx \quad \text{at } x = 0,$$

proving the lemma.

III.2. Local parametrization

To define totally characteristic Fourier integral operators using oscillatory integrals, by analogy with the work of Hörmander [10], it is necessary to show that boundary canonical transformations can be parametrized by phase functions with the special dependence on the one phase variable effectively dual to the normal variable. To do this we recall the definition of the stretch of a manifold with corner, introduced in Section II.4, and more particularly the stretched product $N \hat{\times} M$ of two manifolds with boundary.

The principal reason that the stretched product enters here is that a boundary canonical transformation from M to N sits as a Lagrangian manifold in $T^*(N \hat{\times} M)$. First observe that the dual to the projection

$$\hat{\pi}: N \hat{\times} M \to N \times M$$

gives a C^∞ mapping

$$T^*N \times T^*M \approx T^*(N \times M) \to T^*(N \hat{\times} M) \tag{2.1}$$

which is an isomorphism away from the corner.

PROPOSITION 2.2. *The identification* $T^*N \times T^*M \approx \tilde{T}^*N \times \tilde{T}^*M$ *over the interior combined with* (2.1) *gives a* C^∞ *map*

$$\tilde{T}^*N \times \tilde{T}^*M \to T^*(N \hat{\times} M) \quad \text{over } \mathring{N} \times \mathring{M} \tag{2.3}$$

which, restricted to the twist (graph χ)$'$ *of the graph of a boundary canonical transformation, extends by continuity and embeds it as a* C^∞ *Lagrangian submanifold of* $T^*(N \hat{\times} M)$ *meeting the boundary* $\partial T^*(N \hat{\times} M)$ *only in the part over* $\partial N \times \partial M \times (-1, 1)$ *of the codimension one boundary and transversally.*

Proof. It is only necessary to resort to local coordinates to see this. Let (x, y), (x', y') be coordinates in N, M with (x, y, ξ, η), (x', y', ξ', η'), (x, y, λ, η), $(x', y', \lambda', \eta')$ the canonically dual coordinates in T^*N, T^*M, $\tilde{T}^*N$, $\tilde{T}^*M$ respectively. Near $\partial N \times \partial M \times [-1, 1] \subset N \hat{\times} M$ local coordinates are given by $w = \frac{1}{2}(x + x')$, $r = (x - x')/2w$, y and y'; let $(w, r, y, y'; \omega, \varrho, \eta, \eta')$ be the corresponding dual coordinates in $T^*(N \hat{\times} M)$. Under the isomorphism (2.3) in the interior the forms

$$\frac{\lambda}{x}\, dx + \frac{\lambda'}{x'}\, dx' + \eta \cdot dy + \eta' \cdot dy' \quad \text{and} \quad \omega\, dw + \varrho\, dr + \eta \cdot dy + \eta' \cdot dy'$$

are identified. So,

$$\begin{aligned} \omega &= (1 + r)\frac{\lambda}{x} + (1 - r)\frac{\lambda'}{x'} \\ \varrho &= w\left(\frac{\lambda}{x} - \frac{\lambda'}{x'}\right). \end{aligned} \tag{2.4}$$

The singularity in (2.4) shows that (2.3) does not itself extend to a smooth map up to the boundary, though as we shall see below its inverse does. The submanifold obtained from graph χ by reflection in the fibres of the second factor is of the form:

$$(\text{graph}\,\chi)' = \{((x, y, \lambda, \eta), (x', y', -\lambda', -\eta'); x' = xa, \lambda' = \lambda + xg, y' = Y', \eta' = H'\} \tag{2.5}$$

with $a > 0$, g, Y' and H' C^∞ functions of (x, y, λ, η). Thus, $r = (1 - a)/(1 + a)$ is C^∞ and bounded away from ± 1 on (graph χ)$'$. From this it follows that

$$\omega = -g(1 + r) \tag{2.6}$$

is also C^∞ as is

$$\varrho = \frac{2\lambda}{1+r} + \frac{2\lambda'}{1-r}. \tag{2.7}$$

So the map (2.3) extends smoothly to

$$(\text{graph}\,\chi)' \leftrightarrow \hat{\Lambda}_\chi \hookrightarrow T^*(N \hat{\times} M) \tag{2.8}$$

as a C^∞ lagrangian submanifold with the stated intersection property.

Suppose that $\hat{\Lambda} \subset T^*(N \hat{\times} M) \setminus 0$ is a Lagrangian submanifold which is the image of the graph of a boundary canonical transformation under the map (2.8). As noted above,

$$\hat{\Lambda} \pitchfork T^*_{\partial_1''(N \hat{\times} M)}(N \hat{\times} M) \tag{2.9}$$

$$\hat{\Lambda} \cap T^*_{\partial_1'(N \hat{\times} M)}(N \hat{\times} M) = \varnothing. \tag{2.10}$$

Now, $\hat{\Lambda}$ actually satisfies a stronger form of (2.9), namely the intersection is symplectically transversal as well:

$$\text{the Hamilton foliation of } T^*_{\partial_1''(N \hat{\times} M)}(N \hat{\times} M) \text{ is not tangent to } \hat{\Lambda}. \tag{2.11}$$

To see this just observe that in the canonical coordinates $(w, r, y, y'; \omega, \varrho, \eta, \eta')$ used in the proof of Proposition 2.2 the Hamilton foliation of the surface, given by $w=0$, consists of the ω-lines. From (2.6) we see immediately that ∂_ω is not tangent to $\hat{\Lambda} \subset \{\omega + g(1+r) = 0\}$.

Now, from (2.11) it follows that if $\hat{\Lambda}$ is the image of a boundary canonical transformation under (2.3) then the quotient

$$\hat{\Lambda}_1 = \hat{\Lambda} \cap T^*_{\partial_1''(N \hat{\times} M)}(N \hat{\times} M)/H \subset T^*(\partial_1''(N \hat{\times} M)) \tag{2.12}$$

by the Hamilton foliation H of $T^*_{\partial_1''(N \hat{\times} M)}(N \hat{\times} M)$ is again a Lagrangian submanifold. From the defining relations (2.4) a further transversality condition can be deduced, involving the fibre bundle F considered in (II.4.15) and Corollary II.4.16.

$$\hat{\Lambda}_1 \text{ meets } F \text{ transversally and the fibres cleanly in } F. \tag{2.13}$$

Indeed, from the coordinates $(w, r, y, y'; \omega, \varrho, \eta, \eta')$ used in the proof of Proposition 2.2 one gets canonical coordinates $(r, y, y'; \varrho, \eta, \eta')$ in $T^*\partial_1''(N \hat{\times} M)$ in terms of which F is defined by

$$F = \{\varrho = 0\} \tag{2.14}$$

and the fibres are the r-lines. From (2.7) $\hat{\Lambda}_1$ meets $\varrho = 0$ transversally and as $r = (1-a)/(1+a)$ on $\hat{\Lambda}$ ∂_r is not tangent to $\hat{\Lambda}_1$.

Finally the Lagrangian $\hat{\Lambda}$ satisfies the usual 'no zeros' condition for a relation. If $\pi_1^*\colon T^*N \to T^*(N \times M)$, $\pi_2^*\colon T^*M \to T^*(N \hat{\times} M)$ are dual to the projections

$$\begin{tikzcd} & N \hat{\times} M \arrow[d] \arrow[ddl, "\pi_1"'] \arrow[ddr, "\pi_2"] & \\ & N \times M \arrow[dl] \arrow[dr] & \\ N & & M \end{tikzcd}$$

then

$$\hat{\Lambda} \text{ does not meet either } \pi_1^*(T^*N) \text{ or } \pi_2^*(T^*M). \tag{2.15}$$

To see this first note that $\pi_1^*(T^*N) \subset T^*(N \hat{\times} M)$ is defined in the coordinates $(w, r, y, y'; \omega, \varrho, \eta, \eta')$ by

$$(1+r)\varrho = \omega w, \quad \eta' = 0 \tag{2.16}$$

and similarly, $\pi_2^*(T^*M) \subset T^*(N \hat{\times} M)$ is defined by

$$(1-r)\varrho = \omega w, \quad \eta = 0. \tag{2.17}$$

Now, $|r| \neq 1$ on $\hat{\Lambda}$ near $w=0$ and from (2.4), (2.5)

$$\varrho = \frac{4\lambda}{1-r^2} + \frac{2xg}{1-r} \tag{2.18}$$

so (2.16) cannot hold on $\hat{\Lambda} \cap \{w=0\}$ since it would imply $\lambda' = 0$, $\eta' = 0$ and χ is a map from $\tilde{T}^*M \setminus 0$ to $\tilde{T}^*N \setminus 0$. Similarly (2.17) cannot occur on $\hat{\Lambda} \cap \{w=0\}$. Away from $w=0$, i.e. $\partial_2''(N \hat{\times} M)$, the condition (2.15) is just the usual statement that $\hat{\Lambda}$ avoids the zero sections of T^*M, T^*N.

The intersection conditions (2.9), (2.10), (2.11), (2.13) and (2.15) characterize Lagrangians which come from (2.8), except for the condition that χ should be a mapping. We therefore introduce:

Definition 2.19. A boundary canonical relation from a manifold with boundary M to a manifold with boundary N is a conic submanifold

$$\hat{\Lambda} \subset T^*(N \hat{\times} M) \setminus 0$$

which is Lagrangian and satisfies (2.9), (2.10), (2.11), (2.13) and (2.15).

We can now give a converse to Proposition 2.2,

PROPOSITION 2.20. *The mapping over the interior*

$$T^*(N \hat{\times} M) \to \tilde{T}^*N \times \tilde{T}^*M \tag{2.21}$$

extends to a C^∞ map of manifolds with corner. This restricts to an isomorphism of each boundary canonical relation $\hat{\Lambda}$ to a submanifold $\tilde{\Lambda}\subset(\tilde{T}^*N\setminus 0)\times(\tilde{T}^*M\setminus 0)$ with the following properties:

$$\tilde{\Lambda} \text{ meets the topological boundary of } \tilde{T}^*N\times\tilde{T}^*M \text{ only in the corner.} \tag{2.22}$$

$$\tilde{\Lambda} \text{ meets each hypersurface } \partial\tilde{T}^*N\times\tilde{T}^*M,\ \tilde{T}^*N\times\partial\tilde{T}^*M \text{ transversally in } \tilde{\Lambda}_1. \tag{2.23}$$

$$\tilde{\Lambda} \text{ is Lagrangian and conic.} \tag{2.24}$$

$$\tilde{\Lambda}_1 \text{ meets } T^*\partial N\times\partial\tilde{T}^*M,\ \partial\tilde{T}^*N\times T^*\partial M \text{ transversally and only in } T^*\partial N\times T^*\partial M. \tag{2.25}$$

Conversely each C^∞ submanifold $\tilde{\Lambda}\subset(T^*N\setminus 0)\times(T^*M\setminus 0)$ with these properties is the image of a boundary canonical relation which is (locally) the twisted graph of a canonical diffeomorphism precisely when the two projections from $\tilde{\Lambda}$ to $\tilde{T}^*N$ and $\tilde{T}^*M$ are (locally) diffeomorphisms.

Proof. In terms of canonical coordinates (x, y, λ, η), $(x', y', \lambda', \eta')$ in $\tilde{T}^*N$, $\tilde{T}^*M$ and $(w, r, y, y'; \omega, \varrho, \eta, \eta')$ in $T^*(N\hat{\times}M)$ the map (2.3) is given by (2.4) and

$$w=\tfrac{1}{2}(x+x'),\quad r=(x-x')/(x+x'). \tag{2.26}$$

Inverting this gives

$$x=w(1+r),\quad x'=w(1-r) \tag{2.27}$$

and

$$\lambda=\tfrac{1}{2}\varrho(1-r^2)+\tfrac{1}{2}w\omega(1+r),\quad \lambda'=-\tfrac{1}{2}\varrho(1-r^2)+\tfrac{1}{2}w\omega(1-r), \tag{2.28}$$

showing that (2.21) is C^∞.

On $\hat{\Lambda}$, $|r|\neq 1$ by (2.10) and ∂_ω is not tangent to $\hat{\Lambda}$ at $w=0$ by (2.11). From this it is clear that (2.21) resticts to an isomorphism of $\hat{\Lambda}$ to $\tilde{\Lambda}$ which has the properties (2.22), (2.23) and (2.24). Finally, since (2.18) holds and $\hat{\Lambda}$ satisfies (2.13), $\tilde{\Lambda}_1$ meets $T^*\partial N\times\partial T^*M$ transversally, i.e. $\lambda=0$, and only inside $\lambda'=0$. Thus, (2.25) also holds.

The converse part of the proposition is similar so, together with the characterization of local boundary canonical transformations, the details are omitted.

The first use we shall make of Proposition 2.20 is in showing the existence of useful parametrizations of boundary canonical relations, which term will be used for the Lagrangians $\hat{\Lambda}$ and $\tilde{\Lambda}$ interchangeably. A preliminary result is needed. Suppose that M is any C^∞ manifold with boundary and $\Lambda\subset T^*M\setminus 0$ is a conic Lagrangian submanifold with

$$\Lambda \pitchfork \partial T^*M. \tag{2.29}$$

If $M_E\supset M$ is an open extension of M across ∂M then, at least locally, Λ can be extended across ∂T^*M to a Lagrangian Λ_E in T^*M_E. To see this just observe that there must always be, near any given $p \in \partial\Lambda$, a Hamilton vector field V on T^*M tangent to Λ and transversal to ∂T^*M. Indeed if $V = H_f$, $f \in C^\infty(T^*M)$ this just requires $f = 0$ on Λ and $df \neq 0$ on ∂T^*M. The existence of such a function follows from (2.29). Extending f across ∂T^*M to $g \in C^\infty(T^*M_E)$ gives an extension Λ_E as the H_g-flow-out of Λ, near p.

Recall that a parametrization of a conic Lagrangian submanifold $\Lambda \subset T^*M \setminus 0$ near $p \in \Lambda$ is provided by a homogeneous C^∞ function

$$\varphi : \Omega \times \gamma \to \mathbf{R}$$

where $\gamma \subset \mathbf{R}^N \setminus 0$ is an open cone with variables denoted $\theta_1, \ldots, \theta_N$, and Ω is an open neighbourhood of $\pi(p) \in M$, such that everywhere on

$$C_\varphi = \{(m, \theta) \in \Omega \times \gamma;\ d_\theta \varphi = 0\} \tag{2.30}$$

the N differentials $d(\partial\varphi/\partial\theta_j)$ are independent and the map

$$C_\varphi \ni (m, \theta) \mapsto (m, d_m \varphi) \in T^*M \tag{2.31}$$

is a local isomorphism onto a neighbourhood of p in Λ. The existence of such parametrizations is, of course, an integral part of the theory of Fourier integral operators as described by Hörmander [10]. The transversality properties of the Lagrangians $\hat{\Lambda}$ allow one to find special parametrizations.

PROPOSITION 2.32. *If $\hat{\Lambda} \subset T^*(N \hat{\times} M)$ is a boundary canonical relation and (w, r, y, y') are coordinates in $N \hat{\times} M$ induced by coordinates (x, y) in N and (x', y') in M then, near each point p in the boundary of $\hat{\Lambda}$ there is a parametrizing phase function of the form*

$$\psi(w, y, y', \mu, \theta) - \mu \frac{1-r}{1+r} \tag{2.33}$$

where ψ is homogeneous of degree one in the phase variables $(\mu, \theta) \in \gamma \subset \mathbf{R}^{1+N}$ and satisfies the non-degeneracy conditions

$$\frac{\partial \psi}{\partial \mu} \neq 0 \quad \text{and} \quad \operatorname{rank}\left(d\frac{\partial\psi}{\partial\theta_1}, \ldots, d\frac{\partial\psi}{\partial\theta_N}, d\left(\mu\frac{\partial\psi}{\partial\mu}\right)\right) = N+1 \tag{2.34}$$

on

$$\tilde{C}_\psi = \{(w, y, y', \mu, \theta);\ d_\theta \psi = 0\}. \tag{2.35}$$

Proof. This can be shown directly by reviewing the construction of parametrizations, instead we shall take as starting point the existence of local parametrizations even when the Lagrangian is non-homogeneous and depends smoothly on a parameter. First, change variable from r to $s=(1-r)/(1+r)$, which is possible since $r \neq \pm 1$ on $\widehat{\Lambda}$. In the new canonical coordinates $(w, s, y, y'; \omega, \sigma, \eta, \eta')$ in $T^*(N \widehat{\times} M)$ the bundle F is still given by $\sigma=0$. Near $w=0$, $\sigma=0$ the transversality condition (2.13) means that the projection of $\widehat{\Lambda} \cap \{\sigma=\varepsilon\}$ along ∂_s is Lagrangian in the variables $(w, y, y'; \omega, \eta, \eta')$ and depends smoothly on ε. If $\psi_1(w, y, y', \varepsilon, \theta)$ is a parametrizing phase function, depending smoothly on ε then the phase function

$$\varphi = \psi_1(w, y, y', \mu, \theta) - \mu(s-\delta) \tag{2.36}$$

parametrizes $\widehat{\Lambda}$ for a suitable choice of constant δ. Clearly this fulfills all the conditions of the Proposition, since ψ_1 can be chosen homogeneous in (μ, θ). Away from $\sigma=0$, but still near $w=0$, the vector field ∂_s has no invariant meaning, but coordinates can always be chosen so that it is not tangent to $\widehat{\Lambda}$, then the same method yields a parametrizing phase function (2.36)

Remark 2.37. In the proof above the variable $w=x+x'$ can be replaced by x without other alteration, so $\widehat{\Lambda}$ has local parametrizations of the form

$$\psi(x, y, y', \mu, \theta) - \mu \frac{1-r}{1+r} \tag{2.37}$$

with ψ satisfying (2.34) and (2.35). Then the parametrizing map

$$\tilde{C}_\psi \ni (x, y, y', \mu, \theta) \mapsto (x, y, x\psi_x + \mu\psi_\mu, \psi_y; x\psi_\mu, y', \mu\psi_\mu, -\psi_{y'}) \tag{2.38}$$

is easily seen to be a local isomorphism of $\tilde{C}_\psi$ and the relation $\tilde{\Lambda}$. Alternatively using the coordinate invariance of parametrization the original variables can be reintroduced so that

$$\psi(x, y, y', x\zeta, \theta) - x'\zeta \tag{2.39}$$

is seen to be a parametrizing phase function for the Lagrangian

$$\Lambda \subset T^*N \times T^*M$$

which is just the image of $\tilde{\Lambda}$. In fact, starting from Λ the standard construction of a parametrization can be modified to give a phase function of the form (2.39), with its special dependence on the one variable ζ 'dual to x'. The approach adopted above, whilst more abstract, is more natural in view of the importance of the manifold $N \widehat{\times} M$ as the carrier of the kernels discussed below.

Finally we observe that $\tilde{\Lambda}$ is locally a boundary canonical transformation if, and only *if, it can be represented by a phase function* (2.39) *satisfying*

$$\det\begin{bmatrix} \psi_{yy'} & \psi_{y\theta} & \psi_{y\mu} \\ \psi_{\theta y'} & \psi_{\theta\theta} & \psi_{\theta\mu} \\ \mu\psi_{\mu y'} & \mu\psi_{\mu\theta} & (\mu\psi_{\mu})_{\mu} \end{bmatrix} \neq 0, \quad \text{on } \tilde{C}_{\varphi}. \tag{2.40}$$

III.3. Oscillatory integrals

As for the pseudodifferential operators of Chapter II the amplitudes of the oscillatory integral operators considered here are of the form $a(z, y', x\xi, \theta)$ with special dependence on the one variable 'dual' to the normal variable to the boundary, x. Suppose that $\varphi \in C^{\infty}(Z \times \mathbf{R}^n \times (\mathbf{R}^{N+1} \setminus 0))$ is real-valued and homogeneous of degree one in $(\mu, \theta) \in \mathbf{R}^{N+1}$. Consider an open cone Γ in which φ is a phase function in the sense that (2.34), (2.35) holds on $\Gamma \cap \{x=0\}$,

$$\partial_{\mu}\varphi > 0 \quad \text{in } \Gamma \tag{3.1}$$

and in $x>0$, in Γ, $\varphi(z, y', x\xi, \theta)$ is a non-degenerate phase function in the usual sense.

Set

$$\mathring{S}^m(\Gamma) = \{a \in S^m(Z \times \mathbf{R}^n \times \mathbf{R}^{N+1};\ a \text{ is in } S^{-\infty} \text{ outside a closed subcone of } \Gamma\}. \tag{3.2}$$

If $a \in \mathring{S}^m(\Gamma)$ and $(z, y') \in K \subset\subset Z \times \mathbf{R}^n$ the integral

$$L(\varphi)\, a = \int e^{i\varphi(z, y', \mu, \theta) - is\mu} a(z, y', \mu, \theta)\, d\mu \tag{3.3}$$

is well-defined in the oscillatory sense for

$$s < \varepsilon(K), \quad s > 1/\varepsilon(K), \quad \varepsilon(K) > 0. \tag{3.4}$$

Indeed, subtracting a term in $S^{-\infty}$ from a allows one to assume that $\varepsilon < d_{\mu}\varphi < 1/\varepsilon$ on supp $(a) \cap K \times \mathbf{R}^{N+1}$. Integration by parts using

$$\partial_{\mu} e^{i\varphi - is\mu} = i(\partial_{\mu}\psi - s)\, e^{i\varphi - is\mu}$$

allows (3.3) to be replaced by a convergent integral with symbol of arbitrarily low order, provided (3.4) holds. In particular,

$$L(\varphi)a \in C^{\infty}([0, \varepsilon] \cup [1/\varepsilon, \infty) \times K;\ \mathcal{S}(\mathbf{R}^N_{\theta})). \tag{3.5}$$

Now note that

$$L(\varphi)\colon S^{-\infty}(Z \times \mathbf{R}^n \times \mathbf{R}^{N+1}) \to C^{\infty}([0, \infty) \times Z \times \mathbf{R}^n;\ \mathcal{S}(\mathbf{R}^N)) \tag{3.6}$$

and for any $a \in \dot{S}^m(\Gamma) L(\varphi)a$ is always rapidly decreasing as $|(s,\theta)| \to \infty$. It follows from Lemma II.1.5 that the map (3.6), restricted to $s \in [0,\varepsilon]$, $\varepsilon > 0$, is surjective, since if $a \in S^{-\infty}$ so is $e^{i\varphi}a$ and conversely. This proves the existence of lacunary symbols for the phase function φ, since $\overline{L(\varphi)\,a} = M(e^{-i\varphi}\tilde{a})$, with M as in (II.1.6).

PROPOSITION 3.7. *If $S^m_{\mathrm{la}}(\Gamma,\varphi) \subset \dot{S}^m(\Gamma)$ consists of the symbols a such that $L(\varphi)a$ is rapidly decreasing as $s \downarrow 0$ and $s \uparrow \infty$ then there is a continuous linear map*

$$B\colon \dot{S}^m(\Gamma) \to S^{-\infty}$$

such that $\mathrm{Id} + B_\varphi\colon \dot{S}^m(\Gamma) \to S^m_{\mathrm{la}}(\Gamma,\varphi)$.

If φ is a phase function with (3.1) holding in Γ, consider the oscillatory integral operator

$$Fu(z) = \int e^{i\varphi(z,y',x\xi,\theta) - ix'\xi}\, a(z,y',x\xi,\theta)\, u(z')\, dz'\, d\xi\, d\theta. \tag{3.8}$$

Making the usual change of variable $s = x'/x$, $\mu = x\xi$ reduces (3.8), formally, to

$$Fu(z) = \int e^{i\varphi(z,y',\mu,\theta) - is\mu}\, a(z,y',\mu,\theta)\, u(xs,y')\, dy'\, ds\, d\mu\, d\theta. \tag{3.9}$$

To investigate (3.9) it is necessary to recall, briefly, the theory of Lagrangian distributions which is to be found, in part implicitly, in Hörmander [10] and for example in Guillemin and Sternberg [7]. In fact following the elementary definition in Chapter I of the Lagrangian distributions associated to the conormal bundle to the boundary it is relatively easy to give a direct definition of the more general space $I^\infty(M,\Lambda)$. Here, $\Lambda \subset T^*M \setminus 0$ is a homogeneous Lagrangian submanifold. The local representation theorem, a straightforward extension of Darboux's theorem shows that near each $\lambda \in \Lambda$ there are $n = \dim \Lambda$ C^∞ functions g_j which are real-valued, homogeneous of degree one and vanish on Λ, simply near λ. In consequence,

$$\{g_i, g_j\} = 0 \quad \text{on } \Lambda. \tag{3.10}$$

To extend this geometric structure to analysis, choose properly supported pseudodifferential operators $G_j \in L^1(M)$ with principal symbols g_j, $j = 1, \dots, n$. Then consider a distribution $u \in \mathcal{E}'(M)$ which has

$$\mathrm{WF}\,(u) \subset \Lambda \tag{3.11}$$

and for which there exists $s \in \mathbf{R}$ such that

$$G_{i_1} \dots G_{i_k} u \in H^s_{\mathrm{loc}}(M), \quad \forall k. \tag{3.12}$$

Definition 3.13. $I^\infty(M, \Lambda)\subset \mathcal{D}'(M)$ consists of those distributions $v\in\mathcal{D}'(M)$ such that (3.11) holds and for each $\varrho\in C_c^\infty(M)$, (3.12) holds for $u=\varrho v$ and all local defining functions g_j for Λ; s depending only on v.

Note that (3.12) is independent of the choice of G_j with $\sigma_1(G_j)=g_j$ since pseudodifferential operators of order zero are bounded on H^s_{loc}, if properly supported, and we assume (3.12) for all local defining functions. A straightforward application of the theory of Fourier integral operators which, for completeness, is outlined below, shows that this definition is actually equivalent to the usual one in terms of oscillatory integrals and local parametrizations.

It is clear that (3.12) is microlocal; that is it holds for u if and only if it holds for all $B_j u$ when the $B_j\in L^0(M)$ form a pseudodifferential operator partition of unity. Thus, it is enough to assume that for a preassigned neighbourhood Λ' of some $\lambda\in\Lambda$,

$$\mathrm{WF}\,(u)\subset\Lambda'.$$

Taking a canonical transformation

$$c\colon T^*M,\Lambda',\lambda\to T^*\mathbf{R}^n, N^*\{x_1=0\}, (0,0,\ldots,0;1,0,\ldots,0)$$

and associated properly supported Fourier integral operator, F, elliptic over WF (u), it is immediate that (3.12) holds for u if and only if it holds for Fu with respect to the Lagrangian $N^*\{x_1=0\}$. This case is analysed in Section I, where it is shown, in equation (I.1.5), that Fu can be written, modulo C^∞, as an oscillatory integral with phase function $x_1\xi$ parametrizing $N^*\{x_1=0\}$. The fact, namely the Lemma of stationary phase, that Fourier integral operators associated to canonical transformations act on such oscillatory integrals, together with the calculus developed in [10] proves the following result, with respect to Definition 3.13.

THEOREM 3.14. *For each $u\in I^\infty(M,\Lambda)$ there exists $m\in\mathbf{R}$ such that if $\varphi_j\in C^\infty(M\times\mathbf{R}^{N_j})$ are local parametrizations of Λ in open cones Γ_j, in the sense of (2.31), covering the whole of Λ then there exist symbols $a_j\in S^{m+\frac14 n-\frac12 N_j}_{1,0}(M\times\mathbf{R}^{N_j})$ and $u_R\in C^\infty(M)$ such that the following sum is locally finite and*

$$\sum_j\int e^{i\varphi_j(m,\theta)}a_j(m,\theta)\,d\theta=u-u_R. \tag{3.15}$$

The symbol isomorphism

$$\sigma_m\colon I^m(M,\Lambda)/I^{m-1}(M,\Lambda)\leftrightarrow S^{m+\frac14 n}(\Lambda, L\otimes\Omega^{\frac12}_\Lambda\otimes\Omega_M^{-\frac12})/S^{m+\frac14 n-1}(\Lambda, L\otimes\Omega^{\frac12}_\Lambda\otimes\Omega_M^{-\frac12}), \tag{3.16}$$

defined by Hörmander, has L the Maslov bundle on Λ, $\Omega^{\frac12}_\Lambda$ the $\frac12$-density bundle on Λ and $\Omega_M^{-\frac12}$ the pull back to Λ of the dual to the $\frac12$-density bundle on the base, M. With respect to the local *coordinates introduced by the parametrizations p_j, and corresponding local trivializations of the bundles over Λ, $\sigma_m(u)$ is just a linear combination with C^∞ coefficients of the a_j on C_{φ_j}.*

Now, suppose that M is a C^∞ manifold with boundary and that $\Lambda \subset T^*M\setminus 0$ is a homogeneous Lagrangian manifold with

$$\Lambda \pitchfork \partial T^*M, \quad N^*\partial M \cap \Lambda = \emptyset. \tag{3.17}$$

It was shown above that, under condition (2.29), such a submanifold Λ can be extended to a Lagrangian submanifold in the cotangent bundle of any extension of M across that boundary.

If Λ_1, Λ_2 are two extensions of Λ, near λ, in extensions M_1, M_2 of M then there always exist parametrizations $\varphi_i \in C^\infty(M_i \times \mathbf{R}^N)$ of Λ_i near λ, with $\varphi_1 = \varphi_2$ over $M \times \mathbf{R}^N$ near the base point. Indeed, in view of the first condition in (3.17), it is possible to introduce local coordinates (x, y) in M so that Λ is given by

$$y_j = \frac{\partial}{\partial \eta_j}\psi(x, y', \eta''), \quad y' = (y_1, \ldots, y_k), \quad \eta'' = (\eta_{k+1}, \ldots, \eta_n), \quad j = k+1, \ldots, n.$$

Then the same is true for Λ_1, Λ_2 in local coordinates in M_1, M_2 extending these coordinates with ψ_1, ψ_2 replacing ψ and both extensions of it into $x<0$. The phase functions

$$\varphi_i = y''\cdot\eta'' - \psi_i(x, y', \eta'')$$

are parametrizations as desired. This means that any $u_1 \in I^m(M_1, \Lambda_1)$ with wavefront set near λ is equal in M to some $u_2 \in I^m(M_2, \Lambda_2)$.

Definition 3.18. Let $\Lambda \subset T^*M\setminus 0$ be a homogeneous Lagrangian submanifold satisfying (3.17). The space $I^m(M, \Lambda) \subset \mathcal{D}'(M)$ consists of those distributions which can be written as the restriction to M of a locally finite sum of distributions in the spaces $I^m(M_E, \Lambda_E)$ corresponding to extensions of M, Λ.

The independence of extension discussed above shows that the symbol isomorphism (3.16) remains valid when M is a manifold with boundary and (3.17) holds if the symbol spaces on the right are interpreted in the obvious way for Λ a manifold with boundary, i.e. as the restrictions to Λ of symbols on an extension. Note that, because of the second condition in (3.17), each element of $I^m(M, \Lambda)$ is C^∞ up to the boundary with respect to any normal variable, so is in $\mathcal{A}'(M)$. In particular one can equally well consider, unambiguously,

$$I^m(M, \Lambda) \subset \dot{\mathcal{D}}'(M),$$

by cutting off at the boundary. Note also that

$$\bigcap_m I^m(M, \Lambda) = C^\infty(M).$$

PROPOSITION 3.19. *The integral in* (3.9):

$$\Phi(a) = \int e^{i\varphi(z,y',\mu,\theta)-is\mu}\,a(z,y',\mu,\theta)\,d\mu\,d\theta$$

defines a linear map

$$\Phi\colon S^m_{\mathrm{la}}(\Gamma,\varphi)\to I^m(Z\,\hat{\times}\,Z,\hat{\Lambda}) \tag{3.20}$$

whenever $\hat{\Lambda}$ *is a boundary canonical relation in* $T^*(Z\,\hat{\times}\,Z)$, *parametrized by* φ *in the sense of Section* 2. *Moreover*, $\Phi(a)$ *vanishes to all orders on the parts* $\partial Z\times\mathring{Z}$ *and* $\mathring{Z}\times\partial Z$ *of the boundary* $Z\,\hat{\times}\,Z$.

Proof. It should be noted that the abuse of notation in (3.20), arising from the fact that $Z\,\hat{\times}\,Z$ is not a manifold with boundary, is minor since the Lagrangian $\hat{\Lambda}$ only meets the boundary away from the corner and does have the transversality properties (3.17) there. Thus, $I^m(Z\,\hat{\times}\,Z,\hat{\Lambda})$ consists of distributions which are C^∞ up to the corner. In fact, since it is shown in Proposition 2.20 that $\varphi(z,y',\mu,\theta)-s\mu$ does parametrize $\hat{\Lambda}$ the proposition has already been proved.

Following this preliminary investigation of (3.9) the space of kernels of interest can be defined directly on $N\,\hat{\times}\,M$.

Definition 3.21. Let M, N be C^∞ manifolds with boundary. If $\hat{\Lambda}\subset T^*(N\,\hat{\times}\,M)$ is a boundary canonical relation in the sense of definition (2.19), the space

$$\mathcal{K}^m(\Lambda)\subset I^m(N\,\hat{\times}\,M,\hat{\Lambda};\hat{\Omega}_N)\hookrightarrow\dot{\mathcal{D}}'(N\,\hat{\times}\,M;\hat{\Omega}_N)$$

consists of those Lagrangian sections of $\hat{\Omega}_N$ associated to $\hat{\Lambda}$ and vanishing to all orders at $N\times\partial M$ and $\partial N\times M$ in $N\,\hat{\times}\,M$.

The residual space $\mathcal{K}^{-\infty}(\hat{\Lambda})=\mathcal{K}^{-\infty}(M,N)$ is clearly independent of $\hat{\Lambda}$. In fact,

$$\mathcal{K}^{-\infty}(M,N)\subset C^\infty(N\,\hat{\times}\,M;\hat{\Omega}_N)$$

consists of those smooth sections vanishing at the part $\mathring{N}\times\partial M\cup\partial N\times\mathring{M}$ of the boundary. Suppose that M and N have the same dimension so are locally diffeomorphic to the same standard manifold Z. Clearly $\mathcal{K}^{-\infty}(M,N)$ is just, locally, the space of kernels of operators in $L_b^{-\infty}(Z)$. Indeed, by the definition of $\hat{\Omega}_N$ any $k\in\mathcal{K}^{-\infty}(M,N)$ is of the form

$$k=\frac{1}{x}\,\alpha\left(x,y,y',\frac{x'}{x}\right)dx'\,dy' \tag{3.22}$$

with $\alpha\in C^\infty$; this proves the following result.

PROPOSITION 3.23. *If M, N are manifolds of the same dimension the injection,*

$$\mathcal{K}^{-\infty}(M, N) \hookrightarrow \mathcal{D}'(N \,\hat{\times}\, M, \hat{\Omega}_N),$$

given by (3.22) *in local coordinates, maps onto the space of Schwartz kernels, of the operators in $L_b^{-\infty}(M, N)$, the elements of which give maps in $L_b^{-\infty}(Z)$ in any local coordinates.*

Even when the dimensions of the two manifolds are different the residual kernels are, essentially, the same in local coordinates except for the differing number of y and y' variables. Since these are basically parameters all the results for operators in $L^{-\infty}(Z)$ apply, simply by the addition of an appropriate number of extra variables.

Similarly it is now possible to define the space of totally characteristic Fourier integral operators associated to a given boundary canonical relation. Each element $F \in I_b^m(M, N; \hat{\Lambda})$ is fixed uniquely by an element of $\mathcal{K}^m(\hat{\Lambda})$ by noting that in any local coordinates $z = (x, y)$, $z' = (x', y')$ on M and N the formal definition (3.9) can be written

$$Fu(z) = \int K(x, y, y', s)\, u(sx, y')\, dy'\, ds, \quad u \in C_c^\infty(N) \tag{3.24}$$

where $K dy' ds \in \mathcal{K}^m(\hat{\Lambda})$. It can be assumed, by adding a term in $L^{-\infty}(M, N)$ that K has compact support in $s \in (0, \infty)$ and then the integral is just the distributional pairing of integrand and 1. The assumption (2.15) on the boundary canonical relation $\hat{\Lambda}$ means that $(x, y) \mapsto K(x, y, \cdot, \cdot) \in \mathcal{D}'(\mathbf{R}^n \times \mathbf{R}_s)$ is C^∞ so any $F \in I^m(M, N; \hat{\Lambda})$ is a map

$$F\colon C_c^\infty(M) \to C^\infty(N).$$

III.4. Operator calculus

The definition of totally characteristic Fourier integral operators given above in terms of the kernels in $\mathcal{K}^m(\hat{\Lambda})$ can be extended to other vector bundles over M and N. If E, F are vector bundles over M, N respectively the space of kernels is just the space of distributional sections of $\hat{\Omega}_N \otimes E' \otimes F$, Lagrangian over $\hat{\Lambda}$ and vanishing to all orders at $\partial N \times M \cup N \times \partial M$. The corresponding space of operators will be denoted $I_b^m(E, F; \hat{\Lambda})$, with M and N understood.

PROPOSITION 4.1. *If $F \in I_b^m(M, N; \hat{\Lambda})$ the adjoint operator $F^* \in I_b^m(\Omega_N, \Omega_M; \hat{\Lambda}^{-1})$.*

Proof. In computing the adjoint it suffices, in view of continuous dependence on the kernel, to assume $F \in L_b^{-\infty}(M, N)$. Then, in local coordinates in which (3.24) holds

$$F^*\varphi(z) = \int \bar{K}\left(tx', y, y', \frac{1}{t}\right) \varphi(tx', y) \frac{dt}{t}\, dy. \tag{4.2}$$

The mapping

$$K(x,y,y',s)\,\frac{dx'}{x}\,dy' \mapsto \frac{1}{t}\,K\!\left(tx',y,y',\frac{1}{t}\right)\frac{dx}{x'}\,dy\cdot dx'\,dy'\cdot(dx\,dy)^{-1} \tag{4.3}$$

is the coordinate version of an isomorphism:

$$I^m(N\,\hat{\times}\,M,\hat{\Lambda}_\chi;\hat{\Omega}_N)\to I^m(M\,\hat{\times}\,N,\hat{\Lambda}^{-1};\hat{\Omega}_M\otimes\Omega_N\otimes\Omega_M^{-1}). \tag{4.4}$$

In fact the only point not immediately clear is the identity of the Lagrangian on the right in (4.4). However, the fact that (4.4) is valid in the interior, which follows from the standard theory of Fourier integral operators, shows this Lagrangian to be $\hat{\Lambda}^{-1}$.

By duality it follows from this proposition that any element $F\in I_b^m(M,N;\Lambda)$ has the mapping property:

$$F\colon \dot{\mathcal{E}}'(M)\to\dot{\mathcal{D}}'(N). \tag{4.5}$$

Next consider the formula for the product of two pseudodifferential operators in $L_b^{-\infty}(Z)$, given in (II.7.9). Writing this out in terms of the kernels of F_1, F_2 of the form (3.24), assuming supports to be compact:

$$F_1F_2\varphi(z)=\int K_1(z,y,y',s)\,K_2(sx,y',y'',s')\,\varphi(ss'x,y'')\,dy''\,ds'\,ds\,dy'.$$

Since K_1, K_2 are rapidly decreasing as $s\to 0,\infty$ the change of variable $t=ss'$ is permissible, so $F=F_1F_2$ is of the form

$$F\varphi(z)=\int K(z,y,y'',t)\,\varphi(tx,y'')\,dy''\,dt \tag{4.6}$$

where

$$K(z,y,y'',t)=\int K_1(z,y,y',s)\,K_2\!\left(sx,y',y'',\frac{t}{s}\right)dy'\,\frac{ds}{s}. \tag{4.7}$$

Noting that an operator in $I_b^m(M,N;\Lambda)$ is properly supported if and only if the two projections from the support of K in $N\,\hat{\times}\,M$, to M and N are proper, let the corresponding subspaces of properly supported operators and kernels be $I_{b,P}^m(M,N;\hat{\Lambda})$ and $\mathcal{K}_P^m(\hat{\Lambda})\subset\dot{\mathcal{D}}'(N\,\hat{\times}\,M)$.

PROPOSITION 4.8. *If* $\chi_1\colon\tilde{T}^*N\to\tilde{T}^*Q$, $\chi_2\colon\tilde{T}^*M\to\tilde{T}^*N$ *are boundary canonical transformations the mapping* (4.7) *extends by continuity to a separately continuous bilinear form*

$$\mathcal{K}^m(\chi_1)\times\mathcal{K}_p^{m'}(\chi_2)\to\mathcal{K}^{m+m'}(\chi_1\chi_2). \tag{4.9}$$

Proof. It is enough to suppose that K_1, K_2 have compact supports away from $s=0,\infty$ in (4.7), since the other terms are certainly in $L^{-\infty}(M,Q)$. Thus, (4.7) reduces to a computation with Lagrangian distributions. Restoring the oscillatory integral representation of the kernels K_i into (4.7) gives

$$K(z, y, y'', t) = \int \exp\,(i\varphi_1(z, y, y', \mu, \theta) - is\mu + i\varphi_2(sx, y', y'', \mu', \theta') - it\mu'/s) \\ \times c_1(z, y', \mu, \theta)\, c_2(sx, y', y'', \mu', \theta')\, d\mu\, d\mu'\, d\theta\, ds'. \tag{4.10}$$

The phase function in (4.10) is non-degenerate, so K is a Lagrangian distribution. It can be checked directly that this phase function parametrizes $\chi_1 \cdot \chi_2$ but, again, this is not necessary since from the standard theory of Fourier integral operators it is known to be true in the interior. This proves (4.9) with the continuity in symbols obvious from (4.10).

Thus, if $F_1 \in I_b^m(N, Q; \chi_1)$ and $F_2 \in I_b^{m'}(M, N; \chi_2)$ then $F_1 \cdot F_2 \in I_b^{m+m'}(M, Q; \chi_1 \cdot \chi_2)$ provided one at least of F_1, F_2 is properly supported. If F_2 is properly supported this follows directly from Proposition 4.8 and (4.7), if F_1 is properly supported then it follows by considering the adjoints.

Observe that the analogue of Proposition II.5.10 holds for these Fourier integral operators. Indeed, the vector field D_s on $T^*(N \hat{\times} M)$, near $\partial N \times \partial M \times [0, \infty] \cap \hat{\Lambda}$ is not tangent to $\hat{\Lambda}$. Thus, one easily deduces the extension of (II.1.9) for the kernels in $\mathcal{K}^m(\hat{\Lambda})$; namely regularity in the t-variable can always be obtained by integration by parts. Thus,

$$F\colon H_c^s(M) \to H_{\mathrm{loc}}^S(N) \tag{4.11}$$

where $S = S(s, m)$. If $A \in L_{b,P}^k(N)$ then using the composition formula above there exists $B \in L_{b,P}^k(M)$ such that

$$A \cdot F = F' \cdot B + G \tag{4.12}$$

with $F' \in I_k^{b,P}(M, N; \chi)$ and $G \in L_b^{-\infty}(M, N)$. If χ is not globally a diffeomorphism then it may be necessary to assume that A in (4.12) has small essential support. In any case, for $u \in \dot{\mathcal{A}}_c(M)$, such that $Bu \in H_c^s(M)$ for all $B \in L_{b,P}^\infty(M)$,

$$A \cdot (Fu) \in H_{\mathrm{loc}}^{S'}(N)$$

with S', sufficiently negative, independent of A. This shows that

$$F\colon \dot{\mathcal{A}}_c(M) \to \dot{\mathcal{A}}(N). \tag{4.13}$$

By duality it follows that

$$F\colon \mathcal{A}_c'(M) \to \mathcal{A}'(N) \tag{4.14}$$

if $F \in I_b^\infty(M, N; \chi)$. Naturally we expect that there is a map

$$(\ )_\partial\colon I_b^m(M, N; \chi) \to I^m(\partial M, \partial N; \partial\chi), \tag{4.15}$$

extending (II.5.25), such that

$$Fu\,|_{\partial N} = F_\partial(u\,|_{\partial M}), \quad \forall u \in \mathcal{A}'_c(M). \tag{4.16}$$

To see this, consider (3.24). For $u \in C_c^\infty(M)$, (4.16) follows directly with

$$(F_\partial \varphi)(y) = \int K(0, y, y', s)\varphi(y')\,dy'\,ds \tag{4.17}$$

in local coordinates. Naturally (4.16) follows in general from the (weak) continuity of boundary values.

PROPOSITION 4.18. *The map* (4.15) *is defined on the kernels by*

$$\mathcal{K}^m(\chi) \ni K \mapsto c_* i^* K \in I^m(\partial M \times \partial N;\, \Omega_{\partial N};\, (\text{graph } \partial\chi)') \tag{4.19}$$

where $i\colon \partial M \times \partial N \times [-1, 1] \hookrightarrow M \mathbin{\hat{\times}} N$ *is the inclusion and* $c\colon \partial M \times \partial N \times [-1, 1] \to \partial M \times \partial N$ *is the projection.*

Proof. Naturally (4.19) is just the invariant form of (4.17). It is really only necessary to check the behaviour of the bundles. Note however that the restriction i^* is well-defined because of the second condition in (3.17), on $M \mathbin{\hat{\times}} N$. The integration c_* is meaningful since K vanishes rapidly at $0, \infty$. Now note that K is a generalized section of $\hat{\Omega}_M$, that is of the form $\varphi\nu/x$ with φ Lagrangian and ν a C^∞ section of the density bundle Ω_M lifted to $M \mathbin{\hat{\times}} N$. Thus the restriction to $\partial M \times \partial N \times [0, \infty]$ gives, invariantly, a section of

$$\Omega_{\partial M} \cdot (N^*\partial N)^{-1} \cdot (N^*\partial M) \tag{4.20}$$

since at ∂M, $\Omega_M \cong \Omega_{\partial M} \otimes (N^*\partial M)$ and x transforms at ∂N as $N^*\partial N$. The 1-form ds transforms as $(N^*\partial N)^{-1} \cdot (N^*\partial M)$ at the component $\partial M \times \partial N \times [0, \infty]$ of the boundary so (4.19) holds independently of coordinates.

Next let us note the symbolic versions of these properties. First define the symbol of a Fourier integral operator in terms of the symbol of its kernel, using (3.16):

$$I_b^m(M, N; \hat{\Lambda}) \to S^{m+\frac{1}{2}(n+1)}(\hat{\Lambda}_\chi;\, L \otimes \Omega^{\frac{1}{2}}_{\hat{\Lambda}} \otimes \hat{\Omega}_M \otimes \Omega^{-\frac{1}{2}}_{M \hat{\times} N})/S^{m+\frac{1}{2}(n+1)-1}. \tag{4.21}$$

If χ is a diffeomorphism this can be simplified in the usual way. The map (2.33) gives an isomorphism

$$\chi'\colon T^*N \to \hat{\Lambda}_\chi. \tag{4.22}$$

Pulling back the various bundles over $\hat{\Lambda}_\chi$ and using the isomorphism

$$\Omega^{\frac{1}{2}}(\tilde{T}^*N) \otimes \hat{\Omega}_M \otimes \Omega^{-\frac{1}{2}}(M \mathbin{\hat{\times}} N) \cong \Omega^{\frac{1}{2}}_M \otimes \Omega_N^{-\frac{1}{2}} \otimes (\tilde{\omega})^{\frac{1}{2}(n+1)}, \tag{4.23}$$

where $\tilde{\omega}$ is the singular 2-form on $\tilde{T}^*N$, gives

$$\sigma_{m,L}\colon I^m(M, N; \chi) \to S^m(\tilde{T}^*N; L\otimes\Omega_M^{\frac{1}{2}}\otimes\Omega_N^{-\frac{1}{2}})/S^{m-1} \tag{4.24}$$

the left- or image-reduced symbol. In local coordinates (4.23) is just the relation

$$(dx\,d\lambda\,dy\,d\eta)^{\frac{1}{2}}\cdot(ds\,dy')\cdot(ds\,dx\,dy\,dy')^{-\frac{1}{2}} = c_n\cdot(dx'\,dy')^{\frac{1}{2}}\cdot(dx\,dy)^{-\frac{1}{2}}\cdot\left(\frac{1}{x}\,dx\,d\lambda\,dy\,d\eta\right)^{\frac{1}{2}}$$

where c_n is a constant and $s = x'/x$.

Naturally one can define, instead of (4.24), the right- or domain-reduced symbol:

$$\sigma_{m,R}\colon I^m(M, N, \chi) \to S^m(\tilde{T}^*M; L\otimes\Omega_M^{\frac{1}{2}}\otimes\Omega_N^{-\frac{1}{2}})/S^{m-1} \tag{4.25}$$

and then one has Egorov's theorem:

$$\sigma_{m,L}\cdot\chi = \sigma_{m,R}\,. \tag{4.26}$$

This is most often seen in combination with the product formula. If $F\in I_b^m(M, N; \chi_1)$ and $G\in I_b^{m'}(N, Q; \chi_2)$ with at least one properly supported then

$$\sigma_{m+m',R}(G\cdot F) = \sigma_{m,R}(F)\cdot\chi_1^*\sigma_{m',R}(G) \tag{4.27}$$

where the product involves the tensorial cancellation

$$\Omega_M^{\frac{1}{2}}\otimes\Omega_N^{-\frac{1}{2}}\otimes\Omega_N^{\frac{1}{2}}\otimes\Omega_Q^{-\frac{1}{2}} = \Omega_M^{\frac{1}{2}}\otimes\Omega_Q^{-\frac{1}{2}} \quad \text{over } \tilde{T}^*M. \tag{4.28}$$

For a pseudodifferential operator, $\sigma_{m,R} = \sigma_{m,L}$ so if $A\in L_{b,P}^m(N)$ and $B\in L_{b,P}^m(M)$ with $A\cdot F = B\cdot F$, F elliptic then (4.27) becomes

$$\chi^*\sigma_m(A) = \sigma_m(B). \tag{4.29}$$

For adjoints one has the usual simple formula

$$\sigma_{m,R}(F^*) = \overline{\sigma_{m,L}(F)}, \tag{4.30}$$

where we use the obvious extension of (4.24) for operators on vector bundles

$$\sigma_{m,L}\colon I^m(E, F; \chi) \to S^m(\tilde{T}^*N; L\otimes\Omega_M^{\frac{1}{2}}\otimes\Omega_N^{-\frac{1}{2}}\otimes E'\otimes F) \tag{4.31}$$

and cancellation amongst the density bundles. Similarly, consider the restriction to the boundary. Here,

$$\sigma_{m,L}(F_\partial) = \sigma_{m,L}(F)\big|_{T^*\partial N}\,. \tag{4.32}$$

Over the inclusion $T^*\partial N \hookrightarrow \partial\tilde{T}^*N \subset \tilde{T}^*N$ the isomorphism

$$\Omega_M^{\frac{1}{2}}\otimes\Omega_N^{-\frac{1}{2}} \cong \Omega_{\partial M}^{\frac{1}{2}}\otimes\Omega_N^{-\frac{1}{2}}\otimes[(N^*\partial M)(N^*\partial N)^{-1}]^{\frac{1}{2}}$$

together with the trivialization

$$N^*\partial M = N^*\partial N \quad \text{over } \partial\hat{\Lambda}_\chi, \tag{4.33}$$

given by the transversality of D_s and $\hat{\Lambda}_\chi$, i.e. $s = dx'/dx = f(y, \eta)$ at $x = x' = 0$, shows that (4.3) is meaningful with

$$\sigma_{m,L}(F_\partial) \in S^m(T^*\partial N; L_\partial \otimes \Omega^{\frac{1}{2}}_{\partial M} \otimes \Omega^{-\frac{1}{2}}_{\partial N}) / S^{m-1}$$

as it should be.

To complete the calculus it should be noted that the symbol map (4.24) can also be given an oscillatory test definition. The usual oscillatory test method (see [6]) defines (4.21) by examining the behaviour for large τ of the pairing of the kernel (with C^∞ section of $\hat{\Omega}_M$ removed) with an oscillatory term:

$$\langle \alpha(x, y, y', s), \varrho(x, y, y', s)\, e^{i\tau\psi(x, y, y', s)} \rangle$$

where $\varrho \in C_c^\infty$ localizes near a chosen point and $\psi \in C^\infty$ is real-valued and such that the Lagrangian $\Lambda_\psi = \text{graph}\, d\psi$ meets $\hat{\Lambda}_\chi$ transversally. The transversality properties of $\hat{\Lambda}_\chi$ show that to test near the boundary ψ can always be chosen in the form

$$\psi = \beta(x, y, y') + \alpha \log s \tag{4.34}$$

with α constant and $\beta \in C^\infty$, $d\beta \neq 0$. It suffices to consider curves

$$f_\tau(x', y') = \varrho\, e^{i\tau\beta}(x')_+^{i\tau\alpha} \in \dot{\mathcal{A}}_c(M) \tag{4.35}$$

in M, where $0 \leqslant x \in C^\infty(M)$ vanishes simply on ∂M, β is real-valued and $d\beta \neq 0$ on the support of ϱ.

PROPOSITION 4.36. *If $F \in I^m(M, N; \chi)$ then*

$$F(f_\tau) = a_\tau e^{i\tau\beta'}(x)_+^{i\tau\alpha} \tag{4.37}$$

where $\beta' \in C^\infty(N)$, $x \in C^\infty(N)$ are real-valued, x vanishes simply on ∂N and $a_\tau \in S^m(M \times \mathbf{R})$.

Proof. Substituting (4.35) into (3.34) gives

$$F(f_\tau) = (x)_+^{i\tau\alpha} \int K(x, y, y', s)\, e^{i\tau\beta(xs, y') + i\tau\alpha \log s}\, dy'\, ds.$$

The standard theory of Lagrangian distributions now gives (4.37).

The details of the formula relating a_τ, modulo S^{m-1}, to $\sigma_{m,L}(F)$ are left to the interested reader to glean from [6].

III.5. Boundary value problems

The operators in $L_b^m(M)$ are totally characteristic and one would not normally expect to prescribe boundary conditions on elements of their kernels. To give a general setting for boundary value problems, transformed by the Fourier integral operators described above, we can easily extend the definition of $L_b^m(M)$.

Definition 5.1. If $k \in \mathbf{N}$ we let $L_b^{m,k}(M)$ be the space of operators $\dot{\mathcal{E}}'(M) \to \dot{\mathcal{D}}'(M)$ locally of the form

$$\sum_{\text{finite}} A_j P_j + Q_j B_j$$

where B_j, $A_j \in L_b^{m-m_j}(M)$ and P_j, Q_j are differential operators of order $m_j \leqslant k$.

First consider the kernels of operators in $L_b^{-\infty,k}(M)$. Directly from the definition we see that in local coordinates these kernels are at most k-fold derivatives on the right or left of kernels in $L_b^{-\infty}$. Certainly $A \in L_b^{-\infty,k}$ has the mapping property demanded in Lemma II.4.1, so is determined by its Schwartz kernel in the open quarter space.

Recall that on $M \hat{\times} M$ the bundle $\hat{\Omega}_M$ has C^∞ sections which are of the form, with respect to local coordinates, $\alpha\nu/x$ with ν a C^∞ density lifted from the first factor and $\alpha \in C^\infty(M \hat{\times} M)$. More generally let $\hat{\Omega}_M^{(k)}$ be the line bundle with sections $w = \alpha\nu/x^{k+1}$, $k \in \mathbf{N}$. It is meaningful to say that such a section vanishes to all orders at the part $\partial M \times \mathring{M} \cup \mathring{M} \times \partial M$ of the boundary of $M \times M$, i.e. that α can be chosen to vanish to all orders there. For such a section, and working in local coordinates $(x, y, y', s) s = x'/x$,

$$w = \sum_{1 \leqslant p \leqslant k} x^{-p} w_p(y, y', s)\, ds\, dy' + w_0 \tag{5.2}$$

where $w_0 \in C^\infty(\hat{\Omega}_M)$ also vanishes to all orders at $\partial M \times \mathring{M} \cup \mathring{M} \times \partial M$. The expansion (5.2) is not coordinate invariant, but the conditions

$$\int_0^\infty s^r w_p(y, y', s)\, ds = 0, \quad \text{on } \partial M \times \partial M \quad \text{for } 0 \leqslant r < p,\ 1 \leqslant p \leqslant k \tag{5.3}$$

are invariant. This can be seen directly, but follows indirectly from the next result.

PROPOSITION 5.4. *The kernels of operators in $L_b^{-\infty,k}(M)$ constitute the space of C^∞ sections over $M \hat{\times} M$ of $\hat{\Omega}_M^{(k)}$ vanishing to all orders on $\partial M \times \mathring{M} \cup \mathring{M} \times \partial M$ and satisfying* (5.3).

Proof. The definition of $L_b^{-\infty,k}(M)$ is local so it suffices to work on Z. Consider $A \cdot D_x^k \in L_b^{-\infty,k}(Z)$ where $A \in L_b^{-\infty}(Z)$. Ignoring a term in $\dot{C}^\infty(Z \times Z)$, which is clearly unimportant,

$$\begin{aligned}(A\cdot D_x^k)\varphi(z) &= \int \alpha(z,y,y',s)\,(D_{x'}^k\varphi)(xs,y')\,ds\,dy' \\ &= \int \alpha(z,y,y',s)x^{-k}D_s^k\varphi(xs,y')\,ds\,dy' \\ &= \int x^{-k}(-D_s)^k\alpha(z,y,y',s)\varphi(xs,y')\,ds\,dy'. \end{aligned} \tag{5.5}$$

Observe that a kernel w satisfying (5.3) can always be written in the form

$$K=\sum_{0\leqslant p\leqslant k} x^{-p}(-D_s)^p\alpha_p \tag{5.6}$$

with $\alpha_p\in\mathcal{K}^{-\infty}$. Indeed the conditions (5.3) are precisely the requirement that the solution of

$$D_s^p\alpha_p=w_p,\quad D_s^j\alpha_p(0)=0\quad 0\leqslant j<p,$$

should be rapidly decreasing as $s\to\infty$ and hence in $\mathcal{K}^{-\infty}$. This shows that all sections of the stated type can be obtained as kernels from $L_b^{-\infty,k}$.

Now the converse follows from (5.6) and the fact that for $A\in L_b^{-\infty}$ there exists B, $E\in L_b^{-\infty}$ such that

$$A\cdot D_x=L_x\cdot B+E. \tag{5.7}$$

To see (5.7) just observe that if the kernel of B is $x^{-1}\beta(z,y',s)$ then that of $D_x\cdot B$ is $x^{-1}(D_x\beta)-x^{-2}D_s(s\beta)$. So if A has kernel $x^{-1}\alpha$ it is only necessary to take $\beta=s^{-1}\alpha$ and E with kernel $x^{-1}(D_x s^{-1}\alpha)$. Iteration of (5.7) shows that

$$G\in L_b^{-\infty,k}(Z)\quad\Leftrightarrow\quad G=\sum_{0\leqslant j\leqslant k}A_j\cdot D_x^j,\quad A_j\in L_b^{-\infty}(Z), \tag{5.8}$$

which completes the proof of the proposition.

We remark that the continuity properties of the operators in $L_b^m(M)$, with respect to their kernels, allow one to deduce immediately the behaviour of the finite order operators.

COROLLARY 5.9. *The kernels of $L_b^{m,k}(M)$ are the generalized sections of $\hat{\Omega}_M^{(k)}$ over $M\hat{\times}M$ which are Lagrangian over the image in $M\hat{\times}M$ of the diagonal, of order m, and which vanish to all orders on the part $\partial M\times\mathring{M}\cup\mathring{M}\times\partial M$ of the boundary.*

It also follows directly from (5.8) that the properly supported elements form a bi-filtered ring:

$$L_b^{m,k}(M)\cdot L_{b,P}^{m',k'}(M)\subset L_b^{m+m',k+k'}(M). \tag{5.10}$$

Away from the boundary, elements of $L_b^{m,k}(M)$ are again just pseudodifferential operators in the usual sense. We briefly discuss the symbolic properties of this space.

As a consequence of Proposition 5.4 the symbol of an element in $L_b^{m,k}(M)$ is well-defined as a formal quotient of a symbol of order m in $\tilde{T}^*M$, a, which vanishes to order k at $T^*\partial M$, by a function homogeneous of degree zero vanishing to precisely order k at ∂T^*M.

$$\sigma_m(A) = \frac{a}{x^k} = \sum_{j\leqslant k} a_j x^{j-k}\lambda^{k-j} \tag{5.11}$$

The calculus clearly extends to these enlarged spaces in a straightforward manner. All differential operators of order k lie in $L_b^{k,k}(M)$ so we can extend the ideas of Section I.3 by extending the condition that an operator be non-characteristic with respect to the boundary.

Definition 5.12. $P \in L_b^{m,k}(M)$ is non-characteristic with respect to ∂M if the leading part of the symbol

$$x^k\sigma_m(A) \in C^\infty(\tilde{T}^*M\diagdown 0)$$

is elliptic on $\partial\tilde{T}^*M \diagdown (T^*\partial M)$ and hence vanishes to precisely kth order at $T^*\partial M$.

Following the computations above this normal ellipticity condition can be restated as follows. $P \in L_b^{m,k}(M)$ is non-characteristic if, locally, it can be written in the form

$$P \equiv AQ \quad \operatorname{mod} L_b^{m,k-1}(M) + L_b^{-\infty,k}(M) \tag{5.13}$$

where Q is a differential operator of order k, non-characteristic, and $A \in L_b^{m-k}(M)$ is elliptic.

PROPOSITION 5.14. *If $P \in L_b^{m,k}(M)$ is non-characteristic then $u \in \dot{\mathcal{E}}'(M)$, $Pu \in \dot{\mathcal{B}}^{(k-1)}(M)$, i.e. $Pu \in \dot{\mathcal{B}}(M)$ with boundary terms only to order $k-1$, implies that*

$$u \in \mathcal{A}'(M) + \dot{\mathcal{A}}(M) \tag{5.15}$$

Proof. Suppose $Pu = f \in \dot{\mathcal{B}}^{(k-1)}(M)$. Using (5.10) and applying a parametrix for A gives

$$P'u = \sum_{j\leqslant k} A_j Q_j u = f - Ru, \quad f \in \dot{\mathcal{B}}^{(k-1)} \tag{5.16}$$

where $A_j \in L^{k-j}$, $A_k = \mathrm{Id}$ and $R \in L^{-\infty,k}$. Thus $Ru \in \dot{\mathcal{A}}$. Now, the terms $\sum_{j<k} A_j Q_j$ in P' map $\dot{\mathcal{A}}_c^{(s)}$ into $\dot{\mathcal{A}}^{(s+k-1)}$ for every $s \in \mathbf{R}$. It follows from the proof of Lemma I.3.2 that P' is an isomorphism, mod $\dot{C}^\infty$, from $\dot{\mathcal{A}}^{(s)}$ to $\dot{\mathcal{A}}^{(s+k)}$. By induction then P' is an isomorphism of $\dot{\mathcal{A}}$ to $\dot{\mathcal{A}}$ mod $\dot{C}^\infty$. We can therefore add a term to u, $u' \in \dot{\mathcal{A}}(M)$, so that

$$P'v = g' \in \dot{\mathcal{B}}^{(k-1)}, \quad v = u - u'. \tag{5.17}$$

The proofs of Lemma I.3.8 and Proposition I.3.12 therefore apply to P' and it follows that $v \in \mathcal{A}'$ as asserted.

Since $\mathcal{A}_c'(M) \cap \dot{\mathcal{A}}_c(M) = C_c^\infty(M)$ it follows from Proposition (5.10) that, at least modulo C^∞, boundary value problems can be stated for non-characteristic operators in $L_b^{m,k}$. Thus, if P is a pseudodifferential operator in $L_b^{m,k}(M; E, F)$ acting on vector bundle sections and is non-characteristic and $f \in \mathcal{A}'(M, F)$ a differential operator, or pseudodifferential operator $B \in L_b^{m,k}(M, E, C)$ can be used to impose boundary conditions

$$RBv \equiv g \in \mathcal{D}'(\partial M, G) \quad \mathrm{mod}\ C^\infty \tag{5.18}$$

and any solution $v \in \dot{\mathcal{D}}'(M)$ to

$$Pv = f \ \mathrm{mod}\ \mathcal{D}'(\partial M, F_{(k-1)}) \tag{5.19}$$

since by (5.11) $Bv \in \dot{\mathcal{B}}(M) + \dot{\mathcal{A}}(M)$ so the boundary condition (5.18), interpreted as acting on the first term, is well-defined modulo C^∞.

In terms of the microdistributions discussed in Section II.8 this problem, (5.18), (5.19) is well-defined with microdistributional data. The basic problem of the theory of the singularities of non-characteristic boundary value problems is the examination of the relationship between $\mathrm{WF}_b\,(f)$, $\mathrm{WF}\,(g)$ and $\mathrm{WF}_b\,(u)$, i.e. the supports of microdistribution solutions. We also remark that the Fourier integral operators discussed above act by conjugation on the space $L_b^{m,k}(M)$. Under such transformation the form of a boundary value problem (5.18), (5.19) is preserved. Thus, the standard transformation method used in the analysis of pseudodifferential operators on boundaryless manifolds can be extended to the case of boundaries.

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