Two theorems of N. Wiener for solutions of quasilinear elliptic equations

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1. Introduction

Relatively little is known about boundary behavior of solutions of quasilinear elliptic partial differential equations as compared to that of harmonic functions. In this paper two results, which in the harmonic case are due to N. Wiener, are generalized to a non-linear situation. Suppose that G is a bounded domain in $\mathbf{R}^n$. We consider functions $u: G \to \mathbf{R}$ which are free extremals of the variational integral

$$\int F(x, \nabla u(x))\, dm(x)$$

in the conformally invariant or borderline case $F(x,h) \approx |h|^n$. For the precise assumptions on the kernel $F: G \times \mathbf{R}^n \to \mathbf{R}$ see Section 2. Equivalently, the F-extremality of u means that u is a weak solution of the corresponding Euler equation

$$\nabla \cdot \nabla_h F(x, \nabla u(x)) = 0 \tag{1.1}$$

with measurable coefficients. Solutions u, *F-extremals*, of the equation (1.1) form a similar basis for a non-linear potential theory as harmonic functions do for the classical potential theory. Especially, the Perron-Wiener-Brelot method can be applied, see Section 2.10. For each bounded function $f: \partial G \to R$ there exist two F-extremals, the *upper Perron solution* $\bar{H}_f$ and the *lower Perron solution* $\underline{H}_f$ with "boundary values" f in G. These functions are defined via sub- and super-solutions as in the classical harmonic case. In 1970, W. Mazja [M] proved, although his formulation was slightly less general, that if f is continuous and if the Wiener condition

$$\int_0 \frac{\varphi(t)^{1/(n-1)}}{t}\,dt = \infty, \tag{1.2}$$

$\varphi(t)=\mathrm{cap}_n(B^n(x_0,2t),\ \complement G\cap \bar{B}^n(x_0,t))$, holds at $x_0\in\partial G$, i.e. $\complement G$ is not thin at x_0, then

$$\lim_{x\to x_0} \underline{H}_f(x) = f(x_0) = \lim_{x\to x_0} \bar{H}_f(x). \tag{1.3}$$

We call the point x_0 *F-regular* if the condition (1.3) holds for each continuous $f\colon \partial G\to\mathbf{R}$. Our first theorem, proved in Section 3, shows that Mazja's result has a converse.

THEOREM 1.4. *If $x_0\in\partial G$ is F-regular, then $\complement G$ is not thin at x_0.*

The proof for Theorem 1.4 shows that the F-regularity can be studied via restrictions of $C^\infty(\mathbf{R}^n)$-functions to the boundary.

The result of Mazja and Theorem 1.4 were proved by N. Wiener [W2] in the classical harmonic case in $\mathbf{R}^n$, $n\geq 2$. His proof strongly employed linearity. Our proof is based on sharp F-capacity estimates and hence in the classical plane harmonic case our version gives a new and geometric proof for Theorem 1.4 based on energy considerations.

The variational interpretation of Mazja's result and Theorem 1.4 is the following. For a function $f\colon \bar{G}\to\mathbf{R}$ in the Sobolev class $W_n^1(G)$, let $u=u_f$ be the unique F-extremal with Sobolev boundary values f, i.e. $u-f\in W_{n,0}^1(G)$. *Then*

$$\lim_{x\to x_0} u(x) = f(x_0)$$

for all continuous functions f if and only if the condition (1.2) *holds at x_0.*

Theorem 1.4 implies the following result, which in the linear case is due to W. Littman, G. Stampacchia and H. F. Weinberger [LSW], see also [P].

COROLLARY 1.5. *F-regularity is independent of the variational kernel F.*

To formulate our second result we call a function $f\colon\partial G\to\mathbf{R}\cup\{-\infty,\infty\}$ *F-resolutive*, if the corresponding upper and lower Perron solutions $\bar{H}_f$ and $\underline{H}_f$ are F-extremals in G and $\bar{H}_f=\underline{H}_f$ there. In Section 4 we extend the well-known potential theoretic theorem of N. Wiener [W3].

THEOREM 1.6. *All continuous functions $f\colon\partial G\to\mathbf{R}$ are F-resolutive.*

Proofs for Wiener's result are usually based on approximation of a continuous function by a difference of two subharmonic functions. Our proof for Theorem 1.6 rests on an important obstacle method, which for $n=2$ gives a new and direct proof for Wiener's result.

The results in Theorems 1.4 and 1.6 can be extended considerably. For instance, it is possible to consider solutions u of an elliptic quasilinear differential equation

$$\nabla \cdot A(x, \nabla u(x)) = 0$$

in divergence form where for a.e. $x \in G$, $|A(x, h)| \leq \gamma |h|^{p-1}$ and $p > n-1$ (for the precise assumption on A see Remark 2.3 in Section 2). The assumption $p > n-1$ is needed in the proof for Theorem 1.4 to guarantee that the p-capacity between two non-degenerate continua in $\mathbf{R}^n$ is positive. However, we restrict our considerations to the case $p=n$.

2. Perron's method

In this section we first present the basic assumptions on variational kernels F and recall some properties of the corresponding F-extremals. The rest of the section is devoted to the Perron-Wiener-Brelot method in a non-linear situation.

2.1. *Variational integrals.* Let G be a bounded domain in $\mathbf{R}^n$. We consider weak solutions, called F-extremals, of an Euler equation

$$\nabla \cdot \nabla_h F(x, \nabla u(x)) = 0 \tag{2.2}$$

where the variational kernel $F\colon G \times \mathbf{R}^n \to \mathbf{R}$ satisfies the assumptions

(a) For each $\varepsilon > 0$ there is a closed set K in G such that $m(G \setminus K) < \varepsilon$ and the restriction $F|K \times \mathbf{R}^n$ is continuous.

(b) For a.e. $x \in G$ the mapping $h \mapsto F(x, h)$ is strictly convex and differentiable in $\mathbf{R}^n$; for a fixed x the gradient of F with respect to h is denoted by $\nabla_h F$.

(c) There are constants $0 < \alpha \leq \beta < \infty$ such that for a.e. $x \in G$

$$\alpha |h|^n \leq F(x, h) \leq \beta |h|^n,$$

whenever $h \in \mathbf{R}^n$.

(d) For a.e. $x \in G$

$$F(x, \lambda h) = |\lambda|^n F(x, h),$$

when $h \in \mathbf{R}^n$ and $\lambda \in \mathbf{R}$.

A typical example is the n-Dirichlet kernel $F(x, h) = |h|^n$. For a thorough analysis of the above assumptions see [GLM1]. If the exponent n in (c) and (d) is replaced by an exponent $p>1$, most of our theory still holds; in the cases $1<p\leq n-1$ the method in Section 3 breaks down.

A function u in $C(G)\cap \operatorname{loc} W_n^1(G)$, i.e. u is ACL^n, is called an F-extremal in G, if for all domains $D\subset\subset G$

$$I_F(u,D)=\inf_{v\in\mathcal{F}_u} I_F(v,D)$$

where

$$I_F(v,D)=\int_D F(x,\nabla v(x))\,dm(x)$$

is the variational integral with the kernel F and

$$\mathcal{F}_u=\{v\in C(\bar{D})\cap W_n^1(D)\colon v=u \text{ in } \partial D\}.$$

A function u in the class $C(G)\cap \operatorname{loc} W_n^1(G)$ is an F-extremal if and only if it is a weak solution of (2.2), i.e.

$$\int_G \nabla_h F(x,\nabla u(x))\cdot\nabla\eta(x)\,dm(x)=0$$

for all $\eta\in C_0^\infty(G)$. For this result see [GLM1, Theorem 3.18].

2.3 *Remark.* In [M], see also the remark [MH] due to L.-I. Hedberg, Mazja considered quasilinear second order elliptic equations in divergence form

$$\nabla\cdot A(x,\nabla u(x))=0 \tag{2.4}$$

where the function $A\colon G\times\mathbf{R}^n\to\mathbf{R}^n$, in addition to the usual measurability conditions, satisfies in the borderline case $p=n$ the assumptions:

(i) There are constants $0<\gamma,\ \sigma<\infty$ such that for a.e. $x\in G$, $|A(x,h)|\leq\gamma|h|^{n-1}$, and $h\cdot A(x,h)\geq\sigma|h|^n$.

(ii) For a.e. $x\in G$, $A(x,\lambda h)=\lambda^{n-1}A(x,h)$, $\lambda>0$.

(iii) For a.e. $x\in G$, $(A(x,h)-A(x,k))\cdot(h-k)>0$ for $h\neq k$.

It is easy to see, cf. [GLM1, pp. 48–49], that the function $A(x,h)=\nabla_h F(x,h)$ satisfies (i)–(iii). Although it is not true that a vector function A satisfying (i)–(iii) is the gradient of a variational kernel F, it is possible in Section 3 to replace the concept of F-capacity by the so called A-potential, see [M, p. 47]. Thus Theorems 1.4 and 1.6 hold for equations (2.4) as well.

2.5. *Properties of F-extremals.* Although F-extremals do not in general form a linear space, λu and $u+\lambda$ are F-extremals whenever u is an F-extremal and $\lambda \in \mathbf{R}$. Moreover, F-extremals satisfy the important *F-comparison principle* [GLM3, Lemma 2.3]: If u and v are F-extremals in G,

$$\overline{\lim_{x\to y}}\, u(x) \le \underline{\lim_{x\to y}}\, v(x)$$

for all $y \in \partial G$ and the left and right hand sides are neither ∞ nor $-\infty$ at the same time, then $u \le v$ in G. Also *Harnack's principle* is true for F-extremals [GLM1, Theorem 4.22]: If u_i is an increasing family of F-extremals in G then either $u_i \to u_0$ uniformly on compact subsets of G and u_0 is an F-extremal in G or $u_i(x) \to \infty$ at each point $x \in G$.

2.6. *Sub-F-extremals.* An upper semi-continuous function $u\colon G \to [-\infty, \infty)$ is called a *sub-F-extremal* in G if u satisfies the F-comparison principle in each domain $D \subset\subset G$, i.e. if $h \in C(\bar{D})$ is an F-extremal in D and $h \ge u$ in ∂D, then $h \ge u$ in D.

A function $v\colon G \to (-\infty, \infty]$ is called a *super-F-extremal,* if $-v$ is sub-F-extremal. For the properties of sub- and super-F-extremals see [GLM1] and [GLM3].

2.7. *Existence of F-extremals.* Suppose that $\varphi \in C(\bar{G}) \cap W_n^1(G)$ is given. Then it is well-known that there is a unique F-extremal u in $C(G) \cap W_n^1(G)$ with "Sobolev boundary values" φ, i.e. $u - \varphi \in W_{n,0}^1(G)$. Here $W_{n,0}^1(G)$ consists of all functions $w \in W_n^1(G)$ which can be approximated in $W_n^1(G)$ by $C_0^\infty(G)$-functions. The existence is easily established via a minimizing sequence as in the proof of Theorem 2.9 and the uniqueness is a consequence of the strict convexity (b), cf. [GLM1, 4.17]. Especially,

$$I_F(u, G) \le I_F(v, G)$$

for all v with $v - \varphi \in C(G) \cap W_{n,0}^1(G)$.

If G is a *regular domain,* i.e. ∂G contains no point components, and if $\varphi \in C(\bar{G}) \cap W_n^1(G)$, then there is a unique F-extremal h in G such that $h \in C(\bar{G}) \cap W_n^1(G)$ and $h|\partial G = \varphi|\partial G$. Especially,

$$I_F(h, G) \le I_F(v, G)$$

for all similar v and $h = u$ where u is the F-extremal with Sobolev boundary values φ. See [GLM1, Theorem 3.24 and Remark 3.21] for these results.

2.8. *Obstacles*. A constructive approach to the theory of sub-F-extremals is provided by a variational obstacle consideration. Let $\varphi\in C^\infty(\mathbf{R}^n)$ and set

$$\mathcal{F}_\varphi=\{u\in C(G)\cap W_n^1(G)\colon u\leq\varphi,\ u-\varphi\in W_{n,0}^1(G)\}.$$

THEOREM 2.9. *There is a unique u_φ in $\mathcal{F}_\varphi$ such that*

$$I_F(u_\varphi, G)\leq I_F(u, G)$$

for all $u\in\mathcal{F}_\varphi$. Moreover, u_φ is a sub-F-extremal.

Proof. Choose a minimizing sequence u_i in $\mathcal{F}_\varphi$. Using [GLM1, Lemma 5.9] in each subdomain D of G we may assume that every u_i satisfies

$$\omega(u_i, D)\leq\max(\omega(\varphi,\partial D),\omega(u_i,\partial D))$$

where $\omega(v,A)=\sup_A v-\inf_A v$ denotes the oscillation of a function v on a set A. From the proof of [GLM1, Lemma 2.10] it follows that the family u_i is equicontinuous in G and passing to a subsequence, if necessary, it is easy to see that $u_i\to u_\varphi$ uniformly in compact subsets of G, where $u_\varphi\in C(G)\cap W_n^1(G)$ minimizes I_F in $\mathcal{F}_\varphi$, cf. [GLM1, p. 53 and p. 62]. Since $W_{n,0}^1(G)$ is weakly closed in $W_n^1(G)$, $u_\varphi-\varphi\in W_{n,0}^1(G)$ and thus $u_\varphi\in\mathcal{F}_\varphi$. The uniqueness of u_φ follows from the strict convexity of F, the assumption (b) in Section 2.1. Finally, the sub-F-extremality of u_φ follows from [GLM1, Theorem 5.17 (ii)].

2.10. *Perron's method*. Let $f\colon\partial G\to[-\infty,\infty]$ be any function. The *lower class* L_f consists of all functions $u\colon G\to[-\infty,\infty)$ for which

(i) u is a sub-F-extremal,
(ii) u is bounded above and
(iii) $\overline{\lim}_{x\to y} u(x)\leq f(y)$ for every $y\in\partial G$.

The *upper class* U_f is defined analogously via super-F-extremals. The *lower* and *upper solutions*

$$\underline{H}_f=\sup_{u\in L_f} u,\quad \bar{H}_f=\inf_{u\in U_f} u$$

are the main objects in Perron's method. Note that $\underline{H}_f\leq\bar{H}_f$, see [GLM3, Lemma 2.5], and that $f\leq g$ implies $\underline{H}_f\leq\underline{H}_g$ and $\bar{H}_f\leq\bar{H}_g$.

THEOREM 2.11 [GLM3, Theorem 2.2]. *The lower solution $\underline{H}_f$ is either an F-extremal, or identically ∞, or identically $-\infty$ in G. One of these three alternatives is also true for the upper solution $\bar{H}_f$.*

The proof for Theorem 2.11 is based on the Poisson-modification of sub-F-extremals, which will be employed in Section 4 in a special case. Suppose that $u\in C(G)\cap W_n^1(G)$ is sub-F-extremal and that $D\subset\subset G$ is a regular domain. Let $h\in C(\bar{D})\cap W_n^1(D)$ denote the unique F-extremal in D with boundary values $h|\partial D=u|\partial D$, cf. Section 2.7. Then the *Poisson-modification*

$$P(u,D)=\begin{cases} h & \text{in } D \\ u & \text{in } G\setminus D\end{cases}$$

of u is a sub-F-extremal in G, see [GLM3, Lemma 2.9]. Especially, $P(u,D)\geq u$ by the F-comparison principle.

3. Wiener's criterion and boundary regularity

3.1. *Condensers and F-capacity*. Let $F\colon G\times\mathbf{R}^n\to\mathbf{R}$ be a variational kernel satisfying the assumptions (a)–(d) of Section 2.1. If we set $F(x,h)=|h|^n$ for $x\in\mathbf{R}^n\setminus G$, then F is defined in $\mathbf{R}^n\times\mathbf{R}^n$ and satisfies the same assumptions in any bounded domain of $\mathbf{R}^n$ as F in G possibly with different α and β but with the same β/α. Hence we may assume that F is defined everywhere.

Let $E=(A,C)$ be a *condenser* in $\mathbf{R}^n$, i.e. A is open in $\mathbf{R}^n$ and C is a compact subset of A. The *F-capacity* of the condenser E is

$$\operatorname{cap}_F E=\inf_{u\in W(E)} I_F(u,A)$$

where $W(E)$ is the set of all continuous ACL-functions $u\colon A\to\mathbf{R}$ such that

$$\overline{\lim_{x\to y}}\, u(x)\leq 0$$

for all $y\in\partial A$ and $u|C\geq 1$. It is well-known that $W(E)$ can be replaced by $W^\infty(E)=C_0^\infty(A)\cap W(E)$ in the definition for $\operatorname{cap}_F E$. In the case $F(x,h)=|h|^n$ for all $x\in\mathbf{R}^n$ we write $\operatorname{cap}_F E=\operatorname{cap} E$ and call $\operatorname{cap} E$ the *(n-)capacity* of the condenser E. If $A\setminus C$ is a regular open set, i.e. each component of $A\setminus C$ is a regular domain, see Section 2.7, then the condenser $E=(A,C)$ is called *regular*. Let $E=(A,C)$ be a regular condenser and fix a variational kernel F. By Section 2.7 there exists a unique $u=u_E\in W(E)$ such that

(i) u has a continuous extension to $\bar{A}$ and $u|\partial A=0$.
(ii) $u|C=1$ if $C\neq\varnothing$.
(iii) $\operatorname{cap}_F E=I_F(u,A)=I_F(u,A\setminus C)$.

The function u is called the *F-capacity function for E*.

The next four lemmata are needed for the proof of Theorem 1.4. The first lemma immediately follows from the assumption (c) in Section 2.1.

LEMMA 3.2. *If E is a condenser, then*

$$\alpha \operatorname{cap} E \leq \operatorname{cap}_F E \leq \beta \operatorname{cap} E$$

where α, β are the structure constants for F.

LEMMA 3.3. *Let C be closed in $\mathbf{R}^n$ and $0<r<s\leq t$. If $C_r=C\cap\bar{B}^n(r)$, then*

$$\operatorname{cap}(B^n(t),C_r)\leq \operatorname{cap}(B^n(s),C_r)\leq a^{n-1}\operatorname{cap}(B^n(t),C_r)$$

where $a=(\log t/r)/(\log s/r)$.

Proof. The left hand side inequality is trivial. To prove the right hand side inequality let $f\colon B^n(s)\to B^n(t)$ be the mapping

$$f(x)=\begin{cases} x, & x\in \bar{B}^n(r)\\ rg(x/r), & x\in B^n(s)\setminus \bar{B}^n(r)\end{cases}$$

where $g(z)=|z|^{a-1}z$. Then f is a quasiconformal mapping and

$$K(f)=K(g)=a^{n-1}.$$

For the calculation of the dilatation $K(g)$ see [V, 16.2]. Since the change of the n-capacity of a condenser under a quasiconformal mapping f is controlled by $K(f)$, we obtain

$$\begin{aligned}\operatorname{cap}(B^n(s),C_r)&\leq K(f)\operatorname{cap}(f(B^n(s)),f(C_r))\\ &=a^{n-1}\operatorname{cap}(B^n(t),C_r)\end{aligned}$$

as desired.

3.4 *Remark*. The second inequality of Lemma 3.2 is sharp; to see this choose $C=\mathbf{R}^n$. The lemma holds as well for the F-capacity but this is not needed in the sequel.

The next lemma follows from the fact that each open set in $\mathbf{R}^n$ can be approximated from inside by regular open sets, cf. [S, Lemma 5.5].

LEMMA 3.5. *Let $E=(A, C)$ be a condenser such that A is a regular domain. Then for each $\varepsilon>0$ there exists a compact set C_ε in A such that $C\subset C_\varepsilon$, $E_\varepsilon=(A, C_\varepsilon)$ is a regular condenser and*

$$\operatorname{cap}_F E_\varepsilon \leqslant \operatorname{cap}_F E+\varepsilon.$$

Suppose that $E=(A, C)$ is a regular condenser and let u be the F-capacity function for E. For $0<\gamma\leqslant 1$ write

$$E_\gamma=(A, \{x\in A: u(x)\geqslant\gamma\}).$$

The following lemma gives the basic estimate for the main result.

LEMMA 3.6. *If $b=\operatorname{cap}_F E$, then*

$$\operatorname{cap}_F E_\gamma \leqslant \frac{nb}{\gamma^{n-1}}.$$

Proof. If $\gamma=1$, then the result follows immediately. Assume $0<\gamma<1$. The function $\min(u/\gamma, 1)$ belongs to $W(E_\gamma)$. Hence

$$\operatorname{cap}_F E_\gamma \leqslant \frac{1}{\gamma^n} I_F(u, \{u<\gamma\}) = \frac{1}{\gamma^n}(b-I_F(u, \{u>\gamma\})).$$

On the other hand

$$I_F(u, \{u>\gamma\}) = (1-\gamma)^n I_F\left(\frac{u-\gamma}{1-\gamma}, \{u>\gamma\}\right) \geqslant (1-\gamma)^n \operatorname{cap}_F E'_\gamma$$

where the assumption (d) of Section 2.1 has been used and E'_γ is the condenser

$$(\{x\in A: u(x)>\gamma\}, C).$$

Since $\operatorname{cap}_F E'_\gamma \geqslant \operatorname{cap}_F E=b$, the above inequalities yield

$$\operatorname{cap}_F E_\gamma \leqslant \frac{1}{\gamma^n}(b-(1-\gamma)^n b) = \frac{b}{\gamma^n}(1-(1-\gamma)^n)$$

and because $(1-\gamma)^n \geqslant 1-n\gamma$, we obtain the desired inequality.

3.7 *Remark.* The upper bound of Lemma 3.6 for $\operatorname{cap}_F E_\gamma$ is essentially sharp, since

$$\operatorname{cap}_F E_\gamma \geqslant \frac{b}{(n/(n-1))^{n-1}\gamma^{n-1}}. \tag{3.8}$$

To prove (3.8) first observe that if $E_i=(A_i, C_i)$, $i=1,2,3$, are three condensers with

$$A_1 \supset A_2 \supset C_2 \supset A_3 \supset C_3 \supset C_1,$$

then

$$\operatorname{cap}_F E_1^{-1/(n-1)} \geq \operatorname{cap}_F E_2^{-1/(n-1)} + \operatorname{cap}_F E_3^{-1/(n-1)}. \tag{3.9}$$

This is a well known inequality for the usual n-capacity, see [G, Lemma 2], and the proof for the general case is similar: Choose any functions $u_i \in W^\infty(E_i)$, $i=2,3$, such that $u_i=0$ on $\mathbf{R}^n \setminus A_i$ and $u_i=1$ in C_i. Set

$$u_1 = a_2 u_2 + a_3 u_3$$

where $a_2, a_3 \geq 0$ and $a_2+a_3=1$. Then $u_1 \in W(E_1)$ and hence

$$\operatorname{cap}_F E_1 \leq I_F(u_1, A_1) = a_2^n I_F(u_2, A_2 \setminus C_2) + a_3^n I_F(u_3, A_3 \setminus C_3).$$

Taking infimums over all such u_2, u_3 yields

$$\operatorname{cap}_F E_1 \leq a_2^n \operatorname{cap}_F E_2 + a_3^n \operatorname{cap}_F E_3. \tag{3.10}$$

If $\operatorname{cap}_F E_i > 0$ for $i=2,3$, then set

$$a_i = \operatorname{cap}_F E_i^{-1/(n-1)} (\operatorname{cap}_F E_2^{-1/(n-1)} + \operatorname{cap}_F E_3^{-1(n-1)})^{-1}$$

and (3.10) implies (3.9). If $\operatorname{cap}_F E_i=0$ for $i=2$ or 3, then $\operatorname{cap}_F E_1=0$ and (3.9) again follows.

To complete the proof for (3.8) write for $0<\gamma<1$

$$E'_\gamma = (\{x \in A\colon u(x) > \gamma\}, C).$$

The function $(u-\gamma)/(1-\gamma)$ belongs to $W(E'_\gamma)$ and hence

$$\operatorname{cap}_F E'_\gamma \leq \frac{1}{(1-\gamma)^n} I_F(u, \{u > \gamma\}) \leq \frac{1}{(1-\gamma)^n} b.$$

By (3.9)

$$b^{-1/(n-1)} \geq \operatorname{cap}_F E_\gamma^{-1/(n-1)} + \operatorname{cap}_F E_\gamma'^{-1/(n-1)},$$

and we obtain

$$\operatorname{cap}_F E_\gamma \geq \frac{b}{(1-(1-\gamma)^{n/(n-1)})^{n-1}}. \tag{3.11}$$

The elementary inequality

$$(1-\gamma)^{n/(n-1)} \geqslant 1-\frac{n}{n-1}\gamma$$

now shows that (3.8) follows from (3.11).

3.12. *Wiener's criterion*. Suppose that G is a bounded domain in $\mathbf{R}^n$ and $F: \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}$ is the continuation, see Section 3.1, of a variational kernel F satisfying (a)–(d) in Section 2.1. A point $x_0 \in \partial G$ is called *F-regular* if for all continuous functions $f: \partial G \to \mathbf{R}$

$$\lim_{x \to x_0} \underline{H}_f(x) = f(x_0) = \lim_{x \to x_0} \bar{H}_f(x). \tag{3.13}$$

For the definition of $\underline{H}_f$ and $\bar{H}_f$ see Section 2.10. Write $C = \mathbf{R}^n \setminus G$ and for $t > 0$ let

$$\varphi(t) = \operatorname{cap}(B^n(x_0, 2t),\ \bar{B}^n(x_0, t) \cap C).$$

The point x_0 satisfies *Wiener's criterion* if

$$W(x_0, C) = \int_0^1 \frac{\varphi(t)^{1/(n-1)}}{t}\, dt = \infty. \tag{3.14}$$

If $W(x_0, C) < \infty$, then C is said to be *thin* at x_0.

For the proof of Theorem 1.4 we still need two lemmata. In the first lemma the Wiener integral $W(x_0, C)$ is simply estimated from below by a Wiener sum.

LEMMA 3.15. *Let $r_1 > 0$ and $r_{i+1} = r_i/2$, $i = 1, 2, \ldots$. Then*

$$\sum_{i=2}^{\infty} a_i^{1/(n-1)} \leqslant \frac{2}{\log 2} \int_0^{r_1} \frac{\varphi(t)^{1/(n-1)}}{t}\, dt$$

where $a_i = \varphi(r_i)$, $i = 2, 3, \ldots$.

Proof. We may assume $x_0 = 0$. For $t \in [r_{i+1}, r_i]$ set $C_t = C \cap \bar{B}^n(t)$. Then Lemma 3.3 yields

$$\begin{aligned}
\varphi(t) &= \operatorname{cap}(B^n(2t), C_t) \geqslant \operatorname{cap}(B^n(2r_i), C_t) \\
&\geqslant \operatorname{cap}(B^n(2r_i), C_{r_{i+1}}) \geqslant 2^{1-n} \operatorname{cap}(B^n(2r_{i+1}), C_{r_{i+1}}) \\
&= 2^{1-n} \varphi(r_{i+1}) = 2^{1-n} a_{i+1}.
\end{aligned}$$

Hence

$$\begin{aligned}\int_0^{r_1} \frac{\varphi(t)^{1/(n-1)}}{t}\,dt &= \sum_{i=1}^{\infty} \int_{r_{i+1}}^{r_i} \frac{\varphi(t)^{1/(n-1)}}{t}\,dt \\ &\geqslant \frac{1}{2} \sum_{i=1}^{\infty} a_{i+1}^{1/(n-1)} \int_{r_{i+1}}^{r_i} \frac{dt}{t} \\ &= \frac{\log 2}{2} \sum_{i=1}^{\infty} a_{i+1}^{1/(n-1)}\end{aligned}$$

as required.

Let F_0 be a closed set in a ball $B^n(2r)$, $r>0$, and suppose that the condenser $E=(B^n(2r), F_0)$ is regular, see Section 3.1. Let u be the F-capacity function for the condenser E. For $\gamma>0$ let

$$A_\gamma = \{x \in B^n(2r)\colon u(x) < \gamma\}.$$

LEMMA 3.16. *There is a constant c_1 depending only on n and α such that the set A_γ contains some sphere $S^{n-1}(t)$, $t \in (r/4, r/2)$, whenever*

$$\gamma \geqslant c_1 \operatorname{cap} E^{1/(n-1)}.$$

Proof. For $0<\gamma<1$ let E_γ be the condenser

$$(B^n(2r), \{x\colon u(x) \geqslant \gamma\}).$$

Suppose that A_γ does not contain any $S^{n-1}(t)$, $r/4<t<r/2$. Then the set $\{x\colon u(x)\geqslant\gamma\}$ meets $S^{n-1}(t)$ for all $t \in (r/4, r/2)$ and the spherical symmetrization yields, cf. [G, Theorem 1] or [S],

$$\operatorname{cap} E_\gamma \geqslant \operatorname{cap} E' = c(n) > 0$$

where E' is the condenser

$$E' = (B^n(2r), \{s\, e_1\colon r/4 \leqslant s \leqslant r/2\})$$

and $c(n)$ depends only on n, see [L] for $n=3$ and [V, Theorem 11.9 and Remark 11.11] for $n\geqslant2$. Observe that the borderline case $p=n$ has been used in the above estimate. For $p<n-1$ the p-capacity of E' vanishes. Thus by Lemma 3.2

$$\operatorname{cap}_F E_\gamma \geqslant \alpha \operatorname{cap} E_\gamma \geqslant \alpha c(n)$$

and Lemma 3.6 gives the estimate

$$\operatorname{cap}_F E_\gamma \leqslant \frac{nb}{\gamma^{n-1}}$$

where $b=\operatorname{cap}_F E$. The last two inequalities imply

$$\gamma \leqslant b^{1/(n-1)} \left(\frac{n}{\alpha c(n)} \right)^{1/(n-1)} = b^{1/(n-1)} c' \tag{3.17}$$

and if we choose $c_1=c'+1$, then for $\gamma \geqslant c_1 b^{1/(n-1)}$ the inequality (3.17) is a contradiction. Consequently A_γ contains $S^{n-1}(t)$ for some $t \in (r/4, r/2)$ as desired.

3.18. *Proof for Theorem* 1.4. We may assume $x_0=0$. Suppose that $W(x_0, C)<\infty$. Let $\varepsilon>0$. We shall fix ε later. Choose $r_1>0$ such that

$$\int_0^{r_1} \frac{\varphi(t)^{1/(n-1)}}{t} dt \leqslant \varepsilon$$

and let $r_{i+1}=r_i/2$, $i=1,2,\ldots$. Set

$$a_i = \operatorname{cap}(B^n(2r_i), C_{r_i})$$

where $C_t=\bar{B}^n(t) \cap C$. Lemma 3.15 yields

$$\sum_{i=2}^{\infty} a_i^{1/(n-1)} \leqslant \frac{2\varepsilon}{\log 2}$$

and hence

$$a_i^{1/(n-1)} \leqslant 2\varepsilon/\log 2$$

for all $i=2,3,\ldots$. Next for each i choose a regular condenser $E_i=(B^n(2r_i), F_i)$ such that $F_i \supset C_{r_i}$ and

$$\operatorname{cap} E_i^{1/(n-1)} \leqslant a_i^{1/(n-1)} + \varepsilon 2^{-i}.$$

See Lemma 3.5. Set $b_i=\operatorname{cap}_F E_i$ and let u_i be the F-capacity function for the condenser E_i, $i=2,3,\ldots$.

By Lemma 3.16 for $\gamma_i=c_1 b_i^{1/(n-1)}$ the set

$$A_i = \{x \in B^n(2r_i) : u_i(x) < \gamma_i\}$$

contains $S^{n-1}(t_i)$ for some $t_i \in (r_i/4, r_i/2)$. On the other hand

$$\begin{aligned}\sum_{i=2}^{\infty} b_i^{1/(n-1)} &\leq \beta^{1/(n-1)} \sum_{i=2}^{\infty} \operatorname{cap} E_i^{1/(n-1)} \\ &\leq \beta^{1/(n-1)} \left(\sum_{i=2}^{\infty} a_i^{1/(n-1)} + \varepsilon \right) \\ &\leq \beta^{1/(n-1)} \left(\frac{2\varepsilon}{\log 2} + \varepsilon \right) < 5\beta^{1/(n-1)} \varepsilon\end{aligned} \tag{3.19}$$

and hence especially

$$b_i^{1/(n-1)} < 5\beta^{1/(n-1)} \varepsilon$$

for all $i=2, 3, \ldots$. If we now choose

$$\varepsilon = (\beta^{1/(n-1)} 5 c_1)^{-1},$$

then $\gamma_i < 1$ for $i=2, 3, \ldots$. In particular, $S^{n-1}(t_2)$ does not meet F_2 and hence not C. Thus $S^{n-1}(t_2)$ is contained in G.

Next let $f\colon \partial G \to \mathbf{R}$ be the continuous boundary function

$$f(x) = \begin{cases} 1, & x \in B^n(t_2) \cap \partial G, \\ 0, & x \in \partial G \setminus B^n(t_2). \end{cases}$$

Now $\underline{H}_f \not\equiv 1$, since G is bounded and hence there is $r>0$ such that $G \subset B^n(r)$ and if we let $f_0(x)=0$, $x \in S^{n-1}(r)$, and $f_0(x)=1$, $x \in B^n(t_2) \cap \partial G$, then the F-extremal $u=\underline{H}_{f_0}$ in $G'=B^n(r) \setminus (B^n(t_2) \cap \bar{G})$ takes the boundary values 0 in $\partial B^n(r)$, see e.g. [GLM2, Remark 2.20], and thus $u<1$ in $B^n(r) \setminus \bar{B}^n(t_2)$. The F-comparison principle yields $\underline{H}_f \leq u$ in G and thus $\underline{H}_f \not\equiv 1$. On the other hand

$$\max_{S^{n-1}(t_2)} \underline{H}_f = M < 1$$

since the Harnack's inequality [GLM1, Theorem 4.15] holds for the non-negative F-extremal $1-\underline{H}_f$. Set

$$v = \frac{\underline{H}_f - M}{1-M}.$$

Then the F-extremal v is ≤ 0 in $S^{n-1}(t_2)$ and we shall show that $W(x_0, C) < \infty$ implies

$$\varlimsup_{\substack{x \to x_0 \\ x \in G}} v(x) < 1. \tag{3.20}$$

This will complete the proof.

To this end observe that $\bar{B}^n(r_3) \supset S^{n-1}(t_2)$ and $u_3(x)=1$ for x in F_3 implies via the F-comparison principle, see Section 2.5, that

$$v(x) \leqslant u_3(x), \quad x \in B^n(t_3) \setminus F_3.$$

Now $u_3(x) < \gamma_3$ for $x \in S^{n-1}(t_3)$ by the selection of ε and hence the F-extremal $v-\gamma_3$ satisfies

$$v-\gamma_3 \leqslant 0$$

in $S^{n-1}(t_3)$. But now $t_3 < r_3/2 = r_4$ and the F-comparison principle again yields

$$v-\gamma_3 \leqslant u_4 \quad \text{in} \quad B^n(t_3) \setminus F_4$$

and since $u_4 < \gamma_4$ in $S^{n-1}(t_4)$ and $t_4 < t_3$, we obtain $v-\gamma_3 < \gamma_4$ in $S^{n-1}(t_4)$. Continuing we have

$$v - \sum_{i=3}^{k} \gamma_i \leqslant \gamma_{k+1} \quad \text{in} \quad S^{n-1}(t_{k+1}),$$

$k=3,4,\ldots$, and thus

$$v \leqslant \sum_{i=3}^{\infty} \gamma_i \tag{3.21}$$

on each $S^{n-1}(t_i)$, $i=3,4,\ldots$. The definition of γ_i and (3.19) yield

$$\sum_{i=3}^{\infty} \gamma_i = c_1 \sum_{i=3}^{\infty} b_i^{1/(n-1)} < 5c_1 \beta^{1/(n-1)} \varepsilon = 1.$$

Hence (3.21) implies (3.20) and $x_0=0$ cannot be an F-regular point of ∂G.

3.22 *Remark*. From the proof for Theorem 1.4 it follows that if $W(x_0, C) < \infty$, then there exists a C^∞-function $f\colon \mathbf{R}^n \to \mathbf{R}$ such that

$$\varliminf_{\substack{x \to x_0 \\ x \in G}} \underline{H}_f(x) \neq f(x_0).$$

Moreover, $\underline{H}_f$ coincides with the F-extremal with "boundary values" f in Sobolev's sense; see Remark 4.6.

3.23. *Corollaries to Theorem* 1.4. As in the classical harmonic case a point $x_0 \in \partial G$ is said to have an *F-barrier* if there exists a sub-F-extremal $w: G \to R$ such that

(a) $\overline{\lim}_{x\to y} w(x) < 0$ for all $y \in \partial G$, $y \neq x_0$,

(b) $\lim_{x\to x_0} w(x) = 0$.

See [GLM3, Section 3].

THEOREM 3.24. *Suppose that G is a bounded domain in $\mathbf{R}^n$, that $F: G\times\mathbf{R}^n \to \mathbf{R}$ satisfies the assumptions* (a)–(d) *of Section* 2.1 *and that $x_0 \in \partial G$. Then the following conditions are equivalent:*

(i) *x_0 is F-regular.*

(ii) $W(x_0, \mathbf{R}^n \setminus G) = \infty$.

(iii) *x_0 has an F-barrier.*

Proof. The equivalence (i)⇔(iii) follows from [GLM3, Theorem 3.7]. Theorem 1.4 implies (i)⇒(ii). Mazja [M, p. 51] proved (ii)⇒(i), although his formulation was slightly less general; he assumed that the boundary function belongs to $W_n^1(G) \cap C(\bar{G})$. For the general case observe that if $f \in W_n^1(G) \cap C(\bar{G})$, then Remark 4.6 yields $\underline{H}_f = \bar{H}_f = u$ where u is the F-extremal with Sobolev-boundary values f, i.e. $u - f \in W^1_{n,0}(G)$. The F-regularity of x_0 now follows by a simple approximation method via the estimate [M, (26)*, p. 51].

Theorem 3.24 immediately implies Corollary 1.5 and the variational interpretation of Theorem 3.24 presented in Introduction can now be handled as follows. If $W(x_0, \mathbf{R}^n \setminus G) = \infty$, then for all functions $f: \bar{G} \to R$ in the class $C(\bar{G}) \cap W_n^1(G)$ the aforementioned result of Mazja implies $\lim_{x\to x_0} u(x) = f(x_0)$ where u is the F-extremal with Sobolev boundary values f, i.e. $u - f \in W^1_{n,0}(G)$. The converse follows from Remark 3.22.

4. Resolutivity

4.1. *Preliminaries*. Let G be a bounded domain and let a variational kernel F satisfy the assumptions (a)–(d) in Section 2.1. Let $f: \partial G \to [-\infty, \infty]$ be any function. We recall that the function f is *F-resolutive* if $\underline{H}_f$ and $\bar{H}_f$ are F-extremals and $\underline{H}_f = \bar{H}_f$ in G.

For the next lemma observe that if f is F-resolutive, then also $\lambda f + \mu$, $\lambda, \mu \in \mathbf{R}$, is F-resolutive.

LEMMA 4.2. *Suppose that the functions $f_i: \partial G \to R$, $i=1,2,\ldots$, are F-resolutive. If $\lim f_i = f$ uniformly on ∂G, then f is F-resolutive.*

Proof. Given any $\varepsilon>0$, there is i_ε such that

$$f_i - \varepsilon < f < f_i + \varepsilon$$

for $i > i_\varepsilon$. Thus

$$\bar{H}_{f_i} - \varepsilon = \underline{H}_{f_i} - \varepsilon = \underline{H}_{f_i - \varepsilon} \leq \underline{H}_f \leq \bar{H}_f \leq \bar{H}_{f_i + \varepsilon} = \bar{H}_{f_i} + \varepsilon$$

for $i > i_\varepsilon$. This yields $\underline{H}_f = \bar{H}_f$ and since $\bar{H}_f$ is finite, it is also an F-extremal.

4.3 *Remark.* Although the F-regularity of a boundary point $x_0 \in \partial G$ is, by Corollary 1.5, independent of F, this does not hold for the F-resolutivity even in the linear case. To see this let G be the unit disk B^2 in $\mathbf{R}^2$ and let $F(x,h) = |h|^2$ be the classical Dirichlet kernel. Choose a homeomorphism h of $\bar{B}^2$ onto itself such that $h|B^2$ is quasiconformal and $h|\partial B^2$ is not absolutely continuous, cf. [BA]. Let $F_1 = h^{\#}F: B^2 \times \mathbf{R}^2 \to \mathbf{R}$ be the kernel induced by the mapping h. For the construction of F_1 see [GLM1, 6.1]. In this case F-extremals are harmonic functions and since the Euler equation (1.1) for F_1 is linear, F_1-extremals also form a linear space. The corresponding harmonic measures μ and μ_1 on ∂B^2 are singular, see [GLM2, Remark 5.6(b)]. Choose a set C in ∂B^2 such that $\mu(C)>0$ but $\mu_1(C)=0$. Now C contains a subset E which is not μ-measurable. By a theorem of M. Brelot [B1, p. 152] the characteristic function χ_E of E is not F-resolutive although, by the same theorem, it is F_1-resolutive.

4.4. *Proof for Theorem* 1.6. Let $f: \partial G \to \mathbf{R}$ be continuous. By Section 2.10

$$\min_{\partial G} f \leq \underline{H}_f \leq \bar{H}_f \leq \max_{\partial G} f,$$

and hence, by Theorem 2.11, $\underline{H}_f$ and $\bar{H}_f$ are F-extremals. Approximating f with smooth functions we can, according to Lemma 4.2, reduce the proof to the following case: $\underline{H}_\varphi = \bar{H}_\varphi$, whenever $\varphi \in C^\infty(\mathbf{R}^n)$. Fix such a φ and let $v \in C(G) \cap W_n^1(G)$ be the unique F-extremal with Sobolev boundary values $v - \varphi \in W_{n,0}^1(G)$, see Section 2.7. We claim that $\underline{H}_\varphi \geq v$ and $\bar{H}_\varphi \leq v$.

To this end let $u = u_\varphi$ be the sub-F-extremal of Theorem 2.9. Especially, $u \leq \varphi$ in G, $u - \varphi \in W_{n,0}^1(G)$ and

$$I_F(u,G) \leq I_F(\varphi, G) = I. \tag{4.5}$$

Choose regular domains $D_1 \subset\subset D_2 \subset\subset \ldots$ such that $G=\bigcup D_i$ and consider the Poisson-modifications

$$u_i = P(u, D_i), \quad i = 1, 2, \ldots,$$

see Section 2.10. Clearly, $u_i - u \in W^1_{n,0}(G)$ and thus $u_i - \varphi \in W^1_{n,0}(G)$. Note also that $u_i \in L_\varphi$, because $u_i \leq \varphi$ in $G \setminus D_i$. Now $u_{i+1} = P(u_i, D_{i+1})$ yields

$$u_1 \leq u_2 \leq \ldots$$

and by Harnack's principle, see Section 2.5, $h = \lim u_i$ is an F-extremal in G. Observe that the case $h \equiv \infty$ is excluded by the bound $u_i \leq \sup_{\partial G} \varphi$.

By the construction $I_F(u_i, G) \leq I_F(u, G)$ and thus (4.5) yields

$$\int_G |\nabla u_i|^n \, dm \leq \frac{\beta}{\alpha} I, \quad i = 1, 2, \ldots .$$

This implies that $h \in W^1_n(G)$ and $h - \varphi \in W^1_{n,0}(G)$ since $W^1_{n,0}(G)$ is weakly closed. By uniqueness, $v = h$. On the other hand, the inequality $u_i \leq \underline{H}_\varphi$ gives

$$v = h = \lim u_i \leq \underline{H}_\varphi .$$

A corresponding construction from above yields $v \geq \bar{H}_\varphi$. This implies the desired result $\underline{H}_\varphi = \bar{H}_\varphi$.

4.6 *Remark.* If $f\colon \bar{G} \to R$ is sufficiently regular, i.e. $f \in C(\bar{G}) \cap W^1_n(G)$, then the F-extremal u with Sobolev boundary values f, $u - f \in W^1_{n,0}(G)$, coincides with Perron's solution $H_f = \underline{H}_f = \bar{H}_f$. This result, mentioned by Brelot [B2, X.2] in the classical harmonic case, is a byproduct of the previous proof.

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