Acta Math., 169 (1992), 1–103

Desingularization of non-dicritical holomorphic foliations and existence of separatrices

by

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Dedicated to Jean Martinet, in memoriam

Introduction

In this paper we complete the reduction of the singularities for non-dicritical holomorphic foliations of [5] and [7], in order to get only the so-called simple singularities. As a consequence, we prove Thom's conjecture about the existence of convergent separatrices, in dimension three. These results where announced in [8].

Let X be a non-singular analytic variety over $\mathbf{C}$. A holomorphic singular foliation $\mathcal{G}$ of codimension one over X is an integrable and inversible $\mathcal{O}_{X,P}$-module of the cotangent sheaf Ω_X such that the quotient $\Omega_X/\mathcal{G}$ has no torsion. This means that each stalk $\mathcal{G}_P$ is generated by a differential 1-form

$$\Omega=\sum_{i=1}^{n} b_i\,dx_i;\quad b_i\in\mathcal{O}_{X,P}$$

such that $\Omega\wedge d\Omega=0$ and g.c.d.$(b_i; i=1,\dots,n)=1$. The singular locus $\operatorname{Sing}\mathcal{G}$ is locally given by

$$\operatorname{Sing}\mathcal{G}=(b_i=0; i=1,\dots,n).$$

It is a closed analytic subset of X of codimension $\geqslant 2$. An irreducible element $f\in\mathcal{O}_{X,P}$ is a *separatrix or an analytic solution* iff f divides $\Omega\wedge df$. This means that $(f=0)$ is contained in a leaf, outside the singular locus. Analogously, a formal separatrix or a formal solution is an irreducible element $f\in\hat{\mathcal{O}}_{X,P}$ (=formal completion of $\mathcal{O}_{X,P}$ along its maximal ideal) such that f divides $\Omega\wedge df$.

The result in this paper concerning Thom's conjecture may be stated as follows:

EXISTENCE OF SEPARATRIX THEOREM (dimension three). *If $\mathcal{G}$ is a germ of holomorphic singular foliation of codimension one over $(\mathbf{C}^3, 0)$ given by $(\Omega=0)$ then one of the following two properties is satisfied:*

(i) *$\mathcal{G}$ has an analytic solution at the origin.*

(ii) *There is an analytic mapping $\sigma^*: (\mathbf{C}^2, 0) \to (\mathbf{C}^3, 0)$ such that $\sigma^*\Omega$ is not identically zero and the foliation given by $(\sigma^*\Omega=0)$ has infinitely many analytic solutions.*

When (ii) holds we call the singularity a "dicritical singularity". In the two dimensional case, the existence of an analytic solution has been proved by Camacho and Sad [4]. In the three dimensional case, Jouanolou [15] gives a counterexample to the existence of a separatrix in the case (ii) above.

Like in the case of varieties (cf. [1], [13]), the reduction of the singularities intends to improve the singularities by blowing-up the ambient space X. More precisely, let $\pi: X' \to X$ be the blowing-up of X with a non-singular center $Y \subset \operatorname{Sing} \mathcal{G}$. Then there is a unique singular foliation $\mathcal{G}'$ over X' such that

$$\mathcal{G}'|_{X'-\pi^{-1}(Y)} = \mathcal{G}|_{X-Y}.$$

We call $\mathcal{G}'$ the strict transform of $\mathcal{G}$ by π. Note that, even if we blow-up repeatedly, we do not necessarily get that $\mathcal{G}'$ has no singular points. This can be easily seen by blowing-up $ydx+xdy$. Thus we can only hope to get "simple singularities", in order to have the following result:

DESINGULARIZATION THEOREM. *Let $\mathcal{G}$ be a non dicritical holomorphic singular foliation over $X=(\mathbf{C}^3, 0)$. Then there is a sequence of "permissible blowing-ups"*

$$X(1) \overset{\pi(1)}{\leftarrow} X(2) \overset{\pi(2)}{\leftarrow} \cdots \overset{\pi(N)}{\leftarrow} X(N) \tag{1}$$

such that the strict transform $\mathcal{G}(N)$ of $\mathcal{G}$ under this sequence has only simple singularities.

Let us explain somehow the above statements. First, let us recall the situation in the case $\dim X=2$. Write

$$\Omega = a\,dx + b\,dy, \quad a(P) = b(P) = 0,$$

and put

$$D = -b\frac{\partial}{\partial x} + a\frac{\partial}{\partial y}.$$

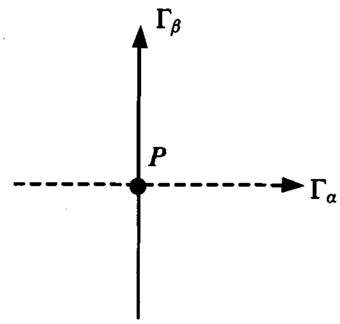


Fig. 1

The point P is a *simple singularity* iff the linear part of D has two distinct eigenvalues $\alpha\neq\beta\neq0$ and $\alpha/\beta\notin\mathbf{Q}_+$ (=strictly positive rational numbers). The simple singularities are persistent under blowing-up. In fact, the blowing-up of a simple singularity produces exactly two other ones, corresponding to the eigendirections. Moreover, there are exactly two formal separatrices Γ_α and Γ_β at P, which are both non-singular and tangent to the corresponding eigendirection. By Briot–Bouquet's Theorem, we know that Γ_β is always convergent. (See Figure 1.)

Now, we can choose a regular system of parameters (x, y) of $\hat{\mathcal{O}}_{X,P}$ and Ω which is written down in one of the following formal normal forms:

(i) $\Omega=xy(dx/x+\lambda\, dy/y)$; $\lambda\in\mathbf{C}$, $\lambda\notin\mathbf{Q}_-$;

(ii) $\Omega=xyy^s(dx/x+(\varepsilon+1/y^s)\,dy/y)$; $s\geq1$, $\varepsilon\in\mathbf{C}$;

(iii) $\Omega=xy(x^py^q)^s\,(dx/x+(\varepsilon+1/(x^py^q)^s)\,(p\,dx/x+q\,dy/y)$; g.c.d.$(p,q)=1$, $s\geq1$;

(cf. Part II). There, we have that $\Gamma_\alpha\cup\Gamma_\beta=(xy=0)$.

Assume now that P is the only singular point of $\mathcal{G}$. Then, the two-dimensional desingularization due to Seidenberg [20] says that there is a finite sequence of blowing-ups at singular points

$$(*)\qquad X(1)\xleftarrow{\pi(1)}X(2)\xleftarrow{\pi(2)}\dots\xleftarrow{\pi(N)}X(N)$$

such that all the singularities in the last step are simple singularities. Let $E(N)$ be the exceptional divisor produced by the sequence $(*)$. The irreducible components of $E(N)$ which are generically transversal to the strict transform $\mathcal{G}(N)$ of $\mathcal{G}$ are called "dicritical components". Thus, the non-dicritical components are leaves of $\mathcal{G}(N)$. Note that a dicritical component produces by blowing-down infinitely many separatrices at P. (See Figure 2.)

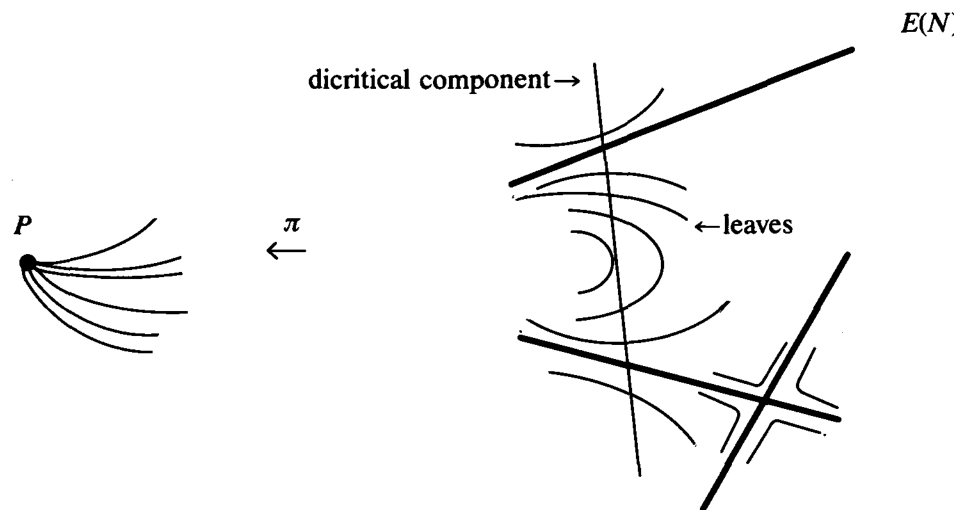


Fig. 2

We say that $\mathscr{G}$ is *non-dicritical* iff $E(N)$ has no dicritical components. This is equivalent to say that $\mathscr{G}$ has only finitely many separatrices at P.

Let us restrict our attention to the non-dicritical case. Given a point $Q \in E(N)$, denote by $e=e(E(N),Q)$ the number of irreducible components of $E(N)$ through Q. If $e=2$, then $\Gamma_\alpha \cup \Gamma_\beta = E(N)$, locally at Q, and hence no other separatrix of $\mathscr{G}(N)$ passes through Q. If $e=1$ and $Q \in \operatorname{Sing} \mathscr{G}$, then either $E(N)=\Gamma_\alpha$ or $E(N)=\Gamma_\beta$, locally at Q, hence there is exactly one separatrix Γ_Q of $\mathscr{G}(N)$ at Q with $\Gamma_Q \neq E(N)$. (See Figure 3.)

By blowing-down these Γ_Q, we obtain a bijection

$$\{\text{formal separatrices of } \mathscr{G} \text{ at } P\} \leftrightarrow \{\text{points } Q \in E(N) \cap \operatorname{Sing} \mathscr{G}(N) \text{ with } e=1\}.$$

By [4], we know that there is always a point Q with $e=1$ such that Γ_Q corresponds to a nonzero eigenvalue, hence Γ_Q is convergent and projects over a convergent separatrix Γ of $\mathscr{G}$ at P. (See Figure 4.)

Fig. 3

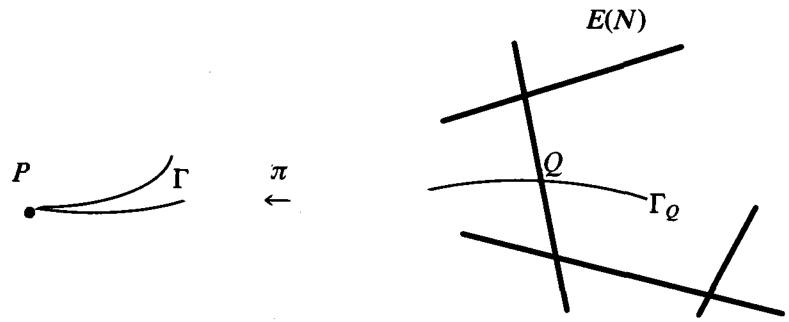


Fig. 4

Thus, as mentioned above, Thom's question about the existence of a convergent separatrix has an affirmative answer in the case $\dim X=2$.

Now, let us consider the case that $\dim X=n\geqslant 3$. Let us fix a normal crossings divisor E of X. Here E plays the role of the exceptional divisor in an intermediary step of the desingularization process, hence in the initial step we shall put $E=\emptyset$. A *dicritical component of* E is an irreducible component of E which is generically transversal to $\mathscr{G}$. Consider a blowing-up $\pi: X'\to X$ with center Y. Note that if the center Y has normal crossings with E, then $E'=\pi^{-1}(E\cup Y)$ is also a normal crossings divisor of X'. We say that Y is a permissible center for $\mathscr{G}$ adapted to E iff, in addition Y satisfies a certain condition of equimultiplicity locally at each point (cf. [5], [7] and Part I). We say that $\mathscr{G}$ *is non-dicritical* iff E has no dicritical components and this remains true after any finite sequence of permissible blowing-ups (this definition is made relatively to E, actually, it deals with the initial singular foliation, before starting the desingularization process). Roughly speaking, to say that $\mathscr{G}$ is dicritical means that for a certain non-degenerate two-dimensional section we can find infinitely many integral curves (cf. [6]). This corresponds to the condition (ii) of the Existence of Separatrix Theorem.

In opposition to the same phenomena in the two dimensional case, the dicriticalness is an obstruction to the existence of a convergent (even a formal) separatrix. In fact, the dicritical foliation given by the differential form

$$\Omega=(x^my-z^{m+1})\,dx+(y^mz-x^{m+1})\,dy+(z^mx-y^{m+1})\,dz,\quad m\geqslant 2,$$

has no separatrices at the origin [15]. Thus, we may reformulate Thom's question about the existence of separatrices as follows:

If $\mathscr{G}$ is non-dicritical, does $\mathscr{G}$ have a convergent separatrix?

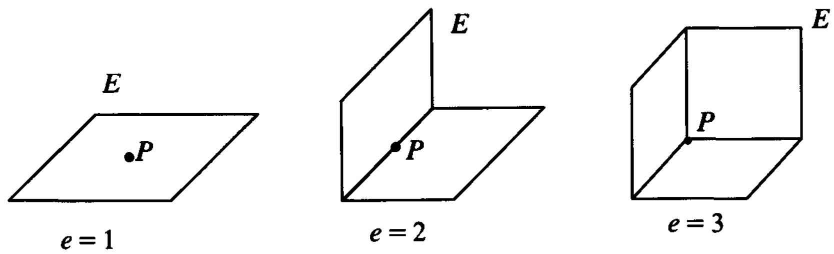


Fig. 5

Assume that $\mathscr{G}$ is non-dicritical and that E is given locally at P by

$$\prod_{i\in A} x_i = 0.$$

Then we can write, in a logarithmic way,

$$\Omega = \left(\prod_{i\in A} x_i\right)\omega; \quad \text{where} \quad \omega = \sum_{i\in A} a_i \frac{dx_i}{x_i} + \sum_{i\notin A} a_i\, dx_i; \quad a_i \in \mathcal{O}_{X,P}$$

and g.c.d.$(a_i; i=1, \ldots, n)=1$. The *adapted multiplicity* $\mu(\mathscr{G}, E; \{P\})$ of $\mathscr{G}$ at P is defined by

$$\mu(\mathscr{G}, E; \{P\}) = \min\{\nu_P(a_i); i\in A\} \cup \{\nu_P(a_i)+1; i\notin A\}$$

where $\nu_P(a_i)$ denotes the order of a_i at the point P. It generalizes the order of the strict transform of a hypersurface, in the case that we begin with $\Omega = df$. The main result in [5] and [7] is stated as follows:

Reduction Theorem ([5], [7]). *Let $\mathscr{G}$ be a non-dicritical holomorphic foliation over $X=(\mathbf{C}^3, 0)$. Then there is a finite sequence of permissible blowing ups*

$$X = X(1) \overset{\pi(1)}{\longleftarrow} X(2) \overset{\pi(2)}{\longleftarrow} \ldots \overset{\pi(N)}{\longleftarrow} X(N) \tag{2}$$

such that $\mu(\mathscr{G}(N), E(N); \{Q\}) \leq 1$ for each point $Q \in X(N)$; where $\mathscr{G}(N)$ is the strict transform of $\mathscr{G}$ and $E(N) \subset X(N)$ is the exceptional divisor of (2).

Assume now that $\dim X = 3$ and fix a point $P \in E$. We want to define the statement: P is a *simple singularity of* $\mathscr{G}$. Put $e = e(E, P)$; we have three possibilities $e = 1, 2$ or 3. (See Figure 5.)

In the case $e=1$, we say that P is a simple singularity iff $\mathscr{G}$ is an analytic cylinder over a two dimensional simple singularity with $e=1$. In particular, in this case Sing $\mathscr{G}$ is

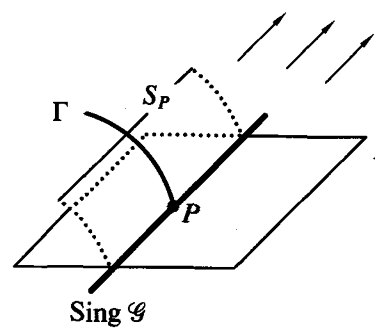


Fig. 6

locally a nonsingular curve contained in E and the formal separatrix of the two dimensinal case produces a formal separatrix S_P at P. (See Figure 6.)

In the case $e=2$, we have two kinds of singularities. The first kind is locally an analytic cylinder over a two dimensional simple singularity with $e=2$. In this case Sing $\mathscr{G}$ is locally the intersection of the two components of E and the only separatrices of $\mathscr{G}$ at P are the irreducible components of E. (See Figure 7.)

Before defining the simple singularities of the second type with $e=2$, let us consider the case $e=3$. Then P is a simple singularity iff $\mu(\mathscr{G}, E; \{P\})=0$ and the singular points near P are simple singularities of the first kind with $e=2$. The singular locus is the union of the intersections of two components of E and the only separatrices of $\mathscr{G}$ at P are the irreducible components of E. In order to verify if P is a simple singularity it is enough to look at any generator of $\hat{\mathscr{G}}=\mathscr{G}_P\hat{\mathcal{O}}_{X,P}$. Hence, it is a formal definition. (See Figure 8.)

Now, assume that $e=2$. Then P is a simple singularity of the second kind with $e=2$

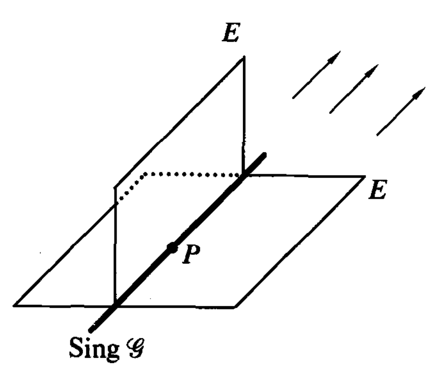


Fig. 7

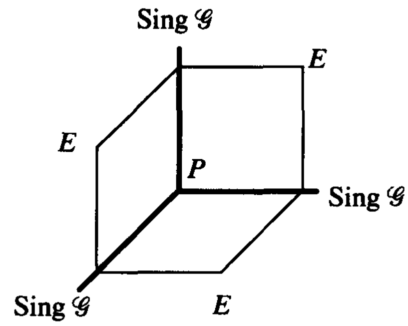


Fig. 8

iff there is a nonsingular formal separatrix S_P at P such that $E \cup S_P$ is a (formal) normal crossings divisor at P and P is a simple singularity for $\mathscr{G}$, relatively to $E \cup S_P$ (i.e. with $e=3$). (See Figure 9.)

In particular, Sing $\mathscr{G}$ is the union of the intersections of two components of $E \cup S_P$ and S_P is the only formal separatrix of $\mathscr{G}$ at P which is not a component of E. Moreover, in this case, the singular points near P are either simple singularities with $e=1$ or simple singularities of the first kind with $e=2$.

The simple singularities and their normal forms are studied in Part II. First of all we define the pre-simple singularities by the following conditions:

(a) Adapted multiplicity less or equal than one.

(b) The directrix (if it exists), has dimension two and has normal crossings with the divisor E.

The directrix is a geometrical invariant which plays a role similar to the strict tangent space of Hironaka [13] (it is defined in [5], [7] and also in the Part I). Hence,

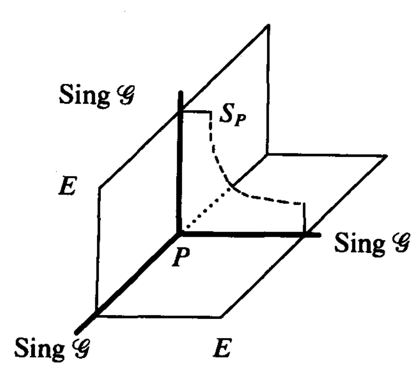


Fig. 9

being a pre-simple singularity is a very geometrical property. Actually, this property is semicontinuous in an evident sense (see Proposition I.2.6). Let us note however that the semicontinuity depends on the non-dicriticalness property.

Now, let P be a pre-simple singularity. Put

$$\hat{\mathscr{D}}(\mathscr{G})_P = \{D \in \hat{\Theta}_{X,P}; \Omega(D) = 0\} \subset \hat{\Theta}_{X,P}$$

where $\hat{\Theta}_{X,P}$ is the $\hat{\mathcal{O}}_{X,P}$-module of the formal vector fields. Then we can find two commuting formal vector fields D_1, D_2 in $\hat{\mathscr{D}}(\mathscr{G})_P$ which produces also $\hat{\mathscr{G}}_P$ by duality. Moreover, one of the following situations occurs:

(A) $D_1 = \partial/\partial z$, $D_2 = x\,\partial/\partial x + a(x,y)\,\partial/\partial y$; with $a(0,0)=0$. (In this case D_1 gives the local analytic triviality.)

(B) $D_1 = x\,\partial/\partial x + a(x,y,z)\,\partial/\partial z$, $D_2 = y\,\partial/\partial y + b(x,y,z)\,\partial/\partial y$; with $a(0,0,0)=b(0,0,0)=0$.

Then, P is a simple singularity iff the eigenvalues of D_1 (in case (A)) or of D_1, D_2 (in case (B)) are non-resonant in a similar sense to the two-dimensional case (quotients not in $\mathbf{Q}_+$). These are diophantic conditions, easily reached after finitely many permissible blowing-ups, if we begin with only pre-simple singularities (see Part III, § 1).

The fact that D_1 and D_2 commute allows us to make a simultaneous jordanization of D_1 and D_2. In particular, we can find a regular system of parameters (x,y,z) of $\hat{\mathcal{O}}_{X,P}$ in which the semisimple parts of D_1 and D_2 are diagonal. After a little additional work we can write down formal normal forms for the pre-simple singularities (see Proposition II.4.4). More particularly, in the case of simple singularities we see that a generator Ω of $\hat{\mathscr{G}}_P$ may be written down either in one of the normal forms (i), (ii), (iii) (in the case (A)) or in one of the following normal forms (in the case (B)):

(iv) $\Omega = xyz(\alpha\, dx/x + \beta\, dy/y + dz/z)$; with $\alpha\cdot\beta \neq 0$ and $-\alpha$, $-\beta$, $-\alpha/\beta \notin \mathbf{Q}_+$.

(v) $\Omega = xyz\cdot z^s(dx/x + \beta\, dy/y + (\varepsilon + 1/z^s)\, dz/z)$; with $s \geq 1$, $0 \neq -\beta \notin \mathbf{Q}_+$.

(vi) $\Omega = xyz(y^p z^q)^s (dx/x + \beta\, dy/y + (\varepsilon + 1/(y^p z^q)^s)(p\, dy/y + q\, dz/z))$; $s \geq 1$, g.c.d.$(p,q)=1$.

(vii) $\Omega = xyz\cdot(x^p y^q z^r)^s (dx/x + \beta\, dy/y + (\varepsilon + 1/x^p y^q z^r)^s)(p\, dx/x + q\, dy/y + r\, dz/z))$; with $s \geq 1$, g.c.d. $(p,q,r)=1$.

Many of the properties we need from simple singularities can be obtained either directly from the formal normal forms, either from the way we obtain the formal normal forms. For instance, the uniqueness property of the formal separatrix S_P, the shape of the singular locus or even the fact that S_P is "convergent" along the exceptional divisor E.

In Part III, we give a proof of the Desingularization Theorem. By the Reduction Theorem, we may assume that we start with adapted multiplicity less or equal than one. The first thing we do is to prove that we can get only pre-simple singularities after finitely many permissible blowing-ups. This is quite difficult, but most of the technics in

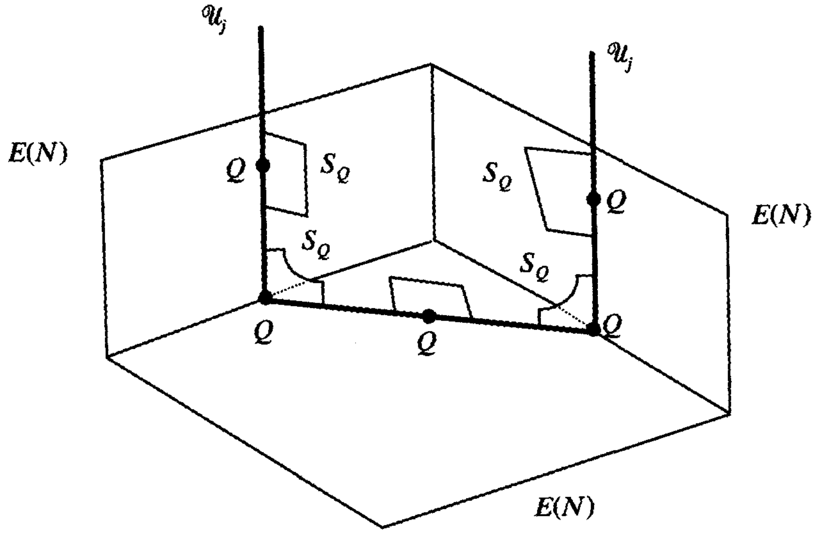


Fig. 10

[7] remain valid. Hence we only give in detail those parts which are either different or may be simplified with respect to the general technics in [7]. Once we have only pre-simple singularities, we finish with a computation of "killing resonancies" along the irreducible components of Sing $\mathcal{G}$.

In Part IV we prove the existence of a convergent separatrix for a non-dicritical holomorphic singular foliation $\mathcal{G}$ over $X=(\mathbf{C}^3,0)$. We begin by taking a desingularization sequence like (1). Now, consider the set

$$\mathcal{U}=\cup\{Y; Y \text{ is an irreducible component of Sing } \mathcal{G}(N) \text{ which is generically contained in only one irreducible component of } E(N)\}.$$

Let us fix a connected component $\mathcal{U}_j$ of $\mathcal{U}$. (See Figure 10.)

Then we have a formal separatrix S_Q at each point $Q\in\mathcal{U}_j$. Assume for a moment that S_Q is convergent. By analytic triviality we may continue in an analytic way S_Q to the points Q' of $\mathcal{U}_j$ with $e(E(N),Q')\leq e(E(N),Q)$. Hence, the only difficult case is to continue S_Q to the points Q' with $e(E(N),Q')=2$, but this can be done (see Proposition II.5.5). Thus, we can "glue" the S_Q in order to obtain a closed hypersurface $S_j(N)\subset X(N)$ which gives locally a separatrix at each point. (See Figure 11.)

Now because of the properness of the sequence (1) then $S_j(N)$ projects over a convergent separatrix $S_j\subset X$ of the foliation $\mathcal{G}$. It remains to show that there is at least one $\mathcal{U}_j$ supporting a convergent separatrix as above. This is done by taking a non-

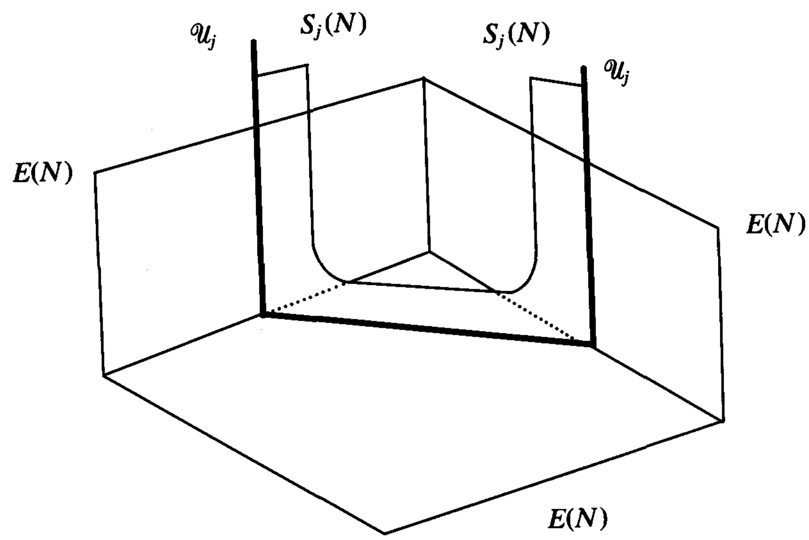


Fig. 11

degenerate plane section of $\mathcal{G}$; by [4], the two-dimensional section has at least one convergent separatrix Γ. Without loss of generality we may assume that the strict transform $\Gamma(N)$ of Γ under (1) is nonsingular and passes through a point $Q \in E(N)$ with $e(E(N), Q)=1$. (See Figure 12.)

Now, by analytic triviality, we see that S_Q is convergent. Thus, the desired $\mathcal{U}_j$ is the connected component of $\mathcal{U}$ passing through Q.

More precisely. Let us denote by $\hat{X}(N)$ the formal completion of $X(N)$ along the inverse image of the origin $\pi^{-1}(0)$, where $\pi=\pi(1)\circ\ldots\circ\pi(N)$. The nature of the formal separatrices S_Q is of such kind that we can construct a coherent hypersurface

$$\hat{S}_j(N) \subset \hat{X}(N)$$

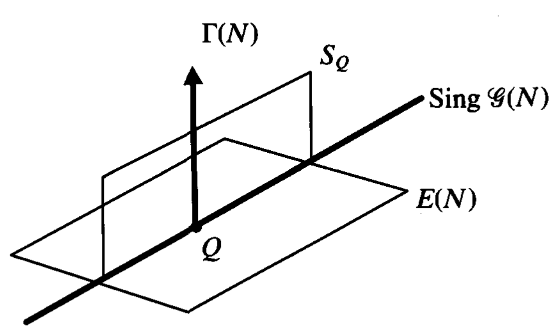


Fig. 12

supported by each connected component $\mathcal{U}_j$ of $\mathcal{U}$, such that $\hat{S}_j(N)$ gives the separatrix S_Q at each point $Q \in \mathcal{U}_j$. Once again, the properness of π assures that $\hat{S}_j(N)$ projects over a formal separatrix $\hat{S}_j$ of $\mathcal{G}$ at the origin. In this way we obtain a bijection

$$\{\text{formal separatrices of } \mathcal{G} \text{ at the origin} \leftrightarrow \{\text{connected components } \mathcal{U}_j \text{ of } \mathcal{U}\}.$$

Open questions and related problems. We give here a list of unsolved problems which seem us to be important ones:

(1) Desingularize holomorphic foliations in higher dimensions and in the dicritical case.

(2) Desingularize a vector field which is tangent either to one or two different foliations of codimension one.

(3) Call "singular holonomy" along $\mathcal{U}_j$ the representation

$$\pi_1(\mathcal{U}_j - \operatorname{Sing} \mathcal{U}_j, Q) \to \operatorname{Trdiff}(\mathcal{G}(N), Q)$$

where $\operatorname{Trdiff}(\mathcal{G}(N), Q)$ means the diffeomorphisms of the restriction of $\mathcal{G}(N)$ to a transversal two-dimensional section at Q into itself. The problem is to understand the non-dicritical singular foliations with the data of the singular holonomy and the holonomy of the components of the exceptional divisor. Some results in this direction may be found in [3] and [19].

(4) Say that $\mathcal{G}$ has the property $\mathcal{P}$ iff it is possible to desingularize $\mathcal{G}$ by only blowing-up points (and an "a posteriori" eventual addition of irreducible components for $E(N)$). In [11] there is a description of such $\mathcal{G}$ which are desingularized after one blowing-up. These foliations have first integrals of Liouville type $\Sigma \lambda_i \operatorname{Log} f_i$. The problem is to describe the foliations having the property $\mathcal{P}$.

(5) Classify the non-dicritical singular foliations in $(\mathbf{C}^3, 0)$ generated by one 1-form with initial part of the type $x\,dx$.

(6) Moduli for simple singularities. As in dimension two, kit is a natural and fascinating problem (see [17]). For example, it is possible to establish some theorems "Poincaré–Siegel" [21] in the "non-resonant" cases (the eigenvalues ratio are not in $\mathbf{Q}$ or even in $\mathbf{R}$) (see [10], [11]). In the resonant case, there is a rigidity result which is a consequence of Ecalle's Theory (see [12]): assume that Ω is formally conjugated to

$$\hat{\Omega} = xyz \cdot (x^p y^q z^r)^s \left(\frac{dx}{x} + \beta \frac{dy}{y} + \left(\varepsilon + \frac{1}{(x^p y^q z^r)^s} \right) \left(p \frac{dx}{x} + q \frac{dy}{y} + r \frac{dz}{z} \right) \right)$$

then, for generic values of the parameters, Ω is holomorphically conjugated to $\hat{\Omega}$ ([9]).

(7) Give the topological classification of simple singularities with a given formal normal form.

Acknowledgements. We wish to express our gratitude to J. Martinet and R. Moussu for very valuable discussions about the subject. One of us (Cano) is very grateful to the Universities of La Bourgogne, Louis Pasteur and Rennes I for his staying there during the preparation of this work. We also thank the referee for his suggestions.

First author partially supported by the DGIGYT.

Second author partially supported by the CNPQ (IMPA).

Part 1. Preliminaries

§ 1. Adapted singular foliations

Most of the concepts and results in this paragraph may be found in [5], [7].

Let X be a nonsingular connected analytic space over $\mathbf{C}$ of dimension n. Fix a normal crossings divisor E of X (always with reduced structure). Let us denote by $\Omega_X[-E]$ the sheaf of germs of meromorphic differential 1-forms over X having at most simple poles along E.

DEFINITION 1.1. *An adapted to E singular foliation of codimension one over X is a pair $(\mathcal{F}, E)$ where $\mathcal{F}$ is an $\mathcal{O}_X$-submodule of $\Omega_X[-E]$ such that:*

(a) *$\mathcal{F}$ is locally free of rank one.*

(b) *$\mathcal{F} \wedge d\mathcal{F} = 0$, where d is the exterior differential.*

(c) *The quotient $\Omega_X[-E]/\mathcal{F}$ is torsion-free.*

Let J_E be the sheaf of ideals defining $E \subset X$. Fix a point P of X. We can choose a regular system of parameters $(x_1, \ldots, x_n)$ of the local ring $\mathcal{O}_{X,P}$ such that

$$J_{E,P} = \left(\prod_{i \in A} x_i\right) \cdot \mathcal{O}_{X,P} \tag{1.1}$$

for a certain set $A \subset \{1, \ldots, n\}$. Then, a basis of the stalk $\Omega_{X,P}[-E]$ is given by

$$\left\{\frac{dx_i}{x_i}\right\}_{i \in A} \cup \{dx_i\}_{i \notin A}. \tag{1.2}$$

Hence, $\mathcal{F}_P$ is generated by a meromorphic differential 1-form

$$\omega=\sum_{i\in A} a_i\frac{dx_i}{x_i}+\sum_{i\notin A} a_i\,dx_i;\quad a_i\in \mathcal{O}_{X,P}. \tag{1.3}$$

such that $\omega\wedge d\omega=0$ and g.c.d.$(a_i; i=1,\dots,n)=1$.

In the case $E=\varnothing$, we find the usual notion of singular foliation of codimension one (cf. [11]). Let us denote by $\mathfrak{F}(X,E)$ the set of adapted to E singular foliations of codimension one over X. Then, we have a bijection

$$\text{hol}: \mathfrak{F}(X,E)\to \mathfrak{F}(X,\varnothing) \tag{1.4}$$

which is defined by the following property:

$$\text{If } (\mathcal{G},\varnothing)=\text{hol}((\mathcal{F},E)),\text{ then } \mathcal{G}|_{X-E}=\mathcal{F}|_{X-E}. \tag{1.5}$$

Moreover, if $\mathcal{F}_P$ is generated by ω as in (1.3), then $\mathcal{G}_P$ is generated by

$$\Omega=\left(\prod_{i\in A^*} x_i\right)\omega \tag{1.6}$$

where the set A^* is given by

$$A^*=\{i\in A; x_i \text{ does not divide } a_i\}. \tag{1.7}$$

Now, fix $(\mathcal{G},\varnothing)\in\mathfrak{F}(X,\varnothing)$ and a point $P\in X$. Assume that $\mathcal{G}_P$ is generated by $\Omega\in\Omega_{X,P}$. We recall that a "separatrix", respectively a "formal separatrix", of $(\mathcal{G},\varnothing)$ at P is a principal prime ideal $f\mathcal{O}_{X,P}$, respectively $f\hat{\mathcal{O}}_{X,P}$, such that

$$f \text{ divides } \Omega\wedge df. \tag{1.8}$$

(cf. [11]). Here $\hat{\mathcal{O}}_{X,P}$ denotes the completion of $\mathcal{O}_{X,P}$ along the maximal ideal. An "invariant analytic space" of $(\mathcal{G},\varnothing)$ is an irreducible closed analytic space K of X such that

$$\Omega|_K=0 \tag{1.9}$$

at the nonsingular points P of K. Any invariant analytic hypersurface $H\subset X$ of $(\mathcal{G},\varnothing)$ defines a separatrix at each point $P\in H$. Conversely, an irreducible hypersurface $H\subset X$ defines an invariant analytic space of $(\mathcal{G},\varnothing)$ iff it defines a separatrix at a point $P\in H$.

Let $(\mathcal{F},E)\in\mathfrak{F}(X,E)$ and fix an irreducible component F of E. We say that F is a "non-dicritical component" of E for $(\mathcal{F},E)$ iff F is an invariant analytic space of hol$((\mathcal{F},E))$. Otherwise, we say that F is a "dicritical component" of E for $(\mathcal{F},E)$. Then, taking the notation of (1.7), we have that

$$A^* = \{i \in A;\ (x_i = 0) \text{ is a non-dicritical component for } (\mathcal{F}, E)\}. \tag{1.10}$$

DEFINITION 1.2. *Given $(\mathcal{F},E) \in \mathfrak{F}(X,E)$ and a point $P \in X$, the adapted order $\nu(\mathcal{F},E;P)$ is the $\mathcal{M}$-adic order of the submodule $\mathcal{F}_P$ of $\Omega_{X,P}[-E]$, where $\mathcal{M}$ is the maximal ideal of $\mathcal{O}_{X,P}$. The singular locus* $\mathrm{Sing}(\mathcal{F},E)$ *is the set of the points $P \in X$ such that $\nu(\mathcal{F},E;P) \geq 1$.*

With the notations of (1.3), we have that

$$\nu(\mathcal{F},E,P) = \min\{\nu_P(a_i);\ i = 1,\dots,n\}. \tag{1.11}$$

Where $\nu_P(a_i)$ is the $\mathcal{M}$-adic order of $a_i \in \mathcal{O}_{X,P}$. The singular locus is a closed analytic subset of X and since $\Omega_X[-E]/\mathcal{F}$ has no torsion, we have

$$\mathrm{Codim}_X \mathrm{Sing}(\mathcal{F},E) \geq 2. \tag{1.12}$$

If $\mathrm{hol}((\mathcal{F},E)) = (\mathcal{G},\varnothing)$, note that

$$\mathrm{Sing}(\mathcal{F},E) \subset \mathrm{Sing}(\mathcal{G},\varnothing) \tag{1.13}$$

and we also have that $\mathrm{Codim}_X \mathrm{Sing}(\mathcal{G},\varnothing) \geq 2$.

Let $Y \subset X$ be a nonsingular analytic subspace of X having normal crossings with E. Let

$$\pi : X' \to X \tag{1.14}$$

be the blowing-up with center Y. Put $E' = \pi^{-1}(E \cup Y)$, with reduced structure. Then $E' \subset X'$ is also a normal crossings divisor of X'. Now, fix $(\mathcal{F},E) \in \mathfrak{F}(X,E)$ and put $(\mathcal{G},\varnothing) = \mathrm{hol}((\mathcal{F},E))$. Then there is a unique $(\mathcal{F}', E')$ in $\mathfrak{F}(X',E')$, respectively $(\mathcal{G}',\varnothing)$ in $\mathfrak{F}(X',\varnothing)$, such that

$$\mathcal{F}'|_{X'-\pi^{-1}(Y)} = \mathcal{F}|_{X-Y}, \quad \text{respectively} \quad \mathcal{G}'|_{X'-\pi^{-1}(Y)} = \mathcal{G}|_{X-Y}, \tag{1.15}$$

under the isomorphism $\pi : X' - \pi^{-1}(Y) \to X - Y$. Moreover

$$(\mathcal{G}',\varnothing) = \mathrm{hol}((\mathcal{F}',E')). \tag{1.16}$$

(cf. [5], [7]).

DEFINITION 1.3. *In the above situation we say that $(\mathcal{F}',E')$ is the adapted strict transform of $(\mathcal{F},E)$ by π and that $(\mathcal{G}',\varnothing)$ is the strict transform of $(\mathcal{G},\varnothing)$ by π.*

Let $J_Y \subset \mathcal{O}_X$ be the sheaf of ideals defining $Y \subset X$. Denote by $\Theta_X[E^*Y]$ the sheaf of germs of vector fields being both tangent to E and Y. Let $\mathcal{U}(\mathcal{F}, E; Y)$ be the image of the bilinear mapping

$$\mathcal{F} \times \Theta_X[E^*Y] \to \mathcal{O}_X \tag{1.17}$$

given by $(\omega, D) \hookrightarrow \omega(D)$.

DEFINITION 1.4. *The adapted multiplicity* $\mu(\mathcal{F}, E; Y)$ *of* $(\mathcal{F}, E)$ *at* Y *is the* J_Y*-adic order of* $\mathcal{U}(\mathcal{F}, E; Y)$.

Remarks 1.5. (a) Take $P \in Y$ and $P' \in \pi^{-1}(Y)$. Let ω be a generator of $\mathcal{F}_P$ and let f be a reduced equation of the exceptional divisor $\pi^{-1}(Y)$ at P'. Put $a = \mu(\mathcal{F}, E; Y)$. Then $\mathcal{F}'_{P'}$ is generated by $f^{-a}\pi^*\omega$ (cf. [5], [7]).

(b) Since Y has normal crossings with E, we can find a regular system of parameters $(x_1, \ldots, x_n)$ of $\mathcal{O}_{X,P}$ and two sets $A, B \subset \{1, \ldots, n\}$ such that

$$J_{E,P} = \left(\prod_{i \in A} x_i\right) \mathcal{O}_{X,P}; \quad J_{Y,P} = \sum_{i \in B} x_i \mathcal{O}_{X,P}. \tag{1.18}$$

If ω generates $\mathcal{F}_P$ as in (1.3), we have explicitly that

$$\mu(\mathcal{F}, E; Y) = \min(\{\nu_Y(a_i); i \notin B - A\} \cup \{\nu_Y(a_i) + 1; i \in B - A\}) \tag{1.19}$$

where $\nu_Y(a_i)$ denotes the $J_{Y,P}$-adic order of $a_i \in \mathcal{O}_{X,P}$. In particular, we can compute $\mu(\mathcal{F}, E; Y)$ at any point $P \in Y$.

(c) The adapted multiplicitity generalizes in a natural way the usual multiplicity of a hypersurface and its behaviour under blowing-up (cf. [7], Introduction).

Consider a point $P \in Y$. Denote by $\varrho(\mathcal{F}, E; Y, P)$ the $\mathcal{M}$-adic order of the ideal $\mathcal{U}(\mathcal{F}, E; Y)_P \subset \mathcal{O}_{X,P}$, where $\mathcal{M}$ is the maximal ideal of $\mathcal{O}_{X,P}$. More explicitly

$$\varrho(\mathcal{F}, E; Y; P) = \min(\{\nu_P(a_i); i \notin B - A\} \cup \{\nu_P(a_i) + 1; i \in B - A\}). \tag{1.20}$$

Note that

$$\varrho(\mathcal{F}, E; Y; P) \geqslant \mu(\mathcal{F}, E; Y) \tag{1.21}$$

and the equality holds outside an analytic subset W of Y, with $W \neq Y$.

DEFINITION 1.6. *Let* Y *be an irreducible closed analytic subspace of* X *and let* $(\mathcal{F}, E) \in \mathfrak{F}(X, E)$. *Fix a point* $P \in Y$. *Then* Y *is a permissible center for* $(\mathcal{F}, E)$ *at* P *iff the following properties hold:*

(a) *$Y\subset \mathrm{Sing}(\mathcal{G},\emptyset)$, where $(\mathcal{G},\emptyset)=\mathrm{hol}((\mathcal{F},E))$.*

(b) *Y is nonsingular at P and has normal crossings with E at P.*

(c) *The equality $\varrho(\mathcal{F},E;Y,P)=\mu(\mathcal{F},E;Y)$ holds.*

A permissible center Y for $(\mathcal{F},E)$ is a permissible center at each point $P\in Y$.

Remarks 1.7. (a) A point $\{P\}\subset \mathrm{Sing}(\mathcal{G},\emptyset)$ is a permissible center.

(b) An analytic subspace $Y\subset \mathrm{Sing}(\mathcal{G},\emptyset)$ is a permissible center outside an analytic subset W of Y, with $W\neq Y$.

Now, we are able to define the "non-dicritical singular foliations". Here we shall give a technical definition which is convenient for our purposes. Another characterizations of this condition are given in [6].

DEFINITION 1.8. *We say that $(\mathcal{F},E)\in\mathfrak{F}(X,E)$ is non-dicritical iff there is no finite sequence*

$$\{X(i),E(i),\mathcal{F}(i),\mathcal{U}(i),Y(i),\pi(i+1)\}_{i=0,1,\dots,N} \tag{1.22}$$

such that:

(a) *$X(0)=X$, $(\mathcal{F}(0),E(0))=(\mathcal{F},E)$.*

(b) *For each $i=0,1,\dots,N$ we have that:*

(b1) *$\mathcal{U}(i)\subset X(i)$ is a nonempty open set.*

(b2) *$Y(i)\subset\mathcal{U}(i)$ is a permissible center for $(\mathcal{F}(i)|_{\mathcal{U}(i)},E(i)\cap\mathcal{U}(i))$.*

(b3) *$\pi(i+1):X(i+1)\to\mathcal{U}(i)$ is the blowing-up with center $Y(i)$.*

(b4) *$(\mathcal{F}(i+1),E(i+1))$ is the adapted strict transform of $(\mathcal{F}(i)|_{\mathcal{U}(i)},E(i)\cap\mathcal{U}(i))$ by $\pi(i+1)$.*

(c) *There is a dicritical component of $E(N)$ for $(\mathcal{F}(N),E(N))$.*

In particular, making $N=0$, we see that if $(\mathcal{F},E)$ is non-dicritical, then E has no dicritical components for $(\mathcal{F},E)$. That is, each irreducible component of E is an invariant hypersurface for $\mathrm{hol}((\mathcal{F},E))$.

THEOREM 1.9 (Stability Theorem). *Let $(\mathcal{F},E)\in\mathcal{F}(X,E)$ and let $Y\subset X$ be a permissible center for $(\mathcal{F},E)$. Let $\pi:X'\to X$ be the blowing-up with center Y and let $(\mathcal{F}',E')$ be the adapted strict transform of $(\mathcal{F},E)$ by π. Fix a point $P\in Y$ and a point $P'\in\pi-1(P)$. Then:*

(a) *$\nu(\mathcal{F}',E';P')\leq\nu(\mathcal{F},E;P)$.*

(b) *If $(\mathcal{F},E)$ is non-dicritical, then $\mu(\mathcal{F}',E';\{P'\})\leq\mu(\mathcal{F},E;\{P\})$.*

Proof. [7], Theorem I.2.7; Theorem I.3.3. □

Hence, in the non-dicritical case, we can use the invariant

$$(r, m) = (\nu(\mathcal{F}, E; P), \mu(\mathcal{F}, E; \{P\})) \tag{1.23}$$

in order to control the behaviour of the singularities under permissible blowing-ups.

Let $Y \subset X$ be a permissible center for $(\mathcal{F}, E) \in \mathfrak{F}(X, E)$. Fix a point $P \in Y$. Then

$$\nu(\mathcal{F}, E; P) \leq \varrho(\mathcal{F}, E; Y, P) = \mu(\mathcal{F}, E; \{P\}) \leq \nu(\mathcal{F}, E; P) + 1. \tag{1.24}$$

We say that Y is *"appropriate"* at P iff

$$\mu(\mathcal{F}, E; Y) = \mu(\mathcal{F}, E; \{P\}). \tag{1.25}$$

PROPOSITION 1.10 (Stationary sequences). *Let $(\mathcal{F}, E) \in \mathfrak{F}(X, E)$ be non-dicritical. Put $(\mathcal{G}, \emptyset) = \mathrm{hol}((\mathcal{F}, E))$. Fix an irreducible curve $\Gamma \subset \mathrm{Sing}(\mathcal{G}, \emptyset)$ and a point $P \in \Gamma$. Consider an infinite sequence*

$$\{X(i), E(i), \mathcal{F}(i), \Gamma(i), P(i), \pi(i+1), r(i), m(i)\}_{i \geq 0} \tag{1.26}$$

defined as follows:

(a) $X(0) = X, (\mathcal{F}(0), E(0)) = (\mathcal{F}, E), \Gamma(0) = \Gamma, P(0) = P$.

(b) $\pi(i+1): X(i+1) \to X(i)$ *is the blowing-up with center* $P(i)$.

(c) $\Gamma(i+1)$ *is the strict transform of* $\Gamma(i)$ *by* $\pi(i+1)$.

(d) $P(i+1) \in \Gamma(i+1) \cap \pi(i+1)^{-1}(P(i))$.

(e) $(\mathcal{F}(i+1), E(i+1))$ *is the adapted strict transform of* $(\mathcal{F}(i), E(i))$ *by* $\pi(i+1)$.

(f) $r(i) = \nu(\mathcal{F}(i), E(i); P(i)), m(i) = \mu(\mathcal{F}(i), E(i); \{P(i)\})$.

Then, the following two conditions are equivalent for any index N:

(A) $\Gamma(N)$ *is nonsingular and has normal crossings with* $E(N)$ *at* $P(N)$ *and for each* $i \geq N$ *we have that* $(r(i), m(i)) = (r(N), m(N))$.

(B) $\Gamma(N)$ *is permissible and appropriate for* $(\mathcal{F}(N), E(N))$ *at* $P(N)$.

Proof. [7], Theorem II.1.1. □

Remark 1.11. There is always an index $N \geq 0$ such that the above condition A is satisfied. Hence, if Γ is not permissible at P, we can achieve this condition by blowing-up the point P finitely many times.

Now, we can state the main result in [5] and [7] as follows:

THEOREM 1.12 (Reduction Theorem). *Assume that* $\dim X = 3$. *Fix a non-dicritical* $(\mathcal{F}, \emptyset) \in \mathfrak{F}(X, \emptyset)$ *and a point* $P \in \mathrm{Sing}(\mathcal{F}, \emptyset)$. *Then there is an open set* $X(0)$ *of* X,

$P \in X(0)$, *and a sequence of permissible blowing-ups*

$$X(0) \overset{\pi(1)}{\leftarrow} X(1) \overset{\pi(2)}{\leftarrow} \dots \overset{\pi(N)}{\leftarrow} X(N) \tag{1.27}$$

such that

$$\mu(\mathscr{F}(N), E(N); \{Q\}) \leq 1, \quad \text{for all} \quad Q \in X(N), \tag{1.28}$$

where $(\mathscr{F}(N), E(N))$ *is the adapted strict transform of* $(\mathscr{F}|_{X(0)}, \varnothing)$ *under the composition* $\pi(1) \circ \dots \circ \pi(N)$. *Moreover, the sequence* (1.27) *may be taken in such a way that*

$$\{P\} = \textit{center of } \pi(1), \tag{1.29}$$

$$\mathrm{Singhol}((\mathscr{F}(N), E(N))) \subset E(N). \tag{1.30}$$

Remarks 1.13. (a) Put $\pi = \pi(1) \circ \dots \circ \pi(N)$. Then (1.29) means that $Z(N) = \pi^{-1}(P)$ is also a normal crossings divisor $Z(N) \subset E(N)$.

(b) In [5] and [7], the above theorem is stated in terms of the germs of X at P. In particular, the condition (1.28) is stated only for the points $Q \in \pi^{-1}(P) = Z(N)$. Nevertheless, the semicontinuity of the adapted multiplicity in the non-dicritical case ([7], Remark I.1.6) allows us to state the result in terms of an open set $X(0) \subset X$, $P \in X(0)$.

§ 2. Pre-simple singularities

Before defining pre-simple singularities, let us recall the notion of "directrix" introduced in [5], [7]. Actually, we shall only consider here the case of adapted order equal to one, which is simpler than the general case.

Given an element $f \in \mathcal{O}_{X,P}$ and an integer $s \geq 0$ such that $\nu_P(f) \geq s$, let us denote by $\mathrm{In}^s(f)$ the image of f in $\mathcal{M}^s/\mathcal{M}^{s+1}$, where $\mathcal{M}$ is the maximal ideal of $\mathcal{O}_{X,P}$. Actually $\mathrm{In}^s f \in \mathrm{Gr}(\mathcal{O}_{X,P})$, where $\mathrm{Gr}(\mathcal{O}_{X,P})$ is the graded ring for the $\mathcal{M}$-adic filtration of $\mathcal{O}_{X,P}$. Note that $\mathrm{Gr}(\mathcal{O}_{X,P})$ is a polynomial ring in the indeterminates $X_i = \mathrm{In}^1(x_i)$, $i = 1, \dots, n$.

Now, consider a non-dicritical $(\mathscr{F}, E) \in \mathfrak{F}(X, E)$ a point $P \in X$ such that

$$m = \mu(\mathscr{F}, E; \{P\}) \leq 1. \tag{2.1}$$

Note that if $r = \nu(\mathscr{F}, E; P)$, then $r \leq m \leq r+1$ and hence either $r = 0$ or $r = 1$. Assume that $r = 1$ and let us write a generator ω of $\mathscr{F}_P$ as in (1.3):

$$\omega = \sum_{i \in A} a_i \frac{dx_i}{x_i} + \sum_{i \notin A} a_i \, dx_i; \quad a_i \in \mathcal{O}_{X,P}. \tag{2.2}$$

Then, the directrix $\mathrm{Dir}(\mathcal{F}, E; P)$ of $(\mathcal{F}, E)$ at P is defined by

$$\mathrm{Dir}(\mathcal{F}, E; P) = \bigcap_{i\in A} (\mathrm{In}^1(a_i) = 0) \subset T_P X, \tag{2.3}$$

where $T_P X$ is the tangent space of X at P. (In the case $r=0$, the directrix is not defined.) Denote by $\mathrm{JDir}(\mathcal{F}, E; P)$ the ideal defining the directrix. Then

$$\mathrm{JDir}(\mathcal{F}, E; P) = \sum_{i\in A} \mathrm{In}^1(a_i)\,\mathrm{Gr}(\mathcal{O}_{X,P}), \tag{2.4}$$

PROPOSITION 2.1. *In the above situation, assume that* $r=m=1$. *Let* $Y\subset X$ *be a permissible center for* $(\mathcal{F}, E)$ *with* $P\in Y$. *Let* $\pi: X'\to X$ *be the blowing-up with center* Y *and let* $(\mathcal{F}, E)$ *be the adapted strict transform of* $(\mathcal{F}, E)$ *by* π. *Then:*

(a) $T_P Y\subset \mathrm{Dir}(\mathcal{F}, E; P)$.

(b) *Let* $P'\in \pi^{-1}(P)$ *be such that* $\nu(\mathcal{F}', E'; P')=\mu(\mathcal{F}', E'; \{P'\})=1$. *Then:*

$$P'\in \mathrm{Proj}(\mathrm{Dir}(\mathcal{F}, E; P)/T_P Y)\subset \mathrm{Proj}(T_P X/T_P Y) = \pi^{-1}(P). \tag{2.5}$$

Proof. [7], Theorem I.4.8 (see also [5], Theorem 4). □

Let us denote by $e=e(E, P)$ the number of irreducible components of E through P. (Actually $e(E, P)$ is the multiplicity of E at P, moreover, with the notation of (1.1) we have $e={}^{\#}A$.) Take a codimension one vector subspace H of $T_P X$. We say that H has *normal crossings with E* iff there are $e+1$ independent linear forms $\varphi_0, \varphi_1, \ldots, \varphi_e$ on $T_P X$ such that

$$H = (\varphi_0 = 0). \tag{2.6}$$

$$T_P E_i = (\varphi_i = 0), \quad i=1, \ldots, e, \tag{2.7}$$

where $E_1, \ldots, E_e$ are the irreducible components of E at P.

DEFINITION 2.2. *Consider a non-dicritical* $(\mathcal{F}, E)\in \mathfrak{F}(X, E)$. *Put* $(\mathcal{G}, \varnothing)=\mathrm{hol}((\mathcal{F}, E))$ *and consider a point* $P\in \mathrm{Sing}(\mathcal{G}, \varnothing)$. *We say that* P *is a pre-simple singularity for* $(\mathcal{F}, E)$ *iff one of the following conditions holds:*

(a) $\nu(\mathcal{F}, E; P)=0$.

(b) $\mu(\mathcal{F}, E; \{P\})=\nu(\mathcal{F}, E; P)=1$ *and* $\mathrm{Dir}(\mathcal{F}, E; P)$ *has normal crossings with* E. *(In particular, the dimension of* $\mathrm{Dir}(\mathcal{F}, E; P)$ *is* $n-1$.)

Remarks 2.3. (a) If P is a pre-simple singularity, then necessarily $e(E,P)\geq 1$. In fact, if $e(E,P)=0$ and $\nu(\mathcal{F},E;P)=0$, then $P\notin \mathrm{Sing}(\mathcal{G},\emptyset)$; if $e(E,P)=0$ and $\nu(\mathcal{F},E;P)=1$, then $\mu(\mathcal{F},E,\{P\})=2$.

(b) If P is a pre-simple singularity with $e(E,P)=n$, then necessarily $\nu(\mathcal{F},E;P)=0$. In fact, if the directrix exits, it cannot have normal crossings with E.

LEMMA 2.4. *Consider a non-dicritical $(\mathcal{F},E)\in\mathfrak{F}(X,E)$. Let $P\in X$ be a point such that*

$$\nu(\mathcal{F},E;P)=\mu(\mathcal{F},E;\{P\})=1. \tag{2.8}$$

Let F be the intersection of all the irreducible components of E through P. Then P is a pre-simple singularity for $(\mathcal{F},E)$ iff

$$\mathrm{Dir}(\mathcal{F},E;P)\not\supset T_P F. \tag{2.9}$$

Proof. The "only if" part is trivial. Conversely, assume that (2.9) holds. Choose a regular system of parameters $(x_1,\dots,x_n)$ of $\mathcal{O}_{X,P}$ such that

$$J_{E,P}=\left(\prod_{i=1,\dots,e} x_i\right)\mathcal{O}_{X,P}. \tag{2.10}$$

Then a generator ω of $\mathcal{F}_P$ is written down as follows:

$$\omega=\sum_{i=1,\dots,e} a_i\frac{dx_i}{x_i}+\sum_{i>e} a_i\,dx_i. \tag{2.11}$$

Denote by $A_i=\mathrm{In}^1(a_i)$, $i=1,\dots,n$; $X_i=\mathrm{In}^1(x_i)$, $i=1,\dots,n$. We can assume without loss of generality that $A_1=X_{e+1}$. Now, the integrability condition $\omega\wedge d\omega=0$ implies that

$$A_1\frac{\partial A_s}{\partial X_{e+1}}=A_s\frac{\partial A_1}{\partial X_{e+1}}=A_s,\quad s=2,\dots,e. \tag{2.12}$$

Hence $A_s=\lambda_s X_{e+1}$, $\lambda_s\in\mathbf{C}$, for all $s=2,\dots,e$ and thus

$$\mathrm{Dir}(\mathcal{F},E;P)=(X_{e+1}=0). \tag{2.13}$$

This ends the proof. □

PROPOSITION 2.5. *Consider a non-dicritical $(\mathcal{F},E)\in\mathfrak{F}(X,E)$. Let $P\in X$ be a pre-simple singularity for $(\mathcal{F},E)$. Consider a permissible center $Y\subset X$ for $(\mathcal{F},E)$. Let $\pi\colon X'\to X$ be the blowing-up with center Y and let $(\mathcal{F}',E')$ be the adapted* *strict transform of* $(\mathcal{F}, E)$ *by* π. *Put* $(\mathcal{G}', \emptyset) = \mathrm{hol}((\mathcal{F}', E')$. *Then each point in* $\pi^{-1}(P) \cap \mathrm{Sing}(\mathcal{G}', \emptyset)$ *is a pre-simple singularity for* $(\mathcal{F}', E')$.

Proof. By Theorem 1.9, the only bad case is $P' \in \pi^{-1}(P)$ with

$$\mu(\mathcal{F}', E'; \{P'\}) = \nu(\mathcal{F}', E'; P') = 1. \tag{2.14}$$

Let F be the intersection of all the irreducible components of E through P as in Lemma 2.4. Define

$$d(\mathcal{F}, E; P) = \dim_{\mathbb{C}}((\mathrm{JDir}(\mathcal{F}, E; P) + JT_P F)/JT_P F)_1 \tag{2.15}$$

where $JT_P F$ is the ideal of $T_P F$ and the subindex 1 means "linear part" (cf. [7],I,(4.2.4) or [5],§4). Since P is a pre-simple singularity then

$$d(\mathcal{F}, E; P) = 1. \tag{2.16}$$

Now by [7], Theorem I.4.8(c), or [5], Theorem 4(iv), we have that

$$d(\mathcal{F}', E'; P') \geqslant d(\mathcal{F}, E; P) = 1. \tag{2.17}$$

This implies that the condition (2.9) of Lemma 2.4 holds. Hence P' is a pre-simple singularity. □

PROPOSITION 2.6. *Consider a non-dicritical* $(\mathcal{F}, E) \in \mathfrak{F}(X, E)$. *Put* $(\mathcal{G}, \emptyset) = \mathrm{hol}((\mathcal{F}, E))$. *Then the set*

$$\mathrm{Sing}^*(\mathcal{F}, E) = \{P \in \mathrm{Sing}(\mathcal{G}, \emptyset);\ P \text{ is not a pre-simple singularity}\} \tag{2.18}$$

is a closed analytic subset of X.

Proof. It is a local statement. Fix $P \in X$ and let $(x_1, \ldots, x_n)$ be a regular system of parameters of $\mathcal{O}_{X,P}$ such that

$$E = \left(\prod_{i \in A} x_i = 0\right), \quad \text{locally at } P. \tag{2.19}$$

Let us consider a generator ω of $\mathcal{F}_P$ given as in (1.3) by

$$\omega = \sum_{i \in A} a_i \frac{dx_i}{x_i} + \sum_{i \notin A} a_i dx_i; \quad a_i \in \mathcal{O}_{X,P}. \tag{2.20}$$

Now, let us put

$$F(s)=\{Q;e(E,Q)\geqslant s\},\quad 0\leqslant s\leqslant e(E,P). \tag{2.21}$$

Given $A'\subset A$, let us define the closed analytic sets

$$E_{A'}=\bigcap_{i\in A'}(x_i=0) \tag{2.22}$$

$$C_{A'}=\mathrm{Sing}(\mathscr{F},E)\cap\{Q;(\partial a_i/\partial x_j)(Q)=0,\ i\in A',j\notin A'\} \tag{2.23}$$

$$D_{A'}=\text{adherence of } C_{A'}\cap(E_{A'}-F(1+{}^{\#}A')). \tag{2.24}$$

In view of Lemma 2.4, we have that

$$\mathrm{Sing}^*(\mathscr{F},E)\cap(E_{A'}-F(1+{}^{\#}A'))=C_{A'}\cap(E_{A'}-F(1+{}^{\#}A')). \tag{2.25}$$

Now, it is enough to prove that

$$\text{If } A'\subset A'',\ {}^{\#}A''={}^{\#}A'+1,\ \text{then } D_{A'}\cap(E_{A'}-F(1+{}^{\#}A''))\subset D_{A''}. \tag{2.26}$$

Since in this case

$$\mathrm{Sing}^*(\mathscr{F},E)=\bigcup_{A'\subset A}D_{A'}. \tag{2.27}$$

In order to prove (2.26), let us reason by contradiction, assuming that (2.26) is not true. We can assume without loss of generality that

$$A''=A=\{1,2,\dots,e\}. \tag{2.28}$$

$$A'=\{2,\dots,e\}. \tag{2.29}$$

$$\text{There is a point } P\in D_{A'}-D_A. \tag{2.30}$$

Then, we can find an analytic branch Γ at P such that $\Gamma\subset D_{A'}$ and $\Gamma\not\subset E_A$. By Proposition 1.10 and Proposition 2.5, blowing-up the point P repeatedly, we may assume without loss of generality that Γ is a permissible center for $(\mathscr{F},E)$. Hence we can take coordinates such that

$$\Gamma=(x_2=\ldots=x_n=0). \tag{2.31}$$

The fact $\Gamma\subset\mathrm{Sing}(\mathscr{F},E)$ implies that $\nu_\Gamma(a_i)\geqslant 1$, $i=1,\dots,n$. Moreover, since P is a pre-simple singularity, then

$$1 \leqslant \mu(\mathcal{F}, E; \Gamma) \leqslant \mu(\mathcal{F}, E; \{P\}) = 1. \tag{2.32}$$

Hence $\mu(\mathcal{F}, E; \Gamma)=1$. Assume first that

$$\nu_\Gamma(a_i) \geqslant 2, \quad \text{for} \quad i = 2, \ldots, e. \tag{2.33}$$

Then, in view of Remark 1.5(a), we find a dicritical component by blowing-up the center Γ. Contradiction. Thus, we may assume that

$$\nu_\Gamma(a_i; i = 2, \ldots, e) = 1. \tag{2.34}$$

Since $\Gamma \subset D_{A'}$, we can write

$$a_i = \sum_{j=2,\ldots,e} \varphi_{ij}(x_1)\, x_j + \Psi_i, \quad i = 2, \ldots, e, \tag{2.35}$$

where $\Psi_i \in (x_2, \ldots, x_n)^2$. Moreover, since P is a pre-simple singularity we can assume without loss of generality that

$$\mathrm{In}^1(a_1) = X_{e+1} = \mathrm{In}^1(x_{e+1}). \tag{2.36}$$

Now, looking at the coefficient of $dx_1 \wedge dx_s \wedge dx_{e+1}$, $s=2, \ldots, e$ in the integrability condition $\omega \wedge d\omega = 0$, we have that

$$x_1\left(-\frac{\partial a_s}{\partial x_1} a_{e+1} + a_s \frac{\partial a_{e+1}}{\partial x_1}\right) + x_s\left(a_1 \frac{\partial a_{e+1}}{\partial x_s} - \frac{\partial a_1}{\partial x_s} a_{e+1}\right) + \left(\frac{\partial a_1}{\partial x_{e+1}} a_s - a_1 \frac{\partial a_s}{\partial x_{e+1}}\right) = 0. \tag{2.37}$$

Looking at the terms of order one with respect to $(x_2, \ldots, x_n)$ in (2.37), we find that

$$\sum_{j=2,\ldots,e} \varphi_{sj}(x_1)\, x_j = 0, \quad s = 2, \ldots, e. \tag{2.38}$$

Then (2.33) holds and we find a contradiction as above. □

Part II. Simple singularities and their normal forms

§ 1. Formal normal forms for abelian Lie algebras of vector fields

Here we shall recall some elementary facts about the theory of formal normal forms for vector fields and abelian Lie algebras of vector fields. Since these results are well known, we shall only sketch the proofs. A good reference about this subject is Martinet's Bourbaki [16].

Let X be a nonsingular analytic space over $\mathbf{C}$ of dimension n. Denote by Θ_X the tangent sheaf of X. Given a point $P \in X$, we shall denote by $\hat{\Theta}_{X,P}$ the $\mathcal{M}$-adic completion of $\Theta_{X,P}$, where $\mathcal{M}$ is the maximal ideal of $\mathcal{O}_{X,P}$. The elements of $\hat{\Theta}_{X,P}$ are called formal vector fields at P. They induce derivations of $\hat{\mathcal{O}}_{X,P}$ in an obvious way. Moreover, we have a canonical inclusion

$$\Theta_{X,P} \subset \hat{\Theta}_{X,P}. \tag{1.1}$$

Put $\hat{\mathcal{M}} = \mathcal{M}\hat{\mathcal{O}}_{X,P}$. Given a formal vector field $D \in \hat{\mathcal{M}}\hat{\Theta}_{X,P}$ and an integer $k \geq 1$, we have an induced derivation

$$\begin{aligned} D^k \colon \mathcal{M}/\mathcal{M}^{k+1} &\to \mathcal{M}/\mathcal{M}^{k+1} \\ f + \mathcal{M}^{k+1} &\hookrightarrow D(f) + \mathcal{M}^{k+1}. \end{aligned} \tag{1.2}$$

(Note that $\hat{\mathcal{M}}/(\hat{\mathcal{M}})^{k+1} = \mathcal{M}/\mathcal{M}^{k+1}$.) We can put the $\mathbf{C}$-linear operator D^k in its Jordan normal form. That is, there is a unique pair of linear operators $D^k{}_S$ and $D^k{}_N$, being respectively semisimple and nilpotent, such that

$$D^k = D^k{}_S + D^k{}_N; \quad D^k{}_S D^k{}_N - D^k{}_N D^k{}_S = 0 \tag{1.3}$$

Actually both $D^k{}_S$ and $D^k{}_N$ are derivations of $\mathcal{M}/\mathcal{M}^{k+1}$ as $\hat{\mathcal{O}}_{X,P}$-module (to see this, it is enough to compute $D^k{}_S$ and $D^k{}_N$ directly in terms of coordinates). By uniqueness of the Jordan decomposition we can take limits

$$D_S = \lim_k D^k{}_S; \quad D_N = \lim_k D^k{}_N \tag{1.4}$$

which are formal vector fields at P such that

$$D = D_S + D_N; \quad [D_S, D_N] = 0 \tag{1.5}$$

where $[\cdot,\cdot]$ denotes the Lie bracket.

The decomposition of (1.5) is called the Jordan decomposition of D. We say that D is *semisimple*, respectively *nilpotent*, if $D = D_S$, respectively $D = D_N$.

PROPOSITION 1.1. *Consider a semisimple formal vector field $D \in \hat{\mathcal{M}}\hat{\Theta}_{X,P}$. Let $(x'_1, \ldots, x'_s)$ be a $\hat{\mathcal{M}}$-regular sequence in $\hat{\mathcal{O}}_{X,P}$ such that*

$$D(x'_i) = \lambda'_i x'_i; \quad \lambda'_i \in \mathbf{C}, \quad i = 1, \ldots, s \tag{1.6}$$

for a certain s, with $0 \leq s \leq n$. *Then there is a regular system of parameters* $(x_1, \dots, x_n)$ *of* $\hat{\mathcal{O}}_{X,P}$ *such that*

$$(1.7)\qquad \begin{gathered} D(x_i) = \lambda_i x_i; \quad \lambda_i \in \mathbf{C}, \quad i = 1, \dots, n \\ x_i = x'_i; \quad \lambda_i = \lambda'_i, \quad i = 1, \dots, s. \end{gathered}$$

Proof. The x'_i, $i=1, \dots, s$, gives a part of a basis of eigenvectors for D^k. Now, it is enough to complete it and to take limits when $k \to \infty$. □

In the situation of the above Proposition 1.1, any regular system of parameters $(x_1, \dots, x_n)$ of $\hat{\mathcal{O}}_{X,P}$ satisfying (1.7) is said to be a *linearizing formal system for D.*

Remark 1.2. Given $D \in \hat{\mathcal{M}}\Theta_{X,P}$, then D is nilpotent iff D^1 is nilpotent. This is evident, since the λ_i, $i=1, \dots, n$, of (1.7) are the eigenvalues of $D^1{}_S$.

Consider $D \in \hat{\mathcal{M}}\hat{\Theta}_{X,P}$. Let $(x_1, \dots, x_n)$ be a linearizing formal system for D_S and let $\lambda = (\lambda_1, \dots, \lambda_n)$ be the corresponding eigenvalues. Take the following notations

$$(1.9)\qquad \text{If } I = (i_1, \dots, i_n) \in \mathbf{N}^n, \text{ then } x^I = x_1^{i_1} \dots x_n^{i_n}.$$

$$(1.10)\qquad \langle \lambda, I \rangle = \sum_{j=1,\dots n} \lambda_j i_j, \quad |I| = \sum_{j=1,\dots,n} i_j.$$

Now, in view of (1.7), we have that

$$(1.11)\qquad \left[D_S, x^I \frac{\partial}{\partial x_i} \right] = (\langle \lambda, I \rangle - \lambda_i) x^I \frac{\partial}{\partial x_1}; \quad i = 1, \dots, n.$$

That is, the monomial formal vector fields $x^I(\partial/\partial x_i)$, $i=1, \dots, n$, are eigenvectors for the operator $[D_S, \cdot]$ with eigenvalues $\langle \lambda, I \rangle - \lambda_i$. Now, write

$$(1.12)\qquad D_N = \sum_{j=1,\dots,n} \sum_{|I| \geq 1} a_{I,j} x^I \frac{\partial}{\partial x_j}.$$

By (1.11), the condition $[D_S, D_N]=0$ is equivalent to say that

$$(1.13)\qquad \text{If } \langle \lambda, I \rangle - \lambda_j \neq 0, \text{ then } a_{I,j} = 0.$$

Hence $D = D_S + D_N$ can be written down as follows:

$$(1.14)\qquad D = \sum_{j=1,\dots,n} \lambda_j x_j \frac{\partial}{\partial x_j} + \sum_{n=1}^{n} \sum_{|I| \geq 1; \langle \lambda, I \rangle = \lambda_j} a_{I,j} x^I \frac{\partial}{\partial x_j}.$$

Remark 1.3. The formula (1.14) has the additional property that the linear operator

$$D_N = \sum_{j=1}^{n} \sum_{|I|\geq 1;\, \langle \lambda, I\rangle = \lambda_j} a_{I,j} x^I \frac{\partial}{\partial x_j} \tag{1.15}$$

is actually a nilpotent operator.

The above formula (1.14) may be generalized to finite dimensional abelian Lie algebras of formal vector fields as follows:

PROPOSITION 1.4. *Let $\mathfrak{G} \subset \hat{\mathcal{M}}\hat{\Theta}_{X,P}$ be a finite dimensional abelian Lie algebra of formal vector fields vanishing at P. Then:*

(a) *There exist two finite dimensional abelian Lie algebras $\mathfrak{G}_S$ and $\mathfrak{G}_N$ in $\hat{\mathcal{M}}\hat{\Theta}_{X,P}$ such that:*

(a1) $\mathfrak{G} \subset \mathfrak{G}_S \oplus \mathfrak{G}_N$.

(a2) $[\mathfrak{G}_S, \mathfrak{G}_N] = 0$.

(a3) If $D \in \mathfrak{G}_S$, *resp.* $D \in \mathfrak{G}_N$, *then D is semisimple, resp. nilpotent.*

(b) *Let $(x'_1, \ldots, x'_s)$ be a $\hat{\mathcal{M}}$-regular sequence in $\hat{\Theta}_{X,P}$ such that:*

$$\text{For all } D \in \mathfrak{G}, \quad D_S(x'_i) = \lambda'_i(D) x'_i; \quad \lambda'_i(D) \in \mathbf{C}, \quad i = 1, \ldots, s. \tag{1.16}$$

for a certain s, $0 \leq s \leq n$. Then, there is a regular system of parameters $(x_1, \ldots, x_n)$ of $\hat{\mathcal{O}}_{X,P}$ such that:

$$\text{For all } D \in \mathfrak{G}, \quad D_S(x_i) = \lambda_i(D) x_i; \quad \lambda_i(D) \in \mathbf{C}, \quad i = 1, \ldots, n. \tag{1.17}$$

$$x_i = x'_i, \quad \lambda_i(D) = \lambda'_i(D); \quad \text{for all} \quad D \in \mathfrak{G} \text{ and } i = 1, \ldots, s. \tag{1.18}$$

Proof. Take

$$\mathfrak{G}_S = \{D_S; D \in \mathfrak{G}\}; \quad \mathfrak{G}_N = \{D_N; D \in \mathfrak{G}\}. \tag{1.19}$$

Thus, we have obviously (a1) and (a3). Given $Z \in \hat{\mathcal{M}}\hat{\Theta}_{X,P}$, consider the linear operator $[Z, \cdot]$, acting on $\hat{\mathcal{M}}\hat{\Theta}_{X,P}$. Working as above, we have a unique decomposition

$$[Z, \cdot] = [Z, \cdot]_S + [Z, \cdot]_N. \tag{1.20}$$

where $[Z, \cdot]_S$ and $[Z, \cdot]_N$ are commuting linear operators that produce the semisimple-nilpotent decomposition of $[Z, \cdot]$, modulo $(\hat{\mathcal{M}})^{k+1}$, for all $k \geq 1$. Moreover, we have that

$$[Z, \cdot]_S = [Z_S, \cdot] \quad \text{and} \quad [Z, \cdot]_N = [Z_N, \cdot]. \tag{1.21}$$

Take two elements $D, D' \in \mathfrak{G}$. The fact that

$$[D, D'] = 0 \tag{1.22}$$

means that D' is an eigenvector for $[D, \cdot]$ with zero eigenvalue. Hence it is so for its semisimple part $[D, \cdot]_S = [D_S, \cdot]$. Thus

$$[D_S, D'] = 0 = -[D', D_S]. \tag{1.23}$$

Now, D_S is an eigenvector for $[D', \cdot]$, hence for $[D'_S, \cdot]$, and

$$[D'_S, D_S] = 0. \tag{1.24}$$

This proves that $\mathfrak{G}_S$ is an abelian Lie algebra. From (1.22), (1.23) and (1.24), we deduce that

$$[D_S, D'_N] = 0; \quad [D_N, D'_N] = 0. \tag{1.25}$$

Hence, $\mathfrak{G}_N$ is an abelian Lie algebra and (a2) holds.

(b) Choose a regular system of parameters $(x''_1, \dots, x''_n)$ of $\hat{\mathcal{O}}_{X,P}$ such that

$$\mathrm{In}^1(x''_i) = \mathrm{In}^1(x'_i), \quad i = 1, \dots, s. \tag{1.26}$$

$$D^1{}_S(\mathrm{In}^1(x''_i)) = \lambda_i(D)\,\mathrm{In}^1(x''_i); \quad \text{for all } D \in \mathfrak{G},\ i = 1, \dots, n. \tag{1.27}$$

Note that (1.27) is always possible by simultaneous reduction to the Jordan form of a set of commuting endomorphisms of a finite dimensional vector space. In particular, (1.27) allows us to define

$$\lambda(D) = (\lambda_1(D), \dots, \lambda_n(D)) \tag{1.28}$$

for all $D \in \mathfrak{G}$. Actually $\lambda: D \hookrightarrow \lambda(D)$ is a linear mapping. Let us fix an element Z of $\mathfrak{G}$, satisfying the following generic property:

$$\langle \lambda(Z), I \rangle = \lambda_j(Z) \quad \Rightarrow \quad (\langle \lambda(D), I \rangle = \lambda_j(D);\ \text{for all } D \in \mathfrak{G}); \quad \text{for all } j = 1, \dots, n. \tag{1.29}$$

We can take a regular system of parameters $(x_1, \dots, x_n)$ of $\hat{\mathcal{O}}_{X,P}$ such that

$$\mathrm{In}^1(x_i) = \mathrm{In}^1(x''_i), \quad i = 1, \dots, n. \tag{1.30}$$

$$Z_S = \sum_{i=1}^{n} \lambda_i(Z)\, x_i \frac{\partial}{\partial x_i}. \tag{1.31}$$

Given $D \in \mathfrak{G}$, let us write

$$D_1 = \sum_{i=1}^{n} \lambda_i(D) x_i \frac{\partial}{\partial x_i}; \quad D_2 = D - D_1. \tag{1.32}$$

Hence D_1 is semisimple, D_2 is nilpotent (since $D_2{}^1$ is nilpotent) and $D=D_1+D_2$. If we show that $[D_1, D_2]=0$ we are done, since then $D=D_1+D_2$ is the Jordan decomposition of D. Note that

$$0 = [Z, D_1+D_2] = [Z, D_1]+[Z, D_2] = [Z, D_2]. \tag{1.34}$$

But in view of the property (1.29) (see also (1.13)), we have that

$$[Z, D_2] = 0 \quad \Rightarrow \quad [D_1, D_2] = 0. \tag{1.35}$$

This ends the proof. □

In particular, the above $(x_1, \ldots, x_n)$ is a common linearizing formal system for D_S, for each $D \in \mathfrak{G}$. Given $D \in \mathfrak{G}$, denote by $\lambda(D)=(\lambda_1(D), \ldots, \lambda_n(D))$ the corresponding eigenvalues, like in (1.28). Take a generic $Z \in \mathfrak{G}$ like in the proof above. The condition $[Z_S, D_N]=0$ means that $D=D_S+D_N$ can be written down as

$$D = \sum_{j=1}^{n} \lambda_j(D) x_j \frac{\partial}{\partial x_j} + \sum_{j=1}^{n} \sum_{I \in \mathcal{H}_j(\mathfrak{G})} a_{I,j}(D) x^I \frac{\partial}{\partial x_j} \tag{1.36}$$

where the set $\mathcal{H}_j(\mathfrak{G})$ is given by

$$\mathcal{H}_j(\mathfrak{G}) = \{I \in \mathbf{N}^n; |I| \geq 2,\ \langle \lambda(D), I \rangle = \lambda_j(D), \text{ for all } D \in \mathfrak{G}\}. \tag{1.37}$$

Note also that the second term on the right hand-side of (1.36) defines a nilpotent operator.

The formula (1.36) will be a key tool in our study of the normal forms for the pre-simple and simple singularities.

§ 2. Dimension two revisited

Let $E \subset X$ be a normal crossings divisor of X. Denote by $\Theta_X[E]$ the sheaf of germs of vector fields which are tangent to each irreducible component of E. Thus $\Theta_X[E] \subset \Theta_X$ and moreover we have a perfect pairing

$$\Omega_X[-E] \times \Theta_X[E] \to \mathcal{O}_X. \tag{2.1}$$

Given a coherent submodule $\mathcal{F}$ of $\Omega_X[-E]$, denote by $\mathcal{D}(\mathcal{F},E)$ the submodule of $\Theta_X[E]$ which annihilates $\mathcal{F}$, i.e., the orthogonal of $\mathcal{F}$ under (2.1). Conversely, given a coherent submodule $\mathcal{D}$ of $\Theta_X[E]$, let $\mathcal{F}(\mathcal{D},E)\subset\Omega_X[-E]$ be the orthogonal of $\mathcal{D}$. Consider $(\mathcal{F},E)\in\mathfrak{F}(X,E)$, the fact that $\mathcal{F}$ is inversible and that $\Omega_X[-E]/\mathcal{F}$ has no torsion implies that

$$\mathcal{F}=\mathcal{F}(\mathcal{D}(\mathcal{F},E),E). \tag{2.2}$$

Moreover, if $(\mathcal{F},E)\in\mathfrak{F}(X,E)$ is non-dicritical and $(\mathcal{G},\varnothing)=\mathrm{hol}((\mathcal{F},E))$, then

$$\mathcal{D}(\mathcal{F},E)=\mathcal{D}(\mathcal{G},\varnothing)\subset\Theta_X[E]. \tag{2.3}$$

Given a point $P\in X$, then (2.1) induces a perfect pairing between $\mathcal{M}$-adic completions

$$\hat{\Omega}_{X,P}[-E]\times\hat{\Theta}_{X,P}[E]\to\hat{\Theta}_{X,P}. \tag{2.4}$$

If $\hat{\mathcal{F}}(\cdot,E)$ and $\hat{\mathcal{D}}(\cdot,E)$ denote the corresponding orthogonality operators, then

$$\hat{\mathcal{F}}(\hat{\mathcal{D}}_P,E)=(\mathcal{F}(\mathcal{D},E)_P)^\wedge,\ \hat{\mathcal{D}}(\hat{\mathcal{F}}_P,E)=(\mathcal{D}(\mathcal{F},E)_P)^\wedge, \tag{2.5}$$

(with evident notations), for each coherent $\mathcal{F}\subset\Omega_X[-E]$ and $\mathcal{D}\subset\Theta_X[X]$.

Assume now that $n=\dim X=2$ and take a non-dicritical $(\mathcal{F},E)\in\mathfrak{F}(X,E)$. Put $(\mathcal{G},\varnothing)=\mathrm{hol}((\mathcal{F},E))$. Consider a point $P\in\mathrm{Sing}(\mathcal{G},\varnothing)$ which is a pre-simple singularity for $(\mathcal{F},E)$. Then $\mathcal{D}(\mathcal{F},E)_P$ is generated by a single germ of vector field $D\in\Theta_{X,P}[E]$ with

$$D\in(\Theta_{X,P}[E])\cap\mathcal{M}\cdot\Theta_{X,P}. \tag{2.6}$$

Moreover, the fact that P is a pre-simple singularity implies that

$$D^1{}_S\neq 0, \tag{2.7}$$

i.e., D^1 has at least one nonzero eigenvalue. Consider an irreducible component F of E at P (it exists since $e(E,P)\geqslant 1$ by Remark I.2.3). Let $x\in\mathcal{O}_{X,P}$ be a generator of the ideal $J_{F,P}$. Since D is tangent to F, then $\mathrm{In}^1(x)$ is an eigenvector of D^1. Hence

$$D^1(\mathrm{In}^1(x))=\lambda\mathrm{In}^1(x),\quad \lambda\in\mathbf{C}. \tag{2.8}$$

Let $\mu\in\mathbf{C}$ be the eigenvalue of D^1 corresponding to an eigenvector of $D^1{}_S$ independent of $\mathrm{In}^1(x)$. Note that $(\lambda,\mu)\neq(0,0)$. Define the invariant $\Lambda(\mathcal{F},E;F;P)$ by

$$\Lambda(\mathcal{F},E;F;P)=\lambda/\mu\in\mathbf{C}\cup\{\infty\}. \tag{2.9}$$

It is intrinsically defined.

DEFINITION 2.1. *In the above situation, we say that P is a simple singularity for $(\mathcal{F}, E)$ iff*

$$\Lambda(\mathcal{F}, E; F; P) \notin \mathbf{Q}_+ \tag{2.10}$$

where $\mathbf{Q}_+ = \{$*strictly positive rational numbers*$\}$.

Remarks 2.2. (a) The above definition does not depend of the chosen irreducible component F of E at P. In fact, if F_1, F_2 are the two irreducible components of F at P (in the case $e(E, P)=2$), then

$$\Lambda(\mathcal{F}, E; F_2; P) = 1/\Lambda(\mathcal{F}, E; F_1; P). \tag{2.11}$$

(b) The simple singularities are stable under blowing-ups. More precisely, let P be a simple singularity for $(\mathcal{F}, E)$, let $\pi: X' \to X$ be the blowing-up with center P and let $(\mathcal{F}', E')$ be the adapted strict transform of $(\mathcal{F}, E)$ by π. Put $(\mathcal{G}', \varnothing) = \mathrm{hol}((\mathcal{F}', E'))$. Then there are exactly two singular points P'_1, P'_2 in $\mathrm{Sing}(\mathcal{G}', \varnothing)$ with $P'_i \in \pi^{-1}(P)$, $i=1,2$. Both P'_1 and P'_2 are simple singularities for $(\mathcal{F}', E')$. The strict transform of each irreducible component of E at P passes through one of these points. Moreover, fix an irreducible component F of E at P and assume that P'_1 is in the strict transform F' of F by π. Then

$$\Lambda(\mathcal{F}', E'; F'; P'_1) = \Lambda(\mathcal{F}, E; F; P) - 1 \tag{2.12}$$

$$\Lambda(\mathcal{F}', E'; \pi^{-1}(P); P'_1) = 1/[\Lambda(\mathcal{F}, E; F; P) - 1]. \tag{2.13}$$

$$\Lambda(\mathcal{F}', E'; \pi^{-1}(P); P'_2) = 1/[[1/\Lambda(\mathcal{F}, E; F; P)] - 1]. \tag{2.14}$$

(c) Let P be a simple singularity with $e(E, P)=2$. Then the only invariant analytic spaces of $(\mathcal{G}, \varnothing)$ through P are the two irreducible components of E at P.

(d) Let P be a simple singularity with $e(E, P)=1$. Then $(\mathcal{G}, \varnothing)$ has exactly two formal separatrices at P. One of them is given by the ideal $J_{E,P}$ of the divisor E, it is of course a convergent one. The other one, say $f \cdot \hat{\mathcal{O}}_{X,P}$, is non singular and transversal to E (i.e., f jointly with a local equation of E define a regular system of parameters of $\hat{\mathcal{O}}_{X,P}$). Classical results say that we can take f to be convergent in the case that

$$\Lambda(\mathcal{F}, E; E; P) \neq \infty. \tag{2.15}$$

(e) Finally, let us recall that Seidenberg's result of desingularization [20] means that in the two-dimensional case we can get a situation with only simple singularities after finitely many blowing-ups.

LEMMA 2.4. *Let P be a pre-simple singularity for $(\mathcal{F},E)$ as above. Then*

$$\Lambda(\mathcal{F},E;F;P)\notin \mathbf{Q}_{+}-\mathbf{N}\cup(1/\mathbf{N}). \tag{2.16}$$

Otherwise $(\mathcal{F},E)$ would be a dicritical singular foliation.

Proof. Let us reason by contradiction, assuming that

$$\Lambda(\mathcal{F},E;F;P)=p/q\in \mathbf{Q}_{+}-\mathbf{N}\cup(1/\mathbf{N}), \tag{2.17}$$

where $p,q\in\mathbf{N}$, $p,q\geqslant 2$ and g.c.d.$(p,q)=1$. Now, let us make induction on $p+q$. We can take a regular system of parameters (x,y) of $\mathcal{O}_{X,P}$ such that a generator Ω of $\mathcal{G}_P$ is given by

$$\Omega=(py+\varphi(x,y))\,dx+(-qx+\Psi(xy))\,dy \tag{2.18}$$

with $\nu_P(\varphi,\psi)\geqslant 2$. Assume that $q<p$, let us blow-up the point P and look at the point P' corresponding to the strict transform of $(y=0)$. Putting $x=x'$, $y=x'y'$, a generator Ω' of $\mathcal{G}'_P$ is given by

$$\Omega'=((p-q)y'+x'\varphi'(x',y'))\,dx'+(-qx'+x'^2\psi'(x',y'))\,dy'. \tag{2.19}$$

We distinguish two cases:

Case 1: $q=m(p-q)$, for some integer $m\geqslant 2$. In this case, blowing-up P' and looking at the point P' corresponding to the strict transform of $x'=0$, a generator Ω'' of $\mathcal{G}''_{P''}$ is given by

$$\Omega''=(-y''+y''^2\varphi''(x'',y''))\,dx''+((m-1)x''+x''^2\psi''(x'',y''))\,dy''. \tag{2.20}$$

If $m-1=1$, we see easily that blowing-up P'' the exceptional divisor is a dicritical component. If $m-1\geqslant 2$, we reason by induction on $m-1$: blowing-up P'' and looking at the point corresponding to $(x''=0)$, then $m-1$ decreases one unit. This is the desired contradiction.

Case 2: otherwise. Then the invariant $p+q$ decreases strictly and we are done by induction. □

The following proposition gives to us the formal normal forms for the non-dicritical pre-simple singularities in dimension two:

PROPOSITION 2.5. *Let P be a pre-simple singularity for $(\mathcal{F},E)$. Then there are a regular system of parameters (x,y) of $\hat{\mathcal{O}}_{X,P}$ and a generator Ω of $\hat{\mathcal{G}}_X$ such that*

$\hat{J}_{E,P} \supset xy\hat{\mathcal{O}}_{X,P}$ and Ω is in one of the following formal normal forms:

(i) $\Omega = xy(dx/x + \lambda\, dy/y)$; $\lambda \in \mathbf{C}$, $\lambda \notin \mathbf{Q}_-$.

(ii) $\Omega = xyy^s(dx/x + (\varepsilon + 1/y^s)\, dy/y)$; $s \geq 1$, $\varepsilon \in \mathbf{C}$.

(iii) $\Omega = xy(x^p y^q)^s (dx/x + (\varepsilon + 1/(x^p y^q)^s)(p\, dx/x + q\, dy/y)$; g.c.d.$(p,q) = 1$, $s \geq 1$.

(iv)* $\Omega = x((my + x^m)\, dx/x - dy)$; $m \geq 1$.

The formal normal forms (i), (ii) *and* (iii) *correspond exactly to the simple singularities; the formal normal form* (iv)* *corresponds to a pre-simple singularity which is not a simple singularity.*

Conversely. Let $(\mathcal{F}, E) \in \mathfrak{F}(X, E)$, assume that P is the only point in $\mathrm{Sing}(\mathcal{G}, \varnothing)$, *where* $(\mathcal{G}, \varnothing) = \mathrm{hol}((\mathcal{F}, E))$. *Assume that there is a regular system of parameters (x, y) of $\hat{\mathcal{O}}_{X,P}$ satisfying*

(a) $1 \neq \hat{J}_{E,P} \supset xy\hat{\mathcal{O}}_{X,P}$;

(b) *a generator Ω of $\hat{\mathcal{G}}_P$ can be written down in either one of the formal normal forms* (i), (ii), (iii), *or* (iv)*;

(c) *if* (iv)*, *then* $\hat{J}_{E,P} = x\hat{\mathcal{O}}_{X,P}$.

Then $(\mathcal{F}, E)$ is non-dicritical and P is a pre-simple singularity for $(\mathcal{F}, E)$.

Proof. Assume first that $e(E,P) = 1$. Take a regular system of parameters (x, y) of $\mathcal{O}_{X,P}$ such that $J_{E,P} = x\mathcal{O}_{X,P}$. Then $\mathcal{F}_P$ is generated by

$$\omega = a\frac{dx}{x} + b\, dy; \quad a, b \in \mathcal{O}_{X,P},\ \text{g.c.d.}(a,b) = 1, \tag{2.21}$$

where $\nu_P(a) \geq 1$ (otherwise $P \notin \mathrm{Sing}(\mathcal{G}, \varnothing)$) and x does not divide a (otherwise E would be a dicritical component for $(\mathcal{F}, E)$). Then $\mathcal{G}_P$ is generated by

$$\Omega = a\, dx + bx\, dy. \tag{2.22}$$

Since P is a pre-simple singularity, one of the following two possibilities holds:

(A) $\nu_P(b) = 0$, i.e. b is a unit of $\mathcal{O}_{X,P}$.

(B) $\nu_P(b) \geq 1$ and $\mathrm{In}^1(a) = \alpha\, \mathrm{In}^1(x) + \beta\, \mathrm{In}^1(y)$ with $\beta \neq 0$.

Consider the possibility (A). Then $b = 1$, up to multiply Ω by b^{-1}. Thus $\mathcal{D}(\mathcal{F}, E)_P$ is generated by the germ of vector field

$$D = x\frac{\partial}{\partial x} - a\frac{\partial}{\partial y}. \tag{2.23}$$

By Proposition 1.1, we have a regular system of parameters $(x, y^\wedge)$ of $\hat{\mathcal{O}}_{X,P}$ which is a linearizing system of parameters for D_S. Assume that $D_S(y^\wedge) = \lambda y^\wedge$, for a certain $\lambda \in \mathbf{C}$.

(Note that $\lambda=-\partial a/\partial y(0)$.) Actually we have that

$$1/\lambda=\Lambda(\mathscr{F},E;E;P). \tag{2.24}$$

Hence $\lambda\notin \mathbf{Q}_{+}-\mathbf{N}\cup(1/\mathbf{N})$ by Lemma 2.4. Consider the following cases:

(A1) $\lambda=0$. Then, in view of (1.14), we can write D as follows:

$$D=x\frac{\partial}{\partial x}-\sum_{j\geq 1}\varepsilon_j\cdot(y^{\wedge})^{j+1}\frac{\partial}{\partial y^{\wedge}}. \tag{2.25}$$

Let s be the first index such that $\varepsilon_s\neq 0$. Note that $s<\infty$, otherwise x divides the coefficients of Ω. Hence, $\hat{\mathscr{F}}_P$ is generated over $\hat{\mathcal{O}}_{X,P}$ by

$$\Omega^{\wedge}=x\cdot y^{\wedge}\cdot(y^{\wedge})^{s}\left[\frac{dx}{x}+u(y^{\wedge})\frac{dy^{\wedge}}{(y^{\wedge})^{s+1}}\right] \tag{2.26}$$

where $u(y^{\wedge})\in\hat{\mathcal{O}}_{X,P}$ is a unit. A coordinate change $y'=y'(y^{\wedge})$ allows us to write

$$u(y^{\wedge})\frac{dy^{\wedge}}{(y^{\wedge})^{s+1}}=\left(\varepsilon+\frac{1}{y'^{s}}\right)\frac{dy'}{y'} \tag{2.27}$$

for a certain residue $\varepsilon\in\mathbf{C}$. Hence, multiplying $\Omega^{\wedge}$ by a unit in $\hat{\mathcal{O}}_{X,P}$, we have the normal form (ii).

(A2) $\lambda=-p/q\in\mathbf{Q}_{-}=-\mathbf{Q}_{+}$; with g.c.d.$(p,q)=1$. We can take

$$D=qx\frac{\partial}{\partial x}-a\frac{\partial}{\partial y};\quad \frac{\partial a}{\partial y}(0)=p. \tag{2.28}$$

By (1.14), we have

$$D=qx\frac{\partial}{\partial x}-py^{\wedge}\frac{\partial}{\partial y^{\wedge}}-\sum_{j\geq 1}\varepsilon_j\cdot(x^{p}y^{\wedge q})^{j}y^{\wedge}\frac{\partial}{\partial y^{\wedge}}. \tag{2.29}$$

Let s be the first index such that $\varepsilon_s\neq 0$. If $s=\infty$, we have the normal form (i). Assume that $s<\infty$. Then, $\hat{\mathscr{F}}_P$ is generated by

$$\Omega^{\wedge}=xy^{\wedge}\cdot(x^{p}y^{\wedge q})^{s}\left(\frac{dx}{x}+\frac{u(x^{p}y^{\wedge q})}{(x^{p}y^{\wedge q})^{s}}\left(p\frac{dx}{x}+q\frac{dy^{\wedge}}{y^{\wedge}}\right)\right), \tag{2.30}$$

where $u(t)\in\mathbf{C}[[t]]$ is a unit. Take coordinate changes $t'=t'(t)$ and $y'=y'(y^{\wedge})$ such that

$$\frac{u(t)}{t^{s}}\cdot\frac{dt}{t}=\left(\varepsilon+\frac{1}{t'^{s}}\right)\frac{dt'}{t'};\quad x^{p}y'^{q}=t'(x^{p}y^{\wedge q}). \tag{2.31}$$

Then, multiplying $\Omega^\wedge$ by a unit, we have the normal form (iii).

(A3) $\lambda=m\in\mathbf{N}_+$. By (1.14), we have

$$D=x\frac{\partial}{\partial x}+my^\wedge\frac{\partial}{\partial y^\wedge}+\mu x^m\frac{\partial}{\partial y^\wedge}. \tag{2.32}$$

Hence, we can take

$$\Omega=x\left((my^\wedge+\mu x^m)\frac{dx}{x}-dy^\wedge\right). \tag{2.33}$$

In the case $\mu=0$, if we blow-up n times following the direction given by $y^\wedge=0$, we see that $(\mathscr{F},E)$ is dicritical. Thus $\mu\neq0$. Multiplying x and Ω by a scalar, we have the normal form (iv)*.

(A4) $\lambda=1/m\in 1/\mathbf{N}_+$, $m\geqslant2$. By (1.14), we have that

$$D=x\frac{\partial}{\partial x}+(1/m)\cdot y^\wedge\frac{\partial}{\partial y^\wedge}. \tag{2.34}$$

Hence, we can take

$$\Omega=x\cdot y^\wedge\left(y^\wedge\frac{dx}{x}-m\frac{dy^\wedge}{y^\wedge}\right). \tag{2.35}$$

Blowing-up m times following $x=0$, we see that $(\mathscr{F},E)$ is dicritical. Thus, this case does not hold.

(A_5) $\lambda\in\mathbf{C}-\mathbf{N}\cup1/\mathbf{N}\cup\mathbf{Q}_-$. Then, by (1.14) $D_N=0$ and D is semisimple. Hence, we have the normal form (i).

Consider now the possibility B. Making the linear coordinate change $y_1=\alpha x+\beta y$, we may assume that $\mathrm{In}^1(a)=\mathrm{In}^1(y)$. Thus $\mathscr{D}(\mathscr{F},E)_P$ is generated by the germ of vector field

$$D=b\cdot x\frac{\partial}{\partial x}-(y+a')\frac{\partial}{\partial y},\quad \nu(bx,a')\geqslant2. \tag{2.36}$$

Let $(x,y^\wedge)$ be a linearizing formal system of parameters for D_S. Note that $D_S(x)=0$ and $D_S(y^\wedge)=1$. By (1.14), we have

$$D=\sum_{j\geqslant1}\varepsilon_j\cdot x^{j+1}\frac{\partial}{\partial x}+y^\wedge\frac{\partial}{\partial y^\wedge}+\sum_{j\geqslant1}\mathscr{N}_j\cdot x^jy^\wedge\frac{\partial}{\partial y^\wedge}. \tag{2.37}$$

Multiplying D by $(1+\Sigma\mathscr{N}_j\cdot x^j)^{-1}$, we can assume that $\hat{\mathscr{F}}_P$ is generated by

$$\Omega = x \cdot x^s \cdot y^\wedge \left(u(x) \frac{dx}{x^{s+1}} + \frac{dy^\wedge}{y^\wedge} \right) \tag{2.38}$$

where $u(x)$ is a unit and $s=\min(j;\varepsilon_j\neq 0)<\infty$ (see (A1)). Now, reasoning as in (A1) and interchanging the role of x and $y^\wedge$, we obtain the normal form (ii).

Assume now that $e(E,P)=2$. Take a regular system of parameters (x,y) of $\mathcal{O}_{X,P}$ such that $J_{E,P}=xy\mathcal{O}_{X,P}$ and a generator ω of $\mathcal{F}_P$ such that

$$\omega = a\frac{dx}{x} + \frac{dy}{y}. \tag{2.39}$$

Then, $\mathcal{D}(\mathcal{F},E)_P$ is generated by the germ of vector field

$$D = x\frac{\partial}{\partial x} - ay\frac{\partial}{\partial y}. \tag{2.40}$$

Now, reasoning as in the case (A) above, we obtain either the normal form (i), (ii), or (iii) (the normal form (iv)* does not appear).

For the second part of the proposition, in the simple cases (i), (ii) or (iii), blowing-up we only obtain simple cases, hence $(\mathcal{F},E)$ is non-dicritical. In the case (iv)*, we obtain either simple cases or a case (iv)* with strictly smaller invariant m; we are done by induction and $(\mathcal{F},E)$ is non-dicritical. □

§ 3. Locally product simple singularities

Let us consider now the study of the case $n=\dim X=3$. Take a non-dicritical $(\mathcal{F},E)\in\mathfrak{F}(X,E)$. Put $(\mathcal{G},\varnothing)=\mathrm{hol}((\mathcal{F},E))$ and let us fix a point $P\in\mathrm{Sing}(\mathcal{G},\varnothing)$ which is a pre-simple singularity for $(\mathcal{F},E)$.

LEMMA 3.1. *In the above situation we have that either $\nu(\mathcal{G},\varnothing;P)=1$ or $\nu(\mathcal{G},\varnothing;P)=2$. In the case $\nu(\mathcal{G},\varnothing;P)=2$ then there is an open set $\mathcal{U}\subset X$, $P\in\mathcal{U}$, such that*

$$\mathcal{U}\cap \mathrm{Sing}^*(\mathcal{F},E)=\varnothing, \quad \text{(see I.(2.18))}, \tag{3.1}$$

$$\mathcal{U}\cap\{Q;\nu(\mathcal{G},\varnothing;Q)\geq 2\}=\{P\}. \tag{3.2}$$

Proof. The first part is a trivial computation. The property of (3.1) is a consequence of Proposition I.2.6. In order to prove (3.2), note that

$$\text{if } e(E,P)=1, \text{ then } \nu(\mathcal{G},\varnothing;P)=1, \tag{3.3}$$

(direct computation). Hence, assume that $e(E,P)\geq 2$. Actually, it is enough to consider the case $e(E,P)=2$; in fact, the case $e(E,P)=3$ follows immediately, (see Remark I.2.3(b)) since the order $\nu(\mathcal{G},\varnothing;Q)=0$, for Q near P, and the points P with $e(E,P)=3$ are isolated points. Take a regular system of parameters (x,y,z) of $\mathcal{O}_{X,P}$ such that $J_{E,P}=xy\mathcal{O}_{X,P}$. We can write a generator ω of $\mathcal{F}_P$ as follows:

$$\omega = a\frac{dx}{x} + b\frac{dy}{y} + c\,dz. \tag{3.4}$$

Note that $\mathcal{G}_P$ is generated by

$$\Omega = ay\,dx + bx\,dy + cxy\,dz. \tag{3.5}$$

Since $\nu(\mathcal{G},\varnothing;P)=2$, then $\nu_P(a,b)\geq 1$. Let us reason by contradiction, assuming that (3.2) does not hold. By (3.3) we must have that

$$\nu_{(x,y)}(a,b) \geq 1. \tag{3.6}$$

Let us distinguish two cases:

(A) $\nu_P(c)\geq 1$. Hence $\mu(\mathcal{F},E;\{P\})=\nu(\mathcal{F},E;P)=1$. Then the condition (3.6) contradicts the fact that $\mathrm{Dir}(\mathcal{F},E;P)$ has normal crossings with E.

(B) $\nu_P(c)=0$. Then $(x=y=0)$ is locally permissible at P for $(\mathcal{F},E)$. In fact, we have

$$\varrho(\mathcal{F},E;(x=y=0);P) = \mu(\mathcal{F},E;(x=y=0)) = 0. \tag{3.7}$$

If we make the blowing-up with center $(x=y=0)$, the condition of (3.6) means that the exceptional divisor is a dicritial component. This contradicts the fact that $(\mathcal{F},E)$ is non-dicritical. $\square$

In this paragraph we shall consider only pre-simple singularities with $\nu(\mathcal{G},\varnothing;P)=1$, and we shall define the simple singularities in this case. The following proposition means that this case is essentially a two dimensional one. In fact, we have an analytic triviality along the singular locus, which respects the exceptional divisor.

Proposition 3.2. *In the above situation, assume that* $\nu(\mathcal{G},\varnothing;P)=1$. *Then there exists a non-singular germ of vector field* $D\in\mathcal{D}(\mathcal{F},E)_P$. *More precisely, we have that*

$$D\in\mathcal{D}(\mathcal{F},E)_P\cap\Theta_{X,P}[E] \quad\text{and}\quad D\notin\mathcal{M}\Theta_{X,P}, \tag{3.8}$$

where $\mathcal{M}$ *is the maximal ideal of* $\mathcal{O}_{X,P}$.

Proof. If $e(E,P)=3$, we have $\nu(\mathcal{G},\varnothing;P)=2$. Then, either $e(E,P)=1$ or $e(E,P)=2$.

(A) Case $e(E,P)=1$. (By (3.3) we always have $\nu(\mathcal{G},\varnothing;P)=1$ in this case.) Assume first that $\nu(\mathcal{F},E;P)=0$. Then there is a regular system of parameters (x,y,z) of $\mathcal{O}_{X,P}$ and a generator ω of $\mathcal{F}_P$ such that

$$J_{E,P}=x\cdot\mathcal{O}_{X,P}. \tag{3.9}$$

$$\omega=a\frac{dx}{x}+b\,dy+dz;\quad \nu_P(a)\geq 1. \tag{3.10}$$

Then the germ of vector field

$$D=\frac{\partial}{\partial y}-b\frac{\partial}{\partial z} \tag{3.11}$$

satisfies (3.8). Note that $D(x)=0$ and hence $D\in\Theta_{X,P}[E]$.

Assume now that $\nu(\mathcal{F},E;P)=1$. Take (x,y,z) and ω such that (3.9) holds and

$$\omega=(z+a')\frac{dx}{x}+b\,dy+c\,dz;\quad \nu_P(a')\geq 2,\ \nu_P(b,c)\geq 1. \tag{3.12}$$

Then $\mathcal{G}_P$ is generated by

$$\Omega=(z+a')\,dx+xb\,dy+xc\,dz. \tag{3.13}$$

Note that $d\Omega$ is non-singular at P, actually

$$d\Omega=xa^*\,dy\wedge dz+b^*\,dz\wedge dx+c^*\,dx\wedge dy \tag{3.14}$$

and $b^*\notin\mathcal{M}$. The integrability condition $\Omega\wedge d\Omega=0$ implies that

$$D=xa^*\frac{\partial}{\partial x}+b^*\frac{\partial}{\partial y}+c^*\frac{\partial}{\partial z} \tag{3.15}$$

is the desired germ of vector field. This situation is named "Kupka–Reeb phenomena" (cf. [11], p. 31). Note that $D(x)=a^*x$ and thus $D\in\Theta_{X,P}[E]$.

(B) Case $e(E,P)=2$. Since $\nu(\mathcal{G},\varnothing;P)=1$, there is a regular system of parameters (x,y,z) of $\mathcal{O}_{X,P}$ and a generator ω of $\mathcal{F}_P$ such that

$$J_{E,P}=xz\mathcal{O}_{X,P}. \tag{3.16}$$

$$\omega=\frac{dx}{x}+b\,dy+c\frac{dz}{z}. \tag{3.17}$$

Hence

$$D=bx\frac{\partial}{\partial x}-\frac{\partial}{\partial y} \tag{3.18}$$

is the desired germ of vector field. □

COROLLARY 3.3. *In the above situation, assume that* $\nu(\mathcal{G},\varnothing;P)=1$ *and let us fix a germ of vector field* $D\in\Theta_{X,P}$ *satisfying* (3.8). *Then there are a regular system of parameters* (x,y,z) *of* $\mathcal{O}_{X,P}$ *and a generator* ω *of* $\mathcal{F}_P$ *such that:*

(a) *Either* $J_{E,P}=x\mathcal{O}_{X,P}$ *or* $J_{E,P}=xy\mathcal{O}_{X,P}$.

(b) $D=\partial/\partial z$.

(c) *Under the identification* $\mathcal{O}_{X,P}=\mathbf{C}\{x,y,z\}$, *we have that*

$$\omega=a\frac{dx}{x}+b\,dy,\quad a,b\in\mathbf{C}\{x,y\};\quad \text{if } e(E,P)=1 \tag{3.19}$$

$$\omega=a\frac{dx}{x}+b\frac{dy}{y},\quad b\in\mathbf{C}\{x,y\};\quad \text{if } e(E,P)=2. \tag{3.20}$$

(d) $\mathrm{Sing}(\mathcal{G},\varnothing)=(x=y=0)$, *locally at* P *(in any regular system of parameters* (x,y,z) *of* $\mathcal{O}_{X,P}$ *as above)*

(e) *Let* Δ *be a germ at* P *of a two-dimensional non-singular subspace of* X *such that* $T_P\Delta$ *is transversal to* $T_P(x=y=0)$. *Then there is a germ of analytic isomorphism between* $(z=0)$ *and* Δ *which sends*

$$(\mathcal{F}|_{z=0},E\cap(z=0))\in\mathfrak{F}((z=0),E\cap(z=0)) \tag{3.21}$$

into

$$(\mathcal{F}|_{\Delta},E\cap\Delta)\in\mathfrak{F}(\Delta,\mathbf{E}\cap\Delta). \tag{3.22}$$

(f) *Making a formal coordinate change in* (x,y) *(which preserves the property that* $\hat{J}_{E,P}\supset xy\hat{\mathcal{O}}_{X,P}$), *then* $\hat{\mathcal{G}}_P$ *is generated by one of the normal forms* (i), (ii), (iii) *or* (iv)* *of Proposition* 2.5.

Proof. Take (x,y,z) satisfying (a) and (b). Then $\mathcal{G}_P$ is generated by

$$\Omega=\alpha(x,y,z)\,dx+\beta(x,y,z)\,dy. \tag{3.23}$$

The integrability condition $\Omega\wedge d\Omega=0$ implies that

$$\alpha(x,y,0)\beta(x,y,z)=\alpha(x,y,z)\beta(x,y,0). \tag{3.24}$$

Hence $\mathcal{G}_P$ is also generated by

$$\Omega_0=\alpha(x,y,0)\,dx+\beta(x,y,0)\,dy. \tag{3.25}$$

The rest of the proof is obvious. □

Remark 3.4. (a) In the above situation, let F be an irreducible component of E through P and let Δ be as in Corollary 3.3 (e). Then both $(\mathcal{F}|_{z=0}, E\cap(z=0))$ and $(\mathcal{F}|_{\Delta}, E\cap\Delta)$ are non-dicritical. Moreover, P is a pre-simple singularity for these adapted singular foliations and

$$(3.26)\qquad \Lambda(\mathcal{F}|_{z=0}, (z=0)\cap E; (z=0)\cap F; P) = \Lambda(\mathcal{F}|_{\Delta}, E\cap\Delta; F\cap\Delta; P).$$

Thus, in this case we define the invariant $\Lambda(\mathcal{F}; E; F; P)$ by

$$(3.27)\qquad \Lambda(\mathcal{F}, E; F; P) = \Lambda(\mathcal{F}|_{\Delta}, E\cap\Delta; F\cap\Delta; P).$$

(b) Let Y be the irreducible component of $\mathrm{Sing}(\mathcal{G}, \varnothing)$ passing through P. The points

$$(3.28)\qquad H = \{Q\in Y; Q \text{ is not pre-simple with } \nu(\mathcal{G}, \varnothing; Q) = 1\}$$

are isolated points. Hence, by connectedness, we have

$$(3.29)\qquad \Lambda(\mathcal{F}, E; F; Q) = \Lambda(\mathcal{F}, E; F; P) \quad \text{for all } Q\in Y-H.$$

DEFINITION 3.5. *Let P be a pre-simple singularity for $(\mathcal{F}, E)$ with $\nu(\mathcal{G}, \varnothing; P)=1$. We say that P is a simple singularity for $(\mathcal{F}, E)$ iff*

$$(3.30)\qquad \Lambda(\mathcal{F}, E; F; P) \notin \mathbf{Q}_+$$

for each irreducible component F of E at P.

Remarks 3.6. (a) The point P is a simple singularity for $(\mathcal{F}, E)$ iff it is a simple singularity for $(\mathcal{F}|_{\Delta}, E\cap\Delta)$, where Δ is as in Corollary 3.3 (e).

(b) Also, P is a simple singularity for $(\mathcal{F}, E)$ iff we have one of the normal forms (i), (ii) or (iii) for a generator of $\hat{\mathcal{G}}_P$.

(c) Conversely, assume that (x, y, z) is a regular system of parameters of $\hat{\mathcal{O}}_{X,P}$ satisfying the following properties:

(c1) $\hat{J}_{E,P}\supset xy\hat{\mathcal{O}}_{X,P}$; $e(E,P)\geq 1$.

(c2) $\hat{\mathcal{G}}_P$ is generated by one of the normal forms (i), (ii), (iii) or (iv)* of Proposition 2.5.

(c3) If (iv)*, then $\hat{J}_{E,P}=x\hat{\mathcal{O}}_{X,P}$.

Then P is a pre-simple singularity for $(\mathcal{F}, E)$ with $\nu(\mathcal{G}, \varnothing; P)=1$.

§ 4. Confluencies of simple singularities

Like in the preceding paragraph, assume that $n=3$, take a non-dicritical $(\mathcal{F}, E) \in \mathfrak{F}(X, E)$, put $(\mathcal{G}, \varnothing)=\mathrm{hol}((\mathcal{F}, E))$ and fix a point $P \in \mathrm{Sing}(\mathcal{G}, \varnothing)$ which is a pre-simple singularity for $(\mathcal{F}, E)$. In this paragraph we shall study the case $\nu(\mathcal{G}, \varnothing; P)=2$. By Lemma 3.1, we can think of P as a "confuence" of pre-simple singularities which are locally a product.

LEMMA 4.1. *Let P be a pre-simple singularity for $(\mathcal{F}, E)$ such that*

$$\nu(\mathcal{G}, \varnothing; P)=2 \quad \textit{and} \quad \mu(\mathcal{F}, E; \{P\})=\nu(\mathcal{F}, E; P)=1. \tag{4.1}$$

(Hence $e(E, P)=2$.) Take a regular system of parameters (x, y, z) of $\mathcal{O}_{X,P}$ with

$$J_{E,P}=xy\mathcal{O}_{X,P}. \tag{4.2}$$

Then, there is a formal coordinate change

$$z^{\wedge}=z+\varphi(x, y); \quad \nu(\varphi(x, y)) \geqslant 1, \tag{4.3}$$

such that $z^{\wedge}\hat{\mathcal{O}}_{X,P}$ is a formal separatrix of $(\mathcal{G}, \varnothing)$.

Proof. We have a generator ω of $\mathcal{F}_P$ given by

$$\omega=a\frac{dx}{x}+b\frac{dy}{y}+c\,dz \tag{4.4}$$

such that $\nu_P(a, b, c) \geqslant 1$ and $\partial a/\partial z(0,0,0)=1$. Let us put

$$a=z+\sum_{i,j\geqslant 1} a_{ij}x^iy^j+za', \quad \nu_P(a') \geqslant 1. \tag{4.5}$$

$$(s, t)=\min\{(i,j); a_{ij} \neq 0\}, \tag{4.6}$$

with respect to the ordering

$$(i,j) \leqslant (i',j') \iff (i+j<i'+j') \text{ or } (i+j=i'+j' \text{ and } i \leqslant i'). \tag{4.7}$$

If $(s, t)=\infty$, then z divides a. Assume the contrary. By the coordinate change

$$z'=z+\varphi_{st}x^sy^t, \quad \varphi_{st}=a_{st}, \tag{4.8}$$

the corresponding invariant (s', t') is strictly bigger than (s, t). By a formal coordinate change

$$z^\wedge = z + \sum_{s,t} \varphi_{st} x^s y^t, \tag{4.9}$$

we may assume that $z^\wedge$ divides a. Multiplying ω by a unit, we have that

$$\omega = z^\wedge \frac{dx}{x} + b\frac{dy}{y} + c\, dz^\wedge; \quad \nu(b,c) \geq 1. \tag{4.10}$$

The integrability condition $\omega \wedge dw=0$ implies that $z^\wedge$ also divides b. Hence

$$\omega = z^\wedge \frac{dx}{x} + z^\wedge b' \frac{dy}{y} + c\, dz^\wedge \tag{4.11}$$

and $z^\wedge$ is a formal separatrix for $(\mathscr{G}, \varnothing)$. □

LEMMA 4.2. *Let P be a pre-simple singularity for $(\mathscr{F}, E)$ such that $\nu(\mathscr{G}, \varnothing; P)=2$. Then there are a regular system of parameters (x,y,z) of $\hat{\mathcal{O}}_{X,P}$ and two formal vector fields D_1 and D_2 such that:*

(a) $xyz\hat{\mathcal{O}}_{X,P} \subset \hat{J}_{E,P}$.

(b) $D_1 = x(\partial/\partial x) - a(\partial/\partial z), D_2 = y(\partial/\partial y) - b(\partial/\partial z)$; *with* $\nu(a,b) \geq 1$.

(c) $D_1, D_2 \in \mathscr{D}(\mathscr{F}, E)^\wedge{}_P$.

(d) $[D_1, D_2]=0$.

Moreover, if $\mathscr{L}$ is the Lie subalgebra of $\hat{\Theta}_{X,P}$ generated by D_1 and D_2, then

$$\hat{\mathscr{F}}_P = \{\omega \in \hat{\Omega}_{X,P}[-E]; \omega(D)=0, \text{ for all } D \in \mathscr{L}\}. \tag{4.12}$$

Proof. Let us consider the following cases:

(A) $e(E,P)=2$ and $\nu(\mathscr{F}, E; P)=0$. Then $\mathscr{F}_P$ is generated by

$$\omega = a\frac{dx}{x} + b\frac{dy}{y} + dz; \quad \nu(a,b) \geq 1 \tag{4.13}$$

in a regular system of parameters of $\mathcal{O}_{X,P}$ such that $J_{E,P} = xy\mathcal{O}_{X,P}$. Take then

$$D_1 = x\frac{\partial}{\partial x} - a\frac{\partial}{\partial z}; \quad D_2 = y\frac{\partial}{\partial y} - b\frac{\partial}{\partial z}. \tag{4.14}$$

The condition (d) is equivalent to $\omega \wedge d\omega=0$ and (4.12) is obvious.

(B) $e(E,P)=3$. Then $\mathscr{F}_P$ is generated by

$$\omega = a\frac{dx}{x} + b\frac{dy}{y} + \frac{dz}{z} \tag{4.15}$$

in a regular system of parameters of $\mathcal{O}_{X,P}$ with $J_{E,P} = xyz\mathcal{O}_{X,P}$. Take then

(4.16) $$D_1 = x\frac{\partial}{\partial x} - az\frac{\partial}{\partial z}; \quad D_2 = y\frac{\partial}{\partial y} - bz\frac{\partial}{\partial z},$$

and let us reason as above.

(C) $e(E,P)=2$ and $\nu(\mathcal{F},E;P)=1$. By Lemma 4.1, we can write a generator ω of $\hat{\mathcal{F}}_P$ as in (4.11):

(4.17) $$\omega = z^\wedge\left(\frac{dx}{x} + b'\frac{dy}{y} + c\frac{dz^\wedge}{z^\wedge}\right)$$

and we can take

(4.18) $$D_1 = z^\wedge\frac{\partial}{\partial z^\wedge} - cx\frac{\partial}{\partial x}; \quad D_2 = y\frac{\partial}{\partial y} - b'x\frac{\partial}{\partial x}.$$

This ends the proof. □

Remark 4.3. The only case in which (x,y,z) is may be not convergent is the case (C) above. By Lemma 4.1, even in this case, the parameters in (x,y,z) defining the irreducible components of E are convergent ones.

PROPOSITION 4.4 (Normal forms). *Let P be a pre-simple singularity for $(\mathcal{F},E)$ with $\nu(\mathcal{G},\varnothing,P)=2$. Then there are a regular system of parameters (x,y,z) of $\hat{\mathcal{O}}_{X,P}$ and a generator Ω of $\hat{\mathcal{G}}_P$ such that $xyz\cdot\hat{\mathcal{O}}_{X,P}\subset\hat{J}_{E,P}$ and Ω is one of the following formal normal forms:*

(iv) $\Omega=xyz(\alpha\, dx/x+\beta\, dy/y+dz/z)$; *with* $\alpha\cdot\beta\neq 0$ *and* $-\alpha, -\beta, -\alpha/\beta\notin\mathbf{Q}_+$

(v) $\Omega=xyz\cdot z^s(dx/x+\beta\, dy/y+(\varepsilon+1/z^s)\, dz/z)$; *with* $s\geq 1$, $0\neq-\beta\notin\mathbf{Q}_+$.

(vi) $\Omega=xyz(y^pz^q)^s(dx/x+\beta\, dy/y+(\varepsilon+1/(y^pz^q)^s)(p\, dy/y+q\, dz/z))$; $s\geq 1$, g.c.d.$(p,q)=1$.

(vii) $\Omega=xyz\cdot(x^py^qz^r)^s(dx/x+\beta\, dy/y+(\varepsilon+1/(x^py^qz^q)^s)(p(dx/x)+q(dy/y)+r(dz/z)))$; *with* $s\geq 1$, g.c.d.$(p,q,r)=1$.

(viii)* $\Omega=xy(\alpha y^m\, dx/x-(mz+y^m)\, dy/y+dz)$; $m\geq 1$, $\alpha\neq 0$.

(ix)* $\Omega=xy(-(pz+x^py^q)\, dx/x-(qz+\beta x^py^q)\, dy/y+dz)$; $p,q\geq 1$, $0\neq\beta\notin\mathbf{Q}_-$.

Proof. Take a regular system of parameters (x,y,z) of $\hat{\mathcal{O}}_{X,P}$ and two formal vector fields

(4.19) $$D_1 = x\frac{\partial}{\partial x} - a\frac{\partial}{\partial z}; \quad D_2 = y\frac{\partial}{\partial y} - b\frac{\partial}{\partial z}$$

as in Lemma 4.2. Let λ,μ be the eigenvalues of D_1, D_2, given by

(4.20) $$\lambda = -\frac{\partial a}{\partial z}(0,0,0); \quad \mu = -\frac{\partial b}{\partial z}(0,0,0).$$

By Proposition 1.4, we can obtain from (x, y, z) a regular system of parameters $(x, y, z^\wedge)$ of $\hat{\mathcal{O}}_{X,P}$ which is both a linearizing system for D_{1S} and D_{2S}. Moreover, if

$$\hat{J}_{E,P} \subset z\hat{\mathcal{O}}_{X,P} \tag{4.21}$$

(in particular, z divides "a" and "b" in (4.19)), then

$$z^\wedge \mathcal{O}_{X,P} = z\hat{\mathcal{O}}_{X,P}. \tag{4.22}$$

Hence the condition $xyz^\wedge \hat{\mathcal{O}}_{X,P} \subset \hat{J}_{E,P}$ holds. In order to simplify the notation, put $(x, y, z) = (x, y, z^\wedge)$. Let us write

$$a = \sum_{i,j,k} a_{ijk} x^i y^j z^k; \quad b = \sum_{i,j,k} b_{ijk} x^i y^j z^k. \tag{4.23}$$

By the formula (1.21), we have that

$$(a_{ijk}, b_{ijk}) \neq (0, 0) \text{ implies that } i + \lambda(k-1) = j + \mu(k-1) = 0. \tag{4.24}$$

Now, let us consider the following cases, corresponding to the possible values of λ, μ:

(A) $\lambda\mu \neq 0; \lambda, \mu \notin \mathbf{Q}_- \cup \mathbf{N}_+$. By (4.24) we have that

$$D_1 = x\frac{\partial}{\partial x} + \lambda z\frac{\partial}{\partial z}; \quad D_2 = y\frac{\partial}{\partial y} + \mu z\frac{\partial}{\partial z}. \tag{4.25}$$

In view of (4.12), then $\hat{\mathcal{G}}_P$ is generated by

$$\Omega = xyz\left(-\lambda\frac{dx}{x} - \mu\frac{dy}{y} + \frac{dz}{z}\right). \tag{4.26}$$

Note that the axis are permissible centers. Blowing-up the axis in an adequate way and making arguments like in the Lemma 2.4, we see that if

$$\lambda, \mu, -\lambda/\mu \notin \mathbf{Q}_+ \tag{4.27}$$

does not hold, then $(\mathcal{F}, E)$ is a dicritical foliation. (The fact that we work with formal coordinates does not worry, in view of the faithfully flatness of $\hat{\mathcal{O}}_{X,P}$ over $\mathcal{O}_{X,P}$.) Hence, we have the normal form (iv).

(B) $\lambda\mu \neq 0$; $\lambda \notin \mathbf{Q}_- \cup \mathbf{N}_+$, $\mu = m \in \mathbf{N}_+$. We can reason like in (A), but (4.27) does not hold and hence we have a dicritical case.

(C) $\lambda\mu \neq 0$; $\lambda \notin \mathbf{Q}_- \cup \mathbf{N}_+$, $\mu = -p/q \in \mathbf{Q}_-$, g.c.d.$(p, q) = 1$. We obtain the normal form (iv) like in (A).

(D) $\lambda\mu \neq 0$; $\lambda = p$, $\mu = q \in \mathbf{N}_+$. By (4.24), we have that

$$D_1 = x\frac{\partial}{\partial x} + (pz + \alpha x^p y^q)\frac{\partial}{\partial z}; \quad D_2 = y\frac{\partial}{\partial y} + (qz + \beta x^p y^q)\frac{\partial}{\partial z}. \tag{4.28}$$

If $\alpha=\beta=0$, we obtain a dicritical case as before. If $\alpha \neq \beta = 0$, blowing-up the axis $(y=z=0)$ repeatedly we see that is is a dicritical case. Thus $\alpha\beta \neq 0$. After a coordinate change $x \hookrightarrow \alpha^{1/p}x$ we may assume that $\alpha=1$. If $\beta \in \mathbf{Q}_-$, blowing-up repeatedly $(x=y=0)$ we obtain a situation like (4.28) but with $\alpha\beta=0$, which is a dicritical case. Hence we have the normal form (ix)*.

(E) $\lambda\mu \neq 0$; $\lambda = -p/q \in \mathbf{Q}_-$, $\mu = m \in \mathbf{N}_+$. We have a dicritical case like in (B).

(F) $\lambda\mu \neq 0$; $\lambda = -p/r \in \mathbf{Q}_-$, $\mu = -q/r \in \mathbf{Q}_-$, q.c.d.$(p,q,r)=1$. By (4.24) there are two formal series $\alpha(t), \beta(t) \in \mathbf{C}[[t]]$, with $\nu(\alpha(t), \beta(t)) \geqslant 1$, such that

$$D_1 = x\frac{\partial}{\partial x} - \frac{p}{r}z\frac{\partial}{\partial z} - \alpha(x^p y^q z^r)\, z\frac{\partial}{\partial z}; \quad D_2 = y\frac{\partial}{\partial y} - \frac{q}{r}z\frac{\partial}{\partial z} - \beta(x^p y^q z^r)\, z\frac{\partial}{\partial z}. \tag{4.29}$$

Thus $\mathscr{G}_P$ is generated by

$$\Omega = xyz\left(r\alpha(x^p y^q z^r)\frac{dx}{x} + r\beta(x^p y^q z^r)\frac{dy}{y} + p\frac{dx}{x} + q\frac{dy}{y} + r\frac{dz}{z}\right). \tag{4.30}$$

If $(\alpha(t), \beta(t)) = (0,0)$, then we obtain the normal form (iv). Assume that $(\alpha(t), \beta(t)) \neq (0,0)$. Put $u = x^p y^q z^r$, then the 1-form

$$\Omega' = x^p y^q z^{r-1}\Omega = xyu\left(r\alpha(u)\frac{dx}{x} + r\beta(u)\frac{dy}{y} + \frac{du}{u}\right) \tag{4.31}$$

is integrable and (x, y, u) is of maximal rank at the generic points. Hence

$$\frac{d\alpha(t)}{dt}\beta(t) = \alpha(t)\frac{d\beta(t)}{dt}. \tag{4.32}$$

Thus, we can write

$$\alpha(t) = (a/r)\cdot\gamma(t); \quad \beta(t) = (\beta/r)\cdot\gamma(t); \quad a, \beta \in \mathbf{C}. \tag{4.33}$$

Put $s = \nu(\gamma(t))$. Note that $1 \leqslant s < \infty$. Then there is a unit $v(t) \in \mathbf{C}[[t]]$ such that $\gamma(t) = t^s \cdot v(t)$. Let $w(t) \in \mathbf{C}[[t]]$ be such that $v(t)w(t) = 1$. Then $\mathscr{G}_P$ is generated by

(4.34)

$$\Omega'' = w(x^p y^q z^r)\,\Omega = xyz(x^p y^q z^r)^s\left(a\frac{dx}{x} + \beta\frac{dy}{y} + \frac{w(x^p y^q z^r)}{(x^p y^q z^r)^s}\left(p\frac{dx}{x} + q\frac{dy}{y} + r\frac{dz}{z}\right)\right).$$

Consider a change of coordinates $u'=u'(u)$ such that

$$\frac{w(u)}{u^s}\cdot\frac{du}{u}=\left(\varepsilon+\frac{1}{u'^s}\right)\frac{du'}{u'}, \tag{4.35}$$

for some $\varepsilon\in\mathbf{C}$. (See also (2.28).) Take $z'=z'(z)$ such that

$$u'(x^py^qz^r)=x^py^qz'^r. \tag{4.36}$$

Multiplying Ω'' by a unit, then $\hat{\mathscr{G}}_P$ is generated by

$$xyz'(x^py^qz'^r)^s\left(\alpha\frac{dx}{x}+\beta\frac{dy}{y}+\left(\varepsilon+\frac{1}{(x^py^qz'^r)^s}\right)\left(p\frac{dx}{x}+q\frac{dy}{y}+r\frac{dz'}{z'}\right)\right). \tag{4.37}$$

Let us assume that $\alpha\neq 0$. Then, multiplying x by $(1/\alpha)^{1/p}$ and the above generator by $(1/\alpha)^{(1+1/p)}$, we have the normal form (vii).

(G) $\lambda=0$, $\mu\neq 0$, $\mu\notin\mathbf{Q}-\mathbf{N}_+$. We have (4.25) like in A, but since $\lambda=0$, then $\nu(\mathscr{G},\varnothing;P)=1$ and this case does not hold.

(H) $\lambda=0$, $\mu\neq 0$, $\mu=m\in\mathbf{N}_+$. By (4.24) we have that

$$D_1=x\frac{\partial}{\partial x}-\alpha y^m\frac{\partial}{\partial y};\quad D_2=y\frac{\partial}{\partial y}+(mz+\beta y^m)\frac{\partial}{\partial z}. \tag{4.38}$$

Then $\hat{\mathscr{G}}_P$ is generated by

$$\Omega=xy\left(\alpha y^m\frac{dx}{x}-(mz+\beta y^m)\frac{dy}{y}+dz\right). \tag{4.39}$$

Note that $\alpha\neq 0$, since otherwise $\nu(\mathscr{G},\varnothing;P)=1$. If $\beta=0$, blowing-up repeatedly $(y=z=0)$ we obtain a dicritical component. Hence $\alpha\beta\neq 0$. We may assume that $\beta=1$ by a coordinate change $y\hookrightarrow\beta^{1/m}y$. We have thus the normal form (viii)*.

(I) $\lambda=0$, $\mu=-p/q\in\mathbf{Q}_-$, g.c.d.$(p,q)=1$. By (4.24), there are two formal series $\alpha(t),\beta(t)\in\mathbf{C}[[t]]$, with $\nu(\alpha(t),\beta(t))\geq 1$, such that

$$D_1=x\frac{\partial}{\partial x}-\alpha(y^pz^q)z\frac{\partial}{\partial z};\quad D_2=y\frac{\partial}{\partial y}-\beta(y^pz^q)z\frac{\partial}{\partial z}. \tag{4.40}$$

Reasoning as in (F), we may assume that $\hat{\mathscr{G}}_P$ is generated by

$$\Omega=xyz(y^pz^q)^s\left(\alpha\frac{dx}{x}+\beta\frac{dy}{y}+\left(\varepsilon+\frac{1}{(y^pz^q)^s}\right)\left(p\frac{dy}{y}+q\frac{dz}{z}\right)\right). \tag{4.41}$$

Note that $\alpha\neq 0$, otherwise $\nu(\mathscr{G},\varnothing;P)=1$. Now, dividing Ω by α and making the coordinate change $y\hookrightarrow(1/\alpha)^{1/ps}y$, we obtain the normal form (vi).

(J) $\lambda=\mu=0$. Then, there are two formal series $\alpha(t), \beta(t) \in \mathbf{C}[[t]]$, with $\nu(\alpha(t), \beta(t)) \geq 1$, such that

$$D_1 = x\frac{\partial}{\partial x} - \alpha(z)\, z\frac{\partial}{\partial z}; \quad D_2 = y\frac{\partial}{\partial y} - \beta(z)\, z\frac{\partial}{\partial z}. \tag{4.42}$$

Then, reasoning as in (F), (I), we obtain the normal form (v). □

Remark 4.5. In the cases (viii)* and (ix)*, we have that

$$\hat{J}_{E,P} = xy\hat{\mathcal{O}}_{X,P}. \tag{4.43}$$

In particular $e(E,P)=2$. This follows since both D_1 and D_2 are not tangent to $(z=0)$, and hence $(z=0)$ is not a component of E.

COROLLARY 4.6. *Let P be a pre-simple singularity for $(\mathcal{F}, E)$ such that $\nu(\mathcal{G}, \emptyset, P)=2$. Then there is a regular system of parameters (x,y,z) of $\mathcal{O}_{X,P}$ such that:*

(a) $(xy=0) \subset E \subset (xyz=0)$, *locally at P.*

(b) *The singular locus* $\mathrm{Sing}(\mathcal{G}, \emptyset)$ *is locally given at P by*

$$\mathrm{Sing}(\mathcal{G}, \emptyset) = (x=y=0) \cup (x=z=0) \cup (y=z=0) \tag{4.44}$$

if we have one of the normal forms (iv), (v), (vii), *or* (ix)*, *and by*

$$\mathrm{Sing}(\mathcal{G}, \emptyset) = (x=y=0) \cup (y=z=0) \tag{4.45}$$

if they have the normal form (viii)*.

Proof. Let us fix a regular system of parameters (x', y', z') of $\hat{\mathcal{O}}_{X,P}$. Assume that

$$\Omega = a\, dx' + b\, dy' + c\, dz' \tag{4.46}$$

is a generator of $\hat{\mathcal{G}}_P$. Let $J \subset \mathcal{O}_{X,P}$ be the ideal defining $\mathrm{Sing}(\mathcal{G}, \emptyset)$ locally at P. Then we have that

$$J\hat{\mathcal{O}}_{X,P} = \mathrm{rad}((a,b,c)\, \hat{\mathcal{O}}_{X,P}), \tag{4.47}$$

where $\mathrm{rad}(\cdot)$ means the radical of the ideal. Assume now that (x', y', z') are given like in Proposition 4.4. Then we have (a) and (b) for (x', y', z') (up to irrelevant reordering). Hence $\mathrm{Sing}(\mathcal{G}, \emptyset)$ is locally given at P by three (or two) irreducible curves $\Gamma_1, \Gamma_2, \Gamma_3$ (or Γ_1, Γ_2). Moreover, if $e(E,P)=3$, then the Γ_i, $i=1,2,3$, are the intersections of two irreducible components of E at P. If $e(E,P)=2$, then Γ_1 is the intersection of the two irreducible components of E at P; the other ones Γ_2 (and eventually) Γ_3, are contained each one in a different component of E and they are transversal to the other one. The corollary follows straigthforward. □

PROPOSITION 4.7. *Let P be a pre-simple singularity for $(\mathcal{F},E)$ such that $\nu(\mathcal{G},\varnothing;P)=2$. Then the following statements are equivalent:*

(A) *There is an open set $\mathcal{U}\subset X, P\in\mathcal{U}$, such that if $Q\in\mathcal{U}\cap\mathrm{Sing}(\mathcal{G},\varnothing)$, with $Q\neq P$, then Q is a simple singularity for $(\mathcal{F},E)$ and $\nu(\mathcal{G},\varnothing;Q)=1$.*

(B) *In the situation of the Proposition* 4.4 *we have one of the normal forms* (iv), (v), (vi), *or* (vii).

Proof. (A)⇒(B). Assume that (B) is not true. Then we have either the normal form (viii)* or (ix)*. In both cases we have that

$$e(E,P)=2 \quad \text{and} \quad \nu(\mathcal{F},E;P)=0. \tag{4.48}$$

Moreover, we can take a regular system of parameters (x,y,z) of $\mathcal{O}_{X,P}$ satisfying the conditions in Lemma 4.2 and in Corollary 4.6 (see Remark 4.3). In particular, we have that

$$E=(xy=0), \quad \text{locally at } P. \tag{4.49}$$

$$\omega=a\frac{dx}{x}+b\frac{dy}{y}+dz \text{ generates } \mathcal{F}_P, \quad \nu_P(a,b)\geq 1. \tag{4.50}$$

$$(y=z=0)\subset \mathrm{Sing}(\mathcal{G},\varnothing). \tag{4.51}$$

$$-\frac{\partial a}{\partial z}(0,0,0)=p\in\mathbf{N}; \quad -\frac{\partial b}{\partial z}(0,0,0)=q\in\mathbf{N}_{+}. \tag{4.52}$$

Now, put $\Delta(\varepsilon)=(x=\varepsilon)$, for ε "near" $0\in\mathbf{C}$. Then $\mathcal{F}|_{\Delta(\varepsilon)}$ is locally generated at $(\varepsilon,0,0)$ by

$$\omega|_{\Delta(\varepsilon)}=b(\varepsilon,0,0)(dy/y)+dz. \tag{4.53}$$

In view of Remark 3.4 (b), the value

$$\frac{\partial b(\varepsilon,0,0)}{\partial z} \tag{4.54}$$

does not depend on ε. Hence, by continuity

$$\frac{\partial b(\varepsilon,0,0)}{\partial z}=\frac{\partial b(0,0,0)}{\partial z}=-q\in\mathbf{N}_{-} \tag{4.55}$$

and we have not a simple singularity. This is the desired contradiction.

(B)⇒(A). Assume first that $e(E,P)=3$. We can take a regular system of parameters (x,y,z) of $\mathcal{O}_{X,P}$ such that

$$E=(xyz=0), \quad \text{locally at } P. \tag{4.56}$$

$$\omega = a\frac{dx}{x}+b\frac{dy}{y}+c\frac{dz}{z} \quad \text{generates } \mathcal{F}_P. \tag{4.57}$$

The vector fields D_1, D_2 of Lemma 4.2 are

$$D_1 = x\frac{\partial}{\partial x}-az\frac{\partial}{\partial z}; \quad D_2 = y\frac{\partial}{\partial y}-bz\frac{\partial}{\partial z} \tag{4.58}$$

and the corresponding eigenvalues λ,μ of (4.20) are given by

$$\lambda = -a(0,0,0); \quad \mu = -b(0,0,0). \tag{4.59}$$

Since neither (viii)* nor (ix)* appear, looking at the proof of Proposition 4.4, we have that

$$\lambda \notin \mathbf{Q}_+ \quad \text{and} \quad \mu \notin \mathbf{Q}_+. \tag{4.60}$$

Note that $\operatorname{Sing}(\mathcal{G},\varnothing)=(x=y=0)\cup(x=z=0)\cup(y=z=0)$. Take a point $P_\varepsilon=(\varepsilon,0,0)$ of $(y=z=0)$. Put $\Delta(\varepsilon)=(x=\varepsilon)$, as above, then $\mathcal{F}|_{\Delta(\varepsilon)}$ is generated by

$$\omega|_{\Delta(\varepsilon)} = b(\varepsilon,0,0)\frac{dy}{y}+\frac{dz}{z}. \tag{4.61}$$

By the Remark 3.4(b), the value $b(\varepsilon,0,0)$ does not depend on ε and thus

$$-b(\varepsilon,0,0) = -b(0,0,0) = \mu \notin \mathbf{Q}_+. \tag{4.62}$$

This implies that P_ε is a simple singularity. We do analogously for the points in $(x=z=0)$. It remains to look at the points in $(x=y=0)$. Note that the integrability condition $\omega\wedge d\omega=0$ implies that

$$b(0,0,z)\frac{\partial a}{\partial z}(0,0,z) = a(0,0,z)\frac{\partial b}{\partial z}(0,0,z). \tag{4.63}$$

Let us distinguish the following cases:

(1) $\lambda\mu\neq 0$. Then, by the proof of Proposition 4.4 we have that

$$-\lambda/\mu \notin \mathbf{Q}_+. \tag{4.64}$$

By (4.63) we have that

$$-a(0,0,\varepsilon)/b(0,0,\varepsilon) = -\lambda/\mu \notin \mathbf{Q}_+ \tag{4.65}$$

for all ε near $0 \in \mathbf{C}$. Now, cutting by $\Delta(\varepsilon)=(z=\varepsilon)$ as above, we obtain a simple singularity.

(2) $\lambda=0, \mu\neq 0$. By (4.63) we have $a(0,0,\varepsilon)=0$ for all ε. We obtain a simple singularity as above.

(3) $\lambda=\mu=0$. Note that

$$(a(0,0,z), b(0,0,z)) \neq (0,0). \tag{4.66}$$

Otherwise, blowing-up the center $(x=y=0)$ we have a dicritical component. By (4.63), there is a series $\gamma(z)$, with $\gamma(0)\neq 0$ such that

$$a(0,0,z) = \gamma(z)\cdot z^s; \quad b(0,0,z) = \beta\cdot\gamma(z)\cdot z^s, \quad \beta \in \mathbf{C}; \ s \geqslant 1. \tag{4.67}$$

Cutting by $\Delta(\varepsilon)=(z=\varepsilon)$, we have that

$$\omega|_{\Delta(\varepsilon)} = a(x,y,\varepsilon)\frac{dx}{x} + b(x,y,\varepsilon)\frac{dy}{y} \tag{4.68}$$

and

$$b(0,0,\varepsilon)/a(0,0,\varepsilon) = \beta. \tag{4.69}$$

Necessarily $-\beta \notin \mathbf{Q}_+$. Otherwise, blowing-up $\omega|_{\Delta(\varepsilon)}$ of (4.68) we obtain a dicritical component after finitely many steps. Hence we have a simple singularity. (Remark: the invariant β in (4.69) is the same one as in the normal form (v); in particular $\beta\neq 0$.)

Assume now that $e(E,P)=2$ and $\nu(\mathcal{F},E;P)=0$. Take a regular system of parameters (x,y,z) of $\mathcal{O}_{X,P}$ satisfying the conditions in Lemma 4.2 and in Corollary 4.6. In particular (4.49) and (4.50) hold and

$$\operatorname{Sing}(\mathcal{G},\varnothing) = (x=y=0)\cup(x=z=0)\cup(y=z=0), \quad \text{locally at } P. \tag{4.70}$$

$$-\frac{\partial a}{\partial z}(0,0,0) = \lambda; \quad -\frac{\partial b}{\partial z}(0,0,0) = \mu; \quad \lambda,\mu \notin \mathbf{Q}_+. \tag{4.71}$$

(For the proof of the property (4.71), we do like in (4.60).) Now, reasoning like in the preceding cases we see that all the points in $\operatorname{Sing}(\mathcal{G},\varnothing)-\{P\}$, near P, are simple singularities.

It remains to study the case $e(E,P)=2$ and $\nu(\mathcal{F},E;P)=1$. Take a regular system of parameters (x, y, z) of $\mathcal{O}_{X,P}$ and a generator ω of $\mathcal{F}_P$ such that the following conditions hold:

$$E=(xy=0), \quad \text{locally at } P. \tag{4.72}$$

$$\operatorname{Sing}(\mathcal{G}, \varnothing)=(x=y=0)\cup(x=z=0)\cup(y=z=0), \quad \text{locally at } P. \tag{4.73}$$

$$\omega=a\frac{dx}{x}+b\frac{dy}{y}+c\,dz, \quad \nu(a, b, c)\geq 1. \tag{4.74}$$

$$\operatorname{In}^1(a)=\operatorname{In}^1(z). \tag{4.75}$$

$$a=z+\varphi'(x, y), \quad \nu(\varphi'(x, y))\geq 2. \tag{4.76}$$

(Note that (4.76) follows from the Implicit Function Theorem.) By the property (4.73) we deduce that

$$a\in(x, z)\subset\mathcal{O}_{X,P}; \quad b\in(y, z)\subset\mathcal{O}_{X,P}. \tag{4.77}$$

Consider a point $(0, 0, \varepsilon)\in(x{=}y{=}0)$. Cutting by $(z{=}\varepsilon)$ we have that

$$\omega|_{z=\varepsilon}=a(x, y, \varepsilon)\frac{dx}{x}+b(x, y, \varepsilon)\frac{dy}{y}. \tag{4.78}$$

The quotient

$$\frac{b(0,0,\varepsilon)}{a(0,0,\varepsilon)}=\frac{b(0,0,\varepsilon)}{\varepsilon} \tag{4.79}$$

does not depend on the value ε "near" $0\in\mathbf{C}$. Reasoning like in (4.69), then

$$-\frac{b(0,0,\varepsilon)}{\varepsilon}\notin\mathbf{Q}_+. \tag{4.80}$$

Hence we have a simple singularity. By the way, note that

$$-\frac{b(0,0,\varepsilon)}{\varepsilon}=\lim_{\varepsilon\to 0}\frac{b(0,0,\varepsilon)}{\varepsilon}=-\frac{\partial b}{\partial z}(0,0,0). \tag{4.81}$$

Consider now a point $(0, \varepsilon, 0)\in(x{=}z{=}0)$. Then

$$\omega|_{y=\varepsilon}=a(x, \varepsilon, z)\frac{dx}{x}+c(x, \varepsilon, z)\,dz. \tag{4.82}$$

The quotient

$$\frac{c(0,\varepsilon,0)}{\dfrac{\partial a}{\partial z}(0,\varepsilon,0)}=\frac{c(0,\varepsilon,0)}{1} \tag{4.83}$$

does not depend on ε near $0 \in \mathbf{C}$. Hence

$$c(0, \varepsilon, 0) = \lim_{\varepsilon \to 0} c(0, \varepsilon, 0) = c(0,0,0) = 0 \notin \mathbf{Q}_-. \tag{4.84}$$

Thus we have a simple singularity.

Finally, let us look at the points $(\varepsilon, 0, 0) \in (y=z=0)$. We have that

$$\omega|_{x=\varepsilon} = b(\varepsilon, y, z)\frac{dy}{y} + c(\varepsilon, 0, 0)\, dz. \tag{4.85}$$

By Corollary 3.3 (b) and since being a pre-simple singularity is an open condition (see Proposition I.2.6), we have that

$$\left(\frac{\partial b}{\partial z}(\varepsilon, 0, 0), c(\varepsilon, 0, 0)\right) \neq (0,0), \quad \text{for} \quad \varepsilon \neq 0. \tag{4.86}$$

Now, the only "bad case" in which $(\varepsilon, 0, 0)$ is not a simple singularity is

$$\frac{\dfrac{\partial b}{\partial z}(x,0,0)}{c(x,0,0)} = \frac{-p}{q} \in \mathbf{Q}_-. \tag{4.87}$$

Let us reason by contradiction assuming that (4.87) holds. Then

$$\frac{\partial b}{\partial z}(x,0,0) = px^s\alpha(x); \quad c(x,0,0) = -qx^s\alpha(x); \quad \alpha(0) \neq 0, \ s \geqslant 1. \tag{4.88}$$

Making $y=z=0$ in the integrability condition $\omega \wedge d\omega = 0$:

$$\left(b\frac{\partial a}{\partial z} - a\frac{\partial b}{\partial z}\right) + \left(a\frac{\partial x}{\partial y} - c\frac{\partial a}{\partial y}\right)y + \left(c\frac{\partial b}{\partial x} - b\frac{\partial c}{\partial x}\right)x = 0 \tag{4.89}$$

and since $b \in (y, z)$, we find that

$$\varphi'(x, 0) = 0. \tag{4.90}$$

Thus, we can write

$$a = z + xy\varphi(x, y). \tag{4.91}$$

Now, by the proof of Lemma 4.1 and by the condition (4.91), we have a formal coordinate change

$$z^\wedge = z + xy\Psi(x, y) \tag{4.92}$$

and a unit $u^\wedge \in \hat{\mathcal{O}}_{X,P}$ such that the following properties hold:

$$\omega^\wedge = u^\wedge \omega = z^\wedge \frac{dx}{x} + z^\wedge b^\wedge \frac{dy}{y} + c^\wedge dz^\wedge. \tag{4.93}$$

$$u^\wedge(0) = 1. \tag{4.94}$$

$$(u^\wedge)^{-1}(z^\wedge b^\wedge) = b. \tag{4.95}$$

$$(u^\wedge)^{-1} c^\wedge = c. \tag{4.96}$$

We shall end the proof by showing that the following condition holds:

$$\frac{b^\wedge|_{y=z^\wedge=0}}{c^\wedge|_{y=z^\wedge=0}} = \frac{-p}{q}. \tag{4.97}$$

Actually, if (4.97) is true, then writting down

$$\omega^\wedge = z^\wedge \left(\frac{dx}{x} + b^\wedge \frac{dy}{y} + c^\wedge \frac{dz^\wedge}{z^\wedge} \right) \tag{4.98}$$

and blowing-up the center $(y=z^\wedge=0)=(y=z=0)$ we find a dicritical component after finitely many steps. Contradiction. Now put

$$f(x,y,z) = \frac{\partial b}{\partial z}(x,y,z) \in \mathbf{C}\{x,y,z\} = \mathcal{O}_{X,P}. \tag{4.99}$$

By (4.95) we have that

$$f(x,0,0) = f|_{y=z=0} = (u^{\wedge -1}|_{y=z=0}) \cdot b^\wedge|_{y=z=0}. \tag{4.100}$$

(Note that we have (4.92).) Then

$$\frac{b^\wedge|_{y=z^\wedge=0}}{c^\wedge|_{y=z^\wedge=0}} = \frac{b^\wedge|_{y=z=0}}{c^\wedge|_{y=z=0}} = \frac{u^\wedge|_{y=z=0}}{u^\wedge{}_{y=z=0}} \cdot \frac{f(x,0,0)}{c(x,0,0)} = \frac{-p}{q}. \tag{4.101}$$

Thus the property (4.97) holds. □

DEFINITION 4.8. *Let P be a pre-simple singularity for $(\mathcal{F},E)$ such that $\nu(\mathcal{G},\varnothing;P)=2$. We say that P is a simple singularity for $(\mathcal{F},E)$ iff the equivalent conditions* (A) *and* (B) *of Proposition* 4.7 *hold.*

Remark 4.9. The normal forms (i), (ii) and (iii) of Proposition 2.5 and the normal forms (iv), (v), (vi) and (vii) of Proposition 4.7 describe exactly the simple singularities in the case $\dim X=3$. The converse is also true: a singularity which can be written in one of the formal normal forms (i)–(vii) is a simple singularity (with evident conditions about the divisor E).

PROPOSITION 4.10. *Assume that each point in* $\mathrm{Sing}(\mathcal{G},\emptyset)$ *is a simple singularity for* $(\mathcal{F},E)$. *Let* $Y\subset X$ *be a permissible center for* $(\mathcal{F},E)$; *let* $\pi: X'\to X$ *be the blowing-up with center* Y *and let* $(\mathcal{F}',E')$ *be the adapted strict transform of* $(\mathcal{F},E)$ *by* π. *Put* $(\mathcal{G},\emptyset)=\mathrm{hol}((\mathcal{F},E))$. *Then each point in* $\mathrm{Sing}(\mathcal{G}',\emptyset)$ *is also a simple singularity for* $(\mathcal{F}',E)$.

Proof. In view of the non-dicriticalness, after blowing-up one of the normal forms (i)–(vii) with center either the origin, or one of the axis in the singular locus, we only get singular points which admit either (i), (ii), (iii), (iv), (v), (vi), or (vii) as normal forms. □

§ 5. Separatrices for simple singularities

Assume that $\dim X=3$. Take a non-dicritical singular foliation $(\mathcal{F},E)\in\mathfrak{F}(X,E)$. Put $(\mathcal{G},\emptyset)=\mathrm{hol}((\mathcal{F},E))$. Let us fix a point $P\in\mathrm{Sing}(\mathcal{G},\emptyset)$ which is a simple singularity for $(\mathcal{F},E)$.

This paragraph is devoted to the description of the convergent and formal separatrices of $(\mathcal{G},\emptyset)$ at P.

PROPOSITION 5.1. *There is a regular system of parameters* (x,y,z) *of* $\hat{\mathcal{O}}_{X,P}$ *such that the following properties hold:*

(a) *If* $\nu(\mathcal{G},\emptyset;P)=1$, *then* $xy\in\hat{J}_{E,P}$ *and the formal separatrices of* $(\mathcal{G},\emptyset)$ *at* P *are exactly* $x\hat{\mathcal{O}}_{X,P}$ *and* $y\hat{\mathcal{O}}_{X,P}$.

(b) *If* $\nu(\mathcal{G},\emptyset;P)=2$, *then* $xyz\in\hat{J}_{E,P}$ *and the formal separatrices of* $(\mathcal{G},\emptyset)$ *at* P *are exactly* $x\hat{\mathcal{O}}_{X,P}$, $y\hat{\mathcal{O}}_{X,P}$ *and* $z\hat{\mathcal{O}}_{X,P}$.

Proof. Consider (x,y,z) giving the corresponding normal form (i)–(vii). We have obviously (a) and (b), except may be for the fact that the "coordinate" separatrices are the only ones. Let Ω be a generator of $\hat{\mathcal{G}}_P$ like in Proposition 2.5 or in Proposition 4.4. Let us distinguish two cases:

(1) $\nu(\mathcal{G},\emptyset;P)=1$ (i.e., we have one of the normal forms (i), (ii) or (iii)). It is a two-dimensional result which follows from Remark 2.3 (d).

(2) $\nu(\mathcal{G},\emptyset;P)=2$ (i.e., we have one of the normal forms (iv), (v), (vi) or (vii)).

Looking at the normal forms, the differential forms

(5.1) $$\Omega|_{x+\lambda y=0}, \quad \lambda \neq 0,$$

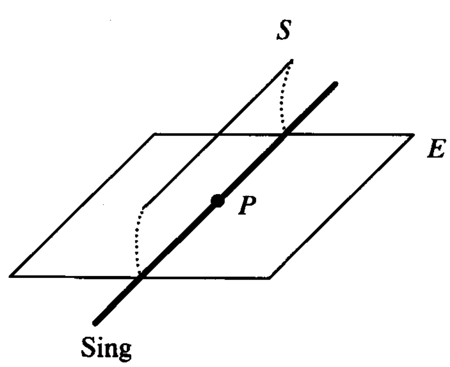
(5.2) $$\Omega|_{y+\mu z=0}, \quad \mu \neq 0,$$

define simple singularities (in dimension two). The only separatrices of (5.1), resp. (5.2), are the intersections of $x+\lambda y=0$, resp. $y+\mu z=0$, with the irreducible components $(x=0)$, $(y=0)$ and $(z=0)$ of E. If $f\hat{\mathcal{O}}_{X,P}$ is a formal separatrix of $\mathcal{G}$ at P, other than the coordinate ones, then it must define a "non-coordinate" separatrix either for (5.1) or (5.2). Contradiction. □

Remark 5.2. The irreducible components of E are convergent separatrices. We have at most one remaining separatrix S which is may be a formal one. The following pictures illustrate the possible cases.

(a) $\nu(\mathcal{G},\varnothing;P)=1$. We have two cases: Figure 13 and Figure 14.

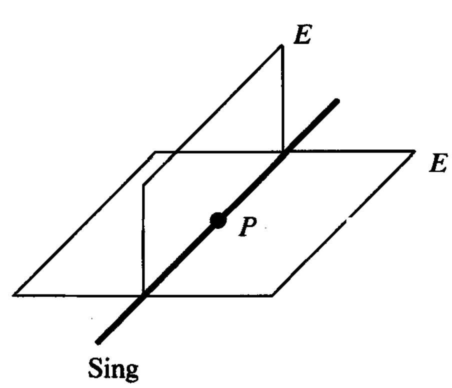
(5.3)



$e(E,P)=1.$

Fig. 13

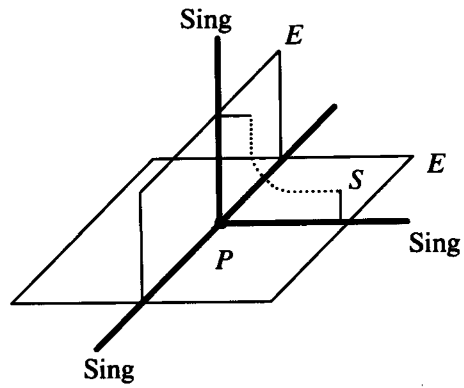
(5.4)



$e(E,P)=2.$

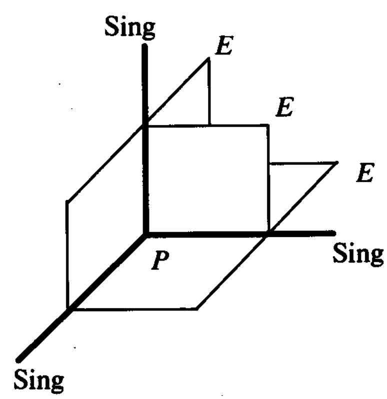
Fig. 14

(b) $\nu(\mathcal{G},\varnothing;P)=2$. We have two cases: Figure 15 and Figure 16.

(5.5) 

$e(E,P)=2.$

Fig. 15

(5.6) 

$e(E,P)=3.$

Fig. 16

The rest of this paragraph is devoted to describe some properties about the convergency of the separatrix S in the most complicated case (5.5) that $\nu(\mathcal{G},\varnothing;P)=2$ and $e(E,P)=2$.

PROPOSITION 5.3. *Assume that* $\nu(\mathcal{G},\varnothing;P)=2$ *and moreover* $\nu(\mathcal{F},E;P)=0$. *Then the three formal separatrices of* $(\mathcal{G},\varnothing)$ *at* P *are in fact convergent ones.*

Proof. There is a regular system of parameters (x,y,z) of $\mathcal{O}_{X,P}$ such that $\mathcal{G}_P$ is generated by

$$\Omega = xy\left(a\frac{dx}{x}+b\frac{dy}{y}+dz\right); \quad \nu_p(a,b)\geq 1 \tag{5.7}$$

and $E=(xy=0)$ locally at P. By Proposition 5.1, there is a formal separatrix of the form

$$f=z-\varphi(x,y), \quad \varphi\in\mathbf{C}[[x,y]]. \tag{5.8}$$

It is enough to prove that φ actually converges. Without loss of generality we can assume that $\nu(\varphi)\geq 2$. Thus, the initial part of Ω is given by

$$xyz\left(\lambda\frac{dx}{x}+\mu\frac{dy}{y}+\frac{dz}{z}\right); \quad \text{where} \quad \lambda=\frac{\partial a}{\partial z}(0,0,0) \text{ and } \mu=\frac{\partial b}{\partial z}(0,0,0). \tag{5.9}$$

Let $\pi: X'\to X$ be the blowing-up with center P. Take a point $P'\in\pi^{-1}(P)$ given by the equations

$$x=x'; \quad y=x'(y'+1); \quad z=x'z'. \tag{5.10}$$

Let $(\mathcal{F}',E')$ be the adapted strict transform of $(\mathcal{F},E)$ by π. Then $\mathcal{F}'_{P'}$ is generated by

$$\omega'=((1+\lambda+\mu)z'+x'\psi)\frac{dx'}{x'}+b'dy'+dz'. \tag{5.11}$$

Because of the non-dicriticalness of $(\mathcal{F},E)$ we have that

$$1+\lambda+\mu\neq 0. \tag{5.12}$$

By Proposition 4.10, then P' is also a simple singularity for $(\mathcal{F}',E')$. Put $(\mathcal{G}',\varnothing)=\mathrm{hol}((\mathcal{F}',E'))$. Since $\nu(\mathcal{G}',\varnothing;P')=1$ (note that $e(E',P')=1$), then $(\mathcal{G}',\varnothing)$ is locally a product along the singular locus

$$\mathrm{Sing}(\mathcal{G}',\varnothing)=(x'=z'=0), \quad \text{locally at } P'. \tag{5.13}$$

Moreover, we also have that

$$1+\lambda+\mu\notin\mathbf{Q}_{-}. \tag{5.14}$$

Consider

$$\varphi'(x',y')=\varphi(x',x'(y'+1))/x'. \tag{5.15}$$

We know that $\varphi(x,y)$ converges iff $\varphi'(x',y')$ do so. Moreover

$$f'=z'-\varphi'(x',y') \tag{5.16}$$

is a formal separatrix for $(\mathscr{G}',\varnothing)$ at P'. Since $(\mathscr{G}',\varnothing)$ is analytically trivial along $(x'=z'=0)$, then f' converges (and hence φ') iff

$$f'|_{y'=0}=z'+\varphi'(x',0)\in \mathbf{C}[[x',z']] \tag{5.17}$$

converges. But $z'-\varphi'(x',0)$ is a separatrix for the differential form

$$(x'\omega')|_{y'=0}=[(1+\lambda+\mu)z'+x'\psi(x',0,z')]\,dx'+x'dz'. \tag{5.18}$$

Now, by (5.12) and (5.14) we can apply Briot–Bouquet's Theorem (cf. [14], p. 295; § 12.6) and hence f' converges. □

Now, let us consider the case $\nu(\mathscr{G},\varnothing;P)=2$, $e(E,P)=2$ and $\nu(\mathscr{F},E;P)=1$. We know that there is a regular system of parameters (x,y,z) of $\mathscr{O}_{X,P}$ such that

$$E=(xy=0),\quad \text{locally at } P. \tag{5.19}$$

$$\mathrm{Sing}(\mathscr{G},\varnothing)=(x=y=0)\cup(x=z=0)\cup(y=z=0). \tag{5.20}$$

Moreover, there is a formal separatrix at P given by

$$f=z+\varphi(x,y);\quad \text{where}\quad \varphi(x,y)\in\mathbf{C}[[x,y]]. \tag{5.21}$$

(Note that actually $\varphi\in xy\cdot\mathbf{C}[[x,y]]$ in view of (5.20).) Let us write

$$\varphi(x,y)=\sum_{i\geqslant 1}\alpha_i(y)\,x^i=\sum_{j\geqslant 1}\beta_j(x)\,y^j \tag{5.22}$$

where $\alpha_i(y)\in\mathbf{C}[[y]]$ and $\beta_j(x)\in\mathbf{C}[[x]]$.

PROPOSITION 5.4. *Assume that we are in the above situation, then $\alpha_i(y)\in\mathbf{C}\{y\}$, for all $i\geqslant 1$, and $\beta_j(x)\in\mathbf{C}\{x\}$, for all $j\geqslant 1$. Moreover, there is a value $\varepsilon>0$ such that $\alpha_i(y)$ converges for $|y|<\varepsilon$ and for all $i\geqslant 1$ and such that $\beta_j(x)$ converges for $|x|<\varepsilon$ and for all $j\geqslant 1$.*

Proof. Up to a reordering of x,y we can take a generator

$$\omega=a\frac{dx}{x}+b\frac{dy}{y}+c\,dz,\quad \text{where}\quad \nu_p(a,b,c)=1, \tag{5.23}$$

of $\mathscr{F}_p$ such that $\mathrm{In}^1(a)=\mathrm{In}^1(z)$. By (4.10) and (4.11) in the proof of the Lemma 4.1, then f

is a separatrix of $(\mathcal{G}, \varnothing)$ at P iff

$$f \text{ divides } a + x\frac{\partial\varphi}{\partial x}\cdot c = a_1. \tag{5.24}$$

Actually, we can write

$$\omega = a_1\frac{dx}{x} + b_1\frac{dy}{y} + c\,df \tag{5.25}$$

and if f divides a_1 then it also divides b_1 and it is a separatrix; the converse is true by the uniqueness property in Proposition 5.1 (b). Now, let us write

$$a = \sum_{i,k} A_{ik}(y)\,x^i z^k. \tag{5.26}$$

$$c = \sum_{i,k} C_{ik}(y)\,x^i z^k. \tag{5.27}$$

Note that $A_{01}(0)=1$. Then there is a common ray of convergence $\varepsilon>0$ for $1/A_{01}(y), A_{ik}(y)$ and $C_{ik}(y)$ for all i, k. Denote by $\mathbf{C}_\varepsilon\{y\}$ the series having a ray of convergence bigger or equal than ε. The condition (5.24) is equivalent to the following equality

(5.28)

$$\sum_{i,k} A_{ik}(y)\,x^i\left(-\sum_{m\geq 1}\alpha_m(y)\,x^m\right)^k + \left(\sum_{m\geq 1} m\alpha_m(y)\,x^m\right)\sum_{i,k} C_{ik}(y)\,x^i\left(-\sum_{m\geq 1}\alpha_m(y)\,x^m\right)^k = 0.$$

The explicit computation of the coefficient of x in (5.28) gives to the following equation

$$-A_{01}(y)\,\alpha_1(y) + A_{10}(y) = 0. \tag{5.29}$$

Hence

$$\alpha_1(y) = A_{10}(y)/A_{01}(y) \in \mathbf{C}_\varepsilon\{y\}. \tag{5.30}$$

Computing the coefficient of x^i in (5.28), we find that

$$-A_{01}(y)\,\alpha_1(y) + \mathcal{R}_i(\alpha_1(y), \ldots, \alpha_{i-1}(y)) = 0 \tag{5.31}$$

where $\mathcal{R}_i \in \mathbf{C}_\varepsilon\{y\}[t_1, \ldots, t_{i-1}]$ is a polynomial. Hence, from (5.31) we find inductively that

$$\alpha_i(y) \in \mathbf{C}_\varepsilon\{y\}, \quad \text{for all } i \geq 1. \tag{5.32}$$

This ends the part of the proof concerning the $\alpha_i(y)$. Now, let us write

$$a=\sum_{j,k} A'_{jk}(x)\,y^j z^k. \tag{5.33}$$

$$c=\sum_{j,k} C'_{jk}(x)\,y^j z^k. \tag{5.34}$$

Note that $A'_{01}(0)=1$. Then, there is a common ray of convergence $\varepsilon'>0$ for $1/A'_{01}(x), A'_{jk}(x)$ and $C'_{jk}(x)$, for all j,k. We can assume without loss of generality that $\varepsilon'=\varepsilon$. The condition (5.24) is equivalent to

(5.35)

$$\sum_{j,k} A'_{jk}(x)\,y^j\left(-\sum_{m\geq 1}\beta_m(x)\,y^m\right)^k+\left(\sum_{m\geq 1} x\beta'_m(x)\,y^m\right)\sum_{j,k} C'_{jk}(x)\,y^j\left(-\sum_{m\geq 1}\beta_m(x)\,y^m\right)^k=0$$

where $\beta'_m(x)$ is the derivative of $\beta_m(x)$. Reasoning as above, we find that

$$-A'_{01}(x)\,\beta_j(x)+\mathcal{R}'_j(\beta_1(x),\ldots,\beta_{j-1}(x),x\beta'_1(x),\ldots,x\beta'_{j-1}(x))=0 \tag{5.36}$$

where $\mathcal{R}'_j\in\mathbf{C}_\varepsilon\{x\}\,[t_1,\ldots,t_{j-1},\mathcal{U}_1,\ldots,\mathcal{U}_{j-1}]$ is a polynomial. Hence we find inductively that

$$\beta_j(x)\in\mathbf{C}_\varepsilon\{x\},\quad \text{for all } j\geq 1. \tag{5.37}$$

This ends the proof of the proposition. □

Assume that we are in the preceding situation. Call $S=(f=0)$ the separatrix at the point P given by (5.21). Put

$$Y_1=(x=z=0);\quad Y_2=(y=z=0). \tag{5.38}$$

Note that a point $Q\in Y_1\cup Y_2, Q\neq P, Q$ near P, is a simple point for $(\mathcal{F},E)$ with $e(E,Q)=1$, and hence with $\nu(\mathcal{G},\varnothing;Q)=1$. Thus there is exactly one formal separatrix S_Q of $\mathcal{G}$ at Q which is different from E. We shall say that Y_1 (respectively Y_2) *supports a convergent separatrix* if S_Q converges for all $Q\in Y_1$ (respectively $Q\in Y_2$), $Q\neq P$, Q near P. By analytic triviality it is enough to test a single point $Q\in Y_1$ (resp. $Q\in Y_2$), $Q\neq P$, Q near P. (See Figure 17.)

PROPOSITION 5.5. *In the above situation, if Y_1 (resp. Y_2) supports a convergent separatrix then S is convergent and hence Y_2 (resp. Y_1) supports a convergent separatrix.*

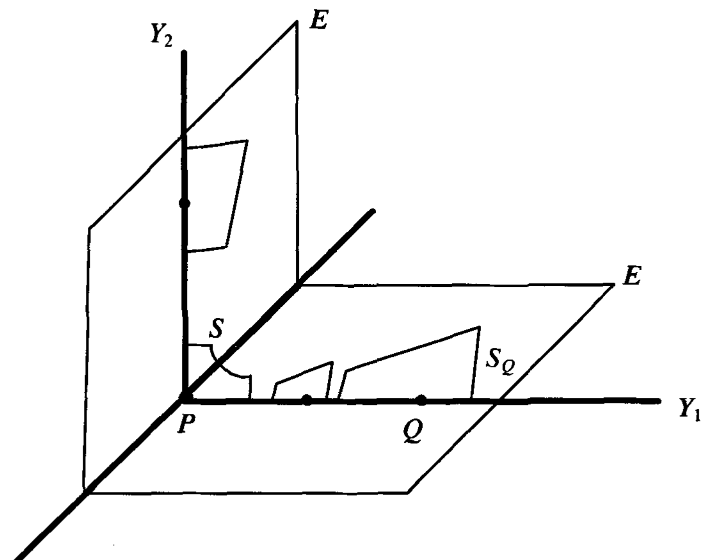


Fig. 17

Proof. Take notations as in the proof of Proposition 5.4. Let $Q=(0,y_0,0)\in Y_1$ be a fixed point of Y_1 with $|y_0|<\varepsilon$. Then

(5.39) $$f_Q(x,y',z)=z+\sum_{i\geq 1}a_i(y'+y_0)x^i$$

is a formal separatrix of $\mathscr{G}$ at Q. Hence it is a convergent one. In particular

(5.40) $$\varphi(x,y_0)=\sum_{i\geq 1}a_i(y_0)x^i\in \mathbf{C}\{x\}.$$

Now, fix $0<\varrho<\varepsilon$. Then for $|y_0|=\varrho$ there is a common ray of convergence $\tau>0$ such that $\varphi(x,y_0)$ converges for $|x|\leq\tau$. Hence we obtain uniform estimates

(5.41) $$|a_i(y_0)|\leq K\cdot(1/\tau)^i,\quad \text{for } |y_0|=\varrho.$$

By the Maximum Modulus Principle, we have that

(5.42) $$|a_i(y_0)|\leq K\cdot(1/\tau)^i,\quad \text{for } |y_0|\leq\varrho.$$

This implies that $\varphi(x,y)$ converges in the polycylinder

(5.43) $$|x|\leq\tau,\quad |y|\leq\varrho.$$

Thus S is convergent. Now, given a point $Q=(x_0,0,0)$ in Y_2, with $|x_0|<<1$, then

$$f_Q(x',y,z)=z+\varphi(x'+x_0,y) \tag{5.44}$$

is a convergent separatrix of $\mathcal{G}$ at Q. This ends the proof. □

Part III. Reduction of the singularities

§ 0. Statement of the results

Let X be a nonsingular connected analytic space over $\mathbf{C}$ of $\dim X=3$. Let $E\subset X$ be a normal crossings divisor of X and let $Z\subset E$ be a compact analytic subspace of E such that each irreducible component of Z is also an irreducible component of E. Let us consider a non-dicritical $(\mathcal{F},E)\in\mathfrak{F}(X,E)$. Put $(\mathcal{G},\varnothing)=\mathrm{hol}((\mathcal{F},E))$. Let us assume that the following two properties hold

$$\mathrm{Sing}(\mathcal{G},\varnothing)\subset E. \tag{0.1}$$

$$\mu(\mathcal{F},E;\{P\})\leqslant 1, \quad \text{for all } P\in X. \tag{0.2}$$

Our goal in this Part III is to give a proof of the following result:

THEOREM 0.1. *In the above situation, there is an open set $X(0)\subset X$, with $Z\subset X(0)$, and a finite sequence of permissible blowing-ups*

$$X(0)\overset{\pi(1)}{\leftarrow}X(1)\overset{\pi(2)}{\leftarrow}\dots\overset{\pi(1)}{\leftarrow}X(N), \tag{0.3}$$

such that each point in $\mathrm{Sing}(\mathcal{G}(N),\varnothing)$ is a simple singularity for $(\mathcal{F}(N),E(N))$, where $(\mathcal{F}(N),E(N))\in\mathfrak{F}(X(N),E(N))$ is the adapted strict transform of $(\mathcal{F}|_{X(0)},E\cap X(0))$ by $\pi(1)\circ\dots\circ\pi(N)$ and $(\mathcal{G}(N),\varnothing)=\mathrm{hol}((\mathcal{F}(N),E(N)))$.

Let us consider the set of singular points over Z which are not pre-simple singularities

$$\mathrm{Sing}^*(\mathcal{F},E;Z)=Z\cap\mathrm{Sing}^*(\mathcal{F},E). \tag{0.4}$$

(Recall I, (2.18)). Given a sequence like (0.3), let us denote

$$Z(N)=[\pi(1)\circ\dots\circ\pi(N)]^{-1}(Z). \tag{0.5}$$

We shall do the proof of the Theorem 0.1 in two steps:

Step A. *The Theorem* 0.1 *is true under the additional assumption*

$$\text{Sing}^*(\mathcal{F}, E; Z) = \varnothing. \tag{0.6}$$

(*That is, all the points in* Sing$(\mathcal{G}, \varnothing)$ *are in fact pre-simple singularities.*)

Step B. *We can get the property* (0.6) *after finitely many permissible blowing-ups.*

The proof of Step A will be given by a computation of "killing resonancies" along the irreducible components of Sing$(\mathcal{G}, \varnothing)$. For the proof of Step B, most of the technics in [7] will remain valid. Hence we shall only give in detail those parts which are either different or may be simplified with respect to the general technics in [7]. The rest will only be sketched.

By Theorem I.1.12, we can get the properties (0.1) and (0.2) after finitely many permissible blowing-ups. This result, jointly with Theorem 0.1 above, allows us to state the following result of desingularization:

THEOREM 0.2 (Desingularization Theorem). *Let X be a nonsingular connected analytic space over* **C** *of* $\dim X=3$, *consider a non-dicritical* $(\mathcal{F},\varnothing)\in\mathfrak{F}(X,\varnothing)$ *and fix a point* $P\in X$. *Then there is an open set* $X(0)\subset X$, *with* $P\in X(0)$, *and a finite sequence of permissible blowing-ups*

$$X(0) \xleftarrow{\pi(1)} X(1) \xleftarrow{\pi(2)} \dots \xleftarrow{\pi(N)} X(N) \tag{0.7}$$

such that each point in Sing$(\mathcal{G}(N),\varnothing)$ *is a simple singularity for* $(\mathcal{F}(N),E(N))$, *where* $(\mathcal{F}(N),E(N))\in\mathfrak{F}(X(N),E(N))$ *is the adapted transform of* $(\mathcal{F}|_{X(0)},\varnothing)$ *by* $\pi(1)\circ\dots\circ\pi(N)$ *and* $(\mathcal{G}(N),\varnothing)=\text{hol}((\mathcal{F}(N),E(N)))$.

§ 1. Proof of Step A

Assume that Sing$^*(\mathcal{F},E;Z)=\varnothing$. Up to consider an open set $X(0)\subset X$, we can assume without loss of generality that Sing$^*(\mathcal{F},E)=\varnothing$ and that Sing$(\mathcal{G},\varnothing)$ has only a finite number of irreducible components.

LEMMA 1.1. *In the above situation, let Y be an irreducible component of* Sing$(\mathcal{G},\varnothing)$. *Then Y has dimension one and it is a permissible center for* $(\mathcal{F},E)$.

Proof. Take a point $P\in Y$. If Y is analytically irreducible at P, then the result comes from Corollary II.3.3 and Corollary II.4.6 by a case by case computation in terms of the normal forms. Now, given two analytic branches Y_1 and Y_2 of $\mathrm{Sing}(\mathcal{G},\varnothing)$ at P, there is an irreducible component F of E such that $Y_1 \subset F$ and $Y_2 \not\subset F$. Since F is a global irreducible component of E, we deduce that Y must be analytically irreducible at P. $\square$

Given an irreducible component Y of $\mathrm{Sing}(\mathcal{G},\varnothing)$, define an invariant $\mathrm{Inv}(\mathcal{F},E;Y)$ as follows:

$$\begin{aligned} &\mathrm{Inv}(\mathcal{F},E;Y)=0, \quad \text{if} \quad \Lambda(\mathcal{F},E;F;Y)\notin \mathbf{Q}_+; \\ &\mathrm{Inv}(\mathcal{F},E;Y)=p+q, \quad \text{if} \quad \Lambda(\mathcal{F},E;F;Y)=p/q \in \mathbf{Q}_+, \end{aligned} \tag{1.1}$$

where F is an irreducible component of E, $Y \subset F$, and the rational number p/q is written in an irreducible way. The above definition is independent of the component F, with $F \supset Y$. Now, put

$$\mathrm{Inv}(\mathcal{F},E)=\max\{\mathrm{Inv}(\mathcal{F},E;Y);\ Y \text{ is an irreducible component of } \mathrm{Sing}(\mathcal{G},\varnothing)\}. \tag{1.2}$$

$$s(\mathcal{F},E)=\#\{Y;\mathrm{Inv}(\mathcal{F},E;Y)=\mathrm{Inv}(\mathcal{F},E)\}. \tag{1.3}$$

Note that $\mathrm{Inv}(\mathcal{F},E)<\infty$. We shall prove Step A by induction on the lexicographical invariant $(\mathrm{Inv}(\mathcal{F},E), s(\mathcal{F},E))$.

If $\mathrm{Inv}(\mathcal{F},E)=0$, we are done, since all the points in $\mathrm{Sing}(\mathcal{G},\varnothing)$ are necessarily simple singularities (see II.(3.31)–(3.32) and Definition II.4.8). Hence the following result completes the proof of Step A.

PROPOSITION 1.1. *Let Y be an irreducible component of* $\mathrm{Sing}(\mathcal{G},\varnothing)$ *such that* $\mathrm{Inv}(\mathcal{F},E;Y)=\mathrm{Inv}(\mathcal{F},E)>0$. *Let $\pi: X' \to X$ be the blowing-up with center Y and let $(\mathcal{F}',E')$ be the adapted strict transform of $(\mathcal{F},E)$ by π. Then*

$$(\mathrm{Inv}(\mathcal{F}',E'), s(\mathcal{F}',E')) < (\mathrm{Inv}(\mathcal{F},E), s(\mathcal{F},E)) \tag{1.4}$$

(*strictly*), *for the lexicographical ordering.*

Proof. It is enough to show that if Y' is an irreducible component of $\mathrm{Sing}(\mathcal{G}',\varnothing)$ with $Y' \subset \pi^{-1}(Y)$, then

$$\mathrm{Inv}(\mathcal{F}',E';Y') < \mathrm{Inv}(\mathcal{F},E;Y). \tag{1.5}$$

Assume that $\mathrm{Inv}(\mathcal{F}',E';Y')=p+q>0$. This implies that $e(E;Y)=1$ (more precisely, by

Proposition II.2.5, the only possible normal form is (iv)* at a generic point of Y). We can distinguish two cases:

(1) $\pi(Y')=Y$. Taking a generic point $P\in Y$, and working eventually with formal coordinates, we have the normal form (iv)* of Proposition II.2.5. Hence $Y=(x=y=0), E=(x=0)$ and $\mathcal{F}_P$ is generated by

$$\omega=(my+x^m)\frac{dx}{x}-dy;\quad m\geqslant 1 \text{ and } m=\operatorname{Inv}(\mathcal{F},E;Y). \tag{1.6}$$

After the blowing-up, at the only point $P'\in\operatorname{Sing}(\mathcal{G}',\varnothing)$, with $\pi(P')=P$, then $\mathcal{F}'_{P'}$ is generated by

$$\omega=((m-1)y'+x'^{m-1})\frac{dx'}{x'}-dy'. \tag{1.7}$$

Hence $\operatorname{Inv}(\mathcal{F}',E';Y')\leqslant m-1<m=\operatorname{Inv}(\mathcal{F},E;Y)$.

(2) $\pi(Y')=\{P\}\subset Y$. Then $\mathcal{D}(\mathcal{F},E)_P$ does not contain a non-singular germ of vector field. Otherwise, computing as above, we contradict the fact $\pi(Y')=\{P\}$. In particular, necessarily $e(E,P)=2$ ($e(E,P)=3$ cannot occur since $e(E,Y)=1$). If E_1,E_2 are the irreducible components of E through P and $Y\subset E_1$, then necessarily

$$Y'=\pi^{-1}(P)=\pi^{-1}(Y)\cap(\text{strict transform of } E_2). \tag{1.8}$$

Hence $e(E',Y')=2$ and thus $\operatorname{Inv}(\mathcal{F}',E';Y')=0$. □

§ 2. Global strategy in the proof of Step B

In this paragraph we shall particularize the ideas and results in [5] and [7] for determining a global algorithm of reduction. In our case, the role of the Samuel stratum $\operatorname{Sam}(\mathcal{F},E;Z)$ in [5] and [7] is played by the set $\operatorname{Sing}^*(\mathcal{F},E;Z)$ defined in (0.4).

General remark 2.1. In the sequel we shall not consider the kind of properties that can be avoided by taking a small enough open set $X(0)\subset X$, with $Z\subset X(0)$. Also note that this remains valid even in an intermediary step $X(m)$ of the sequence (0.3), since if $\mathcal{U}\subset X(m)$ is an open set with $Z(m)\subset\mathcal{U}$, by compactness we find an open set $W\subset X(0), Z\subset W$ such that

$$(\pi(1)\circ\ldots\circ\pi(m))^{-1}(W)\subset\mathcal{U}. \tag{2.1}$$

Hence, we shall in fact work in terms of germs around Z, but without saying it explicitly.

DEFINITION 2.2. *Let Y be an irreducible component of* $\mathrm{Sing}(\mathcal{G},\emptyset)$. *We say that Y is appropriate for $(\mathcal{F},E)$ relatively to Z iff the following two properties hold*:

(a) $Y\cap \mathrm{Sing}^*(\mathcal{F},E;Z)\neq\emptyset$.

(b) $\mu(\mathcal{F},E;Y)=1$.

Moreover, if $\dim Y=1$ *we shall say that Y is an appropriate curve. We shall denote by $S^*(\mathcal{F},E;Z)$ the union of all the appropriate curves. (Hence an appropriate curve is exactly an irreducible component of $S^*(\mathcal{F},E;Z)$.)*

Remark 2.3. If $Y\subset X$ is a permissible center for $(\mathcal{F},E)$ such that $Y\cap \mathrm{Sing}^*(\mathcal{F},E;Z)\neq\emptyset$, then Y is appropriate for $(\mathcal{F},E)$ relatively to Z. To see this, pick a point $P\in Y\cap\mathrm{Sing}^*(\mathcal{F},E;Z)$. Then, we have

$$1=\nu(\mathcal{F},E;P)\leq\varrho(\mathcal{F},E\colon Y,P)=\mu(\mathcal{F},E;Y)\leq\mu(\mathcal{F},E;\{P\})=1 \tag{2.2}$$

and we are done.

LEMMA 2.4. *Let Y be a permissible center for $(\mathcal{F},E)$. Let $\pi\colon X'\to X$ be the blowing-up with center Y. Let $(\mathcal{F}',E')$ be the adapted strict transform of $(\mathcal{F},E)$ by π and put $Z'=\pi^{-1}(Z)$. Then*

$$S^*(\mathcal{F}',E';Z')\subset S^*(\mathcal{F},E;Z)'\cup Y' \tag{2.3}$$

where $S^*(\mathcal{F},E;Z)$ is the strict transform of the adherence of $S^*(\mathcal{F},E;Z)-Y$ by π and Y' is a non singular curve which is contained in the exceptional divisor $\pi^{-1}(Y)$.

Proof. [7], Theorem II.1.9. □

DEFINITION 2.5. *We say that $S^*(\mathcal{F},E;Z)$ has weak normal crossings with E iff the following properties hold*:

(a) *Each irreducible component of $S^*(\mathcal{F},E;Z)$ is nonsingular.*

(b) *$S^*(\mathcal{F},E;Z)\subset E$ (this always holds).*

(c) *Given a point $P\in Z$, there are at most two irreducible components of $S^*(\mathcal{F},E;Z)$ through P.*

(d) *Assume that Y_1* and *Y_2 are two irreducible components of $S^*(\mathcal{F},E;Z)$ and $P\in Y_1\cap Y_2\cap Z$. Then $T_PY_1\neq T_PY_2$ and there are two irreducible components E_1 and E_2 of E such that $Y_i\subset E_i$, $i=1,2$, but $Y_2\not\subset E_1$.*

The pictures in Figure 18 illustrate some situations of weak normal crossings.

THEOREM 2.6. (a) *The property that $S^*(\mathcal{F},E;Z)$ has weak normal crossings is stable under permissible blowing-up.*

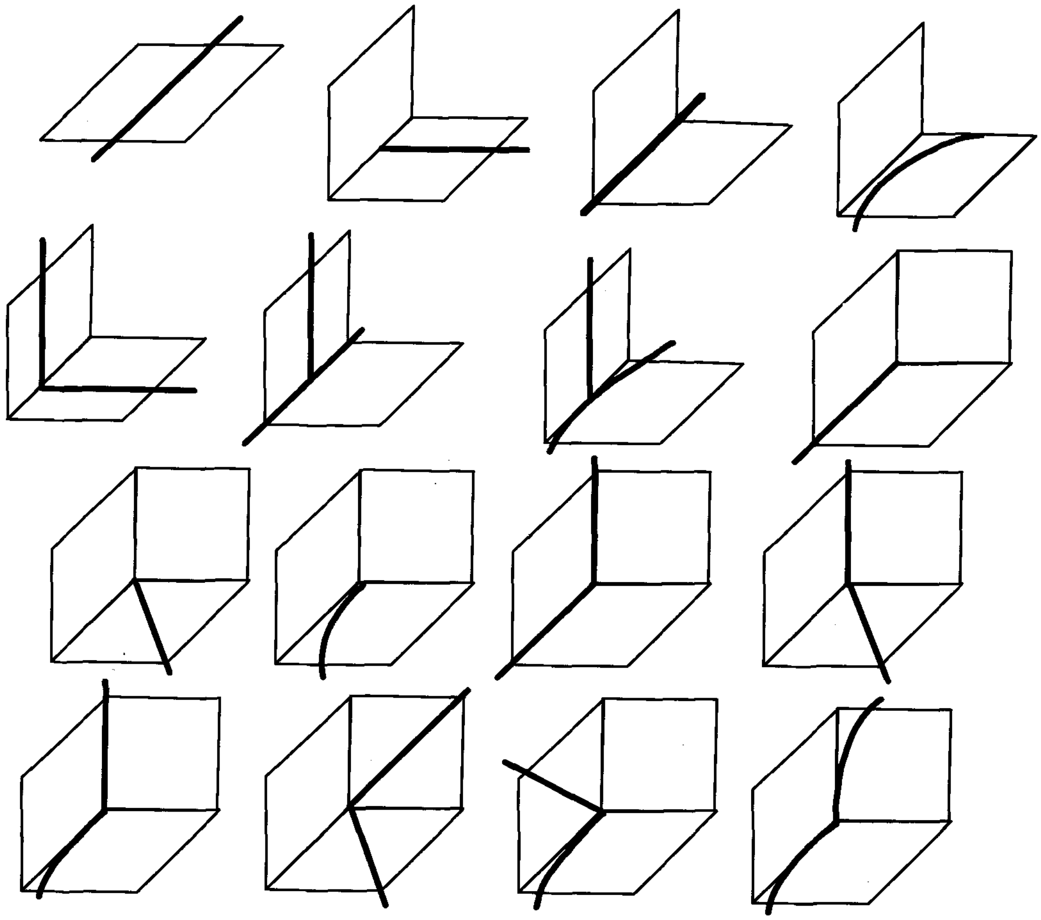


Fig. 18

(b) *After finitely many permissible blowing-ups centered at points we can assume that $S^*(\mathcal{F}, E; Z)$ has weak normal crossings with E.*

Proof. (a) Follows from Lemma 2.4.

(b) It is a consequence of the standard results on desingularization of curves and of Lemma 2.4. (For more details, see [7], Theorem II.2.5.) □

Hence, in the sequel we shall assume that the property

(2.4) "$S^*(\mathcal{F}, E; Z)$ has weak normal crossings with E"

is satisfied, jointly with the properties (0.1) and (0.2).

DEFINITION 2.7. *Let $k \geq 2$ be an integer. A cycle σ of order k for $(\mathcal{F}, E)$ is an application*

$$\begin{aligned} \sigma: \mathbf{Z}/(k) &\to \{\text{appropriate curves}\} \times \mathrm{Sing}^*(\mathcal{F}, E; Z) \\ i &\hookrightarrow \sigma(i) = (Y_i, P_i) \end{aligned} \tag{2.5}$$

satisfying the following properties:

(a) *$Y_i \neq Y_j$, if $i \neq j$.*

(b) *$P_i \in Y_i \cap Y_{i+1}$, for all $i \in \mathbf{Z}/(k)$.*

Denote by $\mathrm{Cycl}^k(\mathcal{F}, E; Z)$ *the set of cycles of order k. The set of cycles* $\mathrm{Cycl}(\mathcal{F}, E; Z)$ *is defined by*

$$\mathrm{Cycl}(\mathcal{F}, E; Z) = \bigcup_{k \geq 2} \mathrm{Cycl}^k(\mathcal{F}, E; Z). \tag{2.6}$$

Remark 2.8. The set of cycles $\mathrm{Cycl}(\mathcal{F}, E; Z)$ is a finite set.

PROPOSITION 2.9. *Let Y be a permissible center for $(\mathcal{F}, E)$. Let $\pi: X' \to X$ be the blowing-up with center Y. Let $(\mathcal{F}', E')$ be the adapted strict transform of $(\mathcal{F}, E)$ by π and put $Z' = \pi^{-1}(Z)$. Then there is an injective application*

$$\phi: \mathrm{Cycl}(\mathcal{F}', E'; Z') \to \mathrm{Cycl}(\mathcal{F}, E; Z). \tag{2.7}$$

Hence $\#\mathrm{Cycl}(\mathcal{F}', E'; Z') \leq \#\mathrm{Cycl}(\mathcal{F}, E; Z)$.

Proof. Any cycle in $\mathrm{Cycl}(\mathcal{F}', E'; Z')$ comes from a cycle in $\mathrm{Cycl}(\mathcal{F}, E; Z)$ by adding eventually the curve Y' of (2.3) instead of the center of π. (For more details, see [7], Proposition II.3.3.) □

As in [7], we shall state an additional assumption, called Hypothesis of Local Control, which will allow us to end the proof of Step B. The Hypothesis of Local Control is the key of the proof of Step B and we shall devote § 3 to show in a detailed way that it always holds.

HYPOTHESIS OF LOCAL CONTROL. *It does not exist an infinite sequence*

$$\{X(i), Z(i), E(i), \mathcal{F}(i), \mathcal{U}(i), Y(i), P(i), \pi(i+1)\}_{i \geq 0} \tag{2.8}$$

such that $(X(0), Z(0), E(0), \mathcal{F}(0)) = (X, Z, E, \mathcal{F})$ and the following properties hold for all $i \geq 0$:

(a) *$\mathcal{U}(i)$ is an open set of $X(i)$, $Y(i)$ is a permissible center for $(\mathcal{F}(i)|_{\mathcal{U}(i)}, E(i) \cap \mathcal{U}(i))$ and $P(i) \in Y(i) \cap \mathcal{U}(i) \cap \mathrm{Sing}^*(\mathcal{F}(i), E(i), Z(i))$.*

(b) $\pi(i+1): X(i+1) \to \mathcal{U}(i)$ *is centered at* $Y(i)$ *and* $(\mathcal{F}(i+1), E(i+1))$ *is the adapted strict transform of* $(\mathcal{F}(i)|_{\mathcal{U}(i)}, E(i) \cap \mathcal{U}(i))$ *by* $\pi(i+1)$.

(c) $\pi(i+1)(P(i+1)) = P(i); Z(i+1) = \pi^{-1}(Z(i) \cap \mathcal{U}(i))$.

(d) *If* $Y(i) = \{P(i)\}$, *then there is no appropriate curve of* $(\mathcal{F}(i), E(i))$ *being permissible at* $P(i)$.

(e) *If* $Y(i) \neq \{P(i)\}$, *then* $Y(i)$ *is an appropriate and permissible curve for* $(\mathcal{F}(i)|_{\mathcal{U}(i)}, E(i) \cap \mathcal{U}(i))$.

Remarks 2.10. (a) Recall that the above statement is made under the assumptions (0.1), (0.2) and (2.4).

(b) The properties (d) and (e) determine what kind of strategy has been followed in order to do the local blowing-up $\pi(i+1)$.

(c) The property (e) is redundant in our context, since if $Y(i) \neq \{P(i)\}$, then $Y(i)$ must to be a curve, $Y(i) \ni P(i)$, which is permissible and hence appropriate by Remark 2.2. We do not eliminate this property (e) in order to keep the parallelism with the corresponding statement in [7], II.(2).

THEOREM 2.11. *Assume that the Hypothesis of Local Control holds. Then, after finitely many permissible blowing-ups we may assume that*

$$\mathrm{Cycl}(\mathcal{F}, E; Z) = \emptyset. \tag{2.9}$$

Proof. By Proposition 2.9, it is enough to destroy a single cycle σ. Put $\sigma(1) = (Y, P), \sigma(2) = (T, Q)$. Now, blow-up repeatedly with the priorities:

(I) If Y is permissible, blow-up Y.

(II) If T is permissible, blow-up T.

(III) If Y is permissible at P, but not at another point, blow-up this point.

(IV) If T is permissible at P, but not at another point, blow-up this point.

(V) Blow-up P.

Then, looking over P, in the direction pointed by Y, we see that if σ is not destroyed, we can produce a sequence like (2.8). This contradicts the Hypothesis of Local Control. (For more details, see [7], Theorem II.3.5.) □

Hence, let us assume in the sequel that (2.9) holds.

DEFINITION 2.12 (Global criteria of blowing-up). *Let* Y *be a permissible center for* $(\mathcal{F}, E)$. *Let* $\pi: X' \to X$ *be the blowing-up with center* Y. *We say that* π *follows the global criteria of blowing-up for* $(\mathcal{F}, E)$ *relatively to* Z *iff the choice of the center* Y *satisfies the following priorities*:

(I) *If there is a point* $Q \in Z - \mathrm{Sing}^*(\mathcal{F}, E; Z)$ *and an appropriate curve* Γ *which is not permissible at* Q, *then* $Y = \{Q\}$ *for one of such points* Q.

(II) *The case* (I) *does not occur and there is an appropriate curve* Γ *which is globally permissible. Then* $Y = \Gamma$ *for one of such* Γ.

(III) *The cases* (I), (II) *do not occur. There is a point* $P \in \mathrm{Sing}^*(\mathcal{F}, E; Z)$ *such that there is exactly one appropriate curve* Γ *passing through* P. *Moreover,* Γ *is not permissible at* P. *Then* $Y = \{P\}$ *for one of such* P.

(IV) *The cases* (I), (II), (III) *do not occur. There is a point* $P \in \mathrm{Sing}^*(\mathcal{F}, E; Z)$ *such that there are exactly two appropriate curves* Γ_1 *and* Γ_2 *passing through* P. *Moreover, neither* Γ_1 *nor* Γ_2 *are permissible at* P. *Then* $Y = \{P\}$ *for one of such points* P.

(V) *The cases* (I), (II), (III) *and* (IV) *do not occur. There is a point* $P \in \mathrm{Sing}^*(\mathcal{F}, E; Z)$ *such that no appropriate curve passes through* P. *Then* $Y = \{P\}$ *for one such point* P.

PROPOSITION 2.13. *In the above situation* (*i.e. under the assumptions* (0.1), (0.2), (2.4) *and* (2.9)), *if* $\mathrm{Sing}^*(\mathcal{F}, E; Z) \neq \emptyset$, *then one of the cases* (I), (II), (III), (IV) *or* (V) *of the Definition* 2.12 *occurs. Hence we can always find a blowing-up which follows the global criteria of blowing-up.*

Proof. Assuming that (I), (II), (III), (IV) and (V) do not occur, we are allowed to find a cycle. Contradiction. (For more details, see [7], Proposition II.4.2.) □

Remark 2.14. Note that the blowing-up π of the Definition 2.12. always respects the local strategy described in the conditions (d) and (e) of the Hypothesis of Local Control.

Now, let us end the proof of Step B under the Hypothesis of Local Control. The arguments are the same ones that in [7], II.(4.2)–(4.5). Thus we shall only sketch them.

We reason by contradiction, assuming that Step B is not true. Hence we can construct an infinite sequence of permissible blowing-ups

$$\mathcal{S}:\ X(0) \overset{\pi(1)}{\leftarrow} X(1) \overset{\pi(2)}{\leftarrow} X(2) \overset{\pi(3)}{\leftarrow} \dots \tag{2.10}$$

such that each $\pi(i)$ respects the global strategy of blowing-up. Note also that the properties (0.1), (0.2), (2.4), (2.9) and the fact $\mathrm{Sing}^*(\mathcal{F}, E; Z) \neq \emptyset$ are satisfied in each intermediary step of the sequence $\mathcal{S}$.

Now, looking at the centers in $\mathcal{S}$ which are points in $\mathrm{Sing}^*(\mathcal{F}, E; Z)$, we can construct a tree, called "tree of bad points", see [7], Definition II.4.3, Remark II.4.4.

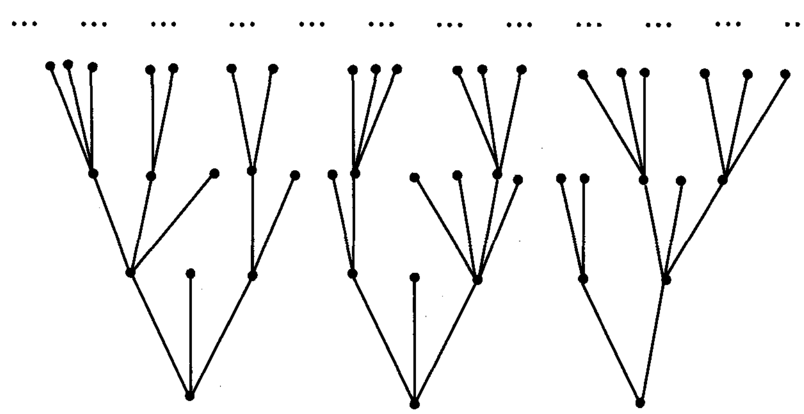


Fig. 19

This tree has an ordering induced by the blowing-ups $\pi(i)$ and moreover, the horizontal levels are finite sets. (See Figure 19.)

Note that it is not possible to blow-up curves in ξ for all the indexes $i>>0$. In fact, if this is possible, we contradict the Hypothesis of Local Control by looking at a point in a curve which has been modified infinitely many times. Hence we always blow-up a point in $\mathrm{Sing}^*(\mathcal{F}, E; Z)$ after finitely many steps (the situation (I) in Definition 2.12 only repeats finitely many times). This implies that the tree of "bad points" above is in fact an infinite tree. Since the horizontal levels are finite sets, we can find in it an infinite "branch". This branch provides an infinite sequence of points which satisfies the conditions of the Hypothesis of Local Control (see Remark 2.14). This is the desired contradiction.

§3. Local control

This paragraph is devoted to prove the Hypothesis of Local Control stated in the preceding §2. Hence to the end of the proof of Step B.

Like in the preceding paragraph, we shall follow the ideas in [7]. But in this case we shall find sligth differences and big simplications with respect to the cases treated in [7], III. The main difference is that we cannot replace the set $\mathrm{Sing}^*(\mathcal{F}, E; Z)$ by the sligthly bigger set

$$\mathrm{Sam}(\mathcal{F}, E; Z) = \{P \in Z;\ \mu(\mathcal{F}, E; \{P\}) = 1,\ \nu(\mathcal{F}, E; P) = 1\} \tag{3.1}$$

as in [7] for the case "adapted multiplicity bigger or equal than two". Note that there are simple singularities with $\mu(\mathcal{F}, E; \{P\})=1, \nu(\mathcal{F}, E; P)=1, e(E, P)=1, 2$, which cannot be destroyed by permissible blowing-ups. In fact the Local Control is achieved in [7] by means of a characteristic polygon (inspired in Hironaka's characteristic polyhedra) which appears to be empty for the points of $\mathrm{Sam}(\mathcal{F}, E; Z)$ which are pre-simple singularities. Hence we lose the Local Control. But for the points in $\mathrm{Sing}^*(\mathcal{F}, E; Z)$ we still have a characteristic polygon which permits the Local Control. In this case our argument is contained in the general argument of [7], III, but it is a simpler one. That is why we do it in a detailed way.

We shall reason by contradiction, assuming that the Hypothesis of Local Control does not hold. Thus, let us *fix* a sequence

$$\{X(i), Z(\mathrm{i}), E(i), \mathcal{F}(i), \mathcal{U}(i), Y(i), P(i), \pi(i+1)\}_{i\geqslant 0} \tag{3.2}$$

satisfying the properties (a), (b), (c), (d) and (e) of the Hypothesis of Local Control stated in the previous paragraph. Also recall that we assume the properties (0.1), (0.2) and (2.4) in the step $i=0$ of (3.2) and hence in all the steps. In order to simplify notation, put

$$e(i) = e(E(i), P(i)), \quad \text{for all } i \geqslant 0. \tag{3.3}$$

THEOREM 3.1. *The Hypothesis of Local Control holds except eventually in the case*:

$$\begin{gathered} \dim \mathrm{Dir}(\mathcal{F}, E; P) = 2 \quad \textit{and} \quad \mathrm{Dir}(\mathcal{F}, E; P) = T_P F \\ \textit{for a certain irreducible component } F \textit{ of } E. \end{gathered} \tag{3.4}$$

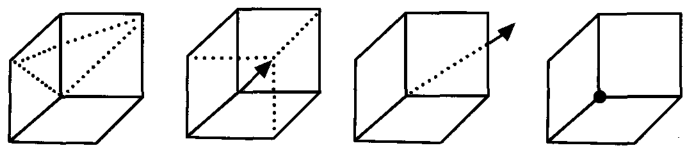
Proof. If $e(0)=1$, then $\dim \mathrm{Dir}(\mathcal{F}, E; P)=2$ and if (3.4) does not hold then P is a pre-simple singularity. Hence $e(0) \geqslant 2$. Consider the following cases:

Case I. $e(0)=3$ and $\mathrm{Dir}(\mathcal{F}, E; P)$ does not contain the tangent space of the intersection of two irreducible components of E, for any two irreducible components of E. If we draw E and the directrix, this case corresponds to one of the schematic situations of Figure 20.

The fact that $Y(0)$ is permissible and appropriate implies that

$$T_P Y(0) \subset \mathrm{Dir}(\mathcal{F}, E; P) \tag{3.5}$$

(see [5], Theorem 4 (i)). Since $Y(0)$ has normal crossings with E and $e(0)=3$ then $Y(0)$ is contained in the intersection of two irreducible components of E. Hence $Y(0)=\{P\}$.



$d=2$ $\quad$ $d=1$ $\quad$ $d=1$ $\quad$ $d=0$; $\quad$ $d=\dim \mathrm{Dir}(\mathscr{F},E;P)$

Fig. 20

Also by [5], Theorem 4 (ii), we may assume that

$$\dim \mathrm{Dir}(\mathscr{F},E;P) \geqslant 1. \tag{3.6}$$

Take a regular system of parameters (x,y,z) of $\mathcal{O}_{X,P}$ such that $E=(xyz=0)$ locally at P. Consider a generator ω of $\mathscr{F}_P$,

$$\omega = a\frac{dx}{x}+b\frac{dy}{y}+c\frac{dz}{z}, \quad \nu_P(a,b,c)=1. \tag{3.7}$$

Put $A=\mathrm{In}^1(a), B=\mathrm{In}^1(b), C=\mathrm{In}^1(c)$. Assume without loss of generality that a regular system of parameters (x',y',z') in $\mathcal{O}_{X(1),P(1)}$ is given by

$$x=x'; \quad y=(y'+\zeta)x'; \quad z=(z'+\xi)x'. \tag{3.8}$$

Now, consider first the case $\dim \mathrm{Dir}(\mathscr{F},E;P)=1$. Then

$$\mathrm{Dir}(\mathscr{F},E;P)=(Y-\zeta X=Z-\xi X=0) \quad (X=\mathrm{In}^1(x), Y=\mathrm{In}^1(y), Z=\mathrm{In}^1(z)). \tag{3.9}$$

In particular $(\zeta,\xi)\neq(0,0)$. Assume that $\xi\neq 0$. Note that

$$A=\varphi(Y-\zeta X, Z-\xi X), \quad B=\Psi(Y-\zeta X, Z-\xi X) \quad \text{and} \quad C=\eta(Y-\zeta X, Z-\xi X).$$

By the non-dicriticalness, we have that

$$A+B+C=\Phi(Y-\zeta X, Z-\xi X)\neq 0. \tag{3.10}$$

If $\zeta\neq 0$, then $\mathscr{F}(1)_{P(1)}$ is generated by

$$\omega' = (\Phi(y',z')+x'(\ldots))\frac{dx'}{x'}+b'dy'+c'dz'. \tag{3.11}$$

Thus $P(1)$ is a pre-simple point since the directrix $(\Phi(Y',Z')+X'(\ldots)=0)$ is transversal to the exceptional divisor $x'=0$, only component of $E(1)$ at $P(1)$. If $\zeta=0$, then $\mathscr{F}(1)_{P(1)}$ is generated by

$$\omega' = (\Phi(y',z') + x'(\dots))\frac{dx'}{x'} + (\psi(y',z') + x'(\dots))\frac{dy'}{y'} + c'dz'. \tag{3.12}$$

First of all, let us show that the following condition does not occur

$$\Phi(y',z') = \lambda y', \ \lambda \neq 0; \quad \psi(y',z') = \mu y'. \tag{3.13}$$

Assume that (3.13) occurs. Then

$$A = \alpha_1 Y + \beta(Y - \xi X); \quad B = \alpha_2 Y; \quad C = \alpha_3 Y - \beta(Y - \xi X), \quad \text{where } \alpha_1 + \alpha_2 + \alpha_3 \neq 0. \tag{3.14}$$

The integrability condition $\omega \wedge d\omega = 0$ implies that

$$A\left(-Z\frac{\partial B}{\partial Z} + Y\frac{\partial C}{\partial Y}\right) + B\left(-X\frac{\partial C}{\partial X} + Z\frac{\partial A}{\partial Z}\right) + C\left(-Y\frac{\partial A}{\partial Y} + X\frac{\partial B}{\partial X}\right) = 0. \tag{3.15}$$

Looking at the coefficient of XY in (3.15), we find that

$$(\alpha_1 + \alpha_2 + \alpha_3)\,\xi = 0. \tag{3.16}$$

Contradiction. Hence (3.13) does not occur. This implies that

$$\mathrm{Dir}(\mathcal{F}(1), E(1); P(1)) \not\supset T_{P(1)}(x' = y' = 0) = (\mathrm{In}^1(x') = \mathrm{In}^1(y') = 0). \tag{3.17}$$

Thus, we can write

$$\omega = a'\frac{dx'}{x'} + b'\frac{dy'}{y'} + c'dz'; \quad \nu_{P(1)}(a', b') = 1, \ \nu_{P(1)}(c') \geq 1, \tag{3.18}$$

where

$$A' = \mathrm{In}^1(a') = Z' + \varphi'(X', Y') \quad (X' = \mathrm{In}^1(x'), \ Y' = \mathrm{In}^1(y'), \ Z' = \mathrm{In}^1(z')). \tag{3.19}$$

Put $B' = \mathrm{In}^1(b')$. Then the integrability condition $\omega' \wedge d\omega' = 0$ implies that $B' = \lambda A'$. Thus $P(1)$ is a pre-simple singularity. Contradiction.

Assume now that $\dim \mathrm{Dir}(\mathcal{F}, E; P) = 2$. Then

$$\mathrm{Dir}(\mathcal{F}, E; P) = (Z + \lambda X + \mu Y = 0) \tag{3.20}$$

where $\lambda \neq 0 \neq \mu$. Note that $A + B + C = \varepsilon(Z + \lambda X + \mu Y) \neq 0$. If $e(1) = 1$, we can reason as in (3.12) and if $e(1) = 2$ we find directly the condition (3.17) and we reason as above. The case $e(1) = 3$ does not occur since by [5], Theorem 4 (ii), we have

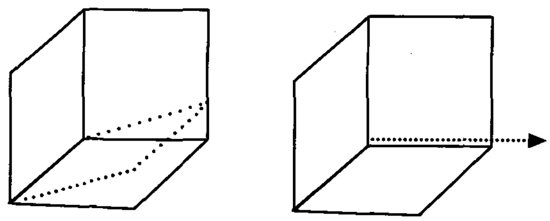


Fig. 21

$$\xi+\lambda+\mu\zeta=0, \tag{3.21}$$

which has not the solution $(\zeta,\xi)=(0,0)$. This ends the proof of Case I.

Case II. $e(0)=3$ and $\mathrm{Dir}(\mathscr{F},E;P)$ contains the tangent space of the intersection of two irreducible components of E. If we draw E and the directrix, this case corresponds to one of the schematic situations of Figure 21.

Consider first the case $\dim Y(0)=1$. By [5], Theorem 4 (ii), necessarily

$$\dim \mathrm{Dir}(\mathscr{F},E;P)=2. \tag{3.22}$$

Thus, put

$$\mathrm{Dir}(\mathscr{F},E;P)=(Z-\xi X=0), \quad \xi\neq 0. \tag{3.23}$$

Then $Y(0)=(x=z=0)$ by [5], Theorem 4 (i). By [5], Theorem 4 (ii), a regular system of parameters (x',y',z') of $\mathcal{O}_{X(1),P(1)}$ is given by

$$x=x'; \quad y=y'; \quad z=(z'+\xi)x'. \tag{3.24}$$

We have that $A=\lambda(Z-\xi X), B=\mu(Z-\xi X), C=\varrho(Z-\xi X)$, with $\lambda+\mu+\varrho\neq 0$. Then $\mathscr{F}(1)_{P(1)}$ is generated by

$$\omega'=((\lambda+\mu)z'+\varphi')\frac{dx'}{x'}+(\mu z'+\psi')\frac{dy'}{y'}+c'dz', \tag{3.25}$$

where $\varphi',\psi'\in(y',x')\mathcal{O}_{X(1),P(1)}$. Thus we have (3.17) and $P(1)$ is a pre-simple point.

Consider the case $Y(0)=\{P\}$. Assume first that $\dim \mathrm{Dir}(\mathscr{F},E;P)=1$. Put

$$\mathrm{Dir}(\mathscr{F},E;P)=(Y=Z=0). \tag{3.26}$$

We have that $A=\varphi(Y,Z)$, $B=\psi(Y,Z)$, $C=\eta(Y,Z)$. A regular system of parameters

(x', y', z') of $\mathcal{O}_{X(1),P(1)}$ is given by

$$x = x'; \quad y = y'x'; \quad z = z'x'. \tag{3.27}$$

Then $\mathcal{F}(1)_{P(1)}$ is generated by

$$\begin{aligned} \omega' &= (\Phi(y', z') + x'(\ldots))\frac{dx'}{x'} + (\psi(y', z') + x'(\ldots))\frac{dy'}{y'} + (\eta(y', z') + x'(\ldots))\frac{dz'}{z'} \\ &= a'\frac{dx'}{x'} + b'\frac{dy'}{y'} + c'\frac{dz'}{z'} \end{aligned} \tag{3.28}$$

where $\Phi = \varphi + \psi + \eta$. Assume first that

$$(\Phi(Y', Z'), \psi(Y', Z'), \eta(Y', Z')) \neq (A', B', C') \tag{3.29}$$

where $A' = \mathrm{In}^1(a')$, etc. In this case we are in one of the situations of the case I above, hence we are done. Otherwise, we repeat the same situation. But this situation cannot be repeated indefinitely, since this would imply that the curve $(y = z = 0)$ is appropriate and permissible at the point P (see the stationary sequences in Proposition I.1.10) in contradiction with the condition (d) of the Hypothesis of Local Control.

Now, consider the case $Y(0) = \{P\}$ and $\dim \mathrm{Dir}(\mathcal{F}, E; P) = 2$. Thus, put

$$\mathrm{Dir}(\mathcal{F}, E; P) = (Z - \zeta Y = 0), \quad \zeta \neq 0. \tag{3.30}$$

If $e(1) = 2$, we find the condition (3.17) and hence $P(1)$ is a pre-simple point. Thus, necessarily $e(1) = 3$ and hence we have equations like (3.27). Then $\mathcal{F}(1)_{P(1)}$ is generated by

$$\omega' = (\lambda(z' - \zeta y') + x'(\ldots))\frac{dx'}{x'} + (\mu(z' - \zeta y') + x'(\ldots))\frac{dy'}{y'} + (\varrho(z' - \zeta y') + x'(\ldots))\frac{dz'}{z'} \tag{3.31}$$

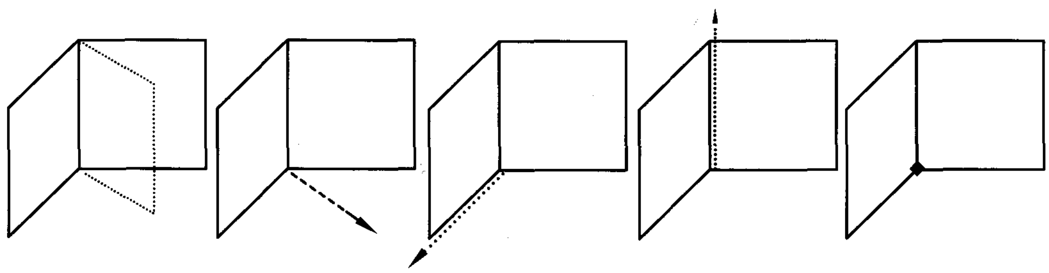
with $(\lambda, \mu, \varrho) \neq (0, 0, 0)$. We see thus that either we repeat the situation or we are in one of the previous cases. But an argument as above says that the situation cannot be repeated indefinitely. This ends the proof of Case II.

Note that there are no more possibilities with $e(0) = 3$.

Case III. $e(0) = 2$. If we draw E and the directrix, this case corresponds to one of the schematic situations of Figure 22.

Choose a regular system of parameters (x, y, z) of $\mathcal{O}_{X,P}$ such that $E = (xy = 0)$ locally at P. If $\dim \mathrm{Dir}(\mathcal{F}, E; P) = 2$, then

$$\mathrm{Dir}(\mathcal{F}, E; P) = (Y - \zeta X = 0), \quad \zeta \neq 0. \tag{3.32}$$



$d=2$ $d=1$ $d=1$ $d=1$ $d=0$

Fig. 22

With computations as above, we see that if $e(1)=3$, then the condition (3.4) does not hold. Hence we are done. If $e(1)=1$ (only possibility), we find a pre-simple point. If $\dim \mathrm{Dir}(\mathcal{F}, E; P)=1$ and (3.17) holds, then we are done. Hence the only possibility is

$$\mathrm{Dir}(\mathcal{F}, E; P) = (X = Y = 0). \tag{3.33}$$

Now, necessarily $Y(0)=\{P\}$ and $e(1)=3$. Moreover $\dim \mathrm{Dir}(\mathcal{F}(1), E(1); P(1)) \leqslant 1$ and thus the condition (3.4) does not hold, hence we are done. This ends the proof of the theorem. □

Thus, in view of the theorem above, we shall assume in the sequel that the condition (3.4) holds at each step of the sequence (3.2).

LEMMA 3.2. *$e(i) \geqslant 2$ for each $i \geqslant 1$.*

Proof. It is enough to prove that $e(1) \geqslant 2$. By the condition (3.4) and by [5], Theorem 4 (ii), we see that $P(1)$ is both in the exceptional divisor of $\pi(1)$ and in the strict transform of the irreducible component F of E. Hence $e(1) \geqslant 2$. □

Hence, in the sequel we shall assume that $e(i) \geqslant 2$, for all $i \geqslant 0$, without loss of generality.

DEFINITION 3.3. *A regular system of parameters (x, y, z) of $\mathcal{O}_{X,P}$ is said to be "prepared" iff the following properties hold:*

$$(xz = 0) \subset E \subset (xyz = 0), \quad \textit{locally at } P. \tag{3.34}$$

$$\mathrm{Dir}(\mathcal{F}, E; P) = (\mathrm{In}^1(z) = 0). \tag{3.35}$$

(A prepared regular system of parameters always exists.)

Given an element $f \in \mathcal{O}_{X,P}$ such that $\nu_P(f) \geq 1$ and a regular system of parameters (x, y, z) of $\mathcal{O}_{X,P}$, let us define the *characteristic polygon* $\Delta^1(f; x, y; z)$ (see [5] and [7]). Let us write

$$f = \sum_{h,i,j} f_{hij} x^h y^i z^j = \sum_{h,i} f_{hi0} x^h y^i + z(\ldots). \tag{3.36}$$

We define the characteristic polygon $\Delta^1(f; x, y; z)$ by

$$\Delta^1(f; x, y; z) = [[\{(h, i); f_{hi0} \neq 0\}]] \subset \mathbf{R}^2, \tag{3.37}$$

where $[[\ldots]]$ means the positive convex hull. That is, if $T \subset \mathbf{R}_0^2$ (where $\mathbf{R}_0 = \{t \in \mathbf{R}; t \geq 0\}$) then

$$[[T]] = \text{convex hull of } T + \mathbf{R}_0^2 \subset \mathbf{R}_0^2. \tag{3.38}$$

Fix a prepared regular system of parameters (x, y, z) of $\mathcal{O}_{X,P}$ and consider a generator ω of $\mathcal{F}_P$

$$\omega = a\frac{dx}{x} + b\, dy + c\frac{dz}{z}, \quad \text{if } e(0) = 2. \tag{3.39}$$

$$\omega = a\frac{dx}{x} + b\frac{dy}{y} + c\frac{dz}{z}, \quad \text{if } e(0) = 3. \tag{3.40}$$

DEFINITION 3.4. *In the above situation, the characteristic polygon*

$$\Delta(\mathcal{F}, E; P; x, y; z) = \Delta$$

is defined to be

$$\Delta = [[\Delta^1(a; x, y; z) \cup \Delta^1(yb; x, y; z) \cup \Delta^1(c; x, y; z)]], \quad \text{if } e(0) = 2, \tag{3.41}$$

$$\Delta = [[\Delta^1(a; x, y; z) \cup \Delta^1(b; x, y; z)]], \quad \text{if } e(0) = 3. \tag{3.42}$$

Remarks 3.5. (a) $\Delta = \Delta(\mathcal{F}, E; P; x, y: z)$ does not depend on the particular choice of ω.

(b) $\Delta \neq \emptyset$. Since otherwise z divides a, b, c.

(c) The vertices of Δ have entire coordinates.

(d) If $(v_1, v_2) \in \Delta$, then $v_1 + v_2 > 1$. Otherwise we contradict either (3.35) or the fact $\nu_P(b) \geq 1$.

(e) $\Delta \subset \{(u, v); u \geq 1)$ iff $(y = z = 0)$ is appropriate and permissible at P.

(f) $\Delta \subset \{(u, v); v \geq 1\}$ iff $(y = z = 0)$ is appropriate and permissible at P.

Let (α,β) be the vertex of Δ which has lowest abcisse α. We shall call it the *main vertex of* Δ. Our argument will be based upon the controll of (α,β).

THEOREM 3.6. *Fix a prepared regular system of parameters* (x,y,z) *of* $\mathcal{O}_{X,P}$. *Let* (α,β) *be the main vertex of the characteristic polygon*

$$\Delta=\Delta(\mathcal{F},E;P;x,y;z). \tag{3.43}$$

Assume also that the following situation does not occur

$$e(0)=3 \quad \text{and} \quad e(1)=2. \tag{3.44}$$

Then, there is a prepared regular system of parameters (x',y',z') *of* $\mathcal{O}_{X(1),P(1)}$ *such that if* (α',β') *is the main vertex of the characteristic polygon*

$$\Delta'=\Delta(\mathcal{F}(1),E(1);P(1);x',y';z') \tag{3.45}$$

then we have that

$$\beta'\leqslant\beta. \tag{3.46}$$

Moreover, we have strict inequality in (3.46) *unless eventually in the following cases*:

(A) $Y(0)=(x=z=0)$.

(B) $Y(0)=\{P(0)\}, P(1)\notin$ *strict transform of* $(x=0)$ *and* $(x'=0)$ *is a local equation of* $\pi(1)^{-1}(P(0))$.

Proof. Assume first that $Y(0)=\{P(0)\}$. Then, by (3.35), a regular system of parameters (x',y',z') of $\mathcal{O}_{X(1),P(1)}$ is given by one of the following equations:

$$T-1,\zeta:\quad x=x';\quad y=(y'+\zeta)x';\quad z=z'x'. \tag{3.47}$$

$$T-2:\quad x=x'y';\quad y=y';\quad z=z'y'. \tag{3.48}$$

Note that if $\zeta\neq 0$, then $e(0)=2$, since (3.44) does not occur. In this case, a coordinate change

$$y_1=y-\zeta x \tag{3.49}$$

does not modify the main vertex (α,β) of Δ and hence we can assume that $\zeta=0$ without loss of generality.

Note that the condition (3.34) of Definition 3.3 holds trivially for (x',y',z'). Moreover, if (3.35) does not hold for (x',y',z'), then we are in the cases studied in Theorem 3.1, i.e. the condition (3.4) does not hold. Hence (x',y',z') is prepared.

Now, we see that

$$\Delta' = [[\sigma_1(\Delta)]], \quad \text{if } T\text{−}1, 0. \tag{3.50}$$

$$\Delta' = [[\sigma_2(\Delta)]], \quad \text{if } T\text{−}2 \tag{3.51}$$

where $\sigma_i, i=1,2$, are the affine mappings given by

$$\sigma_1(u, v) = (u+v-1, v) \tag{3.52}$$

$$\sigma_2(u, v) = (u, u+v-1). \tag{3.53}$$

Note that $\alpha<1$ (actually $\alpha=0$) since otherwise $(x=z=0)$ would be an appropriate and permissible curve at P, in contradiction with condition (d) of the Hypothesis of Local Control. Thus, in the case T−2, we have that

$$\beta' = \alpha+\beta-1<\beta. \tag{3.54}$$

If T−1, 0, since the polygon moves horizontally, we have that

$$\beta' \leqslant \beta. \tag{3.55}$$

Note that this case corresponds to the case (B) in the statement.

Assume now that $Y(0)$ has dimension one. By [5], Theorem 4 (i), we have that

$$T_P Y(0) \subset \mathrm{Dir}(\mathscr{F}, E; P) = (\mathrm{In}^1(z) = 0). \tag{3.56}$$

Since $Y(0)$ has normal crossings with E and $(z=0)$ is an irreducible component of E, then

$$Y(0) \subset (z=0). \tag{3.57}$$

We have two possibilities:

$$Y(0) = (x=z=0). \tag{3.58}$$

$$Y(0) \text{ is transversal to } (x=0). \tag{3.59}$$

Assume first that $Y(0)=(x=z=0)$. Then, by (3.35) and [5], Theorem 4 (ii), a regular system of parameters (x', y', z') of $\mathcal{O}_{X(1), P(1)}$ is given by the equation

$$T\text{−}3\colon \quad x=x'; \quad y=y'; \quad z=x'z'. \tag{3.60}$$

The condition (3.34) of Definition 3.3 holds trivially for (x', y', z'). Also (3.35) holds as above. Hence (x', y', z') is prepared. Also, we have that

$$\Delta' = [[\sigma_3(\Delta)]] \tag{3.61}$$

where

$$\sigma_3(u, v) = (u-1, v). \tag{3.62}$$

In particular, we have that

$$(\alpha', \beta') = (\alpha-1, \beta). \tag{3.63}$$

Note that we are in the case (A) of the statement. Note also that this situation does not occur indefinitely, since α cannot drop indefinitely. Assume now that $Y(0)$ is transversal to $(x=0)$. If $e(0)=3$, then $Y(0)=(y=z=0)$, since $Y(0)$ has normal crossings with E. In the case $e(0)=2$, then

$$Y(0) = \left(z = y + \sum_{i\geq 1} \zeta_i x^i = 0\right). \tag{3.64}$$

But the coordinate change

$$y_1 = y + \sum_{i\geq 1} \zeta_i x^i \tag{3.65}$$

does not modify neither the fact that (x, y_1, z) is prepared nor the main vertex of the characteristic polygon. Hence, we may assume that

$$Y(0) = (y = z = 0). \tag{3.66}$$

Then, a regular system of parameters (x', y', z') of $\mathcal{O}_{X(1), P(1)}$ is given by

$$T{-}4: \quad x = x'; \quad y = y'; \quad z = y'z'. \tag{3.67}$$

We see as above that (x', y', z') is prepared and

$$\Delta' = [[\sigma_4(\Delta)]], \tag{3.68}$$

where

$$\sigma_4(u, v) = (u, v-1). \tag{3.69}$$

Hence

$$\beta' = \beta - 1 < \beta. \tag{3.70}$$

This ends the proof. □

COROLLARY 3.7. *If the situation*

$$e(i)=3, \quad e(i+1)=2 \tag{3.71}$$

occurs only for finitely many indexes $i \geqslant 0$, then the Hypothesis of Local Control holds.

Proof. We can assume without loss of generality that (3.71) does not occur at all. In this case, the preceding proof shows us how to choose a prepared regular system of parameters $(x(i), y(i), z(i))$ at each step $i \geqslant 0$, and hence an invariant $\beta(i)$, for each $i \geqslant 0$. Also we can assume without loss of generality that

$$\beta(i)=\beta(0) \quad \text{for all } i \geqslant 0. \tag{3.72}$$

$$e(i)=e(0) \quad \text{for all } i \geqslant 0. \tag{3.73}$$

Now, up to an eventual coordinate change

$$y(0) \hookrightarrow y(0)+\sum_{i \geqslant 1} \zeta_i (x(0))^i \tag{3.74}$$

which does not modify our argument, and in view of the conditions (A) and (B) of the preceding theorem we can assume without loss of generality that

$$P(i) \in \text{strict transform of } (y(0)=z(0)=0). \tag{3.75}$$

This implies, in particular, that we never do the coordinate change of (3.49). Hence, for the characteristic polygons $\Delta(i), i \geqslant 0$, we have that

$$\Delta(i)=[[\sigma_{\varepsilon(i)} \circ \ldots \circ \sigma_{\varepsilon(i)}(\Delta(0))]] \tag{3.76}$$

where $\varepsilon(1)=1$ or 3 (notations as in the previous proof). Assume that $\Delta(0)$ has a vertex $(\gamma(0), \delta(0))$ with

$$\delta(0)<1 \quad (\text{hence } \delta(0)=0). \tag{3.77}$$

Then, the corresponding vertex $(\gamma(i), \delta(i))$ in $\Delta(i)$ is given by

$$(\gamma(i), \delta(i))=(\gamma(0)-i, 0), \quad i \geqslant 0 \tag{3.78}$$

in view of (3.76). This is a contradiction, since $\gamma(0)$ cannot drop indefinitely. Thus, we have

$$\Delta(0) \subset \{(u, v) ; v \geqslant 1\} \tag{3.79}$$

and this implies that $(y(0)=z(0)=0)$ is appropriate and permissible at $P(0)$. Now, since the transformation $T-3$ does not repeat indefinitely, we may assume without loss of generality that $Y(0)=\{P(0)\}$. But this contradicts the property (d) of the Hypothesis of Local Control. □

Now, in order to end the proof of the Hypothesis of Local Control, we shall prove that the situation (3.7.1) occurs in fact only finitely many times.

Take a prepared regular system of parameters (x, y, z) of $\mathcal{O}_{X,P}$. Let (α, β) be the main vertex of $\Delta=\Delta(\mathcal{F}, E; P; x, y; z)$ and let ω be a generator of $\mathcal{F}_P$

$$\omega = a\frac{dx}{x}+b\,dy+c\frac{dz}{z}, \quad \text{if } e(0)=2. \tag{3.80}$$

$$\omega = a\frac{dx}{x}+b\frac{dy}{y}+c\frac{dz}{z}, \quad \text{if } e(0)=3. \tag{3.81}$$

Let us write

$$a=\varphi(x, y)+z(\ldots). \tag{3.82}$$

$$yb=\Psi(x, y)+z(\ldots), \text{ if } e(0)=2; \quad b=\Psi(x, y)+z(\ldots), \text{ if } e(0)=3. \tag{3.83}$$

$$c=\varrho(x, y)+z(\ldots). \tag{3.84}$$

Thus, we have that

$$\varphi=\lambda_1 x^\alpha y^\beta+\phi'; \quad \Psi=\lambda_2 x^\alpha y^\beta+\Psi'; \quad \varrho=\lambda_3 x^\alpha y^\beta+\varrho' \tag{3.85}$$

where:

$$\varphi', \Psi', \varrho' \in (x^\alpha y^{\beta+1}, x^{\alpha+1})\,\mathbf{C}[[x, y]]. \tag{3.86}$$

$$(\lambda_1, \lambda_2, \lambda_3) \neq (0, 0, 0). \tag{3.87}$$

Definition 3.8. *In the above situation, we say that the main vertex* (α, β) *of* Δ *is non-resonant iff one of the following conditions is satisfied:*

(a) $\lambda_3 \neq 0$.

(b) $\lambda_3=0, p\lambda_1+q\lambda_2 \neq 0$ *for all* $p, q \in \mathbf{N}_+$.

Lemma 3.9. *Assume that the condition* (3.44) *does not hold and that we are in the situation of Theorem* 3.6. *Let* (x', y', z') *be obtained from* (x, y, z) *as in the proof of Theorem* 3.6. *Assume also that we have*

$$(\alpha',\beta') = \sigma_i(\alpha,\beta), \quad i = 1,2,3 \text{ or } 4 \tag{3.88}$$

if $\pi(1)$ *is given respectively by* $(T-1,\zeta), T-2, T-3$ *or* $T-4$. *Then the main vertex* (α',β') *of* Δ^1 *is non resonant if the main vertex* (α,β) *of* Δ *is non-resonant.*

Proof. The coordinate changes $y \hookrightarrow y_1$ in the proof of Theorem 3.6 do not modify the values $(\lambda_1,\lambda_2,\lambda_3)$. Hence we can assume without loss of generality that $\pi(1)$ is given either by $(T-1,0)$, or by $T-2$, or by $T-3$, or by $T-4$. In particular monomials are transformed into monomials. The condition in (3.88) implies that the monomials who contribute to (α',β') are exactly the transformed monomials of the ones who contributed to (α,β). Hence, the corresponding values $(\lambda'_1,\lambda'_2,\lambda'_3)$ are given exactly by:

$$\begin{aligned}
&(\lambda'_1,\lambda'_2,\lambda'_3) = (\lambda_1+\lambda_2+\lambda_3,\lambda_2,\lambda_3), && \text{if } T-1,0.\\
&(\lambda'_1,\lambda'_2,\lambda'_3) = (\lambda_1,\lambda_1+\lambda_2+\lambda_3,\lambda_3), && \text{if } T-2.\\
&(\lambda'_1,\lambda'_2,\lambda'_3) = (\lambda_1+\lambda_3,\lambda_2,\lambda_3), && \text{if } T-3.\\
&(\lambda'_1,\lambda'_2,\lambda'_3) = (\lambda_1,\lambda_2+\lambda_3,\lambda_3), && \text{if } T-4.
\end{aligned} \tag{3.89}$$

Now the proof is straigthforward. □

THEOREM 3.10. *The situation*

$$e(i) = 3, \quad e(i+1) = 2 \tag{3.90}$$

occurs only for finitely many indexes $i \geqslant 0$. *Hence the Hypothesis of Local Control always holds.*

Proof. Assume that $e(0)=3, e(1)=2$. Fix a prepared regular system of parameters (x,y,z) of $\mathcal{O}_{X,P}$. Since $e(1)=2$, necessarily $Y(0)=\{P\}$ (actually if (3.4) holds and $\dim Y(0)=1$, then $e(1)=3$ by [5], Theorem 4(ii); now, looking at the proof of Theorem 3.1, we see that $Y(0)=\{P\}$). A regular system of parameters (x',y',z') of $\mathcal{O}_{X(1),P(1)}$ is given by

$$T-1,\zeta\colon \quad x = x', \quad y = (y'+\zeta)x'', \quad z = z'x'; \quad \zeta \neq 0. \tag{3.91}$$

Moreover, the non-dicriticalness of $(\mathcal{F},E)$ shows easily that (x',y',z') is prepared (recall that (3.4) always holds). Hence, we have (in view of Theorem 3.6 and the above argument) a procedure to choose a prepared regular system of parameters $(x(i),y(i),z(i))$ at each step $i \geqslant 0$, even if (3.90) holds. Hence, we have an invariant $\beta(i)$ for each $i \geqslant 0$, where $(\alpha(i),\beta(i))$ is the main vertex of the corresponding characteristic polygon $\Delta(i)$.

Now, let us reason by contradiction, assuming that (3.90) holds infinitely many times. Also, in order to simplify notations, assume now that $e(0)=2$ and $e(1)=3$.

Given an index $i \geq 0$, put

$$i_- = \max\{j \geq 0; j \leq i, e(j) = 2\}. \tag{3.92}$$

$$i_+ = \min\{j \geq 0; j > i, e(j) = 2\}. \tag{3.93}$$

Both i_- and i_+ are well-defined and they are finite numbers. Also note that

$$i_- < i_+ \quad \text{and} \quad (i_+)_- = i_+. \tag{3.94}$$

Assume that the following statements are true:

$$\beta(i_+ - 1) + 1 \geq \beta(i_+), \quad \text{for all } i \geq 0. \tag{3.95}$$

(3.96) If $\beta(i_+ - 1) + 1 = \beta(i_+)$, then the vertex $(\alpha(i_+), \beta(i_+))$ is non-resonant.

(3.97) If $(\alpha(i_+ - 1), \beta(i_+ - 1))$ is non-resonant, then $\beta(i_+ - 1) \geq \beta(i_+)$.

Let us show that we find then a contradiction. If $i_+ = i_- + 1$, then

$$\beta(i_-) \geq \beta(i_+) \tag{3.98}$$

by Theorem 3.6. If $i_+ > i_- + 1$, then

$$\beta(i_-) = \beta(i_- + 1) + 1 \geq \beta(i_+ - 1) + 1 \geq \beta(i_+) \tag{3.99}$$

by Theorem 3.6, Remark 3.5 (c) and (3.95). In any case, we have that

$$\beta(i_-) \geq \beta(i_+). \tag{3.100}$$

Now, by (3.94) it is enough to show that given $M \geq 0$ there is $i \geq M$ such that in (3.100) we have strict inequality. Fix an index $i \geq 0$, and assume without loss of generality that $e(i)=3, e(i+1)=2$. Assume also that

$$\beta(i_-) = \beta(i_+). \tag{3.101}$$

This implies that

$$\beta(i_-) = \beta(i_- + 1) + 1 \geq \beta(i_+ - 1) + 1 \geq \beta(i_+) = \beta(i_-). \tag{3.102}$$

Hence, by (3.96) the vertex $(\alpha(i_+), \beta(i_+))$ is non-resonant. Now, let $j = \max\{h \geq i_+;$

$e(h)=2\}$. We have that

$$\beta(i_+) \geqslant \beta(i_+ + 1) \geqslant \ldots \geqslant \beta(j). \tag{3.103}$$

If one of the inequalities in (3.103) is strict, we are done. Hence, we can assume that

$$\beta(i_+) = \beta(i_+ + 1) = \ldots = \beta(j). \tag{3.104}$$

In particular, the condition (3.88) of Lemma 3.9 holds. This implies that $(\alpha(j), \beta(j))$ is non-resonant. Note that $j = j_-$. We have that

$$\beta(j_-) = \beta(j_- + 1) + 1 \geqslant \beta(j_- + 2) + 1 \geqslant \ldots \geqslant \beta(j_+ - 1) + 1. \tag{3.105}$$

If one of the inequalities in (3.105) is strict, we are done by (3.95). Otherwise, the condition (3.88) of Lemma 3.9 holds and thus $(\alpha(j_+ - 1), \beta(j_+ - 1))$ is non-resonant. Hence, by (3.97) we have that

$$\beta(j_-) \geqslant \beta(j_+) + 1 \tag{3.106}$$

and we are done.

Now it suffices to prove the statements (3.95), (3.96) and (3.97). Let us simplify notations as in the beginning of this proof. Assume now that

$$e(0) = 3, \quad e(1) = 2 \tag{3.107}$$

and put $(x, y, z) = (x(0), y(0), z(0))$, $(x', y', z') = (x(1), y(1), z(1))$, $\Delta = (0)$, $\Delta' = \Delta(1)$, $(\alpha, \beta) = (\alpha(0), \beta(0))$, $(\alpha', \beta') = (\alpha(1), \beta(1))$. Note that (x', y', z') is given by $(T-1, \zeta), \zeta \neq 0$, like in (3.91). Consider a generator ω of $\mathcal{F}_P$

$$\omega = a\frac{dx}{x} + b\frac{dy}{y} + c\frac{dz}{z}. \tag{3.108}$$

Put

$$a = \varphi(x, y) + z(\ldots); \quad b = \psi(x, y) + z(\ldots); \quad c = \varrho(x, y) + z(\ldots). \tag{3.109}$$

Note that $(\alpha, \beta) = (0, \beta)$ (otherwise $(x = z = 0)$ is appropriate and permissible at P). Hence

$$\varphi = \lambda_1 y^\beta + \varphi'; \quad \psi = \lambda_2 y^\beta + \psi'; \quad \varrho = \lambda_3 y^\beta + \varrho; \quad \text{where} \quad \varphi', \psi', \varrho' \in (y^{\beta+1}, x) \tag{3.110}$$

and $(\lambda_1, \lambda_2, \lambda_3) \neq (0, 0, 0)$. Now put

$$
(3.111)\quad
\begin{gathered}
y^* = y - \zeta x; \quad \phi(x,y) = \varphi(x,y) + \psi(x,y) + \varrho(x,y) = \sum \phi_{hi} x^h y^i; \\
\phi^*(x,y^*) = \phi(x, y^* + \zeta x) = \sum \phi^*_{hi} x^h (y^*)^i; \quad \psi(x,y) = \sum \psi_{hi} x^h y^i; \\
\psi^*(x,y^*) = \psi(x, y^* + \zeta x) = \sum \psi^*_{hi} x^h (y^*)^i, \quad \varrho(x,y) = \sum \varrho_{hi} x^h y^i; \\
\varrho^*(x,y^*) = \varrho(x, y^* + \zeta x) = \sum \varrho^*_{hi} x^h (y^*)^i.
\end{gathered}
$$

An easy computation shows that

$$(3.112)\qquad \Delta = [[\{(h,i); (\phi_{hi}, \psi_{hi}, \varrho_{hi}) \neq (0,0,0)\}]]$$

$$(3.113)\qquad \Delta' = [[\{(h+1-1,i); (\phi^*_{hi}, \varrho^*_{hi}) \neq (0,0)\} \cup \{(h+i-1, i+1); \psi^*_{hi} \neq 0\}]].$$

Now, put

$$(3.114)\qquad \delta = \min\{h+i; (\phi_{hi}, \psi_{hi}, \varrho_{hi}) \neq 0,0,0)\}.$$

We obviously have that

$$(3.115)\qquad \delta = \min\{h+i; (\phi^*_{hi}, \psi^*_{hi}, \varrho^*_{hi}) \neq (0,0,0)\}.$$

From (3.111) and (3.113), we deduce that

$$(3.116)\qquad \alpha' = \delta - 1.$$

Note that $\delta \leqslant \beta$. Assume first that $\delta < \beta$. Hence, for a certain $t \geqslant 1$ we have that

$$(3.117)\qquad (\phi^*_{t,(\delta-t)}, \psi^*_{t,(\delta-t)}, \varrho^*_{t,(\delta-t)}) \neq (0,0,0)$$

(note that if $t=0$, we contradict (3.110)). But (3.113), jointly with (3.117) implies that

$$(3.118)\qquad (\delta-1, \delta-t+1) \in \Delta'.$$

Hence $\beta' \leqslant \delta - t + 1 \leqslant \delta < \beta$ and (3.97) (and a fortiori (3.95), (3.96)) holds.

Hence, let us assume that $\delta = \beta$. If $\lambda_3 \neq 0$ or $\lambda_1 + \lambda_2 + \lambda_3 \neq 0$, we find that

$$(3.119)\qquad (\phi^*_{0\beta}, \varrho^*_{0\beta}) \neq (0,0)$$

and hence

$$(3.120)\qquad (\beta-1, \beta) \in \Delta'.$$

This implies, jointly with (3.91), that $\beta' \leqslant \beta$. But the above holds if (α, β) is non-resonant, hence (3.97) (and a fortiori (3.95) and (3.96)) holds. Assume that $\lambda_3=0$ and that $\lambda_1+\lambda_2=0$ (this is a resonant case, hence (3.97) always holds). Then, in particular $\lambda_2\neq 0$ and by (3.113) we see that

$$(\beta-1,\beta+1)\in\Delta'. \tag{3.121}$$

This implies that $\beta'\leqslant\beta+1$. Hence (3.95) holds. Assume now that $\beta'=\beta+1$ (only case for (3.96)). By (3.113), this implies that

$$(\phi^*_{hi},\varrho^*_{hi})\neq(0,0),\ h+i=\beta \text{ implies that } i\geqslant\beta+1. \tag{3.122}$$

$$\psi^*_{hi}\neq 0, h+i=\beta \text{ implies that } i\geqslant\beta. \tag{3.123}$$

Hence, the only possibility is that

$$\begin{aligned}\phi^* &= 0\cdot x^{\beta-1}y^{\beta+1}+(\phi^*)'\\ y\psi^* &= \lambda_2 x^{\beta-1}y^{\beta+1}+(\psi^*)'\\ \varrho^* &= 0\cdot x^{\beta-1}y^{\beta+1}+(\varrho^*)',\end{aligned} \tag{3.124}$$

where

$$(\phi^*)',(\psi^*)',(\varrho^*)'\in(x^{\beta-1}y^{\beta+2},x^{\beta}). \tag{3.125}$$

Note that $\mathscr{F}(1)_{P(1)}$ is generated by

(3.126)

$$\omega=(\phi^*(x',y')+z'(\dots))\frac{dx'}{x'}+(\zeta+y')^{-1}(\psi^*(x',y')+z'(\dots))\,dy'+(\varrho^*(x',y')+z'(\dots))\frac{dz'}{z'}.$$

Now, (3.126) jointly with (3.124) show that the corresponding values $(\lambda'_1,\lambda'_2,\lambda'_3)$ in this case are

$$(\lambda'_1,\lambda'_2,\lambda'_3)=(0,1,0). \tag{3.127}$$

Hence the main vertex $(\beta-1,\beta+1)$ is non-resonant. This proves (3.96) and the proof is ended. □

Part IV. Existence of separatrices

§ 1. Existence of a convergent separatrix

Let X be a nonsingular connected analytic space over $\mathbf{C}$ of $\dim X=3$. Fix a point $P\in X$ and a non-dicritical $(\mathscr{F},\varnothing)\in\mathfrak{F}(X,\varnothing)$. This paragraph is devoted to the proof of the following result:

THEOREM 1.1. *The singular foliation* $(\mathscr{F},\varnothing)$ *has a convergent separatrix* S *at the point* P.

First of all, let us fix a sequence of desingularization

$$X(0) \xleftarrow{\pi(1)} X(1) \xleftarrow{\pi(2)} \dots \xleftarrow{\pi(N)} X(N) \tag{1.1}$$

as given in the Desingularization Theorem of Part III. In particular, let us recall that:

(a) $X(0) \subset X$ is an open set with $P \in X(0)$.

(b) For each $i, 0 \leqslant i \leqslant N$:

(b1) Let $(\mathscr{F}(i), E(i)) \in \mathfrak{F}(X(i), E(i))$ be the adapted strict transform of $(\mathscr{F}|_{X(0)}, \varnothing)$ under the composition $\pi(1) \circ \dots \circ \pi(i)$.

(b2) Put $(\mathscr{G}(i), \varnothing) = \mathrm{hol}((\mathscr{F}(i), E(i)))$.

(b3) $Z(i) = (\pi(1) \circ \dots \circ \pi(i))^{-1}(P)$.

(Note that $Z(i) \subset E(i)$ and each irreducible component of $Z(i)$ is also an irreducible component of $E(i)$.)

(c) Each point $Q \in \mathrm{Sing}(\mathscr{G}(N), \varnothing)$ is a simple singularity for $(\mathscr{F}(N), E(N))$.

Note that if we replace $X(0)$ by any open set $X'(0)$ with $P \in X'(0) \subset X(0)$, we obtain also a sequence of desingularization like (1.1).

LEMMA 1.2 (Up to make $X(0)$ smaller). *There is a closed analytic curve* $\Gamma \subset X(0)$ *with* $P \in \Gamma$ *which is irreducible at* P *such that:*

(a) $\Gamma \not\subset \mathrm{Sing}(\mathscr{F},\varnothing)$.

(b) Γ *is an invariant analytic space for* $(\mathscr{F},\varnothing)$.

Proof. By the Transversality Theorem in [18], VI.2, there is a germ $\Delta \subset X(0)$ of a non-singular analytic surface at P, such that the restriction $\mathscr{F}|_\Delta$ has an isolated singularity at P. Hence $(\mathscr{F}|_\Delta, \varnothing) \in \mathfrak{F}(\Delta, \varnothing)$. In particular no curve in Δ is contained in $\mathrm{Sing}(\mathscr{F},\varnothing)$. The main theorem in [4] implies that there is an invariant curve Γ of $\mathscr{F}|_\Delta$ passing through P. Obviously Γ is also an invariant curve for $(\mathscr{F},\varnothing)$. □

Now, fix a curve Γ as in the preceding Lemma 1.2. Denote by $\Gamma(i)$ the strict transform of Γ by $\pi(1) \circ \dots \circ \pi(i)$, for $0 \leqslant i \leqslant N$. Let $P(i)$ be the only point in $E(i) \cap \Gamma(i)$. (Note that $P(i) \in Z(i)$.) Moreover $P(i) \in \mathrm{Sing}(\mathscr{G}(i), \varnothing)$, since otherwise the only leaf of $(\mathscr{G}(i), \varnothing)$ locally at $P(i)$ is the divisor $E(i)$ and this contradicts the fact that $\Gamma(i) \not\subset E(i)$ is contained in a leaf.

LEMMA 1.3. *We can assume without loss of generality that the two following*

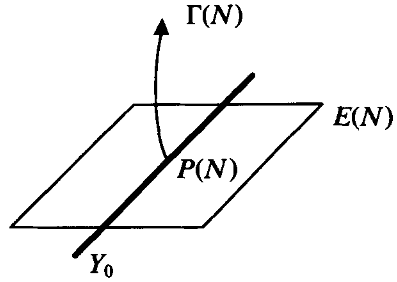


Fig. 23

properties are satisfied:

(a) $\Gamma(N)$ *is non-singular and has normal crossing with* $E(N)$.

(b) $e(E(N), P(N))=1$.

Proof. By standard results on desingularization of curves we may blow-up the point $P(N)$ finitely many times in order to have the property (a). Note that $P(N) \in \mathrm{Sing}(\mathscr{G}(N), \emptyset)$ permissible center for $(\mathscr{F}(N), E(N))$. Now, (b) is a consequence of (a) and the fact that $e(\Gamma(N), E(N))=0$. □

Let us consider the analytic subset of $X(N)$ defined by

(1.2) $\mathscr{V} = \bigcup\{$irreducible components Y of $\mathrm{Sing}(\mathscr{G}(N), \emptyset)$ with $e(E(N), Y) = 1\}$.

Recall that each irreducible component Y of $\mathrm{Sing}(\mathscr{G}(N), \emptyset)$ is a non-singular curve and $e(E(N), Y) \geq 1$ and moreover it has normal crossings with $E(N)$ (Part II).

Let us show that $\mathscr{V} \neq \emptyset$. Looking at the point $P(N)$ we have that

$$e(E(N), P(N)) = 1. \tag{1.3}$$

This implies that there is a unique irreducible component Y_0 of $\mathrm{Sing}(\mathscr{G}(N), \emptyset)$ such that $P(N) \in Y_0$ and necessarily $e(E(N), Y_0)=1$. (See Figure 23.)

Now, let us consider the decomposition of $\mathscr{V}$ into connected components

$$\mathscr{V} = \mathscr{V}_0 \cup \mathscr{V}_1 \cup \ldots \cup \mathscr{V}_k \tag{1.6}$$

and let us assume that $\mathscr{V}_0$ is the connected component of $\mathscr{V}$ which contains Y_0.

LEMMA 1.4. *For each point* $Q \in \mathscr{V}_0$ *there is an open set* $\mathscr{U} \subset X(N)$, $Q \in \mathscr{U}$, *and an element* $f \in \Gamma(\mathscr{U}, \mathscr{O}_{X(N)})$ *such that the following properties hold:*

(a) $(f=0)$=*is an irreductible hypersurface of* $\mathscr{U}$.

(b) $(f=0) \cap E(N) \cap \mathscr{U} = \mathscr{V}_0 \cap \mathscr{U}$.

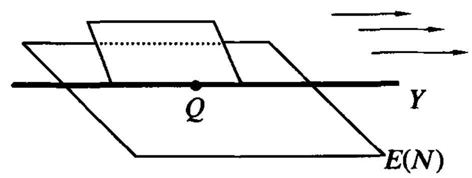


Fig. 24

(c) *For each* $Q' \in \mathcal{V}_0 \cap \mathcal{U}$, *then the germ of* f *at* Q' *gives the only separatrix of* $(\mathcal{G}(N), \varnothing)$ *at* Q' *which is not an irreducible component of* $E(N)$.

(d) *For each point* $R \in Z(N) - \mathcal{V}_0$, *then there is an open set* $W \subset X(N)$, $W \ni R$, *such that* $W \cap (f=0) = \varnothing$.

Proof. Let us consider the set

$$\mathcal{H} = \{Q \in \mathcal{V}_0;\ \text{we can find } f \text{ and } \mathcal{U} \text{ satisfying (a)–(d)}\}. \tag{1.7}$$

Obviously $\mathcal{H} \subset \mathcal{V}_0$ is an open set of $\mathcal{V}_0$. Let us show now that $\mathcal{H}$ is a closed set of $\mathcal{V}_0$. Let $\mathfrak{H} \subset \mathcal{V}_0$ be the closure of $\mathcal{H}$. Pick a point $Q \in \mathfrak{H} - \mathcal{H}$. We have two possibilities:

(a) $e(E(N), Q) = 1$.

(b) $e(E(N), Q) = 2$.

In the first case, there is exactly one irreducible component Y of $\operatorname{Sing}(\mathcal{G}(N), \varnothing)$ such that $Q \in Y$. In fact $Y \subset \mathcal{V}_0$. (See Figure 24.)

If there is no convergent separatrix at Q, it is the same for the points $Q' \in Y$, Q' near Q, in view of the local analytic triviality along Y proved in Part II. This contradicts the fact $Q \in \mathfrak{H}$. Hence, there is a convergent separatrix at Q. The local triviality allows us to find easily $\mathcal{U}, f$ with the properties (a)–(d). Hence $Q \in \mathcal{H}$. Contradiction.

Assume now that $e(E(N), Q) = 2$. Then there are exactly two irreducible components Y_1 and Y_2 of $\mathcal{V}$ such that $Q \in Y_1 \cap Y_2$. In fact $Y_1, Y_2 \subset \mathcal{V}_0$. (See Figure 25.)

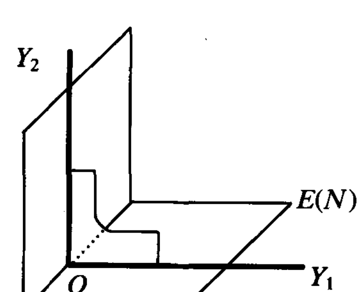


Fig. 25

Up to change the order, for $Q' \in Y_1$ arbitrarily near of Q, we have $Q' \in \mathscr{H}$. Hence, by Proposition II.5.5, we know that the only formal separatrix at Q is in fact a convergent one. This allows us to find without difficulty $\mathscr{U}$, f with the properties (a)–(d). Contradiction.

Hence $\mathfrak{H}=\mathscr{H}$. Moreover, consider the analytic local triviality at $P(N)$ along Y_0. Since $\Gamma(N)$ is contained in a separatrix, we deduce that this separatrix is in fact a convergent one. This implies that $P(N) \in \mathscr{H}$. By connectedness we find that $\mathscr{H}=\mathscr{V}_0$. □

COROLLARY 1.5. *There is an irreductible hypersurface $S(N) \subset X(N), S(N) \not\subset E(N)$, which is an invariant variety for $(\mathscr{G}(N), \varnothing)$.*

Proof. Given $Q \in \mathscr{V}_0$, let $(\mathscr{U}_Q, f_Q)$ be the data of Lemma 1.4. Up to make $X(0)$ smaller, there is a finite number of points $Q_1, \ldots, Q_k$ such that

$$\mathscr{V}_0 \subset \bigcup_{i=1,\ldots,k} \mathscr{U}_{Qi}. \tag{1.8}$$

Given $R \in Z(N) - \mathscr{V}_0$, denote by $W_{R,i}$ the corresponding data in Lemma 1.4 to the point $Q_j, i=1, \ldots, k$. Put

$$W_R = \bigcap_{i=1,\ldots,k} W_{R,i}. \tag{1.9}$$

By compactness of $Z(N)$, we can cover $Z(N)$ by a finite number of open sets W_R and $\mathscr{U}_R$. In particular, up to make $X(0)$ smaller, we may assume that

$$X(N) \subset (\textstyle\bigcup_R W_R) \cup (\bigcup_{i=1,\ldots,K} \mathscr{U}_{Qi}). \tag{1.10}$$

Now, let us define $S(N)$ by

$$S(N) \cap \mathscr{U}_{Qi} = (f_{Qi} = 0) \tag{1.11}$$

$$S(N) \cap W_R = \varnothing. \tag{1.12}$$

Note that (1.12) does not contradict (1.11). Also, by the uniqueness of the separatrices we see that

$$(f_{Qj} = 0) \cap \mathscr{U}_{Qi} \cap \mathscr{U}_{Qj} = (f_{Qi} = 0) \cap \mathscr{U}_{Qi} \cap \mathscr{U}_{Qj} \tag{1.13}$$

and hence $S(N)$ is a well defined closed hypersurface of $X(N)$. Obviously $S(N)$ is non-singular. Since $\mathscr{V}_0$ is connected and

$$S(N) \cap E(N) = \mathscr{V}_0 \tag{1.14}$$

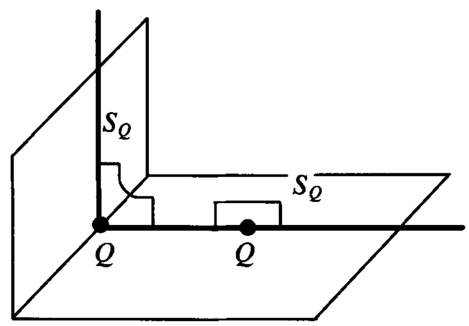


Fig. 26

then $S(N)$ is connected. Hence $S(N)$ is irreducible. Moreover since $S(N)$ is an invariant variety at $P(N)$, then $S(N)$ is invariant at each point. □

Now, in the above situation, let us put

$$S = (\pi(1) \circ \ldots \circ \pi(N))\,(S(N)) \subset X(0). \tag{1.15}$$

By the Proper Mapping Theorem and the fact $S(N) \not\subset E(N)$, we see that S is an irreducible hypersurface of $X(0)$, with $P \in S$. Looking at a point Q in $S - \mathrm{Sing}(\mathcal{F}, \varnothing)$, we see that S is an invariant variety. This ends the proof of the Theorem 1.1.

§ 2. Formal separatrices

Let us fix a sequence of desingularization (1.1) like in the previous paragraph. Now, consider a point $Q \in \mathcal{U}$, where $\mathcal{U}$ is defined as in (1.2). We know that there is exactly one (formal or convergent) separatrix S_Q of $(\mathcal{G}(N), \varnothing)$ at Q, such that S_Q is not an irreducible component of $E(N)$. We have two possibilities:

(I) $(E(N), Q) = 1$, and hence $\mathcal{U}$ is given locally at Q by a non singular curve contained in $E(N)$.

(II) $(E(N), Q) = 2$, and hence $\mathcal{U}$ is given locally at Q by two non singular curves having normal crossings with $E(N)$.

Moreover

$$S_Q \cap E(N) = \mathcal{U}, \quad \text{formally at } Q. \tag{2.1}$$

(See Figure 26.)

The arguments in the preceding paragraph show that if S_Q is convergent, then the connected component $\mathcal{U}_j$ of $\mathcal{U}$ which contains Q determines a convergent separatrix S_j of $(\mathcal{F},\varnothing)$ at the point $P\in X(0)$.

Conversely, let S be a convergent separatrix of $(\mathcal{F},\varnothing)$ at the point $P\in X(0)$ and consider the strict transform $S(N)\subset X(N)$. Note that

$$S(N)\cap E(N) \tag{2.2}$$

is connected. Pick a point $Q\in S(N)\cap E(N)$. Then $Q\in\mathcal{U}$ and

$$S(N)=S_Q,\quad \text{locally at } Q. \tag{2.3}$$

Thus $S(N)$ is the invariant variety of $(\mathcal{G}(N),\varnothing)$ which has been constructed from the component $\mathcal{U}_j=S(N)\cap E(N)$ of $\mathcal{U}$. Hence we have a bijection:

$$\begin{bmatrix}\text{convergent separatrices}\\ \text{of } (\mathcal{F},\varnothing) \text{ at } P\end{bmatrix} \leftrightarrow \begin{bmatrix}\text{connected components}\\ \mathcal{U}_j \text{ of } \mathcal{U} \text{ such that } S_Q \text{ is}\\ \text{convergent for a } Q\in\mathcal{U}_j\end{bmatrix}. \tag{2.4}$$

In this paragraph we shall extend the above bijection to the formal separatrices. More precisely, we shall prove the following result:

THEOREM 2.1. *There is a bijection*

$$\begin{bmatrix}\textit{formal or convergent}\\ \textit{separatrices of } (\mathcal{F},\varnothing)\\ \textit{at } P\end{bmatrix} \leftrightarrow \begin{bmatrix}\textit{connected components}\\ \textit{of } \mathcal{U}\end{bmatrix} \tag{2.5}$$

such that each separatrix S of $(\mathcal{F},\varnothing)$ at P corresponds to $S(N)\cap E(N)$, where $S(N)$ is the strict transform of S under $\pi(1)\circ\ldots\circ\pi(N)$.

Note that, with our definition of simple singularities, the formal separatrices are "disjoint" after desingularization, in the sense that two of them never have a common formal curve. Note also that if S is a formal separatrix of $(\mathcal{F},\varnothing)$, then we can do the correspondence

$$S\hookrightarrow S(N)\cap E(N) \tag{2.6}$$

exactly as in the convergent case. Hence, the only remaining problem is to show that the map of (2.5) is a surjective map. In other words, we have to show that the formal separatrices S_Q "glue" along a connected component of $\mathcal{U}$ in such a way that they project to a formal separatrix at $P\in X(0)$. The nature of the formal coordinates defining S_Q, which was investigated in Part II, will make us able to do that.

Let us express our results in terms of the theory of formal analytic spaces (cf. [2]). At each step $i\geqslant 0$ of the sequence (1.1), put

$$\hat{X}(i) = \text{formal completion of } X(i) \text{ along } Z(i). \tag{2.7}$$

This means that $\hat{X}(i)$ is a locally ringed space $(Z(i), \mathcal{O}_{\hat{X}(i)})$, whose structural sheaf is

$$\mathcal{O}_{\hat{X}(i)} = \varprojlim \frac{\mathcal{O}_{X(i)}}{(\mathcal{I}_{Z(i)})^m} \tag{2.8}$$

where $\mathcal{I}_{Z(i)}$ is the sheaf of ideals defining $Z(i)\subset X(i)$. The sequence of morphisms in (1.1) gives rise to a sequence of morphisms of formal analytic spaces

$$\hat{X}(0) \overset{\hat{\pi}(1)}{\leftarrow} \hat{X}(1) \overset{\hat{\pi}(2)}{\leftarrow} \dots \overset{\hat{\pi}(N)}{\leftarrow} \hat{X}(N). \tag{2.9}$$

DEFINITION 2.2. *Given an index $i\geqslant 0$, an irreducible hypersurface $\mathcal{H}$ of $\hat{X}(i)$ is a coherent sheaf of ideals $\mathcal{H}\subset\mathcal{O}_{\hat{X}(i)}$ such that*:

(a) *$\mathcal{H}$ is locally free of rank one.*

(b) *For each point $P\in Z(i)$ then $\mathcal{H}_P=f\mathcal{O}_{\hat{X}(i),P}$ and f is an irreducible element of the formal completion $\hat{\mathcal{O}}_{\hat{X}(i),P}$* of *$\mathcal{O}_{\hat{X}(i),P}$ along its maximal ideal.*

(c) *The support* $\mathrm{Supp}(\mathcal{O}_{\hat{X}(i)}/\mathcal{H})$ *is connected.*

Remarks 2.3. (a) The above definition is not a standard one. Actually, for this special sequence (1.1), the conditions (b) and (c) of the above definition are equivalent to say that there is no decomposition $\mathcal{H}=\mathcal{H}_1\mathcal{H}_2$.

(b) In the case $i=0$, we have $Z(0)=\{P\}$. Hence

$$\hat{X}(0) = (\{P\}, \mathcal{O}_{\hat{X}(0)}) \tag{2.10}$$

and $\mathcal{O}_{\hat{X}(0),P}$ is the formal completion of $\mathcal{O}_{X(0),P}$ along its maximal ideal. In particular, if (x, y, z) is a regular system of parameters of $\mathcal{O}_{X(0),P}$, then

$$\mathcal{O}_{\hat{X}(0),P} = \mathbf{C}[[x, y, z]]. \tag{2.11}$$

Hence, any irreducible hypersurface of $\hat{X}(0)$ is given by an irreducible formal power series $f\in\mathbf{C}[[x, y, z]]$.

(c) For each point $Q\in Z(i)$, we have that

$$\mathcal{O}_{X(i),Q} \subset \mathcal{O}_{\hat{X}(i),Q} \subset \hat{\mathcal{O}}_{\hat{X}(i),Q} = \hat{\mathcal{O}}_{X(i),Q}. \tag{2.12}$$

(d) Each irreducible hypersurface $S(i)$ of $X(i)$ produces in an evident way an irreducible hypersurface $\hat{S}(i)$ of $\hat{X}(i)$. To see this, note that

$$S(0) = \pi(1) \circ \ldots \circ \pi(i) (S(i)) \tag{2.13}$$

is an irreducible hypersurface of $X(0)$ (except if $S(i)$ is an irreducible component of $E(i)$), hence it is given by an irreducible convergent series $f \in \mathcal{O}_{X(0),P}$, and then $f \in \hat{\mathcal{O}}_{X(0),P}$ is also an irreducible formal power series. Now, $S(i)$ is the strict transform of $S(0)$; this allows us to verify the properties (b) and (c) of Definition 2.2. In the case that $S(i)$ is an irreducible component of $E(i)$, we see directly that $\hat{S}(i)$ is an irreducible hypersurface of $\hat{X}(i)$.

LEMMA 2.4. *Fix an index i, $0 \leq i \leq N-1$, and let us consider an irreducible hypersurface $\mathcal{H}$ of $\hat{X}(i)$ which is not an irreducible component of $E(i)$. Let $F \subset E(i+1)$ be the exceptional divisor of $\pi(i+1)$ and let $\hat{\mathcal{I}}_F$ be the corresponding irreducible hypersurface in $\hat{X}(i+1)$. Then, there is a unique irreducible hypersurface $\mathcal{H}'$ of $\hat{X}(i+1)$ and an integer $\nu \geq 0$ such that*

$$\mathcal{H}\mathcal{O}_{\hat{X}(i+1)} = ((\hat{\mathcal{I}}_F)^{\nu}\, \mathcal{H}')\, \mathcal{O}_{\hat{X}(i+1)}. \tag{2.14}$$

Moreover $\mathcal{H}'$ is not an irreducible component of $E(i+1)$.

Proof. The hypersurface $\mathcal{H}'$ is locally given by the strict transform of the formal power series generating locally $\mathcal{H}$. The details are straightforward. □

DEFINITION 2.5. *In the above situation $\mathcal{H}'$ will be called the strict transform of $\mathcal{H}$ by $\hat{\pi}(i+1)$.*

PROPOSITION 2.6. *Fix an index i, $0 \leq i \leq N-1$, and let us consider an irreducible hypersurface $\mathcal{H}$ of $\hat{X}(i)$ which is not an irreducible component of $E(i)$. Let $\mathcal{H}'$ be the strict transform of $\mathcal{H}$ by $\hat{\pi}(i+1)$. Then, the following conditions are equivalent:*

(i) *There is a point $Q \in \operatorname{Supp}(\mathcal{O}_{\hat{X}(i)}/\mathcal{H})$ such that $\mathcal{H}_Q = f\mathcal{O}_{\hat{X}(i),Q}$, where f gives a formal separatrix of $(\mathcal{G}(i), \emptyset) = \operatorname{hol}(\mathcal{F}(i), E(i))$ at the point Q.*

(ii) *For each point $Q \in \operatorname{Supp}(\mathcal{O}_{\hat{X}(i)}/\mathcal{H})$ then $\mathcal{H}_Q = f\mathcal{O}_{\hat{X}(i),Q}$, where f gives a formal separatrix of $(\mathcal{G}(i), \emptyset) = \operatorname{hol}(\mathcal{F}(i), E(i))$ at the point Q.*

(iii) *There is a point $Q' \in \operatorname{Supp}(\mathcal{O}_{\hat{X}(i+1)}/\mathcal{H}')$ such that $\mathcal{H}'_{Q'} = f'\mathcal{O}_{\hat{X}(i+1),Q'}$, where f' gives a formal separatrix of $(\mathcal{G}(i+1), \emptyset)$ at the point Q'.*

Proof. (i)⇔(ii). One sees that (i) gives an open and closed property.

(i)⇔(iii). Take $Q' \in \pi^{-1}(Q)$. Now, we can compute in terms of formal equations and formal strict transforms. □

LEMMA 2.7. *Let $\mathcal{U}\subset\mathbf{C}_n$ be a Stein open set and consider*

$$Z=\left(\prod_{i\in C} x_i=0\right)\cap\mathcal{U} \tag{2.15}$$

where $x_1, \dots, x_n$ are the coordinates in $\mathbf{C}^n$. Let $\hat{\mathcal{U}}$ be the formal completion of $\mathcal{U}$ along Z and let $\mathcal{H}$ be an irreducible hypersurface in $\hat{\mathcal{U}}$. Then $\mathcal{U}$ is generated by a single global section.

Proof. Let $\mathcal{I}_Z$ be the ideal defining Z. Consider

$$\mathcal{H}^{(m)}=\frac{\mathcal{H}+(\mathcal{I}_Z)^m}{(\mathcal{I}_Z)^m}\subset\frac{\mathcal{O}_{\hat{\mathcal{U}}}}{(\mathcal{I}_Z)^m}=\frac{\mathcal{O}_{\mathcal{U}}}{(\mathcal{I}_Z)^m}. \tag{2.16}$$

Now, standard arguments over Stein spaces show that $\mathcal{H}^{(m)}$ is generated by a single global section $\sigma^{(m)}$. If $m'>m$, then $\sigma^{(m')}$ induces in $\mathcal{H}^{(m)}$ a generating global section $(\sigma^{(m)})'$. Now, up to multiply $\sigma^{(m')}$ by a unit in the global sections of $\mathcal{O}_{\hat{\mathcal{U}}}$, we can assume that

$$(\sigma^{(m)})'=\sigma^{(m)}. \tag{2.17}$$

Thus, $\sigma^{(m)}\to\sigma$, where σ is a generating global section of $\mathcal{H}$. □

The following result is analogous to Chow's Theorem in Analytic Geometry. In the Appendix we shall give a direct proof of it.

THEOREM 2.8. *Let us fix an index i, $0\leq i\leq N-1$, and an irreducible hypersurface $\mathcal{H}(i+1)$ of $\hat{X}(i+1)$ which is not an irreducible component of $E(i+1)$. Then there is a unique irreducible hypersurface $\mathcal{H}(i)$ of $\hat{X}(i)$ such that $\mathcal{H}(i+1)$ is the strict transform of $\mathcal{H}(i)$ by $\hat{\pi}(i+1)$.*

Now, we can end the proof of Theorem 2.1 as follows. In Part II we have seen that the formal separatrices are given at the simple singularities by elements in the local rings of $\hat{X}(N)$ (see Proposition II.5.4 and the description in the proof above of the local sections). Chose a connected component $\mathcal{V}_j$ of $\mathcal{V}$. Thus, repeating the arguments of Lemma 1.4 and Corollary 1.5 in this context we have an irreducible hypersurface $\hat{S}(N)\subset\hat{X}(N)$ whose support is $Z\cap\mathcal{U}_j$, such that $\hat{S}(N)$ is a formal separatrix at each point. By Theorem 2.8, then $\hat{S}(N)$ is the strict transform of an irreducible hypersurface $\hat{S}(0)\subset\hat{X}(0)$ which, by the Proposition 2.6, gives the desired formal separatrix at the point P. □

Appendix: Proof of Theorem IV.(2.8)

In order to simplify notation, let us put

$$X = X(i),\quad \hat{X} = \hat{X}(i),\quad Z = Z(i),\quad Y = Y(i),\quad E = E(i),\quad \pi = \pi(i),\quad \hat{\pi} = \hat{\pi}(i).$$

$$X' = X(i+1),\quad \hat{X} = \hat{X}(i+1),\quad Z' = Z(i+1),\quad E' = E(i+1),\quad \mathcal{H}' = \mathcal{H}(i+1).$$

A standard argument allows us to glue together the local solutions in X and hence we can restrict ourselves to the following case:

$$X = \{(x_1, \ldots, x_n) \in \mathbf{C}^n;\ |x_i| < 1\} \subset \mathbf{C}^n.$$

$$E = \left(\prod_{i \in A} x_i = 0\right);\quad Z = \left(\prod_{i \in C} x_i = 0\right),\ \text{for some } C \subset A$$

$$Y = (x_i = 0; i \in B),\quad \text{for some } B \subset \{1, \ldots, n\}.$$

We can give a description of the blowing-up $\pi : X' \to X$ as follows

$$X' = \bigcup_{i \in B} \mathcal{U}_i.$$

$$\mathcal{U}_i = \{(y_1^{(i)}, \ldots, y_n^{(i)}) \in \mathbf{C}^n;\ |y_j^{(i)}| < 1 \text{ if } j = i \text{ or } j \notin B\}.$$

$$y_j^{(i)} = x_j,\ \text{if } j \notin B - \{i\};\quad y_j^{(i)} = x_j/x_i,\ \text{if } j \in B - \{i\}.$$

In particular, the $\mathcal{U}_i$ are Stein spaces. Note that

$$\mathcal{U}_i \cap Z' = \left(\prod_{j \in C'_i} y_j^{(i)} = 0\right) \subset \mathcal{U}_i$$

where

$$C'_i = C \cup \{i\},\ \text{if either } B \cap C \neq \varnothing \text{ or } C = \varnothing;\quad C'_i = C,\ \text{if } B \cap C = \varnothing \text{ and } C \neq \varnothing.$$

(The case $C = \varnothing$ corresponds to the first step in the sequence (1.1); note that in this case $B = \{1, \ldots, n\}$.)

On the other hand, we know that for a fixed $a, b \in B$, $a \neq b$, the following morphisms are bijective

$$\Gamma(X, \mathcal{O}_X) \to \Gamma(X', \mathcal{O}_{X'}) \to \Gamma(\mathcal{U}_a \cup \mathcal{U}_b, \mathcal{O}_{X'})$$

and also

$$\Gamma(Z, \mathcal{O}_{\hat{X}}) \to \Gamma(Z', \mathcal{O}_{\hat{X}'}) \to \Gamma(\mathcal{U}_a \cup \mathcal{U}_b, \mathcal{O}_{\hat{X}'}). \tag{A.1}$$

Now, since the $\mathcal{U}_i$ are Stein spaces, by the preceding lemma, there is a section

$$f_i \in \Gamma(\mathcal{U}_i \cap Z', \mathcal{O}_{\hat{X}'})$$

which generates $\mathcal{H}'$ in $\mathcal{U}_i \cap Z'$. In view of (A.1), it is enough to show that we can choose a unit $u_a \in \Gamma(\mathcal{U}_a \cap Z', \mathcal{O}_{\hat{X}'})^*$ and an integer $t \geq 0$ such that

$$(x_a)^t u_a f_a$$

extends to a section in $\Gamma((\mathcal{U}_a \cup \mathcal{U}_b) \cap Z', \mathcal{O}_{\hat{X}'})^*$. But note that

$$f_b = f_a u_{ab}; \quad u_{ab} \in \Gamma(\mathcal{U}_a \cap \mathcal{U}_b \cap Z', \mathcal{O}_{\hat{X}'})^*.$$

Assume that the following property is true:

(A.2) There is an integer $t \geq 0$ and two elements $u_a \in \Gamma(\mathcal{U}_a \cap Z', \mathcal{O}_{\hat{X}'})^*$ and $u_b \in \Gamma(\mathcal{U}_b \cap Z', \mathcal{O}_{\hat{X}'})^*$ such that $u_{ab} = (y_b^{(a)})^{-t} u_a (u_b)^{-1}$.

Then, we are done, since (A.2) implies that

$$(x_b)^t u_b f_b = (x_a)^t u_a f_a \tag{A.3}$$

and thus $(x_a)^t u_a f_a$ extends to a section in $\Gamma((\mathcal{U}_a \cup \mathcal{U}_b) \cap Z', \mathcal{O}_{\hat{X}'})$.

It remains to prove (A.2). First of all, let us give a description of the rings of sections that we need. Given $i \in B$ and $j \in C'_i$, put

$$W_j^{(i)} = \{(y_1^{(i)}, \ldots, y_n^{(i)} \in \mathcal{U}_i;\ y_j^{(i)} = 0\} \subset \mathbf{C}^n$$

and consider it as a set in $\mathbf{C}^{n-1}$ with coordinates $y_s^{(i)}, s \neq j$. Let us denote by $\mathcal{O}(W_j^{(i)})$ the ring of holomorphic functions defined in $W_j^{(i)}$. Then, we have that

$$\Gamma(\mathcal{U}_i \cap Z', \mathcal{O}_{\hat{X}'}) = \bigcap_{j \in C'_i} \mathcal{O}(W_j^{(i)})[[y_j^{(i)}]] \tag{A.4}$$

where the intersection is taken inside the formal power series ring

$$\mathbf{C}[[y_1^{(i)}, \ldots, y_n^{(i)}]].$$

Let us consider the two indices a, b. Then

$$\mathcal{U}_a \cap \mathcal{U}_b \cap Z' = \bigcup_{j \in C'_a - \{b\}} W_{jb}^{(a)},$$

where $W_{jb}^{(a)} = W_j^{(a)} - \{y_b^{(a)} = 0\} \subset W_j^{(a)}$. Let us denote by $\mathcal{O}(W_{jb}^{(a)})$ the ring of holomorphic functions defined in $W_{jb}^{(a)}$. Note that each element in $\mathcal{O}(W_{jb}^{(a)})$ is defined by a Laurent series and thus

$$\mathcal{O}(W_{jb}^{(a)}) \subset \mathbf{C}[[y_1^{(a)}, \ldots, y_{j-1}^{(a)}, y_{j+1}^{(a)}, \ldots, y_n^{(a)}; (y_b^{(a)})^{-1}]].$$

Then, we have that

$$\Gamma(\mathcal{U}_a \cap \mathcal{U}_b \cap Z', \mathcal{O}_{\hat{X}'}) = \bigcap_{j \in C'_a - \{b\}} \mathcal{O}(W_{jb}^{(a)})[[y_j^{(a)}]], \tag{A.5}$$

where the intersection is taken inside the ring

$$\mathbf{C}[[y_1^{(a)}, \ldots, y_n^{(a)}; (y_b^{(a)})^{-1}]].$$

Moreover, the units in (A.4) and (A.5) are given by

$$\Gamma(\mathcal{U}_i \cap Z', \mathcal{O}_{\hat{X}'})^* = \bigcap_{j \in C'_i} [(\mathcal{O}(W_j^{(i)})[[y_j^{(i)}]])^*]$$

$$\Gamma(\mathcal{U}_a \cap \mathcal{U}_b \cap Z', \mathcal{O}_{\hat{X}'})^* = \bigcap_{j \in C'_a - \{b\}} [(\mathcal{O}(W_{jb}^{(a)})[[y_j^{(a)}]])^*].$$

Now, given an index $j \in C'_a - \{b\}$, let us write

$$u_{ab} = \sum_{s \geq 0} \mu_s^{(abj)} (y_j^{(a)})^s; \quad \mu_s^{(abj)} \in \mathcal{O}(W_{jb}^{(a)}).$$

Moreover, since u_{ab} is a unit, necessarily

$$\mu_0^{(abj)} \in \mathcal{O}(W_{jb}^{(a)})^*.$$

In particular, it defines a function

$$\mu_0^{(abj)} \colon W_{jb}^{(a)} \to \mathbf{C}^* = \mathbf{C} - \{0\}. \tag{A.6}$$

Note that the fundamental group of $W_{jb}^{(a)}$ is $\mathbf{Z}$. Hence (1.6) defines a map $\mathbf{Z} \to \mathbf{Z}$. Let $-t$ be the image of 1 under this map. Then

$$\mu_0^{(abj)} (y_b^{(a)})^t$$

defines the zero map between fundamental groups. Thus, we have a lifting

$$\begin{tikzcd} & \mathbf{C} \arrow[d, "\exp"] \\ W_{jb}^{(a)} \arrow[ur, dashed, "\nu_0^{(abj)}"] \arrow[r, "\mu_0^{(abj)}(y_b^{(a)})^t"'] & \mathbf{C}^* \end{tikzcd}$$

i.e., there is an element $\nu_0^{(abj)} \in \mathcal{O}(W_{jb}^{(a)})$ such that

$$\mu_0^{(abj)}(y_b^{(a)})^t = \exp(\nu_0^{(abj)}). \tag{A.7}$$

Now, let us write

$$\nu_{abj} = \sum_{s \geq 0} \nu_s^{(abj)}(y_j^{(a)})^s$$

and let us consider the equation

$$u_{ab}(y_b^{(a)})^t = \exp(\nu_{abj}). \tag{A.8}$$

In view of (A.7), by identifying indeterminate coefficients in $(y_j^{(a)})^s$ we see that (A.8) has a unique solution ν_{abj} and

$$\nu_{abj} \in \mathcal{O}(W_{jb}^{(a)})\,[[y_j^{(a)}]]. \tag{A.9}$$

Now, put

$$R_b^{(a)} = \bigcap_{j \in C'_a - \{b\}} W_{jb}^{(a)}$$

and denote by $\boldsymbol{\mu}_0^{(abj)}$ the restriction of $\mu_0^{(abj)}$ to $R_b^{(a)}$. Note that

$$\boldsymbol{\mu}_0^{(abj)} = \boldsymbol{\mu}_0^{(abj')}, \quad \text{for all } j, j' \in C'_a - \{b\}.$$

Put $\boldsymbol{\mu}_0^{(ab)} = \boldsymbol{\mu}_0^{(abj)}$ for some $j \in C'_a - \{b\}$. Then, we have that

$$\boldsymbol{\mu}_0^{(ab)}(y_b^{(a)})^t = \exp(\nu_0^{(abj)}),$$

where $\boldsymbol{\nu}_0^{(abj)}$ means the restriction of $\nu_0^{(abj)}$ to $R_b^{(a)}$. Thus, up to modify in a trivial way $\nu_0^{(abj)}$, we may assume that

$$\boldsymbol{\nu}_0^{(ab)} = \boldsymbol{\nu}_0^{(abj)} = \boldsymbol{\nu}_0^{(abj')}, \quad \text{for all } j, j' \in C'_a - \{b\}.$$

Hence

$$\boldsymbol{\mu}_0^{(ab)}(y_b^{(a)})^t = \exp(\mathbf{v}_0^{(ab)}). \tag{A.10}$$

But (A.8) is also obtained in a unique way from (A.10) by identifying coefficients. This shows that

$$\nu_{abj} = \nu_{abj'}, \quad \text{for all } j,j' \in C'_a - \{b\}. \tag{A.11}$$

Put $\nu_{ab} = \nu_{abj}$, then by (A.9) and (A.11) we have that

$$\nu_{ab} \in \Gamma(\mathcal{U}_a \cap \mathcal{U}_b \cap Z', \mathcal{O}_{\hat{X}'})$$

and hence

$$\exp(\nu_{ab}) \in \Gamma(\mathcal{U}_a \cap \mathcal{U}_b \cap Z', \mathcal{O}_{\hat{X}'})^*.$$

Now, writing ν_{ab} as a Laurent series in $y_b^{(a)}$ and since $y_b^{(a)} = (y_b^{(a)})^{-1}$, we can decompose

$$\nu_{ab} = \omega_a - \omega_b,$$

where

$$\omega_a \in \Gamma(\mathcal{U}_a \cap Z', \mathcal{O}_{\hat{X}'}); \quad \omega_b \in \Gamma(\mathcal{U}_b \cap Z', \mathcal{O}_{\hat{X}'}).$$

Put

$$u_a = \exp(\omega_a) \in \Gamma(\mathcal{U}_a \cap Z', \mathcal{O}_{\hat{X}'})^*; \quad u_b = \exp(\omega_b) \in \Gamma(\mathcal{U}_b \cap Z', \mathcal{O}_{\hat{X}'})^*.$$

Thus, we have that

$$u_{ab}(y_b^{(a)})^t = u_a(u_b)^{-1}.$$

The fact that $t \geq 0$ follows easily from the identification (A.3). This ends the proof.

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Received November 21, 1989
Received in revised form September 21, 1990