HYPERGEOMETRIC FUNCTIONS OF TWO VARIABLES.

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Summary.

The systematic investigation of contour integrals satisfying the system of partial differential equations associated with Appell's hypergeometric function F_1 leads to new solutions of that system. Fundamental sets of solutions are given for the vicinity of all singular points of the system of partial differential equations. The transformation theory of the solutions reveals connections between the system under consideration and other hypergeometric systems of partial differential equations. Presently it is discovered that any hypergeometric system of partial differential equations of the second order (with two independent variables) which has only three linearly independent solutions can be transformed into the system of F_1 or into a particular or limiting case of this system. There are also other hypergeometric systems (with four linearly independent solutions) the integration of which can be reduced to the integration of the system of F_1.

Introduction.

1. The system of partial differential equations

$$\begin{aligned} x(1-x)r + y(1-x)s + \{\gamma - (\alpha+\beta+1)x\}p - \beta y q - \alpha\beta z = 0 \\ y(1-y)t + x(1-y)s + \{\gamma - (\alpha+\beta'+1)y\}q - \beta' x q - \alpha\beta' z = 0 \end{aligned} \tag{1}$$

in which x and y are the independent variables, z the unknown function of x and y, and $p = \dfrac{\partial z}{\partial x}$, $q = \dfrac{\partial z}{\partial y}$, $r = \dfrac{\partial^2 z}{\partial x^2}$, $s = \dfrac{\partial^2 z}{\partial x \partial y}$, $t = \dfrac{\partial^2 z}{\partial y^2}$ Monge's well-known notation for partial derivatives, has been investigated by many writers.

Appell (1880) introduced this system of partial differential equations in connection with the hypergeometric series in two variables[1]

$$F_1(\alpha, \beta, \beta', \gamma;\ x, y) = \sum\sum \frac{(\alpha)_{m+n}\,(\beta)_m\,(\beta')_n}{(\gamma)_{m+n}\ m!\ n!}\,x^m\,y^n \qquad (|x| < 1,\ |y| < 1) \tag{2}$$

which is a solution of (1).

Actually, definite integrals and series representing solutions of (1) were considered before Appell by Pochhammer (1870, 1871). Pochhammer regarded his integrals and series as functions of one complex variable only; what is now considered as the other variable appeared as a parameter in Pochhammer's work. Accordingly his integrals and series appeared as solutions of an ordinary homogeneous linear differential equation of the third order.

Soon after the publication of Appell's first note on the subject Picard (1880) discovered the connection between Pochhammer's integrals and Appell's function F_1, and since then several authors investigated the integration of (1) by means of definite integrals of the Pochhammer type. References to relevant literature will be found in the monograph by Appell and Kampé de Fériet (1926, p. 53 et seq.) where there is also a summary of the results obtained. From the Pochhammer-Picard integral ten different solutions of (1) have been derived and these solutions are represented by not less than sixty convergent series of the form

$$x^{\varkappa}(1-x)^{\varrho}\,y^{\varkappa'}(1-y)^{\varrho'}(x-y)^{\sigma}\,F_1(\lambda, \mu, \mu', \nu;\ t, t') \tag{3}$$

where $\lambda, \mu, \mu', \nu, \varkappa, \varkappa', \varrho, \varrho', \sigma$ depend on $\alpha, \beta, \beta', \gamma$ and t, t' are rational functions of x and y (Le Vavasseur 1893, Appell and Kampé de Fériet 1926 pp. 61—64). Each integral is represented by six series of the form (3) thus exhibiting certain transformations of Appell's series F_1.

2. The tableau of the sixty solutions (3) is impressive, but not exhaustive. It is of course possible to express any solution of (1) as a linear combination of three linearly independent solutions, and in so far as the tableau does contain three linearly independent solutions, it may be said to contain the general solution. Yet, if we look for three linearly independent solutions represented by series *convergent in the same domain*, we discover the gaps. For instance, among

[1] In this and in all similar sums the summation with respect to m and n runs from 0 to ∞ and $(\alpha)_n = \dfrac{\Gamma(\alpha+n)}{\Gamma(\alpha)}$ throughout.

the sixty series $z_1, \ldots \ldots, z_{60}$ there are only two distinct solutions convergent in the neighbourhood of $x=0$, $y=1$, viz. the solutions $z_4 = z_{10n+4}$ and $z_1 = z_{10n+7}$, $n = 1, 2, \ldots, 5$, so that the general solution in this neighbourhood cannot be represented in terms of the sixty series of the tableau. The reason for this omission will appear later; for the present it is sufficient to remark that there must be certain solutions of fundamental importance which should be added to the list of the sixty solutions (3).

Among the series missing from the tableau there are sixty convergent series of the form

$$x^{\varkappa}(1-x)^{\varrho}\, y^{\varkappa'}(1-y)^{\varrho'}(x-y)^{\sigma}\, G_2(\lambda, \mu, \mu', \nu;\, t, t') \tag{4}$$

which satisfy (1). Here $\lambda, \mu, \mu', \nu, \varkappa, \varkappa', \varrho, \varrho', \sigma, t, t'$ have similar meanings as in (3) and G_2 is the series introduced by Horn (1931 p. 383)

$$G_2(\alpha, \alpha', \beta, \beta';\, x, y) = \sum\sum (\alpha)_m\,(\alpha')_n\,(\beta)_{n-m}\,(\beta')_{m-n}\,\frac{x^m}{m!}\,\frac{y^n}{n!}. \tag{5}$$

We shall see later that the sixty series (4) represent fifteen distinct new solutions. There are many more series, among them series involving particular cases of Appell's series F_2, F_3 and of Horn's series H_2. All the latter series do not define new solutions but are merely transformations of solutions of the form 3) or (4).

Borngässer (1932 p. 31) obtained a solution of the form (4), viz.

$$y^{-\beta'}\, G_2\left(\beta, \beta', 1+\beta'-\gamma,\, \alpha-\beta';\, -x, -\frac{1}{y}\right) \tag{6}$$

by assuming power-series expansions of the solution of (1) in the vicinity of the singular point $x=0$, $y=\infty$. As far as I can see no attempt has been made to derive (6) or the other solutions of the form (4) from the integral representations, though it would seem natural to expect contour integrals to yield readily all significant solutions. Also the integral representations would be expected to enable one to construct the fundamental systems of solutions and their transformation theory.

The present work was undertaken with the purpose of filling this gap and obtaining all significant solutions of (1) from the integration of this system of partial differential equations by means of contour integrals. Not only were the expectations with regard to fundamental systems of solutions of (1) and with regard to the transformation theory of (1) fully justified, but the work lead to conclusions of more general importance. The results regarding fundamental systems are important for the general theory of systems of partial differential equations in the complex domain in that they indicate the procedure to be followed in case of a (hitherto untractable) singular point which is the intersection of more than two singular manifolds, or in case of a singular point at which two singular manifolds touch each other, or, lastly, at a singular point of a singular manifold. Again, the transformation theory reveals connections between (1) and other hypergeometric systems of partial differential equations and leads to an important general theorem in the theory of hypergeometric functions of two variables: *any hypergeometric system of partial differential equations of the second order which has only three linearly independent solutions can be transformed into* (1) *or into a particular or limiting case of the system* (1). Besides, there are also other hypergeometric systems (with four linearly independent solutions) of the second order the integration of which can be reduced to the integration of (1).

Eulerian Integrals.

3. All the results mentioned in the last paragraph follow readily from some simple, to the point of triviality simple, observations on integrals of the Eulerian type. It seems worth while to set forth these observations in considerable detail, because they have often been overlooked in the past, and because they are significant whenever the integration of linear differential equations by means of contour integrals leads to integrands with five or more singularities.

Picard (1881) has proved that the integral

$$z = c \int u^{\beta+\beta'-\gamma} (u-1)^{\gamma-\alpha-1} (u-x)^{-\beta} (u-y)^{-\beta'} \, du \tag{7}$$

satisfies the system (1) provided that the path of integration is either a closed contour (closed that is to say on the Riemann surface of the integrand) or an open path which ends in zeros of $u^{\beta+\beta'-\gamma}(u-1)^{\gamma-\alpha-1}(u-x)^{-\beta-1}(u-y)^{-\beta'-1}$ (Cf also Appell and Kampé de Fériet 1926 p. 55 et seq.).

The simplest types of paths are (i) open paths joining two of the five singularities 0, 1, x, y, ∞ of the integrand without encircling any other singularity, (ii) loops beginning and ending at one and the same singularity and encircling one and only one of the other singularities, and (iii) double loops (closed on the Riemann surface of the integrand) slung round two of the singularities, it being understood that the other three are outside the double-loop. Owing to the multiplicative character of the branchpoints of the integrand of (7), contours of the type (ii) and (iii) are equivalent to paths of the type (i) if the values of the parameters α, β, β', γ are such that the integrals along the latter paths are convergent; and therefore the same solutions are derived from all the three types of paths. Clearly there are $\binom{5}{2} = 10$ different simple paths joining two of the five singularities, and the integrals along these give precisely the ten solutions which can be represented by sixty series of the type (3). As far as I can see these are the only contours used by previous writers. I have not been able to examine Le Vavasseur's *Thèse*, but the account given of it by Appell and Kampé de Fériet would seem to indicate that Le Vavasseur, like the other authors, used only the contours described above.

There are more involved types of contours, for instance double circuits inside each loop of which there are two singularities of the integrand, but such contours have never been used for the integration of (1). Of course it is obvious that the more involved contours (being closed contours or equivalent with closed contours) are permissible contours for (7); but it appears that apart from Pochhammer nobody realised that solutions determined by such contours are as fundamental as solutions determined by the simple contours. Jordan who in the first edition of his *Cours d'Analyse* introduced double-loop integrals independently from Pochhammer does not even mention the more involved types; nor are they known to Nekrasoff (1891) who at the same time as and independently from Goursat, Jordan, and Pochhammer developed a theory of integrating linear differential equations by definite integrals. Pochhammer (1890) seems to be the only one who recognised the importance of some of these contours — and even he failed to apply them to what at that time he called "hypergeometric functions of the third order" and what are in effect the solutions of (1) considered as functions of one variable only, for instance as functions of x.

4. In order to classify double-circuit integrals, let us consider instead of (7) the more general integral

$$\int (u-a_1)^{\beta_1} (u-a_2)^{\beta_2} \ldots\ldots (u-a_n)^{\beta_n} H(u)\, du \tag{8}$$

in which $H(u)$ is a one-valued analytic function whose only singularities lie in some or all of the points $a_1, \ldots, a_n$. There is no loss of generality in assuming that the integrand of (8) is regular at infinity, for this can always be achieved by a bilinear transformation of the variable of integration.

The integrand has n finite singularities $a_1, \ldots, a_n$. Correspondingly, there are $\binom{n}{2}$ double circuits of the Pochhammer-Jordan type each slung round two of the n singularities. In the usual manner (cf for instance Whittaker and Watson 1927, § 12.43) we use $(a_1 +; a_2 +; a_1 -; a_2 -)$ as the symbol of a double circuit which starting, say, at a point P between a_1 and a_2 encircles first a_1 then a_2 in the positive (counterclockwise) direction and then a_1 and again a_2 in the negative direction, returning to P: the double circuit is assumed to be such that no other singularity of the integrand is encircled. Thus arg $(u - a_1)$ and arg $(u - a_2)$, having first each increased by 2π, and then again decreased by the same amount, return to their initial values; so do the phases of $u - a_3, \ldots, u - a_n$, and the double circuit (also called double loop) is closed on the Riemann surface of the integrand.

The fundamental importance of the $\binom{n}{2}$ simple double circuits for the integration of linear differential equations or systems of partial differential equations is generally recognised: this importance is due to the comparatively simple behaviour of the integrals taken along these double loops when the singularities of the integrand are variable. If any of the singularities outside of the double loop encircles any other singularity outside the double loop, the integral remains unchanged; if one of the singularities inside of the double circuit encircles the other one inside the (other loop of the) double circuit, the integral returns to its initial value multiplied by a constant factor: it is only when one of the singularities inside the double circuit encircles one of the singularities outside the double circuit, or conversely, that the system of $\binom{n}{2}$ integrals (8) (which are essentially functions of the cross ratios of the $a_1, \ldots, a_n$) undergoes a more involved linear substitution. Accordingly, if the integral taken along $(a_1 +; a_2 +; a_1 -; a_2 -)$, say, is regarded as a function of a_1, then a_2 will be a multiplicative branch point of this function so that the integral will represent a fundamental solution for the neighbourhood of a_2: in this case $a_3, \ldots, a_n$ will be singularities of a more complex type. The same integral regarded as a function of a_n, say, will be regular at $a_3, \ldots, a_{n-1}$, and its only singularities will be a_1 and a_2.

5. Let us now divide the n singularities into three groups of respectively p, $q-p$, $n-q$ elements $(0 < p < q < n)$, and number the singularities of the integrand so that $a_1, \ldots, a_p$ shall compose the first group, $a_{p+1}, \ldots, a_q$ the second group, and $a_{q+1}, \ldots, a_n$ the third. Let P, Q, N be closed curves such that $a_1, \ldots, a_p$ lie inside P but outside Q and N, that $a_{p+1}, \ldots, a_q$ lie inside Q but outside P and N, and $a_{q+1}, \ldots, a_n$ inside N but outside P and Q. The double loop $(P+;\, Q+;\, P-;\, Q-)$ which is supposed to lie entirely outside N will be denoted by $(a_1, \ldots, a_p +;\, a_{p+1}, \ldots, a_q +;\, a_1, \ldots, a_p -;\, a_{p+1}, \ldots, a_q -)$ where the semicolons separate different groups, the commas different elements of the same group. All singularities not mentioned in the symbol of the double circuit are supposed to lie outside the contour. The particular case $p=1$ corresponds to the more general double circuits studied (but not applied to the present problem) by Pochhammer (1890).

The foregoing representation of our contour is not yet symmetrical and does not adequately reflect the intrinsic properties of this type of double circuit. By a deformation of the contour by "pulling it over infinity" (which it will be remembered is a regular point of the integrand) it can be seen that the double loop slung round P and Q is essentially equivalent to the double loop slung round P and R, or to the double loop slung round Q and R. Following Felix Klein (1933 p. 66 et seq.) we obtain a more symmetrical intrinsic representation of our contour if we "pull across infinity" only one of the loops. The result is a trefoil loop which encircles P in the positive direction, intersects itself, encircles Q in the positive direction, intersects itself again, encircles R in the positive direction, intersects itself a last time and then returns to its beginning. It is easy to see that this contour which we shall call a trefoil or triple loop is closed on the Riemann surface of the integrand, is invariant against deformation of the contour and against bilinear transformations of the variable of integration, and that it is equivalent to the former double circuit. The trefoil loop will be denoted by $(P+;\, Q+;\, R+)$ or $(a_1, \ldots, a_p +;\, a_{p+1}, \ldots, a_q +;\, a_{q+1}, \ldots, a_n +)$. This trefoil loop should carefully be distinguished from the simple circuit enclosing P, Q, and R in the positive direction: the symbol of the simple circuit in our notation is $(P, Q, R+)$ or $(a_1, \ldots, a_n +)$, and the integral taken along this simple circuit vanishes by virtue of Cauchy's theorem, since the integrand is regular everywhere outside the contour (including infinity).

The properties of the integral (8) taken along such a trefoil loop are easily understood. When one of the singularities encircles another singularity contained in the same loop of the trefoil as the first one, the value of the integral does not change at all: the integral regarded as function of a_1 is regular at $a_2, \ldots, a_p$. When a whole group of singularities encircles another whole group, the integral will return to its original value multiplied by a constant factor. This is of importance when p (or $q-p$, or $n-q$) is equal to unity, for then, regarding the integral as function of a_1 (or a_q, or a_n respectively), we see that it acquires only a constant factor when a_1 (or a_q, or a_n) encircles one of the two groups contained in the two other loops of the trefoil loop. Lastly, when one group encircles both the other groups, the integral does not change its value.

Beside the triple loops there are also quadruple, quintuple, etc. loops with corresponding properties of the functions represented by integrals along such loops. There are reasons for believing that the more complicated types of multiple loops are of very little importance for the integration of linear differential equations. Such a conclusion is strongly suggested by the observation that with suitable restrictions on the mutual position of the singularities, and using convergent power series expansions of the integrand, the evaluation of (8) along a trefoil loop can always be reduced to the evaluation of Euler integrals of the first kind, but such reduction is not possible in case of quadruple and more complicated loops.

6. At first it may seem strange that the only trefoil loops applied in the past to the integration of differential equations are those for which two of the three numbers p, $q-p$, $n-q$ are equal to unity — and this in spite of the fact that the foregoing considerations are not essentially new. The situation may be clarified to a certain extent by specific consideration of the cases of low n. It then transpires that $n=5$ (the case of the system of F_1) is the first case in which the more general trefoil loop gives any results not obtainable by simple Jordan-Pochhammer double circuits, and we at once understand that taking as a model the well known case $n=4$ (the case of the classical hypergeometric function) significant solutions of the more complicated equations could be overlooked.

In the hierarchy of Eulerian integrals in the general sense of Klein (1933 p. 87) the lowest case is $n=2$, for there must be at last two branch points. A bilinear transformation throws these branch points to 0 and ∞, and the integral becomes

$$\int u^{-\gamma}\, du.$$

No trefoil loop is possible, and hence a Pochhammer double circuit slung around the two singularities gives always a vanishing integral (as is well known).

In the case $n = 3$, a bilinear transformation carries the singularities to 0, 1, ∞, and we have the canonical form

$$\int u^{-\gamma}(u-1)^{\gamma-\alpha-1}\,du.$$

There is essentially only one trefoil loop, for the canonical form it is $(0+\ 1+;\ \infty+)$, it is equivalent to a double circuit $(0+;\ 1+;\ 0-;\ 1-)$, and leads to the Euler integral of the first kind.

Next, in the case of four singularities, $n = 4$, three of the singularities may be assumed to be at 0, 1, ∞ and we have the canonical form

$$\int u^{\beta-\gamma}(u-1)^{\gamma-\alpha-1}(u-x)^{-\beta}\,du,$$

that is integrals of the hypergeometric type. In this case too there is only one kind of trefoil loop, containing respectively 1, 1, and 2 singularities within its three loops. It is equivalent to a Pochhammer double circuit encircling only two of the four singularities. Clearly there are $\binom{4}{2} = 6$ different trefoil loops in this case, giving the six well-known branches of Riemann's P-function.

The fact that in the two best known cases $n = 3$ and $n = 4$ we have to consider only double circuits slung round two singularities, or contours equivalent to such double circuits, seems to have lead to the view (never expressed but tacitly underlying all previous work on the system of F_1) that in every case such simple double circuits would give all solutions of fundamental importance. Yet, already the next case, $n = 5$, shows that this view is untenable.

7. In the case $n = 5$ we obtain a canonical form by throwing three of the singularities to 0, 1 and ∞ by a bilinear transformation of u. The canonical form is

$$\int u^{\beta+\beta'-\gamma}(u-1)^{\gamma-\alpha-1}(u-x)^{-\beta}(u-y)^{-\beta'}\,du. \tag{7}$$

There are now two essentially different types of trefoil loops: one has 1, 1, and 3 the other 1, 2, and 2 singularities respectively within its three loops.

There are $\binom{5}{2} = 10$ different trefoil loops of the type $1, 1, 3$. Each of these ten trefoil loops is equivalent to a double circuit encircling two singularities only. For suitable values of the parameters α, β, β' γ each double loop in its turn is equivalent to an open path joining two singularities. Thus the ten 1, 1, 3 type trefoils give the ten solutions whose sixty expansions constitute the table of Le Vavasseur and Appell and Kampé de Fériet.

There are also trefoil loops of the type 1, 2, 2, that is trefoil loops whose three loops encircle respectively 1, 2 and 2 singularities. Each trefoil loop of this type is equivalent to a double circuit whose one loop encircles one singularity only, while the other loop encircles two singularities. With suitable values of the parameters each of these double circuits is in its turn equivalent to a simple loop which begins and ends at one and the same singularity and encircles two other singularities. Among the $5 \cdot \binom{4}{2}$ simple loops obtainable in this way there are always equivalent pairs, for instance the loop beginning and ending at 0 and encircling 1 and ∞ is equivalent to the loop beginning and ending at 0 and encircling x and y. So we obtain $\frac{1}{2} \cdot 5 \cdot \binom{4}{2} = 15$ distinct new solutions. From the character at the branch points of these solutions it is easy to see that they are not identical with any of the old solutions (though either set can be expressed as linear combinations of solutions of the other set) and it remains to discover the nature of these solutions. The discussion of the properties of these solutions will show that they are as significant as the well-known solutions (3), and it will transpire that they are precisely the 15 solutions whose 60 convergent expansions are of the form (4).

The System of F_1 and Equivalent Systems.

8. The results of the above considerations will now be applied to the integration of the system of partial differential equations associated with F_1, that is to the system (1). In doing so exceptional values of the parameters giving rise to logarithmic solutions will tacitly be excluded. Results for these exceptional cases and for the logarithmic solutions which they involve can be obtained by simple limiting processes carried out in the formulae to be derived for the general case.

Except in section 14, we shall write throughout a, b, c for any permutation of 0, 1, ∞ so that for instance (a, b) stands for any of the six points (0, 1), (0, ∞), (1, 0), (1, ∞), (∞, 0) or (∞, 1). In using this generic notation, by which we shall gain much in brevity, we make the convention that general statements must receive appropriate (and in every case easily obtainable) interpretation when the symbol involved represents ∞. We shall say, for instance, that a certain solution z remains unchanged when x encircles a and mean that this is true of z itself if $a = 0$ or $a = 1$, but true of $x^{\beta} z$ if $a = \infty$. The additional factor x^{β} arises from (7) when this integral is re-written in such a fashion as to make $\frac{1}{x}$ appear as variable instead of x A similar convention holds for b, the appropriate factor in this case being $y^{\beta'}$. The convention is very similar to the one by which statements such as "is analytic at infinity" are interpreted in complex function theory, and has a similar purpose.

9. First we have to discuss the singularities of our system (1). This system of partial differential equations has seven *singular manifolds* or *singular curves*, viz. $x = 0$, $x = 1$, $x = \infty$, $y = 0$, $y = 1$, $y = \infty$, and $x = y$. The singular curves can either be derived from the partial differential equations themselves in the well-known manner, or obtained from the integral (7). In the latter case they emerge as the conditions for the coincidence of two singularities of the integrand. The seven singular curves produce by their various intersections two types of *singular points*.

There are six singular points represented by $x = a$, $y = b$ where (a, b) stands for $(0, 1)$, $(0, \infty)$, $(1, 0)$, $(1, \infty)$, $(\infty, 0)$ or $(\infty, 1)$. Each of these six singular points is the intersection of two singular curves, $x = a$ and $y = b$, and belongs to the simplest type of singular points of systems of partial differential equations.

There are also three singular points $x = a$, $y = a$ that is to say $(0, 0)$, $(1, 1)$ or (∞, ∞), and these are of a more complex type. At each of the singular points (a, a) three singular curves intersect, viz. $x = a$, $y = a$ and $x = y$. For this reason it is impossible to expand the general solution of (1) in power series convergent in the entire four-dimensional neighbourhood of (a, a). Instead, we shall construct fundamental systems of solutions valied in hypercones whose vertex is at (a, a), whose "axis" is one of the singular curves through (a, a) (these singular curves are, of course, two-dimensional manifolds in the four-dimensional space of the two complex variables), and which extends unto another singular curve. For every singular point (a, a) there will be three such hypercones and hence three different fundamental systems. Between them the three systems describe the behaviour of the general solution in the neighbourhood of (a, a) completely.

In order to have a short notation for the solutions of (1), we shall denote by $[P; Q; N]$ the integral (7) taken along the contour $(P+; Q+; N+)$ and multiplied by a suitable constant. Now, the triple loop can be replaced by a double circuit encirling any two of the three groups P, Q, N and accordingly we shall denote the solution $[P; Q; N]$ more briefly, if less symmetrically, by $[Q; N]$, $[N; P]$ or $[P; Q]$, disregarding any constant factors. Instead of $[P; Q]$ for instance we shall also write $[a_1, \ldots, a_p; a_{p+1}, \ldots, a_q]$.

10. In the neighbourhood of an intersection (a, b) of two singular manifolds there is firstly one solution, $[c; a, x] = [c; b, y]$ which is manifestly regular at $x = a$, $y = b$. Like all the following statements about the behaviour of solutions, this follows immediately from the general properties of Eulerian integrals as developed in the earlier sections of this paper, and is to be interpreted appropriately (cf section 8) if a or b is ∞. A second solution for the neighbourhood of (a, b) is $[a; x]$ which is regular at $y = b$ and has a multiplicative branch-manifold at $x = a$; and a third solution is $[b; y]$ which is regular at $x = a$ and has a multiplicative branch-manifold at $y = b$. The behaviour of these three solutions at the singular curves shows that the solutions are linearly independent and thus they form a fundamental system for the neighbourhood of (a, b). Clearly each of the three solutions can be expanded in powers of $x - a$ and $y - b$ (in powers of $\frac{1}{x}$ if $a = \infty$ and in powers of $\frac{1}{y}$ if $b = \infty$).

The situation is different with regard to a neighbourhood of an intersection of three singular manifolds $x = y = a$. There is one solution, $[b; c]$, which is regular at (a, a) and in the entire neighbourhood of this singular point, but there is no other solution valid in the entire neighbourhood of (a, a).

In order to obtain two more solutions in this case let us fix our attention to the neighbourhood of (a, a) "near" $x = a$, i. e. let us assume that both $|x - a|$ and $|y - a|$ are small, and $|x - a| < |y - a|$. Then we have the solution $[a; x]$ which has a multiplicative branch-manifold at $x = a$, and remains unaltered when y encircles both a and x. Of course, $[a; x]$ andergoes a more involved transformation when y encircles a only or x only, but this cannot happen as long as (x, y) remains in the hypercone $|x - a| < |y - a|$ of the four-dimensional neighbourhood of (a, a). We have also the solution $[y; a, x]$ which is regular at

$x=a$ and is merely multiplied by a constant factor when y encircles both a and x. By a similar argument as before, $[b;\ c]$, $[a;\ x]$ and $[y;\ a,\ x]$ constitute a fundamental system of solutions for the neighbourhood of $(a,\ a)$ in the hypercone $|x-a|<|y-a|$.

In the hypercone $|x-a|>|y-a|$, or "near" $y=a$, the corresponding fundamental system is $[b;\ c]$, $[a;\ y]$ and $[x;\ a,\ y]$. Finally, "near" $x-y=0$, i. e. when $|x-y|<|x-a|$ and $|x-y|<|y-a|$ we have the fundamental system $[b;\ c]$, $[x;\ y]$ and $[a;\ x,\ y]$ which undergoes simple transformations when x and y encircle each other or when both of them encircle a.

11. It is now possible to see the fundamental importance of the new type of double circuits. Take for instance the vicinity of $(a,\ b)$. Using only simple double circuits (or the equivalent open paths joining two of the singularities) the solutions $[a;\ x]$ and $[b;\ y]$ are readily obtained, two of the solutions registered by previous writers. However, the simplest solution of all, which is one-valued in the neighbourhood of the singular point $(a,\ b)$ and regular at that point, i. e. $[c;\ a,\ x]=[c;\ b,\ y]=[a,\ x;\ b,\ y]$, cannot be obtained at all from simple double circuits, and consequently does not appear in the Le Vavasseur table of solutions. If we take for instance the point $(0,\ \infty)$, we find in the table (Appell and Kampé de Fériet, 1926, p. 62 et seq.) only two distinct solutions, $z_4=z_{10n+4}$ and $z_9=z_{10n+9}$ $(n=1,\ 2,\ \ldots,\ 5)$ convergent for small x and large y. The third solution, (6), was discovered by Borngässer (1932) who used different methods (integration of the differential equation by power series).

In any of the hypercones in the neighbourhood of $(a,\ a)$ there are again two solutions which can be represented by simple double circuits or equivalent open paths; these solutions are accordingly known. But as far as I know nobody succeeded as yet in finding the third solution for this case explicitly, for the integration of systems of partial differential equations by power series becomes difficult in the neighbourhood of an intersection of three singular curves. Take for instance the neighbourhood of $(0,\ 0)$ "near" $x=0$, or more precisely the domain $|x|<|y|<1$. We find in the table two solutions, z_1 and z_{14}, but the third solution does not appear in the literature known to me.

From these considerations it is seen that the employment of triple loops of all possible types (or of equivalent double circuits of all possible types) is essential for the success of integrating a system of partial differential equations by contour integrals: at the same time it seems that quadruple and yet higher loops can safely by left aside.

12. A point of great interest in the analytic theory of systems of partial differential equations is the transformation theory of the solutions, and contour integral representations of the solutions are notoriously the best tool for developing such a transformation theory. The transformations of the solutions of the form (3) have been discussed in detail by Le Vavasseur (see also Appell and Kampé de Fériet, 1926, pp. 65—68). A complete transformation theory will embrace all 25 solutions which occur in our 15 fundamental systems. The best plan is to express all 25 solutions in terms of 3 arbitrarily chosen linearly independent solutions, and to collect the results in a matrix equation which represents the 25×1 column matrix of the 25 solutions as the product of a 25×3 transformation matrix with the 3×1 column matrix of the selected set of three linearly independent solutions. From this the expression of any of the 15 fundamental systems in terms of any of the remaining 14 systems can be derived by the elementary rules of matrix algebra.

I do not propose to give here a detailed theory of the 25 solutions; only a few of the more important properties of the two types of solutions will be enumerated in the following paragraphs.

13. We consider those solutions of (1) which are expressed by integrals along a simple double circuit or, in case of suitable values of the parameters, along corresponding open paths. These are the ten solutions of Picard (1881) and Goursat (1882). A typical one is

$$F_1(\alpha, \beta, \beta', \gamma; x, y) = \frac{\Gamma(\gamma)}{\Gamma(\alpha)\, \Gamma(\gamma - \alpha)} \int_1^\infty u^{\beta+\beta'-\gamma} (u-1)^{\gamma-\alpha-1} (u-x)^{-\beta} (u-y)^{-\beta'} \, du$$

valid if $\Re(\alpha) > 0$, $\Re(\gamma - \alpha) > 0$, $|\arg(1-x)| < \pi$, $|\arg(1-y)| < \pi$. The observation that the substitutions $u = \frac{v}{v-1}$, $u = x + (1-x)v$, $u = y + (1-y)v$, $u = \frac{v-x}{v-1}$, $u = \frac{v-y}{v-1}$ result in integrals of the same type leads to the expression of each of the solutions in six different ways in terms of F_1. The transformation theory of these solutions was given by Le Vavasseur.

Beside the six expansions in terms of F_1, there are other expansions in terms of hypergeometric series other than F_1. The following four will be needed later.

The expansion of $(u-x)^{-\beta}$ in powers of x, and of

$$(u-y)^{-\beta'} = (1-y)^{-\beta'} u^{-\beta'} \left\{1 - \frac{u-1}{u} \frac{y}{y-1}\right\}^{-\beta'}$$

in powers of $\frac{y}{y-1}$, and the expansions in which the role of x and y are interchanged lead to (Appell and Kampé de Fériet, 1926, p. 24 equations (29 and (29'))

$$\begin{aligned} F_1(\alpha, \beta, \beta', \gamma; x, y) &= (1-y)^{-\beta'} F_3\left(\alpha, \gamma-\alpha, \beta, \beta', \gamma; x, \frac{y}{y-1}\right) \\ &= (1-x)^{-\beta} F_3\left(\gamma-\alpha, \alpha, \beta, \beta', \gamma; \frac{x}{x-1}, y\right) \end{aligned} \tag{9}$$

where F_3 is Appell's series

$$F_3(\alpha, \alpha', \beta, \beta', \gamma; x, y) = \sum\sum \frac{(\alpha)_m (\alpha')_n (\beta)_m (\beta')_n}{(\gamma)_{m+n}\, m!\, n!} x^m y^n. \tag{10}$$

Again, the expansion of $(u-x)^{-\beta}$ in powers of x, and of

$$\left(x\frac{u-y}{y}\right)^{-\beta'} = u^{-\beta'}\left\{1 - \left(1-\frac{x}{y}\right) - \frac{x}{u}\right\}^{-\beta'}$$

in powers of x and $1-\frac{x}{y}$ lead to the first of the two transformations (Appell and Kampé de Fériet, 1926, p. 35 (9))

$$\begin{aligned} F_1(\alpha, \beta, \beta', \gamma; x, y) &= \left(\frac{x}{y}\right)^{\beta'} F_2\left(\beta+\beta', \alpha, \beta', \gamma, \beta+\beta'; x, 1-\frac{x}{y}\right) \\ &= \left(\frac{y}{x}\right)^{\beta} F_2\left(\beta+\beta', \alpha, \beta, \gamma, \beta+\beta'; y, 1-\frac{y}{x}\right) \end{aligned} \tag{11}$$

in which F_2 is Appell's series

$$F_2(\alpha, \beta, \beta', \gamma, \gamma'; x, y) = \sum\sum \frac{(\alpha)_{m+n} (\beta)_m (\beta')_n}{(\gamma)_m (\gamma')_n\, m!\, n!} x^m y^n. \tag{12}$$

Next we expand

$$\left(1-\frac{y}{x}\right)^{\beta'}(u-y)^{-\beta'}=u^{-\beta'}\left\{1+\frac{y(u-x)}{(x-y)u}\right\}^{-\beta'}$$

in powers of $\frac{y}{x-y}$ and then $(u-x)^{-\beta+n}$ in powers of x to obtain the first of the two transformations

$$\begin{aligned} F_1(\alpha,\beta,\beta',\gamma;\,x,y) &= \left(1-\frac{y}{x}\right)^{-\beta'} H_2\left(\beta,\alpha,\beta',1-\beta,\gamma;\,x,\frac{y}{x-y}\right) \\ &= \left(1-\frac{x}{y}\right)^{-\beta} H_2\left(\beta',\alpha,\beta,1-\beta',\gamma;\,y,\frac{x}{y-x}\right) \end{aligned} \tag{13}$$

in which H_2 is Horn's series (Horn, 1931, p. 383)

$$H_2(\alpha,\beta,\gamma,\delta,\varepsilon;\,x,y)=\sum\sum\frac{(\alpha)_{m-n}(\beta)_m(\gamma)_n(\delta)_n}{(\epsilon)_m\,m!\,n!}x^m y^n. \tag{14}$$

Lastly, put $u=x+(1-x)v$, expand

$$\left(\frac{u-y}{1-x}\right)^{-\beta'}=\left(v-\frac{x}{x-1}-\frac{y}{1-x}\right)^{-\beta'}$$

in powers of $\frac{y}{1-x}$ and then $\left(v-\frac{x}{x-1}\right)^{\beta-\gamma-n}$ in powers of $\frac{x}{x-1}$ to obtain the transformation

$$F_1(\alpha,\beta,\beta',\gamma;\,x,y)=(1-x)^{-\alpha}\sum\sum\frac{(\alpha)_{m+n}(\gamma-\beta+n)_m(\beta')_n}{(\gamma)_{m+n}\,m!\,n!}\left(\frac{x}{x-1}\right)^m\left(\frac{y}{1-x}\right)^n. \tag{15}$$

There is a corresponding transformation obtained by interchanging x and y, and at the same time β and β'.

14. Turning now to integrals taken along a double-loop involving three singularities of the integrand, we shall depart from the notation of section 8 and shall denote by a, b, c, d, e, any permutation of the five singularities of the integrand of (7), that is any permutation of $0, 1, x, y, \infty$. The typical solution is $[a;\,b,\,c]=[a;\,d,\,e]$, and there are obviously 15 solutions of this type.

In oder to reduce solutions of this type to convergent infinite series, we remark that there is no loss of generality in assuming that the real part of the exponent of $u-a$ in (7) exceeds -1 (or that the real part of the exponent of $\frac{1}{u}$ exceeds 1 if a happens to be ∞); and under this assumption the double circuit may be replaced by a single loop, so that apart from a constant factor the solution is

$$\int_a^{(b,\, c+)} u^{\beta+\beta'-\gamma}(u-1)^{\gamma-\alpha-1}(u-x)^{-\beta}(u-y)^{-\beta'}\,du \tag{16}$$

where the contour of integration is a loop starting at a, encircling b and c in positive direction and returning to a so that d and e are outside the loop. Alternatively, the solution may be represented by an integral extended over the loop $a \ldots (d,\, e+) \ldots a$ with a slightly different constant factor.

By a linear transformation of u we throw a to -1, one of the singularities inside the loop, say b, to 0 and one of the singularities outside the loop, say d, to ∞. The linear transformation effecting this is

$$v = -\frac{u-b}{u-d}\,\frac{a-d}{a-b}.$$

With the notation

$$t = -\frac{c-b}{c-d}\,\frac{a-d}{a-b}, \qquad t' = -\frac{e-d}{e-b}\,\frac{a-b}{a-d}$$

(16) reduces to a constant multiple of

$$x^{\varkappa}(1-x)^{\varrho}y^{\varkappa'}(1-y)^{\varrho'}(x-y)^{\sigma}\int_{-1}^{(0,\, t+)} v^{\lambda+\mu'-1}(v+1)^{-\mu'-\nu}(v-t)^{-\lambda}(1-t'v)^{-\mu}\,du$$

where $\lambda, \mu, \mu', \nu, \varkappa, \varrho, \varkappa', \varrho'$ depend on $\alpha, \beta, \beta', \gamma$, and t and t' are rational functions of x and y. It remains to show that this is of the form (4).

If $|t| < 1$ and $|t'| < 1$ we may take the unit circle $|v| = 1$ as the contour of integration. The expansions

$$(v-t)^{-\lambda} = v^{-\lambda}\sum\frac{(\lambda)_m}{m!}\left(\frac{t}{v}\right)^m \text{ and } (1-t'v)^{-\mu} = \sum\frac{(\mu)_n}{n!}(t'v)^n$$

are uniformly convergent along the contour so that term-by-term integration is permissible and gives for the integral

$$\sum\sum\frac{(\lambda)_m(\mu)_n}{m!\;n!}t^m\,t'^n\int_{-1}^{(0+)} v^{\mu'-m+n-1}(v+1)^{-\mu'-\nu}\,dv.$$

Now,

$$\frac{1}{2\pi i}\int_{-1}^{(0+)} v^{\mu'-m+n-1}(v+1)^{-\mu'-\nu}\,dv = \frac{\Gamma(1-\mu'-\nu)}{\Gamma(1-\mu'+m-n)\,\Gamma(1-\nu-m+n)}$$

and this is

$$\frac{\Gamma(1-\mu'-\nu)}{\Gamma(1-\mu')\,\Gamma(1-\nu)}\,(\mu')_{n-m}\,(\nu)_{m-n},$$

so that finally

$$\int_{-1}^{(0,t+)} v^{\lambda+\mu'-1}(v+1)^{-\mu'-\nu}(v-t)^{-\lambda}(1-t'v)^{-\mu}\,dv$$

$$=\frac{2\pi i\,\Gamma(1-\mu'-\nu)}{\Gamma(1-\mu')\,\Gamma(1-\nu)}\,G_2(\lambda,\mu,\mu',\nu;t,t')$$

and the solutions under consideration are of the form (4).

15. The last integral is equivalent with the integral representation

$$G_2(\alpha,\alpha',\beta,\beta';x,y)=$$
$$\frac{\Gamma(1-\beta)\,\Gamma(1-\beta')}{2\pi i\,\Gamma(1-\beta-\beta')}\int_{-1}^{(0,x+)} u^{\beta-1}(u+1)^{-\beta-\beta'}\left(1-\frac{x}{u}\right)^{-\alpha}(1-yu)^{-\alpha'}\,du \qquad (17)$$
$$\Re(\beta+\beta')<1$$

from which certain transformations of G_2 follow readily.

First put $u=x+(x+1)v$ in (17) to obtain an integral of the same type and from it the transformation

$$G_2(\alpha,\alpha',\beta,\beta';x,y)=(1+x)^{-\beta'}(1-xy)^{-\alpha'}\,G_2\left(1-\alpha-\beta,\alpha',\beta,\beta';\frac{-x}{x+1},\,y\,\frac{1+x}{1-xy}\right). \qquad (18)$$

Similarly, the substitution $u=\dfrac{v}{vy+y+1}$ leads to the transformation

$$G_2(\alpha,\alpha',\beta,\beta';x,y)=(1+y)^{-\beta}(1-xy)^{-\alpha}\,G_2\left(\alpha,1-\alpha'-\beta',\beta,\beta';x\,\frac{1+y}{1-xy},\frac{-y}{y+1}\right) \qquad (19)$$

which could also be obtained by interchanging α,β,x and α',β',y in (18); and the substitution $u=\dfrac{(x+1)v+x(y+1)}{y+1+y(x+1)v}$ leads to

$$G_2(\alpha,\alpha',\beta,\beta';x,y)=(1+x)^{-\beta'}(1+y)^{-\beta}(1-xy)^{1-\alpha-\alpha'}\times$$
$$\times\,G_2\left(1-\alpha-\beta,1-\alpha'-\beta',\beta,\beta';-x\,\frac{1+y}{1+x},-y\,\frac{1+x}{1+y}\right) \qquad (20)$$

which can also be obtained by combining (18) and (19).

Thus we see that every integral (16) can be expressed in four different ways in the form (4). There are fifteen distinct integrals of the type (16) and hence sixty different series (4). It is quite clear that the number of series of the form (4) which are solutions of (1) must be equal to the number of series of the form (3) satisfying (1), for (2) and (6) satisfy the same system of partial differential equations and thus to every series of the form (3) we must have one

$$x^{\varkappa}(1-x)^{\varrho}\,y^{\varkappa'}(1-y)^{\varrho'}(x-y)^{\sigma}\,t'^{-\mu'}\,G_2\left(\mu,\mu',1+\mu'-\nu,\lambda-\mu';-t,-\frac{1}{t'}\right)$$

satisfying the same system of partial differential equations; and the last series is of the form (4).

Like in the case of F_1 there are numerous expansions of (17) in terms of convergent hypergeometric series other than G_2. The following two will be needed later.

In (17) we use the expansions

$$\left(1-\frac{x}{u}\right)^{-\alpha}=(1+x)^{-\alpha}\sum\frac{(\alpha)_m}{m!}\left(\frac{x}{x+1}\,\frac{u+1}{u}\right)^m$$

$$(1-uy)^{-\alpha'}=(1+y)^{-\alpha'}\sum\frac{(\alpha')_n}{n!}\left(y\,\frac{u+1}{y+1}\right)^n$$

and obtain

$$G_2=(1+x)^{-\alpha}(1+y)^{-\alpha'}\sum\sum\frac{(\alpha)_m\,(\alpha')_n}{m!\,n!}\left(\frac{x}{x+1}\right)^m\left(\frac{y}{y+1}\right)^n\times$$

$$\times\frac{\Gamma(1-\beta)\,\Gamma(1-\beta')}{2\pi i\,\Gamma(1-\beta-\beta')}\int_{-1}^{(0+)}u^{\beta-m-1}(u+1)^{-\beta-\beta'+m+n}\,du$$

and hence

$$G_2(\alpha,\alpha',\beta,\beta';x,y)=\\(1+x)^{-\alpha}(1+y)^{-\alpha'}F_2\left(1-\beta-\beta',\alpha,\alpha',1-\beta,1-\beta';\frac{x}{x+1},\frac{y}{y+1}\right).\tag{21}$$

Combining the same expansion of $(1-uy)^{-\alpha'}$ with

$$\left(1-\frac{x}{u}\right)^{-\alpha}=\sum\frac{(\alpha)_m}{m!}\left(\frac{x}{u}\right)^m$$

we find in a similar manner

$$G_2(\alpha, \alpha', \beta, \beta'; x, y) = (1+y)^{-\alpha'} H_2\left(\beta', \alpha, \alpha', 1-\beta-\beta', 1-\beta; -x, \frac{-y}{y+1}\right) \qquad (22)$$

with a corresponding transformation when the role of x and y are interchanged.

16. Summing up our results regarding solutions of (1), we see that there are exactly 25 distinct integrals of the type (7) extended over a triple loop or an equivalent contour. Each of these 25 integrals represents a solution, the 25 solutions suffice to construct all fundamental systems of solutions, and, conversely, all the 25 are needed for this purpose. 10 of the 25 solutions can be represented, each in 6 different ways, in the form (3); and the other 15, each in 4 different ways, in the form (4): besides, there are numerous other expansions for each of the 25 solutions. It remains to compare this result with that arrived at by previous writers.

There is no need to discuss in detail the results arrived at by earlier authors: they found only the 10 solutions represented by 60 series (3) and we know already how these solutions fit in with our present theory, and wherein the older results are incomplete. It is necessary, however, to consider in somewhat greater detail the investigations of Professor Horn and of his pupil Dr Borngässer. We start with the latter, the comparison of whose results with ours is facilitated by the transformations of sections 8 to 12. We revert to the notations of those sections in that we use a, b, c for a permutation of $0, 1, \infty$.

Borngässer (1932) attempts to find power series solutions of the system of partial differential equations associated with F_1. For the neighbourhood of (a, a), an intersection of three singular manifolds, he finds only one solution. This is in every case of the form (3) and corresponds to our fundamental solution $[b; c]$: in fact it is the only solution in power series which is convergent in an entire neighbourhood of the singular point (a, a). Borngässer's method gives no further solution for such a neighbourhood. For the neighbourhood of (a, b), which is an intersection of two singular manifolds, Borngässer finds the complete fundamental systems. Though his solutions appear in various forms, they can always be transformed into the form (3) or (4).

Dealing with $(0, \infty)$, Borngässer (1932, p. 31) first finds the expansion (6), which is our solution $[1; 0, x]$, then

$$y^{-\alpha} \sum_{m=0}^{\infty} \sum_{n=m}^{\infty} \frac{(\beta)_m (\alpha)_n (\alpha + 1 - \gamma)_{n-m}}{(\alpha + 1 - \beta')_n \, m! \, (n-m)!} x^m y^{-n}$$

which is obviously

$$y^{-\alpha} F_1\left(\alpha, \beta, \alpha + 1 - \gamma, \alpha + 1 - \beta'; \frac{x}{y}, \frac{1}{y}\right)$$

or z_9 of Le Vavasseur's table, and finally

$$x^{\beta'+1-\gamma} y^{-\beta'} \sum_{n=0}^{\infty} \sum_{m=n}^{\infty} \frac{(\beta + \beta' + 1 - \gamma)_m (\beta')_n (\alpha + 1 - \gamma)_{m-n}}{(\beta' + 2 - \gamma)_m n! (m - n)!} x^m y^{-n}$$

which is

$$x^{\beta'+1-\gamma} y^{-\beta'} F_1\left(\beta + \beta' + 1 - \gamma, \alpha + 1 - \gamma, \beta', \beta' + 2 - \gamma; x, \frac{x}{y}\right)$$

or z_{14} of the table: the two last solutions are both of the form (3) and are expansions of our $[y; \infty]$ and $[0; x]$ respectively.

For the neighbourhood of $(1, 0)$ Borngässer (p. 41) finds the series

$$F_2(\alpha, \beta, \beta', \alpha + \beta + 1 - \gamma, \gamma - \beta; 1 - x, y)$$

which can be transformed by means of (21) into

$$x^{-\beta}(1 - y)^{-\beta'} G_2\left(\beta, \beta', \gamma - \alpha - \beta, \beta + 1 - \gamma; \frac{1 - x}{x}, \frac{y}{1 - y}\right)$$

or $[\infty; 0, y]$; and the two series

$$y^{\beta+1-\gamma} F_2(\alpha + \beta + 1 - \gamma, \beta, \beta + \beta' + 1 - \gamma, \alpha + \beta + 1 - \gamma, 2 + \beta - \gamma; 1 - x, y)$$

and

$$(1 - x)^{\gamma-\alpha-\beta} F_2(\gamma - \beta, \gamma - \alpha, \beta', \gamma + 1 - \alpha - \beta, \gamma - \beta; 1 - x, y)$$

which in their turn can be transformed by means of (11) into

$$x^{-\beta} y^{\beta+1-\gamma} F_1\left(\beta + \beta' + 1 - \gamma, \beta, \alpha + 1 - \gamma, 2 + \beta - \gamma; \frac{y}{x}, y\right)$$

and

$$(1 - x)^{\gamma-\alpha-\beta}(1 - y)^{-\beta'} F_1\left(\gamma - \alpha, \gamma - \beta - \beta', \beta', \gamma - \alpha - \beta + 1; 1 - x, \frac{1 - x}{1 - y}\right)$$

and are thus seen to be identical with z_{15} and z_{16} of the table, and to be expansions of our $[0; y]$ and $[1; x]$, showing that in this case too Borngässer's fundamental system is identical with ours.

Lastly, for the neighbourhood of $(1, \infty)$ Borngässer has the series

$$y^{-\beta'} H_2\left(\alpha - \beta', \beta, \beta', \beta + \beta' + 1 - \gamma, \alpha + \beta + 1 - \gamma; 1 - x, -\frac{1}{y}\right)$$

which can be transformed by means of (22) into

$$(y-1)^{-\beta'} G_2\left(\beta, \beta', \gamma-\alpha-\beta, \alpha-\beta'; x-1, \frac{1}{y-1}\right),$$

an expansion of $[0; 1, x]$, and the two series

$$y^{-\alpha} \sum_{m=0}^{\infty} \sum_{n=m}^{\infty} \frac{(\beta)_m (\alpha)_n (\alpha+\beta+1-\gamma)_n}{(\alpha+\beta+1-\gamma)_m (\alpha+1-\beta')_n\, m!\,(n-m)!} (x-1)^m y^{-n}$$

and

$$(1-x)^{\gamma-\alpha-\beta} y^{-\beta'} H_2\left(\gamma-\beta-\beta', \gamma-\alpha, \beta', \beta+\beta'+1-\gamma, \gamma+1-\alpha-\beta; 1-x, -\frac{1}{y}\right)$$

which can be transformed by means of (15) and (13) into respectively

$$(y-1)^{-\alpha} F_1\left(\alpha, \beta, \gamma-\beta-\beta', \alpha+1-\beta'; \frac{x-1}{y-1}, \frac{1}{1-y}\right)$$

and

$$x^{\alpha-\gamma}(1-x)^{\gamma-\alpha-\beta}(y-1)^{-\beta'} F_1\left(\gamma-\alpha, 1-\alpha, \beta', \gamma+1+\alpha-\beta; \frac{x-1}{x}, \frac{x-1}{x}\frac{y}{y-1}\right),$$

or z_{39} and z_{46} of the tableau, and are expansions of $[\infty; y]$ and $[1; x]$.

In every case the series found by Borngässer are expansions of the solutions of the fundamental systems of section 10.

17. The comparison with Horn's results is somewhat more difficult. Horn, who deals with the more involved problem of the neighbourhood of an intersection (a, a) of three singular manifolds, proves the existence of the fundamental system of solutions, derives certain results regarding the form of series representing these solutions, but does not obtain explicit formulae. He puts $x = t \cos \varphi$, $y = t \sin \varphi$, finds the ordinary differential equations satisfied by z as a function of t (a differential equation, incidentally, very nearly related to Pochhammer's differential equation for hypergeometric functions of order three), and discusses the behaviour of the solution of this ordinary differential equation in the neighbourhood of the singularities $t = 0$, $t = \infty$.

In his earlier paper Horn (1935) obtained series which represent the solutions asymptotically as $t \to \infty$. In his later paper (1938) there is a more thorough investigation of all three intersections of three singular manifolds. The series which have been proved to represent the solutions asymptotically turn out to be convergent series (in fact they are equivalent to our series) and to represent the solutions in their domain of convergence. It is clearly sufficient to discuss the result of the later paper, and since all three singularities (a, a) are of the same character, it will be sufficient to deal with one of them, say $(0, 0)$ or $t = 0$. For this singular point Horn (1938, p. 450) first finds the well-known solution (2) which is our $[1; \infty]$, and then proves the existence of two more solutions of the form $z = t^{-\gamma} \sum z_n t^n$ where the z_n are functions of φ.

In order to find explicit representations of Horn's solutions $\sum z_n t^{n-\gamma}$, let us assume for the moment that we are "near" $x = 0$, i.e. that $|\tan\varphi| < 1$. Then we have the solutions $[0; x]$ and $[y; 0, x]$ of section 10 which are, apart from constant factors and under suitable assumptions about the parameters,

$$\int_0^x u^{\beta+\beta'-\gamma}(1-u)^{\gamma-\alpha-1}(x-u)^{-\beta}(y-u)^{-\beta'}\,du$$

and

$$\int_y^{(0,x+)} u^{\beta+\beta'-\gamma}(1-u)^{\gamma-\alpha-1}(u-x)^{-\beta}(u-y)^{-\beta'}\,du.$$

We put $x = t \cos\varphi$, $y = t \sin\varphi$ and $u = tv \cos\varphi$ in the first integral, $u = -tv \sin\varphi$ in the second, whereupon both are easily seen to be of the form $\sum z_n(\varphi) t^{n-\gamma}$. Similarly, $[0; y]$ and $[x; 0, y]$, and also $[x; y]$ and $[0; x, y]$ have expansions of the form $\sum z_n(\varphi) t^{n-\gamma}$, but the coefficients $z_n(\varphi)$ will be different for each of the three pairs of solutions, that is in each of the three hypercones which have their (common) vertex at the singular point, and their axis at one of the singular manifolds passing through this singular point.

This change of the $z_n(\varphi)$ from one hypercone to another is latent in Horn's analysis. He writes

$$z_n = p_0 f_n + q_0 g_n$$

where f_n and g_n are homogeneous polynomials of degree n in $\cos\varphi$ and $\sin\varphi$ and are determined by recurrence relations, while p_0 and q_0 satisfy a system of two ordinary differential equations in which φ is the independent variable. From our point of view the sequence of the functions $z_n(\varphi)$ appears as a system of contiguous hypergeometric functions of $\tan\varphi$, and the nth member of the sequence can be expressed in terms of the first two members with coefficients which are homogeneous polynomials in $\cos\varphi$ and $\sin\varphi$. The system of two ordinary differential equations satisfied by p_0 and q_0 is a hypergeometric system and has the singularities $\tan\varphi = 0, 1, \infty$ or $\varphi = n\pi, (n+\frac{1}{4})\pi, (n+\frac{1}{2})\pi$ (n is an integer) corresponding to $y = 0$, $x = y$, $x = 0$, respectively. Our three sets of solutions correspond to the fundamental systems in the neighbourhood of respectively $\tan\varphi = 0$ ("near" $y = 0$), $\tan\varphi = 1$ ("near" $x = y$), and $\tan\varphi = \infty$ ("near" $x = 0$). When passing from one hypercone to another, we pass from the neighbourhood of one singularity in φ to the neighbourhood of another, and hence from one fundamental system (p_0, q_0) to another.

The comparison thus shows that the representation by definite integrals leads more directly to explicit formulae and indicates the structure of the solutions more clearly,

18. In yet another respect the solution of differential equations by definite integrals is of great advantage: it discovers and elucidates connections between various systems of differential equations. Compare, for instance, (7) and (17), and for the sake of comparison replace u by $-u$ in the latter. (7) represents the general solution of the system of linear partial differential equations associated with F_1, and (17), with an arbitrary contour of integration and an arbitrary constant factor, the general solution of the system of linear partial differential equations associated with G_2. Now, (7) and (17) are indentical, except for the notation, and this suggests at once the equivalence of the system of F_1 with the system of G_2.

Moreover, the comparison of (7) with (17) suggests a transformation which will transform (1) into the system of G_2. It is

$$A = \beta,\ A' = \beta',\ B = \beta' - \gamma + 1,\ B' = \alpha - \beta',\ X = -x,\ Y = -\frac{1}{y},\ Z = y^{\beta'} z, \tag{23}$$

and it changes (1) into

$$\begin{gathered} X(1+X)R - Y(1+X)S + \{1 - B + (A + B' + 1)X\}P - A\,Y\,Q + AB'Z = 0 \\ -X(1+Y)S + Y(1+Y)T - A'XP + \{1 - B' + (A' + B + 1)Y\}Q + A'BZ = 0, \end{gathered} \tag{24}$$

where the symbols $P = \frac{\partial Z}{\partial X}$ etc. have the obvious meanings. Now, (24) is exactly the system of $G_2(A, A', B, B'; X, Y)$ as given by Horn (1931, p. 405) except that B and B' are interchanged, thus correcting a misprint in Horn's paper. (23) is, of course, not the only transformation of (1) into (24). Counting also the interchanging of β, x with β', y, there are in point of fact 120 transformations of

(1) into (24), corresponding to the 120 transformations of (1) into a similar system and an equal number of transformations of (24) into a system of the same form.

Horn (1931, p. 391) attempted the integration of (24) and found two series which are obviously of the form (3), but failed to draw any conclusions from his result. Borngässer (1932, pp. 32, 33) found four solutions of the form (3) and remarks that his results indicate a certain connection between F_1 and G_2. The conjectured connection exists, of course, and follows immediately from our integrals, for relations like

$$[0;\ 1,\ x] = c_1\,[0;\ 1] + c_2\,[1;\ x] + c_3\,[0;\ x]$$

express G_2 as a linear combination of three functions F_1. Instead of the connection between F_1 and G_2 it is more fruitful to consider the relationship between the systems of partial differential equations associated with these functions, and we have seen that our integrals suggest, and the substitution (23) verifies in an elementary manner, complete equivalence, not only a certain connection. The theory of (1) settles finally and completely the integration of (24) too.

Our results indicate also connections between (1) and certain particular cases of the systems of partial differential equations satisfied respectively by F_2, F_3 and H_2. The connection between the systems of F_1 and F_2 is well known (Appell and Kampé de Fériet, 1926, § XVII), and the systems of F_3 and H_2 are known to be equivalent to that of F_2 (ibid. § XIII and Borngässer, 1932, pp. 33 and 34).

Furthermore, there are several hypergeometric systems of partial differential equations the integration of which can be reduced to the integration of particular cases of (1), that is to (1) with particular values of the parameters. In the rest of this paper these systems are enumerated and reduced to (1). The results are mostly new and the ease with which they follow is another indication of the power of our method, and of its appositeness in this problem.

The System of G_1.

19. Horn's series (1931, p. 383)

$$G_1(\alpha, \beta, \beta'; x, y) = \sum\sum (\alpha)_{m+n}\,(\beta)_{n-m}\,(\beta')_{m-n}\,\frac{x^m}{m!}\,\frac{y^n}{n!} \tag{25}$$

satisfies the system of partial differential equations (Borngässer, 1931, p. 47)

$$L_1[z] \equiv x(x+1)r - ys - y^2 t + \{(\alpha+\beta'+1)x + 1 - \beta\} p + (\beta' - \alpha - 1) y q + \alpha \beta' z = 0$$

$$\tag{26}$$

$$L_2[z] \equiv -x^2 r - xs + y(y+1)t + (\beta - \alpha - 1) xp + \{(\alpha+\beta+1)y + 1 - \beta'\} q + \alpha\beta z = 0.$$

An integral respresentation of G_1 is

$$G_1(\alpha, \beta, \beta'; x, y = \frac{\Gamma(1-\beta)\,\Gamma(1-\beta')\,\Gamma(\beta+\beta')}{(2\pi i)^2} \times$$

$$\times \int\limits_C u^{\beta-1}(1-u)^{\beta'-1}\left(1 - \frac{1-u}{u}x - \frac{u}{1-u}y\right)^{-\alpha} du \tag{27}$$

where C is a double loop $(0;\,1)$ slung around $u=0$ and $u=1$ so that $\left|\frac{1-u}{u}x\right| + $ $+ \left|\frac{u}{1-u}y\right| < 1$ everywhere on C. (27) can be proved by expanding the integrand in powers of x and y and integrating term by term.

The integral representation suggests the solution of the differential equations (26) by integrals of the form

$$z = \int u^{\alpha+\beta-1}(1-u)^{\alpha+\beta'-1}\,U^{-\alpha}\,du \quad \text{where} \quad U \equiv u(1-u) - (1-u)^2 x - u^2 y. \tag{28}$$

Substituting this integral in the first equation (26) we obtain

$$L_1[z] = \int u^{\alpha+\beta-1}(u-1)^{\alpha+\beta'-1}\,U^{-\alpha-2}\,[\alpha(\alpha+1)\{(u-1)^4 x(x+1) - u^2(u-1)^2 y - u^4 y^2\}$$
$$+ \alpha\{(1-\beta+(\alpha+\beta'+1)x)(u-1)^2 + (\beta'-\alpha-1)u^2 y\}\,U + \alpha\beta'\,U^2]\,du$$

which gives after some reduction

$$L_1[z] = -\alpha\int u^{\alpha+\beta-1}(u-1)^{\alpha+\beta'}\,U^{-\alpha-2}\,[(\alpha+1)\{(u-1)^2 x - u^2 y\} +$$
$$+ \{\beta' u + (1-\beta)(1-u)\}\,U]\,du$$

$$= \alpha\int u^{\alpha+\beta-1}(u-1)^{\alpha+\beta'}\,U^{-\alpha-2}\left[(\alpha+1)u(u-1)\frac{\partial U}{\partial u} + \right.$$
$$\left. + (\alpha+\beta)(1-u)\,U - (\alpha+\beta'+1)u\,U\right] du$$

$$= -\alpha\int\frac{\partial}{\partial u}\{u^{\alpha+\beta}(u-1)^{\alpha+\beta'+1}\,U^{-\alpha-1}\}\,du.$$

Similarly,

$$L_2[z] = -\alpha\int\frac{\partial}{\partial u}\{u^{\alpha+\beta+1}(u-1)^{\alpha+\beta'}\,U^{-\alpha-1}\}\,du.$$

It is thus seen that (28) is a solution of (26) whenever the integration is taken over a closed contour (closed that is to say on the Riemann surface of the integrand) or else over a contour equivalent to such a closed contour.

20. The integral (28) is of the type (7), thereby indicating a connexion between the systems of partial differential equations (26) and (1). To exhibit this connection more clearly, let us denote by ξ and η the two roots of the quadratic equation $U = 0$ in u. Clearly,

$$\xi = \frac{1 + 2x + (1 - 4xy)^{\frac{1}{2}}}{2(1 + x + y)}, \quad \eta = \frac{1 + 2x - (1 - 4xy)^{\frac{1}{2}}}{2(1 + x + y)}$$

where the square root is defined uniquely for instance by the convention that it reduces to unity when $xy = 0$, and that it is a continuous function of xy. We now have

$$U = -(1 + x + y)(u - \xi)(u - \eta)$$

and hence (28) changes into

$$\{-(1 + x + y)\}^{\alpha} z = \int u^{\alpha+\beta-1}(u - 1)^{\alpha+\beta'-1}(u - \xi)^{-\alpha}(u - \eta)^{-\alpha}\, du.$$

This is, except for the notation, identical with (7) and gives rise to the theorem:

If z is a solution of (26) then $(1 + x + y)^{\alpha} z$ regarded as a function of ξ and η satisfies the system of partial differential equations associated with

$$F_1(1 - \beta - \beta', \alpha, \alpha, \alpha - \beta + 1; \xi, \eta).$$

The reduction thus effected of the integration of (26) to that of (1) is of some importance. Hitherto the integration of (26) presented great difficulties owing to the point of contact at $x = y = -\frac{1}{2}$ of the two singular manifolds $x + y + 1 = 0$ and $4xy - 1 = 0$ of (26). This difficulty has now been removed, for the transformation of x and y into ξ and η transforms the singular point $x = y = -\frac{1}{2}$ into the point $\xi = \eta = \infty$ where three singular manifolds $\xi = \infty$, $\eta = \infty$, $\xi = \eta$ intersect without having a common tangent. Thus the transformation suggested by our contour integral dissolves the point of contact (that is exceptional intersection) of two singular manifolds into an ordinary intersection of three singular manifolds. The integration of the system (1) has been accomplished in the earlier sections of this paper and in particular the fundamental systems belonging to the singular point (∞, ∞) have been discussed in section 10.

Another interesting consequence of our theorem is the transformation

$$G_1(\alpha,\beta,\beta';x,y) = (1+x+y)^{-\alpha} \times$$
$$\times F_2\left(1-\beta-\beta', \alpha, \alpha, 1-\beta, 1-\beta'; \frac{1+2x-(1-4xy)^{\frac{1}{2}}}{2(1+x+y)}, \frac{1+2y-(1-4xy)^{\frac{1}{2}}}{2(1+x+y)}\right) \tag{29}$$

which arises from the following simple consideration. There is only one solution of (26) which is regular at $x=y=0$, viz. G_1 itself. Now, $x=y=0$ corresponds to $\xi=1, \eta=0$, and there is only one solution of the system of $F_1(1-\beta-\beta', \alpha, \alpha, \alpha-\beta+1; \xi, \eta)$ regular at $\xi=1$, $\eta=0$ (Borngässer, 1932, p. 41), viz. the F_2-function on the right hand side of (29) which corresponds to $[\infty; 0, \eta]$ in the notation of section 10 (cf. also section 16). Hence the two sides of (29) are equal except possibly for a constant factor which turns out to be unity on putting $x=y=0$.

The System of G_3.

21. Horn's series (1931, p. 383)

$$G_3(\alpha,\alpha';x,y) = \sum\sum \frac{(\alpha)_{2n-m}(\alpha')_{2m-n}}{m!\ n!} x^m y^n \tag{30}$$

satisfies the system of partial differential equations (Borngässer, 1932, p. 36)

$$\begin{aligned} & x(4x+1)r-(4x+2)ys+y^2t+\{(4\alpha'+6)x+1-\alpha\}p-2\alpha'yq+\alpha'(\alpha'+1)z=0 \\ & x^2r-x(4y+2)s+y(4y+1)t-2\alpha xp+\{(4\alpha+6)x+1-\alpha'\}q+\alpha(\alpha+1)z=0. \end{aligned} \tag{31}$$

The integration of this system presented hitherto considerable difficulties owing to the presence of a cusp, at $x=y=-\frac{1}{3}$, of its singular manifold $h\equiv 27x^2y^2-18xy-4x-4y-1=0$. There is also an apparent singular manifold $f\equiv 15xy+4x+4y+1=0$ which intersects $h=0$ at the cusp. We shall see that in this case as in the previous one the contour integral method is effective in removing the difficulties.

G_3 is represented by an integral of the form

$$z = \int u^{2\alpha+\alpha'-1}(1-u)^{\alpha+2\alpha'-1}\, U^{-\alpha-\alpha'}\, du \tag{32}$$

where

$$U = u(1-u)-(1-u)^3x-u^3y,$$

and in the same way as in former cases it can be proved that (32), multiplied by an arbitrary constant and extended over any contour closed on the Riemann surface of the integrand, is a solution of (31). Yet this case is different from any of the previously discussed cases in that (32) does not represent the general solution of (31). In fact, since $u = \infty$ is a regular point of the integrand, the integrand of (32) has only five singularities, and consequently there are only three linearly independent integrals (32), whereas (31) is known to possess four linearly independent solutions.

The reason for this discrepancy is of course a particular solution of (31) which cannot be represented by an integral of the form (32). It is not difficult to trace this solution: it is $x^\varrho y^\sigma$ where $\varrho = -\frac{1}{3}(\alpha + 2\alpha')$ and $\sigma = -\frac{1}{3}(2\alpha + \alpha')$. The general solution of (31) is a linear combination of $x^\varrho y^\sigma$ and of three linearly independent integrals of the form (32).

The occurrence of the elementary solution $x^\varrho y^\sigma$ of the hypergeometric system (31) is rather curious, and I do not know of any case in which such solutions have been detected previously.

22. It is easily seen that (32) is of the type (7). In fact, let us denote the roots of the cubic equation $U = 0$ in u by ξ, η and ζ so that

$$U = (x - y)(u - \xi)(u - \eta)(u - \zeta).$$

With this notation (32) may be written in the form

$$z = (x - y)^{-\alpha-\alpha'} \int u^{2\alpha+\alpha'-1}(1 - u)^{\alpha+2\alpha'-1}\{(u - \xi)(u - \eta)(u - \zeta)\}^{-\alpha-\alpha'}\, du,$$

and if we put

$$u = \frac{v\zeta}{v + \zeta - 1}, \quad X = \xi\frac{\zeta - 1}{\zeta - \xi}, \quad Y = \eta\frac{\zeta - 1}{\zeta - \eta},$$

the last integral transforms into

$$z = \{(x - y)(\zeta - \xi)(\zeta - \eta)\}^{-\alpha-\alpha'}\zeta^{\alpha}(\zeta - 1)^{\alpha'} \times$$
$$\times \int v^{2\alpha+\alpha'-1}(1 - v)^{\alpha+2\alpha'-1}\{(v - X)(v - Y)\}^{-\alpha-\alpha'}\, dv;$$

and the integral appearing here is clearly of the form (7) and gives rise to the theorem:

If z is a solution of (31) then there exists a constant c such that $(z - c\,x^\varrho y^\sigma)\cdot$ $\cdot\{(x - y)(\zeta - \xi)(\zeta - \eta)\}^{\alpha+\alpha'}\zeta^{-\alpha}(\zeta - 1)^{-\alpha'}$ regarded as a function of X and Y satis-

fies the system of partial differential equations associated with

$$F_1(1-\alpha-\alpha', \alpha+\alpha', \alpha+\alpha', \alpha'+1; X, Y).$$

Broadly speaking, this theorem reduces the integration of the system (31) to the integration of (1) and also indicates a certain connexion between (31) and (26). The system of G_3 has interesting features such as the cusp on its singular manifold and the presence of an apparent singular manifold and these features would justify a more detailed investigation of (31). However, the purpose of the present paper is merely to point out the connexion between the system of G_3 and various other hypergeometric systems and a thorough study of (31) must be left for a future occasion.

The Systems of H_6 and H_3.

23. Horn's series (1931, p. 383)

$$H_6(\alpha, \beta, \gamma; x, y) = \sum\sum \frac{(\alpha)_{2m-n}(\beta)_{n-m}(\gamma)_n}{m!\, n!} x^m y^n \tag{33}$$

satisfies the system of partial differential equations

$$\begin{aligned} &x(4x+1)r-(4x+1)ys+y^2t+\{(4\alpha+6)x+1-\beta\}p-2\alpha y q+\alpha(\alpha+1)z=0 \\ &-x(y+2)s+y(y+1)t-\gamma x p+\{(\beta+\gamma+1)y+1-\alpha\}q+\beta\gamma z=0. \end{aligned} \tag{34}$$

The integration of this system presented hitherto considerable difficulties owing to a point of contact at $x=-\frac{1}{4}$, $y=-2$ of the two singular manifolds $4x+1=0$ and $xy^2-y-1=0$. We shall see that in this case too the contour integral method helps to surmount the difficulties.

(33) has the integral representation

$$\begin{aligned} H_6(\alpha, \beta, \gamma; x, y) = \Gamma(1-\alpha)\,\Gamma(1-\beta)\,\Gamma(\alpha+\beta)\times \\ \times(2\pi i)^{-2}\int_C t^{\alpha-1}(1-t)^{\beta-1}\left(1-\frac{t^2x}{1-t}\right)^{-\alpha-\beta}\left(1-\frac{1-t}{t}y\right)^{-\gamma}dt \end{aligned} \tag{35}$$

where C is a double loop encircling $t=0$ and $t=1$ so that $t=\dfrac{y}{y+1}$ and $t=\dfrac{-1+(4x+1)^{\frac{1}{2}}}{2x}$ are inside the double loop while $t=\dfrac{-1-(4x+1)^{\frac{1}{2}}}{2x}$ is outside of it. With $t = \frac{y u}{y u + 1}$, (35) suggests the integral

$$z = y^{\alpha} \int u^{\alpha+\gamma-1} (u - 1)^{-\gamma} (1 + y u - x y^2 u^2)^{-\alpha-\beta}\, d u \tag{36}$$

which in fact can be shown to satisfy (34) whenever the path of integration is a contour closed on the Riemann surface of the integrand or else a path derived from such a closed contour.

As in the case of G_3, the system of partial differential equations (34) has an elementary solution which does not admit of a representation by a contour integral (36), and the general solution of (34) is in fact a linear combination of $x^{-\alpha-\beta} y^{-\alpha-2\beta}$ and of three linearly independent integrals of the form (36).

24. Let ξ and η be the roots of the quadratic equation $1 + y u - x y^2 u^2 = 0$ in u so that

$$1 + y u - x y^2 u^2 = - x y^2 (u - \xi)(u - \eta),$$

and (36) becomes

$$z = x^{-\alpha-\beta} y^{-\alpha-2\beta} \int u^{\alpha+\gamma-1} (u - 1)^{-\gamma} \{(u - \xi)(u - \eta)\}^{-\alpha-\beta}\, d u.$$

Comparison of this integral with (7) leads to the result:

For any solution z of (34) there exists a constant c such that $x^{\alpha+\beta} y^{\alpha+2\beta} z - c$ regarded as a function of ξ and η satisfies the system of partial differential equations associated with $F_1(\alpha + 2\beta, \alpha + \beta, \alpha + \beta, \alpha + 2\beta - \gamma + 1; \xi, \eta)$.

This theorem reduces the integration of (34) to that of (1) and also indicates a connection between the systems of H_6 and G_1.

25. The system of Horn's series (1931, p. 383)

$$H_3(\alpha, \beta, \gamma; x, y) = \sum \sum \frac{(\alpha)_{2m+n} (\beta)_n}{(\gamma)_{m+n}\, m!\, n!} \tag{37}$$

needs no special discussion. It is easy to see that one of the fundamental solutions of (34) in the neighbourhood of $x = y = 0$ is

$$x^{\beta} H_3(\alpha + 2\beta, \gamma, \beta + 1; -x, x y)$$

so that the systems of H_3 and H_6 are equivalent. In fact, *If z is a solution of the system associated with $H_3(\alpha, \beta, \gamma; x, y)$ then $x^{\gamma-1} z$ is a solution of the system associated with $H_6\left(\alpha-2\gamma+2, \gamma-1, \beta; -x, -\frac{y}{x}\right)$ and vice versa.*

Combining this result with that of the preceding section, we have the following result:

For any solution z of the system of partial differential equations associated with $H_3(\alpha, \beta, \gamma; x, y)$ there exists a constant c such that $y^\alpha z - c$ regarded as a function of the roots ξ and η of the quadratic equation $x - yu + y^2 u^2 = 0$ in u satisfies the system of partial differential equations associated with $F_1(\alpha, \alpha-\gamma+1, \alpha-\gamma+1, \alpha-\beta+1; \xi, \eta)$.

Confluent Hypergeometric Functions.

26. All the hypergeometric series discussed so far are complete hypergeometric series of the second order in two variables, in the terminology of Appell and Kampé de Fériet (1926, Chapter IX). Beside these there are also confluent hypergeometric series of the second order in two variables, and it is well known that the confluent series are limiting cases of the complete series. Clearly, a full theory of the solutions of (1) contains, in nuce, the features of the solutions of any system of partial differential equations which is a limiting case of (1) or of any system which has been shown to be reducible to (1). For this reason we shall merely enumerate the confluent hypergeometric series which are so connected with (1). The solution of the systems of partial differential equations associated with any of these series by contour integrals should present no difficulties, it being understood that the limiting procedure which leads to the confluent series may change the Eulerian integrals into Laplace integrals.

Leaving apart series which reduce to elementary functions or to hypergeometric series of one variable, we find that the seven confluent hypergeometric series of the second order in two variables

$$\Phi_1, \Phi_2, \Phi_3, \Gamma_1, \Gamma_2, H_6, H_8 \tag{38}$$

introduced by Humbert (Φ) and Horn (Γ, H) respectively are limiting cases of one of the series F_1, G_2, H_3, H_6 (cf. Appell and Kampé de Fériet, 1926, pp. 124, 125 for the Φ and Borngässer, 1932, pp. 19 and 20 for the Γ and the H). Clearly, then, the system of partial differential equations associated with any of the functions (38) will be the corresponding limiting case of the system of one of the functions F_1, G_2, H_3, H_6. Now, the integration of the systems of the last four functions all depend on the integration of (1) and so the integration of the system of partial differential equations associated with any of the functions (38) will depend on the integration of a limiting case of (1). There are also several interconnecxions between the systems of the functions (38) themselves: so for instance the systems of Φ_1, Φ_2 and Γ_2 are equivalent to each other.

27. Summing up, we see that the theory developed in sections 8 to 15 settles the integration of the system of partial differential equations associated with any of the functions

$$F_1,\ G_1,\ G_2,\ G_3,\ H_3,\ H_6,\ \Phi_1,\ \Phi_2,\ \Phi_3,\ \Gamma_1,\ \Gamma_2,\ H_6,\ H_8,$$

that is thirteen out of the thirty-four hypergeometric systems of the second order in two variables. The reduction to (1) of these thirteen systems is valid for arbitrary values of the parameters. Besides, the integration of all other hypergeometric systems of the second order and in two variables (with the only possible exception of H_5) can be reduced to the integration of (1) provided that the parameters appearing in those systems are suitably specialised (that is satisfy one or two relations).

Most of the hypergeometric systems of the second order in two variables have four linearly independent integrals. There are, however, eight of them (Borngässer, 1932, p. 9) which have only three linearly independent integrals: they are the systems associated with the series

$$F_1,\ G_1,\ G_2,\ \Phi_1,\ \Phi_2,\ \Phi_3,\ \Gamma_1 \text{ and } \Gamma_2. \tag{39}$$

Now, all the eight series (39) are among those whose systems are reducible to (1), and hence we have the general result:

Any hypergeometric system of partial differential equations of the second order in two independent variables which has only three linearly independent integrals can be transformed into (1) *or into a particular or a limiting case of* (1).

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