Partitioning pairs of countable ordinals

by

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We show that the pairs of countable ordinals can be colored with uncountably many colors so that every uncountable set contains pairs of every color. This gives a definitive limitation on any form of a Ramsey Theorem for the uncountable which reduces the set of colors on some uncountable square. The first such limitation was given by Sierpiński [21] for only two colors. This was later improved by Laver (see [13]) to three colors and then by Galvin and Shelah [4] to four colors (see also Blass [1]). Our method is not based on the existence of certain uncountable linear orderings (an approach still of interest) as was the case with [21], [13], [4] and [1], but on a fine analysis of the concept of a special Aronszajn tree. This analysis will give us also a new proof of the existence of an uncountable linear ordering whose square is the union of countably many chains and many other facts about the uncountable.

All sections of this paper can be read independently from each other, but for a fuller understanding of our methods and definitions, a reading of the first three sections might be necessary. The last section contains a list of most of the recent applications of our methods as well as various other remarks concerning the previous uses of the Continuum Hypothesis in coloring pairs of countable ordinals. It should be pointed out that the main purpose of this paper is to be an exposition of our *method* of minimal walks in the realm of all *countable* ordinals because it is this case which is most often relevant to the Ramsey Problem for the uncountable. This is one of the reasons why many of the results, especially those concerning larger squares, are not stated in their full generality. Interested readers should not have any problems in formulating them in any generality they might wish to consider.

(1) Supported by a grant from Bell Companies.

§ 1. Minimal walks

For an ordinal θ (usually regular and uncountable) put

$$[\theta]^2 = \{\langle \alpha, \beta \rangle : \alpha < \beta < \theta\}.$$

In this article we shall quite often recursively define certain functions

$$a : [\theta]^2 \to X$$

where X is either an ordinal or a set of certain (finite) sequences of ordinals. For technical reasons we shall always need to implicitly assume that

$$a(\alpha, \alpha) = 0 \quad \text{or} \quad a(\alpha, \alpha) = \langle\ \rangle$$

in the first case or in the second case, respectively. For a partially ordered set P, by σP we shall denote the set

$$\{s : \alpha \to P : \alpha \in \mathrm{Ord} \text{ and } s \text{ is strictly increasing}\}$$

and consider σP as a tree under the ordering $\subset$. A mapping s with domain an ordinal, denoted by $l(s)$, will be called a *sequence*. Thus for sequences s and t, $s \subset t$ means s is an initial part of t. For sequences s and t put

$$\Delta(s, t) = \min\{\delta : s(\delta) \neq t(\delta)\},$$

where we assume that $\Delta(s, t) = l(s)$ if $s \subseteq t$. If X is a set of sequences with ranges in a linearly ordered set A, $<_A$ by $<_0$ and $<_1$ we denote the right and left lexicographical ordering on X, respectively, defined by

$$s <_0 t \text{ iff } s \supset t \quad \text{or} \quad s(\Delta(s, t)) <_A t(\Delta(s, t)),$$

and

$$s <_1 t \text{ iff } s \subset t \quad \text{or} \quad s(\Delta(s, t)) <_A t(\Delta(s, t)).$$

For an ordinal θ, set

$$\mathbf{Q}_\theta = {}^{\omega}\theta, <_0.$$

The linearly ordered set $\mathbf{Q}_\theta$ has been quite often used in standard recursive constructions of Aronszajn trees of height θ^+. A tree T constructed in such a way would usually consist of sequences from $\sigma \mathbf{Q}_\theta$ which have maximal elements. This is done in order to ensure T to be *special*. In this section we shall present a new and more canonical construction of such trees T, but let us first mention a result which says that for certain cardinals θ there is no need in taking special care about the elements of T since the whole tree $\sigma\mathbf{Q}_\theta$ will be special, anyway.

(1.1) *$\sigma\mathbf{Q}_\theta$ is the union of θ antichains iff* $\mathrm{cf}\,\theta \neq \omega$.

Proof. Assume first that $\mathrm{cf}\,\theta = \omega$ and let A_ξ $(\xi<\theta)$ be given antichains of $\sigma\mathbf{Q}_\theta$. Fix a sequence $\theta_i \uparrow \theta$, $(i<\omega)$. For t in $\mathbf{Q}_\theta$ let t^* denote the Dedekind cut of $\mathbf{Q}_\theta$ determined by the decreasing sequence

$$t, t^\frown 0, t^\frown 00, \ldots$$

Note that the cofinality of the maximal term of t is equal to the cofinality of the cut t^*. Thus if $i<\omega$ and if $\mathrm{cf}(\max(t)) = \theta_i^+$, then any u in $\sigma\mathbf{Q}_\theta$ with $\sup < t^*$ can be extended to a v with $\sup < t^*$ with the property that no extension of v with $\sup < t^*$ is in

$$B_i = \bigcup_{\xi<\theta_i} A_\xi.$$

So recursively on $i<\omega$ we can build a sequence t_i in $\mathbf{Q}_\theta$ and an $\subset$-increasing sequence v_i in $\sigma\mathbf{Q}_\theta$ such that:

(1) $t_{i+1} < t_i^*$

(2) $\mathrm{cf}(\max(t_i)) = \theta_i^+$,

(3) $\sup v_i < t_i^*$,

(4) no extension of v_i with $\sup < t_i^*$ is in B_i.

Let $v = \bigcup_i v_i$. Then

$$v \notin \bigcup_{\xi<\theta} A_\xi.$$

Assume now $\mathrm{cf}\,\theta > \omega$. By Theorem 14 of [24] it suffices to show that

$$\sigma\mathbf{Q}_\theta \restriction \{\delta < \theta^+ : \mathrm{cf}\,\delta = \mathrm{cf}\,\theta\}$$

is the union of θ antichains. Let $\lambda = \mathrm{cf}\,\theta$, and let v be a given member of $\sigma\mathbf{Q}_\theta$ of length δ with $\mathrm{cf}\,\delta = \lambda$. Fix a sequence $\delta_\alpha \uparrow \delta$, $(\alpha<\lambda)$. For $\alpha<\lambda$, set

$$s_\alpha = v(\delta_\alpha).$$

Let $n(v)$ be the minimal $n<\omega$ such that

$$\{\alpha<\lambda : |s_\alpha| = n\}$$

has size λ. By refining the sequence s_α, we may assume that for some $s(v)$ of length $i<n(v)$, we have

(5) $s(v)\subset s_\alpha$ for all $\alpha<\lambda$,

(6) $s_\alpha(i)<s_\beta(i)$ for $\alpha<\beta<\lambda$.

Let

$$\theta(v)=\sup\{s_\alpha(i)\colon \alpha<\lambda\}$$

It is now easily checked that if $u\subset v$ are members of $\sigma\mathbf{Q}_\theta$ with lengths of cofinality λ, then

$$\langle n(v), s(v), \theta(v)\rangle \neq \langle n(u), s(u), \theta(u)\rangle.$$

Since there exist only θ such triples, we finish.

We shall say that $\langle C_\alpha\colon \alpha<\theta\rangle$ is a *c-sequence* on θ iff C_α is a closed and unbounded subset of α for all $\alpha<\theta$. We shall always implicitly assume that $C_{\alpha+1}=\{\alpha\}$ although many of our definitions will be valid without this assumption. To every c-sequence $\langle C_\alpha\colon \alpha<\theta\rangle$ we associate

$$\varrho_0=\varrho_0(C_\alpha\colon \alpha<\theta)\colon [\theta]^2\to\mathbf{Q}_\theta$$

defined as follows

$$\varrho_0(\alpha,\beta)=\langle \operatorname{tp}(C_\beta\cap\alpha)\rangle^\frown\varrho_0(\alpha,\min(C_\beta\setminus\alpha)).$$

Note that $\varrho_0(\cdot,\alpha)\colon \alpha\to\mathbf{Q}_\theta$ is strictly increasing for all α, whence

$$T(\varrho_0)=\{\varrho_0(\cdot,\beta)\restriction\alpha\colon \alpha\leqslant\beta<\theta\}$$

is a subtree of $\sigma\mathbf{Q}_\theta,\subset$. For $t\in\mathbf{Q}_\theta$ and $\alpha<\theta$, set

$$F_t(\alpha)=\{\xi\leqslant\alpha\colon \varrho_0(\xi,\alpha)\subseteq t\}.$$

Note that since $\varrho_0(\xi,\alpha)$ is a sequence with domain some integer, the inclusion $\varrho_0(\xi,\alpha)\subseteq t$ just means that $\varrho_0(\xi,\alpha)$ is an initial part of t. Note also that by our convention, $\alpha\in F_t(\alpha)$, and that $F_{\varrho_0(\alpha,\beta)}(\beta)$ is just the trace of the minimal walk from β to α along the sequence $\langle C_\alpha\colon \alpha<\theta\rangle$. If ξ is an *ordinal* $\leqslant\alpha$ then we shall use $F_\xi(\alpha)$ as another way of denoting $F_t(\alpha)$ where $t=\varrho_0(\xi,\alpha)$. In case we consider F_ξ generated by some other function rather than ϱ_0 we shall always add an upper index to F to make the distinction. So the present $F_t(\alpha)$ will later be denoted by $F_t^0(\alpha)$.

The following is a list of straightforward but useful facts about the function ϱ_0.

(1.2) Let $\alpha<\beta, \gamma<\theta$ *and* $t=\varrho_0(\alpha,\beta)=\varrho_0(\alpha,\gamma)$. *Suppose that* $C_\eta\cap\alpha=C_\xi\cap\alpha$ *for all* ξ *and* η *with* $\varrho_0(\eta,\beta)=\varrho_0(\xi,\gamma)\sqsubseteq t$. *Then* $\varrho_0(\cdot,\beta)$ *and* $\varrho_0(\cdot,\gamma)$ *agree below* α.

(1.3) $|T(\varrho_0)\cap{}^{\alpha}\mathbf{Q}_\theta|\leq|\{C_\beta\cap\alpha\colon \alpha\leq\beta<\theta\}|+\aleph_0$ *for all* $\alpha<\theta$.

(1.4) *Suppose* $\alpha<\gamma<\theta$ *and* t *is an initial part of* $\varrho_0(\alpha,\gamma)$. *Then* $t\sqsubseteq\varrho_0(\beta,\gamma)$ *for all* β *with* $\alpha\leq\beta\leq\min F_t(\gamma)$.

(1.5) *Assume* $0<\alpha=\bigcup\alpha<\beta,\gamma<\theta$ *and* $\varrho_0(\cdot,\beta)$ *and* $\varrho_0(\cdot,\gamma)$ *agree below* α. *Then* $\varrho_0(\alpha,\beta)=\varrho_0(\alpha,\gamma)$.

(1.6) $\varrho_0(\cdot,\beta)\restriction\alpha\mapsto\varrho_0(\bigcup\alpha,\beta)$ *is a strictly increasing mapping from* $T(\varrho_0)\setminus\{\langle\ \rangle\}$ *into* $\mathbf{Q}_\theta$.

(1.7) $T(\varrho_0)$ *has a* θ*-branch iff there exist club* $C\subseteq\theta$ *and* $\xi<\theta$ *such that for all* $\xi\leq\alpha\leq\theta$ *there is* $\beta\geq\alpha$ *such that* $C\cap\alpha=C_\beta\cap[\xi,\alpha)$.

Proof. We are assuming here (and everywhere) that θ is regular and uncountable although we actually need only cf $\theta>\omega$. Clearly, only the direct implication needs a proof. Let $b\subseteq T(\varrho_0)$ be a θ-branch. Fix a limit nonzero $\alpha<\theta$ and let $\gamma_\alpha\geq\alpha$ be such that

$$\varrho_0(\cdot,\gamma_\alpha)\restriction\alpha\in b.$$

Let $t_\alpha=\varrho_0(\alpha,\gamma_\alpha)$. Let $\beta_\alpha\in F_{t_\alpha}(\gamma_\alpha)$ be maximal with the property that $C_{\beta_\alpha}\cap\alpha$ is unbounded in α. Then $C_\xi\cap\alpha$ is bounded in α for all $\xi\in F_{t_\alpha}(\gamma_\alpha)$ above β_α, so let $h(\alpha)<\alpha$ be an upper bound for all of them. By the pressing down lemma fix a ξ such that $h''A\subseteq\xi$ for some unbounded $A\subseteq\lim\cap\theta$. Let

$$C=\bigcup\{C_{\beta_\alpha}\cap[\xi,\alpha)\colon\alpha\in A\}.$$

Then ξ and C are as required.

The following is an immediate corollary of (1.7).

(1.8) *The following are equivalent for an inaccessible* $\theta>\omega$:

(a) θ *is weakly compact.*

(b) *For any c-sequence* $\langle C_\alpha\colon \alpha<\theta\rangle$ *on* θ *there is a club* $C\subseteq\theta$ *such that for all* $\alpha<\theta$ *there is* $\beta\geqslant\alpha$ *with* $C\cap\alpha=C_\beta\cap\alpha$.

Notice that this gives a rather short *combinatorial* proof of many properties of a weakly compact cardinal such as: θ reflects stationary sets, θ is not first inaccessible, θ is not first Mahlo, ..., etc.

Let T be a tree of height θ and let $f\colon T\to T$. Then f is *regressive* iff $f(t)<_T t$ for all nonminimal t in T. The tree T is *special* iff there is a regressive map $f\colon T\to T$ such that $f^{-1}(t)$ is the union of $<\theta$ antichains for all t in T. By [24; Theorem 14] for $\theta=\varkappa^+$, a tree T of height θ is special iff T is the union of $\varkappa$ antichains. Thus in this case our definition reduces to the standard one, but the point is that our definition makes sense even if θ is a limit cardinal while the standard one does not.

(1.9) *The following are equivalent for inaccessible* $\theta>\omega$.

(a) θ *is Mahlo*.

(b) *There are no special* θ*-Aronszajn trees*.

Proof. Only the implication from (b) to (a) needs a proof. So, assume θ is not Mahlo and let $C\subseteq\theta$ be a club consisting of singular cardinals. Fix a c-sequence $\langle C_\alpha\colon \alpha<\theta\rangle$ such that:

(1) $C_{\alpha+1}=\{\alpha\}$,

(2) $C_\alpha=(\max(C\cap\alpha),\alpha)$ if $\operatorname{cf}\alpha=\alpha$,

(3) $\operatorname{tp} C_\alpha<\min C_\alpha$ if $\operatorname{cf}\alpha<\alpha$.

Let $\varrho_0=\varrho_0(C_\alpha\colon \alpha<\theta)$ and $T=T(\varrho_0)$. We shall show that T is special, and to end this it suffices to define a regressive $f\colon T\restriction C\to T$ so that $f^{-1}(t)$ is the union of $<\theta$ antichains for all t in T.

So let $\varrho_0(\cdot,\gamma)\restriction\alpha$ in $T\restriction C$ be given. Let $t=\varrho_0(\alpha,\gamma)$ and let $\beta\in F_t(\gamma)$ be maximal with property

$$\sup(C_\beta\cap\alpha)=\alpha.$$

Then $C_\xi\cap\alpha$ is bounded in α for all $\xi\in F_t(\gamma)$ above β, so let δ be an upper bound for all of them. By (3) the type of $C_\beta\cap\alpha$ is $<\alpha$ so the ordinal code ε of $\langle\delta,\operatorname{tp}(C_\beta\cap\alpha)\rangle$ is $<\alpha$. Finally, put

$$f(\varrho_0(\cdot,\gamma)\restriction\alpha)=\varrho_0(\cdot,\gamma)\restriction\varepsilon.$$

This defines regressive $f\colon T\upharpoonright C\to T$. It is now easily seen that if t in T has height ε, then $f^{-1}(t)$ contains no chain of type $\varepsilon+2$, so it must be the union of $<\theta$ antichains.

Assume now that $\langle C_\alpha\colon \alpha<\theta\rangle$ satisfies the following condition reminiscent of the $\square$-principle of Jensen [8].

(i) If α is a limit point of C_β, then $C_\alpha=C_\beta\cap\alpha$.

Then an easy pressing down argument shows that $T(\varrho_0)$ has the following property:

(*) For any regular $\varkappa<\theta$ and $A\subseteq T(\varrho_0)$ of size $\varkappa$ there is a $B\subseteq A$ of size $\varkappa$ such that $s\cap t$ and $u\cap v$ are comparable for all $s\neq t$ and $u\neq v$ in B.

This property of trees is of independent interest and has already been considered in the literature (see, for example, [25]). Note that under (i) the condition of (1.7) reduces to

(ii) There is no club $C\subseteq\theta$ such that $C_\alpha=C\cap\alpha$ whenever α is a limit point of C.

A c-sequence satisfying (i) and (ii) is called a $\square(\theta)$-sequence. A well-known result of Jensen [8] says that, in L, $\square(\theta)$-sequences exist on any regular non-weakly compact cardinal. The following fact, apparently first noticed by Jensen himself, is an immediate consequence of the proof of this result.

(1.10) *If θ is regular and not weakly compact in L, then there is a constructible $\square(\theta)$-sequence.*

Proof. We assume the reader is familiar with [8; §6] and prove that the $\square(\theta)$-sequence $\langle C_\alpha\colon \alpha<\theta\rangle$ constructed in the proof of Theorem 6.1 of [8] remains such in the real world. This sequence is constructed by first fixing a set $B\subseteq\theta$ and a first order formula φ (with predicates ε, B, D) such that J_θ satisfies $\varphi[D,B]$ for all $D\subseteq\theta$ but for all $\alpha<\theta$ there is a $D\subseteq\alpha$ such that J_θ satisfies $\neg\varphi[D,B\cap\alpha]$. The limit ordinals $<\theta$ are split into two sets Q and its complement $\bar{Q}$ in such a way that if α is a limit point of C_β, then α and β are in the same set. So, if in the real world we have a club C in θ such that $C_\alpha=C\cap\alpha$ for all α in $\lim(C)$, then either $\lim(C)$ is a subset of Q, or else it is a subset of $\bar{Q}$. The second case is impossible since $\operatorname{tp} C_\alpha<\alpha$ for all α in $\bar{Q}\setminus\{\omega\}$ (see Cases 1–4 of §6 and Case 4 of §5 in [8]). Thus, $\lim(C)$ is a subset of Q. By pp. 288–290 of [8], for each α in Q there exist fixed $\beta>\alpha$ and $D\subseteq\alpha$ such that D and $B\cap\alpha$ are in J_β and J_α satisfies $\neg\varphi[D,B\cap\alpha]$. Moreover, if $\bar{\alpha}$ is a limit point of C_α, then there is a (uniquely determined) elementary embedding $\pi=\pi_{\bar{\alpha}\alpha}$ of $J_{\bar{\beta}}$ into J_β such that $\pi\restriction\bar{\alpha}=\mathrm{id}$, $\pi(\bar{\alpha})=\alpha$, $\pi(\bar{D})=D$ and $\pi(B\cap\bar{\alpha})=B\cap\alpha$. Thus, $\pi_{\bar{\alpha}\alpha}$'s for $\alpha<\bar{\alpha}$ in $\lim(C)$ form a directed system with well-founded direct limit since $\mathrm{cf}\,\theta>\omega$. The transitive collapse of this limit is equal to some J_ξ which contains a set $D\subseteq\theta$ for which J_θ, satisfies $\neg\varphi[D,B]$, contradicting the choice of B and φ.

Thus we have the following corollary.

(1.11) *If θ is regular and not weakly compact in L, then there is a constructible θ-Aronszajn tree with the property* (*).

(1.12) *If θ is regular and not Mahlo in L, then there is a constructible special θ-Aronszajn tree with the property* (*).

Proof. Work in L. By [8], we can pick a $\square(\theta)$-sequence $\langle C_\alpha:\alpha<\theta\rangle$ such that for some closed and unbounded $C\subseteq\theta$, $\mathrm{tp}\,C_\alpha<\alpha$ for all α in C. Let $T=T(\varrho_0)$ for $\varrho_0=\varrho_0(C_\alpha:\alpha<\theta)$. Then working as in (1.9) one shows that there is regressive $f: T\restriction C\to T$. So, T is special. But being special is upward absolute, so we are done.

Note that if we drop the requirement for the property (*) from (1.12), then we do not need the $\square$-result of [8] since we could simply choose a constructible c-sequence with properties (1), (2) and (3) of (1.9).

Let us now restrict to the case $\theta=\varkappa^+$ for some infinite $\varkappa$. In this case we can choose a c-sequence $\langle C_\alpha:\alpha<\varkappa^+\rangle$ so that $\mathrm{tp}\,C_\alpha\leq\varkappa$ for all $\alpha<\varkappa^+$. Then by (1.6) there is a strictly increasing mapping from $T(\varrho_0)$ into $\mathbf{Q}_\varkappa$ whence $T(\varrho_0)$ is special. So if, moreover, we can choose such $\langle C_\alpha:\alpha<\varkappa^+\rangle$ with the additional property

$$|\{C_\beta\cap\alpha:\alpha\leq\beta<\varkappa^+\}|\leq\varkappa \quad \text{for all} \quad \alpha<\varkappa^+,$$

the tree $T(\varrho_0)$ is a very canonical special $\varkappa^+$-Aronszajn tree. However, the main advantage of the case $\theta=\varkappa^+$ is that we could assume $\langle C_\alpha:\alpha<\varkappa^+\rangle$ to be a $\square_\varkappa$-sequence ([8]) in which case $\varrho_0:[\varkappa]^2\to\mathbf{Q}_\varkappa$ has many useful and interesting properties such as (see [25; Theorem 5]):

(iii) $|\varrho_0''[A]^2|=|A|$ for every infinite $A\subseteq\varkappa^+$ of size $\leq\varkappa$.

Moreover, in this case we can also use ϱ_0 to step up many combinatorial properties from $\varkappa$ to $\varkappa^+$. To illustrate this let us choose one of the principles considered in [5].

Exactly the same argument will show that we can also step up many other properties such as the principle $H_n(\varkappa)$ of [5], to reprove the Theorem 7 of [25], or to get a uniform counterexample to $CC_\alpha(\mathrm{Ord})$ of [5], ..., etc.

(1.13) *Assume* $\square_\varkappa$. *Suppose that for some regular* $\lambda>\omega$ *and finite* $n\geq 2$ *there is a* $c\colon [\varkappa]^n\to\varkappa$ *such that for all* $A\subseteq\varkappa$ *of size* λ *there exist* $X\in[A]^n$ *and* $\alpha\in A\setminus X$ *with* $c(X)=\alpha$. *Then* $\varkappa^+$ *and* λ *satisfy the same property with* n *replaced by* $n+1$.

Proof. Fix a $\square_\varkappa$-sequence $\langle C_\alpha\colon\alpha<\varkappa^+\rangle$ and let $\varrho_0=\varrho_0(C_\alpha\colon\alpha<\varkappa^+)$. Without loss of generality we may assume that c maps $[\mathbf{Q}_\varkappa]^n$ into $\mathbf{Q}_\varkappa$ since clearly $\mathbf{Q}_\varkappa$ has size $\varkappa$.

Fix an $X\in[\varkappa^+]^{n+1}$ and let $\alpha<\beta<\gamma$ be the last three elements of X. Let $t=\varrho_0(\alpha,\gamma)\cap\varrho_0(\beta,\gamma)$ and let $\delta\in[\beta,\gamma]$ be determined by $\varrho_0(\delta,\gamma)=t$. If $\beta<\delta$ set

$$d(X)=\varrho_0(\cdot,\delta)^{-1}(c(\{\varrho_0(\xi,\delta)\colon\xi\in X\cap\delta\})),$$

when this is defined; otherwise set $d(X)=0$. A simple pressing down argument shows that $d\colon[\varkappa^+]^{n+1}\to\varkappa^+$ is as required.

We finish this section with an interesting variation of ϱ_0. So, let $\langle C_\alpha\colon\alpha<\theta\rangle$ be a c-sequence and define

$$\varrho_2=\varrho_2(C_\alpha\colon\alpha<\theta)\colon[\theta]^2\to\omega$$

as follows

$$\varrho_2(\alpha,\beta)=\varrho_2(\alpha,\min(C_\beta\setminus\alpha))+1.$$

Thus, $\varrho_2(\alpha,\beta)=|\varrho_0(\alpha,\beta)|$, i.e., $\varrho_2(\alpha,\beta)$ is the number of steps of the minimal walk from β to α. To state an interesting property of ϱ_2, let the *distance* between two sequences of integers s and t be equal to the supremum of absolute values of $s(\xi)-t(\xi)$ for ξ in $l(s)\cap l(t)$.

(1.14) *If* $\langle C_\alpha\colon\alpha<\theta\rangle$ *is a* $\square(\theta)$*-sequence, then* $\varrho_2=\varrho_2(C_\alpha\colon\alpha<\theta)$ *has the following properties:*

(a) *The distances between* $\varrho_2(\cdot,\alpha)$ *and* $\varrho_2(\cdot,\beta)$ *is finite for all* $\alpha<\beta<\theta$.

(b) *For any* θ*-sequence* t *of integers there is a* $\delta<\theta$ *such that the distance between* t *and* $\varrho_2(\cdot,\delta)$ *is infinite.*

(c) $T(\varrho_2)$ *is a* θ*-Aronszajn tree.*

Proof. (a) is proved inductively on α and β, and it is an immediate consequence of the property (i) of $\langle C_\alpha : \alpha<\theta\rangle$ which gives that $\varrho_0(\cdot,\alpha)\subset\varrho_0(\cdot,\beta)$ whenever α is a limit point of C_β.

To show (b) pick an integer n such that the set S of all $\xi<\theta$ for which $t(\xi)=n$ is unbounded. Now find a closed enough ordinal $\delta<\theta$ such that there is an arbitrarily long walk from δ to an element of $S\cap\delta$. This will use the property (ii) of $\langle C_\alpha : \alpha<\theta\rangle$.

(c) is an immediate consequence of (a), (b) and (1.3).

§ 2. The functions ϱ_1 and ϱ

In this section we shall assume $\theta=\varkappa^+$ and we shall define two new closely related functions ϱ_1 and ϱ. Again we shall start with a c-sequence $\langle C_\alpha : \alpha<\varkappa^+\rangle$ but with some additional properties.

For an ordinal ν let ν^+ denote the minimal infinite cardinal above ν. Let $\varkappa$ be an infinite cardinal and let $\langle C_\alpha : \alpha<\varkappa^+\rangle$ be a fixed c-sequence such that

$$\operatorname{tp} C_\alpha \leq \varkappa \quad \text{for all} \quad \alpha<\varkappa^+.$$

Define $\varrho_1=\varrho_1(C_\alpha : \alpha<\varkappa^+) : [\varkappa^+]^2\to\varkappa$ by

$$\varrho_1(\alpha,\beta)=\max\{\operatorname{tp}(C_\beta\cap\alpha),\varrho_1(\alpha,\min C_\beta\setminus\alpha)\}.$$

Thus, $\varrho_1(\alpha,\beta)=\max(\text{range } \varrho_0(\alpha,\beta))$.

(2.1) *If $\alpha<\beta<\varkappa^+$ and $\nu<\varkappa$, then*
(a) $|\{\xi\leq\alpha : \varrho_1(\xi,\alpha)\leq\nu\}|<\nu^+$,
(b) $|\{\xi\leq\alpha : \varrho_1(\xi,\alpha)\neq\varrho_1(\xi,\beta)\}|<\varrho_1(\alpha,\beta)^+$.

Proof. The proof is by induction on α and β, respectively. Since the proofs of (a) and (b) are very similar let us prove only (a). So let $A\subseteq\alpha$ be a set of type ν^+. We have to find a $\xi\in A$ such that $\varrho_1(\xi,\alpha)>\nu$. This will certainly be true if $\operatorname{tp}(C_\alpha\cap\xi)>\nu$ for some $\xi\in A$. So assume

$$\operatorname{tp}(C_\alpha\cap\xi)\leq\nu \text{ for all } \xi\in A.$$

Then there exist $\eta\in C_\alpha$ and $B\subseteq A$ of size ν^+ such that $\eta=\min(C_\alpha\setminus\xi)$ for all $\xi\in B$. Hence

$$\varrho_1(\xi,\alpha)=\max\{\operatorname{tp}(C_\alpha\cap\eta),\varrho_1(\xi,\eta)\}$$

for all $\xi \in B$. By induction hypothesis pick a $\xi \in B$ such that

$$\varrho_1(\xi, \eta) > \nu \geq \mathrm{tp}(C_\alpha \cap \eta) = \mathrm{tp}(C_\alpha \cap \xi).$$

Then $\varrho_1(\xi, \alpha) = \varrho_1(\xi, \eta) > \nu$. This finishes the proof.

Note that $\varrho_1(\cdot, \alpha)$ is not necessarily $1-1$. This can be corrected as follows, where the operations in question are ordinal multiplication and exponentiation.

$$\bar{\varrho}_1(\alpha, \beta) = 2^{\varrho_1(\alpha, \beta)} \cdot (2 \cdot \mathrm{tp}\,\{\xi \leq \alpha : \varrho_1(\xi, \beta) = \varrho_1(\alpha, \beta)\} + 1).$$

The following is an easy consequence of (2.1).

(2.2) *If* $\alpha < \beta < \varkappa^+$ *and* $\nu < \varkappa$, *then*
(a) $\bar{\varrho}_1(\cdot, \alpha)$ *is* $1-1$,
(b) $|\{\xi \leq \alpha : \bar{\varrho}_1(\xi, \alpha) \neq \bar{\varrho}_1(\xi, \beta)\}| < \bar{\varrho}_1(\alpha, \beta)^+$.

Assume now $\langle C_\alpha : \alpha < \varkappa^+ \rangle$ is a $\square_\varkappa$-sequence, i.e. that $\langle C_\alpha : \alpha < \varkappa^+ \rangle$ moreover satisfies the property (i) of § 1. For $\alpha < \beta < \varkappa^+$ let

$$\eta_{\alpha\beta} = \text{maximal limit point of } C_\beta \cap (\alpha + 1),$$

if any exists; otherwise let $\eta_{\alpha\beta} = 0$. Now we are ready to define

$$\varrho = \varrho(C_\alpha : \alpha < \varkappa^+) : [\varkappa^+]^2 \to \varkappa$$

by

$$\varrho(\alpha, \beta) = \max\{\mathrm{tp}(C_\beta \cap \alpha), \varrho(\alpha, \min C_\beta \setminus \alpha), \varrho(\xi, \alpha), (\xi \in C_\beta \cap [\eta_{\alpha\beta}, \alpha))\}.$$

(2.3) *If* $\alpha < \beta < \gamma < \varkappa^+$, $\nu < \varkappa$, *and* $0 < \delta = \bigcup \delta < \varepsilon < \varkappa^+$, *then*
(a) $|\{\xi \leq \alpha : \varrho(\xi, \alpha) \leq \nu\}| < \nu^+$,
(b) $\varrho(\alpha, \gamma) \leq \max\{\varrho(\alpha, \beta), \varrho(\beta, \gamma)\}$,
(c) $\varrho(\alpha, \beta) \leq \max\{\varrho(\alpha, \gamma), \varrho(\beta, \gamma)\}$,
(d) *there is a* $\zeta < \delta$ *such that* $\varrho(\xi, \varepsilon) \geq \varrho(\xi, \delta)$ *for all* $\zeta \leq \xi < \delta$.

Proof. The proof of (a) is almost identical to the proof of (2.1) (a). Let us prove (b) and (c) simultaneously by induction on γ. First of all note that the condition (i) of § 1 implies

(iv) If α is a limit point of C_β, then $\varrho(\cdot, \alpha) \subset \varrho(\cdot, \beta)$.

Let $\nu=\max\{\varrho(\alpha,\beta),\varrho(\beta,\gamma)\}$, and let

$$\xi_\alpha=\min(C_\gamma\setminus\alpha) \quad\text{and}\quad \xi_\beta=\min(C_\gamma\setminus\beta).$$

We have to show $\varrho(\alpha,\gamma)\leq\nu$.

Case 1. $\alpha<\eta_{\beta\gamma}$. Then by (iv)

$$\varrho(\alpha,\gamma)=\varrho(\alpha,\eta_{\beta\gamma}).$$

On the other hand, $\varrho(\alpha,\beta),\varrho(\eta_{\beta\gamma},\beta)\leq\nu$ so by (c), $\varrho(\alpha,\eta_{\beta\gamma})\leq\nu$ and we are done.

Case 2. $\alpha\geq\eta_{\beta\gamma}$. Then $\eta_{\alpha\gamma}=\eta_{\beta\gamma}$ so if $\xi_\alpha=\xi_\beta$, $\varrho(\alpha,\gamma)\leq\nu$ follows easily from $\varrho(\beta,\gamma)\leq\nu$. If $\xi_\alpha<\xi_\beta$, then $\varrho(\xi_\alpha,\beta)\leq\nu$, and so by (c), $\varrho(\alpha,\xi_\alpha)\leq\nu$. Similarly one checks $\varrho(\xi,\alpha)\leq\nu$ for all $\xi\in C_\gamma\cap[\eta_{\alpha\gamma},\alpha)$. Thus, $\varrho(\alpha,\gamma)\leq\nu$. This proves (b).

Assume now $\nu=\max\{\varrho(\alpha,\gamma),\varrho(\beta,\gamma)\}$ and let ξ_α and ξ_β be as above.

Case 1. $\alpha<\eta_{\beta\gamma}$. Then again by (iv),

$$\varrho(\alpha,\eta_{\beta\gamma})=\varrho(\alpha,\gamma)\leq\nu.$$

But $\varrho(\eta_{\beta\gamma},\beta)\leq\nu$ so by (b), $\varrho(\alpha,\beta)\leq\nu$.

Case 2. $\alpha\geq\eta_{\beta\gamma}$. If $\xi_\alpha=\xi_\beta=\xi$, then $\varrho(\alpha,\xi),\varrho(\beta,\xi)\leq\nu$, so by (c), $\varrho(\alpha,\beta)\leq\nu$. If $\xi_\alpha<\xi_\beta$ then $\varrho(\xi_\alpha,\beta)\leq\nu$ and $\varrho(\alpha,\xi_\alpha)\leq\nu$ so by (b), $\varrho(\alpha,\beta)\leq\nu$. This proves (c).

To prove (d) let $t=\varrho_0(\delta,\varepsilon)$. Let $\mu\in F_t(\varepsilon)$ be maximal with the property that $C_\mu\cap\delta$ is unbounded in μ and let $\zeta<\delta$ be an upper bound of

$$C_\xi\cap\delta,\quad \xi\in F_t(\varepsilon)\setminus(\mu+1).$$

Then $\varrho_0(\mu,\varepsilon)\subseteq\varrho_0(\xi,\varepsilon)$ for all ξ in $[\zeta,\mu)$, whence

$$\varrho(\xi,\varepsilon)\geq\varrho(\xi,\mu)\quad\text{for all}\quad \xi\in[\zeta,\mu).$$

But this finishes the proof since $\varrho(\cdot,\delta)\subseteq\varrho(\cdot,\mu)$ by (iv).

Again $\varrho(\cdot,\alpha)$ is not necessarily $1-1$ and we can correct this by putting

$$\bar\varrho(\alpha,\beta)=2^{\varrho(\alpha,\beta)}\cdot(2\cdot\operatorname{tp}\{\xi\leq\alpha:\varrho(\xi,\alpha)\leq\varrho(\alpha,\beta)\}+1).$$

(2.4) *The function* $\bar\varrho\colon[\varkappa^+]^2\to\varkappa$ *also satisfies* (2.3) (a)–(d) *and, moreover,* $\bar\varrho(\cdot,\alpha)$ *is* $1-1$ *for all* $\alpha<\varkappa^+$.

(2.5) *Assume* $\xi<\alpha<\beta<\varkappa^+$ *and* $\varrho(\xi,\alpha)>\varrho(\alpha,\beta)$. *Then* $\varrho(\xi,\alpha)=\varrho(\xi,\beta)$.

Proof. By (2.3) (b) and (c),

$$\varrho(\xi,\beta)\leqslant\max\{\varrho(\xi,\alpha),\varrho(\alpha,\beta)\}=\varrho(\xi,\alpha),$$
$$\varrho(\xi,\alpha)\leqslant\max\{\varrho(\xi,\beta),\varrho(\alpha,\beta)\}.$$

But the last maximum must be equal to $\varrho(\xi,\beta)$ because $\varrho(\xi,\alpha)$ is bigger than $\varrho(\alpha,\beta)$.

Note that we have actually shown that any $a\colon[\varkappa^+]^2\to\varkappa$ satisfying (2.3) (a), (b) and (c) also satisfies (2.1) (a) and (b). The next result shows that, at least in the cases of a successor of a regular cardinal, some sort of a converse to this is true. We do not give the proof since it is very similar (and easier) than the proof of (2.3).

(2.6) *Suppose $\varkappa$ is regular and $a\colon[\varkappa^+]^2\to\varkappa$ satisfies*
(a) $|\{\xi\leqslant\alpha\colon a(\xi,\alpha)\leqslant\nu\}|<\varkappa$ *for all* $\alpha<\varkappa^+$ *and* $\nu<\varkappa$,
(b) $|\{\xi\leqslant\alpha\colon a(\xi,\alpha)\neq a(\xi,\beta)\}|<\varkappa$ *for all* $\alpha<\beta<\varkappa^+$.
Define $\varrho_a\colon[\varkappa^+]^2\to\varkappa$ *by*

$$\varrho_a(\alpha,\beta)=\min\{\nu\colon a(\alpha,\beta)\leqslant\nu \text{ and } \forall\xi<\alpha(a(\xi,\alpha)\neq a(\xi,\beta)\to a(\xi,\alpha),a(\xi,\beta)\leqslant\nu)\}.$$

Then for all $\alpha<\beta<\gamma<\varkappa^+$ *and* $\nu<\varkappa$,
(c) $|\{\xi\leqslant\alpha\colon\varrho_a(\xi,\alpha)\leqslant\nu\}|<\varkappa$,
(d) $\varrho_a(\alpha,\gamma)\leqslant\max\{\varrho_a(\alpha,\beta),\varrho_a(\beta,\gamma)\}$,
(e) $\varrho_a(\alpha,\beta)\leqslant\max\{\varrho_a(\alpha,\gamma),\varrho_a(\beta,\gamma)\}$.

We shall now see that the restriction to $\varkappa$ regular in (2.6) is essential. For $\tau\colon[\varkappa^+]^2\to\varkappa$ by $T(\tau)$ we denote the tree

$$\{\tau(\cdot,\beta)\restriction\alpha\colon\alpha\leqslant\beta<\varkappa^+\},\ \subset.$$

(2.7) *If* $\tau\colon[\varkappa^+]^2\to\varkappa$ *satisfies* (2.6) (c), (d) *and* (e) (*with* τ *in place of* ϱ_a), *and if* $2^{\varkappa}=\varkappa$, *then* $T(\tau)$ *is a* $\varkappa^+$*-Aronszajn tree.*

Proof. It suffices to show that levels of $T(\tau)$ have size $\leqslant\varkappa$. So let $\alpha<\varkappa^+$ and let $t=\tau(\cdot,\beta)\restriction\alpha$ be a given element of the αth level of $T(\tau)$. By (2.5), t is uniquely determined by its restriction on

$$\{\xi<\alpha\colon\tau(\xi,\alpha)\leqslant\tau(\alpha,\beta)\}=\{\xi<\alpha\colon\tau(\xi,\beta)\leqslant\tau(\alpha,\beta)\}.$$

But there exist only $\varkappa$ such restrictions.

(2.8) *Suppose now* $\operatorname{cf}\varkappa=\omega$ *and there exists* $\tau\colon[\varkappa^+]^2\to\varkappa$ *satisfying* (2.6) (c), (d) *and* (e). *Then there is an order type* φ *of size* $\varkappa^+$ *and density* $\varkappa$ *so that every* $\psi\leq\varphi$ *of size* $\varkappa$ *is the union of* $\aleph_0$ *wellorderings.*

Proof. Fix an increasing sequence $\langle\varkappa_n\colon n<\omega\rangle$ converging to $\varkappa$. For $\alpha<\varkappa^+$ define $f_\alpha\in{}^\omega\varkappa$ by

$$f_\alpha(n)=\operatorname{tp}\{\xi\leq\alpha\colon\tau(\xi,\alpha)\leq\varkappa_n\},$$

and set $\varphi=\operatorname{tp}(\{f_\alpha\colon\alpha<\varkappa^+\},<_0)$. We claim that φ is as required. To see this first note that

$$f_\alpha(n)<f_\beta(n)\quad\text{for all } n \text{ with } \varkappa_n\geq\tau(\alpha,\beta).$$

Thus, if

$$F_n(\alpha)=\{\xi\leq\alpha\colon\tau(\xi,\alpha)\leq\varkappa_n\},$$

then $f_\eta(n)\neq f_\xi(n)$ for all $\eta\neq\xi$ in $F_n(\alpha)$. So $\{f_\xi\colon\xi\in F_n(\alpha)\}$, $<_0$ is a well-ordering. But this clearly finishes the proof.

It is well-known that an order type as in (2.8) does not exist if we collapse a supercompact cardinal to ω_2. This shows that our use of a $\square_\varkappa$-sequence in the definition of ϱ is in some sense necessary.

The definition of ϱ can also be given in the following form (resulting to the same function).

$$\varrho(\alpha,\beta)=\sup\{\operatorname{tp}(C_\beta\cap\alpha),\varrho(\alpha,\min(C_\beta\setminus\alpha)),\varrho(\xi,\alpha),(\xi\in C_\beta\cap\alpha)\}.$$

In this case a proof that the supremum is always $<\varkappa$ (even if $\varkappa$ is singular) must be given. This is done by a straight-forward induction on β using the property (i) of the $\square_\varkappa$-sequence $\langle C_\alpha\colon\alpha<\varkappa^+\rangle$ (which gives the property (iv) of ϱ). The advantage of the present definition is that for regular $\varkappa$ it makes sense even if $\langle C_\alpha\colon\alpha<\varkappa^+\rangle$ is not necessarily a $\square_\varkappa$-sequence but only a c-sequence with the property $\operatorname{tp} C_\alpha\leq\varkappa$ for all α. In this case the proof of (2.3) shows that ϱ has the properties (a), (b) and (c).

Finally, we note that the function ϱ has also a very strong stepping-up potential which has been already used in several recent applications (see [29]).

§ 3. Aronszajn trees and Countryman types

We shall say that a linearly ordered set A, $<$ has a *Countryman type* if A is uncountable and A^2 with the product ordering is the union of $\aleph_0$ chains. It is easily seen that if A is

Countryman then, in fact, any finite power A^n is the union of $\aleph_0$ chains. In this section we shall see that both ϱ_0 and ϱ_1 (and hence ϱ) give us a Countryman type.

From now on in this section we fix a c-sequence $\langle C_\alpha : \alpha<\omega_1\rangle$ on ω_1 such that $\operatorname{tp} C_\alpha \in \{0, 1, \omega\}$ for all $\alpha<\omega_1$. Let $\varrho_0=\varrho_0(C_\alpha : \alpha<\omega_1)$ and $\varrho_1=\varrho_1(C_\alpha : \alpha<\omega_1)$, i.e.,

$$\varrho_0(\alpha,\beta) = \langle \operatorname{tp}(C_\beta \cap \alpha)\rangle^\frown \varrho_0(\alpha, \min C_\beta \setminus \alpha),$$

$$\varrho_1(\alpha,\beta) = \max\{\operatorname{tp}(C_\beta \cap \alpha), \varrho_1(\alpha, \min C_\beta \setminus \alpha)\}.$$

The following two facts have already been proved in §§ 1 and 2.

(3.1) *$T(\varrho_0)$ and $T(\varrho_1)$ are Aronszajn trees.*

(3.2) *$T(\varrho_0)$ is special since $\varrho_0(\cdot,\beta)\restriction\alpha \mapsto \varrho_0(\bigcup\alpha, \beta)$ is a strictly increasing map from $T(\varrho_0)\setminus\{\langle\ \rangle\}$ into $\mathbf{Q}_\omega$.*

(3.3) *$T(\varrho_0), <_0$ and $T(\varrho_1), <_1$ are both Countryman types.*

The proof of (3.3) will be given below in a slightly more general form. When considering $a: [\omega_1]^2 \to \omega$ we shall as always assume $a(\alpha,\alpha)=0$ for all $\alpha<\omega_1$ and write a_α for $a(\cdot,\alpha)$. By $a_\alpha =^* a_\beta \restriction \alpha$ we denote the fact

$$\{\xi<\alpha : a(\xi,\alpha) \neq a(\xi,\beta)\} \quad \text{is finite.}$$

Thus (2.1) is saying that $(\varrho_1)_\alpha$ is finite-to-one and that $(\varrho_1)_\alpha =^* (\varrho_1)_\beta \restriction \alpha$ for $\alpha<\beta<\omega_1$. Suppose u and v are two functions with domains sets of ordinals. Then we write

$$u \cong v$$

iff there is a strictly increasing map h from $\operatorname{dom}(u)$ onto $\operatorname{dom}(v)$ such that $u(\alpha)=v(h(\alpha))$ for all $\alpha \in \operatorname{dom}(u)$. Note that there exist only countably many isomorphism types of finite functions from ordinals into ω.

Suppose $a: [\omega_1]^2 \to \omega$ satisfies (2.1) (a) and (b) and let $A=\{a_\alpha : \alpha<\omega_1\}$. Assume that we have a chain-decomposition $h: A^2 \to \omega$ of A^2 with the product ordering. Then a chain decomposition of $(T(a))^2$ is defined as follows. Let $\langle a_\beta \restriction \alpha, a_\delta \restriction \gamma\rangle$ be a given member of $(T(a))^2$. Then we set

$$\langle a_\beta \restriction \alpha, a_\delta \restriction \gamma\rangle \in K = K(i_0, i_1, i_2, u, v) \quad \text{iff}$$

$$h(a_\alpha, a_\gamma) = i_0, \quad \varrho_a(\alpha,\beta) = i_1, \quad \varrho_a(\gamma,\delta) = i_2 \quad \text{(see (2.6))},$$

$$a_\beta \restriction \{\xi \leq \alpha : a(\xi,\alpha) \leq i_1\} \cong u \quad \text{and} \quad a_\delta \restriction \{\xi \leq \gamma : a(\xi,\gamma) \leq i_2\} \cong v.$$

It is straightforward to check that K is a chain in $(T(a))^2$. Since there are only $\aleph_0$ parameters this gives a chain-decomposition of $(T(a))^2$ into $\aleph_0$ chains.

(3.4) *Suppose a_α is finite-to-one and $a_\alpha =^* a_\beta \restriction \alpha$ for all $\alpha<\beta<\omega_1$. Then $T(a)$, $<_1$ is a Countryman type.*

Proof. By the above remark, it suffices to decompose $\{\langle a_\alpha, a_\beta\rangle : \alpha<\beta<\omega_1\}$ into $\aleph_0$ chains. For $\alpha<\beta<\omega_1$ set $n_{\alpha\beta}=\varrho_a(\alpha,\beta)$ (see (2.6)) and

$$F_{\alpha\beta}=\{\xi\leq\alpha : a(\xi,\alpha)\leq n_{\alpha\beta}\} \quad (=\{\xi\leq\alpha : a(\xi,\beta)\leq n_{\alpha\beta}\}).$$

Clearly, to get the decomposition it suffices to assume that for some $\alpha<\beta$ and $\gamma<\delta$, we have $a_\alpha <_1 a_\gamma$ and

$$n_{\alpha\beta}=n_{\gamma\delta}=n, \quad a_\alpha\restriction F_{\alpha\beta}\cong a_\gamma\restriction F_{\gamma\delta}, \quad a_\beta\restriction F_{\alpha\beta}\cong a_\delta\restriction F_{\gamma\delta},$$

and then prove that $a_\beta <_1 a_\delta$. To end this set

$$\xi_{\alpha\gamma}=\Delta(a_\alpha, a_\gamma) \quad \text{and} \quad \xi_{\beta\delta}=\Delta(a_\beta, a_\delta).$$

Claim. $\xi_{\alpha\gamma}=\xi_{\beta\delta}=\xi$.

To see this note that $F=F_{\alpha\beta}\cap\xi_{\alpha\gamma}$ is an initial part of both $F_{\alpha\beta}$ and $F_{\gamma\delta}$, whence $a_\beta\restriction F=a_\delta\restriction F$. Then $a_\beta\restriction\xi_{\alpha\gamma}=a_\delta\restriction\xi_{\alpha\gamma}$, and so $\xi_{\beta\delta}\geq\xi_{\alpha\gamma}$. Similarly one shows $\xi_{\alpha\gamma}\geq\xi_{\beta\delta}$.

If $\xi\in F_{\alpha\beta}\setminus F_{\gamma\delta}$ then

$$a_\gamma(\xi)=a_\delta(\xi)>n\geq a_\beta(\xi)$$

which means $a_\beta <_1 a_\delta$.

If $\xi\in F_{\gamma\delta}\setminus F_{\alpha\beta}$ then

$$a_\alpha(\xi)>n\geq a_\gamma(\xi)$$

contradicting $a_\alpha <_1 a_\gamma$.

If $\xi\notin F_{\alpha\beta}\cup F_{\gamma\delta}$, then

$$a_\beta(\xi)=a_\alpha(\xi)<a_\gamma(\xi)=a_\delta(\xi),$$

which gives $a_\beta <_1 a_\delta$. This completes the proof.

In the proof that $T(\varrho_0)$, $<_0$ is Countryman the role of the finite sets $F_{\alpha\beta}$ will be played by the sets $F(\alpha,\beta)\subseteq\alpha+1$ defined as follows:

$$\xi\in F(\alpha,\beta) \quad\text{if}\quad \xi\in F(\min C_\beta\setminus\xi,\alpha) \quad\text{and}\quad \min C_\beta\setminus\xi<\alpha, \quad\text{or}$$
$$\xi\in F(\alpha,\min C_\beta\setminus\xi) \quad\text{and}\quad \min C_\beta\setminus\xi\geqslant\alpha,$$

where we agree that always $F(\alpha,\alpha)=\{\alpha\}$. An easy induction on β shows that for all $\xi\leqslant\alpha<\beta<\omega_1$,

$$\varrho_0(\xi,\alpha)=\varrho_0(\min F(\alpha,\beta)\setminus\xi,\alpha)^\frown\varrho_0(\xi,\min F(\alpha,\beta)\setminus\xi),$$
$$\varrho_0(\xi,\beta)=\varrho_0(\min F(\alpha,\beta)\setminus\xi,\beta)^\frown\varrho_0(\xi,\min F(\alpha,\beta)\setminus\xi).$$

It should now be clear that the finite sets $F(\alpha,\beta)$ can indeed take the role of the sets $F_{\alpha\beta}$ in the proof of (3.4) and give us a decomposition of $(T(\varrho_0))^2$ into $\aleph_0$ chains.

§ 4. Coloring pairs of countable ordinals

In this section we give several proofs of the main result of this article. Our partitions will have one of the following forms:

(1) $a\colon[\omega_1]^2\to\omega_1$ such that $\alpha<a(\alpha,\beta)\leqslant\beta$ and $a''[A]^2$ contains a club for all uncountable $A\subseteq\omega_1$.

(2) $b\colon[\omega_1]^2\to\omega_1$ such that $\alpha\leqslant b(\alpha,\beta)\leqslant\beta$ and $b''[B]^2$ is cobounded for all uncountable $B\subseteq\omega_1$.

(3) $c\colon[\omega_1]^2\to\omega$ and $c''[C]^2=\omega$ for all uncountable $C\subseteq\omega_1$.

(4) $d\colon[\omega_1]^2\to\omega_1$ such that $d(\alpha,\beta)<\alpha$ for nonzero α and $d''[D]^2=\omega_1$ for all uncountable $D\subseteq\omega_1$.

Any of the forms will be sufficient for the main result of this paper. For example, to get d from a one simply composes a with a decomposition of ω_1 into $\aleph_1$ stationary sets. So it remains to see how to get b and d from the existence of c. For this fix an $e\colon[\omega_1]^2\to\omega$ such that e_α is $1-1$ for all α and define

$$b(\alpha,\beta)=\max\{\alpha,e_\beta^{-1}(c(\alpha,\beta))\},$$
$$d(\alpha,\beta)=e_\alpha^{-1}(c(\alpha,\beta)),$$

where we put $b(\alpha,\beta)=\alpha$ and $d(\alpha,\beta)=0$ when the corresponding term on the right hand side is undefined. Then it is straightforward to check that b and d have the desired properties.

From now on we fix a c-sequence $\langle C_\alpha : \alpha<\omega_1\rangle$ on ω_1 such that $\operatorname{tp} C_\alpha \in \{0,1,\omega\}$ for all $\alpha<\omega_1$. Let ϱ_0 and ϱ_1 be the corresponding functions. For $\xi<\alpha<\beta<\omega_1$ and $n<\omega$ set

$$\sigma_0(\alpha,\beta)=\max\{\eta\leq\alpha : \varrho_0(\eta,\alpha)=\varrho_0(\eta,\beta)\},$$

$$\Delta(\alpha,\beta)=\Delta(\varrho_1(\cdot,\alpha),\varrho_1(\cdot,\beta)),$$

$$\sigma_1(\alpha,\beta)=\varrho_1(\Delta(\alpha,\beta),\beta),$$

$$F^0_\xi(\alpha)=\{\eta\leq\alpha : \varrho_0(\eta,\alpha)\subseteq\varrho_0(\xi,\alpha)\},$$

$$F^1_n(\alpha)=\{\xi\leq\alpha : \varrho_1(\xi,\alpha)\leq n\}.$$

(Note that σ_0 is well-defined by (1.5).) Finally, define our first two partitions $a_0\colon[\omega_1]^2\to\omega_1$ and $a_1\colon[\omega_1]^2\to\omega_1$ by

$$a_0(\alpha,\beta)=\min(F^0_{\sigma_0(\alpha,\beta)}(\beta)\setminus\alpha),$$

$$a_1(\alpha,\beta)=\min(F^1_{\sigma_1(\alpha,\beta)}(\beta)\setminus\alpha).$$

(4.1) *$a_0''[A]^2$ contains a club for all uncountable $A\subseteq\omega_1$.*

(4.2) *$a_1''[A]^2$ contains a club for all uncountable $A\subseteq\omega_1$.*

Proof. The proofs of (4.1) and (4.2) are very similar so we prove only (4.1). So let $A\subseteq\omega_1$ be uncountable, let λ be a large enough regular cardinal and let $M\prec H_\lambda$ be countable such that $A,\langle C_\alpha : \alpha<\omega_1\rangle\in M$. It suffices to find $\alpha<\beta$ in A such that

$$a_0(\alpha,\beta)=M\cap\omega_1=\delta.$$

Fix a $\beta\in A$ above δ and let $t=\varrho_0(\delta,\beta)$. Then for all $\xi\in F^0_t(\beta)$ above δ, $C_\xi\cap\delta$ is bounded in δ so we fix a bound $\gamma<\delta$ for all of them. Then $t\subseteq\varrho_0(\eta,\beta)$ for all η in $[\gamma,\delta)$. Let

$$B_0=\{\alpha\in A : \varrho_0(\cdot,\beta)\restriction\gamma\subset\varrho_0(\cdot,\alpha)\}.$$

Then $B_0\in N$ and $\beta\in B_0$, so B_0 is uncountable. Since $T(\varrho_0)$ is Aronszajn there must be a $v\in T(\varrho_0)\restriction\delta$ such that

$$\varrho_0(\cdot,\beta)\restriction\gamma\subset v\not\subset\varrho_0(\cdot,\beta)$$

and

$$C = \{\alpha \in B_0 : v \subset \varrho_0(\cdot, \alpha)\}$$

is uncountable. Let

$$\varepsilon = \max \{\eta : v(\eta) \neq \varrho_0(\eta, \beta)\}.$$

Then for all α in $C \cap N$,

$$\sigma_0(\alpha, \beta) = \varepsilon \text{ and } F_\iota^0(\beta) \subseteq F_\varepsilon^0(\beta)$$

So if α in $C \cap N$ is above $F_\varepsilon^0(\beta) \cap \delta$ then

$$a_0(\alpha, \beta) = \min (F_\varepsilon^0(\beta) \setminus \alpha) = \min F_\iota^0(\beta) = \delta.$$

This completes the proof.

Let T be a special Aronszajn tree and let $a: T \to \omega$ be an antichain-decomposition. Let $<$ be a fixed well ordering of T such that $\mathrm{ht}(s) < \mathrm{ht}(s)$ implies $s < t$, and let

$$[T]^2 = \{\langle s, t\rangle : s < t \text{ and } s, t \in T\}.$$

We also assume that for all $s, t \in T$,

$$s \wedge t = \max \{u \in T : u \leqslant_T s \text{ and } u \leqslant_T t\}$$

exists. For $n < \omega$ and t in T set

$$F_n(t) = \{s \leqslant_T t : s = t \text{ or } a(s) \leqslant n\}.$$

Then $F_n(t)$ is a finite chain in T. Finally, define $b: [T]^2 \to T$ by

$$b(s, t) = \min \{u \in F_{a(s \wedge t)}(t) : s < u\}.$$

Then the proof of (4.1) also gives the following.

(4.3) *$b''[A]^2$ intersects club many levels of T for any uncountable $A \subseteq T$.*

For $A \subseteq \omega_1$ set

$$K_A = \{\langle \alpha, \beta\rangle \in [\omega_1]^2 : a_0(\alpha, \beta) \in A\}.$$

Let $\mathcal{G}_A$ denote the graph $\langle \omega_1, K_A\rangle$. Then an easy adjustment of the proof of (4.1) gives the following.

(4.4) *If $A \Delta B$ is stationary in ω_1, then $\mathcal{G}_A$ and $\mathcal{G}_B$ contain no isomorphic uncountable subgraphs.*

For two sets of ordinals E and F we define a relation $R=R(E,F)\subseteq F^2$ as follows:

$$\alpha R \beta \quad \text{iff} \quad [\alpha,\beta]\cap E\subseteq F \quad \text{and} \quad [\beta,\alpha]\cap E\subseteq F.$$

Then R is an equivalence relation on F with convex equivalence classes. Set

$$\operatorname{osc}(E,F)=|F/R(E,F)|,$$

that is, $\operatorname{osc}(E,F)$ is the cardinality of the set of all equivalence classes of R on F. Now, we are ready to define two new partitions $c_0\colon [\omega_1]^2\to\omega$ and $c_1\colon [\omega_1]^2\to\omega$ as follows:

$$c_0(\alpha,\beta)=\text{the exponent of 2 in } \operatorname{osc}\left(F^0_{\sigma_0(\alpha,\beta)}(\alpha), F^0_{\sigma_0(\alpha,\beta)}(\beta)\right),$$

$$c_1(\alpha,\beta)=\text{the exponent of 2 in } \operatorname{osc}\left(F^1_{\sigma_1(\alpha,\beta)}(\alpha), F^1_{\sigma_1(\alpha,\beta)}(\beta)\right).$$

(4.5) $c_0''[A]^2=\omega$ *for all uncountable* $A\subseteq\omega_1$.

(4.6) $c_1''[A]^2=\omega$ *for all uncountable* $A\subseteq\omega_1$.

Proof. The proofs of (4.5) and (4.6) are very similar, so we prove only (4.6). For $E,F\in[\omega_1]^{<\omega}$ put

$$E\leqslant\cdot F \quad \text{iff} \quad E \text{ is an initial part of } F.$$

Then $[\omega_1]^{<\omega}$, $<\cdot$ is a tree and we shall consider any $T\subseteq[\omega_1]^{<\omega}$ as a tree under this ordering. Let n denote a positive integer. Then we say that $T\subseteq[\omega_1]^{<\omega}$ is an *n-tree* iff

(a) T has a root,

(b) height$(T)=n+1$,

(c) if E has height $<n$ in T then the set $\operatorname{succ}(E)$ of all immediate successors of E in T has size $\aleph_1$ and $\{F\setminus E\colon F\in\operatorname{succ}(E)\}$ is a disjoint family.

If T is an n-tree then by T^* we denote the last (nth level) of T.

(4.7) *Suppose S and T are n-trees with roots E_0 and F_0, respectively. Let $l=\operatorname{osc}(E_0,F_0)$. Then for each $k\in(l,l+n]$ there exist uncountable $X\subseteq S^*$ and $Y\subseteq T^*$ such that $\operatorname{osc}(E,F)=k$ for all $E\in X$ and $F\in Y$ with $\max E<\max F$.*

Proof. The proof is by induction on n. The case $n=1$ follows immediately from the definition of an n-tree. So assume the result is true for $n=m$ and prove it for $n=m+1$.

To end this pick an E_1 in the first level of S and an F_1 in the first level of T so that

$$\max F_0 < \min(E_1 \setminus E_0) \leq \max(E_1 \setminus E_0) < \min(F_1 \setminus F_0).$$

Set

$$U = \{E \in S: E_1 \leq \cdot E\} \quad \text{and} \quad V = \{F \in T: F_1 \leq \cdot F\}.$$

Then U and V are m-trees with roots E_1 and F_1, respectively such that

$$\operatorname{osc}(E_1, F_1) = l+1.$$

By the induction hypothesis for each $k \in (l+1, l+1+m]$ we can find uncountable $X \subseteq U^* \subseteq S^*$ and $Y \subseteq V^* \subseteq T^*$ such that

$$\operatorname{osc}(E, F) = k \quad \text{for all} \quad E \in X \text{ and } F \in Y \text{ with} \quad \max E < \max F.$$

The same conclusion for $k=l+1$ follows from the case $n=1$. This finishes the proof.

We are now ready to finish the prof of (4.6). So let $A \subseteq \omega_1$ be uncountable and let $i<\omega$. We have to find $\alpha<\beta$ in A such that $c_1(\alpha, \beta)=i$.

Let $n=5 \cdot 2^i$ and let $\{M_k: k \leq n\}$ be an $\in$-chain of countable elementary submodels of H_λ containing everything relevant. Let

$$\delta_k = M_k \cap \omega_1, \quad (k \leq n).$$

Fix a $\bar{\beta} \in A$ above δ_n and $\bar{m}<\omega$ such that

$$\{\delta_k: k \leq n\} \subseteq F^1_{\bar{m}}(\bar{\beta}).$$

Fix a $\gamma<\delta_0$ above $F^1_{\bar{m}}(\bar{\beta}) \cap \delta_0$, and let $t=\varrho_1(\cdot, \bar{\beta}) \restriction \gamma$. Define

$$B_0 = \{\alpha \in A: t \subset \varrho_1(\cdot, \alpha)\}.$$

Then $B_0 \in M_0$ and $\bar{\beta} \in B_0$ so B_0 is uncountable. Using

$$F^1_{\bar{m}}(\bar{\beta}) \cap [\delta_{k-1}, \delta_k)$$

$(\delta_{-1}=0, \delta_{n+1}=\omega_1)$ as parameters and elementarity of the M_k's, by downward induction one shows the existence of an $(n+1)$-tree S_0 in M_0 such that

$$S_0^* \subseteq \{F^1_{\bar{m}}(\alpha): \alpha \in B_0\}.$$

Since $T(\varrho_1)$ is Aronszajn, by shrinking S_0, we may assume that there exist $u \in T(\varrho_1) \restriction \delta_0$ and uncountable $C \subseteq B_0$ so that:

(a) $t \subset u \subset \varrho_1(\cdot, \alpha)$ for all $\alpha \in C$,

(b) $S_0^* = \{F_{\bar{m}}^1(\alpha) : \alpha \in C\}$,

(c) $u \not\subset \varrho_1(\cdot, \bar{\beta})$.

Let $v = \varrho_1(\cdot, \beta) \restriction l(u)$. Then again for the same reasons we can find an $(n+1)$-tree T_0 in M_0 and uncountable $D \subseteq B_0$ such that:

(d) $v \subset \varrho_1(\cdot, \beta)$ for all $\beta \in D$,

(e) $T_0^* = \{F_{\bar{m}}^1(\beta) : \beta \in D\}$.

Let $\varepsilon = \Delta(u, v)$ and let $m = v(\varepsilon)$. Then $\bar{m} < m$ and $\sigma_1(\alpha, \beta) = m$ for all $\alpha \in C$ and $\beta \in D$ with $\alpha < \beta$. So we can find $(n+1)$-trees S and T such that

$$S^* \subseteq \{F_m^1(\alpha) : \alpha \in C\} \quad \text{and} \quad T^* \subseteq \{F_m^1(\beta) : \beta \in D\}.$$

Let E_0 and F_0 be roots of S and T, respectively and let $l = \operatorname{osc}(E_0, F_0)$. Pick a j such that

$$2^i(2j+1) \in (l, l+n+1].$$

By (4.7) we can find $F_m^1(\alpha)$ in S^* and $F_m^1(\beta)$ in T^* such that $\alpha < \beta$ and

$$\operatorname{osc}(F_m^1(\alpha), F_m^1(\beta)) = 2^i(2j+1).$$

Since $m = \sigma_1(\alpha, \beta)$ this means that $c_1(\alpha, \beta) = i$. This completes the proof.

We conclude this section with a corollary to the proof of (4.6). Let $\bar{\varrho} = \bar{\varrho}(C_\alpha : \alpha < \omega_1)$ be the function from (2.4). For $n < \omega$ and $\alpha < \omega_1$ we set

$$P_n(\alpha) = \{\xi \leqslant \alpha : \bar{\varrho}(\xi, \alpha) \leqslant n\}.$$

To the function $\bar{\varrho}$ we now associate a sequence

$$d_n : [\omega_1]^2 \to \omega, \quad (n < \omega)$$

of partitions as follows. First of all we put $d_n(\alpha, \beta) = 0$ iff

$$\text{the } i\text{th member of } P_n(\alpha) \leqslant \text{the } i\text{th member of } P_n(\beta)$$

for all i for which such members exist. Otherwise, let

$$d_n(\alpha,\beta) = \max\{1, \operatorname{osc}(P_n(\alpha), P_n(\beta)) - 1\}.$$

By the properties (b) and (c) of $\bar{\varrho}$, it follows easily that $d_n(\alpha,\beta)=0$ for all $n \geqslant \bar{\varrho}(\alpha,\beta)$. Thus since the $(\bar{\varrho})_\gamma$'s are one-to-one, $0 \in d_n[A]^2$ for any set A of size $>n$. On the other hand, we have the following

(4.8) *For all $i>0$ and uncountable $A, B \subseteq \omega_1$ there is an m such that for all $n \geqslant m$ there exist $\alpha \in A$ and $\beta \in B$ with $\alpha<\beta$ so that $d_n(\alpha,\beta)=i$.*

§ 5. Coloring larger squares

In this section we make slight modifications of partitions a_0 and c_0 of § 4 which will work on some higher cardinals as well. Roughly speaking, we have only to redefine $\sigma_0(\alpha,\beta)$ in such a way that we no longer make the implicit assumption that the clubs relevant to $\sigma_0(\alpha,\beta)$ agree below the maximum of their intersections.

So let θ be a fixed regular cardinal and let $\langle C_\alpha : \alpha<\theta\rangle$ be a c-sequence on θ. We shall need a slight modification $o\colon [\theta]^2 \to \mathrm{Card}$ of the function osc of § 4 defined as follows:

$$o(\alpha,\beta) = \operatorname{osc}(C_\alpha \setminus (\max(C_\alpha \cap C_\beta)+1), C_\beta \setminus (\max(C_\alpha \cap C_\beta)+1).$$

Now we make the following assumption about θ and the C_α's.

(5.1) *$\theta=\varkappa^+$ for some regular $\varkappa$ and $\operatorname{tp} C_\alpha \leqslant \varkappa$ for all $\alpha<\theta$.*

The first partition $a\colon [\varkappa^+]^2 \to \varkappa$ of this section is defined by

$$a(\alpha,\beta) = \max\{\xi \in F^0_{\varrho_0(\alpha,\beta)}(\beta) \colon o(\eta,\xi) \geqslant 2 \text{ for the } \eta \leqslant \alpha \text{ with } \varrho_0(\eta,\alpha) = \varrho_0(\xi,\beta)\},$$

if this set is nonempty; otherwise $a(\alpha,\beta)=\beta$. Let

$$S_\varkappa = \{\delta < \varkappa^+ \colon \operatorname{cf}\delta = \varkappa\}.$$

(5.2) *$a''[A]^2$ contains a club relative to $S_\varkappa$ for all $A \subseteq \varkappa^+$ of size $\varkappa^+$.*

Proof. Let $A \subseteq \varkappa^+$ have size $\varkappa^+$ and let M be an elementary submodel of H_λ containing everything relevant such that

$$\delta = M \cap \varkappa^+ \in S_\varkappa.$$

Furthermore, we assume M is the union of an $\in$-chain of submodels with the same property. It suffices to find $\alpha<\beta$ in A such that $a(\alpha,\beta)=\delta$. Fix a $\beta\in A$ above δ and let $t=\varrho_0(\delta,\beta)$. Then

$$\max(C_\xi\cap\delta)<\delta \quad \text{for all} \quad \xi\in F_t^0(\beta)\setminus(\delta+1).$$

For $s\subset t$ put

$$\gamma_s=\max(C_\xi\cap\delta), \text{ where } \varrho_0(\xi,\beta)=s.$$

We can now choose submodels $M_0\in M_1\in M$ of H_λ containing $\langle c_\alpha:\alpha<\varkappa^+\rangle$, A, $\{\gamma_s: s\subset t\}$ such that

$$\delta_i=M_i\cap\varkappa^+\in S_\varkappa,\ (i<2) \quad \text{and} \quad C_\delta\cap[\delta_0,\delta_1)\neq\emptyset.$$

For $\alpha\in A$, let $\delta(\alpha,t)$ be the unique ξ such that $\varrho_0(\xi,\alpha)=t$, if it exists. Let B_0 be the set of all $\alpha\in A$ for which $\delta(\alpha,t)$ exists, and let

$$B=\{\alpha\in B_0\colon \gamma_s=\max(C_\xi\cap\delta(\alpha,t)) \text{ for all } s\subset t \text{ and } \xi\leq\alpha \text{ with } \varrho_0(\xi,\alpha)=s\}.$$

Then $B\in M_0$ and $\beta\in B$. The elementarity of M_0 now gives us that for all large enough $\gamma<\varkappa^+$ there is an $\alpha\in B$ such that $\delta(\alpha,t)>\gamma$ and

$$\max(C_{\delta(\alpha,t)}\cap\gamma)=\gamma_t,$$

where $\gamma_t=\max(C_\delta\cap\delta_0)$. Applying this in M_1 for

$$\bar\gamma=\max(C_\delta\cap\delta_1)$$

fix an $\alpha\in B\cap M_1$ such that $\delta(\alpha,t)>\bar\gamma$ and

$$\gamma_t=\max(C_{\delta(\alpha,t)}\cap\bar\gamma).$$

Then

$$t\subseteq\varrho_0(\alpha,\beta), \quad o(\delta(\alpha,t),\delta)=2,$$

and

$$o(\eta,\xi)=1 \quad \text{for all} \quad \xi\in F^0_{\varrho_0(\alpha,\beta)}(\beta)\setminus(\delta+1)=F_t^0(\beta)\setminus(\delta+1)$$

and

$$\eta\leq\alpha \quad \text{with} \quad \varrho_0(\eta,\alpha)=\varrho_0(\xi,\beta).$$

Hence $a(\alpha,\beta)=\delta$. This finishes the proof.

It should be clear that in the above proof of (5.2) we can replace $\varkappa^+$ by any regular uncountable θ and $S_\varkappa$ by any stationary $S\subseteq\theta$ with property $C_\alpha\cap S=\emptyset$ for all limit $\alpha<\theta$. It is also clear that the exact analogue of (4.4) holds in this situation as well.

In order to consider an analogue of c_0 of §4 in the present situation we now make the following assumption.

(5.3) *θ is regular and uncountable and there is no club $C\subseteq\theta$ such that for all $\alpha<\theta$ there is a $\beta\geq\alpha$ so that $C\cap\alpha\subseteq C_\beta\cap\alpha$.*

Clearly, any successor cardinal $\theta=\varkappa^+$ and $\langle C_\alpha\colon\alpha<\theta\rangle$ with the property $\operatorname{tp} C_\alpha\leq\varkappa$ for all $\alpha<\theta$ satisfy the condition of (5.3). The function

$$\sigma_2\colon[\theta]^2\to\mathbf{Q}_\theta$$

is defined by

$$\sigma_2(\alpha,\beta)=\min\{t\subseteq\varrho_0(\alpha,\beta)\colon o(\eta,\xi)\geq 2 \text{ if } \varrho_0(\eta,\alpha)=\varrho_0(\xi,\beta)=t\},$$

where the min is taken in $\mathbf{Q}_\theta$ with respect to $\subset$, and where we set $\sigma_2(\alpha,\beta)=\langle\ \rangle$ if the set on the right hand side is empty. Finally define $c\colon[\theta]^2\to\text{Card}$ by

$$c(\alpha,\beta)=o(\min F^0_{\sigma_2(\alpha,\beta)}(\alpha),\ \min F^0_{\sigma_2(\alpha,\beta)}(\beta)).$$

(5.4) $c''[A]^2\supseteq\omega\setminus 2$ *for all $A\subseteq\theta$ of size θ.*

Proof. Fix an $A\subseteq\theta$ of size θ and an integer $n\geq 2$. We have to find $\alpha<\beta$ in A such that $c(\alpha,\beta)=n$. Pick an $\bar{M}<H_\lambda$ containing everything relevant such that

$$\bar{\delta}=\bar{M}\cap\theta\in\theta.$$

Fix a $\bar{\beta}\in A$ above $\bar{\delta}$ and let $\bar{t}=\varrho_0(\bar{\delta},\bar{\beta})$. Let

$$\delta\in F^0_{\bar{t}}(\bar{\beta})$$

be maximal with the property

$$\sup(C_\delta\cap\bar{\delta})=\bar{\delta},$$

and let $t=\varrho_0(\delta,\bar\beta)(\sqsubseteq\bar t)$. Then for each $s\sqsubset t$, if $\xi\leq\bar\beta$ is determined by $\varrho_0(\xi,\bar\beta)=s$, then

$$\gamma_s=\max(C_\xi\cap\bar\delta)$$

is an ordinal $<\bar\delta$. We may assume $\bar M$ contains a large enough regular cardinal μ and an $\in$-increasing continuous θ-chain $\mathcal{M}$ of elementary submodels of H_μ containing $\{\gamma_s: s\sqsubset t\}$, $\langle C_\alpha:\alpha<\theta\rangle$ and A such that $M\cap\theta\in\theta$ for all $M\in\mathcal{M}$. Let

$$C=\{M\cap\theta: M\in\mathcal{M}\}.$$

Then C is a club in θ and by the assumption (5.3) no endsection of $C\cap\bar\delta$ is contained in C_δ. So we can find an $\in$-subchain $\{M_i: i\leq n\}$ of $\mathcal{M}\cap\bar M$ such that if $\delta_i=M_i\cap\theta$ then

$$C_\delta\cap[\delta_{i-1},\delta_i)\neq\varnothing$$

and $C_i\cap\delta_i$ is bounded in δ_i for $i\leq n$, $(\delta_{-1}=0)$.

For $\alpha\in A$, let $\delta(\alpha,t)$ be the unique $\xi\leq\alpha$ determined by $\varrho_0(\xi,\alpha)=t$, if it exists. Let B_0 be the set of all α in A for which $\delta(\alpha,t)$ exists, and let

$$B=\{\alpha\in B_0:\gamma_s=\max(C_\xi\cap\delta(\alpha,t))\text{ for all }s\sqsubset t\text{ and }\xi\leq\alpha\text{ with }\varrho_0(\xi,\alpha)=s\}.$$

Then $B\in M_0$ and $\bar\beta\in B$.

Let us now extend the notion of an n-tree of § 4 to the present context by simply replacing the requirement of uncountable splitting by the requirement of θ-splitting. Let

$$\varepsilon_i=\max(C_\delta\cap\delta_i)$$

for $i\leq n$. Using the δ_i's and ε_i's as parameters, by downward induction along the models M_i, one can show the existence of an $(n+1)$-tree S in M_0 with root $\{\varepsilon_0\}$ such that

(a) If F is an immediate successor of E in S then $|F\setminus E|=2$.

(b) For all F^* in S^* there is an $\alpha\in B$ such that if

$$F^*=\{\varepsilon_0,\delta_0^*,\varepsilon_1^*,\delta_1^*,\ldots,\varepsilon_n^*,\delta_n^*,\delta(\alpha,t)\}$$

is the increasing enumeration of F^*, then

$$\varepsilon_i^*=\max(C_{\delta(\alpha,t)}\cap\delta_i^*)$$

for all $i\leq n$.

Since S in $M_0 \in \dots \in M_n$ is a θ-splitting tree we can inductively find a top node $\bar{F}$ of S in M_n and an $\bar{\alpha}$ in $B \cap M_n$ as in (b) such that if

$$\bar{F} = \{\varepsilon_0, \delta_0, \bar{\varepsilon}_1, \bar{\delta}_1, \dots, \bar{\varepsilon}_n, \bar{\delta}_n, \delta(\bar{\alpha}, t)\}_<,$$

then

$$\varepsilon_i < \bar{\delta}_{i-1} \leq \bar{\varepsilon}_i < \delta_i$$

for all $0 < i \leq n$. Then it follows immediately that

$$o(\delta(\bar{\alpha}, t), \delta(\bar{\beta}, t)) = n.$$

On the other hand, by the definition of B (and $\gamma_s, s \subset t$), we have that $o(\eta, \xi) = 1$ for all η and ξ for which $\varrho_0(\eta, \bar{\alpha}) = \varrho_0(\xi, \bar{\beta})$ is a proper initial part of t. Thus

$$\sigma_2(\bar{\alpha}, \bar{\beta}) = t,$$

and so $c(\bar{\alpha}, \bar{\beta}) = n$. This completes the proof.

§ 6. Concluding remarks

The key idea of our coloring can roughly be stated as follows: If the set of interesting places is stationary then in any unbounded set we can find $\alpha < \beta$ such that walking from β to α along the C_ξ's we pass through at least one interesting place. The functions σ_0, σ_1 (and σ_2) are designed in such a way in order to pick one interesting place from our (finite) trace. We should note, however, that in most cases the exact kind of our walk is not so important as it may seem at first. For example, the partitions of § 4 could also be defined as follows.

$$b(\alpha, \beta) = \min (F_{\sigma(\alpha, \beta)}(\beta) \setminus \alpha),$$

$$c(\alpha, \beta) = \text{the exponent of 2 in } \operatorname{osc}(F_{\sigma(\alpha, \beta)}(\alpha), F_{\sigma(\alpha, \beta)}(\beta)),$$

where

$$F_n(\alpha) = \{\xi \leq \alpha : e(\xi, \alpha) \leq n\},$$

and

$$\sigma(\alpha, \beta) = \min \{n : r_\alpha(n) \neq r_\beta(n)\},$$

for any $e:[\omega_1]^2 \to \omega$ with e_α's $1-1$, and any $1-1$ sequence $\{r_\alpha : \alpha<\omega_1\} \subseteq {}^\omega\omega$. We have chosen the most canonical forms in §§4 and 5 because these are the forms which are likely to have further applications and modifications.

The work of the present paper started in May 1984 by a discovery of a new proof of the existence of a Countryman type. (The Countryman problem was originally solved by Shelah [14] using an Aronszajn-type construction ([11]).) The main result was obtained in September 1984 and the whole work was completed by the Fall of 1984 and circulated as [26]. Since then several papers appeared using the method of [26] to some other related problems concerning the uncountable ([16], [17], [19], [20], [18], [6].) Since more applications are to be expected, we have decided to present [26] in the present more explicit form.

It is readily seen that any of our partitions

$$p:[\theta]^2 \to I$$

(when reduced in one of the forms (3) and (4) of §4) has the following property.

(6.1) *For any finite n and disjoint* $\{F_\alpha : \alpha<\theta\} \subset [\theta]^n$ *there exist* $i: n^2 \to I$ *and unbounded* $A \subseteq \theta$ *such that for* $\langle \alpha,\beta \rangle \in A^2$ *and* $\langle k,l \rangle \in n^2 \setminus \Delta$,

$$p(F_\alpha(k), F_\beta(l)) = i(k,l).$$

In [19], Shelah and Steprans give an interesting application of (6.1) to Group Theory removing the assumption of CH from previous works. Incidentally, the same proprety of p has been also used in [28] for a quite different purpose. The property (6.1) is telling us that we have not much freedom in getting different colors on $(F_\alpha(k), F_\beta(l))$ for $k \neq l$. This was also the case with the partitions from [21] and [4]. The first example of a set of reals X of regular cardinality θ and a partition

$$p:[X]^2 \to \omega,$$

with a complete freedom in calculating $p(F_\alpha(k), F_\beta(l))$ for different k and l was given by the author in [23]. Using the methods of [26], Shelah [18] constructs partitions with similar properties for many cardinals θ above the continuum.

The main result of Shelah [16] shows that certain extensions of a whole group of results including ours are impossible. But [16] also contains a different-style presentation of our partition a_0 from §4 with certain generalizations. In particular, [16] gives attempts of extending (5.2) to successors of singular cardinals. Concerning this let us note that the partition of [27] can be used in stepping-up the partitions of the present paper in order to get an analogue of (5.2) for any $\varkappa$ which is the ωth successor of a countable product of cardinals $\geqslant 2$.

Our partitions p have also the following interesting property.

(6.2) *For any finite n and unbounded $A \subseteq \theta$ there is a disjoint $\{F_\alpha : \alpha < \theta\} \subset [A]^n$ such that for all i in I and unbounded $B \subseteq \theta$ there is an arbitrarily large finite $C \subseteq B$ such that $p''F_\alpha \times F_\beta = \{i\}$ for all $\alpha < \beta$ in C.*

This is probably most easily seen for the partition b of (4.4). In this case we first find an antichain $\{t_\alpha : \alpha < \omega_1\}$ of T and for each α an F_α in $[A]^n$ such that $t_\alpha <_T t$ for all t in F_α. It is now clear that b on $\{F_\alpha : \alpha < \omega_1\}$ behaves the same way as b on $\{t_\alpha : \alpha < \omega_1\}$, so we finish as in the proof of (4.4). In [17] Shelah shows that by identifying certain colors of such a p the resulting partition

$$q: [\theta]^2 \to I$$

has the following property.

(6.3) *For any finite n, unbounded $A \subseteq \theta$, and $h: [n]^2 \to I$ there is an increasing $f: n \to A$ such that $p(f(i), f(j)) = h(i, j)$ for all $i < j < n$.*

Namely, fix an one-to-one $\{r_\alpha : \alpha < \theta\} \subseteq {}^\omega 2$ and let $\{h_i : i \in I\}$ enumerate all

$$h: [{}^n 2]^2 \to I,$$

where $n < \omega$. Let $n_i < \omega$ correspond to h_i. For $\alpha < \beta < \theta$ set

$$q(\alpha, \beta) = h_{p(\alpha,\beta)}\{r_\alpha \upharpoonright n_{p(\alpha,\beta)}, r_\beta \upharpoonright n_{p(\alpha,\beta)}\}.$$

Then it is easily seen that q satisfies (6.3) when p has the property (6.2). The paper [20] gives an interesting application of (6.3) to the Banach Space Theory removing the use of diamond from previous constructions.

An interesting application of the methods of [26] also appeared in [6] where Hajnal, Kanamori and Shelah gave a new characterization of Mahlo cardinals in terms of the existence of infinite min-homogeneous sets for certain regressive partitions. We note that the characterization of [6] can also be achieved more directly by an application of the more natural characterization (1.9) (=Theorem 7 of [26]). For this one has to consider the following partitions

$$p:[T]^2 \to \omega \quad \text{and} \quad q:[T]^3 \to T$$

associated with a special θ-Aronszajn tree T and the regressive map $f: T \to T$:

$$p(s,t) = \min\{m: \text{height } f^m(s \wedge t) = 0\},$$

and

$$q(s,t,u) = f^n(t \wedge u),$$

where n is the minimal m such that $f^m(t \wedge u)$ has height less than the height of s, if such an m exists; otherwise $q(s,t,u)$ is the minimal point of T below $t \wedge u$. (Here f^m denotes the mth iterate of the regressive map f and $f^0 = \text{id}$.)

After the first limitation to the Ramsey Theorem for the uncountable given by Sierpiński [21] (and rediscovered by Kurepa [12]) several other much stronger limitations appeared using mainly the Continuum Hypothesis or the negation of the Souslin Hypothesis. One of the strongest is the following CH-result of Erdös, Hajnal and Milner [2].

(6.4) *There is a* $p:[\omega_1]^2 \to \omega_1$ *such that for every infinite* $A \subseteq \omega_1$ *and uncountable* $B \subseteq \omega_1$ *there is an* $\alpha \in A$ *so that* $\{p(\alpha,\beta): \beta \in B\} = \omega_1$.

It turns out that (6.4) is just a reformulation of an earlier CH-proposition P_3 of Sierpiński [22] which we state in the following form.

(6.5) *There is a sequence* $f_n: \omega_1 \to \omega_1$, $(n<\omega)$ *such that for any uncountable* $A \subseteq \omega_1$ *there is an* m *so that* $f_n''A = \omega_1$ *for all* $n \geq m$.

To deduce (6.5) from (6.4) define $f_n: \omega_1 \setminus \omega \to \omega_1$ by

$$f_n(\beta) = p(n,\beta).$$

To deduce (6.4) from (6.5) fix an $e: [\omega_1]^2 \to \omega$ such that

(6.6) e_α *is finite-to-one and* $e_\alpha =^* e_\beta \restriction \alpha$ *for* $\alpha < \beta < \omega_1$,

and define

$$p(\alpha,\beta) = f_{e_\beta(\alpha)}(\beta).$$

The proposition (6.5) does not change if we assume that the f_n's map ω_1 into ω rather than into ω_1. To see this suppose $g_n\colon \omega_1 \to \omega$ $(n<\omega)$ satisfy (6.5) and define $f_n\colon \omega_1 \to \omega_1$, $(n<\omega)$ by

$$f_n(\alpha) = \bar{e}_\alpha^{-1}(g_n(\alpha))$$

(see (2.2) for the definition of $\bar{e}$ from e) when this makes sense; otherwise $f_n(\alpha)=0$. Then the f_n's also satisfy (6.5) which is easily checked. This reformulation of (6.5) is of some interest since in this form it is an easy consequence of the existence of an uncountable Luzin set [22]. To see this fix an $1-1$ sequence $\{r_\alpha\colon \alpha<\omega_1\} \subseteq {}^\omega\omega$ with no uncountable nowhere dense set and define $f_n\colon \omega_1 \to \omega$, $(n<\omega)$ by

$$f_n(\alpha) = r_\alpha(n).$$

Then it is easily checked that the f_n's satisfy (6.5).

In the above stepping-up we don't really need to assume $e_\alpha =^* e_\beta \restriction \alpha$ for $\alpha<\beta<\omega_1$ but just the weaker condition that $T(e)$ is Aronszajn. But we have seen in §§ 3 and 4 that (6.6) is of independent interest, so let us give a brief historical remark concerning this proposition. This proposition is implicit in many constructions of Aronszajn trees [11] but the closest implicit forms of (6.6) we are aware of are those of Galvin [3] and Warren [30], while the first explicit construction of (6.6) is in Kunen [9], [10]. So let us explain this in more detail. Warren [30] essentially proves the following proposition.

(6.7) *For all $\alpha<\omega_1$ there is a disjoint partition $\alpha\times\omega = A_\alpha \cup B_\alpha$ such that:*

(a) *If $\alpha = \bigcup \alpha$ and $n<\omega$ then for some $\eta<\alpha$, $\{\langle \xi, n\rangle : \eta \leq \xi < \alpha\} \subset A_\alpha$.*

(b) *For all $\xi<\alpha$ there is an $m<\omega$ such that $\{\langle \xi, n\rangle : m \leq n < \omega\} \subset B_\alpha$.*

(c) *$A_\alpha \Delta (A_\beta \cap (\alpha\times\omega))$ is finite for all $\alpha<\beta<\omega_1$.*

To get the e_α's from the A_α's set

$$e_\alpha(\xi) = \min\{m\colon \{\langle \xi, n\rangle : m \leq n < \omega\} \subseteq B_\alpha\}.$$

To get the A_α's from the e_α's set

$$A_\alpha = \{\langle \xi, n\rangle : \xi < \alpha \text{ and } n < e_\alpha(\xi)\}.$$

Galvin [3] ([7]) proves the following.

(6.8) $\in\restriction\omega_1$ *can be represented as a union of an increasing sequence* $\{<_i: i<\omega\}$ *of tree orderings of heights* $\leqslant\omega$.

To get the e_α's from the $<_i$'s set

$$e_\alpha(\xi)=\min\{i\colon \xi<_i\alpha\}.$$

To get the $<_i$'s from the e_α's set

$$\alpha<_i\beta \quad\text{iff}\quad e_\beta(\alpha)\leqslant i \quad\text{and}\quad \forall\xi<\alpha(e_\alpha(\xi)\neq e_\beta(\xi)\rightarrow e_\alpha(\xi), e_\beta(\xi)\leqslant i).$$

Thus (6.6), (6.7) and (6.8) are all "equivalent". But we should not ignore the fact that they are telling us quite different things about the uncountable. It should be noted, however, that these connections have not been realized before. For example, it has not been noticed before (cf. [3] and [7]) that (6.8) doesn't really change if we strengthen its conclusion to: height $(\omega_1,<_i)$ is finite for all i. Namely, if we define $<_i$'s from e_α's as above and assume e_α's are, in fact, one-to-one (which we can by the obvious stretching-up procedure of (2.2)), then $(\omega_1,<_i)$ has height $\leqslant i+2$ for all i. Finally, we note that the proofs of [30], [3] and [9] all work only for successors of regular cardinals, so our (2.1) seems to be new.

In the final remark of this section we show (once again) the usefulness of the ideas of §§ 1 and 2 by defining a name for a Souslin tree in the standard poset

$$^{\omega}\omega,\supseteq$$

for adding one Cohen real. The first such name was given in Shelah [15] using an involved "morass-type" construction. To commence, fix an $e\colon[\omega_1]^2\rightarrow\omega$ as in (6.6) and by stretching-up (as in (2.2)) assume that the e_α's are, in fact, one-to-one. Now for each real $r\in{}^{\omega}\omega$ we define another $e_r\colon[\omega_1]^2\rightarrow\omega$ by

$$e_r(\alpha,\beta)=r(e(\alpha,\beta)).$$

Clearly,

$$T(e_r)=\{e_r(\cdot,\beta)\restriction\alpha\colon \alpha\leqslant\beta<\omega_1\}$$

is still a tree of height ω_1 with countable levels. Note that

$$T(e_{\mathrm{id}})=T(e)\quad\text{and}\quad T(e_{\mathrm{const}})\cong\omega_1,\in.$$

Our point is that

(6.9) *$T(e_r)$ is Souslin if r is Cohen.*

The proof of (6.9) is straight-forward. For example, to show that every antichain is countable fix an uncountable $A \subseteq \omega_1$ and $p \in {}^n\omega$. Assume that

$$F_n(\alpha) = \{\xi \leq \alpha : e(\xi, \alpha) \leq n\}, \quad (\alpha \in A).$$

form an increasing Δ-system with root F and that the e_α's agree on F. Fix now $\alpha < \beta$ in A and extend p to a q which corrects the finite disagreement of e_α and e_β.

Identifying α with $(e_r)_\alpha$ we may consider the induced tree ordering $<_r$ on ω_1 and get the same conclusion. That is, we may define $<_r$ on ω_1 by

$$\alpha <_r \beta \quad \text{iff} \quad \forall \xi < \alpha (r(e(\xi, \alpha)) = r(e(\xi, \beta))$$

and get

(6.10) *$\omega_1, <_r$ is Souslin if r is Cohen.*

To have a nicer forcing relation let us now assume that e also satisfies (2.3) (b) and (c). For example, we can put $e = \bar{\varrho}$ (see (2.4)). Now for $p \in {}^{\varphi}\omega$ and $\alpha < \beta < \omega_1$ we set

$$\alpha <_p \beta \quad \text{iff} \quad e(\alpha, \beta) \in |p| \quad \text{and} \quad \forall \xi < \alpha (e(\xi, \alpha) \in |p| \rightarrow p(e(\xi, \alpha)) = p(e(\xi, \beta))).$$

Clearly, the properties (2.3) (b) and (c) of e give us the following facts about $<_p$.

(6.11) (a) *$\omega_1, <_p$ is a tree of height $\leq |p| + 1$.*

(b) *$p \subseteq q$ implies $<_p \subseteq <_q$.*

(c) *$<_r = \bigcup \{<_{r \restriction n} : n < \omega\}$ for r in ${}^\omega\omega$.*

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Received June 30, 1986