NORMALISABLE TRANSFORMATIONS IN HILBERT SPACE AND SYSTEMS OF LINEAR INTEGRAL EQUATIONS.

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§ 1. Introduction.

Originally it was assumed in the theory of Hilbert space that the space considered, which we shall call $\mathfrak{R}$, was complete (that is, from $\|f_m - f_n\| \to 0$ for $m, n \to \infty$ follows the existence of an element $f \in \mathfrak{R}$ satisfying $\|f - f_n\| \to 0$ as $n \to \infty$) and separable (that is, there exists an enumerable set of elements lying everywhere dense in $\mathfrak{R}$). It was, however, pointed out by F. Rellich that the most important part of the theory of linear transformations in $\mathfrak{R}$ maintains its validity when the condition of separability is dropped, and, if we confine ourselves to completely continuous self-adjoint or normal transformations, even the completeness of $\mathfrak{R}$ is not necessary.[1] We shortly recall some definitions. A linear transformation K, defined for all elements f of the (not necessarily separable, and not necessarily complete) Hilbert space $\mathfrak{R}$ is called completely continuous when, for every bounded infinite set $\{f\}$, the set $\{Kf\}$ contains a sequence converging to an element $g \in \mathfrak{R}$. It is not difficult to prove that every completely continuous linear transformation K is bounded, that is, $\|Kf\| \leq M\|f\|$ for every $f \in \mathfrak{R}$, where $M \geq 0$ does not depend on f. The bounded linear transformation K, defined for all elements $f \in \mathfrak{R}$, is called normal when the adjoint K^* is also defined for all elements $f \in \mathfrak{R}$ (so that therefore the relation

$$(1) \qquad (Kf, g) = (f, K^* g)$$

holds for arbitrary $f, g \in \mathfrak{R}$), and when, moreover, $KK^* = K^*K$. If K is its own adjoint, K is called self-adjoint. Evidently every self-adjoint transformation is normal. In the case that the space $\mathfrak{R}$ is complete, it is a well-known theorem that every bounded linear transformation possesses a uniquely determined bounded

[1] F. RELLICH, Spektraltheorie in nicht-separablen Räumen, Math. Annalen 110 (1934), p. 342—356.

adjoint K^*, and that, if, moreover, K is completely continuous, the same holds for K^*. When, however, the space $\mathfrak{R}$ is not complete, this theorem is no longer true.

F. Rellich has proved now the following

Theorem A. *Let $\mathfrak{R}$ be a (not necessarily separable, and not necessarily complete) Hilbert space in which the normal transformation K is defined. Supposing now that both K and K^* are completely continuous, and that K is not identical with the null-transformation, this transformation K has at least one characteristic element φ_1 with characteristic value $\lambda_1 \neq 0$. Moreover, the same element φ_1 is also a characteristic element of K^* with characteristic value $\bar{\lambda}_1$ (by $\bar{\lambda}_1$ we mean the conjugate complex number of λ_1). Furthermore, there exists an orthonormal (finite or enumerable) sequence of characteristic elements $\varphi_i (i = 1, 2, \ldots)$ satisfying*

$$K\varphi_i = \lambda_i \varphi_i, \quad K^*\varphi_i = \bar{\lambda}_i \varphi_i, \quad \lambda_i \neq 0$$

such that, if $\alpha_i = (f, \varphi_i)$ for an arbitrary $f \in \mathfrak{R}$,

$$\lim_{n\to\infty} \| Kf - \sum_{i=1}^{n} \lambda_i \alpha_i \varphi_i \| = \lim_{n\to\infty} (Kf - \sum_{i=1}^{n} \lambda_i \alpha_i \varphi_i,\ Kf - \sum_{i=1}^{n} \lambda_i \alpha_i \varphi_i)^{1/2} = 0 \tag{2}$$

and

$$\lim_{n\to\infty} \| K^*f - \sum_{i=1}^{n} \bar{\lambda}_i \alpha_i \varphi_i \| = \lim_{n\to\infty} (K^*f - \sum_{i=1}^{n} \bar{\lambda}_i \alpha_i \varphi_i,\ K^*f - \sum_{i=1}^{n} \bar{\lambda}_i \alpha_i \varphi_i)^{1/2} = 0. \tag{3}$$

In the present paper we shall introduce generalizations of the notions hitherto defined, and this will lead to the proof of a theorem which contains Theorem A as a special case. As an additional result we shall see that the assumption of the complete continuity of K^*, which is essential in Rellich's proof, is superfluous. Our method of proof differs considerably from that adopted by Rellich. Moreover, we shall show that this theorem may be used to obtain expansion theorems for certain systems of linear integral equations, a result which generalizes earlier results of J. Ernest Wilkins[1] (who in his turn generalized investigations of G. A. Bliss[2] and W. T. Reid[3]) and the present author.[4]

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[1] J. Ernest Wilkins, Definitely self-conjugate adjoint integral equations, Duke Math. Journal 11 (1944), p. 155—166.

[2] G. A. Bliss, Definitely self-adjoint boundary value problems, Transactions Am. Math. Soc. 44 (1938), p. 413—428.

[3] W. T. Reid, Expansion problems associated with a system of linear integral equations, Transactions Am. Math. Soc. 33 (1931), p. 475—485.

[4] A. C. Zaanen, On the theory of linear integral equations VIII, Proc. Kon. Ned. Akad. v. Wetensch. (Amsterdam) 50 (1947), p. 465—473 and p. 612—617 (= Indagationes Math. 9 (1947), p. 271—279 and p. 320—325).

We consider a bounded, positive, self-adjoint transformation H, that is, a bounded linear transformation H satisfying $(Hf, g) = (f, Hg)$ and $(Hf, f) \geq 0$ for arbitrary $f, g \in \mathfrak{R}$, and we shall denote by $[\mathfrak{L}]$ the set of all elements h for which $Hh = 0$, while the set of all elements g orthogonal to $[\mathfrak{L}]$ (that is, $(g, h) = 0$ for all $h \in [\mathfrak{L}]$) will be called $[\mathfrak{M}]$. Assuming now that every $f \in \mathfrak{R}$ is expressible in the form $f = g + h$, where $g \in [\mathfrak{M}]$, $h \in [\mathfrak{L}]$, the projection E on $[\mathfrak{M}]$ is defined by $g = Ef$. Furthermore we shall write $N(f)$ for the non-negative number $(Hf, f)^{1/2}$. It is important to observe that, since the identical transformation I is evidently bounded, self-adjoint and positive, we obtain a special case by taking $H = I$. In this case the set $[\mathfrak{L}]$ contains only the nullelement, $E = I$ and $N(f) = \|f\|$. Returning to the general case, we shall call two bounded linear transformations K and $\tilde{K}$ each other's *H-adjoints* when

$$(4) \qquad (HKf, g) = (Hf, \tilde{K}g)$$

holds for arbitrary $f, g \in \mathfrak{R}$. When, moreover, $HK\tilde{K} = H\tilde{K}K$, the transformation K will be called *normalisable* (relative to H). Two elements $f, g \in \mathfrak{R}$ will be said to be *H-orthogonal* when $(Hf, g) = 0$, and the sequence φ_i of elements $\varphi_i \in \mathfrak{R}$ $(i = 1, 2, \ldots)$ will be termed *H-orthonormal* when $(H\varphi_i, \varphi_j) = 1$ for $i = j$ and $= 0$ for $i \neq j$.

We shall prove now, besides other theorems, the following theorem (obtained y joining together the contents of the Theorems 10, 12 and 16):

Theorem B. *Let $\mathfrak{R}$ be a (not necessarily separable, and not necessarily complete) Hilbert space in which the normalisable transformation K is defined. Supposing now that the transformation $T = EK$ is completely continuous, and that $P = HK$ is not identical with the nulltransformation, the transformation $T = EK$ has at least one characteristic element φ_1 with characteristic value $\lambda_1 \neq 0$. Moreover, the same element φ_1 is also a characteristic element of $\tilde{T} = E\tilde{K}$ with characteristic value $\bar{\lambda}_1$. Furthermore there exists an H-orthonormal (finite or enumerable) sequence of characteristic elements φ_i $(i = 1, 2, \ldots)$ satisfying*

$$T\varphi_i = \lambda_i \varphi_i, \quad \tilde{T}\varphi_i = \bar{\lambda}_i \varphi_i, \quad \lambda_i \neq 0$$

such that, if $\alpha_i = (Hf, \varphi_i)$ for an arbitrary $f \in \mathfrak{R}$,

$$(5) \quad \lim_{n \to \infty} N(Kf - \sum_{i=1}^{n} \lambda_i \alpha_i \varphi_i) = \lim_{n \to \infty} (H(Kf - \sum_{i=1}^{n} \lambda_i \alpha_i \varphi_i),\ Kf - \sum_{i=1}^{n} \lambda_i \alpha_i \varphi_i)^{1/2} = 0$$

and

$$(6)\quad \lim_{n\to\infty} N(\tilde{K}f - \sum_{i=1}^{n} \bar{\lambda}_i \alpha_i \varphi_i) = \lim_{n\to\infty} (H(\tilde{K}f - \sum_{i=1}^{n} \bar{\lambda}_i \alpha_i \varphi_i),\ \tilde{K}f - \sum_{i=1}^{n} \bar{\lambda}_i \alpha_i \varphi_i)^{1/2} = 0.$$

In the special case that any element, satisfying $Hf = 0$, satisfies also $Kf = 0$, the transformation K itself has at least one characteristic element ψ_1 with characteristic value λ_1, and there exists an H-orthonormal (finite or enumerable) sequence of characteristic elements ψ_i, satisfying $K\psi_i = \lambda_i \psi_i$ such that, if $\alpha_i = (Hf, \psi_i)$ for an arbitrary $f \in \mathfrak{R}$, the relations (5) and (6) hold with φ_i replaced by ψ_i.

As we have already observed, we obtain a special case by taking $H = I$. From (1) and (4) we infer that in this case any *H-adjoint* $\tilde{K}$ of K is identical with the adjoint K^*, which implies that the notions of normalisable and normal transformations become identical as well. Since also $E = I$, we see that $T = EK = K$ and $\tilde{T} = E\tilde{K} = K^*$; it follows therefore that in this case Theorem B becomes identical with Rellich's Theorem A, except for the assumption about the complete continuity of K^* in Rellich's theorem, which is superfluous.

There exists a close connection between normalisable and normal transformations as will be shown by introducing a factorspace $\mathfrak{Z} = \mathfrak{R}/[\mathfrak{L}]$, the elements $[f]$ of which are classes of elements of the space $\mathfrak{R}$. The element $[f] \in \mathfrak{Z}$ contains besides the element $f \in \mathfrak{R}$ all elements $g \in \mathfrak{R}$ for which $Hg = Hf$. Addition and multiplication with complex numbers α are defined by $[f_1] + [f_2] = [f_1 + f_2]$, $\alpha[f] = [\alpha f]$, while $([f_1], [f_2]) = (Hf_1, f_2)$. In general, the space $\mathfrak{Z}$ will not be complete, even in the case that $\mathfrak{R}$ is complete. By adjunction of ideal elements, however, we shall obtain the complete space $\bar{\mathfrak{Z}}$, the closure of $\mathfrak{Z}$. Defining now, for a normalisable transformation K satisfying the conditions of Theorem B, the transformation $[K]$ in $\mathfrak{Z}$ by $[K][f] = [Kf]$, it will be shown that $[K]$ is a bounded normal transformation in $\mathfrak{Z}$. Defining $[K]$ also for those elements of the closure $\bar{\mathfrak{Z}}$ which do not belong to $\mathfrak{Z}$ (this is possible in virtue of the boundedness of $[K]$), we shall prove (Theorem 25):

Theorem C. *If the normalisable transformation K satisfies the conditions of Theorem B, the transformation $[K]$ in the space $\bar{\mathfrak{Z}}$, corresponding with K in the way described, is a completely continuous normal transformation in $\bar{\mathfrak{Z}}$.*

We shall also pay attention to bounded linear transformations K which, without being normalisable, possess an H-adjoint $\tilde{K}$. An analogue of Theorem B (obtained by joining together the contents of the Theorems 5, 6 and 20) will be proved.

In § 13 we consider normalisable transformations of the form $K = AH$, where one at least of the bounded linear transformations H and A is completely continuous, and where A satisfies $HAHA^*H = HA^*HAH$, and we prove in Theorem 26 that in this case the convergence of the expansions $\Sigma \lambda_i \alpha_i \varphi_i$ and $\Sigma \bar{\lambda}_i \alpha_i \varphi_i$ relative to the norm $N(f)$, as expressed by (5) and (6), may be replaced by ordinary convergence relative to the norm $\|f\|$. The sums of the expansions, however, are not necessarily equal to Kf and $\tilde{K}f$, but

$$Kf = \Sigma \lambda_i \alpha_i \psi_i + h, \quad \tilde{K}f = \Sigma \bar{\lambda}_i \alpha_i \psi_i + k,$$

where $Hh = Hk = 0$.

Finally, in § 14, we indicate the aforementioned applications to the theory of systems of linear integral equations in the space $L_2^{(m)}(\Delta)$ of all functions $f(x)$ with complex values, having the property that $|f(x)|^2$ is Lebesgue-integrable over the m-dimensional interval Δ.

The special case of completely continuous symmetrisable transformations (that is, bounded linear transformations which are their own H-adjoints) has been treated before[1], and its implications for the theory of one linear integral equation with symmetrisable kernel have been investigated in detail.[2] Some of the proofs for normalisable transformations resemble more or less closely the corresponding proofs for symmetrisable transformations. Nevertheless, it seemed advisable to us to include these proofs in the present paper, partly because it is always difficult to know where to draw the line as regards the use of the phrase: "The reader, by comparison with the corresponding theorem, will easily find that . . .", but mainly in order to make an independent whole of the contents of the present paper.

§ 2. Some Preliminary Considerations.

We suppose that $\mathfrak{R}$ is a (not necessarily separable, and not necessarily complete) Hilbert space. We shall not assume that this space has necessarily infinite dimension, so that it may also be a unitary space. The following notations will be used:

[1] A. C. Zaanen, Ueber vollstetige symmetrische und symmetrisierbare Operatoren, Nieuw Arch. v. Wisk. (2), 22 (1943), p. 57—80.

A. C. Zaanen, On the theory of linear integral equations I, Proc. Kon. Ned. Akad. v. Wetensch. (Amsterdam) 49 (1946), p. 194—204 (= Indagationes Math. 8 (1946), p. 91—101).

[2] A. C. Zaanen, On the theory of linear integral equations II—VI, Proc. Kon. Ned. Akad. v. Wetensch. (Amsterdam) 49 (1946), p. 205—212, 292—301, 409—423, 571—585, 608—621 (= Indagationes Math. 8 (1946), p. 102—109, 161—170, 264—278, 352—366, 367—380).

$f, g, \varphi, \psi, \ldots,$	elements of $\mathfrak{R}$,
$\alpha, \beta, \lambda, \mu, \ldots,$	complex numbers,
$\bar{\alpha}, \bar{\beta}, \bar{\lambda}, \bar{\mu}, \ldots,$	the conjugate complex numbers of $\alpha, \beta, \lambda, \mu, \ldots,$
$(f, g),$	the scalar product of f and g,
$\|f\|,$	the non-negative number $(f, f)^{1/2}$,
$K, T, E, U, \ldots,$	bounded linear transformations in $\mathfrak{R}$, that is (for K), $\|Kf\| \leq \leq M\|f\|$ for a certain $M \geq 0$ and $K(\alpha f + \beta g) = \alpha Kf + \beta Kg$ for arbitrary α, β, f, g,
$\|K\|, \|T\|, \ldots,$	the bounds of $K, T, \ldots,$ that is (for K), the smallest number $M \geq 0$ satisfying $\|Kf\| \leq M\|f\|$ for every $f \in \mathfrak{R}$,
$K^*, T^*, \ldots,$	the adjoint transformations of $K, T, \ldots,$ as far as they exist in $\mathfrak{R}$ (it is well-known that, when $\mathfrak{R}$ is complete, every bounded linear transformation K has a uniquely determined, bounded adjoint K^* with the same bound as K); we have therefore (for K) the relation $(Kf, g) = (f, K^* g)$ for arbitrary f, g,
$H,$	a bounded, positive, self-adjoint transformation, that is, a bounded linear transformation satisfying $(Hf, g) = (f, Hg)$ and $(Hf, f) \geq 0$ for arbitrary f, g,
$N(f),$	the non-negative number $(Hf, f)^{1/2}$,
$I,$	the identical transformation; $If = f$ for every $f \in \mathfrak{R}$,
$O,$	the nulltransformation; $Of = 0$ for every $f \in \mathfrak{R}$,
$[\mathfrak{L}], [\mathfrak{M}], \ldots,$	closed linear manifolds in $\mathfrak{R}$, that is (for $[\mathfrak{L}]$), a subset of $\mathfrak{R}$ having the properties that $f, g \in [\mathfrak{L}]$ implies $\alpha f + \beta g \in [\mathfrak{L}]$ for arbitrary α, β, and $f_n \in [\mathfrak{L}]$ $(n = 1, 2, \ldots)$, $\lim f_n = f$ implies $f \in [\mathfrak{L}]$.

We suppose that the bounded, positive, self-adjoint transformation H is defined in $\mathfrak{R}$, and that $H \neq O$. Then the set of all elements $h \in H$, satisfying $Hh = 0$, is a closed linear manifold $[\mathfrak{L}]$, not identical with the space $\mathfrak{R}$ itself. It is not difficult to see that the set of all elements $g \in \mathfrak{R}$, orthogonal to $[\mathfrak{L}]$ (that is, $(g, h) = 0$ for every $h \in [\mathfrak{L}]$), is also a closed linear manifold, which we shall denote by $[\mathfrak{M}]$. We shall assume now that, for every $f \in \mathfrak{R}$, there exists a decomposition $f = g + h$, where $g \in [\mathfrak{M}]$, $h \in [\mathfrak{L}]$. Then, evidently, this decomposition is unique (It is well-known that, when $\mathfrak{R}$ is complete, a decomposition of this kind always exists. The same is true, even when $\mathfrak{R}$ is not complete, in the special case that H is definite, that is, $Hf = 0$ only for $f = 0$, since in this case $[\mathfrak{L}]$ contains only the nullelement). The manifold $[\mathfrak{L}]$ not being identical with the whole space $\mathfrak{R}$, the manifold $[\mathfrak{M}]$ contains not only the nullelement, but other elements as well. Defining, when $f = g + h$ $(g \in [\mathfrak{M}],\ h \in [\mathfrak{L}])$, the projection E on $[\mathfrak{M}]$ by $g = Ef$, we have therefore $E \neq O$. The projection on $[\mathfrak{L}]$ is $I - E$. We observe that $E = I$ in the special case that H is definite.

Lemma 1. $H = HE$.

Proof. On account of $Hh = 0$ for every $h \in [\mathfrak{L}]$ we have $H(I - E)f = 0$ for every $f \in \mathfrak{R}$, hence $Hf = HEf$ or $H = HE$.

Lemma 2. *For any element $f \in \mathfrak{R}$, the relations $Hf = 0$ and $N(f) = (Hf, f)^{1/2} = 0$ are equivalent.*

Proof. It is trivial that $Hf = 0$ implies $N(f) = 0$. To show the converse, we use the inequality

$$|(Hf, g)| \leq (Hf, f)^{1/2} \cdot (Hg, g)^{1/2} = N(f) \cdot N(g), \tag{7}$$

which is proved in a similar way as Schwarz's inequality $|(f, g)| \leq \|f\| \cdot \|g\|$. Taking now $g = Hf$ in (7), we obtain

$$\|Hf\|^2 \leq N(f) \cdot N(Hf),$$

which shows that $N(f) = 0$ implies $Hf = 0$.

§ 3. *H*-adjoints.

When the bounded linear transformations K and $\tilde{K}$, defined in $\mathfrak{R}$, satisfy the relation

$$(HKf, g) = (Hf, \tilde{K}g) \tag{8}$$

for arbitrary $f, g \in \mathfrak{R}$, we shall call $\tilde{K}$ an *H-adjoint* of K. Generally $\tilde{K}$ is not uniquely determined, since, if $\tilde{K}$ is an H-adjoint of K, and the bounded linear transformation $\tilde{K}_1$ satisfies $E\tilde{K}_1 = E\tilde{K}$, then $\tilde{K}_1$ is also an H-adjoint of K, as follows on account of Lemma 1 from

$$\begin{aligned}(HKf, g) = (Hf, \tilde{K}g) = (f, H\tilde{K}g) = (f, HE\tilde{K}g) &= \\ = (f, HE\tilde{K}_1 g) = (f, H\tilde{K}_1 g) &= (Hf, \tilde{K}_1 g).\end{aligned}$$

Conversely, if $\tilde{K}$ and $\tilde{K}_1$ are both H-adjoints of K, we have $E\tilde{K} = E\tilde{K}_1$, since $(HKf, g) = (Hf, \tilde{K}g) = (Hf, \tilde{K}_1 g)$ implies $(f, H\tilde{K}g) = (f, H\tilde{K}_1 g)$ for arbitrary $f, g \in \mathfrak{R}$, hence $H(\tilde{K} - \tilde{K}_1)g = 0$ or $(\tilde{K} - \tilde{K}_1)g \in [\mathfrak{L}]$ for every $g \in \mathfrak{R}$, so that $E(\tilde{K} - \tilde{K}_1) = O$ or $E\tilde{K} = E\tilde{K}_1$.

It is evident that, when $\tilde{K}$ is an H-adjoint of K, then K is also an H-adjoint of $\tilde{K}$, since (8) implies

$$(H\tilde{K}g, f) = (\tilde{K}g, Hf) = (g, HKf) = (Hg, Kf)$$

for arbitrary $f, g \in \mathfrak{R}$.

Finally we observe that, in the special case that $H = I$, the relation (8) becomes

$$(Kf, g) = (f, \tilde{K}g).$$

Any H-adjoint of K, if it exists, is therefore in this case identical with the uniquely determined adjoint K^*.

Before proving now several theorems on H-adjoints, we prove the following

Lemma 3. *When, in $\mathfrak{R}$, the adjoint K^* of the bounded linear transformation K is defined, it is bounded.*

Proof. If $\mathfrak{R}$ is complete, the theorem is well-known; we shall suppose therefore that $\mathfrak{R}$ is not complete. Then, by adjunction of 'ideal' elements ('limits' of fundamental sequences possessing not already a limit in $\mathfrak{R}$), we may obtain the complete space $\overline{\mathfrak{R}}$, the closure of $\mathfrak{R}$. For $f \in \overline{\mathfrak{R}}$, and $\notin \mathfrak{R}$, $f = \lim f_n$, $f_n \in \mathfrak{R}$, Kf is defined to be $\lim Kf_n$. It is easy to prove that this definition is legitimate. Then K is bounded in $\overline{\mathfrak{R}}$, so that, on account of the completeness of $\overline{\mathfrak{R}}$, K^* is also bounded in $\overline{\mathfrak{R}}$, and therefore certainly in $\mathfrak{R}$.

Theorem 1. *Let the bounded linear transformations K and $\tilde{K}$ be H-adjoints. Then any $f \in \mathfrak{R}$, satisfying $Hf = 0$ (equivalent with $f \in [\mathfrak{L}]$) satisfies also $EKf = 0$ and $E\tilde{K}f = 0$.*

Proof. From (8) and $Hf = 0$ follows $(HKf, g) = 0$ for every $g \in \mathfrak{R}$, hence $HKf = 0$. Then $Kf \in [\mathfrak{L}]$, so that $EKf = 0$.

From

$$(Kg, Hf) = (HKg, f) = (Hg, \tilde{K}f) = (g, H\tilde{K}f)$$

and $Hf = 0$ follows $(g, H\tilde{K}f) = 0$ for every $g \in \mathfrak{R}$, hence $H\tilde{K}f = 0$. Then $\tilde{K}f \in [\mathfrak{L}]$, so that $E\tilde{K}f = 0$.

A bounded linear transformation K which is its own H-adjoint, so that it satisfies the relation

$$(HKf, g) = (Hf, Kg) \tag{9}$$

for arbitrary $f, g \in \mathfrak{R}$, is called *symmetrisable* (to the left, and relative to H). Since (9) is equivalent with

$$(HKf, g) = (f, HKg),$$

we may also say that K is symmetrisable whenever HK is self-adjoint.[1] In the special case that $H = I$, a symmetrisable transformation is therefore the same as a self-adjoint transformation.

Theorem 2. *Let the bounded linear transformations K and $\tilde{K}$ be H-adjoints. Then $K\tilde{K}$ and $\tilde{K}K$ are symmetrisable, in other words, $HK\tilde{K}$ and $H\tilde{K}K$ are self-adjoint. Furthermore the self-adjoint transformations $HK\tilde{K}$ and $H\tilde{K}K$ are positive.*

Proof. Using (8) several times, we have, for arbitrary $f, g \in \mathfrak{R}$,

$$(HK\tilde{K}f, g) = (H\tilde{K}f, \tilde{K}g) = (\tilde{K}f, H\tilde{K}g) = (f, HK\tilde{K}g) = (Hf, K\tilde{K}g)$$

and

$$(H\tilde{K}Kf, g) = (\tilde{K}Kf, Hg) = (Kf, HKg) = (HKf, Kg) = (Hf, \tilde{K}Kg),$$

which shows that $K\tilde{K}$ and $\tilde{K}K$ are symmetrisable.

Furthermore

$$(HK\tilde{K}f, f) = (H\tilde{K}f, \tilde{K}f) \geq 0$$

and

$$(H\tilde{K}Kf, f) = (\tilde{K}Kf, Hf) = (Kf, HKf) \geq 0,$$

because H is positive.

Corollary. *Let the bounded linear transformation K possess the adjoint K^* (K^* is bounded by Lemma 3). Then KK^* and K^*K are self-adjoint and positive.*

For abbreviation we shall write, whenever K and $\tilde{K}$ are H-adjoints, $EK = T$ and $E\tilde{K} = \tilde{T}$. Since by Lemma 1 we have $H = HE$, it follows from (8) that

$$(HEKf, g) = (HKf, g) = (Hf, \tilde{K}g) = (f, H\tilde{K}g) = (f, HE\tilde{K}g) = (Hf, E\tilde{K}g)$$

or

$$(HTf, g) = (Hf, \tilde{T}g), \tag{10}$$

so that T and $\tilde{T}$ also are H-adjoints. We observe that, whenever K is symmetrisable, the same is therefore true of $T = EK$. As a consequence of Theorem 2 we have now

Theorem 3. *Let the bounded linear transformations K and $\tilde{K}$ be H-adjoints, let $T = EK$ and $\tilde{T} = E\tilde{K}$. Then $HT\tilde{T}$ and $H\tilde{T}T$ are positive, self-adjoint transformations.*

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[1] Symmetrisable transformations K such that both H and K are of integral type with bounded kernels $H(x, y)$ and $K(x, y)$ were introduced for the first time by J. Marty, Valeurs singulières d'une équation de Fredholm, Comptes Rendus de l'Acad. des sc. (Paris) 150 (1910), p. 1499—1502.

Theorem 4. *Let the bounded linear transformations K and $\tilde{K}$ be H-adjoints. Then, for every $f \in \mathfrak{R}$,*

$$(11) \qquad \|HKf\| \leq \|\tilde{K}\| \cdot \|H\|^{1/2} \cdot N(f)$$

and

$$(12) \qquad \|H\tilde{K}f\| \leq \|K\| \cdot \|H\|^{1/2} \cdot N(f).$$

Proof. Taking $g = Hf$ in (7), we obtain

$$\|Hf\|^2 \leq (Hf, f)^{1/2} \cdot (H^2 f, Hf)^{1/2}.$$

But

$$(H^2 f, Hf) \leq \|H^2 f\| \cdot \|Hf\| \leq \|H\| \cdot \|Hf\|^2,$$

so that

$$\|Hf\|^2 \leq \|H\|^{1/2} \cdot (Hf, f)^{1/2} \cdot \|Hf\|$$

or

$$(13) \qquad \|Hf\| \leq \|H\|^{1/2} \cdot (Hf, f)^{1/2} = \|H\|^{1/2} \cdot N(f).$$

Furthermore

$$\|HKf\|^2 = (HKf, HKf) = (Hf, \tilde{K}HKf) \leq \|Hf\| \cdot \|\tilde{K}HKf\| \leq \|\tilde{K}\| \cdot \|Hf\| \cdot \|HKf\|$$

or

$$\|HKf\| \leq \|\tilde{K}\| \cdot \|Hf\|,$$

so that by (13)

$$\|HKf\| \leq \|\tilde{K}\| \cdot \|H\|^{1/2} \cdot N(f).$$

This disposes of (11). The inequality (12) follows now also, since K and $\tilde{K}$ may be interchanged.

§ 4. Introduction of a Factorspace.

Two elements f and $g \in \mathfrak{R}$ will be called *H-orthogonal* when $(Hf, g) = 0$; and the system Q of elements will be called *H-orthonormal* when, for $\varphi \in Q$, $\psi \in Q$, we have

$$(H\varphi, \psi) = \begin{cases} 1 & \text{for } \varphi = \psi, \\ 0 & \text{for } \varphi \neq \psi. \end{cases}$$

The elements $f_1, f_2, \ldots, f_n$ will be called *H-independent* when $H \sum_{i=1}^{n} \alpha_i f_i = 0$ implies $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$. Evidently, when $f_1, f_2, \ldots, f_n$ are H-independent, they are linearly independent.

Lemma 4. *When the elements $\varphi_1, \ldots, \varphi_n$ are H-orthonormal, they are H-independent.*

Proof. From $H\sum_{i=1}^{n} a_i \varphi_i = 0$ follows, for $k = 1, \ldots, n$,

$$\sum_{i=1}^{n} a_i (H\varphi_i, \varphi_k) = 0,$$

hence, since

$$(H\varphi_i, \varphi_k) = \begin{cases} 1 & \text{for } i = k, \\ 0 & \text{for } i \neq k, \end{cases}$$

$a_1 = \cdots = a_n = 0$.

Lemma 5. *Given the finite or enumerable set V of H-independent elements $f_n (n = 1, 2, \ldots)$, there exists an H-orthonormal set Q of elements $\varphi_n (n = 1, 2, \ldots)$ such that the linear manifold $\mathfrak{L}(V)$ of all finite linear combinations $\sum_i a_i f_i$ is identical with the linear manifold $\mathfrak{L}(Q)$ of all finite linear combinations $\sum_i \beta_i \varphi_i$.*

Proof. The H-orthonormal sequence Q of elements $\varphi_1, \varphi_2, \ldots$ originates from the sequence $f_1, f_2, \ldots$ by a process, wholly similar to Schmidt's well-known orthogonalization process, in the following way:

$$g_1 = f_1, \quad \varphi_1 = g_1 / N(g_1),$$
$$g_2 = f_2 - (Hf_2, \varphi_1)\varphi_1, \quad \varphi_2 = g_2 / N(g_2),$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

Generally, if $\varphi_1, \ldots, \varphi_{n-1}$ are already defined,

$$g_n = f_n - \sum_{i=1}^{n-1} (Hf_n, \varphi_i)\varphi_i, \quad \varphi_n = g_n / N(g_n).$$

To justify this definition of φ_n we have to show that $N(g_n) \neq 0$. Now, since $f_1, \ldots, f_n$ are H-independent, so are $\varphi_1, \ldots, \varphi_{n-1}, f_n$, hence $Hg_n \neq 0$. This, however, by Lemma 2, is equivalent with $N(g_n) \neq 0$. It is easy to see that the sequence Q is H-orthonormal. Moreover, φ_n depends linearly on $f_1, \ldots, f_n$, and f_n depends linearly on $\varphi_1, \ldots, \varphi_n$; hence $\mathfrak{L}(V) = \mathfrak{L}(Q)$.

We shall introduce now a Hilbert space $\mathfrak{Z}$ with elements $[f]$ that are classes of elements of the space $\mathfrak{R}$. The following definitions of $[f]$ are, by Lemma 2, equivalent:

1°. $[f]$ contains f and all elements g for which $Hg = Hf$,

2°. $[f]$ contains f and all elements g for which $N(f-g) = 0$.

The class $[f]$ contains in particular the element Ef, since $HEf = Hf$ by Lemma 1. The nullclass $[0]$ consists of all elements h for which $Hh = 0$, that is, all elements $h \in [\mathfrak{L}]$. We shall write $f \equiv g \pmod{[\mathfrak{L}]}$, or shortly $f \equiv g$, whenever $f - g \in [\mathfrak{L}]$, in other words, whenever $[f] = [g]$.

Furthermore we define

$$[f] + [g] = [f + g],$$

$$\alpha [f] = [\alpha f] \text{ for arbitrary complex } \alpha,$$

$$([f], [g]) = (Hf, g);$$

hence

$$\|[f]\| = ([f], [f])^{1/2} = (Hf, f)^{1/2} = N(f).$$

No contradiction can arise from these definitions, since $f \equiv f_1$ and $g \equiv g_1$ imply $f + g \equiv f_1 + g_1$, $\alpha f \equiv \alpha f_1$, and

$$(Hf, g) = (Hf_1, g) = (f_1, Hg) = (f_1, Hg_1) = (Hf_1, g_1).$$

Finally $0 = \|[f]\| = N(f)$ if and only if $Hf = 0$, that is, if and only if $[f] = [0]$.

With these definitions the space $\mathfrak{Z}$ is therefore a Hilbert space. It is evidently some factorspace of $\mathfrak{R}$ relative to $[\mathfrak{L}]$, so that we may write $\mathfrak{Z} = \mathfrak{R}/[\mathfrak{L}]$. We observe that, even in the case that the space $\mathfrak{R}$ is complete, the space $\mathfrak{Z}$ is generally not complete. If the transformation H is definite, that is, if $Hf = 0$ only for $f = 0$, there is a one-to-one correspondence between the elements $f \in \mathfrak{R}$ and the elements $[f] \in \mathfrak{Z}$. In the special case that $H = I$, the spaces $\mathfrak{R}$ and $\mathfrak{Z}$ may be regarded as identical.

Lemma 6. *The system $\{f\}$ in the space $\mathfrak{R}$ is H-orthonormal if and only if the system $\{[f]\}$ in the space $\mathfrak{Z}$ is orthonormal.*

Proof. $(Hf, g) = 0$ is equivalent with $([f], [g]) = 0$, and $(Hf, f) = 1$ is equivalent with $\|[f]\|^2 = 1$.

Lemma 7. *The elements $f_1, \ldots, f_n$ in the space $\mathfrak{R}$ are H-independent if and only if the elements $[f_1], \ldots, [f_n]$ in the space $\mathfrak{Z}$ are linearly independent.*

Proof. $H\sum_{i=1}^{n} \alpha_i f_i = 0$ is equivalent with $\sum_{i=1}^{n} \alpha_i [f_i] = [0]$; if therefore one of these relations implies $\alpha_1 = \cdots = \alpha_n = 0$, the same is true of the other.

Lemma 8. *Let $v_i (i = 1, \ldots, n)$ be an H-orthonormal system in $\mathfrak{R}$, and let the unitary space, determined by this system, be called $\mathfrak{R}_n$. Then, if the linear transformation U, defined in $\mathfrak{R}_n$ (that is, $f \in \mathfrak{R}_n$ implies $Uf \in \mathfrak{R}_n$), has the property that the system $U v_i (i = 1, \ldots, n)$ is also H-orthonormal, there exists an H-orthonormal system $\varphi_i (i = 1, \ldots, n)$ in $\mathfrak{R}_n$ such that*

$$U \varphi_i = \mu_i \varphi_i, \quad |\mu_i| = 1 \qquad (i = 1, \ldots, n).$$

Proof. We observe first that $Hf = 0$ for an element $f \in \mathfrak{R}_n$ implies $f = 0$. This follows from the fact that every $f \in \mathfrak{R}_n$ can be written in the form $f = \sum_{i=1}^{n} \alpha_i v_i$, so that $Hf = 0$ implies $\sum_{i=1}^{n} \alpha_i H v_i = 0$ or $\sum_{i=1}^{n} \alpha_i (H v_i, v_j) = 0$ $(j = 1, \ldots, n)$ or $\alpha_j = 0 (j = 1, \ldots, n)$. Introducing the unitary space $\mathfrak{Z}_n$, corresponding with $\mathfrak{R}_n$ in the same way as the Hilbert space $\mathfrak{Z}$ corresponds with the whole space $\mathfrak{R}$, there exists therefore a one-to-one correspondence between the elements $[f]$ of $\mathfrak{Z}_n$ and the elements f of $\mathfrak{R}_n$. Furthermore we define the linear transformation $[U]$ in $\mathfrak{Z}_n$ by $[U][f] = [Uf]$. Then $[U]$ transforms the orthonormal system $[v_i]$ $(i = 1, \ldots, n)$ into the orthonormal system $[U v_i]$ $(i = 1, \ldots, n)$, so that, by a well-known theorem, $[U]$ is a unitary transformation in $\mathfrak{Z}_n$, that is $[U][U]^* = [I]$, where $[I]$ is the identical transformation in $\mathfrak{Z}_n$. It follows, using another well-known theorem on unitary transformations in unitary spaces, that there exists an orthonormal system $[\varphi_i]$ $(i = 1, \ldots, n)$ in $\mathfrak{Z}_n$ such that

$$[U][\varphi_i] = \mu_i [\varphi_i], \quad |\mu_i| = 1 \qquad (i = 1, \ldots, n),$$

so that in the original space $\mathfrak{R}_n$ there exists an H-orthonormal system $\varphi_i (i = 1, \ldots, n)$ such that

$$U \varphi_i = \mu_i \varphi_i, \quad |\mu_i| = 1 \qquad (i = 1, \ldots, n).$$

§ 5. Singular Values of H-adjoints.

The linear transformation K, defined for all $f \in \mathfrak{R}$, is said to be completely continuous when every bounded, infinite set of elements of $\mathfrak{R}$ contains a sequence f_n such that the sequence $K f_n$ converges to an element $g \in \mathfrak{R}$. In the case that $\mathfrak{R}$ is complete, it is sufficient to require that the sequence $K f_n$ converges, since in this case the limitelement $g \in \mathfrak{R}$ exists by hypothesis. It is not difficult to prove that every completely continuous linear transformation is bounded.

Theorem 5. *Let the bounded linear transformations K and $\tilde{K}$ be H-adjoints; let $T = EK$ and $\tilde{T} = E\tilde{K}$. Supposing now that T is completely continuous, and that $P = HK \neq O$, there exist two elements u and v, both $\neq 0$, and a positive number λ such that*

$$Tu = \lambda v, \quad \tilde{T} v = \lambda u.$$

The number λ will ce called a singular value of T and $\tilde{T}$.[1]

Proof.[2] On account of $P \neq O$ there is an element $f_0 \neq 0$ such that $Pf_0 \neq 0$. This implies $Hf_0 \neq 0$ (since from $Hf_0 = 0$ would follow, by Theorem 1, $EKf_0 = 0$, hence $Pf_0 = HKf_0 = HEKf_0 = 0$), so that also $N(f_0) \neq 0$. Writing $\bar{f}_0 = f_0 / N(f_0)$ and $f_1 = T\bar{f}_0$, we find

$$Hf_1 = HT\bar{f}_0 = HEK\bar{f}_0 = HK\bar{f}_0 = P\bar{f}_0 \neq 0,$$

hence $N(f_1) \neq 0$. The sequences of elements f_n and $\bar{f}_n$ $(n = 0, 1, 2, \ldots)$ are now defined by

$$\bar{f}_n = f_n / N(f_n),$$
$$f_{2n+1} = T\bar{f}_{2n}, \quad f_{2n+2} = \tilde{T}\bar{f}_{2n+1}$$
$$(n = 0, 1, 2, \ldots).$$

To justify this definition, we have to prove that $N(f_n) \neq 0$ for every value of n. This, however, is a consequence of $N(f_0) \neq 0$, $N(f_1) \neq 0$,

$$N(f_{2n+1}) = (Hf_{2n+1}, \bar{f}_{2n+1}) = (HT\bar{f}_{2n}, \bar{f}_{2n+1}) = (H\bar{f}_{2n}, \tilde{T}\bar{f}_{2n+1}) =$$
$$(H\bar{f}_{2n}, f_{2n+2}) \leq N(\bar{f}_{2n}) \cdot N(f_{2n+2}) = N(f_{2n+2}) \qquad (n = 0, 1, 2, \ldots)$$

and

$$N(f_{2n}) = (Hf_{2n}, \bar{f}_{2n}) = (H\tilde{T}\bar{f}_{2n-1}, \bar{f}_{2n}) = (H\bar{f}_{2n-1}, T\bar{f}_{2n}) =$$
$$(H\bar{f}_{2n-1}, f_{2n+1}) \leq N(\bar{f}_{2n-1}) \cdot N(f_{2n+1}) = N(f_{2n+1}) \qquad (n = 1, 2, \ldots).$$

The sequence of numbers $N(f_n)$ $(n = 1, 2, \ldots)$ is therefore non-descending. Furthermore we observe that $N(f_n) = (H\bar{f}_{n-1}, f_{n+1})$ implies

$$N(f_{n-1}) \cdot N(f_n) = (Hf_{n-1}, f_{n+1}) \qquad (n = 1, 2, \ldots), \tag{14}$$

and, since on account of this relation $(Hf_{n-1}, f_{n+1}) = (f_{n-1}, Hf_{n+1})$ is real, also

$$N(f_{n-1}) \cdot N(f_n) = (Hf_{n+1}, f_{n-1}) \qquad (n = 1, 2, \ldots). \tag{15}$$

[1] Some authors use the name of singular value for the reciprocal value of λ.

[2] Part of the idea of this proof is derived from the proof that an integral equation with a non-vanishing Hermitian kernel has at least one characteristic value $\neq 0$, as given in O. D. Kellogg, On the existence and closure of sets of characteristic functions, Math. Annalen 86 (1922), p. 14—17.

We shall prove now that the sequence $f_{2n+1} = T\bar{f}_{2n}$ contains a converging subsequence. For this purpose we observe first that the sequence Hf_n is bounded since, on account of Theorem 4,

$$\|Hf_n\| = \begin{cases} \|HT\bar{f}_{n-1}\| \leq \|\tilde{T}\| \cdot \|H\|^{1/2} \cdot N(\bar{f}_{n-1}) = \|\tilde{T}\| \cdot \|H\|^{1/2} & \text{for } n \text{ odd,} \\ \|H\tilde{T}\bar{f}_{n-1}\| \leq \|T\| \cdot \|H\|^{1/2} \cdot N(\bar{f}_{n-1}) = \|T\| \cdot \|H\|^{1/2} & \text{for } n \text{ even.} \end{cases}$$

This enables us to show that the assumption that $\lim \|\bar{f}_{2n}\| = \infty$ leads to a contradiction. Indeed, supposing that $\lim \|\bar{f}_{2n}\| = \infty$, there is an infinite number of values of the index $2n$ for which $\|\bar{f}_{2n+2}\| \geq \|\bar{f}_{2n}\|$, so that for a certain subsequence $\bar{f}_j$ $(j = 2n_1, 2n_2, \ldots)$ we have $\|\bar{f}_{j+2}\| \geq \|\bar{f}_j\|$. Since T is completely continuous and $\|\bar{f}_j / \|\bar{f}_j\|\| = 1$, the sequence $T\bar{f}_j / \|\bar{f}_j\| = f_{j+1} / \|\bar{f}_j\|$ contains a subsequence $f_{k+1} / \|\bar{f}_k\|$ converging to an element f. We have then

$$Hf = \lim Hf_{k+1} / \|\bar{f}_k\| = 0$$

on account of $\lim \|\bar{f}_k\| = \infty$ and the boundedness of Hf_{k+1}. Furthermore

$$\tilde{T}f = \lim \tilde{T}f_{k+1} / \|\bar{f}_k\| = \lim N(f_{k+1}) \cdot \tilde{T}\bar{f}_{k+1} / \|\bar{f}_k\| = \lim N(f_{k+1}) \cdot f_{k+2} / \|\bar{f}_k\|.$$

But, in virtue of $N(f_{k+2}) \geq N(f_{k+1}) \geq N(f_1)$ and $\|\bar{f}_{k+2}\| \geq \|\bar{f}_k\|$, we find

$$\begin{aligned} \left\| N(f_{k+1}) \cdot f_{k+2} / \|\bar{f}_k\| \right\| &= \left\| N(f_{k+1})\, N(f_{k+2}) \cdot \bar{f}_{k+2} / \|\bar{f}_k\| \right\| \\ &= N(f_{k+1})\, N(f_{k+2}) \cdot \|\bar{f}_{k+2}\| / \|\bar{f}_k\| \geq N^2(f_1), \end{aligned}$$

so that

$$\tilde{T}f = \lim N(f_{k+1}) f_{k+2} / \|\bar{f}_k\| \neq 0.$$

This, however, is in contradiction with $Hf = 0$, since, by Theorem 1, $Hf = 0$ implies $\tilde{T}f = E\tilde{K}f = 0$. The relation $\lim \|\bar{f}_{2n}\| = \infty$ being therefore impossible, we may conclude that the sequence $\bar{f}_{2n}$ contains a bounded subsequence $\bar{f}_i$. Then, on account of the complete continuity of T, the sequence $f_{i+1} = T\bar{f}_i$ contains a subsequence f_l converging to an element $t \in \mathfrak{R}$. From

$$N^2(f_l) = (Hf_l, f_l) \leq \|Hf_l\| \cdot \|f_l\| \leq \|H\| \cdot \|f_l\|^2$$

it follows further that the sequence of numbers $N(f_l)$ is bounded, so that, since $N(f_l)$ is a subsequence of the non-descending sequence $N(f_n)$ $(n = 1, 2, \ldots)$, the whole sequence $N(f_n)$ is also bounded. Consequently $\lambda = \lim N(f_n)$ exists, and $\lambda > 0$. Then $\lim \bar{f}_l = t/\lambda$, so that

$$(16) \qquad \begin{cases} \lim f_{l+1} = \lim \tilde{T}\bar{f}_l = \tilde{T}t/\lambda = u, \\ \lim f_{l+2} = \lim T\bar{f}_{l+1} = Tu/\lambda = v. \end{cases}$$

The relations (14) and (15) imply

$$\lim (H f_l, f_{l+2}) = \lim (H f_{l+2}, f_l) = \lim N(f_l) \cdot N(f_{l+1}) = \lambda^2,$$

hence

$$N^2(t-v) = \lim N^2(f_l - f_{l+2}) =$$
$$\lim [N^2(f_l) - (H f_l, f_{l+2}) - (H f_{l+2}, f_l) + N^2(f_{l+2})] = \lambda^2 - \lambda^2 - \lambda^2 + \lambda^2 = 0,$$

so that also $H(t-v) = 0$ by Lemma 2, which implies, on account of Theorem 1, $\tilde{T}(t-v) = 0$ or $\tilde{T} t = \tilde{T} v$. This being so, we infer from (16) that

$$T u = \lambda v, \quad \tilde{T} v = \lambda u.$$

Both u and v are $\neq 0$ on account of $N(u) = \lim N(f_{l+1}) = \lambda > 0$ and $N(v) = \lim N(f_{l+2}) = \lambda > 0$. This completes the proof.

Theorem 6. *Let the bounded linear transformations K and $\tilde{K}$ be H-adjoints, and let $T = EK$ be completely continuous and $P = HK \neq O$. Supposing now, moreover, that any $f \in \mathfrak{R}$, satisfying $Hf = 0$, satisfies also $Kf = \tilde{K} f = 0$, the transformations K and $\tilde{K}$ have a singular value $\lambda > 0$; in other words, there exist two elements y and z, both $\neq 0$, and a positive number λ such that*

$$K y = \lambda z, \quad \tilde{K} z = \lambda y.$$

Proof. Since by hypothesis $Kh = \tilde{K} h = 0$ for all $h \in [\mathfrak{L}]$ (we recall that $[\mathfrak{L}]$ is the set of all elements h satisfying $Hh = 0$), we have

$$K(I-E)f = \tilde{K}(I-E)f = 0$$

for all $f \in \mathfrak{R}$, or $K = KE$, $\tilde{K} = \tilde{K} E$. Furthermore, by the previous theorem, there exist two elements u and v, both $\neq 0$, and a positive number λ such that

$$T u = \lambda v, \quad \tilde{T} v = \lambda u.$$

Then

$$K \tilde{K} v = K E \tilde{K} v = K \tilde{T} v = \lambda K u,$$
$$\tilde{K} K u = \tilde{K} E K u = \tilde{K} T u = \lambda \tilde{K} v.$$

Defining $y = \lambda^{-1} \tilde{K} v$, $z = \lambda^{-1} K u$, we have therefore

$$K y = \lambda z, \ \tilde{K} z = \lambda y.$$

From $u \neq 0$ follows $E \tilde{K} v = \tilde{T} v = \lambda u \neq 0$, hence $\tilde{K} v \neq 0$ or $y = \lambda^{-1} \tilde{K} v \neq 0$. Then also $z \neq 0$, since $z = 0$ would imply $y = \lambda^{-1} \tilde{K} z = 0$.

Remark. We observe that

$$E y = \lambda^{-1} E \tilde{K} v = \lambda^{-1} \tilde{T} v = u,$$
$$E z = \lambda^{-1} E K u = \lambda^{-1} T u = v.$$

Theorem 7. *Under the same assumptions as in the preceding theorem, the relations $u = E y$, $v = E z$ and $y = \lambda^{-1} \tilde{K} v$, $z = \lambda^{-1} K u$ define a one-to-one correspondence between all pairs of elements y, z (both $\neq 0$) satisfying*

$$K y = \lambda z, \quad \tilde{K} z = \lambda y \qquad (\lambda \neq 0), \tag{17}$$

and all pairs of elements u, v (both $\neq 0$) satisfying

$$T u = \lambda v, \quad \tilde{T} v = \lambda u \qquad (\lambda \neq 0). \tag{18}$$

Proof. Whenever $K y = \lambda z$, $\tilde{K} z = \lambda y$; λ, y and $z \neq 0$, we have, writing $u = E y$, $v = E z$,

$$T u = E K u = E K E y = E K y = \lambda E z = \lambda v,$$
$$\tilde{T} v = E \tilde{K} v = E \tilde{K} E z = E \tilde{K} z = \lambda E y = \lambda u,$$
$$u \neq 0, \text{ since } K u = K E y = K y = \lambda z \neq 0,$$
$$v \neq 0, \text{ since } \tilde{K} v = \tilde{K} E z = \tilde{K} z = \lambda y \neq 0.$$

With every pair of elements $y, z \neq 0$, satisfying (17), corresponds therefore the pair of elements $u = E y$, $v = E z$, both $\neq 0$, satisfying (18). We shall show now that with different pairs y_1, z_1 and y_2, z_2, satisfying (17), cannot correspond the same pair u, v, satisfying (18). For this purpose we suppose that $\lambda \neq 0$, $y_1, z_1, y_2, z_2 \neq 0$,

$$K y_1 = \lambda z_1, \ \tilde{K} z_1 = \lambda y_1 \text{ and } K y_2 = \lambda z_2, \ \tilde{K} z_2 = \lambda y_2,$$
$$y_1 \neq y_2, \ E y_1 = E y_2.$$

Then $z_1 \neq z_2$ (since $z_1 = z_2$ would imply $\lambda y_1 = \tilde{K} z_1 = \tilde{K} z_2 = \lambda y_2$ or $y_1 = y_2$), hence

$$K(y_1 - y_2) = \lambda (z_1 - z_2) \neq 0.$$

But from $E y_1 = E y_2$ and $K = K E$ follows

$$K(y_1 - y_2) = K E (y_1 - y_2) = K(E y_1 - E y_2) = 0,$$

so that we arrive at a contradiction.

It remains to prove that, if $u, v \neq 0$ satisfy (18), there exist elements $y, z \neq 0$, satisfying (17), such that $E y = u$, $E z = v$. As we have seen in the preceding theorem, the elements $y = \lambda^{-1} \tilde{K} v$, $z = \lambda^{-1} K u$ fulfil these conditions.

§ 6. Characteristic Values of Normalisable Transformations.

Theorem 8. *When the bounded linear transformations K and $\tilde{K}$ are H-adjoints, then $HK\tilde{K} = HT\tilde{T}$ and $H\tilde{K}K = H\tilde{T}T$.*

Proof. From $H = HE$ follows $HK = HEK = HT$ and $H\tilde{K} = HE\tilde{K} = H\tilde{T}$, hence

$$(HK\tilde{K}f, g) = (HT\tilde{K}f, g) = (H\tilde{K}f, \tilde{T}g) = (H\tilde{T}f, \tilde{T}g) = (HT\tilde{T}f, g)$$

for arbitrary $f, g \in \mathfrak{R}$, so that $HK\tilde{K} = HT\tilde{T}$. The relation $H\tilde{K}K = H\tilde{T}T$ is proved in a similar way.

Definition. *Let the bounded linear transformations K and $\tilde{K}$ be H-adjoints. Then K will be called normalisable (relative to H) when $HK\tilde{K} = H\tilde{K}K$.*

It follows from Theorem 8 that we may also say that K is normalisable whenever $HT\tilde{T} = H\tilde{T}T$, and this shows that, even though the H-adjoint $\tilde{K}$ of K may not be uniquely determined, our definition is nevertheless independent of the particular choice of $\tilde{K}$. We observe that, in the special case that $H = I$, we have $\tilde{K} = K^*$; in this case, therefore, K is normalisable when $KK^* = K^*K$, which shows that a bounded linear transformation which is normalisable relative to I is simply a bounded normal transformation.

Theorem 9. *The bounded linear transformation K is normalisable if and only if $T\tilde{T} = \tilde{T}T$.*

Proof. If $T\tilde{T} = \tilde{T}T$, K is evidently normalisable. Conversely, if K is normalisable, we have $H(T\tilde{T} - \tilde{T}T)f = 0$ or, by the definition of E, $E(T\tilde{T} - \tilde{T}T)f = 0$ for every $f \in \mathfrak{R}$. But, since $E^2 = E$, we have $ET = E^2K = EK = T$ and $E\tilde{T} = E^2\tilde{K} = E\tilde{K} = \tilde{T}$. Hence $(T\tilde{T} - \tilde{T}T)f = 0$ for every $f \in \mathfrak{R}$ or $T\tilde{T} = \tilde{T}T$.

If K is a bounded linear transformation in $\mathfrak{R}$, and $Kf = \lambda f$ for an element $f \neq 0$, this element is called a characteristic element of K, belonging to the characteristic value λ.[1] The set of all characteristic elements, belonging to the same characteristic value λ, is a closed linear manifold in $\mathfrak{R}$, and the dimension (that is, the maximal number of linearly independent elements) of this closed linear manifold is called the multiplicity of the characteristic value λ. We shall assume the following lemma to be known:

[1] Some authors call λ an eigenvalue of K, and reserve the name of characteristic value for the reciprocal value of λ.

Lemma 9. *The number of characteristic values of a completely continuous linear transformation is finite or enumerable, and in this latter case the characteristic values tend to* 0. *The multiplicity of every characteristic value* $\neq 0$ *is finite.*

Lemma 10. *Let the linear transformation* A *be completely continuous, and* $Hf = 0$ *imply* $Af = 0$. *Supposing now that* $\lambda \neq 0$ *is a characteristic value of* A, *there exist a positive integer* n *and an* H*-orthonormal system* $\varphi_1, \ldots, \varphi_n$ *such that the set of all characteristic elements of* A, *belonging to the characteristic value* λ, *is identical with the set of all linear combinations* $\sum_{i=1}^{n} \alpha_i \varphi_i$.

Proof. By Lemma 9 the muliplicity of the characteristic value λ is a finite integer $n \geq 1$. There exist therefore n linearly independent elements $\chi_1, \ldots, \chi_n$ such that the set of all characteristic elements of A, belonging to the characteristic value λ, is identical with the set of all elements $\sum_{i=1}^{n} \alpha_i \chi_i$. The elements $\chi_1, \ldots, \chi_n$ are H-independent since, by hypothesis, $H \sum_{i=1}^{n} \alpha_i \chi_i = 0$ implies

$$A \sum_{i=1}^{n} \alpha_i \chi_i = 0 \quad \text{or} \quad \lambda \sum_{i=1}^{n} \alpha_i \chi_i = 0,$$

hence $\alpha_1 = \cdots = \alpha_n = 0$ on account of $\lambda \neq 0$ and the linear independence of $\chi_1, \ldots, \chi_n$. The existence of an H-orthonormal system $\varphi_1, \ldots, \varphi_n$ with the required property follows therefore from Lemma 5.

Now we come to one of our main theorems:

Theorem 10. *Let the bounded linear transformation* K *be normalisable,* $T = EK$ *be completely continuous, and* $P = HK \neq O$. *Then* T *has at least one characteristic value* $\lambda_1 \neq 0$ *with characteristic element* φ_1. *Moreover,* $\tilde{T} = E\tilde{K}$ *has the characteristic value* $\bar{\lambda}_1$ *with the same characteristic element* φ_1.

Proof. By Theorem 5 there exist a number $\lambda > 0$ and elements $u, v \neq 0$ such that

$$Tu = \lambda v, \quad \tilde{T} v = \lambda u.$$

Then $T\tilde{T} v = \lambda T u = \lambda^2 v$, which shows that v is a characteristic element of $T\tilde{T}$ with characteristic value λ^2. The transformation $T\tilde{T}$ is completely continuous (T is completely continuous and $\tilde{T}$ is bounded), and $Hf = 0$ implies $T\tilde{T}f = 0$ ($Hf = 0$ implies $\tilde{T}f = 0$ by Theorem 1, hence certainly $T\tilde{T}f = 0$). The set of all elements v, satisfying $T\tilde{T}v = \lambda^2 v$, is therefore, by Lemma 10, identical with the set $\mathfrak{N}_n$ of all linear combinations $\sum_{i=1}^{n} \alpha_i v_i$, where $n \geq 1$, and the system $v_1, \ldots, v_n$ is H-orthonormal.

From $T\tilde{T} = \tilde{T}T$ and $T\tilde{T}v_i = \lambda^2 v_i$ $(i = 1, \ldots, n)$ follows

$$T\tilde{T}(\lambda^{-1}Tv_i) = \lambda^{-1}T\tilde{T}Tv_i = \lambda^{-1}T(T\tilde{T}v_i) = \lambda^2(\lambda^{-1}Tv_i),$$

which shows that the elements $\lambda^{-1}Tv_i$ $(i = 1, \ldots, n)$ are also characteristic elements of $T\tilde{T}$, belonging to the characteristic value λ^2. The transformation $U = \lambda^{-1}T$ transforms, therefore, every element of $\mathfrak{N}_n$ into an element of $\mathfrak{N}_n$. Furthermore

$$(HUv_i, Uv_j) = \lambda^{-2}(HTv_i, Tv_j) = \lambda^{-2}(H\tilde{T}Tv_i, v_j) = \lambda^{-2}(HT\tilde{T}v_i, v_j) = (Hv_i, v_j),$$

so that the system Uv_i $(i = 1, \ldots, n)$ is H-orthonormal. Then, by Lemma 8, there exists an H-orthonormal system $\varphi_1, \ldots, \varphi_n$ in $\mathfrak{N}_n$ such that

$$U\varphi_i = \mu_i\varphi_i, \ |\mu_i| = 1 \qquad (i = 1, \ldots, n),$$

hence, since $U = \lambda^{-1}T$,

$$T\varphi_i = \lambda_i\varphi_i, \ |\lambda_i| = \lambda \qquad (i = 1, \ldots, n).$$

This shows that T has at least one characteristic value $\lambda_1 \neq 0$ with characteristic element φ_1.

To prove that $\tilde{T}\varphi_i = \bar{\lambda}_i\varphi_i$ we observe that, since $\varphi_i \in \mathfrak{N}_n$, we have

$$\tilde{T}T\varphi_i = T\tilde{T}\varphi_i = \lambda^2\varphi_i = \lambda_i\bar{\lambda}_i\varphi_i,$$

so that from $\lambda_i\varphi_i = T\varphi_i$ follows

$$\lambda_i\tilde{T}\varphi_i = \tilde{T}T\varphi_i = \lambda_i\bar{\lambda}_i\varphi_i.$$

Hence $\tilde{T}\varphi_i = \bar{\lambda}_i\varphi_i$.

Considering the case that $H = I$, we obtain the following

Corollary. *Let the completely continuous linear transformation $K \neq O$ be normal. Then K has at least one characteristic value $\lambda_1 \neq 0$ with characteristic element φ_1. Moreover, K^* has the characteristic value $\bar{\lambda}_1$ with the same characteristic element φ_1.*

Theorem 11. *Let the bounded linear transformation K be symmetrisable, $T = EK$ be completely continuous, and $P = HK \neq O$. Then T has at least one real characteristic value $\lambda_1 \neq 0$.*

Proof. Since now $T = \tilde{T}$, there exist, by the preceding theorem, a complex number $\lambda_1 \neq 0$ and an element $\varphi_1 \neq 0$ satisfying

$$T\varphi_1 = \lambda_1 \varphi_1, \quad T\varphi_1 = \tilde{T}\varphi_1 = \bar{\lambda}_1 \varphi_1.$$

This shows that $\lambda_1 = \bar{\lambda}_1$, so that λ_1 is real.

Considering again the case that $H = I$, we obtain the following:

Corollary. *If the completely continuous linear transformation $K \neq O$ is self-adjoint, then K has at least one characteristic value $\lambda_1 \neq 0$.*

Theorem 12. *Let the bounded linear transformation K be normalisable, $T = EK$ be completely continuous, and $P = HK \neq O$. Supposing now, moreover, that any $f \in \mathfrak{R}$, satisfying $Hf = 0$, satisfies also $Kf = 0$, the transformation K has at least one characteristic value $\lambda_1 \neq 0$.*

Proof. By Theorem 10 the transformation $T = EK$ has at least one characteristic value $\lambda_1 \neq 0$ with characteristic element φ_1:

$$T\varphi_1 = \lambda_1 \varphi_1.$$

Then, since now $K = KE$ by our additional hypothesis (compare the proof of Theorem 6), we have

$$KK\varphi_1 = KEK\varphi_1 = KT\varphi_1 = \lambda_1 K\varphi_1,$$

so that $\psi_1 = \lambda_1^{-1} K\varphi_1$ satisfies

$$K\psi_1 = \lambda_1 \psi_1.$$

Theorem 13. *Under the same assumptions as in the preceding theorem, the relations*

$$\varphi = E\psi, \quad \psi = \lambda^{-1} K\varphi$$

define a one-to-one correspondence between all elements $\psi \neq 0$ satisfying

$$K\psi = \lambda\psi \qquad (\lambda \neq 0)$$

and all elements $\varphi \neq 0$ satisfying

$$T\varphi = \lambda\varphi \qquad (\lambda \neq 0).$$

Proof. The proof of this theorem is similar to that of Theorem 7.

§ 7. Some Properties of $T\tilde{T}$.

Theorem 14. *If the bounded linear transformation A is symmetrisable (that is, if HA is self-adjoint), and if $Hf = 0$ implies $Af = 0$, then the characteristic values of A are real, and characteristic elements, belonging to different characteristic values, are H-orthogonal. If, moreover, HA is positive, the characteristic values of A are non-negative.*

Proof. Let $f \neq 0$ and $Af = \lambda f$. If $(Hf, f) = 0$, so that by Lemma 2 also $Hf = 0$, we have by hypothesis $Af = 0$ or $\lambda f = 0$. Hence, since $f \neq 0$, $\lambda = 0$. If $(Hf, f) \neq 0$, we find

$$\lambda(Hf, f) = (H\lambda f, f) = (HAf, f) = (f, HAf) = (f, H\lambda f) = \bar{\lambda}(Hf, f)$$

or $\lambda = \bar{\lambda}$, which shows that λ is real.

Let now $\lambda \neq \mu$, $f \neq 0$, $g \neq 0$, $Af = \lambda f$ and $Ag = \mu g$. Then

$$\lambda(Hf, g) = (HAf, g) = (f, HAg) = \bar{\mu}(f, Hg) = \mu(Hf, g)$$

or $(\lambda - \mu)(Hf, g) = 0$, from which follows, since $\lambda - \mu \neq 0$, that $(Hf, g) = 0$.

Finally, if HA is positive, $f \neq 0$, $Af = \lambda f$, we have $\lambda = 0$ for $(Hf, f) = 0$ as already proved, and, for $(Hf, f) > 0$,

$$\lambda(Hf, f) = (H\lambda f, f) = (HAf, f) \geq 0,$$

hence $\lambda \geq 0$. This shows that now the characteristic values of A are non-negative.

Supposing now again that the bounded linear transformations K and $\tilde{K}$ are H-adjoints, that $P = HK \neq O$ and $T = EK$ is completely continuous, the results of the last theorem can be applied to the completely continuous symmetrisable transformation $T\tilde{T}$, since $Hf = 0$ implies $\tilde{T}f = 0$ (Theorem 1), hence certainly $T\tilde{T}f = 0$, and $HT\tilde{T}$ is positive (Theorem 3). The characteristic values of $T\tilde{T}$ being therefore non-negative, we shall denote an arbitrary one of them by $|\lambda^2|$. It is evident that $T\tilde{T}$ has at least one characteristic value $\neq 0$, since, by Theorem 5, there exist elements u and $v \neq 0$ and a number $\lambda > 0$ such that $Tu = \lambda v$, $\tilde{T}v = \lambda u$, hence $T\tilde{T}v = \lambda Tu = \lambda^2 v$. On account of Lemma 9 it is possible now to range the characteristic values $\neq 0$ into a sequence $|\lambda_i|^2$ $(i = 1, 2, \ldots)$ such that every one of them occurs in this sequence as many times as denoted by its multiplicity, while, moreover, $|\lambda_1| \geq |\lambda_2| \geq \cdots$. By Lemma 10 it is possible then to choose in the unitary space of all characteristic elements belonging to a certain characteristic value $|\lambda|^2 \neq 0$ with multiplicity n an H-orthonormal system consisting of n elements such that this system determines this unitary space. Doing this for all characteristic values $\neq 0$, we may range the elements of all these H-orthonormal systems into a sequence v_i $(i = 1, 2, \ldots)$ such that for every value of i the element v_i belongs to the characteristic value $|\lambda_i|^2$, hence

$$T\tilde{T}v_i = |\lambda_i|^2 v_i. \tag{19}$$

Evidently the whole sequence v_i is also H-orthonormal, since for $|\lambda_m| = |\lambda_n|$ the relation $(Hv_m, v_n) = 0$ follows from our definition of the sequence v_i, and for $|\lambda_m| \neq |\lambda_n|$ this relation follows from the H-orthogonality of characteristic elements belonging to different characteristic values. We observe that as a result of these remarks every characteristic element of $T\tilde{T}$ with characteristic value $|\lambda|^2 \neq 0$ is a linear combination of those elements v_j from the sequence v_i for which $|\lambda_j|^2 = |\lambda|^2$.

Writing now $\tilde{T}v_i = |\lambda_i| u_i$, so that $Tu_i = |\lambda_i| v_i$ by (19), the sequence u_i is also H-orthonormal, as follows from

$$(Hu_i, u_j) = |\lambda_i \lambda_j|^{-1}(H\tilde{T}v_i, \tilde{T}v_j) = |\lambda_i \lambda_j|^{-1}(HT\tilde{T}v_i, v_j) =$$
$$= |\lambda_i|^2 \cdot |\lambda_i \lambda_j|^{-1}(Hv_i, v_j) = \begin{cases} 1 & \text{for } i = j, \\ 0 & \text{for } i \neq j. \end{cases}$$

§ 8. A Maximum-property of the Characteristic Values of a Normalisable Transformation.

To prepare the way for the proof of an Expansion Theorem for normalisable transformations, which will be given in the next paragraph, we shall prove in the present paragraph that the characteristic values of the normalisable transformations T and K, considered in Theorems 10 and 12, possess a certain maximum-property.

We suppose therefore that the bounded linear transformation K is normalisable, that $P = HK \neq O$, and $T = EK$ is completely continuous. Then it follows from Theorem 10 that the sequences v_i and $|\lambda_i|$, satisfying

$$T\tilde{T}v_i = |\lambda_i|^2 v_i,$$

which we introduced in the preceding paragraph, may be identified now with the H-orthonormal sequence φ_i of characteristic elements of T and with the absolute values of the characteristic values λ_i of T, satisfying

$$T\varphi_i = \lambda_i \varphi_i, \quad \tilde{T}\varphi_i = \bar{\lambda}_i \varphi_i.$$

Theorem 15. 1°. *$|\lambda_n| = \max N(Kf)/N(f)$ for all elements f satisfying the conditions $N(f) \neq 0$ and $(Hf, \varphi_1) = \cdots = (Hf, \varphi_{n-1}) = 0$. For $f = \varphi_n$ the maximum is attained.*

2°. *$N(Kf) = 0$ if and only if $(Hf, \varphi_i) = 0$ for every value of i.*

In the special case that $Hf = 0$ implies $Kf = 0$, the characteristic elements φ of T may be replaced in both parts of the theorem by the characteristic elements ψ of K, corresponding with the elements φ by Theorem 13.

Proof. 1°. Let $N(f) \neq 0$ and $(Hf, \varphi_1) = \cdots = (Hf, \varphi_{n-1}) = 0$. If $N(Kf) = 0$, the inequality $N(Kf)/N(f) \leq |\lambda_n|$ is certainly satisfied; if, however, $N(Kf) \neq 0$, we have $Pf = HKf \neq 0$, and we define the sequences f_k and $\bar{f}_k$ $(k = 0, 1, 2, \cdots)$ in the same way as in Theorem 5 by

$$f_0 = f, \quad \bar{f}_k = f_k / N(f_k),$$
$$f_{2k+1} = T\bar{f}_{2k}, \quad f_{2k+2} = \tilde{T}\bar{f}_{2k+1}$$
$$(k = 0, 1, 2, \ldots).$$

We observe first that $(Hf_k, \varphi_1) = \cdots = (Hf_k, \varphi_{n-1}) = 0$ for every value of k. This is proved by induction; the relations hold for $k = 0$ by hypothesis, and supposing them to be true for $k - 1$ (k odd), we find

$$(Hf_k, \varphi_i) = (HT\bar{f}_{k-1}, \varphi_i) = (H\bar{f}_{k-1}, \tilde{T}\varphi_i) = \lambda_i (Hf_{k-1}, \varphi_i)/N(f_{k-1}) = 0$$

for $i = 1, \ldots, n-1$. The proof for even k is similar. In the same way as in Theorem 5 we find now elements $u, v \neq 0$ and a positive number λ such that $Tu = \lambda v$, $\tilde{T}v = \lambda u$, where $v = \lim f_{l+2}$ $(l = k_1, k_2, \ldots)$. This implies

$$(Hv, \varphi_1) = \cdots = (Hv, \varphi_{n-1}) = 0. \tag{20}$$

Now, since $T\tilde{T}v = \lambda^2 v$, the element v is a linear combination of those elements φ_j from the sequence φ_i for which $|\lambda_j| = \lambda$; the relations (20) show that $\varphi_1, \ldots, \varphi_{n-1}$ are not among these elements; hence $\lambda \leq |\lambda_n|$. Finally, the non-descending sequence of numbers $N(f_k)$ $(k = 1, 2, \ldots)$ having the limit λ, we conclude that

$$N(Kf)/N(f) = N(K\bar{f}_0) = N(T\bar{f}_0) = N(f_1) \leq \lambda \leq |\lambda_n|.$$

For $f = \varphi_n$ the maximumvalue $|\lambda_n|$ is attained on account of

$$N(K\varphi_n)/N(\varphi_n) = N(K\varphi_n) = N(T\varphi_n) = N(\lambda_n \varphi_n) = |\lambda_n| N(\varphi_n) = |\lambda_n|.$$

2°. We suppose first that $N(Kf)=0$. Then, by Lemma 2, $HKf = HTf = 0$, so that

$$(Hf, \varphi_i) = (Hf, \bar{\lambda}_i^{-1}\tilde{T}\varphi_i) = \lambda_i^{-1}(Hf, \tilde{T}\varphi_i) = \lambda_i^{-1}(HTf, \varphi_i) = 0$$

for every value of i.

Let now conversely $(Hf, \varphi_i)=0$ for every value of i. If $N(f)=0$, then, by Theorem 4, $HKf=0$ or $N(Kf)=0$; we shall assume therefore that $N(f)\neq 0$. Supposing first that the number of characteristic values λ_i is infinite, so that $\lim \lambda_i = 0$, the relations $(Hf, \varphi_i)=0$ imply, by what we have proved in 1°,

$$N(Kf) \leq |\lambda_i|\, N(f) \qquad (i = 1, 2, \ldots);$$

hence $N(Kf)=0$. If the number N of characteristic values λ_i is finite, the existence of an element f such that $N(Kf)\neq 0$ and $(Hf, \varphi_i)=0$ for $i=1,\ldots,N$ implies, as the proof of 1° shows, the existence of a characteristic element v of $T\tilde{T}$ with characteristic value $\lambda\neq 0$, and with the property that $(Hv, \varphi_i)=0$ for $i=1,\ldots,N$. This, however, is impossible since v must be a linear combination of some of the elements φ_i.

In the special case that $Hf=0$ implies $Kf=0$, the elements φ may be replaced in both parts of the theorem by the corresponding elements ψ, since then $(Hf, \psi_i) = (Hf, \varphi_i)$, the system ψ_i is H-orthonormal, and $K\psi_n = KE\psi_n = K\varphi_n$.

Remark. Since

$$N^2(Kf) = (HKf, Kf) = (Hf, \tilde{K}Kf) = (f, H\tilde{K}Kf) =$$
$$= (f, HK\tilde{K}f) = (Hf, K\tilde{K}f) = (H\tilde{K}f, \tilde{K}f) = N^2(\tilde{K}f)$$

we may replace $N(Kf)$ by $N(\tilde{K}f)$ in the last theorem.

§ 9. Expansion Theorem for Normalisable Transformations.

Under the same assumptions about K and T as in the preceding paragraph, we shall prove now the following

Theorem 16 (Expansion Theorem). *If* $\alpha_i = (Hf, \varphi_i)$ $(i = 1, 2, \ldots)$ *for an arbitrary element* $f \in \mathfrak{R}$, *then*

$$(21)\quad \lim_{n\to\infty} N\Big(Kf - \sum_{i=1}^{n} \lambda_i \alpha_i \varphi_i\Big) = \lim_{n\to\infty}\Big(H\Big(Kf - \sum_{i=1}^{n} \lambda_i \alpha_i \varphi_i\Big),\ Kf - \sum_{i=1}^{n} \lambda_i \alpha_i \varphi_i\Big)^{1/2} = 0$$

and

$$(22)\quad \lim_{n\to\infty} N\Big(\tilde{K}f - \sum_{i=1}^{n} \bar{\lambda}_i \alpha_i \varphi_i\Big) = \lim_{n\to\infty}\Big(H\Big(\tilde{K}f - \sum_{i=1}^{n} \bar{\lambda}_i \alpha_i \varphi_i\Big),\ \tilde{K}f - \sum_{i=1}^{n} \bar{\lambda}_i \alpha_i \varphi_i\Big)^{1/2} = 0.$$

Furthermore

$$(HKf, f) = \sum \lambda_i |\alpha_i|^2, \quad (H\tilde{K}f, f) = \sum \bar{\lambda}_i |\alpha_i|^2.$$

In the special case that $Hf = 0$ implies $Kf = 0$, the elements φ_i may be replaced by the corresponding elements ψ_i, and then $\alpha_i = (Hf, \varphi_i) = (Hf, \psi_i)$.

Proof. Writing $r_{n+1} = f - \sum_{i=1}^{n} \alpha_i \varphi_i$, we have $(H r_{n+1}, \varphi_1) = \cdots = (H r_{n+1}, \varphi_n) = 0$, hence $\left(H r_{n+1}, \sum_{i=1}^{n} \alpha_i \varphi_i\right) = 0$, from which follows immediately

$$N^2(f) = (Hf, f) = \left(H \sum_{i=1}^{n} \alpha_i \varphi_i, \sum_{i=1}^{n} \alpha_i \varphi_i\right) + (r_{n+1}, r_{n+1}) = N^2\left(\sum_{i=1}^{n} \alpha_i \varphi_i\right) + N^2(r_{n+1}),$$

so that $N(r_{n+1}) \leq N(f)$.

Supposing now that the number of characteristic values λ_i is infinite, and that $N(r_{n+1}) \neq 0$, we find in virtue of Theorem 15, 1°, since $(H r_{n+1}, \varphi_1) = \cdots = (H r_{n+1}, \varphi_n) = 0$, that

$$N(K r_{n+1}) \leq |\lambda_{n+1}| \, N(r_{n+1}) \leq |\lambda_{n+1}| \, N(f). \tag{23}$$

If $N(r_{n+1}) = 0$, then also $HK r_{n+1} = 0$ (Theorem 4) or $N(K r_{n+1}) = 0$, so that (23) is true in this case as well. Hence

$$N\left(Kf - \sum_{i=1}^{n} \lambda_i \alpha_i \varphi_i\right) = N\left(Kf - \sum_{i=1}^{n} \alpha_i T \varphi_i\right) = N\left(Kf - \sum_{i=1}^{n} \alpha_i K \varphi_i\right) =$$
$$= N(K r_{n+1}) \leq |\lambda_{n+1}| \, N(f),$$

and this shows, since $\lim |\lambda_{n+1}| = 0$, that

$$\lim_{n \to \infty} N\left(Kf - \sum_{i=1}^{n} \lambda_i \alpha_i \varphi_i\right) = 0.$$

If the number N of characteristic values λ_i is finite, we find on account of Theorem 15, 2° the relation $N(K r_{N+1}) = 0$ or

$$N\left(Kf - \sum_{i=1}^{N} \lambda_i \alpha_i \varphi_i\right) = 0.$$

Writing therefore $\lambda_i = 0$ for $i \geq N + 1$, we see that (21) holds also in this case.

This disposes of (21). The formula (22) is proved in a similar way, using Theorem 15 with K replaced by $\tilde{K}$.

From

$$\left|\left(H\left(Kf - \sum_{i=1}^{n} \lambda_i \alpha_i \varphi_i\right), g\right)\right| \leq N\left(Kf - \sum_{i=1}^{n} \lambda_i \alpha_i \varphi_i\right) \cdot N(g) \to 0$$

as $n \to \infty$ we infer that

$$\lim_{n\to\infty} \left| (H K f, g) - \sum_{i=1}^{n} \lambda_i \alpha_i (H \varphi_i, g) \right| = 0$$

or

$$(H K f, g) = \sum \lambda_i \alpha_i (H \varphi_i, g),$$

hence in particular, for $g = f$,

$$(H K f, f) = \sum \lambda_i |\alpha_i|^2.$$

The relation

$$(H \tilde{K} f, f) = \sum \bar{\lambda}_i |\alpha_i|^2$$

is proved in a similar way.

Evidently, if $Hf = 0$ implies $Kf = 0$, we have $\alpha_i = (Hf, \varphi_i) = (Hf, E\psi_i) = (Hf, \psi_i)$, and the elements φ_i may be replaced by the corresponding elements ψ_i in (21) and (22).

Remark. By taking $H = I$, we obtain, as we remarked already in the Introduction, Rellich's Expansion Theorem for completely continuous normal transformations.

§ 10. Continuation on the Closure $\overline{\mathfrak{R}}$.

Supposing the Hilbert space $\mathfrak{R}$ to be not complete, we may continue the bounded linear transformations H, K, $\tilde{K}$, P, T, $\tilde{T}$ and E, about which we make the same assumptions as in the preceding two paragraphs, on the closure $\overline{\mathfrak{R}}$ of $\mathfrak{R}$ in such a way as to leave their bounds unchanged. It is easy to prove that the relations $H = HE$, $P = HK = HT$, $T = EK$, $\tilde{T} = E\tilde{K}$, $T\tilde{T} = \tilde{T}T$, holding in the space $\mathfrak{R}$, remain true in the space $\overline{\mathfrak{R}}$. In the particular case that, in the space $\mathfrak{R}$, the relation $Hf = 0$ implies $Kf = 0$, we have found $K = KE$. This relation, therefore, remains also true after continuation on $\overline{\mathfrak{R}}$.

We shall prove now that the theorems in the preceding two paragraphs remain valid for all elements $f \in \overline{\mathfrak{R}}$, so that it is not necessary to restrict ourselves to elements $f \in \mathfrak{R}$.

Theorem 17. 1°. $|\lambda_n| = \max N(Kf)/N(f)$ *for all elements* $f \in \overline{\mathfrak{R}}$ *satisfying the conditions* $N(f) \neq 0$ *and* $(Hf, \varphi_1) = \cdots = (Hf, \varphi_{n-1}) = 0$. *For* $f = \varphi_n$ *the maximum is attained.*

2°. $N(Kf) = 0$ *for an element* $f \in \overline{\mathfrak{R}}$ *if and only if* $(Hf, \varphi_i) = 0$ *for every value of* i.

In the special case that, in the space $\mathfrak{R}$, the relation $Hf = 0$ implies $Kf = 0$, the elements φ may be replaced in both parts of the theorem by the corresponding elements ψ.

Proof. 1°. Let $f \in \overline{\mathfrak{R}}$, $N(f) \neq 0$ and $(Hf, \varphi_1) = \cdots = (Hf, \varphi_{n-1}) = 0$, and let the sequence $g_i \in \mathfrak{R}$ be such that $\lim g_i = f$, so that also $\lim Hg_i = Hf$. Then the elements $f_i = g_i - \sum_{k=1}^{n-1} (Hg_i, \varphi_k)\varphi_k$ belong also to the space $\mathfrak{R}$ and $(Hf_i, \varphi_1) = \cdots = (Hf_i, \varphi_{n-1}) = 0$. Hence, in virtue of Theorem 15,

$$N(Kf_i)/N(f_i) \leq |\lambda_n|. \tag{24}$$

But, since $\lim_{i\to\infty} (Hg_i, \varphi_k) = (Hf, \varphi_k) = 0$ for $k = 1, \ldots, n-1$, we have $\lim f_i = \lim g_i = f$, so that $\lim Kf_i = Kf$, $\lim N(Kf_i) = N(Kf)$ and $\lim N(f_i) = N(f)$. We conclude therefore from (24) that

$$N(Kf)/N(f) \leq |\lambda_n|.$$

We have already proved in Theorem 15 that the maximum value $|\lambda_n|$ is attained for $f = \varphi_n$.

2°. That $N(Kf) = 0$ implies $(Hf, \varphi_i) = 0$ for every value of i is proved in the same way as in Theorem 15.

Let now conversely $(Hf, \varphi_i) = 0$ for every value of i. If $N(f) = 0$, or if the number of characteristic values λ_i is infinite, we may again repeat the proof as given in Theorem 15. Let us assume therefore that the number N of characteristic values λ_i is finite, let $f \in \overline{\mathfrak{R}}$, $N(f) \neq 0$ and $(Hf, \varphi_1) = \cdots = (Hf, \varphi_N) = 0$. Then, supposing again the sequence $g_i \in \mathfrak{R}$ to be such that $\lim g_i = f$, we find that the elements $f_i = g_i - \sum_{k=1}^{N} (Hg_i, \varphi_k)\varphi_k$ belong to the space $\mathfrak{R}$, and $(Hf_i, \varphi_1) = \cdots = (Hf_i, \varphi_N) = 0$. Hence, in virtue of Theorem 15, $N(Kf_i) = 0$. But, since $\lim_{i\to\infty} (Hg_i, \varphi_k) = (Hf, \varphi_k) = 0$ for $k = 1, \ldots, N$, we have $\lim f_i = \lim g_i = f$, so that $N(Kf) = \lim N(Kf_i) = 0$.

That the elements φ may be replaced by the corresponding elements ψ in the case that, in the space $\mathfrak{R}$, $Hf = 0$ implies $Kf = 0$, is proved in the same way as in Theorem 15.

Theorem 18. *The statements in Theorem 16 remain true for an arbitrary element $f \in \overline{\mathfrak{R}}$.*

Proof. The proof of Theorem 16 remains unchanged.

§ 11. Expansion Theorem for H-adjoints.

In this paragraph we shall suppose again that the bounded linear transformations K and $\tilde{K}$ are H-adjoints, that $P = HK \neq O$, and that $T = EK$ is completely continuous (but no longer that K is normalisable). We have already introduced for this case, in § 7, the sequence $|\lambda_i|$ of singular values of T and $\tilde{T}$, and the H-orthonormal sequences of elements u_i and v_i, satisfying

$$T u_i = |\lambda_i| v_i, \quad \tilde{T} v_i = |\lambda_i| u_i,$$

and we further observe that, in the special case that $Hf = 0$ implies $Kf = \tilde{K}f = 0$, Theorem 7 establishes a one-to-one correspondence between the H-orthonormal sequences u_i, v_i and the H-orthonormal sequences y_i, z_i, satisfying

$$K y_i = |\lambda_i| z_i, \quad \tilde{K} z_i = |\lambda_i| y_i.$$

It is possible now to prove theorems analogous to those in the paragraphs 8, 9 and 10. Since the proofs are also analogous, we shall omit them and only mention the theorems.

Theorem 19. 1°. $|\lambda_n| = \max N(Kf)/N(f)$ *for all elements* $f \in \overline{\mathfrak{R}}$ *satisfying the conditions* $N(f) \neq 0$ *and* $(Hf, u_1) = \cdots = (Hf, u_{n-1}) = 0$. *For* $f = u_n$ *the maximum is attained.*

$|\lambda_n| = \max N(\tilde{K}f)/N(f)$ *for all elements* $f \in \overline{\mathfrak{R}}$ *satisfying the conditions* $N(f) \neq 0$ *and* $(Hf, v_1) = \cdots = (Hf, v_{n-1}) = 0$. *For* $f = v_n$ *the maximum is attained.*

2°. $N(Kf) = 0$ *for an element* $f \in \overline{\mathfrak{R}}$ *if and only if* $(Hf, u_i) = 0$ *for every value of* i.

$N(\tilde{K}f) = 0$ *for an element* $f \in \overline{\mathfrak{R}}$ *if and only if* $(Hf, v_i) = 0$ *for every value of* i.

In the special case that, in the space $\mathfrak{R}$, $Hf = 0$ *implies* $Kf = \tilde{K}f = 0$, *the elements* u, v *may be replaced in both parts of the theorem by the corresponding elements* y, z.

Theorem 20 (Expansion Theorem). *If* $\beta_i = (Hf, u_i)$ *and* $\gamma_i = (Hf, v_i)$ $(i = 1, 2, \ldots)$ *for an arbitrary element* $f \in \overline{\mathfrak{R}}$, *then*

$$\lim_{n \to \infty} N\left(Kf - \sum_{i=1}^{n} |\lambda_i| \beta_i v_i\right) = 0$$

and

$$\lim_{n \to \infty} N\left(\tilde{K}f - \sum_{i=1}^{n} |\lambda_i| \gamma_i u_i\right) = 0.$$

Furthermore

$$(HKf, f) = \sum |\lambda_i| \beta_i \bar{\gamma}_i, \quad (H\tilde{K}f, f) = \sum |\lambda_i| \bar{\beta}_i \gamma_i.$$

In the special case that, in the space $\mathfrak{R}$, $Hf = 0$ *implies* $Kf = 0$, *the elements* u_i, v_i *may be replaced by the corresponding elements* y_i, z_i, *and then* $\beta_i = (Hf, y_i)$; $\gamma_i = (Hf, z_i)$.

§ 12. The Relation between Normalisable Transformations in $\mathfrak{R}$ and Normal Transformations in the Factorspace $\mathfrak{Z}$.

We shall consider here certain linear transformations in the factorspace $\mathfrak{Z} = \mathfrak{R}/[\mathfrak{L}]$, introduced in § 4. If A is a linear transformation in $\mathfrak{R}$, we define the linear transformation $[A]$ in $\mathfrak{Z}$ by $[A][f] = [Af]$. This definition, however, is only then without contradiction if $[f] = [g]$ implies $[Af] = [Ag]$, or, in other words, if $Hf = 0$ implies $HAf = 0$. We shall consider, therefore, in this paragraph only linear transformations A having this property. It is not difficult to see that, conversely, with every linear transformation $[A]$, defined for all $[f] \in \mathfrak{Z}$, corresponds a class of linear transformations A in the space $\mathfrak{R}$, satisfying the condition that $Hf = 0$ implies $HAf = 0$, and such that $[Af] = [A][f]$ for each of the transformations A. The equality $[A_1] = [A_2]$ in $\mathfrak{Z}$ holds therefore if and only if $HA_1 = HA_2$ in $\mathfrak{R}$.

If the bounded linear transformations K and $\tilde{K}$ in $\mathfrak{R}$ are H-adjoints, $Hf = 0$ implies $HKf = H\tilde{K}f = 0$ by Theorem 4; this shows, by what we have just seen, that the linear transformations $[K]$ and $[\tilde{K}]$ exist in $\mathfrak{Z}$. The same is true of $[K\tilde{K}]$ and $[\tilde{K}K]$. The proof for $[K\tilde{K}]$ is as follows: $Hf = 0$ implies $H\tilde{K}f = 0$, and this in its turn implies $HK\tilde{K}f = 0$; the proof for $[\tilde{K}K]$ is similar. Furthermore

$$[K][\tilde{K}][f] = [K][\tilde{K}f] = [K\tilde{K}f] = [K\tilde{K}][f]$$

for arbitrary $[f] \in \mathfrak{Z}$, so that $[K][\tilde{K}] = [K\tilde{K}]$. In the same way we obtain $[\tilde{K}][K] = [\tilde{K}K]$.

We shall suppose now, as in the paragraphs 8, 9 and 10, that the bounded linear transformation K is normalisable, that $P = HK \neq O$, and $T = EK$ is completely continuous. Then the linear transformation $[K] = [T]$ in the space $\mathfrak{Z}$ exists, and it is bounded, since on account of Theorem 15 we have

$$\|[K]\| = \underset{[f] \in \mathfrak{Z}}{\text{l.u.b.}} \|[K][f]\| / \|[f]\| = \underset{[f] \in \mathfrak{Z}}{\text{l.u.b.}} \|[Kf]\| / \|[f]\| = \underset{f \in \mathfrak{R}}{\text{l.u.b.}} N(Kf) / N(f) = |\lambda_1|.$$

In the same way we find that $[\tilde{K}] = [\tilde{T}]$ exists, and that $\|[\tilde{K}]\| = |\lambda_1|$.

Furthermore we see that $[\tilde{K}] = [K]^*$ on account of

$$([K][f], [g]) = ([Kf], [g]) = (HKf, g) = (Hf, \tilde{K}g) = ([f], [\tilde{K}g]) = ([f], [\tilde{K}][g])$$

for arbitrary $[f], [g] \in \mathfrak{Z}$. Finally, observing that $[K\tilde{K}] = [\tilde{K}K]$ in virtue of $HK\tilde{K} = H\tilde{K}K$, we obtain

$$[K][K]^* = [K][\tilde{K}] = [K\tilde{K}] = [\tilde{K}K] = [\tilde{K}][K] = [K]^*[K].$$

The result is therefore that $[K]$ is a bounded normal transformation in $\mathfrak{Z}$.

If $\lambda \neq 0$ is a characteristic value of $T = EK$, that is, if $\lambda\varphi = EK\varphi$ where $\lambda\varphi \neq 0$, we have $\lambda[\varphi] = [EK][\varphi] = [K][\varphi]$, where $[\varphi] \neq [0]$ since $H\varphi \neq 0$ on account of $EK\varphi \neq 0$. We see therefore that with any characteristic element φ of T, belonging to the characteristic value $\lambda \neq 0$, corresponds the characteristic element $[\varphi]$ of $[K]$, also belonging to the characteristic value λ. We shall prove that this correspondence is a one-to-one correspondence, and this will enable us to enunciate theorems for $[K]$, analogous to the theorems for T in the paragraphs 8, 9 and 10.

Theorem 21. *There is a one-to-one correspondence between all characteristic elements φ of $T = EK$, belonging to characteristic values $\neq 0$, and all characteristic elements $[\varphi]$ of $[K]$, belonging to characteristic values $\neq 0$. Corresponding elements have the same characteristic value.*

Proof. We have seen already that with the characteristic element φ of T, belonging to the characteristic value $\lambda \neq 0$, corresponds the characteristic element $[\varphi]$ of $[K]$, also belonging to the characteristic value λ. We shall prove now that no two different characteristic elements φ_1 and φ_2 of T correspond with the same characteristic element of $[K]$. For this purpose we shall suppose that

$$\lambda\varphi_1 = T\varphi_1,\ \mu\varphi_2 = T\varphi_2,\ \lambda\varphi_1 \neq 0,\ \mu\varphi_2 \neq 0,\ \varphi_1 \neq \varphi_2,\ [\varphi_1] = [\varphi_2],$$

and show that in this case we obtain a contradiction. Indeed, from $\lambda\varphi_1 = T\varphi_1$ and $\mu\varphi_2 = T\varphi_2$ follows $\lambda[\varphi_1] = [K][\varphi_1]$ and $\mu[\varphi_2] = [K][\varphi_2]$, hence $\lambda[\varphi_1] = \mu[\varphi_2]$ or $\lambda = \mu$ on account of $[\varphi_1] = [\varphi_2] \neq [0]$. Since $\varphi_1 \neq \varphi_2$, the relation $\lambda = \mu$ implies $T(\varphi_1 - \varphi_2) = \lambda(\varphi_1 - \varphi_2) \neq 0$. On the other hand, we derive from $[\varphi_1] = [\varphi_2]$ the relation $H(\varphi_1 - \varphi_2) = 0$, so that also $T(\varphi_1 - \varphi_2) = 0$, in contradiction with $T(\varphi_1 - \varphi_2) \neq 0$.

It remains only to show that, if $\lambda[\varphi] = [K][\varphi]$ where $\lambda[\varphi] \neq [0]$, there exists an element ψ satisfying the relations $\lambda\psi = T\psi$ and $[\psi] = [\varphi]$. For this purpose we observe that $\lambda[\varphi] = [K][\varphi] = [K][E\varphi]$ implies $\lambda\varphi = KE\varphi + h$, where the element h satisfies $Hh = 0$, and therefore also $Eh = 0$. We obtain therefore $\lambda E\varphi = EKE\varphi = TE\varphi$, which shows that $\psi = E\varphi$ is the required element.

Remark. In the same way we may prove that there exists a similar one-to-one correspondence between the characteristic elements of $\tilde{T} = E\tilde{K}$, belonging to characteristic values $\neq 0$, and those of $[\tilde{K}] = [K]^*$.

This theorem shows that, since T and $\tilde{T}$ have at least one characteristic value $\neq 0$ by Theorem 10, the same is true of $[K]$ and $[\tilde{K}]$. If now $[\varphi_i]$ is the orthonormal sequence of characteristic elements of $[K]$, corresponding with the H-orthonormal sequence φ_i of characteristic elements of T, so that

$$[K][\varphi_i] = \lambda_i[\varphi_i], \quad [\tilde{K}][\varphi_i] = \bar{\lambda}_i[\varphi_i],$$

we immediately get the analogues for $[K]$ and $[\tilde{K}]$ of the Theorems 15—18 for T and $\tilde{T}$. We observe that the statements in those theorems were the result of the complete continuity of $T = EK$, whereas their analogues for $[K]$ and $[\tilde{K}]$ are the result of the established correspondence between the sequences φ_i and $[\varphi_i]$.

Theorem 22. 1°. $|\lambda_n| = \max \|[K][f]\| / \|[f]\| = \max \|[\tilde{K}][f]\| / \|[f]\|$ *for all* $[f] \neq [0]$ *satisfying the conditions* $([f], [\varphi_1]) = \cdots = ([f], [\varphi_{n-1}]) = 0$. *For* $[f] = [\varphi_n]$ *the maximum is attained.*

2°. $[K][f] = [\tilde{K}][f] = [0]$ *if and only if* $([f], [\varphi_i]) = 0$ *for every value of* i, *or, stated in a different way, the orthogonal complement of the closed linear manifold determined by* $[\varphi_1], [\varphi_2], \ldots$ *is identical with the set of all elements* $[f]$ *satisfying* $[K][f] = [\tilde{K}][f] = [0]$.

Proof. Follows immediately from Theorem 15 since, for any element $f \in \mathfrak{R}$ belonging to the class of elements $[f] \in \mathfrak{Z}$, we have the relations

$$\|[K][f]\| / \|[f]\| = N(Kf) / N(f), \quad \|[\tilde{K}][f]\| / \|[f]\| = N(\tilde{K}f) / N(f)$$

and $([f], [\varphi_i]) = (Hf, \varphi_i)$, while $[f] \neq [0]$ is equivalent with $N(f) \neq 0$, and $[K][f] = [0]$ with $N(Kf) = 0$.

Theorem 23 (Expansion Theorem). *If* $\alpha_i = ([f], [\varphi_i])$ $(i = 1, 2, \ldots)$ *for an arbitrary element* $[f] \in \mathfrak{Z}$, *then*

$$[K][f] = \sum \lambda_i \alpha_i [\varphi_i],$$

$$[\tilde{K}][f] = \sum \bar{\lambda}_i \alpha_i [\varphi_i],$$

$$([K][f], [f]) = \sum \lambda_i |\alpha_i|^2, \quad ([\tilde{K}][f], [f]) = \sum \bar{\lambda}_i |\alpha_i|^2.$$

Proof. Follows immediately from Theorem 16.

In the same way as we have obtained the complete Hilbert space $\overline{\mathfrak{R}}$ from the space $\mathfrak{R}$ by adjunction of 'ideal' elements, we may obtain the complete Hilbert space $\overline{\mathfrak{Z}}$ from the space $\mathfrak{Z}$. We shall use the notation $[f]$ also to denote elements of $\overline{\mathfrak{Z}}$, not belonging to $\mathfrak{Z}$, although for elements of this kind there are no longer elements $f \in \mathfrak{R}$ corresponding with it. The bounded linear transformations $[K]$ and $[\tilde{K}]$ may be continued now on the closure $\overline{\mathfrak{Z}}$ in such a way as to leave their bounds $\|[K]\| = \|[\tilde{K}]\| = |\lambda_1|$ unchanged.

Theorem 24. *The statements in the Theorems 22 and 23 remain true for all elements* $[f] \in \overline{\mathfrak{Z}}$, *so that it is not necessary to restrict ourselves to elements* $[f] \in \mathfrak{Z}$.

Proof. The proof is similar to those of the Theorems 17 and 18.

Theorem 25. *The bounded normal transformations* $[K]$ *and* $[\tilde{K}]$ *are completely continuous in the space* $\overline{\mathfrak{Z}}$.

Proof. We might give a proof depending on a general theorem about the spectral representation of bounded normal transformations[1], but we prefer to give a more 'elementary' proof. For this purpose we recall that a sequence $[f]_n$ of elements belonging to the complete Hilbert space $\overline{\mathfrak{Z}}$ is called weakly convergent when the sequence of complex numbers $([f]_n, [g])$ converges for every element $[g] \in \overline{\mathfrak{Z}}$. It is well-known that every bounded infinite set of elements $[f] \in \overline{\mathfrak{Z}}$ contains a weakly convergent sequence $[f]_n$. It follows therefore from the definition of a completely continuous transformation in § 5 that to prove the complete continuity of $[K]$ in $\overline{\mathfrak{Z}}$ it is sufficient to show that, if $[f]_n$ is a bounded, weakly converging sequence, the sequence $[K][f]_n$ converges.

Let now $\mathfrak{L}[\varphi_1, \ldots, \varphi_k]$ be the unitary space determined by $[\varphi_1], \ldots, [\varphi_k]$. Then, for any $[f]$ belonging to the orthogonal complement of $\mathfrak{L}[\varphi_1, \ldots, \varphi_k]$, we have by Theorem 22 the inequality $\|[K][f]\| \leq |\lambda_{k+1}| \cdot \|[f]\|$. In the case that the total number N of characteristic values λ_i is finite, the same theorem shows that $[K][f] = [0]$ for any $[f]$ in the orthogonal complement of $\mathfrak{L}[\varphi_1, \ldots, \varphi_N]$.

Given the bounded, weakly converging sequence $[f]_n$, we shall prove that $[K][f]_n$ converges. Let $\|[f]_n\| \leq M$. Then, since we may write $[f]_n = [g]_n + [h]_n$, where $[g]_n \in \mathfrak{L}[\varphi_1, \ldots, \varphi_k]$ and $[h]_n$ belongs to the orthogonal complement of $\mathfrak{L}[\varphi_1, \ldots, \varphi_k]$, so that $\|[f]_n\|^2 = \|[g]_n\|^2 + \|[h]_n\|^2$, we see that also $\|[h]_n\| \leq M$. By what we have just proved, the element $[K][h]_n$ satisfies therefore the inequality

$$\|[K][h]_n\| \leq |\lambda_{k+1}| \cdot \|[h]_n\| \leq |\lambda_{k+1}| M$$

[1] A bounded normal transformation in a Hilbert space of infinite dimension is completely continuous if and only if its spectrum converges to 0.

(in the case that the total number N of characteristic values λ_i is finite, we have even, for $k = N$, the relation $\|[K][h]_n\| = 0$); hence, given $\varepsilon > 0$, we may take the index k so large that $\|[K][h]_n\| < \varepsilon/3$ for every value of n.

As regards the elements $[g]_n$, it is not difficult to see that they converge. Indeed, since $[g]_n = \sum_{i=1}^{k} ([f]_n, [\varphi_i])[\varphi_i]$ and $[f]_n$ converges weakly, we have

$$\lim_{n \to \infty} ([f]_n, [\varphi_i]) = \alpha_i \qquad (i = 1, \ldots, k);$$

hence $\lim [g]_n = \sum_{i=1}^{k} \alpha_i [\varphi_i]$. The elements $[K][g]_n$ converge then as well, so that

$$\|[K][g]_n - [K][g]_m\| < \varepsilon/3 \quad \text{for} \quad m, n > n_0.$$

This shows that, for $m, n > n_0$,

$$\|[K][f]_n - [K][f]_m\| \leq$$
$$\leq \|[K][g]_n - [K][g]_m\| + \|[K][h]_n\| + \|[K][h]_m\| < \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon;$$

the sequence $[K][f]_n$ converges therefore. This completes the proof for $[K]$; that for $[\tilde{K}]$ is similar.

§ 13. Normalisable Transformations of a Special Kind.

In this paragraph we shall suppose that the Hilbert space $\mathfrak{R}$ is complete, that H is a bounded, positive, self-adjoint transformation and A is a bounded linear transformation in $\mathfrak{R}$. Then, as we already remarked in § 2, the adjoint A^* exists in $\mathfrak{R}$, and is also bounded. It follows now from

$$(HAHf, g) = (AHf, Hg) = (Hf, A^*Hg)$$

that the transformations $K = AH$ and $\tilde{K} = A^*H$ are H-adjoints. We observe that if $f \in \mathfrak{R}$ satisfies $Hf = 0$, then also $Kf = \tilde{K}f = 0$. If, moreover, $HK\tilde{K} = H\tilde{K}K$, that is, if

$$HAHA^*H = HA^*HAH, \tag{25}$$

the transformation $K = AH$ is normalisable.

Making now the assumptions that (25) is satisfied and that one at least of the transformations A and H is completely continuous, the transformation $K = AH$ is therefore normalisable and completely continuous, so that the theorems in §§ 8—9 hold. It is possible, however, to prove somewhat more in this special case:

Theorem 26 (Expansion Theorem). *If ψ_i is the H-orthonormal sequence of characteristic elements of $K = AH$, belonging to the sequence of characteristic values $\lambda_i \neq 0$, and if $\alpha_i = (Hf, \psi_i)$ $(i = 1, 2, \ldots)$ for an arbitrary element $f \in \mathfrak{R}$, then*

$$Kf = \sum \lambda_i \alpha_i \psi_i + h, \quad \tilde{K} f = \sum \bar{\lambda}_i \alpha_i \psi_i + k,$$

where $Hh = Hk = 0$. For $n \geq 2$ we have

$$K^n f = \sum_i \lambda_i^n \alpha_i \psi_i, \quad \tilde{K}^n f = \sum_i \bar{\lambda}_i^n \alpha_i \psi_i.$$

Proof. It is well-known that, since H is bounded, self-adjoint and positive, there exists a uniquely determined, bounded, self-adjoint and positive transformation $H^{1/2}$, having the property that $(H^{1/2})^2 = H$. On account of $(H\psi_i, \psi_j) = (H^{1/2}\psi_i, H^{1/2}\psi_j)$ we see therefore that the sequence $H^{1/2}\psi_i$ is orthonormal, which implies that, writing $\alpha_i = (g, H^{1/2}\psi_i)$ for an arbitrary $g \in \mathfrak{R}$, the sums $s_k = \sum_{i=1}^{k} \alpha_i H^{1/2}\psi_i$ converge to an element p. Taking $g = H^{1/2} f$, we find then $\sum \alpha_i H^{1/2}\psi_i = p$, where $\alpha_i = (H^{1/2} f, H^{1/2}\psi_i) = (Hf, \psi_i)$. From this we derive

$$A H^{1/2} p = A H^{1/2} \sum \alpha_i H^{1/2} \psi_i = \sum \alpha_i A H \psi_i = \sum \lambda_i \alpha_i \psi_i.$$

The convergence of the series $\sum \lambda_i \alpha_i \psi_i$ is thus established, and this enables us now to make $n \to \infty$ in the relation

$$\lim_{n\to\infty} N\left(Kf - \sum_{i=1}^{n} \lambda_i \alpha_i \psi_i\right) = 0,$$

proved in Theorem 16. Writing $Kf - \sum \lambda_i \alpha_i \psi_i = h$, we obtain then $N(h) = 0$; hence, $N(h) = 0$ being equivalent with $Hh = 0$,

$$Kf = \sum \lambda_i \alpha_i \psi_i + h,$$

where $Hh = 0$.

From this we infer

$$K^2 f = \sum \lambda_i \alpha_i K\psi_i + Kh = \sum \lambda_i^2 \alpha_i \psi_i + Kh,$$

but, since $Hh = 0$, we have $Kh = AHh = 0$; hence

$$K^2 f = \sum \lambda_i^2 \alpha_i \psi_i.$$

The relation

$$K^n f = \sum \lambda_i^n \alpha_i \psi_i$$

for $n > 2$ follows easily by induction.

The relations for $\tilde{K} f$ and $\tilde{K}^n f$ $(n \geq 2)$ are proved in a similar way.

Remarks. 1°. It is not difficult to prove that the elements h and k in this theorem are not necessarily identical with the nullelement.

2°. It may be proved easily that the relation $HAHA^*H = HA^*HAH$ is equivalent with $H^{1/2}AHA^*H^{1/2} = H^{1/2}A^*HAH^{1/2}$, and this latter equality is evidently equivalent with saying that $H^{1/2}AH^{1/2}$ is normal. It is not difficult to show now that, under the mentioned conditions, the normal transformation $H^{1/2}AH^{1/2}$ has the same sequence λ_i of characteristic values $\neq 0$ as the normalisable transformation $K = AH$. Indeed, let $K\psi = AH\psi = \lambda\psi \neq 0$. Then, writing $H^{1/2}\psi = \chi$, we have $H^{1/2}AH^{1/2}\chi = H^{1/2}AH\psi = \lambda H^{1/2}\psi = \lambda\chi$, where $\lambda\chi \neq 0$ since $AH^{1/2}\lambda\chi = \lambda AH\psi = \lambda^2\psi \neq 0$. Conversely, if $H^{1/2}AH^{1/2}\chi = \lambda\chi \neq 0$, we find, writing $\psi = \lambda^{-1}AH^{1/2}\chi$, so that $H^{1/2}\psi = \lambda^{-1}H^{1/2}AH^{1/2}\chi = \chi$, the relation

$$K\psi = AH^{1/2}H^{1/2}\psi = AH^{1/2}\chi = \lambda\psi,$$

where $\lambda\psi \neq 0$ since $H^{1/2}\lambda\psi = \lambda\chi \neq 0$. This shows that K and $H^{1/2}AH^{1/2}$ have the same characteristic values $\neq 0$, and that with the H-orthonormal sequence ψ_i of characteristic elements of K corresponds the orthonormal sequence $H^{1/2}\psi_i$ of characteristic elements of $H^{1/2}AH^{1/2}$.

3°. In the special case that $HAH = HA^*H$, in particular when A is self-adjoint, we have $(HK)^* = (HAH)^* = HA^*H = HAH = HK$; in this case, therefore, K is symmetrisable, so that all characteristic values λ_i are real.

4°. Supposing no longer that $K = AH$ is normalisable, we may prove, in a similar way as we did the last theorem, the following Expansion Theorem for the H-adjoints $K = AH$ and $\tilde{K} = A^*H$:

Theorem 27. *If y_i and z_i are the H-orthonormal systems and $|\lambda_i|$ is the sequence of non-negative numbers, mentioned in Theorem 20, and $\beta_i = (Hf, y_i)$, $\gamma_i = (Hf, z_i)$ $(i = 1, 2, \ldots)$ for an arbitrary element $f \in \mathfrak{R}$, then*

$$Kf = \sum |\lambda_i| \beta_i z_i + h',$$

$$\tilde{K}f = \sum |\lambda_i| \gamma_i y_i + k',$$

where $Hh' = Hk' = 0$.

§ 14. Applications to Linear Integral Equations.

In the present paragraph we shall give, finally, some indications of how the contents of the preceding paragraphs may be applied to the theory of linear integral equations. Let a_i, b_i $(i = 1, \ldots, m)$ be real, and such that $a_i < b_i$ $(i = 1, \ldots, m)$. Then $\Delta = [a_1, b_1; \ldots; a_m, b_m]$ is an interval in the m-dimensional

Euclidean space. The point $(x_1, \ldots, x_m)$ in this space will be denoted by x. Furthermore we shall denote the function space of all functions $f(x)$, with complex values, such that $|f(x)|^2$ is summable (in the sense of Lebesgue) over $\varDelta$, by $L_2^{(m)}(\varDelta)$ or $L_2(\varDelta)$ or shortly by L_2. As well-known, L_2 is a complete Hilbert space if addition and multiplication with complex numbers are defined in the usual way, and the scalar product of f and g as the integral of $f(x)\overline{g(x)}$ over $\varDelta$. Convergence in this Hilbert space of the series $\sum_{i=1}^{\infty} f_i(x)$ to $f(x)$ means that

$$\lim_{n \to \infty} \int_{\varDelta} \left| f(x) - \sum_{i=1}^{n} f_i(x) \right|^2 d\,x = 0;$$

it is equivalent therefore with saying that $\sum_{i=1}^{\infty} f_i(x)$ converges in mean to $f(x)$, and we shall write $f(x) \backsim \sum_{i=1}^{\infty} f_i(x)$ in this case. The interval $[a_1, b_1; \ldots; a_m, b_m; a_1, b_1; \ldots; a_m, b_m]$ in $2\,m$-dimensional Euclidean space will be denoted by $\varDelta \times \varDelta$, and the function space of all functions $f(x, y)$ $(x, y \in \varDelta)$, with complex values, for which $|f(x, y)|^2$ is summable over $\varDelta \times \varDelta$, by $L_2^{(2\,m)}$.

Furthermore, when $\mathfrak{R}$ is a complete Hilbert space (elements $f, g, \ldots$; α complex), it is well-known that the set $\mathfrak{R}^n$ of all elements $\{f\} = \{f^1, \ldots, f^n\}$, when the fundamental operations and the scalar product in it are defined by

$$\{f\} + \{g\} = \{f^1 + g^1, \ldots, f^n + g^n\},$$

$$\alpha\{f\} = \{\alpha f^1, \ldots, \alpha f^n\},$$

$$(\{f\}, \{g\}) = \sum_{i=1}^{n} (f^i, g^i),$$

(where the letters i and j will denote indices, and not exponents), is also a complete Hilbert space. The following lemma's are now easy to prove:

Lemma 11. *If, for $x \in \varDelta$, the functions $A_{ij}(x)$ $(i, j = 1, \ldots, n)$ are complex-valued measurable functions, the transformation A, defined by $\{g\} = A\{f\}$, where*

$$g^i(x) = \sum_{j=1}^{n} A_{ij}(x)\, f^j(x) \qquad (i = 1, \ldots, n),$$

is a bounded linear transformation in $[L_2^{(m)}(\varDelta)]^n$ if and only if all functions $A_{ij}(x)$ are bounded in $\varDelta$. In this case the adjoint $\{h\} = A^*\{f\}$ is given by

$$h^i(x) = \sum_{j=1}^{n} \overline{A_{ji}(x)}\, f^j(x) \qquad (i = 1, \ldots, n).$$

We have $A \neq O$ *if and only if one at least of the functions* $A_{ij}(x) \neq 0$ *on a set of positive measure;* A *is self-adjoint if and only if the matrix* $\| A_{ij}(x) \|$ *is Hermitian, that is, if and only if* $A_{ij}(x) = \overline{A_{ji}(x)}$ *for almost every* $x \in \varDelta$*; and, supposing* A *to be bounded and self-adjoint, it is positive if and only if*

$$\sum_{i,j=1}^{n} A_{ij}(x)\, \overline{f^i(x)}\, f^j(x) \geq 0$$

for arbitrary $\{f\} \in [L_2]^n$ *and for almost every* $x \in \varDelta$.

Lemma 12. *If, for* $(x, y) \in \varDelta \times \varDelta$*, the functions* $A_{ij}(x, y)$ $(i, j = 1, \ldots, n)$ *are complex-valued and measurable, and if the integrals*

$$(26) \qquad \int_{\varDelta \times \varDelta} | A_{ij}(x, y) |^2 \, dx\, dy \qquad (i, j = 1, \ldots, n)$$

are finite (in other words, if $A_{ij}(x, y) \in L_2^{(2m)}(\varDelta)$*), the linear "integral transformation"* A *in* $[L_2^{(m)}(\varDelta)]^n$*, defined by* $\{g\} = A\{f\}$*, where*

$$g^i(x) = \sum_{j=1}^{n} \int_{\varDelta} A_{ij}(x, y) f^j(y)\, dy \qquad (i = 1, \ldots, n),$$

is completely continuous. The adjoint $\{h\} = A^*\{f\}$ *is given by*

$$h^i(x) = \sum_{j=1}^{n} \int_{\varDelta} \overline{A_{ji}(y, x)}\, f^j(y)\, dy \qquad (i = 1, \ldots, n).$$

We have $A \neq O$ *if and only if one at least of the integrals* (26) *does not vanish, that is, if and only if one at least of the functions* $A_{ij}(x, y) \neq 0$ *on a set of positive measure in* $\varDelta \times \varDelta$. A *is self-adjoint if and only if the "matrix-kernel"* $\| A_{ij}(x, y) \|$ *is Hermitian, that is, if and only if* $A_{ij}(x, y) = \overline{A_{ji}(y, x)}$ *almost everywhere in* $\varDelta \times \varDelta$*. Supposing* A *to be self-adjoint, it is positive if and only if*

$$\sum_{i,j=1}^{n} \int_{\varDelta \times \varDelta} A_{ij}(x, y)\, \overline{f^i(x)}\, f^j(y)\, dx\, dy \geq 0$$

for arbitrary $\{f\} \in [L_2]^n$.

The theory in § 13 may be applied now to several types of integral transformations:

I. The normal integral transformation K in $[L_2]^n$ with matrix-kernel $\|K_{ij}(x, y)\|$, where all $K_{ij}(x, y) \in L_2^{(2m)}$. The adjoint transformation K^* being determined by the matrix-kernel $\|K^*_{ij}(x, y)\| = \|\overline{K_{ji}(y, x)}\|$, we have therefore, since $KK^* = K^*K$,

$$\sum_{l=1}^{n} \int_{\Delta} K_{il}(x, z) \overline{K_{jl}(y, z)} \, dz = \sum_{l=1}^{n} \int_{\Delta} \overline{K_{li}(z, x)} K_{lj}(z, y) \, dz$$

for almost every point $(x, y) \in \Delta \times \Delta$. Since (by Lemma 12) K is completely continuous, Theorem 12 (with $H = I$) shows now that if one at least of the functions $K_{ij}(x, y) \neq 0$ on a set of positive measure in $\Delta \times \Delta$, the system of homogeneous linear integral equations

$$\sum_{j=1}^{n} \int_{\Delta} K_{ij}(x, y) \, \psi^j(y) \, dy - \lambda \psi^i(x) = 0 \qquad (i = 1, \ldots, n)$$

has a non-trivial solution with $\lambda \neq 0$, while from Theorem 26 it follows that, if $\psi^i_k(x)$ $(i = 1, \ldots, n;\ k = 1, 2, \ldots)$ is the orthonormal sequence of characteristic "functionsets" of this system of equations, belonging to the sequence of characteristic values $\lambda_k \neq 0$, and if

$$\alpha_k = (\{f\}, \{\psi_k\}) = \sum_{i=1}^{n} \int_{\Delta} f^i(x) \overline{\psi^i_k(x)} \, dx$$

for an arbitrary $\{f\} = \{f^1(x), \ldots, f^n(x)\} \in [L_2]^n$, then

$$(27) \qquad \sum_{j=1}^{n} \int_{\Delta} K_{ij}(x, y) \, f^j(y) \, dy \backsim \sum_k \lambda_k \, \alpha_k \, \psi^i_k(x) \qquad (i = 1, \ldots, n),$$

$$(28) \qquad \sum_{j=1}^{n} \int_{\Delta} K^*_{ij}(x, y) f^j(y) \, dy \backsim \sum_k \overline{\lambda}_k \, \alpha_k \, \psi^i_k(x) \qquad (i = 1, \ldots, n).$$

We observe that the expressions on the left in (27) and (28) vanish if (and only if) $\{f\}$ is orthogonal to all $\{\psi_k\}$.

Besides the expansions (27) and (28), it is, however, possible to prove an expansion theorem for the element $K_{ij}(x, y)$ of the matrix-kernel as well.

Theorem 28. *We have*

$$(29) \qquad K_{ij}(x, y) \backsim \sum_k \lambda_k \, \psi^i_k(x) \, \overline{\psi^j_k(y)} \qquad (i, j = 1, \ldots, n),$$

$$(30) \qquad \sum_{i,j=1}^{n} \int_{\Delta \times \Delta} |K_{ij}(x, y)|^2 \, dx \, dy = \sum_k |\lambda_k|^2.$$

Proof. From $K_{ij}(x, y) \in L_2^{(2m)}$ follows that the element $\{k\} = \{k^1(x), \ldots, k^n(x)\}$, where $k^i(x) = K_{ij}(x, y)$ and j is fixed, belongs to the space $[L_2^{(m)}]^n$ for almost every $y \in \varDelta$. We shall show now that the relations $(\{k\}, \{\psi_k\}) = \lambda_k \overline{\psi_k^j(y)}$ and $(\{k\}, \{g\}) = 0$ for every $\{g\} \in [L_2]^n$ orthogonal to all $\{\psi_k\}$, hold for almost every $y \in \varDelta$, so that it will be possible to write

$$\{k\} = \sum (\{k\}, \{\psi_k\}) \{\psi_k\} = \sum_k \lambda_k \overline{\psi_k^j(y)} \{\psi_k\}$$

in the terminology of Hilbert space. Indeed,

$$(\{k\}, \{\psi_k\}) = \sum_{i=1}^n \int_\varDelta K_{ij}(x, y) \overline{\psi_k^i(x)}\, dx = \sum_{i=1}^n \int_\varDelta \overline{K_{ji}^*(y, x) \psi_k^i(x)}\, dx = \lambda_k \overline{\psi_k^j(y)}$$

for almost every $y \in \varDelta$, and, if $(\{g\}, \{\psi_k\}) = 0$ for all values of k, so that by (28) $\sum_{j=1}^n \int_\varDelta K_{ij}^*(x, y) g^j(y)\, dy = 0$ almost everywhere, we have

$$(\{k\}, \{g\}) = \sum_{i=1}^n \int_\varDelta K_{ij}(x, y) \overline{g^i(x)}\, dx = \sum_{i=1}^n \int_\varDelta \overline{K_{ji}^*(y, x) g^i(x)}\, dx = 0$$

for almost every $y \in \varDelta$.

The relation $\{k\} = \sum (\{k\}, \{\psi_k\}) \{\psi_k\}$ implies

$$\left\| \{k\} - \sum_{k=1}^p (\{k\}, \{\psi_k\}) \{\psi_k\} \right\|^2 = \sum_{k=p+1} |(\{k\}, \{\psi_k\})|^2,$$

hence

$$\sum_{i=1}^n \int_\varDelta \left| K_{ij}(x, y) - \sum_{k=1}^p \lambda_k \overline{\psi_k^j(y)} \psi_k^i(x) \right|^2 dx = \sum_{k=p+1} |\lambda_k|^2 |\psi_k^j(y)|^2$$

for almost every $y \in \varDelta$. Summing from $j = 1$ to $j = n$ and integrating over y, we see that

$$\sum_{i,j=1}^n \int_{\varDelta \times \varDelta} \left| K_{ij}(x, y) - \sum_{k=1}^p \lambda_k \psi_k^i(x) \overline{\psi_k^j(y)} \right|^2 dx\, dy = \sum_{k=p+1} |\lambda_k|^2.$$

For $p = 0$ we have (30), and, making $p \to \infty$, we find (29).

Similar results hold for the system of equations with iterated matrix-kernel $\| K_{ij}^{(p)}(x, y) \|$, where

$$K_{ij}^{(1)}(x, y) = K_{ij}(x, y),$$

$$K_{ij}^{(p)}(x, y) = \sum_{l=1}^n \int_\varDelta K_{il}(x, z) K_{lj}^{(p-1)}(z, y)\, dz \qquad (p > 1).$$

It is not difficult to prove that $K_{ij}^{(p)}(x, y) \in L_2^{(2m)}(\Delta)$, that the transformation K^p in $[L_2]^n$, corresponding with $\| K_{ij}^{(p)}(x, y) \|$, is also normal, that λ_k^p $(k = 1, 2, \ldots)$ is the sequence of all characteristic values $\neq 0$ of K^p, and that $\{\psi_k\}$ is a corresponding sequence of characteristic elements.

Finally, if $\| K_{ij}(x, y) \|$ satisfies the condition that all $K_{ij}(x, y)$ are *continuous in mean*, that is,

$$\int_\Delta | K_{ij}(x, y) |^2 \, dy \text{ is finite for } i, j = 1, \ldots, n, \text{ and for every } x \in \Delta,$$

and

$$\lim_{x_2 \to x_1} \int_\Delta | K_{ij}(x_2, y) - K_{ij}(x_1, y) |^2 \, dy = 0 \qquad (i, j = 1, \ldots, n),$$

it follows in a well-known way that the convergence in mean in (27) and (28) may be replaced by uniform convergence, and that, for $p \geq 2$, the series $\sum_k \lambda_k^p \psi_k^i(x) \overline{\psi_k^j(y)}$ converges uniformly in $\Delta \times \Delta$ to $K_{ij}^{(p)}(x, y)$.

II. The integral transformation K in $[L_2]^n$ with matrix-kernel $\| K_{ij}(x, y) \| = \| A_{ij}(x, y) \| \cdot \| h_{ij}(y) \|$ (the dot means that the matrix-product is to be taken), the following conditions being satisfied:

(a) All $h_{ij}(y)$ are bounded and measurable in Δ, and one at least of them is $\neq 0$ on a set of positive measure,

(b) $h_{ij}(y) = \overline{h_{ji}(y)}$ and $\sum_{i,j=1}^{n} h_{ij}(y) \bar{\alpha}_i \alpha_j \geq 0$ for any system of complex numbers $\alpha_1, \ldots, \alpha_n$ and every $y \in \Delta$; the matrix $\| h_{ij}(y) \|$ is therefore Hermitian and of positive type,

(c) All $A_{ij}(x, y) \in L_2^{(2m)}(\Delta)$,

(d) When the bounded linear transformation $\{g\} = H\{f\}$ in $[L_2]^n$ is determined by

$$g^i(x) = \sum_{j=1}^{n} h_{ij}(x) f^j(x) \qquad (i = 1, \ldots, n),$$

and A is the integral transformation with matrix-kernel $\| A_{ij}(x, y) \|$, so that therefore $K = AH$, then

$$HAHA^*H = HA^*HAH.$$

Since, by Lemma 11, the transformation H is self-adjoint, positive and $\neq O$, and the transformation A is, by Lemma 12, completely continuous, we see that, by condition (d), the transformation $K = AH$ is completely continuous and normalisable (relative to H).

Integral transformations of this kind were considered by J. Ernest Wilkins[1], who, however, supposed, instead of condition (a), all $h_{ij}(y)$ to be continuous on Δ, and, instead of condition (c), all $A_{ij}(x, y)$ to be bounded in $\Delta \times \Delta$ with their discontinuities "regularly distributed", while finally, instead of condition (d), he supposed that $HAH = HA^*H$, in other words, that $K = AH$ is symmetrisable. He proved some extremizing properties for the characteristic values of the system of linear integral equations, associated with the transformation K, and obtained an expansion theorem for functions of the form $\sum_{j=1}^{n} \int_{\Delta} K_{ij}(x, y) f^j(y)\, dy$. The present author[2] relaxed conditions (a) and (c) to the form quoted above, but retained condition (d) in the form $HAH = HA^*H$. He succeeded in finding an expansion theorem for the elements of the matrix-kernel $\| K_{ij}(x, y) \|$ as well. Here we shall relax condition (d) to $HAHA^*H = HA^*HAH$, which is equivalent, therefore, to the generalization from a symmetrisable K to a normalisable K.

Before stating results we recall the well-known fact that every bounded, positive, self-adjoint transformation H in a Hilbert space possesses a uniquely determined "positive square root" $H^{1/2}$. The question may be raised now what can be said about this root $H^{1/2}$ when H is defined as in condition (d). The answer is given in the following lemma, which may be proved along well-known lines:

Lemma 13. 1°. *There exists a uniquely determined matrix $\| h_{ij}^{(1/2)}(x) \|$, which is Hermitian and of positive type for all $x \in \Delta$, such that all functions $h_{ij}^{(1/2)}(x)$ are bounded and measurable in Δ, and*

$$\| h_{ij}^{(1/2)}(x) \| \cdot \| h_{ij}^{(1/2)}(x) \| = \| h_{ij}(x) \|.$$

2°. *If all functions $h_{ij}(x)$ are continuous in Δ, the same holds for all functions $h_{ij}^{(1/2)}(x)$.*

3°. *The uniquely determined, bounded, positive, self-adjoint transformation $\{g\} = H^{1/2}\{f\}$ in $[L_2]^n$ is determined by*

$$g^i(x) = \sum_{j=1}^{n} h_{ij}^{(1/2)}(x) f^j(x) \qquad (i = 1, \ldots, n).$$

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[1] Cf. p. 198, footnote 1.

[2] Cf. p. 198, footnote 4.

Considering now the system of homogeneous linear integral equations

$$(31)\qquad \sum_{j=1}^{n} \int_{\Delta} K_{ij}(x, y)\, \psi^j(y)\, dy - \lambda\, \psi^i(x) = 0 \qquad (i = 1, \ldots, n),$$

Theorem 12 shows that if one at least of the functions $P_{ij}(x, y) \neq 0$ on a set of positive measure in $\Delta \times \Delta$, where $\| P_{ij}(x, y) \| = \| h_{ij}(x) \| \cdot \| K_{ij}(x, y) \|$, this system has a non-trivial solution with $\lambda \neq 0$. Let us denote by λ_k $(k = 1, 2, \ldots)$ the sequence of all characteristic values of (31) and by $\psi_k^i(x)$ $(i = 1, \ldots, n)$ a corresponding H-orthonormal sequence of characteristic functionsets. These functionsets satisfy therefore the relations

$$\sum_{i,j=1}^{n} \int_{\Delta} h_{ij}(x)\, \overline{\psi_k^i(x)}\, \psi_k^j(x)\, dx = 1$$

$$\sum_{i,j=1}^{n} \int_{\Delta} h_{ij}(x)\, \overline{\psi_k^i(x)}\, \psi_l^j(x)\, dx = 0 \quad \text{for} \quad k \neq l,$$

or, writing $\sum_{j=1}^{n} h_{ij}(x)\, \psi_k^j(x) = \chi_k^i(x)$ (so that $H\{\psi_k\} = \{\chi_k\}$),

$$\sum_{i=1}^{n} \int_{\Delta} \psi_k^i(x)\, \overline{\chi_l^i(x)}\, dx = \begin{cases} 1 & \text{for } k = l, \\ 0 & \text{for } k \neq l. \end{cases}$$

Then, by Theorem 26, if

$$\alpha_k = (H\{f\}, \{\psi_k\}) = (\{f\}, H\{\psi_k\}) = (\{f\}, \{\chi_k\}) = \sum_{i=1}^{n} \int_{\Delta} f^i(x)\, \overline{\chi_k^i(x)}\, dx$$

for an arbitrary $\{f\} = \{f^1(x), \ldots, f^n(x)\} \in [L_2]^n$, we have

$$(32)\qquad \sum_{j=1}^{n} \int_{\Delta} K_{ij}(x, y) f^j(y)\, dy \sim \sum_{k} \lambda_k\, \alpha_k\, \psi_k^i(x) + p^i(x) \qquad (i = 1, \ldots, n),$$

where $\{p\} = \{p^1(x), \ldots, p^n(x)\}$ satisfies $H\{p\} = \{0\}$, that is

$$\sum_{j=1}^{n} h_{ij}(x)\, p^j(x) = 0 \qquad (i = 1, \ldots, n).$$

Besides the expansion (32), we shall prove now an expansion theorem for the elements $K_{ij}(x, y)$ of the matrix-kernel as well.

Theorem 29. *We have*

$$K_{ij}(x, y) - p_{ij}(x, y) \sim \sum_{k} \lambda_k\, \psi_k^i(x)\, \overline{\chi_k^j(y)} \qquad (i, j = 1, \ldots, n),$$

where $p_{ij}(x, y) \in L_2^{(2m)}(\Delta)$ $(i, j = 1, \ldots, n)$ *satisfies*

$$\sum_{q=1}^{n} h_{iq}(x)\, p_{qj}(x, y) = 0. \tag{33}$$

Proof. We observe first that by Theorem 26, Remark 2°, $N = H^{1/2} A H^{1/2}$ has the same sequence λ_k of characteristic values $\neq 0$ as $K = AH$, and that $\{\Psi_k\} = H^{1/2}\{\psi_k\}$ is a corresponding orthonormal sequence of characteristic elements. Hence, by Theorem 28,

$$N_{iq}(x, y) \sim \sum_k \lambda_k \Psi_k^i(x) \overline{\Psi_k^q(y)} \qquad (i, q = 1, \ldots, n),$$

with $\Psi_k^i(x) = \sum_{j=1}^{n} h_{ij}^{(1/2)}(x)\, \psi_k^j(x)$, so that

$$\sum_{r=1}^{n} h_{ir}^{(1/2)}(x)\, K_{rj}(x, y) = \sum_{r, s=1}^{n} h_{ir}^{(1/2)}(x)\, A_{rs}(x, y)\, h_{sj}(y) =$$

$$= \sum_{q=1}^{n} N_{iq}(x, y)\, h_{qj}^{(1/2)}(y) \sim \sum_k \lambda_k \Psi_k^i(x) \Big(\sum_{q=1}^{n} \overline{\Psi_k^q(y)\, h_{jq}^{(1/2)}(y)}\Big)$$

or

$$\sum_{r=1}^{n} h_{ir}^{(1/2)}(x)\, K_{rj}(x, y) \sim \sum_k \lambda_k \Psi_k^i(x)\, \overline{\chi_k^j(y)}. \tag{34}$$

Let us consider now the matrix-kernel $\| D_{ij}(x, y) \|$, belonging to the transformation $D = A H^{1/2}$. Then, for every i $(i = 1, \ldots, n)$ and for almost every $x \in \Delta$, $\{d_i\} = \{d_i^1(y), \ldots, d_i^n(y)\}$, where $d_i^j(y) = D_{ij}(x, y)$, belongs to $[L_2]^n$. Hence, by Bessel's inequality (the system of functionsets $\overline{\Psi_k^i(x)}$ is orthonormal),

$$\sum_k |(\{d_i\}, \{\overline{\Psi}_k\})|^2 \leq \| \{d_i\} \|^2$$

or, since

$$(\{d_i\}, \{\overline{\Psi}_k\}) = \sum_{j=1}^{n} \int_\Delta d_i^j(y)\, \Psi_k^j(y)\, dy = \sum_{j=1}^{n} \int_\Delta D_{ij}(x, y)\, \Psi_k^j(y)\, dy =$$

$$= \sum_{j=1}^{n} \int_\Delta \sum_{q, r=1}^{n} A_{iq}(x, y)\, h_{qj}^{(1/2)}(y)\, h_{jr}^{(1/2)}(y)\, \psi_k^r(y)\, dy = \sum_{r=1}^{n} \int_\Delta K_{ir}(x, y)\, \psi_k^r(y)\, dy = \lambda_k\, \psi_k^i(x),$$

$$\sum_k |\lambda_k|^2\, |\psi_k^i(x)|^2 \leq \sum_{j=1}^{n} \int_\Delta |D_{ij}(x, y)|^2\, dy \tag{35}$$

for almost every $x \in \Delta$.

After this we observe that in the Hilbert space $[L_2^{(2m)}(\Delta)]^{n^2}$ the elements

$$\{f_k\} \equiv \{f_k^{ij}(x, y) = \lambda_k \psi_k^i(x) \overline{\Psi_k^j(y)}\} \qquad (i, j = 1, \ldots, n)$$

are orthogonal on account of the orthogonality of the system $\{\Psi_k\}$ in $[L_2^{(m)}(\Delta)]^n$. Furthermore, by (35),

$$\sum_k \|\{f_k\}\|^2 = \sum_k |\lambda_k|^2 \Big(\sum_{i,j=1}^n \int_{\Delta\times\Delta} |\psi_k^i(x)|^2 \cdot |\Psi_k^j(y)|^2 \, dx \, dy\Big) =$$

$$= \sum_k |\lambda_k|^2 \Big(\sum_{i=1}^n \int_\Delta |\psi_k^i(x)|^2 \, dx\Big) \leq \sum_{i,j=1}^n \int_{\Delta\times\Delta} |D_{ij}(x, y)|^2 \, dx \, dy < \infty,$$

which shows, since

$$\Big\|\sum_{k=p}^q \{f_k\}\Big\|^2 = \sum_{k=p}^q \|\{f_k\}\|^2$$

by the orthogonality, that $\sum \{f_k\}$ converges in $[L_2^{(2m)}(\Delta)]^{n^2}$. This implies that, for $i, j = 1, \ldots, n$, the series $\sum_k \lambda_k \psi_k^i(x) \overline{\Psi_k^j(y)}$ converges in mean. Denoting the sumfunction by $f_{ij}(x, y)$, we have therefore

$$f_{ij}(x, y) \sim \sum_k \lambda_k \psi_k^i(x) \overline{\Psi_k^j(y)},$$

so that, writing

$$\sum_{q=1}^n f_{iq}(x, y) \, h_{qj}^{(1/2)}(y) = K_{ij}(x, y) - p_{ij}(x, y),$$

we have also

$$K_{ij}(x, y) - p_{ij}(x, y) \sim \sum_k \lambda_k \psi_k^i(x) \overline{\chi_k^j(y)} \qquad (i, j = 1, \ldots, n). \tag{36}$$

The only thing that remains to be proved is $\sum_{q=1}^n h_{iq}(x) \, p_{qj}(x, y) = 0$. From (36) we deduce

$$\sum_{r=1}^n h_{ir}^{(1/2)}(x) \, K_{rj}(x, y) - \sum_{r=1}^n h_{ir}^{(1/2)}(x) \, p_{rj}(x, y) \sim \sum_k \lambda_k \Psi_k^i(x) \overline{\chi_k^j(y)},$$

hence, comparing this with (34),

$$\sum_{r=1}^n h_{ir}^{(1/2)}(x) \, p_{rj}(x, y) = 0 \quad \text{or} \quad \sum_{r=1}^n h_{ir}(x) \, p_{rj}(x, y) = 0.$$

This completes the proof.

Similar results hold for the system of equations with iterated matrix-kernel $\|K_{ij}^{(p)}(x, y)\|$.

Theorem 30. *For $p \geq 2$ we have*

$$\sum_{j=1}^{n} \int_{\Delta} K_{ij}^{(p)}(x, y) f^j(y)\, dy \sim \sum_k \lambda_k^p \alpha_k \psi_k^i(x) \qquad (i = 1, \ldots, n), \tag{37}$$

where α_k has the same meaning as in (32);

$$K_{ij}^{(p)}(x, y) \sim \sum_k \lambda_k^p \psi_k^i(x) \overline{\chi_k^j(y)} \qquad (i, j = 1, \ldots, n). \tag{38}$$

Proof. Formula (37) follows from the last part of Theorem 26. To prove (38), we observe that, by Schwarz's inequality,

$$K_{qj}(z, y) - p_{qj}(z, y) \sim \sum_k \lambda_k \psi_k^q(z) \overline{\chi_k^j(y)}$$

implies

$$\sum_{q=1}^{n} \int_{\Delta} K_{iq}(x, z) K_{qj}(z, y)\, dz - \sum_{q=1}^{n} \int_{\Delta} K_{iq}(x, z) p_{qj}(z, y)\, dz \sim$$

$$\sim \sum_k \lambda_k \overline{\chi_k^j(y)} \left(\sum_{q=1}^{n} \int_{\Delta} K_{iq}(x, z) \psi_k^q(z)\, dz \right),$$

so that, since by (33)

$$\sum_{q=1}^{n} \int_{\Delta} K_{iq}(x, z) p_{qj}(z, y)\, dz = \sum_{q, r=1}^{n} \int_{\Delta} A_{ir}(x, z) h_{rq}(z) p_{qj}(z, y)\, dz = 0,$$

we have

$$K_{ij}^{(2)}(x, y) \sim \sum_k \lambda_k^2 \psi_k^i(x) \overline{\chi_k^j(y)}.$$

The proof for $p > 2$ follows by induction.

Finally, if all $A_{ij}(x, y)$ are continuous in mean in $\Delta \times \Delta$, and all $h_{ij}(x)$ are continuous in Δ (so that by Lemma 13 all $h_{ij}^{(1/2)}(x)$ are continuous in Δ as well), it is not difficult to prove that in (32) the convergence in mean may be replaced by uniform convergence, while the functions $p^i(x)$ $(i = 1, \ldots, n)$ are now continuous as well. Moreover, for $p \geq 2$, the series $\sum_k \lambda_k^p \psi_k^i(x) \overline{\chi_k^j(y)}$ converges uniformly in $\Delta \times \Delta$ to $K_{ij}^{(p)}(x, y)$.

III. The integral transformation K in $[L_2]^n$ with matrix-kernel $\| K_{ij}(x, y) \| = \| A_{ij}(x) \| \cdot \| H_{ij}(x, y) \|$, the following conditions being satisfied:

(a) All $H_{ij}(x, y) \in L_2^{(2m)}(\Delta)$, and one at least of them is $\neq 0$ on a set of positive measure in $\Delta \times \Delta$,

(b) $H_{ij}(x, y) = \overline{H_{ji}(y, x)}$ almost everywhere in $\varDelta \times \varDelta$, and

$$\sum_{i,j=1}^{n} \int_{\varDelta \times \varDelta} H_{ij}(x, y)\, \overline{f^i(x)}\, f^j(y)\, dx\, dy \geq 0$$

for every $\{f\} \in [L_2]^n$; the bounded linear transformation H in $[L_2]^n$, determined by the matrix-kernel $\| H_{ij}(x, y) \|$, is therefore self-adjoint and positive,

(c) All $A_{ij}(x)$ are bounded and measurable in $\varDelta$,

(d) When the bounded linear transformation $\{g\} = A\{f\}$ in $[L_2]^n$ is determined by

$$g^i(x) = \sum_{j=1}^{n} A_{ij}(x)\, f^j(x) \qquad (i = 1, \ldots, n),$$

and H is the transformation defined in condition (c), so that therefore $K = A\,H$, then

$$H\,A\,H\,A^*\,H = H\,A^*\,H\,A\,H.$$

Since, by Lemma 12, the transformation H is completely continuous, self-adjoint, positive and $\neq O$, and the transformation A is, by Lemma 11, bounded, we see that, by condition (d), the transformation $K = A\,H$ is completely continuous and normalisable (relative to H).

Considering now the system of homogeneous linear integral equations

$$\sum_{j=1}^{n} \int_{\varDelta} K_{ij}(x, y)\, \psi^j(y)\, dy - \lambda\, \psi^i(x) = 0 \qquad (i = 1, \ldots, n), \tag{39}$$

Theorem 12 shows that this system has a non-trivial solution with $\lambda \neq 0$, if only one at least of the functions $P_{ij}(x, y) \neq 0$ on a set of positive measure in $\varDelta \times \varDelta$, where

$$P_{ij}(x, y) = \sum_{q=1}^{n} \int_{\varDelta} H_{iq}(x, z)\, K_{qj}(z, y)\, dz \qquad (i, j = 1, \ldots, n).$$

Let us denote by λ_k $(k = 1, 2, \ldots)$ the sequence of all characteristic values of (39) and by $\psi_k^i(x)$ $(i = 1, \ldots, n)$ a corresponding H-orthonormal sequence of characteristic functionsets. These functionsets satisfy therefore the relations

$$\sum_{i,j=1}^{n} \int_{\varDelta \times \varDelta} H_{ij}(x, y)\, \overline{\psi_k^i(x)}\, \psi_k^j(y)\, dx\, dy = 1,$$

$$\sum_{i,j=1}^{n} \int_{\varDelta \times \varDelta} H_{ij}(x, y)\, \overline{\psi_k^i(x)}\, \psi_l^j(y)\, dx\, dy = 0 \quad \text{for} \quad k \neq l,$$

or, writing $\sum_{j=1}^{n} \int_{\Delta} H_{ij}(x, y)\, \psi_k^j(y)\, dy = \chi_k^i(x)$ (so that $H\{\psi_k\} = \{\chi_k\}$),

$$\sum_{i=1}^{n} \int \psi_k^i(x)\, \overline{\chi_l^i(x)}\, dx = \begin{cases} 1 & \text{for } k = l, \\ 0 & \text{for } k \neq l. \end{cases}$$

Then, by Theorem 26, if

$$\alpha_k = (H\{f\}, \{\psi_k\}) = (\{f\}, H\{\psi_k\}) = (\{f\}, \{\chi_k\}) = \sum_{i=1}^{n} \int_{\Delta} f^i(x)\, \overline{\chi_k^i(x)}\, dx$$

for an arbitrary $\{f\} \in [L_2]^n$, we have

$$\sum_{j=1}^{n} \int_{\Delta} K_{ij}(x, y)\, f^j(y)\, dy \sim \sum_k \lambda_k\, \alpha_k\, \psi_k^i(x) + p^i(x) \qquad (i = 1, \ldots, n), \tag{40}$$

where $\{p\} = \{p^1(x), \ldots, p^n(x)\}$ satisfies $H\{p\} = \{0\}$, that is

$$\sum_{j=1}^{n} \int_{\Delta} H_{ij}(x, y)\, p^j(y)\, dy = 0 \qquad (i = 1, \ldots, n).$$

Moreover, if $\| K_{ij}^{(p)}(x, y) \|$ is again the p-th iterated kernel, we have for $p \geq 2$

$$\sum_{j=1}^{n} \int_{\Delta} K_{ij}^{(p)}(x, y)\, f^j(y)\, dy \sim \sum_k \lambda_k^p\, \alpha_k\, \psi_k^i(x) \qquad (i = 1, \ldots, n). \tag{41}$$

Besides the expansions (40) and (41) we shall prove now an expansion theorem for the elements of the iterated matrix-kernel $\| K_{ij}^{(p)}(x, y) \|$ $(p \geq 2)$ as well. Since, in the general case that we consider here, the transformation $H^{1/2}$ is not determined by a matrix-kernel with elements belonging to $L_2^{(2m)}(\Delta)$, it seems not to be possible to obtain an expansion for the elements $K_{ij}(x, y)$ themselves. The same fact causes some peculiar difficulties in the proof of the now following theorem.

Theorem 31. *We have*

$$K_{ij}^{(2)}(x, y) - p_{ij}(x, y) \sim \sum_k \lambda_k^2\, \psi_k^i(x)\, \chi_k^j(y) \qquad (i, j = 1, \ldots, n),$$

where $p_{ij}(x, y) \in L_2^{(2m)}(\Delta)$ satisfies

$$\sum_{q=1}^{n} \int_{\Delta} H_{iq}(x, z)\, p_{qj}(z, y)\, dz = 0 \tag{42}$$

for almost every point $(x, y) \in \Delta \times \Delta$.

Furthermore, for $p \geq 3$,

$$K_{ij}^{(p)}(x, y) \sim \sum_k \lambda_k^p \psi_k^i(x) \overline{\chi_k^j(y)} \qquad (i, j = 1, \ldots, n).$$

Proof. The proof is divided into several parts.

1°. If, in the separable Hilbert space $\mathfrak{R}$, the transformation K is bounded and linear, and φ_p $(p = 1, 2, \ldots)$ is a complete orthonormal system, K is said to be of finite norm, if $N^2(K) = \sum_{p,q=1}^{\infty} |k_{pq}|^2 < \infty$, where $k_{pq} = (K\varphi_q, \varphi_p)$. Then we have[1]:

If A is bounded and linear, and K is of finite norm, then AK and KA are of finite norm, and

$$N(AK) \leq \|A\| \cdot N(K), \quad N(KA) \leq \|A\| \cdot N(K).$$

Indeed,

$$\sum_{p=1}^{\infty} |(AK\varphi_q, \varphi_p)|^2 = \|AK\varphi_q\|^2 \leq \|A\|^2 \cdot \|K\varphi_q\|^2 = \|A\|^2 \sum_{r=1}^{\infty} |(K\varphi_q, \varphi_r)|^2,$$

hence summing over q, $N^2(AK) \leq \|A\|^2 \cdot N^2(K)$ or $N(AK) \leq \|A\| \cdot N(K)$. Finally, since evidently $N(K^*) = N(K)$, we have $N(KA) = N(A^* K^*) \leq \|A^*\| \cdot N(K^*) = \|A\| \cdot N(K)$.

2°. If, in the Hilbert space $[L_2^{(m)} \varDelta)]^n$, the transformation K is determined by the matrix-kernel $\|K_{ij}(x, y)\|$, where all $K_{ij}(x, y) \in L_2^{(2m)}(\varDelta)$, then K is of finite norm. Indeed, if $\{\varphi_p\}$ is an arbitrary complete orthonormal system in $[L_2]^n$, then

$$N^2(K) = \sum_{p=1}^{\infty} \|K\{\varphi_p\}\|^2 = \sum_{p=1}^{\infty} \Big[\sum_{i=1}^{n} \int_{\varDelta} \Big|\sum_{j=1}^{n} \int_{\varDelta} K_{ij}(x, y)\, \varphi_p^j(y)\, dy\Big|^2 dx\Big] =$$

$$= \sum_{i=1}^{n} \int_{\varDelta} dx \Big[\sum_{p=1}^{\infty} \Big|\sum_{j=1}^{n} \int_{\varDelta} K_{ij}(x, y)\, \varphi_p^j(y)\, dy\Big|^2\Big] = \sum_{i=1}^{n} \int_{\varDelta} \Big[\sum_{j=1}^{n} \int_{\varDelta} |K_{ij}(x, y)|^2\, dy\Big] dx =$$

$$= \sum_{i,j=1}^{n} \int_{\varDelta \times \varDelta} |K_{ij}(x, y)|^2\, dx\, dy.$$

Conversely, if K is of finite norm, so that, on account of $\sum_{p,q=1}^{\infty} |k_{pq}|^2 < \infty$, the series $\sum_{p,q=1}^{\infty} k_{pq}\, \varphi_p^i(x)\, \overline{\varphi_q^j(y)}$ converges in mean in $L_2^{(2m)}(\varDelta)$ to a function $K_{ij}(x, y)$, it is not difficult to see that K is determined by the matrix-kernel $\|K_{ij}(x, y)\|$.

[1] F. SMITHIES, The Fredholm theory of integral equations, Duke Math. Journal 8 (1941), p. 107—130, Lemma 2.6.

3°. Writing $B = AHA$, so that $K^2 = BH$, we see without difficulty that $HBHB^*H = HB^*HBH$ on account of condition (d), hence also $H^{1/2}BHB^*H^{1/2} = H^{1/2}B^*HBH^{1/2}$, which shows that $N = H^{1/2}BH^{1/2}$ is normal. By Theorem 26, Remark 2°, N has the same sequence λ_k^2 of characteristic values $\neq 0$ as $K^2 = BH$, and $\{\Psi_k\} = H^{1/2}\{\psi_k\}$ is a corresponding orthonormal sequence of characteristic elements. Furthermore, by 1°, N is of finite norm, so that, by 2°, N is determined by a matrix-kernel $\| N_{ij}(x, y) \|$ with elements belonging to $L_2^{(2m)}(\Delta)$. Hence, by Theorem 28,

$$N_{ij}(x, y) \sim \sum_k \lambda_k^2 \Psi_k^i(x) \overline{\Psi_k^j(y)} \qquad (i, j = 1, \ldots, n).$$

We shall show now that $\sum_k \lambda_k^2 \Psi_k^i(x) \overline{\chi_k^j(y)}$ converges in mean as well. Indeed, from $\left\| \sum_{k=p}^{q} \alpha_k \{\chi_k\} \right\|^2 = \left\| H^{1/2} \sum_{k=p}^{q} \alpha_k \{\Psi_k\} \right\|^2 \leq \| H^{1/2} \|^2 \cdot \left\| \sum_{k=p}^{q} \alpha_k \{\Psi_k\} \right\|^2$ we deduce, taking $\alpha_k = \overline{\lambda_k^2 \Psi_k^i(x)}$, that

$$\sum_{j=1}^{n} \int_{\Delta} \left| \sum_{k=p}^{q} \lambda_k^2 \Psi_k^i(x) \overline{\chi_k^j(y)} \right|^2 dy \leq \| H^{1/2} \|^2 \sum_{j=1}^{n} \int_{\Delta} \left| \sum_{k=p}^{q} \lambda_k^2 \Psi_k^i(x) \overline{\Psi_k^j(y)} \right|^2 dy,$$

from which the result follows immediately. Hence

$$C_{ij}(x, y) \sim \sum_k \lambda_k^2 \Psi_k^i(x) \overline{\chi_k^j(y)}. \tag{43}$$

4°. We shall show now that the transformation C, corresponding with the matrix-kernel $\| C_{ij}(x, y) \|$, satisfies $C = NH^{1/2}$ (hence $C = H^{1/2}BH = H^{1/2}K^2$). Let, for this purpose, $\{f\}$ and $\{g\}$ be two arbitrary elements of $[L_2]^n$, and write $\{t\} = H^{1/2}\{f\}$. Then, denoting the inner product in the Hilbert space $L_2^{(2m)}(\Delta)$ by $(\ldots, \ldots)_{2m}$, we have

$$\begin{aligned}
(C\{f\}, \{g\}) &= \sum_{i,j=1}^{n} \int_{\Delta \times \Delta} C_{ij}(x, y) \overline{g^i(x)} f^j(y)\, dx\, dy = \sum_{i,j=1}^{n} (C_{ij}(x, y), g^i(x) \overline{f^j(y)})_{2m} \\
&= \lim_{p \to \infty} \sum_{i,j=1}^{n} \Big(\sum_{k=1}^{p} \lambda_k^2 \Psi_k^i(x) \overline{\chi_k^j(y)},\ g^i(x) \overline{f^j(y)} \Big)_{2m} \\
&= \lim_{p \to \infty} \sum_{i,j=1}^{n} \Big[\sum_{k=1}^{p} \lambda_k^2 \int_{\Delta} \Psi_k^i(x) \overline{g^i(x)}\, dx \cdot \int_{\Delta} f^j(y) \overline{\chi_k^j(y)}\, dy \Big] \\
&= \lim_{p \to \infty} \sum_{i=1}^{n} \Big[\sum_{k=1}^{p} \lambda_k^2 \int_{\Delta} \Psi_k^i(x) \overline{g_i(x)}\, dx \cdot (\{f\}, H^{1/2}\{\Psi_k\}) \Big]
\end{aligned}$$

$$= \lim_{p\to\infty} \sum_{i,j=1}^{n} \Big[\sum_{k=1}^{p} \lambda_k^2 \int_\Delta \Psi_k^i(x)\, \overline{g^i(x)}\, dx \cdot \int_\Delta t^j(y)\, \overline{\Psi_k^j(y)}\, dy \Big]$$

$$= \lim_{p\to\infty} \sum_{i,j=1}^{n} \Big(\sum_{k=1}^{p} \lambda_k^2\, \Psi_k^i(x)\, \overline{\Psi_k^j(y)},\ g^i(x)\, \overline{t^j(y)} \Big)_{2m}$$

$$= \sum_{i,j=1}^{n} (N_{ij}(x, y),\ g^i(x)\, \overline{t^j(y)})_{2m} = (N\{t\}, \{g\}),$$

hence $C\{f\} = N\{t\} = N H^{1/2}\{f\}$ or $C = N H^{1/2} = H^{1/2} K^2$.

5°. We consider now the matrix-kernel $\| D_{ij}(x, y) \|$, belonging to the transformation $D = B H^{1/2}$. Then, for every i $(i = 1, \ldots, n)$ and for almost every $x \in \Delta$, $\{d_i\} = \{d_i^1(y), \ldots, d_i^n(y)\}$, where $d_i^j(y) = D_{ij}(x, y)$, belongs to $[L_2]^n$. Hence, by Bessel's inequality (the system of functionsets $\overline{\Psi_k^i(x)}$ is orthonormal),

$$\sum_k |(\{d_i\}, \{\overline{\Psi}_k\})|^2 \le \| \{d_i\} \|^2,$$

or, since

$$(\{d_i\}, \{\overline{\Psi}_k\}) = \sum_{j=1}^{n} \int_\Delta D_{ij}(x, y)\, \Psi_k^j(y)\, dy = \sum_{j=1}^{n} \int_\Delta K_{ij}^{(2)}(x, y)\, \psi_k^j(y)\, dy = \lambda_k^2\, \psi_k^i(x),$$

$$\sum_k |\lambda_k|^4\, |\psi_k^i(x)|^2 \le \sum_{j=1}^{n} \int_\Delta |D_{ij}(x, y)|^2\, dy \tag{44}$$

for almost every $x \in \Delta$.

In the same way as we proved in Theorem 29 the convergence in mean of $\sum_k \lambda_k\, \psi_k^i(x)\, \overline{\Psi_k^j(y)}$ by using (35), we may prove now the convergence in mean of $\sum_k \lambda_k^2\, \psi_k^i(x)\, \overline{\Psi_k^j(y)}$ by using (44). After that, as in 3°, we see that $\sum_k \lambda_k^2\, \psi_k^i(x)\, \overline{\chi_k^j(y)}$ converges in mean as well. Hence, denoting the sumfunction by $K_{ij}^{(2)}(x, y) - p_{ij}(x, y)$, we have

$$K_{ij}^{(2)}(x, y) - p_{ij}(x, y) \sim \sum_k \lambda_k^2\, \psi_k^i(x)\, \overline{\chi_k^j(y)} \qquad (i, j = 1, \ldots, n). \tag{45}$$

The only thing that remains to be proved is (42). From (45) we deduce as in 3°, and bearing in mind that $C = H^{1/2} K^2$ has the kernel $\| C_{ij}(x, y) \|$,

$$C_{ij}(x, y) - q_{ij}(x, y) \sim \sum_k \lambda_k^2\, \Psi_k^i(x)\, \overline{\chi_k^j(y)}, \tag{46}$$

where, when $\| p_{ij}(x, y) \|$ corresponds with the transformation P, $\| q_{ij}(x, y) \|$ corresponds with $H^{1/2} P$. Comparing (43) and (46), we see that $H^{1/2} P = O$, hence $H P = O$, which is equivalent with (42).

6°. For $p \geq 3$, the proof of the expansion for $K_{ij}^{(p)}(x, y)$ is similar to the proof of Theorem 30.

Theorem 32. *If the transformation* $H^{1/2}$ *is determined by a matrix-kernel with elements belonging to* $L_2^{(2m)}(\Delta)$, *then*

$$K_{ij}(x, y) - p_{ij}(x, y) \sim \sum_k \lambda_k \psi_k^i(x) \overline{\chi_k^j(y)} \qquad (i, j = 1, \ldots, n), \tag{47}$$

where $p_{ij}(x, y) \in L_2^{(2m)}(\Delta)$ *satisfies*

$$\sum_{q=1}^{n} \int_{\Delta} H_{iq}(x, z)\, p_{qj}(z, y)\, dz = 0 \tag{48}$$

for almost every point $(x, y) \in \Delta \times \Delta$.

Furthermore, for $p \geq 2$,

$$K_{ij}^{(p)}(x, y) \sim \sum_k \lambda_k^p \psi_k^i(x) \overline{\chi_k^j(y)} \qquad (i, j = 1, \ldots, n). \tag{49}$$

Proof. The proof is similar to that of the preceding theorem, using the fact that in the case which we consider now, $H^{1/2}$ is of finite norm. We remark that it is not difficult to prove that $H^{1/2}$ is of finite norm if and only if $\sum_k \mu_k$ converges, where μ_k $(k = 1, 2, \ldots)$ is the sequence of characteristic values $\neq 0$ of the transformation H.

Finally, if all $H_{ij}(x, y)$ are continuous in $\Delta \times \Delta$, it is possible to prove that in (40), (41), (47) and (49) the convergence in mean may be replaced by uniform convergence, while (48) holds now for every point $(x, y) \in \Delta \times \Delta$. Moreover, when $H\{p\} = \{0\}$ implies $\{p\} = \{0\}$, and the functions $A_{ij}(x)$ are either all continuous in Δ or have the property that the determinant of the matrix $\|A_{ij}(x)\|$ is $\neq 0$ for almost every $x \in \Delta$, the functions $p^i(x)$ in (40) and $p_{ij}(x, y)$ in (47) vanish identically, while in the latter of these two cases the series $\sum_k \chi_k^i(x) \overline{\chi_k^j(y)}$ converges to $H_{ij}(x, y)$, uniformly in $\Delta \times \Delta$.

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