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Precise damping conditions for global asymptotic stability for nonlinear second order systems

by

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1. Introduction

The global asymptotic stability of the rest point for nonlinear equations has been treated by Levin and Nohel, Salvadori, Thurston and Wong, Artstein and Infante, and Ballieu and Peiffer. These studies have been generalized to scalar variational problems in [8] and to variational systems in [9]. Here we shall unify and further extend this work, by obtaining necessary and sufficient conditions for the asymptotic stability of solutions of quasi-variational systems in terms of the damping functions of the systems treated.

Throughout the paper we thus consider vector unknowns $u: J\to \mathbf{R}^N$, where J is a half open interval of the form $[R,\infty)$. The typical system which we shall study then has the form

$$(\nabla\mathcal{L}(t,u,u'))'-\nabla_u\mathcal{L}(t,u,u')=Q(t,u,u'), \tag{1.1}$$

where $\mathcal{L}(t,u,p)=G(u,p)-F(t,u)$ and where

$$G: \mathbf{R}^N\times\mathbf{R}^N\to\mathbf{R}, \quad F: J\times\mathbf{R}^N\to\mathbf{R}, \quad Q: J\times\mathbf{R}^N\times\mathbf{R}^N\to\mathbf{R}^N$$

are given continuously differentiable functions. The most important of the conditions which will be imposed on (1.1) are that

$$G(u,\cdot)\text{ is strictly convex in }\mathbf{R}^N;\quad G(u,0)=0,\quad \nabla G(u,0)=0, \tag{1.2}$$

$$(\nabla_u F(t,u),u)>0\quad\text{for }u\neq 0;\quad F(t,0)=0, \tag{1.3}$$

$$(Q(t,u,p),p)\leqslant 0. \tag{1.4}$$

Here $(\cdot,\cdot)$ denotes the inner product in $\mathbf{R}^N$ and

$$\nabla=\left(\frac{\partial}{\partial p_1},\dots,\frac{\partial}{\partial p_N}\right),\quad \nabla_u=\left(\frac{\partial}{\partial u_1},\dots,\frac{\partial}{\partial u_N}\right).$$

The function F represents a restoring potential, this being analytically described by (1.3), while Q represents a general nonlinear damping, expressed by (1.4). In Section 2 we shall give a complete set of hypotheses, while explicit examples are given at the end of the introduction and also in references [4] and [9].

Since $\nabla G(u,0)=\nabla_u G(u,0)=\nabla_u F(t,0)=Q(t,u,0)=0$ it is clear that the rest state $u\equiv 0$ is a solution of (1.1). We shall say that the rest state is *a global attractor* for the system if any bounded solution u, defined on some interval J, has the property

$$u(t),\ u'(t)\to 0 \quad \text{as } t\to\infty.$$

This concept of asymptotic stability is defined with respect to *bounded* solutions rather than with respect to *all* solutions. The motivation for this point of view is that the concepts of boundedness and of stability are essentially different and accordingly should be treated separately (for the boundedness of solutions see [2], [3], [10] and [13]).

Our approach depends on the construction of an appropriate Liapunov function for the system (1.1), based on the general theory of variational identities introduced in [7]. In general (1.1) involves coupling of the second derivatives in each equation of the system. This makes the determination of an appropriate Liapunov function far from transparent and causes additional problems when G is not suitably smooth or presents other singularities. By introducing a Liapunov function directly in terms of a variational identity, these difficulties are avoided.

To present our results, we consider first the important special case of (1.1) given by

$$u''+A(t,u,u')u'+f(u)=0, \tag{1.5}$$

for which the conclusions can be stated in simple form. Here A is a continuous $N\times N$ damping matrix for which there exist non-negative measurable functions σ, δ: $J\to\mathbf{R}$ such that

$$(A(t,u,p)p,p)\geqslant \text{Pos. Const.}\,|A(t,u,p)p|\cdot|p| \tag{1.6}$$

and

$$\sigma(t)|p|\leqslant|A(t,u,p)p|\leqslant\delta(t)|p| \tag{1.7}$$

hold for all $t\in J$ and $u,p\in\mathbf{R}^N$. Furthermore f is a continuous gradient of a scalar potential F, with

$$(f(u),u)>0 \quad \text{for } u\neq 0.$$

The system (1.5) is naturally identified with (1.1) by the choice $\mathcal{L}(t,u,p)=\frac{1}{2}|p|^2-F(u)$ and $Q(t,u,p)=-A(t,u,p)p$. The scalar case of (1.5) was the main object of the studies [1], [2], [5], [12] and [13].

Levin and Nohel have shown for the linear scalar case of (1.5), i.e. when $A=a(t)$ and $f(u)=u$, that if $a(t)$ is small, or more precisely if it is in $L^1(J)$, then there exist solutions which are oscillatory and moreover do not tend to a limit as $t\to\infty$. On the other hand, Ballieu and Peiffer have shown that when $a(t)$ is sufficiently large then again there exist solutions which do not tend to zero as $t\to\infty$ (Artstein and Infante have given a specific example of this behavior).

The delicacy of the problem is evident from these results: both for small a, due to resonant behavior, and for large a, due to overdamping, the rest state loses asymptotic stability. Moreover the question of asymptotic stability is separate from the question of oscillation of solutions, as is clear from the example of Bessel's equation

$$u''+\frac{\beta}{t}u'+u=0,$$

for which the rest state is a global attractor for all $\beta>0$, but whose solutions are oscillatory when $\beta<1$ and monotone when $\beta\geqslant 1$. Reflecting this situation, and as a corollary of our general results for the system (1.1), the following conclusions hold for the system (1.5).

THEOREM A. (i) *Suppose $\sigma\delta$ is bounded on J. If σ is continuous and of bounded variation, or if $\log\sigma$ is uniformly Lipschitz continuous, then a sufficient condition for the rest state of* (1.5) *to be a global attractor is that*

$$\sigma\notin L^1(J),$$

while a necessary condition is that

$$\delta\notin L^1(J).$$

(ii) *Suppose $1/\sigma\delta$ is bounded on J. If $1/\delta$ is continuous and of bounded variation, or if $1/\delta$ is absolutely continuous and $|(1/\delta)'|\leqslant\text{Const.}\sqrt{\sigma/\delta}$, then a sufficient condition for the rest state to be a global attractor is that*

$$\frac{1}{\delta}\notin L^1(J),$$

while a necessary condition is that

$$\frac{1}{\sigma}\notin L^1(J).$$

When σ,δ fail to satisfy the hypotheses of Theorem A, for example when the product $\sigma\delta$ is at the same time neither bounded from zero nor bounded above, the situation is more delicate. In this case we have the following more general conditions for the stability of solutions of (1.5).

THEOREM B. *The rest state of* (1.5) *is a global attractor if one of the following conditions* (a), (b), (c) *is satisfied.*

(a) *There exist positive numbers c and η such that*

$$t\sigma(t) \geqslant c, \qquad \int_R^t \delta(s)s^{\eta-2}\,ds \leqslant \text{Const.}\,t^{\eta}.$$

(b) *There exists a non-negative continuous function $k=k(t)$ of bounded variation on J such that*

$$k \notin L^1(J), \quad k(t) \leqslant \text{Const.}\,\sigma(t) \quad \textit{in } J, \tag{1.8}$$

and either

$$\text{(i) } \delta k \in L^{\infty}(J) \quad \textit{or} \quad \text{(ii) } \delta k^2 \in L^1(J).$$

(c) *There exists a non-negative bounded absolutely continuous function $k=k(t)$ satisfying* (1.8) *and* (i) *or* (ii), *such that*

$$|k'| \leqslant \text{Const.}\,\sqrt{k\sigma} \quad \textit{a.e. in } J.$$

Sufficiency for the first part of Theorem A follows from (b) and (c) with $k=\sigma$, and for the second part from (b) and (c) with $k=1/\delta$. Necessity for the first part of Theorem A comes from Corollary 1 of Section 5, and for the second part from Theorem D below.

Part (a) of Theorem B is the case $\mu=1$ of Corollary 5 in Section 4. Parts (b) and (c) are Corollary 1 in Section 4, with $\mu=1$ and with the respective decompositions $d=\delta$, $e=0$ for (i) and $d=0$, $e=\delta$ for (ii). Case (b) then corresponds to the application of Theorem 4.1, and (c) to the application of Theorem 4.2 with the parameter values $m=2$, $\nu=1$, $\lambda=\frac{1}{2}$.

Theorems A and B are new in both the scalar and vector case of (1.5). When (1.5) is scalar, several subcases of Theorems A and B are already known: in particular, Smith [12] obtained, in the linear case, the first part of Theorem A when $\sigma=\delta$ and σ is decreasing; Artstein and Infante found (a) when $\sigma(t)$=Constant and $\eta=2$; and Ballieu and Peiffer obtained (b) in the following special cases: when $\sigma(t)$=Constant and $\delta k \in L^{\infty}(J)$, and when $\delta(t)$=Constant and k is decreasing. Finally, Salvadori obtained asymptotic stability for the vector system (1.5) when σ and δ are bounded both from zero and above, a special case of (a).

From Section 5 we also obtain

THEOREM C. *Let $\delta \in L^1(J)$. Then there are no solutions of* (1.5) *except $u\equiv 0$ which approach a limit in $\mathbf{R}^N$ as $t\to\infty$.*

Theorem C is complemented by the following result (cf. [10], Section 4).

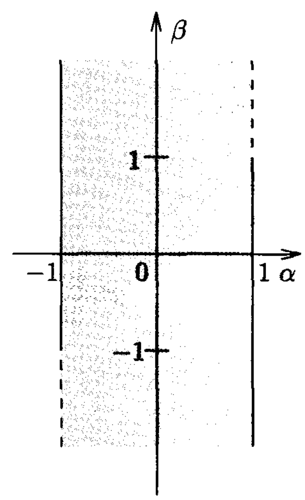

Fig. 1. Asymptotic stability for the system (1.5) when $A = t^{\alpha}(\log t)^{\beta} I$. The rest state is a global attractor if and only if (α, β) is in the shaded region.

THEOREM D. *Let* $1/\sigma \in L^1(J)$. *Then every bounded solution of* (1.5) *approaches a limit as* $t \to \infty$, *and the set of attainable limits is dense in* $\mathbf{R}^N$.

The matrix $A = t^{\alpha}(\log t)^{\beta} I$ serves as an example to illustrate these results. For this case the rest state of (1.5) is a global attractor if and only if either $|\alpha| < 1$; or $\alpha = 1$, $\beta \leqslant 1$; or $\alpha = -1$, $\beta \geqslant -1$. When $\alpha = 1$ and $\beta > 1$ or when $\alpha > 1$ each solution of (1.5) approaches a limit, and the set of attainable limits is $\mathbf{R}^N$. On the other hand, if $\alpha = -1$ and $\beta < -1$ or if $\alpha < -1$ no solution of (1.5), with the exception of $u \equiv 0$, can approach a limit as $t \to \infty$. See Figure 1.

A direct generalization of condition (1.7), for which results similar to Theorems A–D also hold is

$$\sigma(t)|p|^{\mu} \leqslant |A(t, u, p)p| \leqslant \delta(t)|p|^{\mu} \tag{1.9}$$

where $\mu > 0$, see particularly the corollaries in Section 4.

For the general system (1.1) our results are entirely analogous to those stated above for (1.5). These further results, which are the main purpose of the paper, are contained in the theorems of Sections 3 and 4, in a series of corollaries to these theorems, and in Section 5.

That a set of corollaries is needed for an adequate statement of the conclusions is due, we believe, to the fact that no single useful condition can encompass the general theory of asymptotic stability. On the other hand, the corollaries present a *unified* theory of asymptotic stability, based on a single proof technique.

There are a number of situations which can be represented by the system (1.1).

When $G(p)=|p|^m/m$, $m>1$, the corresponding system is

$$|u'|^{m-4}\big[|u'|^2\delta_{ij}+(m-2)u_i'u_j'\big]u_j''+f_i(t,u)=Q_i(t,u,u'),\quad f=\nabla F, \tag{1.10}$$

where δ_{ij} denotes the Kronecker symbol and the repeated index j is to be summed from 1 to N; note that when $m\neq 2$ this system is singular at points where $u'=0$. A similar example is $G(p)=\sum_{i=1}^N |p_i|^m/m$, $m>1$, the related system then being

$$|u_i'|^{m-2}u_i''+f_i(t,u)=Q_i(t,u,u'). \tag{1.11}$$

The second derivatives are uncoupled here, but again when $m\neq 2$ the system is singular if at least one $u_i'=0$. If $m=2$ both (1.10) and (1.11) reduce to the important form

$$u''+f(t,u)=Q(t,u,u').$$

When $G(p)=\sqrt{1+|p|^2}-1$, that is, when G is the mean curvature operator, the system (1.1) becomes

$$(1+|u'|^2)^{-3/2}[(1+|u'|^2)\delta_{ij}-u_j'u_j']u_j''+f_i(t,u)=Q_i(t,u,u'). \tag{1.12}$$

The system (1.1) also arises as the *radial* version of the partial differential system in $\mathbf{R}^n$

$$\operatorname{div}\nabla G(Du)+f(t,u)=0,\quad t=|x|,$$

where Du denotes the Jacobian matrix $(\partial u_i/\partial x_j)$, $i=1,\dots,N$, $j=1,\dots,n$ and where $G(Du)$ has the special form $\overline{G}(|\nabla u_1|,\dots,|\nabla u_N|)$. To place this in the context of (1.1) we take $G(p)=\overline{G}(|p_1|,\dots,|p_N|)$ and $Q=-(n-1)\nabla G/t$.

The general system (1.1) may be considered as the motion equation for a holonomic dynamical system with N degrees of freedom, whose Lagrangian $\mathcal{L}$ is defined by an action energy $T=G(u,p)$ and a potential $U=F(t,u)$, and whose dynamics are governed by a general nonlinear damping term $Q=Q(t,u,p)$. Here the variables u_i represent appropriate Lagrangian coordinates.

A final example worth noting arises from the Euler–Lagrange system for extremals of the variational integral

$$\int_J g(t)\mathcal{L}(t,u,u')\,dt,$$

where $\mathcal{L}$ has the form given earlier and $g\colon J\to\mathbf{R}$ is continuously differentiable, positive and non-decreasing. Extremals for this functional satisfy (1.1) with

$$Q(t,u,p)=-\frac{g'(t)}{g(t)}\nabla G(u,p).$$

Here the condition $(Q,p)\leqslant 0$ is a consequence of the fact that $g(t)>0$, $g'(t)\geqslant 0$ together with (1.2), namely $(\nabla G(u,p),p)\geqslant 0$. For further generalizations to variational problems we refer to the work of Leoni, Manfredini and Pucci [4].

It is also possible to treat the non-homogeneous analogue of (1.1), namely

$$(\nabla G(u,u'))'-\nabla_u G(u,u')+f(t,u)=Q(t,u,u')+e(t),$$

and the corresponding analogue of (1.5),

$$u''+A(t,u,u')u'+f(u)=e(t).$$

In the final section of the paper we show that the same asymptotic stability results hold for these cases as for homogeneous systems, provided the non-homogeneous term can be decomposed in the form

$$e(t)=e_1(t)+e_2(t),$$

where $e_1\in L^1(J)$ and e_2 is of bounded variation on J with $e_2(t)\to 0$ as $t\to\infty$.

The system (1.10) is further discussed at the end of Section 5. Various other examples and applications which illustrate our results have been given in [1], [4], [5] and [9].

The following section contains preliminary material, including the statement of the important identity (2.8) for solutions of (1.1). In Sections 3 and 4 we present our main sufficiency results for global asymptotic stability. Section 3 covers functions G which are merely assumed to be strictly convex, while in Section 4 we allow G to have a more specialized structure, of the type possessed in particular by the systems (1.10)–(1.12). Section 5 contains necessary conditions for global asymptotic stability. Finally, Theorem 5.1 is a backward uniqueness result of interest in itself.

2. Preliminaries

We consider vector solutions $u=(u_1,\dots,u_N)$ of the quasi-variational ordinary differential system

$$(\nabla G(u,u'))'-\nabla_u G(u,u')+f(t,u)=Q(t,u,u'),\quad J=[R,\infty), \tag{2.1}$$

where ∇ denotes the gradient operator with respect to the variable p and

$$f(t,u)=\nabla_u F(t,u).$$

It will be supposed throughout the paper that

$$G\in C^1(\mathbf{R}^N\times\mathbf{R}^N;\mathbf{R}),\quad F\in C^1(J\times\mathbf{R}^N;\mathbf{R}),\quad Q\in C(J\times\mathbf{R}^N\times\mathbf{R}^N;\mathbf{R}^N),$$

and also, except in Section 5, that the following natural conditions hold:

(H_1) $G(u,\cdot)$ is strictly convex in $\mathbf{R}^N$ for all $u\in\mathbf{R}^N$, with $G(u,0)=0$ and $\nabla G(u,0)=0$. Finally, for every $U>0$ and $p_0>0$ there exists a non-negative constant such that

$$(\nabla_u G(u,p),u)\leqslant \text{Const.}\,(\nabla G(u,p),p) \quad \text{for all } |u|\leqslant U \text{ and } |p|\geqslant p_0. \tag{2.2}$$

(H_2) $F(t,0)=0$. For all u_0, U with $0<u_0\leqslant U$ there exist a constant $\varkappa>0$ and a non-negative function $\psi\in L^1(J)$ such that

$$(f(t,u),u)\geqslant\varkappa \quad \text{when } t\in J \text{ and } |u|\in[u_0,U], \tag{2.3}$$

$$F_t(t,u)\leqslant\psi(t) \quad \text{when } t\in J \text{ (a.e) and } |u|\leqslant U. \tag{2.4}$$

(H_3) $(Q(t,u,p),p)\leqslant 0$ for all $t\in J$, $u\in\mathbf{R}^N$ and $p\in\mathbf{R}^N$.

A condition of the type (2.2) first appears in [4]; of course this condition automatically holds in the important case when G is independent of u. Moreover (2.2) is implied by the somewhat simpler condition $(\nabla_u G(u,p),u)\leqslant\text{Const.}\,G(u,p)$, since for any function satisfying (H_1) one has $0\leqslant G(u,p)\leqslant(\nabla G(u,p),p)$. A simple example of this type occurs when $G(u,p)=g(u)\bar{G}(u,p)$ with $g(u)>0$ in $\mathbf{R}^N$ and $\bar{G}$ satisfying (H_1).

Condition (2.3) implies that $(f(t,u),u)>0$ and $F(t,u)>0$ for $t\in J$ and $u\neq 0$. If F does not depend on t, then (2.3) follows from the condition $(f(u),u)>0$ for $u\neq 0$, while (2.4) is irrelevant. Finally, when $N=1$ any continuous function f is of gradient type, with $F(t,u)=\int_0^u f(t,s)\,ds$.

When f has the form $f(t,u)=l(t)\chi(u)$ with $l\colon J\to\mathbf{R}^+$ and $\chi\colon\mathbf{R}^N\to\mathbf{R}^N$, then (2.3) is equivalent to the simple condition

$$l(t)\geqslant\text{Const.}>0 \quad \text{for } t\in J, \quad (\chi(u),u)>0 \quad \text{for } u\neq 0.$$

Property (2.4) then holds if and only if $(l')^+\in L^1(J)$.

In what follows, we consider (weak) solutions of (2.1) on J, namely vector functions $u\colon J\to\mathbf{R}^N$ of class C^1, for which

$$\nabla G(u(t),u'(t))\in C^1(J;\mathbf{R}^N) \tag{2.5}$$

and which satisfy the system (2.1) in J. Conditions (2.3) and (H_3) imply that $f(t,0)=Q(t,u,0)=0$. Moreover from (H_1) it follows that $\nabla G(u,0)=\nabla_u G(u,0)=0$ so that $u\equiv 0$ is a solution of (2.1) on J.

Let $H(u,\cdot)$ be the Legendre transform of $G(u,\cdot)$, namely

$$H(u,p)=(\nabla G(u,p),p)-G(u,p). \tag{2.6}$$

If $u=u(t)$ is a weak solution of (2.1) on J, then we have

$$\{H(u,u')+F(t,u)\}' = (Q(t,u,u'),u')+F_t(t,u). \tag{2.7}$$

Under the minimal conditions which we are assuming here, namely that G is of class C^1 and is strictly convex in the variable p, this identity is proved in a recent paper of the authors, *On the derivation of Hamilton's equations*; see also [4]. (It is worth noting as well that (2.7) is directly and easily verified for weak solutions u which are also assumed to be of class $C^2(J,\mathbf{R}^N)$; this additional degree of smoothness is *not* always satisfied however, e.g. when $G(p)=|p|^m/m$, $m>2$.)

In view of (2.5) and (2.7) the following identity holds for solutions of (2.1) and for any pair of scalar functions $\varphi,\omega\in C^1(J;\mathbf{R})$,

$$\begin{aligned}\{\omega[H(u,u')+F(t,u)]&+\varphi(\nabla G(u,u'),u)\}'\\ =\omega F_t(t,u)&+\omega'[H(u,u')+F(t,u)]-\varphi(f(t,u),u)\\ &+\varphi[(\nabla G(u,u'),u')+(\nabla_u G(u,u'),u)]+\varphi'(\nabla G(u,u'),u)\\ &+\omega(Q(t,u,u'),u')+\varphi(Q(t,u,u'),u).\end{aligned} \tag{2.8}$$

This formula was originally discovered as a special case of the main identity in [7], see particularly the remark on p. 685. Note that (2.8) reduces to (2.7) when $\omega=1$ and $\varphi=0$.

The identity (2.8) can also be usefully written in the notation of analytical dynamics, by putting

$$v=\nabla\mathcal{L}=\nabla G$$

and

$$\mathcal{H}=(p,v)-\mathcal{L}=H+F,$$

so that v is the conjugate variable to p and $\mathcal{H}$ is the Hamiltonian associated with the Lagrangian $\mathcal{L}$. Then it becomes

$$\{\omega\mathcal{H}+\varphi(u,v)\}'=\omega'\mathcal{H}-\omega\mathcal{L}_t+(\varphi u,\nabla_u\mathcal{L})+((\varphi u)',v)+(\omega u'+\varphi u,Q).$$

In the sequel we shall use the properties that $H(u,0)=0$ and $H(u,p)>0$ for $p\neq 0$. The first follows from the condition $G(u,0)=0$ and the latter is a standard consequence of the strict convexity of $G(u,\cdot)$. Moreover $\nabla G(u,p)\neq 0$ for $p\neq 0$ since $\nabla G(u,0)=0$. Another consequence of these facts is the following useful

LEMMA 2.1. *For every u, $p\in\mathbf{R}^N$ we have*

$$|\nabla G(u,p)|\leqslant H(u,p)+\max_{|v|=1} G(u,v). \tag{2.9}$$

Proof. Fix $u \in \mathbf{R}^N$. Obviously (2.9) is valid when $p=0$. Take $p \neq 0$ so that

$$v = \frac{\nabla G(u,p)}{|\nabla G(u,p)|}$$

is well defined. Then from the convexity condition

$$G(u,p) - G(u,v) \leqslant (\nabla G(u,p), p-v)$$

we immediately obtain, using (2.6),

$$|\nabla G(u,p)| = (\nabla G(u,p), v) \leqslant (\nabla G(u,p), p) - G(u,p) + G(u,v) = H(u,p) + G(u,v),$$

which yields (2.9).

3. Sufficient conditions for stability, Part 1

The purpose of this section is to study the asymptotic stability of bounded solutions of (2.1). We shall always assume, without further mention, that (H_1)–(H_3) are valid. Moreover, for this section we suppose the additional hypotheses:

(C_1) For every $U>0$ there exist a measurable control set $I \subset J$, two non-negative measurable damping functions $\sigma, \delta: I \to \mathbf{R}$, and two continuous increasing weight functions $\phi, \varrho: \mathbf{R}_0^+ \to \mathbf{R}_0^+$ with $\phi(0) = \varrho(0) = 0$, such that

$$|(Q(t,u,p),p)| \geqslant \sigma(t)\phi(|p|)|p| \tag{3.1}$$

for $t \in I$, $|u| \leqslant U$ and $p \in \mathbf{R}^N$; and

$$(Q(t,u,p),u) \leqslant \delta(t)\varrho(|p|) \tag{3.2}$$

for $t \in I$, $|u| \leqslant U$ and all sufficiently small $p \in \mathbf{R}^N$.

(C_2) For every $U>0$ there exists a positive constant $\gamma \geqslant 1$ such that

$$|Q(t,u,p)| \cdot |p| \leqslant \gamma |(Q(t,u,p),p)| \quad \text{for all } t \in I,\ |u| \leqslant U \text{ and } p \in \mathbf{R}^N.$$

Although I, σ, δ and ϕ, ϱ, γ may depend on U, for simplicity we do not specifically indicate this dependence.

Conditions (C_1) and (C_2) are automatically satisfied, for example, by the damping function $Q = -a(t)b(|p|)p$ when $a, b \geqslant 0$ and $sb(s)$ is increasing. Note that (C_2) also is satisfied automatically when $N=1$ and indeed whenever $-Q$ and p have the same direction, with $\gamma = 1$.

THEOREM 3.1. *Suppose that for every $U>0$ there is a continuous function k of bounded variation on J such that*

$$k \notin L^1(J), \quad k(t)=0 \quad \text{for } t \in J\backslash I \tag{3.3}$$

and

$$0 \leqslant k(t) \leqslant \beta\sigma(t) \quad \text{for } t \in I \tag{3.4}$$

for some positive constant β. Assume also that there exists $M>0$ such that

$$\int_{(R,t)\cap I} \delta(s)k(s)\exp\left(-\int_s^t k(r)\,dr\right)ds \leqslant M \quad \text{for } t \in J. \tag{3.5}$$

Then the rest state is a global attractor for the system (2.1).

Proof. We first give the proof when the BV function k is of class $C^1(J)$.

Let u be a solution of (2.1) with $|u(t)| \leqslant L$ on J for some $L>0$. We take $U=L$ and let ψ, σ, δ, etc., be the associated functions and constants in (H_1)–(H_3) and (C_1)–(C_2), and similarly let k be the associated function in the theorem.

By (2.7) the expression

$$H(u,u')+F(t,u)-\int_R^t (Q(s,u,u'),u')\,ds-\int_R^t F_t(s,u)\,ds \tag{3.6}$$

is constant. Since the first three terms in (3.6) are non-negative, it follows from (2.4) that

$$(Q(t,u(t),u'(t)),u'(t)) \in L^1(J). \tag{3.7}$$

Therefore also $F_t(t,u(t)) \in L^1(J)$ and in turn there exists a number $l \geqslant 0$ such that

$$H(u,u')+F(t,u) \to l \quad \text{as } t \to \infty. \tag{3.8}$$

First assume $l=0$. Since $H(u,p)>0$ for $p \neq 0$, and since (again by the strict convexity of the function $G(u,\cdot)$)

$$\inf\{H(u,p) : |u| \leqslant L, |p| \geqslant 1\} > 0,$$

it follows easily that $u'(t) \to 0$ as $t \to \infty$. Moreover by (2.3), see Lemma 2 of [9], there is a positive function ω_L defined on $(0,L]$ such that $F(t,u) \geqslant \omega_L(|u|)$ for any $t \in J$. Hence also $u(t) \to 0$ as $t \to \infty$, completing the proof when $l=0$.

Now assume for contradiction that $l>0$ in (3.8). The proof relies on the main identity (2.8) with the choice $\varphi=\alpha\omega'$, where ω is the positive non-decreasing function defined by

$$\omega(t)=\exp\left(\int_R^t k(s)\,ds\right) \tag{3.9}$$

and α is a positive number which will be determined later. Then, recalling that $f=\nabla_u F$, the identity (2.8) reduces to

$$\begin{aligned}
&\{\omega[H(u,u')+F(t,u)+\alpha k(\nabla G(u,u'),u)]\}' \\
&\quad =\omega F_t(t,u)+\omega'[H(u,u')+F(t,u)-\alpha(f(t,u),u)] \\
&\qquad +\alpha\omega'[(\nabla G(u,u'),u')+(\nabla_u G(u,u'),u)]+\alpha\omega''(\nabla G(u,u'),u) \\
&\qquad +\omega(Q(t,u,u'),u')+\alpha\omega'(Q(t,u,u'),u),
\end{aligned} \tag{3.10}$$

where

$$\omega'=k\omega, \quad \omega''=(k^2+k')\omega. \tag{3.11}$$

By (3.8) the function $H(u,u')$ is bounded on J. Hence by Lemma 2.1 also $|\nabla G(u,u')|$ is bounded, say

$$|\nabla G(u,u')|\leqslant C \quad \text{for } t\in J. \tag{3.12}$$

We take R_1 so large that

$$H(u,u')+F(t,u)\geqslant \tfrac{3}{4}l \quad \text{on } J_1 \quad \text{and} \quad \int_{J_1}\psi(s)\,ds\leqslant \tfrac{1}{4}l, \tag{3.13}$$

where $J_1=[R_1,\infty)$.

Let $\mathcal{R}(t)$ denote the right hand side of the identity (3.10). Our purpose, and the principal effort in the proof, will be to obtain an appropriate estimate for $\mathcal{R}(t)$ when $t\in J_1$ and α is suitably small.

First, since $k(t)=k'(t)=0$ on $J\setminus I$ it follows from (3.10), (3.11), (2.4) and (H_3) that

$$\mathcal{R}(t)\leqslant \omega\psi(t), \quad t\in J_1\setminus I \quad \text{(a.e.)}. \tag{3.14}$$

On the remaining set $I'=I\cap J_1$ we partition $\mathcal{R}(t)$ in the form

$$\frac{1}{\omega}\mathcal{R}=F_t+k(H+F)+\sum_1^5 \mathcal{R}_i,$$

where

$$\begin{aligned}
\mathcal{R}_1&=\tfrac{1}{5}(Q,u')-\alpha k(f,u)\\
\mathcal{R}_2&=\tfrac{1}{5}(Q,u')+\alpha k\{(\nabla G,u')+(\nabla_u G,u)\}\\
\mathcal{R}_3&=\tfrac{1}{5}(Q,u')+\alpha k^2(\nabla G,u)\\
\mathcal{R}_4&=\tfrac{1}{5}(Q,u')+\alpha k'(\nabla G,u)\\
\mathcal{R}_5&=\tfrac{1}{5}(Q,u')+\alpha k(Q,u),
\end{aligned}$$

the notation being obvious, and u, u' clearly standing for $u(t)$, $u'(t)$. The important term (Q,u') lies behind the desired estimates; accordingly this term is partitioned into each of the expressions $\mathcal{R}_1$ through $\mathcal{R}_5$.

Again $F_t \leqslant \psi$ from (2.4), while from (3.8)

$$|H+F-l| = \varepsilon(t) \to 0 \quad \text{as } t \to \infty. \tag{3.15}$$

The remaining terms will be treated one-by-one.

1. We assert that $\mathcal{R}_1 \leqslant -\alpha\varkappa k$ for some constant $\varkappa>0$, provided that α is suitably small. To see this, first let $p_1>0$ be fixed so that $H(u,p)\leqslant\frac{1}{4}l$ when $|u|\leqslant L$ and $|p|\leqslant p_1$. Clearly this can be done since H is continuous and $H(u,0)=0$. Then by (3.13) we have

$$F(t,u(t)) \geqslant \tfrac{1}{2}l \quad \text{in } I_1 = \{t\in I' : |u'(t)| \leqslant p_1\}.$$

On the other hand, by (2.4) and (3.13),

$$\begin{aligned} F(t,u(t)) &= F(R_1,u(t)) + \int_{R_1}^{t} F_t(s,u(t))\,ds \\ &\leqslant F(R_1,u(t)) + \int_{R_1}^{\infty} \psi(s)\,ds \leqslant F(R_1,u(t)) + \tfrac{1}{4}l \end{aligned}$$

for $t\in J_1$. Thus

$$F(R_1,u(t)) \geqslant \tfrac{1}{4}l \quad \text{in } I_1.$$

Since $F(R_1,0)=0$ it follows that there exists a number $u_0>0$ such that $|u(t)|\geqslant u_0$ for $t\in I_1$. Denote by $\varkappa=\varkappa(u_0,L)>0$ the constant given in (2.3) corresponding to u_0 and $U=L$, so that now

$$(f(t,u(t)),u(t)) \geqslant \varkappa > 0 \quad \text{for } t\in I_1. \tag{3.16}$$

Since $(Q,u')\leqslant 0$ by (H_3), this gives the required estimate when $t\in I_1$.

In the remaining set $I'\setminus I_1$ we have $|u'(t)|>p_1$. Hence by (3.1) and (3.4)

$$|(Q,u')| \geqslant \sigma\phi(|u'|)\cdot|u'| \geqslant \frac{k}{\beta}\phi(|u'|)\cdot|u'| \geqslant 5\alpha\varkappa k \quad \text{in } I'\setminus I_1,$$

provided that

$$\alpha \leqslant \frac{p_1}{5\beta\varkappa}\phi(p_1); \tag{3.17}$$

here $\phi(|u'|)\geqslant\phi(p_1)>0$ since $\phi(0)=0$ and ϕ is increasing. Since $(f(t,u),u)\geqslant 0$ by (2.3), the required estimate is now valid for all $t\in I'$, provided of course (3.17) is satisfied.

2. We claim that $\mathcal{R}_2\leqslant\frac{1}{8}\alpha k\varkappa$ on I', provided α is suitably small. Indeed

$$(\nabla G,u') + (\nabla_u G,u) \leqslant \tfrac{1}{8}\varkappa$$

if $|u'|$ is suitably small, say $|u'|\leqslant p_2$, since $\nabla G(u,0)=\nabla_u G(u,0)=0$ and the solution is bounded. Hence in this case the claim holds. Otherwise, if $|u'|>p_2$, then by (2.2) there is a constant $b>0$ such that

$$(\nabla G, u')+(\nabla_u G, u)\leqslant (1+b)\cdot(\nabla G, u').$$

Moreover by (3.4) and (3.12) we have

$$\begin{aligned} k(\nabla G, u') &\leqslant kC|u'| \leqslant \beta C\sigma|u'| \leqslant \frac{\beta C\sigma\phi(|u'|)\cdot|u'|}{\phi(p_2)} \\ &= C_2\sigma\phi(|u'|)\cdot|u'| \leqslant C_2|(Q,u')| \end{aligned}$$

by (3.1), where $C_2=\text{Constant}>0$. Hence if

$$\alpha\leqslant\frac{1}{5C_2(1+b)} \tag{3.18}$$

then $\mathcal{R}_2\leqslant 0$ when $|u'|>p_2$, completing step *2*.

3. Again $\mathcal{R}_3\leqslant\frac{1}{8}\alpha k\varkappa$ on I'. This is treated exactly as in step *2*, the only exception being that (3.18) is replaced by a corresponding bound

$$\alpha\leqslant\frac{1}{5C_3}, \tag{3.19}$$

where $C_3=\beta CL\sup k/p_3\phi(p_3)$ and p_3 is an appropriate positive constant analogous to p_2 in step *2*. Here we use the condition that k is bounded, since $\mathcal{R}_3$ includes the factor k^2 rather than k.

4. By (H_3) and (3.12) obviously $\mathcal{R}_4\leqslant\alpha CL|k'|$ in I'.

5. We claim that $\mathcal{R}_5\leqslant\varepsilon_5(\alpha)\alpha\delta k$, where $\varepsilon_5(\alpha)\to 0$ as $\alpha\to 0$. Let

$$I_5=\{t\in I' : |u'(t)|\geqslant\Lambda_5\alpha\},\quad \Lambda_5=5L\gamma\sup_{t\in J}k(t).$$

In this set, by (C_2),

$$\alpha k(Q,u)\leqslant\alpha Lk|Q|\leqslant\frac{1}{5\gamma}|Q|\cdot|u'|\leqslant\frac{1}{5}|(Q,u')|$$

so that $\mathcal{R}_5\leqslant 0$ in I_5. In $I'\setminus I_5$ we have $|u'(t)|<\Lambda_5\alpha$. Hence

$$\mathcal{R}_5\leqslant\alpha k\delta\varrho(|u'(t)|)\leqslant\varepsilon_5(\alpha)\alpha\delta k$$

by (3.2), where $\varepsilon_5(\alpha)=\varrho(\Lambda_5\alpha)$. Thus the claim is proved for all $t\in I'$.

Combining the previous estimates, including (3.14) and (3.15), now yields, for almost all $t\in J_1$ and all $\alpha>0$ satisfying (3.17), (3.18) and (3.19)

$$\mathcal{R}\leqslant\omega\{\psi+\alpha CL|k'|+\varepsilon_5(\alpha)\alpha\delta k\}+\omega'\{l+\varepsilon-\alpha\varkappa+2\cdot\tfrac{1}{8}\alpha\varkappa\},\tag{3.20}$$

where we introduce the agreement $\delta k=0$ on $J\setminus I$. Recall that $k=0$ on $J\setminus I$ by assumption $(3.3)_2$.

Now fix $\alpha>0$ so small that (3.17), (3.18) and (3.19) hold and also so that

$$\varepsilon_5(\alpha)\leqslant\frac{\varkappa}{8M},\tag{3.21}$$

where M is the constant given in (3.5). Moreover take $R_2\geqslant R_1$ such that

$$\varepsilon=\varepsilon(t)\leqslant\tfrac{1}{8}\alpha\varkappa\quad\text{for all } t\geqslant R_2\tag{3.22}$$

and

$$\int_{R_2}^{\infty}\psi(s)\,ds\leqslant\alpha\frac{\varkappa}{8},\quad\int_{R_2}^{\infty}|k'(s)|\,ds\leqslant\frac{\varkappa}{8CL}.\tag{3.23}$$

Condition $(3.23)_2$ can be attained since $k\in\mathrm{BV}(J)$ and so $k'\in L^1(J)$. Then from (3.10), (3.20) and (3.22) we see that the function

$$\begin{aligned}\Psi(t)=\omega(t)\Big\{&H(u,u')+F(t,u)+\alpha k(\nabla G(u,u'),u)-\Big(l+\alpha\frac{\varkappa}{8}\Big)+\alpha\varkappa-\alpha\frac{\varkappa}{4}\\&-\frac{1}{\omega(t)}\int_{R_2}^{t}\psi(s)\omega(s)\,ds-\frac{\alpha CL}{\omega(t)}\int_{R_2}^{t}|k'(s)|\omega(s)\,ds\\&-\frac{\varepsilon_5(\alpha)\alpha}{\omega(t)}\int_{R_2}^{t}\delta(s)k(s)\omega(s)\,ds\Big\}\end{aligned}\tag{3.24}$$

is decreasing in $J_2=[R_2,\infty)$.

We claim that there exists a sequence (t_n) with $t_n\nearrow\infty$ such that

$$k(t_n)\nabla G(u(t_n),u'(t_n))\to 0\quad\text{as } n\to\infty.\tag{3.25}$$

By (3.12) and the boundedness of k, this follows immediately if either $\liminf_{t\to\infty}k(t)=0$ or $\liminf_{t\to\infty}|\nabla G(u(t),u'(t))|=0$. But if neither of these occurs, then for some sufficiently large constant $\bar{t}$ there holds

$$k(t)\geqslant k_0>0\quad\text{and}\quad|\nabla G(u,u')|\geqslant C_0>0\quad\text{for } t\geqslant\bar{t}.$$

In turn, by $(3.3)_2$ we must have $I\supset[\bar{t},\infty)$ and so using (3.4) and the strict convexity of $G(u,\cdot)$, see (H_1), we have

$$\sigma(t)\geqslant\frac{k_0}{\beta}\quad\text{and}\quad|u'(t)|\geqslant p_0>0\quad\text{for } t\geqslant\bar{t}.$$

Hence by (3.1) we obtain

$$|(Q,u')| \geqslant \sigma\phi(|u'|)\cdot|u'| \geqslant \frac{k_0 p_0 \phi(p_0)}{\beta} > 0 \quad \text{in } [\bar{t},\infty),$$

which contradicts (3.7).

Now, along the sequence (t_n), by (3.5), (3.15), (3.21)–(3.24) there results

$$\Psi(t_n) \geqslant \omega(t_n)\Big\{\Big(l-\alpha\frac{\varkappa}{8}\Big)-\alpha L k(t_n)|\nabla G(u(t_n),u'(t_n))|-\Big(l+\alpha\frac{\varkappa}{8}\Big)+\alpha\varkappa-5\alpha\frac{\varkappa}{8}\Big\}.$$

But then using (3.25) it follows that $\Psi(t_n)\geqslant\frac{1}{16}\alpha\varkappa\omega(t_n)$ for n sufficiently large. Hence $\Psi(t_n)\to\infty$ as $n\to\infty$ by (3.9) and $(3.3)_1$, which contradicts the fact that Ψ is decreasing in J_2. Therefore the case $l\neq 0$ cannot occur in (3.8), and the proof is complete when $k\in C^1(J)$.

We now turn to the general case when $k\in \mathrm{BV}(J)$. The proof is based on the following lemma, whose demonstration will be given in the Appendix.

LEMMA A. *Let k be a non-negative continuous function of bounded variation on J. Then for every constant $\theta>1$ there exists a function $\bar{k}\in C^1(J)$ and an open set $E\subset J$ such that*

$$\text{(i)}\ \ \theta k \geqslant \bar{k} \geqslant \begin{cases} k & \text{in } J\setminus E \\ 0 & \text{in } E \end{cases}; \quad \text{(ii)}\ \ \operatorname{Var}\bar{k}\leqslant\theta\operatorname{Var}k; \quad \text{(iii)}\ \ \int_E k\leqslant 1.$$

Now fix $\theta>1$ and let $\bar{k}$ and E be as given in the lemma. We assert that $\bar{k}$ satisfies (3.3)–(3.5) with β replaced by $\bar{\beta}=\theta\beta$. Indeed by (i) it is clear that $(3.3)_2$ is satisfied. Also $(3.3)_1$ holds thanks to (i) and (iii). Similarly, in view of (i) and (iii),

$$\int_R^t \delta(s)\bar{k}(s)\exp\Big(-\int_s^t \bar{k}(r)\,dr\Big)\,ds \leqslant \theta e\int_R^t \delta(s)k(s)\exp\Big(-\int_s^t k(r)\,dr\Big)\,ds,$$

and so (3.5) holds. This completes the proof.

The condition that $k\in\mathrm{BV}(J)$ in Theorem 3.1 can be replaced by an interesting alternative hypothesis.

THEOREM 3.2. *Suppose that the control set I in (C_1) is the entire interval J and that for every $U>0$ there is a bounded function k on J such that*

$$k\notin L^1(J), \tag{3.3$'$}$$

$$0<k(t)\leqslant\beta\sigma(t)\ \ \textit{in } J, \quad \log k\in\mathrm{Lip}(J), \tag{3.4$'$}$$

for some constant $\beta>0$. *Assume also that* (3.5) *holds. Then the rest state is a global attractor for the system* (2.1).

Proof. Clearly $k\in\mathrm{Lip}(J)$, so $k\in\mathrm{AC}(J)$ with $|k'|\leqslant\mathrm{Const.}\,k$ a.e. in J. The previous proof now applies word-for-word, except for step *4*. For this step, however, we have the alternative estimate

$$\alpha k'(\nabla G, u)\leqslant \mathrm{Const.}\,\alpha k|\nabla G| \quad \text{a.e. in } J_1.$$

Therefore, using the argument of steps *2* and *3*, we easily derive that for α suitably small

$$\mathcal{R}_4\leqslant \tfrac{1}{8}\alpha k\varkappa \quad \text{a.e. in } J_1.$$

In turn the second term in the first braces of (3.20) should be deleted, while at the same time an extra term $\frac{1}{8}\alpha\varkappa$ should be inserted in the second braces. The corresponding function Ψ again is decreasing in J_2. The rest of the proof follows exactly as before.

Theorem 3.1 has several interesting corollaries. Corresponding results can also be obtained from Theorem 3.2, but we leave their statement to the interested reader.

COROLLARY 1. *Suppose that the hypotheses of Theorem* 3.1 *are satisfied, except in place of* (3.5) *we assume that the function* δ *allows a decomposition* $\delta=d+e$ *with*

$$d\in L^\infty(I) \quad \textit{and} \quad ek\in L^1(I). \tag{3.5$'$}$$

Then the rest state is a global attractor.

Proof. Clearly

$$\int_R^t k(s)\exp\left(-\int_s^t k(r)\,dr\right)ds = 1-\exp\left(-\int_R^t k(r)\,dr\right)\leqslant 1.$$

Hence (3.5)$'$ implies (3.5), with $M=\|d\|_{L^\infty(I)}+\|ek\|_{L^1(I)}$, and Theorem 3.1 can be applied.

COROLLARY 2. *Define* $\bar\sigma(t)=\sigma(t)$ *for* $t\in I$ *and* $\bar\sigma(t)=0$ *for* $t\in J\setminus I$, *for any* σ *given in* (C_1). *Assume also that for every* $U>0$,

$$\sigma\notin L^1(I), \quad \delta\in L^\infty(I), \quad \bar\sigma\in\mathrm{CBV}(J).$$

Then the rest state is a global attractor.

Proof. Take $k(t)=\bar\sigma(t)$. Then (3.3) and (3.4) hold (with $\beta=1$), while (3.5)$'$ is satisfied with $d=\delta$, $e=0$. Thus Corollary 1 can be applied.

COROLLARY 3. *Condition* (3.5) *in Theorem* 3.1 *can be replaced by*

$$\liminf_{t\to\infty} \int_{(R,t)\cap I} \delta(s)k(s)\exp\left(-\int_s^t k(r)\,dr\right) ds = M < \infty, \tag{3.5$''$}$$

provided that $k(t)\to 0$ *as* $t\to\infty$.

Proof. We choose the sequence (t_n) in the proof of Theorem 3.1 so that

$$\int_{(R,t_n)\cap I} \delta(s)k(s)\exp\left(-\int_s^{t_n} k(r)\,dr\right) ds \leqslant 2M.$$

Then (3.25) holds as before, by (3.12) and the assumption that $k(t)\to 0$ as $t\to\infty$. The rest of the proof is the same as that in Theorem 3.1.

COROLLARY 4. *Suppose that the control set* I *in* (C_1) *is the entire interval* J, *and that for every* $U>0$ *there exist positive numbers* c *and* η *such that*

$$t\sigma(t) \geqslant c, \quad \liminf_{t\to\infty} \int_R^t \left(\frac{s}{t}\right)^\eta \delta(s)\,\frac{ds}{s} = M' < \infty. \tag{3.26}$$

Then the rest state is a global attractor.

Proof. Taking $k(t)=\eta/t$, and $\beta=\eta/c$, we immediately derive all the assumptions of Corollary 3, with $M=M'\eta$ in (3.5)$''$.

The case $\eta=1$ in Corollary 4 is particularly interesting. Moreover, the result of Corollary 4 is in a sense best possible. For example, $(3.26)_2$ is satisfied when δ is bounded but fails when $\delta(t)\geqslant t^\varepsilon$ for some $\varepsilon>0$, while there exist equations with $\delta(t)\geqslant t^\varepsilon$ for which the rest state is *not* a global attractor (see [9], Section 5).

Remarks. Condition (2.4) can be weakened to

$$F_t(t,u) \leqslant \psi(t)\{1+F(t,u)\}, \quad |u|\leqslant U. \tag{3.27}$$

Indeed if this condition holds, then by integration

$$\log\{1+F(t,u)\} \leqslant \int_R^\infty \psi(s)\,ds + \log\{1+F(R,u)\} \leqslant \text{Const.}$$

Hence $1+F(t,u)$ is also bounded above for $t\in J$ and $|u|\leqslant U$ and we get $F_t(t,u)\leqslant$ Const. $\psi(t)$, which is essentially condition (2.4). That is (3.27) is in fact equivalent to (2.4).

The preliminary change of variable $t \mapsto \tau = \tau(t) = \int_R^t q(s)\,ds$, with $q>0$ and with $q \notin L^1(J)$, can occasionally be helpful in applications, particularly when properties (2.3), (2.4) are not satisfied. For example (see [2], p. 327) the system

$$u'' + q^2(t) f(u) = Q(t, u, u')$$

transforms to

$$\ddot{u} + f(u) = \frac{1}{q^2(t(\tau))}\{Q(t(\tau), u, q\dot{u}) - q'(t(\tau))\dot{u}\}.$$

Various results which can be obtained in this connection will be treated in a paper to appear later.

4. Sufficient conditions for stability, Part 2

We suppose as before that conditions (H_1)–(H_3) and (C_2) are satisfied. However, in place of assumption (C_1) we shall require the following more specific hypothesis in which the functions ϕ, ϱ are assumed to be algebraic.

$(C_1)'$ There exist exponents $0 < \mu \leqslant \nu$ such that assumption (C_1) is satisfied with the specific functions

$$\phi(s) = \min\{1, s^\nu\}, \quad \varrho(s) = s^\mu. \tag{4.1}$$

The first main result of the section is the following

THEOREM 4.1. *Suppose that for every $U>0$ there is a continuous function k of bounded variation on J such that*

$$k \notin L^1(J), \quad k(t) = 0 \quad \textit{for } t \in J \setminus I \tag{4.2}$$

and for some $\beta > 0$

$$0 \leqslant k(t) \leqslant \beta\sigma(t) \quad \textit{for } t \in I. \tag{4.3}$$

Assume also that there exists $M>0$ for which

$$\int_{(R,t)\cap I} \delta(s) k^{\mu+1}(s) \exp\left(-\int_s^t k(r)\,dr\right) ds \leqslant M \quad \textit{for } t \in J. \tag{4.4}$$

Then the rest state is a global attractor for the system (2.1).

Proof. The first part of the proof is the same as in Theorem 3.1, while steps *1–4* are also essentially as before.

For step *5* the only change is in the estimate for the set $I'\backslash I_5$. We assert that on this set there holds

$$\alpha k(Q,u)\leqslant -\alpha^{1/2}L\gamma(Q,u')+\alpha^{1+\mu/2}\delta k^{\mu+1}. \tag{4.5}$$

Since this is obvious if $u'=0$, we may suppose $u'\neq 0$. Then by Young's inequality

$$\alpha k\leqslant \alpha^{1/2}\{|u'|+|u'|^{-\mu}(\alpha^{1/2}k)^{\mu+1}\}$$

and so by (3.2) with $\varrho(s)=s^{\mu}$, see $(\mathrm{C}_1)'$,

$$\alpha k(Q,u)\leqslant \alpha^{1/2}L|Q|\cdot|u'|+\alpha^{1+\mu/2}\delta k^{\mu+1}.$$

This gives the required result (4.5) in view of (C_2) and (H_3). To apply (3.2) the derivative $|u'(t)|$ must be sufficiently small, but this is certainly true in $I'\backslash I_5$ provided α is small enough.

In view of (4.5) we thus have

$$\mathcal{R}_5\leqslant\left(\tfrac{1}{5}-\alpha^{1/2}L\gamma\right)(Q,u')+\alpha^{1+\mu/2}\delta k^{\mu+1}\quad\text{in } I'\backslash I_5.$$

Taking $\alpha\leqslant(5L\gamma)^{-2}$ therefore gives

$$\mathcal{R}_5\leqslant \alpha^{1+\mu/2}\delta k^{1+\mu}\quad\text{in } I'\backslash I_5. \tag{4.6}$$

Since we have already shown that $\mathcal{R}_5\leqslant 0$ in I_5, the estimate (4.6) then holds in all of I'.

The rest of the proof follows exactly that for Theorem 3.1, except that in (3.20) the term $\varepsilon_5(\alpha)\alpha\delta k$ should be replaced by $\alpha^{1+\mu/2}\delta k^{1+\mu}$. Of course in the estimates for the corresponding function Ψ we use (4.4) instead of (3.5).

As in Section 3, the condition that $k\in\mathrm{BV}(J)$ can be replaced by other hypotheses. It is clear, for example, that we could suppose $\log k\in\mathrm{Lip}(J)$ in parallel with the result of Theorem 3.2. Another alternative is given by the following theorem, in which we assume that:

For every $U>0$ there exists a positive constant $\Theta=\Theta(U)>0$ and an exponent $m>1$ such that

$$|\nabla G(u,p)|\leqslant\Theta|p|^{m-1}\quad\text{when } |u|\leqslant U \text{ and } |p|\leqslant 1. \tag{4.7}$$

Remark. Obviously (4.7) holds when $G(p)=|p|^m/m$, with $\Theta=1$. It is also satisfied when $G(p)=\sqrt{1+|p|^2}-1$, with $\Theta=1$ and $m=2$. Another example, in which G depends explicitly on u, is $G(u,p)=(1+|u|^a)|p|^m/m$ where $a>0$, $m>1$. In general m can be thought of as being independent of U, as in the examples above. However, this is not strictly necessary, as the function $G(u,p)=|p|^{2+|u|^2}$ shows.

THEOREM 4.2. *Suppose that for every $U>0$ there is a bounded absolutely continuous function k on J such that* (4.2)–(4.4) *are satisfied. Assume also that*

$$|k'| \leqslant B\sigma^{\lambda} k^{1-\lambda} \quad a.e.\ on\ I, \tag{4.8}$$

where B is a positive constant and

$$\lambda = \begin{cases} \dfrac{m-1}{\nu+1}, & if\ \nu > m-2 \\ 1, & if\ \nu \leqslant m-2 \ \ (and\ m>2). \end{cases} \tag{4.9}$$

Then the rest state is a global attractor for the system (2.1).

Observe that the two values of λ in (4.9) coincide when $\nu=m-2$. In case $\nu=m-1$ we can write $\lambda=1/q$ and $1-\lambda=1/m$, so q is the Hölder conjugate of m. Finally, the limiting case $m\to1$, $\lambda\to0$ corresponds to the condition $\log k\in\mathrm{Lip}(J)$ noted above.

Proof of Theorem 4.2. The previous proof applies essentially word-for-word, except for step *4*. Here, in I' we see by (4.8) that

$$\begin{aligned} \alpha k'(\nabla G, u) &\leqslant \alpha B L \sigma^{\lambda} k^{1-\lambda} |\nabla G| \\ &\leqslant \alpha B L \sigma \begin{cases} \Theta(k/\sigma)^{1-\lambda} |u'|^{m-1}, & \text{if } |u'|<1 \\ \beta^{1-\lambda} C, & \text{if } |u'|\geqslant 1, \end{cases} \end{aligned} \tag{4.10}$$

by (4.7) and by (3.12), (4.3). We now proceed somewhat as in steps *2* and *3* of the proof of Theorem 3.1. First, if $|u'(t)|\geqslant1$ then from (H_3), (3.1) and $(4.1)_1$ we have $(Q,u')\leqslant-\sigma$. Thus by $(4.10)_2$ if

$$\alpha \leqslant \frac{\beta^{\lambda-1}}{5BCL},$$

then $\mathcal{R}_4\leqslant0$ at such values of t. Second, if $|u'(t)|<1$ but $\nu\leqslant m-2$ (so that $m>2$ and $\lambda=1$), then by $(4.10)_1$

$$\alpha k'(\nabla G, u) \leqslant \alpha B L \Theta \sigma |u'|^{m-1} \leqslant \alpha B L \Theta \sigma |u'|^{\nu+1}$$

while by (H_3), (3.1) and $(4.1)_1$ once more, $(Q,u')\leqslant-\sigma|u'|^{\nu+1}$. Hence if

$$\alpha \leqslant (5B\Theta L)^{-1},$$

then $\mathcal{R}_4\leqslant0$ also for the second set of values of t.

In the remaining case we have $\nu>m-2$ and $|u'(t)|<1$. Let

$$\begin{aligned} I_4' &= \{t\in I' : \Lambda_4(k/\sigma)^{1-\lambda}\alpha \leqslant |u'(t)|^{\nu-m+2},\ |u'(t)|<1\} \\ I_4'' &= \{t\in I' : |u'(t)|^{\nu-m+2} < \Lambda_4(k/\sigma)^{1-\lambda}\alpha,\ |u'(t)|<1\}, \end{aligned}$$

where $\Lambda_4 = 5BL\Theta$. In I_4' it is easy to see that $\mathcal{R}_4 \leqslant 0$, using the usual estimates for the term (Q, u'). On the other hand, in I_4'' we have

$$|u'(t)| < [\Lambda_4 (k/\sigma)^{1-\lambda} \alpha]^{1/(\nu - m + 2)} \quad (\text{since } \nu - m + 2 > 0).$$

Consequently in this set, by $(4.10)_1$ and (H_3),

$$\mathcal{R}_4 \leqslant \alpha BL\Theta \sigma^\lambda k^{1-\lambda} [\Lambda_4 (k/\sigma)^{1-\lambda} \alpha]^{\lambda'} = \text{Const.}\, \alpha^{1+\lambda'} k,$$

where $\lambda' = (m-1)/(\nu - m + 2) = \lambda/(1-\lambda) > 0$. By choosing α sufficiently small we then get

$$\mathcal{R}_4 \leqslant \tfrac{1}{8} \alpha \varkappa k$$

as in steps *2* and *3*. Hence in (3.20) the second term on the right side is dropped and another term $\frac{1}{8}\alpha\varkappa k$ is added in the second braces.

The rest of the proof is essentially the same as for Theorem 4.1.

Remark. While Theorem 4.2 requires the full strength of assumption $(\mathrm{C}_1)'$ as well as condition (4.7), the result of Theorem 4.1 actually uses only the second part of $(\mathrm{C}_1)'$, i.e. the condition $\varrho(s) = s^\mu$.

The following corollaries are helpful in the application of Theorems 4.1 and 4.2 to specific systems.

COROLLARY 1. *Suppose that the hypotheses of Theorem* 4.1 *or of Theorem* 4.2 *are satisfied, except that in place of* (4.4) *we assume the function* δ *has a decomposition* $\delta = d + e$ *with*

$$dk^\mu \in L^\infty(I) \quad \text{and} \quad ek^{\mu+1} \in L^1(I). \tag{4.4$'$}$$

Then the rest state is a global attractor.

Proof. It is easily seen, as in the proof of Corollary 1 of Theorem 3.1, that $(4.4)'$ implies (4.4). Hence Theorem 4.1 and Theorem 4.2 can be applied.

COROLLARY 2. *Define* $\bar\sigma(t) = \sigma(t)$ *for* $t \in I$ *and* $\bar\sigma(t) = 0$ *for* $t \in J \setminus I$, *for any* σ *given in* $(\mathrm{C}_1)'$. *Assume also that for every* $U > 0$

$$\sigma \notin L^1(I), \quad \delta\sigma^\mu \in L^\infty(I), \quad \bar\sigma \in \mathrm{CBV}(J).$$

Then the rest state is a global attractor.

Proof. Take $k = \bar\sigma$ and apply Corollary 1 in the case of Theorem 4.1.

COROLLARY 3. *Assume $I=J$ and that for every $U>0$*

$$\sigma \notin L^1(J), \quad \delta\sigma^\mu \in L^\infty(J), \quad \log\sigma \in \mathrm{Lip}(J).$$

Then the rest state is a global attractor.

Proof. Take $k=\sigma$ and apply Corollary 1 in the case of Theorem 4.2. (Note that σ is Lipschitz continuous, and so also absolutely continuous as is required for Theorem 4.2.)

COROLLARY 4. *Assume $I=J$. Suppose for every $U>0$ that δ admits the decomposition $\delta=d+e$, with $d>0$, and*

$$\frac{1}{d} \notin L^{1/\mu}(J), \quad \frac{1}{d}\left(1+\frac{1}{\sigma^\mu}\right) \in L^\infty(J), \quad \frac{e}{d} \in L^1(J).$$

Suppose also that

$$d^{-1/\mu} \in \mathrm{CBV}(J) \tag{4.11}$$

or that (4.7) *holds and*

$$d^{-1/\mu} \in \mathrm{AC}(J), \quad |(d^{-1/\mu})'| \leqslant \mathrm{Const.}\,(d^{-1/\mu})^{1-\lambda}\sigma^\lambda. \tag{4.12}$$

Then the rest state is a global attractor.

Proof. Take $k=d^{-1/\mu}$ so that (4.2)–(4.4)$'$ are satisfied and k is bounded. If (4.11) holds, then we apply Corollary 1 in the case of Theorem 4.1. If (4.12) holds then (4.8) is satisfied and Corollary 1 in the case of Theorem 4.2 can be applied.

COROLLARY 5. *Suppose $I=J$ and that for every $U>0$ there exist positive numbers c and η such that*

$$t\sigma(t) \geqslant c, \quad \liminf_{t\to\infty} \int_R^t \left(\frac{s}{t}\right)^\eta \delta(s)\,\frac{ds}{s^{\mu+1}} < \infty.$$

Then the rest state is a global attractor.

The proof is parallel to that of Corollaries 3 and 4 in Section 3. Just as for Corollary 4 in Section 3, the second condition of Corollary 5 is in a sense best possible. That is, it is satisfied when

$$\delta(t) \leqslant \mathrm{Const.}\,t^\mu$$

but fails when $\delta(t) \geqslant \mathrm{Const.}\,t^{\mu+\varepsilon}$ for some $\varepsilon>0$, while there are equations with $\delta(t) \geqslant \mathrm{Const.}\,t^{\mu+\varepsilon}$ for which the rest state is not stable.

Remarks. The last condition in Corollary 5 is obviously satisfied if $\delta(t)/t^{\mu+1} \in L^1(J)$. Similarly the last condition in Corollary 4 of Section 3 holds if $\delta(t)/t \in L^1(J)$.

For the system (1.5), the last condition of Corollary 5 becomes, since $\mu=1$,

$$\liminf_{t\to\infty} \int_R^t \left(\frac{s}{t}\right)^\eta \delta(s)\,\frac{ds}{s^2} < \infty.$$

Taking $\eta=2$ yields

$$\liminf_{t\to\infty} \frac{1}{t^2} \int_R^t \delta(s)\,ds < \infty. \tag{4.13}$$

For the scalar case of (1.5), and under the assumption $\sigma(t)\geqslant$Constant>0, condition (4.13) is essentially due to Artstein and Infante [1].

There are several special cases of Corollaries 2–4 which are of interest, e.g. when $\delta\leqslant$Const.$\,\sigma$ and when $\mu=m-1$. We leave these to the reader.

5. Necessary conditions for stability

In this section we shall obtain several necessary conditions for global asymptotic stability of the rest state $u\equiv 0$ for the system (2.1).

We shall not require the full strength of the hypotheses (H_1)–(H_3). In fact, only the following standing assumptions will be made, beyond the basic continuity and smoothness conditions stated at the outset of Section 2,

$$G(u,0)=0, \quad F(t,0)=0.$$

Recall finally that $H(u,p)=(\nabla G(u,p),p)-G(u,p)$, so in particular $H(u,0)=0$ for all u.

THEOREM 5.1. *Let u be a solution of* (2.1) *on J such that*

$$u(t)\to 0 \quad \textit{and} \quad u'(t)\to 0 \quad \textit{as } t\to\infty.$$

Suppose that for every $t\in J$ and for all u and p sufficiently small,

$$H(u,p)>0, \quad p\neq 0, \tag{5.1}$$

$$F(t,u)\geqslant 0, \quad 0\leqslant F_t(t,u)\leqslant \psi(t) \quad \textit{with } \psi\in L^1(J), \tag{5.2}$$

$$0\leqslant -(Q(t,u,p),p)\leqslant \hat\delta(t)H(u,p), \tag{5.3}$$

where

$$\hat\delta\in L^1(J). \tag{5.4}$$

Then $u\equiv 0$ in J.

Proof. Take $R_1\geqslant R$ such that $|u(t)|$ and $|u'(t)|$ are small enough in $J_1=[R_1,\infty)$ to allow the application of (5.1)–(5.3) along the solution.

By (2.7) the expression

$$H(u,u')+F(t,u)-\int_{R_1}^{t}(Q(s,u,u'),u')\,ds-\int_{R_1}^{t}F_t(s,u)\,ds \tag{5.5}$$

is constant. Since the first three terms are non-negative by (5.1), $(5.2)_1$ and $(5.3)_1$, from $(5.2)_2$ it follows that

$$(Q(t,u,u'),u')\in L^1(J_1)$$

and so

$$H(u,u')+F(t,u)=l-\int_{t}^{\infty}(Q(s,u,u'),u')\,ds-\int_{t}^{\infty}F_t(s,u)\,ds \tag{5.6}$$

for some l.

We claim that $F(t,u(t))\to 0$ as $t\to\infty$. Indeed by $(5.2)_2$ we obtain

$$F(t,u(t))\leqslant F(r,u(t))+\int_r^t \psi(s)\,ds,\quad R_1\leqslant r\leqslant t.$$

Since $u(t)\to 0$, this gives by $(5.2)_1$ and the fact that $F(t,0)=0$

$$0\leqslant \limsup_{t\to\infty} F(t,u(t))\leqslant \int_r^\infty \psi(s)\,ds.$$

The claim is proved by letting $r\to\infty$.

Now let $t\to\infty$ in (5.6) to obtain $l=0$. Hence in turn, because of (5.3) and $(5.2)_2$

$$\begin{aligned} H(u,u')+F(t,u)&=-\int_{t}^{\infty}(Q(s,u,u'),u')\,ds-\int_{t}^{\infty}F_t(s,u)\,ds\\ &\leqslant\int_t^\infty \hat{\delta}(s)H(u,u')\,ds=X(t). \end{aligned} \tag{5.7}$$

Since $H(u,u')\to 0$ as $t\to\infty$ and since $\hat{\delta}\in L^1(J)$, it is clear that $X(t)\to 0$ as $t\to\infty$. By (5.7) and $(5.2)_1$

$$X'=-\hat{\delta}H\geqslant -\hat{\delta}X,$$

so that for any $t\in J_1$ and $r>t$

$$X(t)\leqslant X(r)\exp\int_t^r \hat{\delta}(s)\,ds.$$

Letting $r\to\infty$ and using (5.4) again, we obtain $X(t)\leqslant 0$ for all $t\in J_1$.

By (5.1), $(5.2)_1$ and (5.7) it follows that $H(u(t),u'(t))\equiv 0$ on J_1. Hence $u'(t)\equiv 0$ on J_1 by (5.1), and so $u(t)\equiv 0$ on J_1 since $u(t)\to 0$ as $t\to\infty$. In particular $u(R_1)=u'(R_1)=0$. By repetition of the previous argument one finds that the subset of J where $u=0$ is both open and closed in J and so is all of J. This completes the proof.

COROLLARY 1. *Suppose that* (5.1)–(5.3) *hold. Then a necessary condition for the rest state to be a global attractor for* (2.1) *is that*

$$\hat{\delta} \notin L^1(J).$$

In fact if $\hat{\delta} \in L^1(J)$ then by Theorem 5.1 there are no solutions whatsoever, except the trivial one $u \equiv 0$, which approach zero together with their derivatives at infinity.

Remarks. To prove Corollary 1 it would in fact be enough to find just one solution (or one bounded solution) which does not tend to zero as $t \to \infty$. Hence Theorem 5.1 is actually a much stronger result than simply that the rest state is not a global attractor.

The hypotheses (5.1) and (5.3) of Theorem 5.1 hold for the function $G(p) = |p|^m/m$, $m > 1$, and for Q satisfying (H_3) and (3.2) of $(C_1)'$ in the stronger form

$$|Q(t,u,p)| \leqslant \delta(t) \cdot |p|^{\mu} \quad \text{for } t \in J,\ |u| \leqslant U \text{ and } |p| \leqslant q, \tag{3.2$'$}$$

with $\mu \geqslant m-1$. Indeed $H(p) = (m-1)|p|^m/m$, giving (5.1), while also for $t \in J$, $|u| \leqslant U$ and $|p| \leqslant q \leqslant 1$

$$-(Q,p) \leqslant |Q| \cdot |p| \leqslant \delta |p|^{1+\mu} \leqslant \hat{\delta} H, \quad \text{with } \hat{\delta} = \frac{m\delta}{m-1}.$$

Hence (5.3) holds and $\hat{\delta} \in L^1(J)$ if and only if $\delta \in L^1(J)$. The simplest example of this behavior is the system (1.5) discussed in the introduction.

The same ideas apply to the mean curvature operator $G(p) = \sqrt{1+|p|^2} - 1$. Here $H(p) = 1 - 1/\sqrt{1+|p|^2}$, giving (5.1) immediately. Moreover for Q satisfying (H_3) and $(3.2)'$ with $\mu = 1$, we have $-(Q,p) \leqslant 4\delta H$ when $|p| \leqslant q \leqslant 1$. Here it is essential to restrict to small p to obtain the validity of (5.3). Clearly a wide class of functions G and Q can be treated in the same way.

Finally (5.2) is a consequence of (H_2) if we add the assumption $F_t(t,u) \geqslant 0$. Of course, if F is independent of t, only $(5.2)_1$ is required.

As a special example, consider the strongly nonlinear systems (1.10) and (1.11) in the introduction, with $F = F(u)$, $(f(u), u) > 0$, and

$$\sigma(t) = t^{\alpha}(\log t)^{\beta}, \quad \delta(t) = \text{Pos. Const.}\, \sigma(t), \tag{5.8}$$

in $(C_1)'$; of course here we replace (3.2) by $(3.2)'$. If $\alpha < 0$ then from Corollary 2 of Section 4 we see that the rest state is a global attractor if

$$\alpha > -1 \quad \text{or} \quad \alpha = -1,\ \beta \geqslant -1. \tag{5.9}$$

On the other hand, by Corollary 1 of Theorem 5.1, condition (5.9) is also necessary for the rest state to be a global attractor provided $\mu \geqslant m-1$. It is doubtful that (5.9) is a necessary condition when $\mu < m-1$.

Next suppose $\alpha > 0$, but that otherwise $(C_1)'$ holds with the conditions given above. Then by Corollary 4 of Section 4 (with $d=\delta$, $e=0$ and (4.11) holding) the rest state is a global attractor if

$$\alpha < \mu \quad \text{or} \quad \alpha = \mu,\ \beta \leqslant \mu.$$

This condition is also necessary, as shown in [10, Theorem 4.4]; that is, if $1/\sigma \in L^{1/\mu}(J)$ then the rest state cannot be a global attractor. Finally, it is easy to check that when $\alpha = 0$ the rest state is always a global attractor.

When $N=1$ the condition $u'(t) \to 0$ as $t \to \infty$ in Theorem 5.1 can be omitted and the proof simplified. For this improvement we require the further structural condition that $H(0,p)$ be strictly increasing for $p>0$ and strictly decreasing for $p<0$. (This is a consequence of the strict convexity of $G(0,p)$.)

THEOREM 5.1′. *Assume $N=1$. Let the hypotheses of Theorem* 5.1 *hold, with the exception that the condition $u'(t) \to 0$ as $t \to \infty$ is omitted. Then $u \equiv 0$ in J.*

Proof. The argument is the same as before, except that now we obtain $l=0$ in a different way and without using the limit assumption on u'.

Suppose then that $N=1$. In the same way as before $F(t,u(t)) \to 0$ as $t \to \infty$. Then by (5.6) it follows that $H(u(t),u'(t)) \to l$ as $t \to \infty$. Since $H(0,p)$ is strictly increasing for $p>0$ and strictly decreasing for $p<0$, it is clear that

$$u'(t) \to p_\infty \text{ (possibly infinite)} \quad \text{as } t \to \infty$$

(note that for any $t_0 \geqslant R$ the values of u' for $t \geqslant t_0$ form an interval of $\mathbf{R}$). This is obviously impossible when $u(t) \to 0$ unless $p_\infty = 0$. Thus $l=0$ and the previous proof applies without further changes.

A result corresponding to Theorem 5.1′ can also be proved for vector solutions. We indicate how this can be accomplished for the system

$$u'' + f(u) = Q(t,u,u'),$$

when F and Q satisfy the conditions (5.2)–(5.4) and Q obeys (C_2). Thus suppose that $u(t) \to 0$ as $t \to \infty$. By Theorem 5.1 it is enough to prove that also $u'(t) \to 0$ as $t \to \infty$ since (5.1) obviously holds. From (2.7) and (5.3) we obtain $\{H(u')+F(u)\}' \leqslant 0$ so that $H(u') = \frac{1}{2}|u'|^2 \to \text{limit} = l$. If $l=0$ we are done, so suppose that $l>0$. Then

$$u'(t) - u'(r) = \int_r^t Q(s,u,u')\,ds - \int_r^t f(u)\,ds.$$

As in the first part of the proof of Theorem 5.1, $(Q,u')\in L^1(J)$. But by (C_2), for t sufficiently large,

$$|Q|\leqslant\frac{|Q|\cdot|u'|}{\frac{1}{2}\sqrt{2l}}\leqslant\frac{2\gamma}{\sqrt{2l}}|(Q,u')|.$$

Hence also $Q(t,u,u')\in L^1(J)$. On the other hand $f(u(t))\to f(0)=0$ as $t\to\infty$, because $u(t)\to 0$. Consequently, for $t\in(r,r+1]$ and $r\to\infty$ we have

$$u'(t)-u'(r)\to 0.$$

Then from the relation $|u'|^2\to 2l\neq 0$ there results

$$|u(r+1)-u(r)|=\left|\int_r^{r+1}u'(t)\,dt\right|\geqslant|u'(r)|-\int_r^{r+1}|u'(t)-u'(r)|\,dt\geqslant\tfrac{1}{2}\sqrt{2l}$$

for r large, which contradicts the fact that $u(t)\to 0$ as $t\to\infty$.

The same idea can be extended to strictly convex action energies $G(p)$, for which $H(p)\to\infty$ as $|p|\to\infty$. Indeed from $H(u')\to l$ it is clear that $|u'(t)|\geqslant\text{Const.}>0$ for large t (when $l>0$). Thus $Q\in L^1(J)$ as above, and in turn $\nabla G(u'(t))-\nabla G(u'(r))\to 0$. Since u' is bounded by the assumption on H, it now follows that $u'(t)-u'(r)\to 0$, and the contradiction is obtained as before.

6. The non-homogeneous case

In this section we consider the non-homogeneous analogue of the system (2.1), namely

$$(\nabla G(u,u'))'-\nabla_u G(u,u')+f(t,u)=Q(t,u,u')+e(t),\quad J=[R,\infty).\tag{6.1}$$

Under the conditions (a) or (b) in the following lemma, we shall show that the asymptotic stability of the rest state $u\equiv 0$ for (6.1) is essentially the same as for the system (2.1).

LEMMA. *Suppose $F(t,u)\geqslant 0$ in $J\times\mathbf{R}^N$, that* (2.4) *and* $(5.3)_1$ *hold, and that either*

(a) *$e'\in L^1(J)$, $e(t)\to 0$ as $t\to\infty$,*

or

(b) *$e\in L^1(J)$ and $|p|\leqslant\text{Const.}\{1+H(u,p)\}$ for $u,p\in\mathbf{R}^N$.*

Then the quantity $H(u,u')+F(t,u)$ tends to a non-negative limit as $t\to\infty$ along any bounded solution of (6.1).

Proof. From (6.1) we obtain, corresponding to (2.7) for the system (2.1),

$$\{H(u,u')+F(t,u)\}'=(Q(t,u,u'),u')+F_t(t,u)+(e,u').$$

In case (a) we write

$$(e,u')=(e,u)'-(e',u).$$

Then by (2.4), $(5.3)_1$ and the fact that $|u|\leqslant L$, say, we have

$$\left\{H(u,u')+F(t,u)+\int_t^\infty \psi(s)\,ds-(e,u)+L\int_t^\infty |e'(s)|\,ds\right\}'\leqslant 0$$

and the result follows at once. In case (b)

$$\{H(u,u')+F(t,u)\}'\leqslant \psi(t)+\text{Const.}\,|e(t)|\{1+H(u,u')\}.$$

Thus by Gronwall's inequality and the fact that $F(t,u)\geqslant 0$, we find that $H(u,u')$ is bounded. In turn

$$\{H(u,u')+F(t,u)\}'\leqslant \psi(t)+\text{Const.}\,|e(t)|$$

and again the conclusion follows.

We can now give the main result of the section.

THEOREM 6.1. *Let the hypotheses of Theorems* 3.1, 3.2, 4.1, *or* 4.2 *hold, and suppose also that e satisfies either condition* (a) *or condition* (b) *of the lemma.*

Then the rest state is a global attractor for the system (6.1).

Proof. We show that the proofs of Theorems 3.1, 3.2, 4.1, and 4.2 are essentially unchanged if either (a) or (b) holds.

Consider first case (a). We use the variational identity for (6.1) corresponding to the identity (3.10) for the problem (2.1). The new identity in fact differs from (3.10) only by the addition, to the left hand side of (3.10), of the term $-(e,u)\omega$ and, to the right hand side, of the quantity

$$(\alpha-1)(e,u)\omega'-(e',u)\omega,$$

see [7]. Clearly

$$|(\alpha-1)(e,u)\omega'|\leqslant \text{Const.}\,|e|\omega',$$

and since $e(t)\to 0$ as $t\to\infty$ one has

$$|(\alpha-1)(e,u)\omega'|\leqslant \frac{\alpha\varkappa}{8}\omega'$$

for large t. Similarly

$$|(e',u)\omega|\leqslant L|e'|\omega,\quad |e'|\in L^1(J),$$

and this term can be treated almost exactly the same as the term Const. $|k'|\omega$ in step *4* of Theorem 3.1.

Next if (b) holds we write the additional term on the left hand side of the identity, namely $\{-(e,u)\omega\}'$, in the form

$$-(e,u)\omega'-(e',u)\omega-(e,u')\omega.$$

These terms cancel several of the additional terms on the right hand side, leaving only the following additional terms on the right:

$$\alpha(e,u)\omega'+(e,u')\omega.$$

Since u and u' are bounded in case (b)—recall that $|p|\leqslant \text{Const.}\,\{1+H(u,p)\}$ and that, by the lemma, $H(u,u')$ is bounded—and since

$$\omega'=k\omega\leqslant \text{Const.}\,\omega,$$

the additional terms are bounded by

$$\text{Const.}\,|e|\omega.$$

Using the fact that $e\in L^1$, this term can also be treated as in step *4* of Theorem 3.1. This completes the proof.

Remarks. The condition $|p|\leqslant \text{Const.}\,\{1+H(u,p)\}$ in case (b) is satisfied when $G(p)=|p|^m/m$, $m>1$, and in particular for the system (1.5). In the scalar case, Levin and Nohel had found the condition $e\in L^1(J)$, $e(t)\to 0$ as $t\to\infty$, whereas here for case (b) only $e\in L^1(J)$ is needed.

It is, finally, worth noting that hypothesis (b) can be generalized without difficulty to the case when the non-homogeneous term e in (6.1) depends not only on t but also on u and p, provided we assume that

$$|e(t,u,p)|\leqslant \psi(t)\frac{H(u,p)+F(t,u)+1}{|p|+1} \tag{6.2}$$

where $\psi\in L^1(J)$.

Appendix

We prove the lemma used in the proof of Theorem 3.1.

Proof of Lemma A. If $k>0$ in J we take $E=\varnothing$, and define $\bar{\bar{k}}$ to be a continuous piecewise linear function with vertices on the graph of the function $\frac{1}{2}(\theta+1)k$, such that

$$k<\bar{\bar{k}}<\theta k, \quad t\in J.$$

(For this construction it is necessary that k be continuous; in particular on any compact set k then has a positive lower bound and the graphs of k, $\frac{1}{2}(\theta+1)k$ and θk are separated by a positive distance.)

Rounding the corners of the graph of $\bar{\bar{k}}$ (say with small circular arcs) then produces the desired function $\bar{k}$, since clearly

$$\operatorname{Var}\bar{k} \leqslant \operatorname{Var}\bar{\bar{k}} \leqslant \operatorname{Var}\left(\tfrac{1}{2}(\theta+1)k\right) \leqslant \theta \operatorname{Var} k. \tag{A1}$$

When the set C of zeros of k is non-empty, the construction is slightly more delicate. Clearly C is closed since k is continuous. Hence $J\backslash C$ is a relatively open set, O, and we can write

$$O = \bigcup I_j,$$

where I_j, $j=1,2,\ldots$, are disjoint open intervals in J, with $I_1=[R,a)$ if $k(R)>0$.

Let J_1, J_2, ... be a sequence of disjoint intervals such that $|J_n|=1$ and $J=\bigcup J_n$. It is not difficult to see that there exists a subsequence of intervals I_i' drawn from (I_j) such that

(1) only a finite number of intervals I_i' intersect any J_n,

(2) $\int_{J_n\backslash O'} k \leqslant 2^{-(n+1)}$, where $O'=\bigcup I_i'$.

To define E, we first choose a closed subinterval G_i in each interval I_i', with the property that

$$\int_{S_i} k \leqslant 2^{-(i+1)}, \quad S_i = I_i'\backslash G_i, \quad i=1,2,\ldots,$$

and put $E=\operatorname{Int} J\backslash\bigcup G_i$ (here E is open, since $\bigcup G_i$ is certainly closed; if $I_1=[R,a)$ we take $I_1'=I_1$ and G_1 in the form $[R,b)$, $b<a$). Clearly

$$E = (C\cup O''\cup S)\backslash\{R\}, \quad \text{where } O''=O\backslash O' \text{ and } S=\bigcup S_i.$$

Since $\int_{O''} k \leqslant \frac{1}{2}$ by (2), it follows that

$$\int_E k = \int_{O''} k + \int_S k \leqslant 1$$

and (iii) is verified.

Next, define $\bar{\bar{k}}$ to be a continuous piecewise linear function on J, all of whose vertices lie in $O'=\bigcup I_i'$, such that:

In G_i, the vertices of $\bar{\bar{k}}$ lie on the graph of $\frac{1}{2}(\theta+1)k$ and $k<\bar{\bar{k}}<\theta k$ (recall that $k>0$ on the closed set G_i),

$\bar{\bar{k}}$ has exactly one vertex $(c_i, 0)$ in each S_i, $i=1,2,\ldots$, and $0\leqslant\bar{\bar{k}}<\theta k$ in S_i.

Clearly $0 \leqslant \bar{\bar{k}} \leqslant \theta k$ in E. Finally, we obtain $\bar{k}$ by rounding the corners $\bar{\bar{k}}$ as before (obviously $\bar{\bar{k}}$ has at most a finite number of corners in any bounded interval of J). Conditions (i) and (ii) now follow as in the first part of the proof.

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