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Dolbeault cohomology of a loop space

by

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0. Introduction

Loop spaces LM of compact complex manifolds M promise to have rich analytic cohomology theories, and it is expected that sheaf and Dolbeault cohomology groups of LM will shed new light on the complex geometry and analysis of M itself. This idea first occurs in [W], in the context of the infinite-dimensional Dirac operator, and then in [HBJ] that touches upon Dolbeault groups of loop spaces; but in all this, both works stay heuristic. Our goal here is rigorously to compute the Dolbeault group $H^{0,1}$ of the first interesting loop space, that of the Riemann sphere $\mathbf{P}_1$. The consideration of $H^{0,1}(L\mathbf{P}_1)$ was directly motivated by [MZ], that among other things features a curious line bundle on $L\mathbf{P}_1$. More recently, the second author classified in [Z] all holomorphic line bundles on $L\mathbf{P}_1$ that are invariant under a certain group of holomorphic automorphisms of $L\mathbf{P}_1$—a problem closely related to describing (a certain subspace of) $H^{0,1}(L\mathbf{P}_1)$. One noteworthy fact that emerges from the present research is that analytic cohomology of loop spaces, unlike topological cohomology (cf. [P, Theorem 13.14]), is rather sensitive to the regularity of loops admitted in the space. Another fact concerns local functionals, a notion from theoretical physics. Roughly, if M is a manifold, a local functional on a space of loops $x: S^1 \to M$ is a functional of form

$$f(x) = \int_{S^1} \Phi(t, x(t), \dot{x}(t), \ddot{x}(t), \dots)\, dt,$$

where Φ is a function on $S^1 \times$ an appropriate jet bundle of M. It turns out that all cohomology classes in $H^{0,1}(L\mathbf{P}_1)$ are given by local functionals. Nonlocal cohomology classes exist only perturbatively, i.e., in a neighborhood of constant loops in $L\mathbf{P}_1$; but none of them extends to the whole of $L\mathbf{P}_1$.

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We fix a smoothness class C^k, $k=1,2,...,\infty$, or Sobolev class $W^{k,p}$, $k=1,2,...$, $1\leqslant p<\infty$. If M is a finite-dimensional complex manifold, consider the space $LM=L_kM$, or $L_{k,p}M$, of maps $S^1=\mathbf{R}/\mathbf{Z}\to M$ of the given regularity. These spaces are complex manifolds modeled on a Banach space, except for $L_\infty M$, which is modeled on a Fréchet space. We shall focus on the loop space(s) $L\mathbf{P}_1$. As on any complex manifold, one can consider the space $C^\infty_{r,q}(L\mathbf{P}_1)$ of smooth (r,q)-forms, the operators

$$\bar\partial_{r,q}\colon C^\infty_{r,q}(L\mathbf{P}_1)\to C^\infty_{r,q+1}(L\mathbf{P}_1)$$

and the associated Dolbeault groups

$$H^{r,q}(L\mathbf{P}_1)=\operatorname{Ker}\bar\partial_{r,q}/\operatorname{Im}\bar\partial_{r,q-1};$$

for all this, see e.g. [L1] and [L2]. On the other hand, let $\mathfrak{F}$ be the space of holomorphic functions $F\colon \mathbf{C}\times L\mathbf{C}\to\mathbf{C}$ that have the following properties:

(1) $F(\zeta/\lambda,\lambda^2y)=O(\lambda^2)$ as $\mathbf{C}\ni\lambda\to 0$;

(2) $F(\zeta,x+y)=F(\zeta,x)+F(\zeta,y)$ if $\operatorname{supp}x\cap\operatorname{supp}y=\varnothing$;

(3) $F(\zeta,y+\mathrm{const})=F(\zeta,y)$.

As we shall see, the additivity property (2) implies that $F(\zeta,y)$ is local in y.

THEOREM 0.1. $H^{0,1}(L\mathbf{P}_1)\approx\mathbf{C}\oplus\mathfrak{F}$.

In the case of $L_\infty\mathbf{P}_1$, examples of $F\in\mathfrak{F}$ are

$$F(\zeta,y)=\zeta^\nu\left\langle\Phi,\prod_{j=0}^m y^{(d_j)}\right\rangle,\qquad(0.1)$$

where Φ is a distribution on S^1, $y^{(d)}$ denotes dth derivative, each $d_j\geqslant d_0=1$ and $0\leqslant\nu\leqslant 2m$. A general function in $\mathfrak{F}$ can be approximated by linear combinations of functions of form (0.1), see Theorem 1.5.

On any, possibly infinite-dimensional, complex manifold X, the space $C^\infty_{r,q}(X)$ can be given the compact-C^∞ topology as follows. First, the compact-open topology on $C^\infty_{0,0}(X)=C^\infty(X)$ is generated by C^0-seminorms $\|f\|_K=\sup_K|f|$ for all compact $K\subset X$. The family of C^ν-seminorms is defined inductively: each $C^{\nu-1}$-seminorm $\|\cdot\|$ on $C^\infty(TX)$ induces a C^ν-seminorm $\|f\|'=\|df\|$ on $C^\infty(X)$. The collection of all C^ν-seminorms, $\nu=0,1,...$, defines the compact-C^∞ topology on $C^\infty(X)$. The compact-C^∞ topology on a general $C^\infty_{r,q}(X)$ is induced by the embedding $C^\infty_{r,q}(X)\subset C^\infty(\bigoplus^{r+q}TX)$. With this topology $C^\infty_{r,q}(X)$ is a separated locally convex vector space, complete if X is first countable. The quotient space $H^{r,q}(X)$ inherits a locally convex topology, not necessarily separated. We note that on the subspace $\mathcal{O}(X)\subset C^\infty(X)$ of holomorphic functions, the compact-C^∞ topology restricts to the compact-open topology. The isomorphism in Theorem 0.1 is topological; it is also equivariant with respect to the obvious actions of the group of C^k-diffeomorphisms of S^1.

There is another group, the group $G\approx\mathrm{PSL}(2,\mathbf{C})$ of holomorphic automorphisms of $\mathbf{P}_1$, whose holomorphic action on $L\mathbf{P}_1$ (by postcomposition) and on $H^{0,1}(L\mathbf{P}_1)$ will be of greater concern to us. Theorems 0.2–0.4 below will describe the structure of $H^{0,1}(L\mathbf{P}_1)$ as a G-module. Recall that any irreducible (always holomorphic) G-module is isomorphic, for some $n=0,1,\ldots$, to the space $\mathfrak{K}_n$ of holomorphic differentials $\psi(\zeta)(d\zeta)^{-n}$ of order $-n$ on $\mathbf{P}_1$; here ψ is a polynomial, $\deg\psi\leqslant 2n$ and G acts by pullback. (For this, see [BD, pp. 84–86], and note that the subgroup $\approx\mathrm{SO}(3)$ formed by $g\in G$ that preserve the Fubini–Study metric is a maximally real submanifold; hence the holomorphic representation theory of G agrees with the representation theory of $\mathrm{SO}(3)$.) The nth isotypical subspace of a G-module V is the sum of all irreducible submodules isomorphic to $\mathfrak{K}_n$. In particular, the 0th isotypical subspace is the space V^G of fixed vectors.

THEOREM 0.2. *If $n\geqslant 1$, the n-th isotypical subspace of $H^{0,1}(L_\infty\mathbf{P}_1)$ is isomorphic to the space $\mathfrak{F}^n$ spanned by functions of form* (0.1), *with $m=n$.*

The isomorphism above is that of locally convex spaces, as $\mathfrak{F}$ or $\mathfrak{F}^n$ have not been endowed with an action of G yet. But in §2 they will be, and we shall see that the isomorphism in question is a G-morphism.—The fixed subspace of $H^{0,1}(L\mathbf{P}_1)$ can be described more explicitly, for any loop space:

THEOREM 0.3. *The space $H^{0,1}(L\mathbf{P}_1)^G$ is isomorphic to the space $C^{k-1}(S^1)^*$ (resp. $W^{k-1,p}(S^1)^*$) if the dual spaces are endowed with the compact-open topology.*

The isomorphisms in Theorem 0.3 are not $\mathrm{Diff}\,S^1$-equivariant. To remedy this, one is led to introduce the spaces $C^l_r(S^1)$ (resp. $W^{l,p}_r(S^1)$) of differentials $y(t)(dt)^r$ of order r on S^1, of the corresponding regularity; $L^p_r=W^{0,p}_r$. Then $H^{0,1}(L\mathbf{P}_1)^G$ will be $\mathrm{Diff}\,S^1$-equivariantly isomorphic to $C^{k-1}_1(S^1)^*$ (resp. $W^{k-1,p}_1(S^1)^*$).

For low-regularity loop spaces one can very concretely represent all of $H^{0,1}(L\mathbf{P}_1)$:

THEOREM 0.4. (a) *If $1\leqslant p<2$, all of $H^{0,1}(L_{1,p}\mathbf{P}_1)$ is fixed by G. Hence it is isomorphic to $L^{p'}(S^1)$, with $p'=p/(p-1)$.*

(b) *If $1\leqslant p<\infty$ then $H^{0,1}(L_{1,p}\mathbf{P}_1)$ is isomorphic to*

$$\bigoplus_{n=0}^{p-1}\mathfrak{K}_n\otimes L^{p/(n+1)}_{n+1}(S^1)^*\approx\bigoplus_{n=0}^{p-1}\mathfrak{K}_n\otimes L^{p_n}_{-n}(S^1),\quad p_n=\frac{p}{p-1-n},$$

and so it is the sum of its first $[p]$ isotypical subspaces. Indeed, the isomorphisms above are $G\times\mathrm{Diff}\,S^1$-equivariant, G and $\mathrm{Diff}\,S^1$ respectively acting on one of the factors $\mathfrak{K}_n$ and L^q_r naturally, and trivially on the other.

Again, the dual spaces are endowed with the compact-open topology.

It follows that the infinite-dimensional space $H^{0,1}(L_{1,p}\mathbf{P}_1)$ can be understood in finite terms, if it is considered as a representation space of S^1. Here S^1 acts on itself (by translations), hence also on $L\mathbf{P}_1$ and on $H^{0,1}(L\mathbf{P}_1)$. One can read off from Theorem 0.4 that each irreducible representation of S^1 occurs in $H^{0,1}(L_{1,p}\mathbf{P}_1)$ with the same multiplicity $[p]^2$. On the other hand, for spaces of loops of regularity at least C^1, in $H^{0,1}(L\mathbf{P}_1)$ each irreducible representation of S^1 occurs with infinite multiplicity, and, somewhat contrary to earlier expectations, it is not possible to associate with this cohomology space even a formal character of S^1. This indicates that Dolbeault groups of general loop spaces LM should be studied as representations of Diff S^1 rather than S^1.

The structure of this paper is as follows. In §§ 1 and 2 we study the space $\mathfrak{F}$ as a G-module. Theorem 1.1 connects it with a similar but simpler space of functions that are required to satisfy only the first two of the three conditions defining $\mathfrak{F}$. This result will be needed in proving the isomorphism $H^{0,1}(L\mathbf{P}_1)\approx\mathbf{C}\oplus\mathfrak{F}$, and also in concretely representing elements of $\mathfrak{F}$. Further, we shall rely on Theorem 1.1 in identifying isotypical subspaces of $\mathfrak{F}$ (Theorems 2.1 and 2.2). This will then prove Theorems 0.2–0.4, modulo Theorem 0.1.

To prove Theorem 0.1, we shall cover $L\mathbf{P}_1$ with open sets

$$LU_a=\{x\in L\mathbf{P}_1 : a\notin x(S^1)\},\quad a\in\mathbf{P}_1,$$

each biholomorphic to $L\mathbf{C}$. Given a cohomology class $[f]\in H^{0,1}(L\mathbf{P}_1)$, represented by a closed $f\in C^\infty_{0,1}(L\mathbf{P}_1)$, we first solve the equation $\bar\partial u_a=f|_{LU_a}$, see §3. If an appropriate normalizing condition is imposed on the solution, u_a will be unique and depend *holomorphically* on a. At this point it is natural to introduce the Čech cocycle

$$\mathfrak{f}=(u_a-u_b : a,b\in\mathbf{P}_1)\in Z^1(\{LU_a : a\in\mathbf{P}_1\},\mathcal{O}). \tag{0.2}$$

It turns out that $\mathfrak{f}$ depends only on the class $[f]$, and the map $[f]\mapsto\mathfrak{f}$ is an isomorphism between $H^{0,1}(L\mathbf{P}_1)$ and a certain space $\mathfrak{H}$ of cocycles (Theorem 3.3).

In §4 we consider the infinitesimal version of (0.2). The function $\partial u_\zeta(x)/\partial\zeta$ is holomorphic in x and ζ, as long as $\zeta\notin x(S^1)$. We write it as

$$\frac{\partial u_\zeta(x)}{\partial\zeta}=F\left(\zeta,\frac{1}{\zeta-x}\right),\quad F\in\mathcal{O}(\mathbf{C}\times L\mathbf{C}),$$

and prove that F satisfies conditions (1), (2) and (3) above (Theorem 4.1). In §5 we prove that the map $H^{0,1}(L\mathbf{P}_1)\ni[f]\mapsto F\in\mathfrak{F}$ has a right inverse and its kernel is one-dimensional, whence Theorem 0.1 follows. In the final §6 we tie together loose ends, and also represent explicitly some Dolbeault classes in $H^{0,1}(L\mathbf{P}_1)$; for $W^{1,p}$ loop spaces with $1\leqslant p<2$, this amounts to a concrete map $L^p(S^1)^*\to C^\infty_{0,1}(L\mathbf{P}_1)$ that induces the isomorphism in Theorem 0.4 (a).

1. The space $\mathfrak{F}$

In this section and the next we shall study the structure of the space $\mathfrak{F}$, independently of any cohomological content. It will be convenient to allow k to be any integer (but only in this section!); when $k<0$, elements of $C^k(S^1)$ and $W^{k,p}(S^1)$ are distributions, locally equal to the $-k$th derivative of functions in $C(S^1)$ and $L^p(S^1)$, respectively. Let $L^-\mathbf{C}$ denote the space $C^{k-1}(S^1)$ (resp. $W^{k-1,p}(S^1)$). We shall write $L^{(-)}\mathbf{C}$ to mean either $L\mathbf{C}$ or $L^-\mathbf{C}$. Consider the space $\widetilde{\mathfrak{F}}$ of those $F\in\mathcal{O}(\mathbf{C}\times L^-\mathbf{C})$ that have properties (1) and (2) of the introduction. We shall refer to (2) as additivity. A function $F\in\mathcal{O}(\mathbf{C}\times L^{(-)}\mathbf{C})$ will be said to be posthomogeneous of degree m if $F(\zeta,\cdot)$ is homogeneous of degree m for all $\zeta\in\mathbf{C}$. Posthomogeneous degree endows the spaces $\mathfrak{F}$ and $\widetilde{\mathfrak{F}}$ with a grading.—All maps below, unless otherwise mentioned, will be continuous and linear.

THEOREM 1.1. *The graded linear map $\widetilde{\mathfrak{F}}\ni\widetilde{F}\mapsto F\in\mathfrak{F}$ given by $F(\zeta,y)=\widetilde{F}(\zeta,\dot{y})$ has a graded right inverse, and its kernel consists of functions $\widetilde{F}(\zeta,x)=\mathrm{const}\int_{S^1}x$.*

First we shall consider functions $E\in\mathfrak{F}$ (resp. $\widetilde{\mathfrak{F}}$) that are independent of ζ. We denote the space of these functions $\mathfrak{E}\subset\mathcal{O}(L\mathbf{C})$ (resp. $\widetilde{\mathfrak{E}}\subset\mathcal{O}(L^-\mathbf{C})$), graded by degree of homogeneity. Additivity of $E\in\mathcal{O}(L^{(-)}\mathbf{C})$ implies $E(0)=0$, which in turn implies property (1) of the introduction. Let

$$E=\sum_{m=1}^{\infty}E_m,\quad E_m(y)=\int_0^1 E(e^{2\pi i\tau}y)e^{-2m\pi i\tau}\,d\tau, \tag{1.1}$$

be the homogeneous expansion of a general $E\in\mathcal{O}(L^{(-)}\mathbf{C})$ vanishing at 0. Consider tensor powers $(L^{(-)}\mathbf{C})^{\otimes m}$ of the vector spaces $L^{(-)}\mathbf{C}$ over $\mathbf{C}$. In particular, $C^\infty(S^1)^{\otimes m}$ is an algebra, and a general $(L^{(-)}\mathbf{C})^{\otimes m}$ is a module over it. Each E_m in (1.1) induces a symmetric linear map

$$\mathcal{E}_m\colon (L^{(-)}\mathbf{C})^{\otimes m}\longrightarrow\mathbf{C},$$

called the polarization of E_m. On monomials, $\mathcal{E}_m$ is defined by

$$\mathcal{E}_m(y_1\otimes\ldots\otimes y_m)=\frac{1}{2^m m!}\sum_{\varepsilon_j=\pm1}\varepsilon_1\ldots\varepsilon_m E_m(\varepsilon_1y_1+\ldots+\varepsilon_my_m), \tag{1.2}$$

see e.g. [He, §2.2], and then extended by linearity. Thus $E_m(y)=\mathcal{E}_m(y^{\otimes m})$.—We shall call $w\in(L^{(-)}\mathbf{C})^{\otimes m}$ *degenerate* if it is a linear combination of monomials $y_1\otimes\ldots\otimes y_m$ with some $y_j=1$.

LEMMA 1.2. (a) *E is additive if and only if*

$$\mathcal{E}_m(y_1\otimes\ldots\otimes y_m)=0\quad \textit{whenever}\ \bigcap_{j=1}^{m}\operatorname{supp}y_j=\varnothing.$$

(b) *$E(y+\mathrm{const})=E(y)$ if and only if $\mathcal{E}_m(w)=0$ whenever w is degenerate.*

Proof. (a) Clearly E is additive precisely when all the E_m are, whence it suffices to prove the claim when E itself is homogeneous, of degree m, say. In this case $\mathcal{E}_n=0$, $n\neq m$. Denoting $\mathcal{E}_m$ by $\mathcal{E}$, it is also clear that the condition on $\mathcal{E}$ implies that E is additive. We show the converse by induction on m, the case $m=1$ being obvious. Let $x,y\in L^{(-)}\mathbf{C}$ have disjoint supports, so that

$$\mathcal{E}((x+y)^{\otimes m})=\mathcal{E}(x^{\otimes m})+\mathcal{E}(y^{\otimes m}). \tag{1.3}$$

Write λx for x and separate terms of different degrees in λ to find $\mathcal{E}(x\otimes...\otimes y)=0$, which settles the case $m=2$. Next, if we already know the claim when m is replaced by $m-1\geqslant 2$, take a $z\in L^{(-)}\mathbf{C}$ with $\operatorname{supp} y\cap\operatorname{supp} z=\varnothing$, and write $x+\lambda z$ for x in (1.3). Considering the terms linear in λ we obtain

$$\mathcal{E}(z\otimes(x+y)^{\otimes(m-1)})=\mathcal{E}(z\otimes x^{m-1})+\mathcal{E}(z\otimes y^{m-1}), \tag{1.4}$$

the last term being 0. The same will hold if $\operatorname{supp} x\cap\operatorname{supp} z=\varnothing$. Since any $z\in L^{(-)}\mathbf{C}$ can be written $z'+z''$ with the support of z' (resp. z'') disjoint from the support of x (resp. y), (1.4) in fact holds for all z. By the induction hypothesis applied to $\mathcal{E}(z\otimes\cdot)$,

$$\mathcal{E}(z\otimes y_2\otimes...\otimes y_m)=0, \quad \text{if } \bigcap_{j=2}^{m}\operatorname{supp} y_j=\varnothing.$$

Suppose now that $\bigcap_{j=1}^{m}\operatorname{supp} y_j=\varnothing$ and write $y_1=y'+y''$ with $y'=0$ near $\bigcap_{j\neq 2}\operatorname{supp} y_j$ and $y''=0$ near $\bigcap_{j\neq 3}\operatorname{supp} y_j$. Then

$$\mathcal{E}(y_1\otimes...\otimes y_m)=\mathcal{E}(y'\otimes...\otimes y_m)+\mathcal{E}(y''\otimes...\otimes y_m)=0.$$

(b) Again we assume that E is m-homogeneous, and again one implication is trivial. So assume that $\mathcal{E}((y+1)^{\otimes m})=\mathcal{E}(y^{\otimes m})$, where $\mathcal{E}=\mathcal{E}_m$. Differentiating both sides in the directions $y_2,...,y_m$ and setting $y=0$ we obtain $\mathcal{E}(1\otimes y_2\otimes...\otimes y_m)=0$, whence the claim follows.

PROPOSITION 1.3. *The graded map $\widetilde{\mathfrak{E}}\ni\widetilde{E}\mapsto E\in\mathfrak{E}$ given by $E(y)=\widetilde{E}(\dot{y})$ has a graded right inverse, and its kernel is spanned by $\widetilde{E}(x)=\int_{S^1}x$.*

We shall write $\int x$ for $\int_{S^1}x$.

Proof. (a) To identify the kernel, because of homogeneous expansions, it will suffice to deal with homogeneous $\widetilde{E}$. So assume that $\widetilde{E}\in\widetilde{\mathfrak{E}}$ is homogeneous of degree m and that $\widetilde{E}(\dot{y})=0$ for all $y\in L\mathbf{C}$. Its polarization $\tilde{\mathcal{E}}$ satisfies $\tilde{\mathcal{E}}(\dot{y}_1\otimes...\otimes\dot{y}_m)=0$. If $m=1$, this implies that $\widetilde{E}(x)=\text{const}\int x$, so from now on we assume that $m\geqslant 2$, and first we prove by induction that $\tilde{\mathcal{E}}(x_1\otimes\ldots\otimes x_m)=\text{const}\prod\int x_j$. Suppose that we already know this for $m-1$. Then

$$\tilde{\mathcal{E}}(\dot{y}\otimes x_2\otimes\ldots\otimes x_m)=c(\dot{y})\prod_{j=2}^{m}\int x_j.$$

With arbitrary $x_1\in L^-\mathbf{C}$ the function $x_1-\int x_1$ is of form $\dot{y}$, so $x_1=\dot{y}+\int x_1$ and

$$\tilde{\mathcal{E}}(x_1\otimes\ldots\otimes x_m)=l(x_1)\prod_{j=2}^{m}\int x_j+\tilde{\mathcal{E}}(1\otimes x_2\otimes\ldots\otimes x_m)\int x_1, \tag{1.5}$$

where $l(x_1)=c(x_1-\int x_1)$ is linear in x_1. If $\int x_1=0$ and $\operatorname{supp} x_1\neq S^1$, then we can choose $x_2,\ldots$ so that $\bigcap_{j=1}^{m}\operatorname{supp} x_j=\varnothing$ but $\int x_j\neq 0$, $j\geqslant 2$. This makes the left-hand side of (1.5) vanish by Lemma 1.2 (a), and gives $l(x_1)=0$. Since any $x_1\in L^-\mathbf{C}$ with $\int x_1=0$ can be written $x_1=x'+x''$ with $\int x'=\int x''=0$ and $\operatorname{supp} x', \operatorname{supp} x''\neq S^1$, it follows that $l(x_1)=0$ whenever $\int x_1=0$. Hence $l(x_1)=\text{const}\int x_1$. In particular, the first term on the right of (1.5) is symmetric in x_j. Therefore the second term must be symmetric too, which implies that this term is const $\prod_{j=1}^{m}\int x_j$. Thus $\widetilde{E}(x)=\text{const}(\int x)^m$.

Yet for $m\geqslant 2$, $\widetilde{E}(x)=\text{const}(\int x)^m$ is additive only if it is identically zero; so indeed $\widetilde{E}(x)=\text{const}\int x$, as claimed.

(b) To construct the right inverse, consider $E\in\mathfrak{E}$ with homogeneous expansion (1.1). We shall construct m-homogeneous polynomials $\widetilde{E}_m\in\widetilde{\mathfrak{E}}$ such that $E_m(y)=\widetilde{E}_m(\dot{y})$. Define $\widetilde{E}_1(x)=E_1(y)$, where y is chosen so that $\dot{y}=x-\int x$. Now assume $m\geqslant 2$. Let us say that an n-tuple of functions $\varrho_\nu: S^1\to\mathbf{C}$ is centered if $\bigcap_{\nu=1}^{n}\operatorname{supp}\varrho_\nu\neq\varnothing$. We start by fixing a C^∞ partition of unity $\sum_{\varrho\in P}\varrho=1$ on S^1 such that each $\operatorname{supp}\varrho$ is an arc of length less than $\frac{1}{4}$. This implies that $\bigcup_{\nu=1}^{n}\operatorname{supp}\varrho_\nu$ is an arc of length less than $\frac{1}{2}$ if $\varrho_1,\ldots,\varrho_n\in P$ are centered. Given $x\in L^-\mathbf{C}$, for each centered $R=(\varrho_1,\ldots,\varrho_n)$ in P construct $y_R\in L\mathbf{C}$ so that $\dot{y}_R=x$ on a neighborhood of $\bigcup_{\nu=1}^{n}\operatorname{supp}\varrho_\nu$, making sure that $y_R=y_Q$ if Q and R agree as sets. For noncentered n-tuples R in P let $y_R\in L\mathbf{C}$ be arbitrary. We shall refer to the y_R as local integrals.

If Q and R are centered tuples in P then

$$y_Q-y_R=c_{QR}=\text{const}\quad\text{on }\Big(\bigcup_{\varrho\in Q}\operatorname{supp}\varrho\Big)\cap\Big(\bigcup_{\varrho\in R}\operatorname{supp}\varrho\Big). \tag{1.6}$$

When the intersection in (1.6) is empty, or one of Q and R is noncentered, fix $c_{QR}\in\mathbf{C}$ arbitrarily. Define

$$v_{QR}=m\int_0^{c_{QR}}(y_R+\tau)^{\otimes(m-1)}\,d\tau\in(L\mathbf{C})^{\otimes(m-1)}, \tag{1.7}$$

and with the polarization $\mathcal{E}_m$ of E_m from (1.2) consider

$$\mathcal{E}_m\bigg(\sum_{R=(\varrho_1,\ldots,\varrho_m)}(\varrho_1\otimes\ldots\otimes\varrho_m)\Big(y_R^{\otimes m}+1\otimes\sum_{S=(\sigma_2,\ldots,\sigma_m)}(\sigma_2\otimes\ldots\otimes\sigma_m)v_{SR}\Big)\bigg); \tag{1.8}$$

we sum over all m-tuples R and $(m-1)$-tuples S in P. (We will not need it, but here is an explanation of (1.8). Say that tensors $w,w'\in L^{(-)}\mathbf{C}^{\otimes m}$ are *congruent*, $w\equiv w'$, if $w-w'$ is the sum of a degenerate tensor and of monomials $x_1\otimes...\otimes x_m$ with $\bigcap_j \operatorname{supp} x_j=\varnothing$. Denote by ∂^m the linear map $(L\mathbf{C})^{\otimes m}\to(L^-\mathbf{C})^{\otimes m}$ defined by $\partial^m(y_1\otimes...\otimes y_m)=\dot y_1\otimes...\otimes\dot y_m$. Then the symmetrization of the argument of $\mathcal{E}_m$ in (1.8) is a solution w of the congruence $\partial^m w\equiv x^{\otimes m}$, in fact it is the unique symmetric solution, up to congruence. It follows that for the $\widetilde{E}_m$ sought, $\widetilde{E}_m(x)$ must be equal to $\mathcal{E}_m(w)$, which, in turn, equals (1.8).)

We claim that the value in (1.8) depends only on x (and $\mathcal{E}_m$), but not on the partition of unity P and the local integrals y_R. Indeed, suppose first that the local integrals y_R are changed to $\hat y_R$, so that the c_{QR} change to $\hat c_{QR}$ and v_{QR} to $\hat v_{QR}$; but we do not change P. There are $c_R\in\mathbf{C}$ such that for all centered R,

$$\hat y_R=y_R+c_R \quad \text{on } \bigcup_{\varrho\in R}\operatorname{supp}\varrho.$$

Let

$$u_R=m\int_0^{c_R}(y_R+\tau)^{\otimes(m-1)}\,d\tau. \tag{1.9}$$

Clearly $\hat c_{QR}=c_{QR}+c_Q-c_R$ if $Q\cup R$ is centered. In this case one computes also

$$\begin{aligned}\frac{1}{m}\hat v_{QR}&=\int_0^{\hat c_{QR}}(\hat y_R+\tau)^{\otimes(m-1)}\,d\tau\\ &=\int_0^{c_{QR}}(\hat y_R-c_R+\tau)^{\otimes(m-1)}\,d\tau-\int_0^{c_R}(\hat y_R-c_R+\tau)^{\otimes(m-1)}\,d\tau\\ &\quad+\int_0^{c_Q}(\hat y_R-c_R+c_{QR}+\tau)^{\otimes(m-1)}\,d\tau.\end{aligned} \tag{1.10}$$

Because of Lemma 1.2 (a), in (1.8) only centered R, and such S that $R\cup S$ is centered, will contribute. When $y_R^{\otimes m}$ is changed to $\hat y_R^{\otimes m}$, the corresponding contributions change by

$$\begin{aligned}\sum_R\mathcal{E}_m\biggl(\int_0^{c_R}&(\varrho_1\otimes...\otimes\varrho_m)\frac{d}{d\tau}(y_R+\tau)^{\otimes m}\,dt\biggr)\\ &=\sum_R\mathcal{E}_m\biggl(\int_0^{c_R}(\varrho_1\otimes...\otimes\varrho_m)(m\otimes(y_R+\tau)^{\otimes(m-1)})\,d\tau\biggr)\\ &=\sum_R\mathcal{E}_m((\varrho_1\otimes...\otimes\varrho_m)(1\otimes u_R)).\end{aligned}$$

When v_{QR} is changed to $\hat v_{QR}$, in view of (1.10), (1.6) and (1.9), the contribution of the terms in the double sum in (1.8) changes by

$$\begin{aligned}\mathcal{E}_m\biggl((m\varrho_1\otimes\varrho_2\sigma_2\otimes...\otimes\varrho_m\sigma_m)\biggl(\int_0^{c_S}(y_S+\tau)^{\otimes(m-1)}\,d\tau-\int_0^{c_R}(y_R+\tau)^{\otimes(m-1)}\,d\tau\biggr)\biggr)&\\ =\mathcal{E}_m((\varrho_1\otimes\varrho_2\sigma_2\otimes...\otimes\varrho_m\sigma_m)(1\otimes u_S-1\otimes u_R)).&\end{aligned}$$

The net change in (1.8) is therefore

$$\mathcal{E}_m\Big(\sum_{R,S}(\varrho_1\otimes\varrho_2\sigma_2\otimes\ldots\otimes\varrho_m\sigma_m)(1\otimes u_S)\Big)=\mathcal{E}_m\Big(\sum_S(1\otimes\sigma_2\otimes\ldots\otimes\sigma_m)(1\otimes u_S)\Big)=0$$

by Lemma 1.2 (b), as needed.

Now to pass from P to another partition of unity P', introduce

$$\Pi=\{\varrho\varrho':\varrho\in P \text{ and } \varrho'\in P'\}.$$

One easily shows that P and Π give rise to the same value in (1.8), hence so do P and P'. Therefore (1.8) indeed depends only on x, and we define $\widetilde{E}_m(x)$ to be this value. We proceed to check that $\widetilde{E}_m$ has the required properties.

If $x=\dot{y}$ then all y_R can be chosen as y, and (1.8) gives $\widetilde{E}_m(\dot{y})=E_m(y)$. Next suppose that $x',x''\in L^-\mathbf{C}$ have disjoint supports, and $x=x'+x''$. If the supports of all $\varrho\in P$ are sufficiently small, then the local integrals y'_R and y''_R of x' and x'', respectively, can be chosen so that for each R one of them is 0. Hence the local integrals $y_R=y'_R+y''_R$ of x will satisfy $y_R^{\otimes m}=y_R'^{\otimes m}+y_R''^{\otimes m}$, whence $\widetilde{E}_m(x)=\widetilde{E}_m(x')+\widetilde{E}_m(x'')$ follows.

To show that $\sum_{m=1}^\infty\widetilde{E}_m$ is convergent and represents a holomorphic function, note that $\widetilde{E}_m(x)$ is the sum of terms

$$\begin{gathered}\mathcal{E}_m(\varrho_1y_R\otimes\ldots\otimes\varrho_my_R),\\ \int_0^1\mathcal{E}_m(\varrho_1c_{SR}\otimes\varrho_2\sigma_2(y_R+c_{SR}\tau)\otimes\ldots\otimes\varrho_m\sigma_m(y_R+c_{SR}\tau))\,d\tau\end{gathered}\tag{1.11}$$

(we have substituted $c_{QR}\tau$ for τ in (1.7)). Since $y_R\in L\mathbf{C}$ and $c_{QR}\in\mathbf{C}$ can be chosen to depend on x in a continuous linear way, each $\widetilde{E}_m$ is a homogeneous polynomial of degree m. Furthermore, let $K\subset L^-\mathbf{C}$ be compact. For each $x\in K$, $m\in\mathbf{N}$ and m-tuples Q and R in P, we can choose y_R and c_{QR} so that all the functions

$$\varrho c_{QR}\quad\text{and}\quad\varrho\varrho'(y_R+c_{QR}\tau),$$

$\varrho,\varrho'\in P$, $0\leqslant\tau\leqslant1$, belong to some compact $H\subset L\mathbf{C}$. By passing to the balanced hull, it can be assumed that H is balanced. If $\lambda>0$, (1.1) implies

$$\max_H|E_m|=\lambda^{-m}\max_{\lambda H}|E_m|\leqslant\lambda^{-m}\max_{\lambda H}|E|=A\lambda^{-m},$$

so that by (1.2),

$$|\mathcal{E}_m(z_1\otimes\ldots\otimes z_m)|\leqslant A\frac{m^m}{m!}\lambda^{-m}\leqslant A\Big(\frac{e}{\lambda}\Big)^m,$$

if each $z_\mu \in H$. Thus each term in (1.11) satisfies this estimate. If $|P|$ denotes the cardinality of P, we obtain, in view of (1.8),

$$\max_K |\widetilde{E}_m| \leqslant (|P|^m + m|P|^{2m-1}) A\left(\frac{e}{\lambda}\right)^m.$$

Choosing $|\lambda| > e|P|^2$ we conclude that $\sum_{m=1}^\infty \widetilde{E}_m$ uniformly converges on K, and, K being arbitrary, $\widetilde{E} = \sum_{m=1}^\infty \widetilde{E}_m$ is holomorphic. By what we have already proved for $\widetilde{E}_m$, $\widetilde{E} \in \widetilde{\mathfrak{E}}$ and $\widetilde{E}(\dot{y}) = E(y)$. The above estimates also show that the map $E \mapsto \widetilde{E}$ is continuous and linear, which completes the proof of Proposition 1.3.

Now consider an $F \in \mathcal{O}(\mathbf{C} \times L^{(-)}\mathbf{C})$ and its posthomogeneous expansion

$$F = \sum_{m=0}^\infty F_m, \quad F_m(\zeta, y) = \int_0^1 F(\zeta, e^{2\pi i \tau} y) e^{-2m\pi i \tau}\, d\tau. \tag{1.12}$$

PROPOSITION 1.4. *The function F satisfies condition* (1) *of the introduction if and only if each F_m is a polynomial in ζ, of degree $\leqslant 2m-2$ (in particular, $F_0 = 0$).*

Proof. As F satisfies (1) precisely when each F_m does, the statement is obvious.

Proof of Theorem 1.1. Apply Proposition 1.3 on each slice $\{\zeta\} \times L^{(-)}\mathbf{C}$. Accordingly, an $\widetilde{F}$ in the kernel is posthomogeneous of degree 1, hence, by Proposition 1.4, independent of ζ. Thus indeed $\widetilde{F}(\zeta, x) = \text{const} \int x$. Further, the slicewise right inverse applied to $F \in \mathfrak{F}$ produces an additive $\widetilde{F}$, which will be holomorphic on $\mathbf{C} \times L\mathbf{C}$, since the map $E \mapsto \widetilde{E}$ is continuous and linear. To see that $\widetilde{F}$ also verifies condition (1) of the introduction, expand F in a posthomogeneous series

$$F(\zeta, y) = \sum_{m=1}^\infty F_m(\zeta, y) = \sum_{m=1}^\infty \sum_{\nu=0}^{2m-2} \zeta^\nu E_{m\nu}(y), \tag{1.13}$$

by Proposition 1.4, so that

$$\widetilde{F}(\zeta, x) = \sum_{m=1}^\infty \sum_{\nu=0}^{2m-2} \zeta^\nu \widetilde{E}_{m\nu}(x),$$

with $\widetilde{E}_{m\nu}$ m-homogeneous. Again by Proposition 1.4, $\widetilde{F}$ satisfies condition (1), and so belongs to $\widetilde{\mathfrak{F}}$.

Theorem 1.1 can be used effectively to describe elements of the space $\mathfrak{F}$. With ulterior motives we switch notation $m = n+1$, and consider a homogeneous polynomial

$\widetilde{E}\in\mathcal{O}(L^-\mathbf{C})$ of degree $n+1\geqslant 1$. Its polarization $\mathcal{E}$ defines a distribution D on the torus $(S^1)^{n+1}=T$. Indeed, denote the coordinates on T by $t_j\in\mathbf{R}/\mathbf{Z}$ and set

$$\left\langle D, \prod_{j=0}^{n} e^{2\pi i\nu_j t_j}\right\rangle = \mathcal{E}(x_0\otimes\ldots\otimes x_n), \quad x_j(\tau)=e^{2\pi i\nu_j\tau},\ \nu_j\in\mathbf{Z}. \tag{1.14}$$

Since $\widetilde{E}$ is continuous,

$$|\mathcal{E}(x_0\otimes\ldots\otimes x_n)|\leqslant c\prod_{j=0}^{n}\|x_j\|_{C^q(S^1)} \quad \text{with some } c>0 \text{ and } q\in\mathbf{N}.$$

Hence (1.14) can be estimated, in absolute value, by $c'\prod_{j=0}^n(1+|\nu_j|)^q$, and it follows by Fourier expansion that D extends to a unique linear form on $C^\infty(T)$. Clearly, D is symmetric, i.e., invariant under permutation of the factors S^1 of T. Also,

$$\mathcal{E}(x_0\otimes\ldots\otimes x_n)=\langle D, x_0\otimes\ldots\otimes x_n\rangle, \tag{1.15}$$

if on the right $x_0\otimes\ldots\otimes x_n$ is identified with the function $\prod_{j=0}^n x_j(t_j)$.

Assume now that $\widetilde{E}\in\widetilde{\mathfrak{E}}$. Lemma 1.2 (a) implies that D is supported on the diagonal of T. The form of distributions supported on submanifolds is in general well understood; in the case at hand, e.g. [Hö, Theorem 2.3.5], gives that D is a finite sum of distributions of form

$$C^\infty(T)\ni\varrho\longmapsto\left\langle\Psi, \left.\frac{\partial^{\alpha_1+\ldots+\alpha_n}\varrho}{\partial t_1^{\alpha_1}\ldots\partial t_n^{\alpha_n}}\right|_{\text{diag}}\right\rangle, \quad \alpha_j\geqslant 0,$$

where Ψ is a distribution on the diagonal of T. In view of Theorem 1.1 and (1.12)–(1.13) we have therefore proved the following result:

THEOREM 1.5. *The restriction of an $(n+1)$-posthomogeneous $F\in\mathfrak{F}$ (resp. $\widetilde{\mathfrak{F}}$) to $\mathbf{C}\times C^\infty(S^1)$ is a finite sum of functions of form*

$$f(\zeta,y)=\zeta^\nu\left\langle\Phi, \prod_{j=0}^{n} y^{(d_j)}\right\rangle, \quad \nu\leqslant 2n,\ d_j\geqslant d_0=1\ (\textit{resp. } 0),$$

where Φ is a distribution on S^1. For a general $F\in\mathfrak{F}$ (resp. $\widetilde{\mathfrak{F}}$) the restriction $F|_{\mathbf{C}\times C^\infty(S^1)}$ is the limit, in the topology of $\mathcal{O}(\mathbf{C}\times C^\infty(S^1))$, of finite sums of the above functions.

2. The G-action on $\mathfrak{F}$

For $g\in G$ let $J_g(\zeta)=d(g\zeta)/d\zeta$. By considering the posthomogeneous expansion (1.12)–(1.13) of $F\in\mathfrak{F}$ (resp. $\widetilde{\mathfrak{F}}$), one checks that the function gF defined by

$$(gF)(\zeta,y)=F(g\zeta, y/J_g(\zeta))\,J_g(\zeta) \tag{2.1}$$

extends to all of $\mathbf{C}\times L^{(-)}\mathbf{C}$, and the extension (also denoted gF) belongs to $\mathfrak{F}$ (resp. $\widetilde{\mathfrak{F}}$). The action thus defined makes $\mathfrak{F}$ and $\widetilde{\mathfrak{F}}$ holomorphic G-modules. The nth isotypical subspace $\mathfrak{F}^n$ (resp. $\widetilde{\mathfrak{F}}^n$) is the subspace of $(n+1)$-posthomogeneous functions. In this section we shall describe the space $\mathfrak{F}^0$, and, for $W^{1,p}$ loop spaces, the spaces $\mathfrak{F}^n$ as well, $n\geqslant 1$.

THEOREM 2.1. *$\mathfrak{F}^0\approx(L^-\mathbf{C})^*/\mathbf{C}$, the dual endowed with the compact-open topology. If $L^-\mathbf{C}$ is interpreted as the space of one-forms on S^1 of the corresponding regularity, then the isomorphism is* Diff S^1*-equivariant.*

Proof. Indeed, the map $(L^-\mathbf{C})^*=\widetilde{\mathfrak{F}}^0\to\mathfrak{F}^0$ associating with $\Phi\in(L^-\mathbf{C})^*$ the function $F(y)=\langle\Phi,\dot y\rangle$ (or $\langle\Phi,dy\rangle$) has one-dimensional kernel and a right inverse by Theorem 1.1.

THEOREM 2.2. *In the case of $W^{1,p}$ loop spaces, $\mathfrak{F}=\bigoplus_{n=0}^{p-1}\mathfrak{F}^n$. Furthermore,*

$$\mathfrak{K}_n\otimes L^{p/(n+1)}(S^1)^*\approx\mathfrak{F}^n,\quad 1\leqslant n\leqslant p-1,$$

as G-modules, G acting on $L^{p/(n+1)}(S^1)^*$ trivially. Indeed, the map $\varphi\otimes\Phi\mapsto F$ given by

$$F(\zeta,y)=\psi(\zeta)\langle\Phi,\dot y^{n+1}\rangle,\quad \varphi(\zeta)=\psi(\zeta)(d\zeta)^{-n},\tag{2.2}$$

induces the isomorphism above. (To achieve Diff S^1*-equivariant isomorphism, replace $L^{p/(n+1)}(S^1)$ by the space $L^{p/(n+1)}_{n+1}(S^1)$ of $(n+1)$-differentials.)*

We shall need a few auxiliary results to prove the theorem.

LEMMA 2.3. *Let $m\geqslant 2$ be an integer and Ψ a distribution on S^1. If the function*

$$C^\infty(S^1)\ni x\longmapsto\langle\Psi,x^m\rangle\in\mathbf{C}\tag{2.3}$$

extends to a homogeneous polynomial E on $L^p(S^1)$, then $\Psi\equiv 0$, or $m\leqslant p$ and Ψ extends to a form Φ on $L^{p/m}(S^1)$. In the latter case the map $E\mapsto\Phi$ is continuous and linear.

Proof. There is a constant C such that

$$|\langle\Psi,x^m\rangle|=|E(x)|\leqslant C\biggl(\int|x|^p\biggr)^{m/p},\quad x\in C^\infty(S^1).\tag{2.4}$$

Let $z\in C^\infty(S^1)$ be real-valued and $x_\varepsilon=(z+i\varepsilon)^{1/m}\in C^\infty(S^1)$. By (2.4),

$$|\langle\Psi,z\rangle|=\lim_{\varepsilon\to 0}|\langle\Psi,x_\varepsilon^m\rangle|\leqslant C\biggl(\int|z|^{p/m}\biggr)^{m/p}.$$

As the same estimate holds for imaginary z, it will hold for a general $z\in C^\infty(S^1)$ too, perhaps with a different C. Therefore Ψ extends to a form Φ on $L^{p/m}(S^1)$. Unless $p\geqslant m$, $\Phi=0$ by Day's theorem [D]. With $z\in L^{p/m}(S^1)$, any choice of measurable mth root $z^{1/m}$, and $y_\varepsilon\in C^\infty(S^1)$ converging to $z^{1/m}$ in L^p,

$$\langle \Phi, z\rangle = \lim_{\varepsilon\to 0}\langle \Phi, y_\varepsilon^m\rangle = \lim_{\varepsilon\to 0} E(y_\varepsilon) = E(z^{1/m}).$$

This shows that Φ is uniquely determined by E, and depends continuously and linearly on E.

In the rest of this section we work with $W^{1,p}$ loop spaces. Write $\mathfrak{E}^n\subset\mathfrak{E}$ and $\widetilde{\mathfrak{E}}^n\subset\widetilde{\mathfrak{E}}$ for the space of $(n+1)$-homogeneous functions.

LEMMA 2.4. *If $m\geqslant 2$ and $E\in\widetilde{\mathfrak{E}}^{m-1}\subset\mathcal{O}(L^p(S^1))$, then $E(x)=\langle\Phi, x^m\rangle$ with a unique $\Phi\in L^{p/m}(S^1)^*$. In particular, $E=0$ if $m>p$. Also, the map $E\mapsto\Phi$ is an isomorphism between $\widetilde{\mathfrak{E}}^{m-1}$ and $L^{p/m}(S^1)^*$.*

Proof. We shall prove this by induction, first assuming $m=2$. By Theorem 1.5 there are distributions Φ_α so that

$$E(x) = \sum_{\alpha=0}^{d}\langle \Phi_\alpha, x x^{(\alpha)}\rangle, \quad x\in C^\infty(S^1).$$

Now any $x^{(\alpha)}x^{(\beta)}$ will be a linear combination of expressions $(x^{(j)}x^{(j)})^{(h)}$, as one easily proves by induction on $|\alpha-\beta|$. It follows that E can be written with distributions Ψ_j as

$$E(x) = \sum_{j=0}^{d}\langle \Psi_j, (x^{(j)})^2\rangle, \quad x\in C^\infty(S^1). \tag{2.5}$$

Next we show that $d=0$.

Indeed, assuming $d>0$, for fixed $x\in C^\infty(S^1)$,

$$E(\cos\lambda x)+E(\sin\lambda x) = \lambda^{2d}\langle \Psi_d, \dot{x}^{2d}\rangle + \sum_{j=0}^{2d-1} c_j(x)\lambda^j \tag{2.6}$$

is a polynomial in λ. For fixed $\lambda\in\mathbf{C}$ the maps $x\mapsto\cos\lambda x$ and $x\mapsto\sin\lambda x$ map the Banach algebra $W^{1,1}(S^1)$ holomorphically into itself, hence into $L^p(S^1)$. Therefore the left-hand side of (2.6) extends to $W^{1,1}(S^1)$, and consequently $\langle\Psi_d, \dot{x}^{2d}\rangle$ also extends. The extension of the latter will be an additive, $2d$-homogeneous polynomial E' on $W^{1,1}(S^1)$, satisfying $E'(x+\text{const})=E'(x)$. By Proposition 1.3 there is therefore a unique additive

$2d$-homogeneous polynomial $\widetilde{E}$ on $W^{0,1}(S^1)=L^1(S^1)$ such that $E'(x)=\widetilde{E}(\dot{x})$. Since the restriction $\widetilde{E}|_{C^\infty(S^1)}$ is also unique,

$$\widetilde{E}(x)=\langle\Psi_d,x^{2d}\rangle,\quad x\in C^\infty(S^1).$$

In particular, the expression on the right continuously extends to $L^1(S^1)$. By virtue of Lemma 2.3, $\Psi_d\equiv 0$. Thus (2.5) reduces to $E(x)=\langle\Psi,x^2\rangle$, $x\in C^\infty(S^1)$, and by another application of Lemma 2.3, Ψ extends to a form Φ on $L^{p/2}(S^1)$.

Now assume that the lemma is known for degree $m-1\geqslant 2$, and consider an $E\in\widetilde{\mathfrak{E}}^{m-1}$, together with its polarization $\mathcal{E}$. For fixed $x_1\in C^\infty(S^1)$ the inductive assumption implies that there is a distribution Θ such that $\mathcal{E}(x_1\otimes\ldots\otimes x_m)=\langle\Theta,\prod_{j=2}^m x_j\rangle$; in particular,

$$\mathcal{E}(x_1\otimes\ldots\otimes x_m)=\mathcal{E}\left(x_1\otimes\prod_{j=2}^m x_j\otimes 1\otimes\ldots\otimes 1\right),\quad x\in C^\infty(S^1).$$

The case $m=2$ now gives a distribution Ψ such that $\mathcal{E}(x_1\otimes\ldots\otimes x_m)=\langle\Psi,\prod_{j=1}^m x_j\rangle$. We conclude by Lemma 2.3: Ψ extends to $\Phi\in L^{p/m}(S^1)^*$, and $\Phi=0$ unless $m\leqslant p$. It is clear that Φ is uniquely determined by E, and the map $\widetilde{\mathfrak{E}}^{m-1}\ni E\mapsto\Phi\in L^{p/m}(S^1)^*$ is an isomorphism.

Proof of Theorem 2.2. To construct the inverse of the map defined by (2.2), write an arbitrary $F\in\mathfrak{F}^n$, $n\geqslant 1$, as

$$F(\zeta,y)=\sum_{\nu=0}^{2n}\zeta^\nu E_\nu(y),\quad E_\nu\in\mathfrak{E}^n,$$

cf. Proposition 1.4, and find the unique $\widetilde{E}_\nu\in\widetilde{\mathfrak{E}}^n$ so that $E_\nu(y)=\widetilde{E}_\nu(\dot{y})$, see Proposition 1.3. By Lemma 2.4 there are unique $\Phi_\nu\in L^{p/(n+1)}(S^1)^*$ such that $\widetilde{E}_\nu(x)=\langle\Phi_\nu,x^{n+1}\rangle$. If $p<n+1$ then $\Phi_\nu=0$, and so $\mathfrak{F}^n=(0)$. Otherwise the map

$$\mathfrak{F}^n\ni F\longmapsto\sum_{\nu=0}^{2n}\zeta^\nu(d\zeta)^{-n}\otimes\Phi_\nu\in\mathfrak{K}_n\otimes L^{p/(n+1)}(S^1)^*$$

is the inverse of the map given in (2.2), so (2.2) indeed induces an isomorphism. Finally, the posthomogeneous expansion of an arbitrary $F\in\mathfrak{F}$ is

$$F=\sum_{n=0}^\infty F_n=\sum_{n=0}^{[p-1]}F_n,$$

which completes the proof.

3. Cuspidal cocycles

In this section we shall construct an isomorphism between $H^{0,1}(L\mathbf{P}_1)$ and a space of holomorphic Čech cocycles on $L\mathbf{P}_1$. We represent $\mathbf{P}_1$ as $\mathbf{C}\cup\{\infty\}$. Constant loops constitute a submanifold of $L\mathbf{P}_1$, which we identify with $\mathbf{P}_1$. If $a,b,...\in\mathbf{P}_1$, set $U_{ab...}=\mathbf{P}_1\backslash\{a,b,...\}$. Thus LU_a, $a\in\mathbf{P}_1$, form an open cover of $L\mathbf{P}_1$, with $LU_\infty=L\mathbf{C}$ a Fréchet algebra. If $g\in G$ then $g(LU_a)=LU_{ga}$.

Suppose that we are given $v:\mathbf{P}_1\to\mathbf{C}$, finitely many $a,b,...\in\mathbf{P}_1$ and a function $u: LU_{ab...}\to\mathbf{C}$. If ∞ is among $a,b,...$, let us say that u is v-cuspidal at ∞ if $u(x+\lambda)\to v(\infty)$ as $\mathbf{C}\ni\lambda\to\infty$, for all $x\in LU_{ab...}$; and in general, that u is v-cuspidal if g^*u is g^*v-cuspidal at ∞ for all $g\in G$ that maps ∞ to one of $a,b,...$. When $v\equiv 0$, we simply speak of cuspidal functions.

PROPOSITION 3.1. *Given a closed $f\in C^\infty_{0,1}(L\mathbf{P}_1)$ and $v\in C^\infty(\mathbf{P}_1)$ such that $\bar\partial v= f|_{\mathbf{P}_1}$, for each $a\in\mathbf{P}_1$ there is a unique v-cuspidal $u_a\in C^\infty(LU_a)$ that solves $\bar\partial u_a=f|_{LU_a}$. Furthermore, $u_a|_{U_a}=v|_{U_a}$, and $u(a,x)=u_a(x)$ is smooth in (a,x) and holomorphic in a.*

Proof. Uniqueness follows since for fixed $g\in G$ and $y\in L\mathbf{C}$, on the line $\{g(y+\lambda): \lambda\in\mathbf{P}_1\}$ the $\bar\partial$-equation is uniquely solvable up to an additive constant, which constant is determined by the cuspidal condition. To construct u_a, fix a $g\in G$ with $g\infty=a$, let

$$Y=\{y\in L\mathbf{C}: y(0)=0\}$$

and

$$P_g:\mathbf{P}_1\times Y\ni(\lambda,y)\longmapsto g(y+\lambda)\in L\mathbf{P}_1,$$

a biholomorphism between $\mathbf{C}\times Y$ and LU_a. Setting $f_g=P_g^*f$, by [L1, Theorem 5.4] on the $\mathbf{P}_1$-bundle $\mathbf{P}_1\times Y$ the equation $\bar\partial u_g=f_g$ has a unique smooth solution satisfying $u_g(\infty,x)=v(a)$. It follows that $u_a=(P_g^{-1})^*(u_g|_{\mathbf{C}\times Y})$ solves $\bar\partial u_a=f|_{LU_a}$. Also, g^*u_a is g^*v-cuspidal at ∞. On U_a both u_a and v solve the same $\bar\partial$-equation, and have the same limit at a; hence $u_a|_{U_a}=v|_{U_a}$.

One can also consider

$$P:\mathbf{P}_1\times G\times Y\ni(\lambda,g,y)\longmapsto g(y+\lambda)\in L\mathbf{P}_1$$

and $f'=P^*f$. Again by [L1, Theorem 5.4], on the $\mathbf{P}_1$-bundle $\mathbf{P}_1\times G\times Y$ the equation $\bar\partial u'=f'$ has a smooth solution satisfying $u'(\infty,g,x)=v(g\infty)$. Uniqueness of u_g implies $u'(\lambda,g,x)=u_g(\lambda,x)$, whence $u_g(\lambda,x)$ depends smoothly on (λ,g,x), and $u_a(x)$ on (a,x). Furthermore, u' is holomorphic on $P^{-1}(x)$ for any x. In particular, if $g\in G$ with $g\infty=a$ is chosen to depend holomorphically on a (which can be done locally), then it follows that $u_a(x)=u'(g^{-1}x(0),g,g^{-1}x-g^{-1}x(0))$ is holomorphic in a.

Since f determines v up to an additive constant, we can uniquely associate with f the Čech cocycle $\mathfrak{f}=(u_a-u_b : a, b\in\mathbf{P}_1)$. The components of $\mathfrak{f}$ are cuspidal holomorphic functions on LU_{ab}. One easily verifies:

PROPOSITION 3.2. *The form f is exact if and only if $\mathfrak{f}=0$. Hence $\mathfrak{f}$ depends only on the cohomology class $[f]\in H^{0,1}(L\mathbf{P}_1)$. The components $h_{ab}([f],x)$ of $\mathfrak{f}$ depend holomorphically on $a,b\in\mathbf{P}_1$ and $x\in LU_{ab}$, and satisfy the transformation formula*

$$h_{ga,gb}([f],gx)=h_{ab}(g^*[f],x),\quad g\in G,\ x\in LU_{ab}. \tag{3.1}$$

Set

$$\Omega=\{(a,b,x)\in\mathbf{P}_1\times\mathbf{P}_1\times L\mathbf{P}_1 : a,b\notin x(S^1)\}.$$

Let $\mathfrak{H}$ denote the space of those holomorphic cocycles $\mathfrak{h}=(\mathfrak{h}_{ab})_{a,b\in\mathbf{P}_1}$ of the covering $\{LU_a\}$ for which $\mathfrak{h}_{ab}(x)$ depends holomorphically on a, b and $x\in LU_{ab}$, and each $\mathfrak{h}_{ab}$ is cuspidal. Then $\mathfrak{H}\subset\mathcal{O}(\Omega)$, with the compact-open topology, is a complete, separated, locally convex space. The action of G on Ω induces a G-module structure on $\mathfrak{H}$:

$$(g^*\mathfrak{h})_{ab}(x)=\mathfrak{h}_{ga,gb}(gx),\quad g\in G. \tag{3.2}$$

Proposition 3.2 implies that the map $[f]\mapsto\mathfrak{f}$ is a monomorphism $H^{0,1}(L\mathbf{P}_1)\to\mathfrak{H}$ of G-modules.

THEOREM 3.3. *The map $[f]\mapsto\mathfrak{f}$ is an isomorphism $H^{0,1}(L\mathbf{P}_1)\to\mathfrak{H}$.*

The proof would be routine if the loop space $L\mathbf{P}_1$ admitted smooth partitions of unity; but a typical loop space does not, see [K]. The proof that we offer here will work only when the loops in $L\mathbf{P}_1$ are of regularity $W^{1,3}$ at least, and we shall return to the case of $L_{1,p}\mathbf{P}_1$, $p<3$, in §6.

Those $g\in G$ that preserve the Fubini–Study metric form a subgroup (isomorphic to) SO(3). Denote the Haar probability measure on SO(3) by dg.

LEMMA 3.4. *Unless $L\mathbf{P}_1=L_{1,p}\mathbf{P}_1$, $p<3$, there is a $\chi\in C^\infty(L\mathbf{P}_1)$ such that $\chi=0$ in a neighborhood of $L\mathbf{P}_1\setminus L\mathbf{C}=\{x:\infty\in x(S^1)\}$, and $\int_{\mathrm{SO}(3)} g^*\chi\, dg=1$.*

Proof. With $c_0\in(0,\infty)$ to be specified later, fix a nonnegative $\varrho\in C^\infty(\mathbf{R})$ such that $\varrho(\tau)=1$ (resp. 0) when $|\tau|<c_0$ (resp. $>2c_0$). For $x\in L\mathbf{C}$ let

$$\psi(x)=\varrho\left(\int_{S^1}(1+|x|^2)^{3/4}\right),$$

and define $\psi(x)=0$ if $x\in L\mathbf{P}_1\setminus L\mathbf{C}$. We claim that ψ vanishes in a neighborhood of an arbitrary $x\in L\mathbf{P}_1\setminus L\mathbf{C}$. This will then also imply that $\psi\in C^\infty(L\mathbf{P}_1)$.

Indeed, suppose $x(t_0)=\infty$. In a neighborhood of $t_0\in S^1$ the function $z=1/x$ is $W^{1,3}$, hence Hölder continuous with exponent $\frac{2}{3}$ by the Sobolev embedding theorem [Hö, Theorem 4.5.12]. In this neighborhood therefore $|x(t)|\geqslant c|t-t_0|^{-2/3}$ and $\int_{S^1}(1+|x|^2)^{3/4}=\infty$. When $y\in L\mathbf{C}$ is close to x, $\int_{S^1}(1+|y|^2)^{3/4}>2c_0$, i.e. $\psi(y)=0$.

Next we show that for every $x\in L\mathbf{P}_1$ there is a $g\in\mathrm{SO}(3)$ with $\psi(gx)>0$. Let $d(a,b)$ denote the Fubini–Study distance between $a,b\in\mathbf{P}_1$; then with some $c>0$,

$$1+|\zeta|^2\leqslant\frac{c}{d(\zeta,\infty)^2}\quad\text{and}\quad\int_{S^1}(1+|x|^2)^{3/4}\leqslant c\int_{S^1}d(x,\infty)^{-3/2}.$$

Hence

$$\int_{\mathrm{SO}(3)}\int_{S^1}(1+|gx(t)|^2)^{3/4}\,dt\,dg\leqslant c\int_{S^1}\int_{\mathrm{SO}(3)}d(gx(t),\infty)^{-3/2}\,dg\,dt=cI,$$

where, for any $\zeta\in\mathbf{P}_1$,

$$I=\int_{\mathrm{SO}(3)}d(g\zeta,\infty)^{-3/2}\,dg=\int_{\mathbf{P}_1}d(\,\cdot\,,\infty)^{-3/2}<\infty,$$

the last integral with respect to the Fubini–Study area form. If c_0 is chosen larger than cI, then indeed $\int_{S^1}(1+|gx|^2)^{3/4}<c_0$ and $\psi(gx)=1$ for some $g\in\mathrm{SO}(3)$.

It follows that $\int_{\mathrm{SO}(3)}\psi(gx)\,dg>0$, and we can take $\chi(x)=\psi(x)/\int_{\mathrm{SO}(3)}\psi(gx)\,dg$.

Proof of Theorem 3.3. Given $\mathfrak{h}\in\mathfrak{H}$, extend $(g^*\chi)\mathfrak{h}_{a,g^{-1}\infty}$ from $LU_{a,g^{-1}\infty}$ to LU_a by zero, and define the cuspidal functions

$$u_a=\int_{\mathrm{SO}(3)}(g^*\chi)\mathfrak{h}_{a,g^{-1}\infty}\,dg,\quad a\in\mathbf{P}_1.$$

Then $u_a-u_b=\int_{\mathrm{SO}(3)}(g^*\chi)\mathfrak{h}_{ab}\,dg=\mathfrak{h}_{ab}$, so that $f=\bar\partial u_a$ on LU_a consistently defines a closed $f\in C^\infty_{0,1}(L\mathbf{P}_1)$. It is immediate that the map $\mathfrak{h}\mapsto[f]\in H^{0,1}(L\mathbf{P}_1)$ is left inverse to the monomorphism $[f]\mapsto\mathfrak{f}$, whence the theorem follows.

4. The map $\mathfrak{H}\to\mathfrak{F}$

Consider an $\mathfrak{h}=(\mathfrak{h}_{ab})\in\mathfrak{H}$. The cocycle relation implies that $d_\zeta\mathfrak{h}_{a\zeta}(x)$ is independent of a; for $\zeta\in\mathbf{C}$ we can write it as

$$d_\zeta\mathfrak{h}_{a\zeta}(x)=F\Big(\zeta,\frac{1}{\zeta-x}\Big)\,d\zeta,\quad x\in LU_\zeta,\tag{4.1}$$

where $F\in\mathcal{O}(\mathbf{C}\times L\mathbf{C})$. Set $F=\alpha(\mathfrak{h})$. Since $\mathfrak{h}_{aa}=0$,

$$\mathfrak{h}_{ab}(x)=\int_a^b F\Big(\zeta,\frac{1}{\zeta-x}\Big)\,d\zeta,\tag{4.2}$$

provided a and b are in the same component of $\mathbf{P}_1\backslash x(S^1)$—which we shall express by saying that x does not separate a and b—, and we integrate along a path within this component. The main result of this section is the following theorem.

THEOREM 4.1. $\alpha(\mathfrak{h})=F\in\mathfrak{F}$.

The heart of the matter will be the special case when $\mathfrak{h}$ is in an irreducible submodule $\approx\mathfrak{K}_n$. A vector that corresponds, in this isomorphism, to $\mathrm{const}(d\zeta)^{-n}\in\mathfrak{K}_n$ is said to be of lowest weight $-n$. Thus, if $\mathfrak{l}$ is of lowest weight $-n\leqslant 0$, then

$$g_\lambda^*\mathfrak{l}=\lambda^{-n}\mathfrak{l}, \quad \text{when } g_\lambda\zeta=\lambda\zeta, \quad \lambda\in\mathbf{C}\setminus\{0\}, \tag{4.3}$$

$$g_\lambda^*\mathfrak{l}=\mathfrak{l}, \quad \text{when } g_\lambda\zeta=\zeta+\lambda, \quad \lambda\in\mathbf{C}. \tag{4.4}$$

Conversely, an $\mathfrak{l}\neq 0$ satisfying (4.3) and (4.4) is a lowest-weight vector and spans an irreducible submodule, isomorphic to $\mathfrak{K}_n$, but we shall not need this fact.

If $\mathfrak{l}\in\mathfrak{H}$ satisfies (4.4), then $\mathfrak{l}_{\infty\zeta}(x)=\mathfrak{l}_{\infty,\zeta+\lambda}(x+\lambda)$ by (3.2), whence $d_\zeta\mathfrak{l}_{\infty\zeta}(x)$ depends only on $\zeta-x$, and $\alpha(\mathfrak{l})$ is of form $F(\zeta,y)=E(y)$. If, in addition, $\mathfrak{l}$ satisfies (4.3), then similarly it follows that $E\in\mathcal{O}(L\mathbf{C})$ is homogeneous of degree $n+1$. We now fix a nonzero lowest-weight vector $\mathfrak{l}\in\mathfrak{H}$, the corresponding $(n+1)$-homogeneous polynomial E and its polarization $\mathcal{E}$, cf. (1.2).

PROPOSITION 4.2. *$\mathcal{E}(1\otimes y_1\otimes\ldots\otimes y_n)=0$, and so $E(y+\mathrm{const})=E(y)$.*

Proof. Since $\mathfrak{l}_{\infty 0}\in\mathcal{O}(LU_{\infty 0})$ is cuspidal and homogeneous of order $-n$,

$$0=\lim_{\lambda\to\infty}\mathfrak{l}_{\infty 0}\left(\frac{1}{\lambda+x}\right)=\lim_{\lambda\to\infty}\lambda^n\mathfrak{l}_{\infty 0}\left(\frac{1}{1+x/\lambda}\right).$$

Thus $\mathfrak{l}_{\infty 0}$ vanishes at 1 to order $\geqslant n+1$. Hence

$$\frac{\partial}{\partial\zeta}\bigg|_{\zeta=0}\mathfrak{l}_{\infty 0}(x-\zeta)=\frac{\partial}{\partial\zeta}\bigg|_{\zeta=0}\mathfrak{l}_{\infty\zeta}(x)=E\left(\frac{1}{x}\right)$$

vanishes at $x=1$ to order $\geqslant n$, and the same holds for $E(x)$. Differentiating E in the directions $y_1,\ldots,y_n$, we obtain at $x=1$, as needed, that $n!\,\mathcal{E}(1\otimes y_1\otimes\ldots\otimes y_n)=0$.

Let $\mathfrak{K}_n\ni\varphi\mapsto\mathfrak{h}^\varphi\in\mathfrak{H}$ denote the homomorphism that maps $(d\zeta)^{-n}$ to $\mathfrak{l}$.

PROPOSITION 4.3.

$$d_\zeta\mathfrak{h}^\varphi_{a\zeta}(x)=\psi(\zeta)E\left(\frac{1}{\zeta-x}\right)d\zeta, \quad \varphi(\zeta)=\psi(\zeta)(d\zeta)^{-n}. \tag{4.5}$$

By homogeneity, the right-hand side can also be written $\varphi(\zeta)E(d\zeta/(\zeta-x))$.

Proof. Denote the form on the left-hand side of (4.5) by ω^φ. In view of (3.2), it transforms under the action of G on $\mathbf{P}_1\times L\mathbf{P}_1$ as

$$g^*\omega^\varphi=\omega^{g\varphi}, \quad g\in G. \tag{4.6}$$

If we show that the right-hand side of (4.5) transforms in the same way, then (4.5) will follow, since it holds when $\psi\equiv 1$, see (4.1). In fact, it will suffice to check the transformation formula for $g\zeta=\lambda\zeta$, $g\zeta=\zeta+\lambda$ and $g\zeta=1/\zeta$, maps that generate G. We shall do this for the last map, the most challenging of the three types. The pullback of the right-hand side of (4.5) by $g\zeta=1/\zeta$ is

$$(g\varphi)(\zeta)\,E\left(\frac{d(g\zeta)}{g\zeta-gx}\right)=(g\varphi)(\zeta)\,E\left(\frac{d\zeta}{\zeta-x}-\frac{d\zeta}{\zeta}\right)=(g\varphi)(\zeta)\,E\left(\frac{d\zeta}{\zeta-x}\right),$$

by Proposition 4.2, which is what we need.

The form $\mathcal{E}$ defines a symmetric distribution D on the torus $T=(S^1)^{n+1}$ as in §1, cf. (1.14). By (1.15), (4.2) and Proposition 4.3,

$$\mathfrak{h}^{\varphi}_{ab}(x)=\int_a^b \psi(\zeta)\left\langle D,\frac{1}{\zeta-x}\otimes\ldots\otimes\frac{1}{\zeta-x}\right\rangle d\zeta,\quad \varphi=\psi(\zeta)(d\zeta)^{-n}, \tag{4.7}$$

provided $x\in L_\infty U_{ab}$ does not separate a and b. To prove Theorem 4.1, we have to understand $\operatorname{supp} D$. Let

$$O=\{x\in C^\infty(S^1):\pm i\notin x(S^1)\}\quad\text{and}\quad O'=\{x\in O:[-i,i]\cap x(S^1)=\varnothing\},$$

where $[-i,i]$ stands for the segment joining $\pm i$.

LEMMA 4.4. *With Δ a symmetric distribution on $T=(S^1)^{n+1}$ and $\nu=0,\ldots,2n-2$, let*

$$I_\nu(x)=\int_{[-i,i]}\left\langle \Delta,\frac{1}{\zeta-x}\otimes\ldots\otimes\frac{1}{\zeta-x}\right\rangle \zeta^\nu\,d\zeta,\quad x\in O'.$$

If each I_ν continues analytically to O then Δ is supported on the diagonal of T.

In preparation for the proof, consider a holomorphic vector field V on O, and observe that VI_ν also continues analytically to O. Such vector fields can be thought of as holomorphic maps $V:O\to C^\infty(S^1)$. Using the symmetry of Δ we compute

$$(VI_\nu)(x)=(n+1)\int_{[-i,i]}\left\langle \Delta,\frac{V(x)}{(\zeta-x)^2}\otimes\frac{1}{\zeta-x}\otimes\ldots\otimes\frac{1}{\zeta-x}\right\rangle \zeta^\nu\,d\zeta,\quad x\in O'. \tag{4.8}$$

Proof of Lemma 4.4, case $n=1$. Let $\bar s_0\neq\bar s_1\in S^1$. To show that Δ vanishes near $\bar s=(\bar s_0,\bar s_1)$, construct a smooth family $x_{\varepsilon,s}\in O$ of loops, where $\varepsilon\in[0,1]$ and $s\in T$ is in a neighborhood of $\bar s$, so that

$$x_{\varepsilon,s}(\tau)=(-1)^j(\varepsilon^2+(\tau-s_j)^2),\quad\text{when } \tau\in S^1 \text{ is near } \bar s_j,\ j=0,1; \tag{4.9}$$

here, perhaps abusively, $\tau - s_j$ denotes both a point in $S^1 = \mathbf{R}/\mathbf{Z}$ and its representative in $\mathbf{R}$ that is closest to 0. Make sure that $x_{\varepsilon,s} \in O'$ when $\varepsilon > 0$. Fix $y_0, y_1 \in C^\infty(S^1)$ so that $y_j \equiv 1$ near $\bar{s}_j$, and (4.9) holds when τ and s_j are in a neighborhood of $\operatorname{supp} y_j$. This forces y_0 and y_1 to have disjoint support. With constant vector fields $V_j = y_j$,

$$(V_1 V_0 I_0)(x) = 2 \int_{[-i,i]} \left\langle \Delta, \frac{y_0}{(\zeta - x)^2} \otimes \frac{y_1}{(\zeta - x)^2} \right\rangle d\zeta, \quad x \in O', \tag{4.10}$$

analytically continues to O. In particular, for $\varepsilon > 0$ and $t = (t_0, t_1) \in T$, setting

$$K_\varepsilon(t, s) = \int_{[-i,i]} \frac{y_0(t_0)\, y_1(t_1)\, d\zeta}{(\zeta - x_{\varepsilon,s}(t_0))^2 (\zeta - x_{\varepsilon,s}(t_1))^2}, \quad s \text{ near } \bar{s},$$

it follows that $\langle \Delta, K_\varepsilon(\,\cdot\,, s)\rangle$ stays bounded as $\varepsilon \to 0$. Therefore, if $\varrho \in C^\infty(T)$ is supported in a sufficiently small neighborhood of $\bar{s}$, then

$$\left\langle \Delta, \varepsilon^4 \int_T K_\varepsilon(\,\cdot\,, s)\, \varrho(s)\, ds \right\rangle \to 0, \quad \varepsilon \to 0. \tag{4.11}$$

On the other hand, we shall show that for such ϱ,

$$\varepsilon^4 \int_T K_\varepsilon(\,\cdot\,, s)\, \varrho(s)\, ds \to c\varrho, \quad \varepsilon \to 0, \tag{4.12}$$

in the topology of $C^\infty(T)$; here $c \neq 0$ is a constant.

It will suffice to verify (4.12) on $\operatorname{supp} y_0 \otimes y_1$, since both sides vanish on the complement. Thus we shall work on small neighborhoods of $\bar{s}$; we can pretend that $\bar{s} \in \mathbf{R}^2$, and work on $\mathbf{R}^2$ instead of T. When $s, t \in \mathbf{R}^2$ are close to $\bar{s}$, the left-hand side of (4.12) becomes

$$\varepsilon^4 y_0(t_0)\, y_1(t_1) \int_{\mathbf{R}^2} \int_{[-i,i]} \frac{\varrho(s)\, d\zeta\, ds}{(\zeta - \varepsilon^2 - (s_0 - t_0)^2)^2 (\zeta + \varepsilon^2 + (s_1 - t_1)^2)^2}. \tag{4.13}$$

Substituting $s = t + \varepsilon u$ and $\zeta = \varepsilon^2 \xi$, we compute that the limit in (4.12) is

$$\begin{aligned} \lim_{\varepsilon \to 0} y_0(t_0)\, y_1(t_1) \int_{\mathbf{R}^2} \int_{[-i/\varepsilon^2, i/\varepsilon^2]} & \frac{\varrho(t + \varepsilon u)\, d\xi\, du}{(\xi - 1 - u_0^2)^2 (\xi + 1 + u_1^2)^2} \\ &= 4\pi i y_0(t_0)\, y_1(t_1) \int_{\mathbf{R}^2} \frac{\varrho(t)\, du}{(2 + u_0^2 + u_1^2)^3} = c\varrho(t), \end{aligned} \tag{4.14}$$

if $y_0 \otimes y_1 = 1$ on $\operatorname{supp} \varrho$. This limit is first seen to hold uniformly. However, since the integral operator in (4.13) is a convolution, in (4.14) in fact all derivatives converge uniformly. Now (4.11) and (4.12) imply that $\langle \Delta, \varrho \rangle = 0$, so that Δ vanishes close to $\bar{s}$.

Proof of Lemma 4.4, *general n.* The base case $n=1$ settled and the statement being vacuous when $n=0$, we prove by induction. Assume that the lemma holds on the n-dimensional torus, and with $y\in C^\infty(S^1)$, consider the holomorphic vector fields $V_\mu(x)=yx^\mu$, $\mu=0,1,2$. (These vector fields continue to all of $L\mathbf{P}_1$, and generate the Lie algebra of the loop group LG.) In view of (4.8), for $x\in O'$,

$$\int_{[-i,i]}\left\langle \Delta, y\otimes\frac{1}{\zeta-x}\otimes\ldots\otimes\frac{1}{\zeta-x}\right\rangle \zeta^\nu\, d\zeta=\frac{1}{n+1}(V_0I_{\nu+2}-2V_1I_{\nu+1}+V_2I_\nu). \tag{4.15}$$

Therefore the left-hand side continues analytically to O, provided $\nu=0,\ldots,2n-4$. If Δ^y denotes the distribution on $(S^1)^n$ defined by $\langle\Delta^y,\varrho\rangle=\langle\Delta,y\otimes\varrho\rangle$, the left-hand side of (4.15) is

$$\int_{[-i,i]}\left\langle \Delta^y, \frac{1}{\zeta-x}\otimes\ldots\otimes\frac{1}{\zeta-x}\right\rangle \zeta^\nu\, d\zeta.$$

The inductive hypothesis implies that Δ^y is supported on the diagonal of $(S^1)^n$. This being true for all y, the symmetric distribution Δ itself must be supported on the diagonal.

COROLLARY 4.5. *The distribution D in* (4.7) *is supported on the diagonal of T.*

Proof of Theorem 4.1. First assume that $\mathfrak{h}\in\mathfrak{H}$ is in an irreducible submodule $\approx\mathfrak{K}_n$, and $\mathfrak{l}\neq 0$ is a lowest-weight vector in this submodule. Thus $\mathfrak{h}=\mathfrak{h}^\varphi$ for some $\varphi\in\mathfrak{K}_n$, $\varphi(\zeta)=\psi(\zeta)(d\zeta)^{-n}$. With $\mathfrak{l}$ we associated an $(n+1)$-homogeneous polynomial E on $L\mathbf{C}$ and a distribution D on $(S^1)^{n+1}$. By Proposition 4.3, $F(\zeta,y)=\psi(\zeta)E(y)$, and so $F(\zeta,y+\text{const})=F(\zeta,y)$ by Proposition 4.2. Since $\deg\psi\leqslant 2n$, $F(\zeta/\lambda,\lambda^2y)=O(\lambda^2)$ as $\lambda\to 0$. Finally, take $x,y\in L\mathbf{C}$ with disjoint supports. If $x,y\in C^\infty(S^1)$, then

$$E(x+y)=\langle D,(x+y)^{\otimes(n+1)}\rangle=\langle D,x^{\otimes(n+1)}\rangle+\langle D,y^{\otimes(n+1)}\rangle=E(x)+E(y),$$

as $\operatorname{supp} D$ is on the diagonal. By approximation, $E(x+y)=E(x)+E(y)$ follows in general, whence F itself is additive. We conclude that $F\in\mathfrak{F}$ if $\mathfrak{h}$ is in an irreducible submodule.

By linearity it follows that $F\in\mathfrak{F}$ whenever $\mathfrak{h}$ is in the span of irreducible submodules. Since this span is dense in $\mathfrak{H}$ (cf. [BD, III.5.7] and the explanation in the introduction connecting representations of G with those of the compact group SO(3)), $\alpha(\mathfrak{h})\in\mathfrak{F}$ for all $\mathfrak{h}\in\mathfrak{H}$.

THEOREM 4.6. *The map α is a G-morphism.*

Proof. It suffices to verify that the restriction of α to an irreducible submodule of $\mathfrak{H}$ is a G-morphism, which follows directly from Proposition 4.3.

5. The structure of $\mathfrak{H}$

The main result of this section is the following theorem:

THEOREM 5.1. *The G-morphism $\alpha\colon\mathfrak{H}\to\mathfrak{F}$ has a right inverse β. Its kernel is one-dimensional, spanned by the G-invariant cocycle*

$$\mathfrak{h}_{ab}(x)=\operatorname{ind}_{ab}x \tag{5.1}$$

(the winding number of $x\colon S^1\to U_{ab}$).

We shall need the following result:

LEMMA 5.2. *With notation as in §1, suppose that $z_1,\dots,z_N\in L^-\mathbf{C}$ are such that no point in S^1 is contained in the support of more than two z_j. If $\widetilde{F}\in\widetilde{\mathfrak{F}}$ then*

$$\widetilde{F}\Big(\zeta,\sum_{j=1}^N z_j\Big)=\sum_{i<j}\widetilde{F}(\zeta,z_i+z_j)-(N-2)\sum_{j=1}^N\widetilde{F}(\zeta,z_j). \tag{5.2}$$

In particular, if $N\geqslant 3$, and, writing $z_0=z_N$, only consecutive $\operatorname{supp} z_j$*'s intersect each other, then*

$$\widetilde{F}\Big(\zeta,\sum_{j=1}^N z_j\Big)=\sum_{j=1}^N\widetilde{F}(\zeta,z_{j-1}+z_j)-\sum_{j=1}^N\widetilde{F}(\zeta,z_j).$$

Proof. It will suffice to verify (5.2) when $\widetilde{F}(\zeta,z)=\widetilde{E}(z)$ is homogeneous, in which case it follows by expressing both sides in terms of the polarization of $\widetilde{E}$, and using Lemma 1.2 (a). The second formula follows from (5.2) by applying additivity to terms with nonconsecutive i and j.

Proof of Theorem 5.1. (a) *Construction of the right inverse.* By Theorem 1.1, for $F\in\mathfrak{F}$ we can choose $\widetilde{F}\in\widetilde{\mathfrak{F}}$, depending linearly on F, so that $F(\zeta,y)=\widetilde{F}(\zeta,\dot{y})$. With $x\in L\mathbf{P}_1$ consider the differential form

$$F\Big(\zeta,\frac{1}{\zeta-x}\Big)\,d\zeta=\widetilde{F}\Big(\zeta,\frac{\dot{x}}{(\zeta-x)^2}\Big)\,d\zeta, \tag{5.3}$$

holomorphic in $\mathbf{C}\setminus x(S^1)$. In fact, it is holomorphic at $\zeta=\infty$ as well, provided $\infty\notin x(S^1)$, since the coefficient of $d\zeta$ vanishes to second order at $\zeta=\infty$. This latter is easily verified when $\widetilde{F}(\zeta,z)=\zeta^\nu\widetilde{E}(z)$ and $\widetilde{E}$ is $(n+1)$-homogeneous, $\nu\leqslant 2n$; in general it follows from the posthomogeneous expansion

$$\widetilde{F}(\zeta,z)=\sum_{n=0}^\infty\widetilde{F}_n(\zeta,z)=\sum_{n=0}^\infty\sum_{\nu=0}^{2n}\zeta^\nu\widetilde{E}_{n\nu}(\zeta).$$

Hence, if $x \in L\mathbf{P}_1$ does not separate a and b, the integral

$$h_{ab}(x) = \int_a^b \widetilde{F}\left(\zeta, \frac{\dot{x}}{(\zeta-x)^2}\right) d\zeta \tag{5.4}$$

is independent of the path joining a and b within $\mathbf{P}_1 \backslash x(S^1)$, and defines a holomorphic function of a, b and x.

We claim that h_{ab} can be continued to a cuspidal cocycle $\mathfrak{h}=(\mathfrak{h}_{ab}) \in \mathfrak{H}$. First we prove a variant. Let $\sigma \in C^\infty(S^1)$ be supported in a closed arc $I \neq S^1$. Given finitely many $a, b, ... \in \mathbf{P}_1$, set

$$W_{ab...} = \{x \in L\mathbf{P}_1 : a, b, ... \notin x(I)\} \supset LU_{ab...}.$$

We shall show that the integrals

$$\int_a^b \widetilde{F}\left(\zeta, \frac{\sigma\dot{x}}{(\zeta-x)^2}\right) d\zeta, \quad x \text{ does not separate } a \text{ and } b, \tag{5.5}$$

can be continued to functions $\mathfrak{k}_{ab}(x)$ depending holomorphically on $a, b \in \mathbf{P}_1$ and $x \in W_{ab}$. The main point will be that, unlike $LU_{ab...}$, the sets $W_{ab...}$ are connected.

If $x_1 \in W_{ab}$, construct a continuous curve $[0,1] \ni \tau \mapsto x_\tau \in W_{ab}$, with x_0 being a constant loop. Cover S^1 with open arcs $J_1, ..., J_N = J_0$, $N \geqslant 3$, so that only consecutive $\bar{J}_j$'s intersect, and no $x_\tau(\bar{J}_i \cup \bar{J}_j)$ separates a and b. Choose a C^∞ partition of unity $\{\varrho_j\}_{j=1}^N$ subordinate to $\{J_j\}_{j=1}^N$. For x in a connected neighborhood $W \subset W_{ab}$ of $\{x_\tau : 0 \leqslant \tau \leqslant 1\}$ define

$$\mathfrak{k}_{ab}(x) = \sum_{j=1}^N \int_a^b \widetilde{F}\left(\zeta, \frac{(\varrho_{j-1}+\varrho_j)\sigma\dot{x}}{(\zeta-x)^2}\right) d\zeta - \sum_{j=1}^N \int_a^b \widetilde{F}\left(\zeta, \frac{\varrho_j\sigma\dot{x}}{(\zeta-x)^2}\right) d\zeta. \tag{5.6}$$

In the first sum we extend $(\varrho_{j-1}+\varrho_j)\sigma\dot{x}/(\zeta-x)^2$ to $S^1 \backslash (J_{j-1} \cup J_j)$ by 0, and integrate along paths in $\mathbf{P}_1 \backslash x(\bar{J}_{j-1} \cup \bar{J}_j)$; we interpret the second sum similarly. The neighborhood W is to be chosen so small that no $x(\bar{J}_i \cup \bar{J}_j)$ separates a and b when $x \in W$.

As above, the integrals in (5.6) are independent of the path, and define a holomorphic function in W. By Lemma 5.2, $\mathfrak{k}_{ab}$ agrees with (5.5) when x is near x_0. Furthermore, the germ of $\mathfrak{k}_{ab}$ at x_1 depends on the curve x_τ only through the choice of the ϱ_j. In fact, it does not even depend on ϱ_j: let $\mathfrak{k}'_{ab}$ be the function obtained if in (5.6) the ϱ_j are replaced by another partition of unity ϱ'_h. It will suffice to show that $\mathfrak{k}_{ab} = \mathfrak{k}'_{ab}$ under the additional assumption that each ϱ'_h is supported in some J_j. In this case, $\mathfrak{k}'_{ab}$ is holomorphic in W and agrees with $\mathfrak{k}_{ab}$ near x_0, hence on all of W.

Therefore, by varying the partition of unity ϱ_j, we can use (5.6) to define $\mathfrak{k}_{ab}(x)$ depending holomorphically on $a, b \in \mathbf{P}_1$ and $x \in W_{ab}$. Also, $\mathfrak{k}_{ab} + \mathfrak{k}_{bc} = \mathfrak{k}_{ac}$ on W_{abc}, since this is so in a neighborhood of constant loops, and W_{abc} is connected.

Now, to obtain a continuation of h_{ab} in (5.4), construct a partition of unity $\sigma_1, \sigma_2, \sigma_3 \in C^\infty(S^1)$ so that $\operatorname{supp}(\sigma_i+\sigma_j)\neq S^1$ and $\bigcap_{j=1}^3 \operatorname{supp}\sigma_j=\varnothing$. Setting $\sigma_0=\sigma_3$, in light of Lemma 5.2 we can rewrite (5.4) as

$$h_{ab}(x)=\sum_{j=1}^3 \int_a^b \widetilde{F}\left(\zeta, \frac{(\sigma_{j-1}+\sigma_j)\dot{x}}{(\zeta-x)^2}\right) d\zeta-\sum_{j=1}^3 \int_a^b \widetilde{F}\left(\zeta, \frac{\sigma_j \dot{x}}{(\zeta-x)^2}\right) d\zeta$$

and continue each term to LU_{ab}, as above. We obtain a holomorphic cocycle $\beta(F)=\mathfrak{h}=(\mathfrak{h}_{ab})$, with $\mathfrak{h}_{ab}$ depending holomorphically on a and b, and one easily checks that each $\mathfrak{h}_{ab}$ is cuspidal. Therefore $\beta(F)\in\mathfrak{H}$. Finally, $\alpha\beta(F)$ can be computed by considering $d_\zeta \mathfrak{h}_{a\zeta}(x)$, with a in the same component of $\mathbf{P}_1\backslash x(S^1)$ as ζ, so that (5.4) gives

$$d_\zeta \mathfrak{h}_{a\zeta}(x)=d_\zeta h_{a\zeta}(x)=\widetilde{F}\left(\zeta, \frac{\dot{x}}{(\zeta-x)^2}\right) d\zeta=F\left(\zeta, \frac{1}{\zeta-x}\right) d\zeta.$$

Thus $\alpha\beta(F)=F$ as needed.

(b) *The kernel of* α. Take an irreducible submodule of $\operatorname{Ker}\alpha$, spanned by a vector $\mathfrak{l}$ of lowest weight $-n\leqslant 0$. Since $F=\alpha(\mathfrak{l})=0$, (4.2) implies that $\mathfrak{l}_{ab}(x)=0$ if x does not separate a and b; hence, by analytic continuation, whenever $\operatorname{ind}_{ab} x=0$. By the cocycle relation $\mathfrak{l}_{ac}(x)=\mathfrak{l}_{bc}(x)$ if $\operatorname{ind}_{ab} x=0$, i.e., if $\operatorname{ind}_{ac} x=\operatorname{ind}_{bc} x$.

Consider the components of $LU_{0\infty}$

$$X_r=\{x\in LU_{0\infty}: \operatorname{ind}_{0\infty} x=r\}, \quad r\in\mathbf{Z}.$$

Let

$$x_1(t)=e^{2\pi i r t} \quad \text{and} \quad y(t)=e^{4\pi i r t}+e^{6\pi i r t-4}. \tag{5.7}$$

We shall presently show that whenever $x\in LU_{0\infty}$ is in a sufficiently small neighborhood of x_1, and $(\varkappa, \lambda)\in\mathbf{C}^2\backslash\{(0,0)\}$, then $z_{\varkappa\lambda}=\varkappa x+\lambda y\in X_r+\mathbf{C}$. It follows that with such x and y we can define $h(\varkappa,\lambda)=\mathfrak{l}_{a\infty}(z_{\varkappa\lambda})$, where a is chosen so that $\operatorname{ind}_{a\infty} z_{\varkappa\lambda}=r$. Thus $h\in\mathcal{O}(\mathbf{C}^2\backslash\{(0,0)\})$, and by Hartogs' theorem it extends to all of $\mathbf{C}^2$; also, it is homogeneous of degree $-n$. It follows that h is constant, indeed zero when $n>0$. In all cases, $\mathfrak{l}_{0\infty}(x)=h(1,0)=h(0,1)$ is independent of x. This being true for x in a nonempty open set, $\mathfrak{l}_{0\infty}$ is constant on X_r. It follows that $\mathfrak{l}_{a\infty}(x)=\mathfrak{l}_{0\infty}(x-a)$ is locally constant, and so is $\mathfrak{l}_{ab}=\mathfrak{l}_{a\infty}-\mathfrak{l}_{b\infty}$. Moreover, $\mathfrak{l}_{ab}=0$ unless $n=0$.

Suppose now that $n=0$, and let $\mathfrak{l}_{0\infty}|_{X_1}=l\in\mathbf{C}$. We have $\mathfrak{l}_{a\infty}(x)=\mathfrak{l}_{0\infty}(x-a)=l$ if $\operatorname{ind}_{a\infty} x=1$. Choose a homeomorphic $x\in L\mathbf{C}$ and $a,b\in\mathbf{C}\backslash x(S^1)$ so that $\operatorname{ind}_{ab} x=1$; say that b is in the unbounded component. Then $\mathfrak{l}_{ab}(x)=\mathfrak{l}_{a\infty}(x)-\mathfrak{l}_{b\infty}(x)=l$, and the same will hold if x is slightly perturbed. It follows that $\mathfrak{l}_{ab}(x)=l$ whenever $\operatorname{ind}_{ab} x=1$, and in this case $\mathfrak{l}_{ba}(x)=-l$. Finally, with a generic $y\in LU_{ab}$ choose $a_0=a, a_1, ..., a_m=b$ in $\mathbf{P}_1\backslash y(S^1)$ so that $\operatorname{ind}_{a_{j-1}a_j} y=\pm 1$. Then

$$\mathfrak{l}_{ab}(y)=\sum_{j=1}^{m}\mathfrak{l}_{a_{j-1}a_j}(y)=l\sum_{j=1}^{m}\operatorname{ind}_{a_{j-1}a_j}y=l\operatorname{ind}_{ab}y.$$

We conclude that any irreducible submodule of $\operatorname{Ker}\alpha$ is spanned by $\mathfrak{h}$ in (5.1), whence $\operatorname{Ker}\alpha$ itself is spanned by $\mathfrak{h}$, as claimed.

We still owe the proof that $\varkappa x+\lambda y\in X_r+\mathbf{C}$ unless $\varkappa=\lambda=0$, for x near x_1 and y given in (5.7). In fact, the general statement follows once we prove it for $r=1$ and $x=x_1$, which we henceforward assume. If $|\varkappa|\geqslant 2|\lambda|$ then $z_{\varkappa\lambda}\in X_1$ by Rouché's theorem. Otherwise consider the polynomial

$$P(\zeta)=\varkappa\zeta+\lambda(\zeta^2+e^{-4}\zeta^3),\quad \zeta\in\mathbf{C}.$$

For fixed $|\zeta|<2$ the equation $P(\eta)=P(\zeta)$ has two solutions with $|\eta|<5$, again by Rouché's theorem. One of the solutions is $\eta=\zeta$. Let $\eta=R(\zeta)$ be the other one, so that R is holomorphic. There are only finitely many ζ with $|\zeta|=|R(\zeta)|=1$. Indeed, otherwise $|R(\zeta)|=1$ would hold for all unimodular ζ, and by the reflection principle R would be rational. However, $P(R(\zeta))=P(\zeta)$ cannot hold with rational $R(\zeta)\neq\zeta$. We conclude that $z_{\varkappa\lambda}(S^1)$ has only finitely many self-intersection points.

Since $P(0)=0$, $\operatorname{ind}_{0\infty} z_{\varkappa\lambda}\geqslant 1$. Drag a point a from 0 to ∞ along a path that avoids multiple points of $z_{\varkappa\lambda}(S^1)$. Each time we cross $z_{\varkappa\lambda}(S^1)$, $\operatorname{ind}_{a\infty} z_{\varkappa\lambda}$ changes by ± 1. It follows that $\operatorname{ind}_{a\infty} z_{\varkappa\lambda}=1$ for some a, which completes the proof.

For the space $L_{1,p}\mathbf{P}_1$, Theorems 2.1, 2.2 and the construction in Theorem 5.1 lead to explicit representations of elements of $\mathfrak{H}$. First there are the multiples of the cocycle (5.1), and then there is the complementary subspace $\beta(\mathfrak{F})=\bigoplus_{n=0}^{p-1}\beta(\mathfrak{F}^n)$, see Theorem 2.2. According to Theorems 2.1 and 2.2 elements of $\mathfrak{F}^n$ are of form

$$F(\zeta,y)=\sum_{\nu=0}^{2n}\zeta^\nu\langle\Phi_\nu,\dot{y}^{n+1}\rangle,\quad \Phi_\nu\in L^{p/(n+1)}(S^1)^*.$$

Following the proof of Theorem 5.1, to compute $\mathfrak{h}=\beta(F)$ we set

$$\widetilde{F}(\zeta,z)=\sum_{\nu=0}^{2n}\zeta^\nu\langle\Phi_\nu,z^{n+1}\rangle.$$

The substitution $\zeta=\xi+c$ shows that

$$R_\nu(a,b,c)=\int_a^b\frac{\zeta^\nu\,d\zeta}{(\zeta-c)^{2n+2}},\quad 0\leqslant\nu\leqslant 2n,\ c\in\mathbf{P}_1\backslash\{a,b\},$$

are rational functions with poles at $c=a,b$, so that

$$\mathfrak{h}_{ab}(x)=\int_a^b \widetilde{F}\left(\zeta,\frac{\dot{x}}{(\zeta-x)^2}\right)d\zeta=\sum_{\nu=0}^{2n}\langle\Phi_\nu,R_\nu(a,b,x)\dot{x}^{n+1}\rangle,$$

when x does not separate a and b. However, the right-hand side makes sense for any $x\in LU_{ab}$ and, as one checks, defines $\mathfrak{h}=\beta(F)$. For example, if F, and hence $\mathfrak{h}$, are of lowest weight, then $\Phi_\nu=0$ for $\nu\geqslant 1$, and

$$\mathfrak{h}_{ab}(x)=\left\langle\Phi_0,\frac{\dot{x}^{n+1}}{2n+1}\left(\frac{1}{(x-a)^{2n+1}}-\frac{1}{(x-b)^{2n+1}}\right)\right\rangle. \tag{5.8}$$

Letting $n=0$ and $\langle\Phi_0,z\rangle=\int_{S^1}z/2\pi i$, formula (5.8) recovers the locally constant cocycle (5.1) as well. Thus we have proved the following result:

THEOREM 5.3. *In the case of $W^{1,p}$ loop spaces, any lowest-weight cocycle in the n-th isotypical subspace $\mathfrak{H}^n\subset\mathfrak{H}$ is of form* (5.8) *with* (*a unique*) $\Phi_0\in L^{p/(n+1)}(S^1)^*$, $0\leqslant n\leqslant p-1$.

6. Synthesis

In this last section we show how the results obtained by now imply the theorems of the introduction. Theorems 0.1 and 0.2 follow from the isomorphism $H^{0,1}(L\mathbf{P}_1)\approx\mathfrak{H}$ of G-modules (Theorem 3.3) and from the isomorphism $\mathfrak{H}\approx\mathbf{C}\oplus\mathfrak{F}$, a consequence of Theorem 5.1. In particular, $H^{0,1}(L\mathbf{P}_1)^G\approx\mathbf{C}\oplus\mathfrak{F}^0$. The latter being isomorphic to the dual of $L^-\mathbf{C}=C^{k-1}(S^1)$ (resp. $W^{k-1,p}(S^1)$) by Theorem 2.1, Theorem 0.3 also follows. Finally, Theorem 0.4 is a consequence of Theorems 2.2 and 2.1.

Seemingly we are done with all the proofs. However, Theorem 3.3 has not yet been proved for loop spaces $L_{1,p}\mathbf{P}_1$, $p<3$, and we still have to revisit spaces of loops of low regularity. This will give us the opportunity to explicitly represent classes in $H^{0,1}(L_{1,p}\mathbf{P}_1)$, in fact, for all $p\in[1,\infty)$.

Generally, given a complex manifold M, $1\leqslant p<\infty$, and a natural number $m\leqslant p$, consider the space $C^\infty_{0,q}((T^*M)^{\otimes m})$ of $(T^*M)^{\otimes m}$-valued $(0,q)$-forms on M. If ω is such a form, $v\in\bigoplus^q T_s^{0,1}M$ and $w\in T_s^{1,0}M$, we can pair $\omega(v)\in(T_s^*M)^{\otimes m}$ with $w^{\otimes m}$, to obtain what we shall denote $\omega(v,w^m)\in\mathbf{C}$. Write LM for the space of $W^{1,p}$-loops in M, and observe that the tangent space $T_x^{0,1}LM$ is naturally isomorphic to the space $W^{1,p}(x^*T^{0,1}M)$ of $W^{1,p}$-sections of the induced bundle $x^*T^{0,1}M\to S^1$ (see [L2, Proposition 2.2] in the case of C^k-loops).

There is a bilinear map

$$I=I_q\colon L^{p/m}(S^1)^*\times C^\infty_{0,q}((T^*M)^{\otimes m})\longrightarrow C^\infty_{0,q}(LM),$$

obtained by the following Radon-type transformation. If

$$(\Phi,\omega)\in L^{p/m}(S^1)^*\times C^\infty_{0,q}((T^*M)^{\otimes m}),$$

$x\in LM$ and $\xi\in\bigoplus^q T^{0,1}_x LM\approx\bigoplus^q W^{1,p}(x^*T^{0,1}M)$, then $\omega(\xi,\dot{x}^m)\in L^{p/m}(S^1)$. Define $I(\Phi,\omega)=f$ by

$$f(\xi)=\langle\Phi,\omega(\xi,\dot{x}^m)\rangle.$$

One verifies that $\bar\partial I(\Phi,\omega)=I(\Phi,\bar\partial\omega)$, whence I_q induces a bilinear map

$$L^{p/m}(S^1)^*\times H^{0,q}((T^*M)^{\otimes m})\longrightarrow H^{0,q}(LM).$$

Henceforward we take $M=\mathbf{P}_1$, $q=1$, $m=n+1$ and ω given on $\mathbf{C}$ by

$$\omega=\frac{-1}{2n+1}\,\frac{\bar\zeta^{2n}\,d\bar\zeta\otimes(d\zeta)^{n+1}}{(1+|\zeta|^{4n+2})^{(2n+2)/(2n+1)}},\quad \zeta\in\mathbf{C},$$

so that $f=I_1(\Phi,\omega)$ is a closed form on $L\mathbf{P}_1$. Explicitly,

$$f(\xi)=\frac{-1}{2n+1}\left\langle\Phi,\frac{\xi\bar{x}^{2n}\dot{x}^{n+1}}{(1+|x|^{4n+2})^{(2n+2)/(2n+1)}}\right\rangle,\quad \xi\in T^{0,1}_x L\mathbf{P}_1. \tag{6.1}$$

To compute its image in $\mathfrak{H}$ under the map of Theorem 3.3, let

$$\theta_a=\frac{1}{2n+1}\left(\frac{\zeta^{-2n-1}}{(1+|\zeta|^{4n+2})^{1/(2n+1)}}-\zeta^{-2n-1}+(\zeta-a)^{-2n-1}\right)(d\zeta)^{n+1}\quad\text{on } U_a.$$

Thus $\bar\partial\theta_a=\omega|_{U_a}$, and the cuspidal functions $u_a=I_0(\Phi,\theta_a)\in C^\infty(LU_a)$ solve $\bar\partial u_a=f|_{LU_a}$. Hence the image of f in $\mathfrak{H}$ is

$$\mathfrak{h}_{ab}(x)=u_a(x)-u_b(x)=\left\langle\Phi,\frac{\dot{x}^{n+1}}{2n+1}\left(\frac{1}{(x-a)^{2n+1}}-\frac{1}{(x-b)^{2n+1}}\right)\right\rangle.$$

Comparing this with Theorem 5.3 we see that by associating a lowest weight $\mathfrak{h}\in\mathfrak{H}^n$ with the functional $\Phi=\Phi_0$ of (5.8), and then $f\in C^\infty_{0,1}(L\mathbf{P}_1)$ of (6.1), the image of f in $\mathfrak{H}$ will be $\mathfrak{h}$. In particular, the class $[f]\in H^{0,1}(L\mathbf{P}_1)$ is also of lowest weight $-n$. Therefore the linear map $\mathfrak{h}\mapsto[f]$, defined for $\mathfrak{h}\in\mathfrak{H}^n$ of lowest weight, can be extended to a G-morphism $\mathfrak{H}^n\to H^{0,1}(L\mathbf{P}_1)$, and then to a G-morphism $\bigoplus_{n=0}^{p-1}\mathfrak{H}^n=\mathfrak{H}\to H^{0,1}(L\mathbf{P}_1)$, inverse to the morphism $H^{0,1}(L\mathbf{P}_1)\to\mathfrak{H}$ of Theorem 3.3. This completes the proof of Theorem 3.3, and now we are really done.

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