ON MEIJER TRANSFORM

BY

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in Lucknow

1. The integral equation

$$h(p) = p \int_0^\infty e^{-px} g(x)\, dx \tag{1.1}$$

is symbolically denoted as

$$h(p) \doteqdot g(x),$$

and $h(p)$ is known as the Laplace transform of $g(x)$.

The inverse of (1.1) is given by

$$g(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{px} \frac{h(p)}{p}\, dp. \tag{1.2}$$

Meijer [1] introduced the generalized Laplace transform

$$F(s) = \int_0^\infty e^{-\frac{1}{2}st} (st)^{-k-\frac{1}{2}} W_{k+\frac{1}{2},m}^{(st)} f(t)\, dt \tag{1.3}$$

and its inverse

$$f(t) = \lim_{\lambda\to\infty} \frac{1}{2\pi i} \cdot \frac{\Gamma(1-k+m)}{\Gamma(1+2m)} \int_{\beta-\lambda i}^{\beta+\lambda i} e^{\frac{1}{2}st} (st)^{k-\frac{1}{2}} M_{k-\frac{1}{2},m}^{(st)} F(s)\, ds, \tag{1.4}$$

where $M_{k,m}^{(z)}$ and $W_{k,m}^{(z)}$ are the two Whittaker functions. (1.3) and (1.4) are symbolically denoted as [2]

$$f(t) \xrightarrow[m]{k+\frac{1}{2}} \varphi(s),$$

where $\varphi(s) \equiv s F(s)$.

For $k = -m$ (1.3) and (1.4) reduce to (1.1) and (1.2), due to the identities

$$e^{-\frac{1}{2}st} \equiv (st)^{-m-\frac{1}{2}} W_{m+\frac{1}{2},m}^{(st)}$$

and

$$e^{\frac{1}{2}st} \equiv (st)^{-m-\frac{1}{2}} M_{-m-\frac{1}{2},m}^{(st)}.$$

In this paper, in Section I, we have determined $f(t)$ given $\varphi(s)$; also in some cases obtained $\varphi(s)$ given $f(t)$. In Section II, we have established a few rules of Meijer transform and have illustrated them by examples. In Section III, we have derived some properties of this transform by utilising the generalization of Goldstein's Theorem ([2], p. 386; [7], p. 106). In Section IV, we have given some more properties of this transform involving self-reciprocal functions in the Hankel transform.

We have imposed on our results, in general, stringent conditions. These conditions may be relaxed. For instance, in § 2.1 (i) the condition $|a|<|s|$ can be waived by making a suitable cut in the s-plane.

Section I

2. In this section we have obtained $f(t)$, the original when $\varphi(s)$, the image, is given. We have determined those which we have used later on. In some cases we have also obtained $\varphi(s)$, given $f(t)$. We have also obtained two recurrence relations of hypergeometric functions just to indicate that such relations can easily be obtained by the help of these and other Meijer transforms.

2.1. (i). Let

$$\varphi(s)=\frac{s^{\mu}}{(s+a)^{\nu}}=\sum_{r=0}^{\infty}\frac{\nu(\nu+1)\dots(\nu+r-1)}{r!}\,\frac{(-)^{r}a^{r}}{s^{r+\nu-\mu}},$$
$$|a|<|s|.$$

Now, interpreting the right hand side, with the help of ([2], p. 387)[1]

$$\frac{\Gamma(n-2k+1)}{\Gamma_{*}(n-k+1\pm m)}\,t^{n}\;\frac{k+\frac{1}{2}}{\overleftarrow{m}}\;s^{-n},\tag{2.1}$$
$$R(n-k+1\pm m)>0 \text{ and } R(s)>0,$$

we obtain

$$\begin{aligned}\frac{s^{\mu}}{(s+a)^{\nu}}\;\frac{k+\frac{1}{2}}{\overleftarrow{m}}\;&\sum_{r=0}^{\infty}\frac{\nu(\nu+1)\dots(\nu+r-1)\,\Gamma(\nu-\mu+r-2k+1)\,(-a)^{r}\,t^{\nu-\mu+r}}{\Gamma_{*}(\nu-\mu+r-k+1\pm m)}\\ &=t^{\nu-\mu}\frac{\Gamma(\nu-\mu-2k+1)}{\Gamma_{*}(\nu-\mu-k+1\pm m)}\,{}_2F_2\left[\begin{matrix}\nu,\,\nu-\mu-2k+1\\ \nu-\mu-k+1\pm m\end{matrix};\;-at\right]\\ &=f(t),\\ &R(\nu-\mu-k+1\pm m)>0,\; R(s)>0 \text{ and } |a|<|s|.\end{aligned}\tag{2.2}$$

[1] The symbol $\Gamma_{*}(\alpha\pm\beta)$ denotes $\Gamma(\alpha+\beta)\,\Gamma(\alpha-\beta)$ and ${}_mF_n\left[\begin{matrix}a\pm b, & c\pm d,\dots\\ \alpha\pm\beta, & \gamma\pm\delta,\dots\end{matrix};\;z\right]$ denotes ${}_mF_n\left[\begin{matrix}a+b,\;a-b,\;c+d,\;c-d,\dots\\ \alpha+\beta,\;\alpha-\beta,\;\gamma+\delta,\;\gamma-\delta,\dots\end{matrix};\;z\right]$.

(ii). If

$$\varphi(s)=\frac{s^{\mu}}{(s^2+a^2)^{\nu}}=\sum_{r=0}^{\infty}\frac{\nu(\nu+1)\dots(\nu+r-1)(-)^r a^{2r}}{s^{2r+2\nu-\mu}},$$

$$|a|<|s|,$$

then

$$\frac{\Gamma(2\nu-\mu-2k+1)\,t^{2\nu-\mu}}{\Gamma(2\nu-\mu-k+1\pm m)}\,{}_3F_4\left[\begin{matrix}\nu,\ \dfrac{2\nu-\mu-2k+1}{2},\ \dfrac{2\nu-\mu-2k+2}{2}\\ \dfrac{2\nu-\mu-k+1\pm m}{2},\ \dfrac{2\nu-\mu-k+2\pm m}{2}\end{matrix};\ -\frac{a^2t^2}{4}\right]$$

(2.3)

$$\xrightarrow[m]{k+\frac{1}{2}}\ \frac{s^{\mu}}{(s^2+a^2)^{\nu}},$$

$$R(2\nu-\mu-k+1\pm m)>0,\ R(s)>0\ \text{and}\ |a|<|s|.$$

(iii). If

$$\varphi(s)=\frac{s^{\mu}}{(s^4+a^4)^{\nu}},$$

then

$$\frac{t^{4\nu-\mu}\,\Gamma(4\nu-\mu-2k+1)}{\Gamma_*(4\nu-\mu-k+1\pm m)}\times$$

$$\times{}_5F_8\left[\begin{matrix}\nu,\ \dfrac{4\nu-\mu-2k+1}{4},\ \dfrac{4\nu-\mu-2k+2}{4},\ \dfrac{4\nu-\mu-2k+3}{4},\ \dfrac{4\nu-\mu-2k+4}{4}\\ \dfrac{4\nu-\mu-k+1\pm m}{4},\ \dfrac{4\nu-\mu-k+2+m}{4},\ \dfrac{4\nu-\mu-k+3\pm m}{4},\ \dfrac{4\nu-\mu-k+4\pm m}{4}\end{matrix};\ -\frac{a^4t^4}{4^4}\right]$$

(2.4)

$$\xrightarrow[m]{k+\frac{1}{2}}\ \frac{s^{\mu}}{(s^4+a^4)^{\nu}},$$

$$R(4\nu-\mu-k+1\pm m)>0,\ R(s)>0\ \text{and}\ |a|<|s|.$$

(iv). If

$$\varphi(s)=s^{\lambda}e^{-a/s}=\sum_{r=0}^{\infty}\frac{(-)^r a^r}{r!\,s^{r-\lambda}},$$

then

(2.5) $$\frac{\Gamma(1-\lambda-2k)\,t^{-\lambda}}{\Gamma_*(1-\lambda-k\pm m)}\,{}_1F_2\left[\begin{matrix}1-\lambda-2k\\1-\lambda-k\pm m\end{matrix};\ -at\right]\ \xrightarrow[m]{k+\frac{1}{2}}\ s^{\lambda}e^{-a/s},$$

$$R(1-\lambda-k\pm m)>0\ \text{and}\ R(s)>0.$$

(v). If

$$\varphi(s)=s^{-\lambda}\,{}_pF_q\left[\begin{matrix}a_1,\dots,a_p\\b_1,\dots,b_q\end{matrix};\ \frac{1}{s}\right],$$

then

$$\frac{\Gamma(\lambda-2k+1)\,t^{\lambda}}{\Gamma_*(\lambda-k+1\pm m)}\,{}_{p+1}F_{q+2}\left[\begin{matrix}a_1,\dots a_p,\ \lambda-2k+1\\b_1,\dots b_q,\ \lambda-k+1\pm m\end{matrix};\ t\right]$$

(2.6)

$$\xrightarrow[m]{k+\frac{1}{2}}\ s^{-\lambda}\,{}_pF_q\left[\begin{matrix}a_1,\dots,a_p\\b_1,\dots,b_q\end{matrix};\ \frac{1}{s}\right],$$

provided that

(a). If $q \geq p$, then $R(\lambda - k + 1 \pm m) > 0$ and $R(s) > 0$,

(b). If $q = p - 1$, then $R(\lambda - k + 1 \pm m) > 0$, $R(s) > 0$ and $|s| > 1$.

(vi). If

$$\varphi(s) = s^{\mu} J_{\nu}\left(\frac{a}{s}\right) = (\tfrac{1}{2}a)^{\nu} \sum_{r=0}^{\infty} \frac{(-)^r (\frac{1}{4}a^2)^r}{\Gamma(\nu + r + 1)\, r!\, s^{2r+\nu-\mu}},$$

then

$$\frac{a^{\nu} \Gamma(\nu - \mu - 2k + 1)\, t^{\nu-\mu}}{2^{\nu} \Gamma_{*}(\nu - \mu - k + 1 \pm m)\, \Gamma(\nu + 1)} \times$$

$$\times\, {}_2F_5\left[\begin{matrix} & \frac{\nu-\mu-2k+1}{2}, & \frac{\nu-\mu-2k+2}{2} \\ \nu+1, & \frac{\nu-\mu-k+1\pm m}{2}, & \frac{\nu-\mu-k+2\pm m}{2} \end{matrix};\ -\frac{a^2t^2}{4^2}\right] \tag{2.7}$$

$$\xrightarrow[m]{k+\frac{1}{2}} s^{\mu} J_{\nu}(a/s),$$

$R(\nu - \mu - k + 1 \pm m) > 0$ and $R(s) > 0$.

(vii). If

$$\varphi(s) = e^{-b/s} s^{\mu} J_{\nu}(a/s)$$

$$= \sum_{r=0}^{\infty} \frac{(-)^r b^r}{\lfloor r} s^{\mu-r} J_{\nu}(a/s).$$

Now, interpreting by the help of (2.7), we obtain

$$\frac{a^{\nu} t^{\nu-\mu}}{2^{\nu} \Gamma(\nu+1)} \sum_{r=0}^{\infty} \frac{(-)^r b^r \Gamma(\nu - \mu + r - 2k + 1)\, t^r}{\Gamma_{*}(\nu - \mu + r - k + 1 \pm m)\, r!} \times$$

$$\times\, {}_2F_5\left[\begin{matrix} & \frac{\nu-\mu+r-2k+1}{2}, & \frac{\nu-\mu+r-2k+2}{2} \\ \nu+1, & \frac{\nu-\mu+r-k+1\pm m}{2}, & \frac{\nu-\mu+r-k+2\pm m}{2} \end{matrix};\ -\frac{a^2t^2}{4^2}\right] \tag{2.8}$$

$$\xrightarrow[m]{k+\frac{1}{2}} e^{-b/s} s^{\mu} J_{\nu}(a/s),$$

$R(\nu - \mu - k + 1 \pm m) > 0$ and $R(s) > 0$.

(viii). If

$$\varphi(s) = s^{\mu} J_{\nu}(a/\sqrt{s}) = \frac{a^{\nu}}{2^{\nu}} \sum_{r=0}^{\infty} \frac{(-)^r (\frac{1}{2}a)^{2r}}{\Gamma(\nu + r + 1)\, r!\, s^{r+\nu/2-\mu}},$$

then

$$\frac{a^{\nu}\,\Gamma(\nu/2-\mu-2k+1)\,t^{\nu/2-\mu}}{2^{\nu}\,\Gamma_{*}(\nu/2-\mu-k+1\pm m)\,\Gamma(\nu+1)}\,{}_1F_3\left[\begin{matrix}\nu/2-\mu-2k+1\\ \nu+1,\ \nu/2-\mu-k+1\pm m\end{matrix};\ -\frac{a^2t}{4}\right] \tag{2.9}$$

$$\xrightarrow[m]{k+\frac{1}{2}}\ s^{\mu}\,J_{\nu}(a/\sqrt{2}),$$

$$R(\nu/2-\mu-k+1\pm m)>0 \quad\text{and}\quad R(s)>0.$$

From (2.9), we can easily obtain, as in (2.8),

$$\frac{a^{\nu}\,t^{\nu/2-\mu}}{2^{\nu}\,\Gamma(\nu+1)}\sum_{n=0}^{\infty}\frac{(-)^n\,b^n\,\Gamma(\nu/2-\mu+n-2k+1)\,t^n}{n!\,\Gamma_{*}(\nu/2-\mu+n-k+1\pm m)}\times$$

$$\times{}_1F_3\left[\begin{matrix}\nu/2-\mu-2k+n+1\\ \nu+1,\ \nu/2-\mu-k+n+1\pm m\end{matrix};\ -\frac{a^2t}{4}\right] \tag{2.10}$$

$$\xrightarrow[m]{k+\frac{1}{2}}\ e^{-\frac{b}{s}}\,s^{\mu}\,J_{\nu}(a/\sqrt{s}),$$

$$R(\nu/2-\mu-k+1\pm m)>0 \quad\text{and}\quad R(s)>0.$$

(ix). If

$$\varphi(s)=e^{\frac{a^2}{4s^2}}\,s^{\lambda}\,D_n\left(\frac{a}{s}\right)$$

$$=\frac{\Gamma(\frac{1}{2})\,2^{\frac{1}{2}n}}{\Gamma(-\frac{1}{2}n)\,\Gamma(\frac{1}{2}-\frac{1}{2}n)}\left[\Gamma(\tfrac{1}{2})\sum_{r=0}^{\infty}\frac{\Gamma(-\frac{1}{2}n+r)\,(\frac{1}{2}a^2)^r}{\Gamma(\frac{1}{2}+r)\,r!\,s^{2r-\lambda}}+\tfrac{1}{2}\Gamma(-\tfrac{1}{2})\sum_{r=0}^{\infty}\frac{\Gamma(\frac{1}{2}-\frac{1}{2}n+r)\,(\frac{1}{2}a^2)^{r+\frac{1}{2}}}{\Gamma(\frac{3}{2}+r)\,r!\,s^{2r-\lambda+1}}\right],$$

then

$$\frac{\Gamma(\frac{1}{2})\,2^{\frac{1}{2}n}\,t^{-\lambda}}{\Gamma(-\frac{1}{2}n)\,\Gamma(\frac{1}{2}-\frac{1}{2}n)}\left\{\frac{\Gamma(-\frac{1}{2}n)\,\Gamma(1-\lambda-2k)}{\Gamma_{*}(1-\lambda-k\pm m)}\,{}_3F_5\left[\begin{matrix}-\frac{1}{2}n,\ \dfrac{1-\lambda-2k}{2},\ \dfrac{2-\lambda-2k}{2}\\ \frac{1}{2},\ \dfrac{1-\lambda-k\pm m}{2},\ \dfrac{2-\lambda-k\pm m}{4}\end{matrix};\ \frac{a^2t^2}{8}\right]+\right.$$

$$\left.+\frac{a^t\,\Gamma(-\frac{1}{2})\,\Gamma(\frac{1}{2}-\frac{1}{2}n)\,\Gamma(2-\lambda-2k)}{\sqrt{2}\,\Gamma(\frac{1}{2})\,\Gamma_{*}(2-\lambda-k\pm m)}\,{}_3F_5\left[\begin{matrix}\frac{1}{2}-\frac{1}{2}n,\ \dfrac{2-\lambda-2k}{2},\ \dfrac{3-\lambda-2k}{2}\\ \frac{3}{2},\ \dfrac{2-\lambda-k\pm m}{2},\ \dfrac{3-\lambda-k\pm m}{2}\end{matrix};\ \frac{a^2t^2}{8}\right]\right\} \tag{2.11}$$

$$\xrightarrow[m]{k+\frac{1}{2}}\ e^{\frac{a^2}{4s^2}}\,s^{\lambda}\,D_n\left(\frac{a}{s}\right),$$

$$R(1-k-\lambda\pm m)>0 \quad\text{and}\quad R(s)>0.$$

(x). Let

$$\varphi(s)=e^{\frac{s^2}{4}}\,D^{(s)}_{-2n}.$$

We know ([3], p. 211) that

$$e^{\frac{s^2}{4}}\,D^{(s)}_{-2n}=\frac{1}{2^{n-\frac{1}{2}}\,\Gamma(n+\frac{1}{2})}\int_0^{\infty}\frac{x^{2n}\,e^{-\frac{1}{2}x^2}}{(x^2+s^2)^n}\,dx.$$

Interpreting the right hand side, by (2.3), with $\mu=0$, and $\nu=n$, we obtain

$$\varphi(s) \underset{m}{\overset{k+\frac{1}{2}}{\longleftarrow}} \frac{1}{2^{n-\frac{1}{2}}\Gamma(n+\frac{1}{2})} \times$$

$$\times \int_0^\infty x^{2n} e^{-\frac{1}{2}x^2} \left\{ \frac{\Gamma(2n-2k+1)\,t^{2n}}{\Gamma_*(2n-k+1\pm m)}\, {}_3F_4 \left[\begin{matrix} n, \frac{2n-2k+1}{2}, n-k+1 \\ \frac{2n-k+1\pm m}{2}, \frac{2n-k+2\pm m}{2} \end{matrix} ; -\frac{x^2t^2}{4} \right] \right\} dx$$

$$= \frac{t^{2n}}{2^{n-\frac{1}{2}}\Gamma(n)\Gamma(n+\frac{1}{2})} \sum_{r=0}^{\infty} \frac{\Gamma(n+r)\Gamma(2n-2k+1+2r)}{\Gamma_*(2n-k+1+r\pm m)\,r!} \left(-\frac{t^2}{4}\right)^r \int_0^\infty e^{-\frac{1}{2}x^2} x^{2n+2r}\,dx$$

$$= \frac{t^{2n}\Gamma(2n-2k+1)}{\Gamma_*(2n-k+1\pm m)}\, {}_4F_4 \left[\begin{matrix} n, n+\frac{1}{2}, \frac{2n-2k+1}{2}, n-k+1 \\ \frac{2n-k+1\pm m}{2}, \frac{2n-k+2\pm m}{2} \end{matrix} ; -\frac{t^2}{2} \right],$$

provided the change of order of integration and summation is justified. Hence

$$(2.12) \qquad \frac{t^{2n}\Gamma(2n-2k+1)}{\Gamma_*(2n-k+1\pm m)}\, {}_4F_4 \left[\begin{matrix} n, n+\frac{1}{2}, \frac{2n-2k+1}{2}, \frac{2n-2k+2}{2} \\ \frac{2n-k+1\pm m}{2}, \frac{2n-k+2\pm m}{2} \end{matrix} ; -\frac{t^2}{2} \right] \underset{m}{\overset{k+\frac{1}{2}}{\longrightarrow}} e^{\frac{s^2}{4}} D_{-2n}^{(s)},$$

$R(n+\frac{1}{2})>0$, $R(2n-k+1\pm m)>0$ and $R(s)>0$.

Regarding the change of order of integration and summation, we note that

(i) the series ${}_3F_4$ is uniformly and absolutely convergent;

(ii) the integral is uniformly and absolutely convergent, if

$R(n+\frac{1}{2})>0$, and

(iii) the integrated series is uniformly convergent.

Hence the change of order of integration and summation is justified.

(xi). If

$$\varphi(s) = s^{\lambda} J_{\mu,\nu}\left(\frac{3a}{s}\right) = \sum_{r=0}^{\infty} \frac{(-a^3)^r a^{\mu+\nu} s^{-(3r+\mu+\nu-\lambda)}}{\Gamma(\mu+r+1)\Gamma(\nu+r+1)\,r!},$$

then

$$\frac{\Gamma(\mu+\nu-\lambda-2k+1)\, a^{\mu+\nu}\, t^{\mu+\nu-\lambda}}{\Gamma_*(\mu+\nu-\lambda-k+1\pm m)\,\Gamma(\mu+1)\,\Gamma(\nu+1)}\times$$

$$\times {}_3F_8\left[\begin{matrix} \frac{\mu+\nu-\lambda-2k+1}{3}, \frac{\mu+\nu-\lambda-2k+2}{3}, \frac{\mu+\nu-\lambda-2k+3}{3} \\ \mu+1,\ \nu+1,\ \frac{\mu+\nu-\lambda-k+1\pm m}{3}, \frac{\mu+\nu-\lambda-k+2\pm m}{3}, \frac{\mu+\nu-\lambda-k+3\pm m}{3}\end{matrix};\ -\frac{a^3t^3}{3^3}\right]$$

$$\text{(2.13)}\qquad \frac{k+\frac{1}{2}}{m}\quad s^{\lambda} J_{\mu,\nu}\left(\frac{3a}{s}\right),$$

$R(\mu+\nu-\lambda-k+1\pm m)>0$ and $R(s)>0$.

(xii). If

$$\varphi(s)=s^{\lambda} H_{\nu}\left(\frac{2a}{s}\right)=s^{\lambda-\nu-1}\sum_{r=0}^{\infty}\frac{(-)^r s^{-2r} a^{2r+\nu+1}}{\Gamma(r+3/2)\,\Gamma(\nu+r+3/2)},$$

then

$$\frac{\Gamma(\nu-\lambda-2k+2)\, a^{\nu+1}\, t^{\nu-\lambda+1}}{\Gamma_*(\nu-\lambda-k+2\pm m)\,\Gamma(3/2)\,\Gamma(\nu+3/2)}\times$$

$$\text{(2.14)}\qquad \times {}_3F_6\left[\begin{matrix} 1, \frac{\nu-\lambda-2k+2}{2}, \frac{\nu-\lambda-2k+3}{2} \\ 3/2,\ \nu+3/2,\ \frac{\nu-\lambda-k+2\pm m}{2}, \frac{\nu-\lambda-k+3\pm m}{2}\end{matrix};\ -\frac{a^2t^2}{4}\right]$$

$$\frac{k+\frac{1}{2}}{m}\quad s^{\lambda} H_{\nu}\left(\frac{2a}{s}\right),$$

$R(\nu-\lambda-k+2\pm m)>0$ and $R(s)>0$.

(xiii). Let

$$\varphi(s)=s^{\lambda}\int_0^{\infty} x^{l-1} e^{-\frac{1}{2}x} W_{k+\frac{1}{2},m}^{(x)}\ {}_pF_q\left[\begin{matrix}\alpha_1,\alpha_2,\dots\alpha_p\\ \beta_1,\beta_2,\dots\beta_q\end{matrix};\ -\frac{x^2}{s^2}\right]dx.$$

We know that ([4], p. 83),

$$\text{(2.15)}\qquad \int_0^{\infty} x^{l-1} e^{-\frac{1}{2}x} W_{k+\frac{1}{2},x}^{(x)}\ {}_pF_q\left[\begin{matrix}\alpha_1,\alpha_2,\dots\alpha_p\\ \beta_1,\beta_2,\dots\beta_q\end{matrix};\ -x^2a^2\right]dx$$

$$=\frac{\Gamma_*(l+\frac{1}{2}\pm m)}{\Gamma(l-k+\frac{1}{2})}\ {}_{p+4}F_{q+2}\left[\begin{matrix}\alpha_1,\dots,\alpha_p, \frac{1}{2}l+\frac{1}{4}\pm\frac{1}{2}m, \frac{1}{2}l+\frac{3}{4}\pm\frac{1}{2}m\\ \beta_1,\dots,\beta_q, \frac{1}{2}l-\frac{1}{2}k+\frac{1}{4}, \frac{1}{2}l-\frac{1}{2}k+\frac{3}{4}\end{matrix};\ -4a^2\right],$$

provided that

(i) $R(l+\frac{1}{2}\pm m)>0$,

(ii) (α) for all values of a when $p+2\le q$,

(β) for $|a|<\frac{1}{2}$ when $p+1=q$, and

(γ) for $p=q$, only when ${}_{p+4}F_{p+2}$ is a terminating series.

Putting $a = \dfrac{1}{s}$, and interpreting with the help of (2.1), we get

$$\frac{\Gamma_*(l+\frac{1}{2}\pm m)\,\Gamma(-\lambda-2k'+1)}{\Gamma(l-k+\frac{1}{2})\,\Gamma_*(-\lambda-k'+1\pm m)\,t^{\lambda}}\times$$

$$\text{(2.16)}\qquad \times\,{}_{p+6}F_{q+6}\left[\begin{matrix}\alpha_1,\alpha_2,\ldots,\alpha_p,\ \frac{1}{2}l+\frac{1}{4}\pm\frac{1}{2}m,\ \frac{1}{2}l+\frac{3}{4}\pm\frac{1}{2}m,\ \dfrac{\lambda-2k'+1}{2},\ -\dfrac{\lambda-2k'+2}{2}\\ \beta_1,\beta_2,\ldots,\beta_q,\ \frac{1}{2}l-\frac{1}{2}k+\frac{1}{4},\ \frac{1}{2}l-\frac{1}{2}k+\frac{3}{4},\ -\dfrac{\lambda-k'+1\pm m}{2},\ -\dfrac{\lambda-k'+2\pm m}{2}\end{matrix};\ -t^2\right]$$

$$\overset{k'+\frac{1}{2}}{\underset{m'}{\rightharpoonup}}\ s^{\lambda}\int_0^{\infty} x^{l-1}e^{-\frac{1}{2}x}\,W^{(x)}_{k+\frac{1}{2},m}\ {}_pF_q\left[\begin{matrix}\alpha_1,\ldots,\alpha_p\\ \beta_1,\ldots,\beta_q\end{matrix};\ -\frac{x^2}{s^2}\right]dx,$$

provided that

(i) $R(1-\lambda-k'\pm m)>0$, $R(l+\frac{1}{2}\pm m)>0$, and $R(s)>0$,

(ii) (α) for all values of $|s|$ when $p<q$, and

(β) for $p=q$, only when ${}_{p+4}F_{q+2}$ is a terminating series.

(xiv). If

$$\varphi(s) = s^{\lambda} J_{\mu}\left(\frac{2x}{s}\right) J_{\nu}\left(\frac{2x}{s}\right)$$

$$= \sum_{r=0}^{\infty} \frac{\Gamma(\mu+\nu+2r+1)\,(-)^r\,x^{2r+\mu+\nu}}{\Gamma(\mu+r+1)\,\Gamma(\nu+r+1)\,\Gamma(\mu+\nu+r+1)\,r!\,s^{2r+\mu+\nu-\lambda}},$$

then

$$\frac{\Gamma(\mu+\nu-\lambda-2k+1)\,x^{\mu+\nu}\,t^{\mu+\nu-\lambda}}{\Gamma_*(\mu+\nu-\lambda-k+1\pm m)\,\Gamma(\mu+1)\,\Gamma(\nu+1)}\times$$

$$\text{(2.17)}\qquad \times\,{}_4F_7\left[\begin{matrix}\dfrac{\mu+\nu+1}{2},\ \dfrac{\mu+\nu+2}{2},\ \dfrac{\mu+\nu-\lambda-2k+1}{2},\ \dfrac{\mu+\nu-\lambda-2k+2}{2}\\ \mu+1,\ \nu+1,\ \mu+\nu+1,\ \dfrac{\mu+\nu-\lambda-k+1\pm m}{2},\ \dfrac{\mu+\nu-\lambda-k+2\pm m}{2}\end{matrix};\ -x^2t^2\right]$$

$$\overset{k+\frac{1}{2}}{\underset{m}{\rightharpoonup}}\ s^{\lambda} J_{\mu}\left(\frac{2x}{s}\right) J_{\nu}\left(\frac{2x}{s}\right),$$

$$R(\mu+\nu-\lambda-k+1\pm m)>0 \ \text{ and } \ R(s)>0.$$

2.2. We will now obtain two recurrence relations by the help of the Meijer transforms that we have obtained. A number of such relations can be obtained.

(i). We have

$$\frac{s}{(s^2-a^2)} = \frac{1}{2}\left[\frac{1}{s-a}+\frac{1}{s+a}\right].$$

The original of r.h.s. is

$$\frac{1}{2}\left\{\frac{t\,\Gamma(2-2k)}{\Gamma_*(2-k\pm m)}\,{}_2F_2\left[\begin{matrix}2-2k,\ 1\\2-k\pm m\end{matrix};\ a t\right]+\frac{t\,\Gamma(2-2k)}{\Gamma_*(2-k\pm m)}\,{}_2F_2\left[\begin{matrix}2-2k,\ 1\\2-k\pm m\end{matrix};\ -a t\right]\right\},$$

and with $\mu=1$, $\nu=1$ and replacing a by $i a$ in (2.3), the original of the l.h.s. is

$$\frac{\Gamma(2-2k)\,t}{\Gamma_*(2-k\pm m)}\ {}_3F_4\left[\begin{matrix}1,\ \frac{2-2k}{2},\ \frac{3-2k}{2}\\ \frac{2-k\pm m}{2},\ \frac{3-k\pm m}{2}\end{matrix};\ \frac{a^2t^2}{4}\right].$$

Hence it follows that

$$2\cdot{}_3F_4\left[\begin{matrix}1,\ 1-k,\ \frac{3}{2}-k\\ \frac{2-k\pm m}{2},\ \frac{3-k\pm m}{2}\end{matrix};\ \frac{a^2t^2}{4}\right] \tag{2.18}$$
$$={}_2F_2\left[\begin{matrix}2-2k,\ 1\\2-k\pm m\end{matrix};\ a t\right]+{}_2F_2\left[\begin{matrix}2-2k,\ 1\\2-k\pm m\end{matrix};\ -a t\right],$$
$$R(2-k\pm m)>0.$$

(ii). We have

$$\frac{4s^3}{s^4-a^4}=\frac{1}{s-a}+\frac{1}{s+a}+\frac{2s}{s^2+a^2}.$$

As in § 2.2 (i), writing the originals of both the sides with the help of (2.2), (2.3) and (2.4), we obtain

$$4\cdot{}_5F_8\left[\begin{matrix}1,\ \frac{2-2k}{4},\ \frac{3-2k}{4},\ \frac{4-4k}{4},\ \frac{5-4k}{2}\\ \frac{2-k\pm m}{4},\ \frac{3-k\pm m}{4},\ \frac{4-k\pm m}{4},\ \frac{5-k\pm m}{4}\end{matrix};\ \frac{a^4t^4}{4^4}\right] \tag{2.19}$$
$$={}_2F_2\left[\begin{matrix}2-2k,\ 1\\2-k\pm m\end{matrix};\ -a t\right]+{}_2F_2\left[\begin{matrix}2-2k,\ 1\\2-k\pm m\end{matrix};\ a t\right]+2\cdot{}_3F_4\left[\begin{matrix}1,\ 1-k,\ \frac{3}{2}-k\\ \frac{2-k\pm m}{2},\ \frac{3-k\pm m}{2}\end{matrix};\ -\frac{a^2t^2}{4}\right],$$
$$R(2-k\pm m)>0.$$

2.3. (i). If $f(t)=t^{\mu}J_{\nu}(2a\sqrt{t})$, then

$$\varphi(s)=s\int_0^{\infty}e^{-\frac{1}{2}st}(st)^{-k-\frac{1}{2}}\,W_{k+\frac{1}{2},m}^{(st)}\,t^{\mu}J_{\nu}(2a\sqrt{t})\,dt.$$

Expanding $J_\nu(z)$ and changing the order of integration and summation, which can easily be justified as in § 2.1 (x), we have

$$\varphi(s) = a^\nu \sum_{r=0}^{\infty} \frac{(-)^r a^{2r} s^{-(\mu+\nu/2+r)}}{\Gamma(\nu+r+1)\, r!} \left[s \int_0^\infty e^{-\frac{1}{2} st} (st)^{\mu+\nu/2+r-k-\frac{1}{2}-1} W_{k+\frac{1}{2},m}^{(st)}\, dt \right].$$

On evaluating the integral ([7], p. 114), we get

$$(2.20) \quad \varphi(s) = \frac{\Gamma_*(\mu+\nu/2-k+1\pm m)\, a^\nu s^{-\mu-\nu/2}}{\Gamma(\mu+\nu/2-2k+1)\,\Gamma(\nu+1)}\, {}_2F_2 \left[\begin{matrix} \mu+\nu/2-k+1\pm m \\ \nu+1,\ \mu+\nu/2-2k+1 \end{matrix} ;\ -\frac{a^2}{s} \right],$$

$$R(\mu+\nu/2+1-k\pm m) > 0 \text{ and } R(s) > 0.$$

If we note that

$$\cos(2u\sqrt{t}) = \left(\frac{\pi}{2}\right)^{\frac{1}{2}} (2u\sqrt{t})^{\frac{1}{2}} J_{-\frac{1}{2}}(2u\sqrt{t}),$$

then from (2.20), with $\eta = \frac{1}{4}$ and $\nu = -\frac{1}{2}$, we obtain

$$(2.21) \quad \frac{\Gamma_*(1-k\pm m)}{\Gamma(1-2k)}\, {}_2F_2 \left[\begin{matrix} 1-k\pm m \\ \frac{1}{2},\ 1-2k \end{matrix} ;\ -\frac{u^2}{s} \right] \ \frac{k+\frac{1}{2}}{m} \ \cos(2u\sqrt{t}),$$

$$R(1-k\pm m) > 0 \text{ and } R(s) > 0.$$

Similarly, from (2.20), with $\mu = \frac{1}{4}$ and $\nu = \frac{1}{2}$ and noting that

$$\sin(2u\sqrt{t}) = \left(\frac{\pi}{2}\right)^{\frac{1}{2}} (2u)^{\frac{1}{2}} t^{\frac{1}{4}} J_{\frac{1}{2}}(2u\sqrt{t}),$$

we get

$$(2.22) \quad \frac{2u\,\Gamma_*(\frac{3}{2}-k\pm m)}{\sqrt{s}\,\Gamma(\frac{3}{2}-2k)}\, {}_2F_2 \left[\begin{matrix} \frac{3}{2}-k\pm m \\ \frac{3}{2},\ \frac{3}{2}-2k \end{matrix} ;\ -\frac{u^2}{s} \right] \ \frac{k+\frac{1}{2}}{m} \ \sin(2u\sqrt{t}),$$

$$R(\tfrac{3}{2}-k\pm m) > 0 \text{ and } R(s) > 0.$$

(ii). If

$$f(t) = e^{-\frac{1}{4}t} D_n(\sqrt{t}),$$

then ([8], p. 353)

$$\begin{aligned} \varphi(s) &= \int_0^\infty e^{-\frac{1}{2}st} (st)^{-k-\frac{1}{2}} W_{k+\frac{1}{2},m}^{(st)} e^{-\frac{1}{4}t} D_n(\sqrt{t})\, dt \\ (2.23) \qquad &= A\, s \int_0^\infty e^{-\frac{1}{2}st} (st)^{-k-\frac{1}{2}} W_{k+\frac{1}{2},m}^{(st)} \left[\int_0^\infty u^n e^{-2u^2} \cos(2u\sqrt{t})\, du \right] dt \\ &= A \int_0^\infty u^n e^{-2u^2} \left[s \int_0^\infty e^{-\frac{1}{2}st} (st)^{-k-\frac{1}{2}} W_{k+\frac{1}{2},m}^{(st)} \cos(2u\sqrt{t})\, dt \right] du \end{aligned}$$

where $A = (-)^{n/2} 2^{n+2} (2\pi)^{-\frac{1}{2}}$ for n even positive integer, $R(1-k\pm m) > 0$ and $R(s) > 0$.

Now, evaluating the t-integral by (2.21), we get

$$\varphi(s)=\frac{(-)^{n/2}\,2^{n+3/2}\,\Gamma_*(1-k\pm m)}{\sqrt{\pi}\,\Gamma(1-2k)}\int_0^\infty u^n e^{-2u^2}\,{}_2F_2\left[\begin{matrix}1-k\pm m\\ \frac12,\,1-2k\end{matrix};\,-\frac{u^2}{s}\right]du.$$

Again expanding ${}_2F_2$ and changing the order of integration and summation, which can be justified as in § 2.1 (x), we have

$$\varphi(s)=(-)^{n/2}\,2^{n+3/2}\sum_{r=0}^{\infty}\frac{(-)^r\,\Gamma_*(1-k+r\pm m)\,s^{-r}}{\Gamma(\frac12+r)\,\Gamma(1-2k+r)\,r!}\int_0^\infty e^{-2u^2}u^{n+2r}\,du$$

(2.24)

$$=\frac{(-)^{n/2}\,2^{n/2}\,\Gamma_*(1-k\pm m)\,\Gamma\left(\frac{n+1}{2}\right)}{\Gamma(\frac12)\,\Gamma(1-2k)}\,{}_3F_2\left[\begin{matrix}1-k\pm m,\,\frac{n+1}{2}\\ \frac12,\,1-2k\end{matrix};\,-\frac{1}{2s}\right],$$

$$R(1-k\pm m)>0,\ R(s)>0\ \text{and}\ |s|>\tfrac12.$$

Regarding the change of order of integration in (2.23), we note that the u-integral is absolutely convergent, the t-integral is absolutely convergent, if $R(1-k\pm m)>0$ and $R(s)>0$, and the repeated integral is absolutely convergent under the same conditions. Hence the change of order of integration is justified.

If n is an odd positive integer, then ([8], p. 353)

$$e^{-\frac14 t}D_n(\sqrt{t})=A\int_0^\infty u^n e^{-2u^2}\sin(2u\sqrt{t})\,du,$$

where

$$A=(-)^{\frac12(n-1)}\,2^{n+2}\,(2\pi)^{-\frac12}.$$

Now, following the method of § 2.3 (ii), we obtain

$$\varphi(s)=(-)^{\frac12(n-1)}\,2^{\frac12(n-1)}\frac{\Gamma_*(\frac32-k\pm m)\,\Gamma\left(\frac{n}{2}+1\right)}{\Gamma(\frac32)\,\Gamma(\frac32-2k)\,s^{\frac12}}\,{}_3F_2\left[\begin{matrix}\frac32-k\pm m,\,\frac{n+2}{2}\\ \frac32,\,\frac32-2k\end{matrix};\,-\frac{1}{2s}\right], \tag{2.25}$$

$$R(\tfrac32-k\pm m)>0,\ R(s)>0\ \text{and}\ |s|>\tfrac12.$$

(iii). If

$$f(t)=\int_0^\infty x^{l-1}e^{-\frac12 x}\,W_{k+\frac12,m}^{(x)}\,{}_pF_q\left[\begin{matrix}\alpha_1,\ldots,\alpha_p\\ \beta_1,\ldots,\beta_q\end{matrix};\,-x^2t^2\right]dx,$$

then from (2.15), we obtain

$$\varphi(s) = s \int_0^\infty e^{-\frac{1}{2}st} (st)^{-k'-\frac{1}{2}} W_{k'+\frac{1}{2}, m'}^{(st)} \times$$

$$\times \left\{ \frac{\Gamma_*(l+\frac{1}{2}\pm m)}{\Gamma(l-k+\frac{1}{2})} {}_{p+4}F_{q+2} \left[\begin{matrix} \alpha_1, \ldots, \alpha_p, \frac{1}{2}l + \frac{1}{4} \pm \frac{m}{2}, \frac{l}{2} + \frac{3}{4} \pm \frac{m}{2} \\ \beta_1, \ldots, \beta_q, \frac{l}{2} - \frac{k}{2} + \frac{1}{4}, \frac{l}{2} - \frac{k}{2} + \frac{3}{4} \end{matrix} ; -4t^2 \right] \right\} dt$$

(2.26)
$$= \frac{\Gamma_*(l+\frac{1}{2}\pm m)\,\Gamma_*(1-k'\pm m')}{\Gamma(l-k+\frac{1}{2})\,\Gamma(1-2k')} \times$$

$$\times {}_{p+8}F_{q+4} \left[\begin{matrix} \alpha_1, \ldots, \alpha_p, \frac{l}{2} + \frac{1}{4} \pm \frac{m}{2}, \frac{l}{2} + \frac{3}{4} \pm \frac{m}{2}, \frac{1-k'\pm m'}{2}, \frac{2-k'\pm m'}{2} \\ \beta_1, \ldots, \beta_q, \frac{l}{2} - \frac{k}{2} + \frac{1}{4}, \frac{l}{2} - \frac{k}{2} + \frac{3}{4}, \frac{1-2k'}{2}, 1-k' \end{matrix} ; -\frac{16}{s^2} \right],$$

provided that

(i) $R(1-k'\pm m') > 0,\ R(l+\frac{1}{2}\pm m) > 0,\ R(s) > 0,$

(ii) for all values of $|s|$, if $p+4 \le q$ and

(iii) for $|s| > 4$ if $p+5 = q$.

(iv). If

$$f(t) = t^\lambda J_\mu(2at) J_\nu(2at)$$

$$= \frac{a^{\mu+\nu} t^{\mu+\nu+\lambda}}{\Gamma(\mu+1)\Gamma(\nu+1)} {}_2F_3 \left[\begin{matrix} \frac{1+\mu+\nu}{2}, \frac{2+\mu+\nu}{2} \\ 1+\mu, 1+\nu, 1+\mu+\nu \end{matrix} ; -4a^2t^2 \right],$$

then

$$\varphi(s) = \frac{\Gamma_*(\mu+\nu+\lambda-k+1\pm m)\, a^{\mu+\nu}}{\Gamma(\mu+1)\Gamma(\nu+1)\Gamma(\mu+\nu+\lambda-2k+1)\, s^{\mu+\nu+\lambda}} \times$$

(2.27)
$$\times {}_6F_5 \left[\begin{matrix} \frac{1+\mu+\nu}{2}, \frac{2+\mu+\nu}{2}, \frac{\mu+\nu+\lambda-k+1\pm m}{2}, \frac{\mu+\nu+\lambda-k+2\pm m}{2} \\ 1+\mu, 1+\nu, 1+\mu+\nu, \frac{\mu+\nu+\lambda-2k+1}{2}, \frac{\mu+\nu+\lambda-2k+2}{2} \end{matrix} ; -16\frac{a^2}{s^2} \right],$$

$R(\lambda+\mu+\nu-k+1\pm m) > 0,\ R(s) > 0$ and $|s| > 4|a|$.

Section II

3. In this section we have established a few rules of Meijer transform. We have also illustrated these rules by examples.

3.1. (i). Rule 1.

If

$$f(t) \overset{1}{\underset{m}{\to}} \varphi(s),$$

then

$$f'(t) \overset{0}{\underset{m}{\to}} s\,\varphi(s), \tag{3.1}$$

provided $e^{-\frac{1}{2}st}\, W_{0,m}^{(st)}\, f(t)$ is zero for $t=0$ and ∞, and the integrals involved are convergent.

We have ([7], p. 110, with $k=-\frac{1}{2}$)

$$(st)^{-1}\, e^{-\frac{1}{2}st}\, W_{1,m}^{(st)} = -\frac{1}{s}\frac{d}{dt}\{e^{-\frac{1}{2}st}\, W_{0,m}^{(st)}\}. \tag{3.2}$$

Also

$$\frac{\varphi(s)}{s} = \int_0^\infty e^{-\frac{1}{2}st}\,(st)^{-1}\, W_{1,m}^{(st)}\, f(t)\, dt.$$

Integrating by parts, and using (3.2), we get

$$f'(t) \overset{0}{\underset{m}{\to}} s\,\varphi(s)$$

under the given conditions.

(ii). Rule 2.

If

$$f(t) \overset{0}{\underset{m}{\to}} \varphi(s),$$

then

$$\int_0^t f(x)\, dx \overset{1}{\underset{m}{\to}} \frac{\varphi(s)}{s}, \tag{3.3}$$

provided $e^{-\frac{1}{2}st}\, W_{0,m}^{(st)} \int_0^t f(x)\, dx$ is zero for $t=0$ and ∞, and the integrals involved are convergent.

We have

$$\frac{\varphi(s)}{s} = \int_0^\infty e^{-\frac{1}{2}st}\, W_{0,m}^{(st)}\, f(t)\, dt.$$

Integrating by parts and using (3.2), we have

$$\int_0^t f(x)\, dx \overset{1}{\underset{m}{\to}} \frac{\varphi(s)}{s}$$

under the given conditions.

(iii). Rule 3.

If

$$t^r f(t) \xrightarrow[m]{k} \varphi^{(s)}_{r,k,m},$$

then

(3.4) $$\varphi^{(s)}_{n,0,m} = s\,\{\varphi^{(s)}_{n+1,1,m} + \varphi'^{(s)}_{n,0,m}\},$$

provided the Meijer transforms involved exist.

We have

$$t^n f(t) \xrightarrow[m]{0} \varphi^{(s)}_{n,0,m}.$$

Applying (R.2) ([2], p. 386), we get

(3.5) $$n\,t^n f(t) + t^{n+1} f'(t) \xrightarrow[m]{0} -s\,\varphi'^{(s)}_{n,0,m}.$$

Again applying (3.1) on

$$t^{n+1} f(t) \xrightarrow[m]{1} \varphi^{(s)}_{n+1,1,m},$$

we get

(3.6) $$(n+1)\,t^n f(t) + t^{n+1} f'(t) \xrightarrow[m]{0} s\,\varphi^{(s)}_{n+1,1,m}.$$

Now, from (3.5) and (3.6) follows

$$\varphi^{(s)}_{n,0,m} = s\,\{\varphi^{(s)}_{n+1,1,m} + \varphi'^{(s)}_{n,0,m}\}.$$

(iv). Rule 4.

If

$$t^r f(t) \xrightarrow[m]{k} \varphi^{(s)}_{r,k,m},$$

then

(3.7) $$\varphi^{(s)}_{n-1,0,m} = s \int_s^\infty x^{-1}\,\varphi^{(x)}_{n,1,m}\,dx,$$

provided the Meijer transforms exist and the integrals converge.

If $t^n f(t) \xrightarrow[m]{1} \varphi^{(s)}_{n,1,m}$, then applying (R.3) ([2], p. 386) we have

$$\int_0^t x^{n-1} f(x)\,dx \xrightarrow[m]{1} \int_s^\infty x^{-1}\,\varphi^{(x)}_{n,1,m}\,dx.$$

Further, if

$$t^{n-1} f(t) \xrightarrow[m]{0} \varphi^{(s)}_{n-1,0,m},$$

then, by Rule 2, we have

$$\int_0^t x^{n-1} f(x)\,dx \xrightarrow[m]{1} \frac{\varphi^{(s)}_{n-1,0,m}}{s}.$$

Hence

$$\varphi_{n-1,0,m}^{(s)} = s \int_s^\infty x^{-1} \varphi_{n,1,m}^{(x)} \, dx.$$

3.2. Applications:–

(i). If

$$f(t) = t^\nu e^{-at}, \quad \text{then}$$

$$\varphi_{\nu,1,m}^{(s)} = \frac{\Gamma_*(\nu+\frac{1}{2}\pm m)}{s^\nu \Gamma(\nu)} {}_2F_1\left[\begin{matrix}\nu+\frac{1}{2}\pm m \\ \nu\end{matrix}; -\frac{a}{s}\right],$$

$$R(\nu+\tfrac{1}{2}\pm m)>0,\ R(s)>0 \ \text{and}\ |a|<|s|.$$

Applying (3.1), we get

$$-a\,t^\nu e^{-at} + \nu\, t^{\nu-1} e^{-at} \ \underset{m}{\overset{0}{\to}}\ \frac{\Gamma_*(\nu+\frac{1}{2}\pm m)}{s^{\nu-1}\Gamma(\nu)} {}_2F_1\left[\begin{matrix}\nu+\frac{1}{2}\pm m \\ \nu\end{matrix}; -\frac{a}{s}\right].$$

Hence

$$a(\nu+\tfrac{1}{2}\pm m)\,{}_2F_1\left[\begin{matrix}\nu+\frac{3}{2}\pm m \\ \nu+2\end{matrix}; -\frac{a}{s}\right] = s\,\nu(\nu+1)\left\{{}_2F_1\left[\begin{matrix}\nu+\frac{1}{2}\pm m \\ \nu+1\end{matrix}; -\frac{a}{s}\right] - {}_2F_1\left[\begin{matrix}\nu+\frac{1}{2}\pm m \\ \nu\end{matrix}; -\frac{a}{s}\right]\right\},$$

$$R(\nu+\tfrac{1}{2}\pm m)>0,\ R(s)>0 \ \text{and}\ |a|<|s|.$$

(ii). We know the following recurrence relations of $J_\nu(z)$.

$$2J'_\nu(z) = J_{\nu-1}(z) - J_{\nu+1}(z), \tag{3.8}$$

$$J'_\nu(z) = J_{\nu-1}(z) - \frac{\nu}{z} J_\nu(z), \tag{3.9}$$

$$J'_\nu(z) = \frac{\nu}{z} J_\nu(z) - J_{\nu+1}(z). \tag{3.10}$$

Also ([2], p. 389)

$$J_\nu(t) \ \underset{m}{\overset{1}{\to}}\ \frac{(\frac{1}{2})^\nu \Gamma_*(\nu+\frac{1}{2}\pm m)}{s^\nu \Gamma(\nu)\Gamma(\nu+1)} {}_4F_3\left[\begin{matrix}\frac{\nu+\frac{1}{2}\pm m}{2}, \frac{\nu+\frac{3}{2}\pm m}{2} \\ \nu+1, \frac{\nu}{2}, \frac{\nu+1}{2}\end{matrix}; -\frac{1}{s^2}\right].$$

Applying Rule 1, we get

$$J'_\nu(t) \ \underset{m}{\overset{0}{\to}}\ \frac{(\frac{1}{2})^\nu \Gamma_*(\nu+\frac{1}{2}\pm m)}{s^{\nu-1}\Gamma(\nu)\Gamma(\nu+1)} {}_4F_3\left[\begin{matrix}\frac{\nu+\frac{1}{2}\pm m}{2}, \nu+\frac{3}{2}\pm m \\ \nu+1, \frac{\nu}{2}, \frac{\nu+1}{2}\end{matrix}; -\frac{1}{s^2}\right],$$

$$R(\nu+\tfrac{1}{2}\pm m)>0,\ R(s)>0 \ \text{and}\ |s|>1.$$

Now, with the help of (3.8), we obtain

(3.11)
$$
{}_4F_3\left[\begin{matrix}\frac{\nu+\frac{1}{2}\pm m}{2}, \frac{\nu+\frac{3}{2}\pm m}{2} \\ \nu+1, \frac{\nu}{2}, \frac{\nu+1}{2}\end{matrix}; -\frac{1}{s^2}\right]
$$
$$
= {}_4F_3\left[\begin{matrix}\frac{\nu+\frac{1}{2}\pm m}{2}, \frac{\nu+\frac{3}{2}\pm m}{2} \\ \nu, \frac{\nu+1}{2}, \frac{\nu+2}{2}\end{matrix}; -\frac{1}{s^2}\right] - \frac{(\nu+\frac{3}{2}\pm m)(\nu+\frac{1}{2}\pm m)}{4s^2\nu(\nu+1)^2(\nu+2)}\, {}_4F_3\left[\begin{matrix}\frac{\nu+\frac{5}{2}\pm m}{2}, \frac{\nu+\frac{7}{2}\pm m}{2} \\ \nu+2, \frac{\nu+3}{2}, \frac{\nu+4}{2}\end{matrix}, -\frac{1}{s^2}\right],
$$

$R(\nu+\frac{1}{2}\pm m) > 0$, $R(s) > 0$ and $|s| > 1$.

Similarly, from (3.9), we get

(3.12)
$$
{}_4F_3\left[\begin{matrix}\frac{\nu+\frac{1}{2}\pm m}{2}, \frac{\nu+\frac{3}{2}\pm m}{2} \\ \nu+1, \frac{\nu}{2}, \frac{\nu+1}{2}\end{matrix}; -\frac{1}{s^2}\right]
$$
$$
= {}_4F_3\left[\begin{matrix}\frac{\nu+\frac{1}{2}\pm m}{2}, \frac{\nu+\frac{3}{2}\pm m}{2} \\ \nu, \frac{\nu+1}{2}, \frac{\nu+2}{2}\end{matrix}; -\frac{1}{s^2}\right] - {}_4F_3\left[\begin{matrix}\frac{\nu+\frac{1}{2}\pm m}{2}, \frac{\nu+\frac{3}{2}\pm m}{2} \\ 1+\nu, \frac{\nu+1}{2}, \frac{\nu+2}{2}\end{matrix}; -\frac{1}{s^2}\right],
$$

$R(\nu+\frac{1}{2}\pm m) > 0$, $R(s) > 0$ and $|s| > 1$.

We may obtain another similar relation from (3.10).

(iii). If
$$
f(t) = e^{-\frac{1}{4}t}\left\{D_n(\sqrt{t}) - \frac{\Gamma(\frac{1}{2})\,2^{\frac{1}{2}n}}{\Gamma(\frac{1}{2}-\frac{1}{2}n)}\right\},
$$

n being positive even integer, then from (2.24)

$$
\varphi_{n,1,m}^{(s)} = \frac{(-2)^{\frac{n-2}{2}}\,\Gamma_*(\frac{3}{2}\pm m)\,\Gamma\left(\frac{n+3}{2}\right)}{s\,\Gamma(\frac{3}{2})}\, {}_3F_2\left[\begin{matrix}\frac{n+3}{2}, \frac{3}{2}\pm m \\ \frac{3}{2}, 2\end{matrix}; -\frac{1}{2s}\right] +
$$
$$
+ \frac{\Gamma(\frac{1}{2})\,2^{\frac{n-4}{2}}\,\Gamma_*(\frac{3}{2}\pm m)}{s\,\Gamma(\frac{1}{2}-\frac{1}{2}n)}\, {}_2F_1\left[\begin{matrix}\frac{3}{2}\pm m \\ 2\end{matrix}; -\frac{1}{4s}\right],
$$

$R(\frac{3}{2}\pm m) > 0$, $R(s) > 0$ and $|s| > \frac{1}{2}$.

Now applying Rule 1, we obtain

$$\frac{e^{-\frac{1}{4}t} D_n'(t^{\frac{1}{2}})}{2t^{\frac{1}{2}}} - \tfrac{1}{4} e^{-\frac{1}{4}t} \left\{ D_n(t^{\frac{1}{2}}) - \frac{\Gamma(\frac{1}{2}) 2^{\frac{1}{2}n}}{\Gamma(\frac{1}{2}-\frac{1}{2}n)} \right\}$$

$$\text{(3.13)} \qquad \underset{m}{\overset{0}{\rightarrow}} \frac{(-2)^{\frac{n-2}{2}} \Gamma_*(\frac{3}{2}\pm m)\, \Gamma\left(\frac{n+3}{2}\right)}{\Gamma(\frac{3}{2})} {}_3F_2\left[\begin{matrix} \frac{3}{2}\pm m, \frac{n+3}{2} \\ \frac{3}{2}, 2 \end{matrix} ; -\frac{1}{2s}\right] +$$

$$+ \frac{\Gamma(\frac{1}{2})\, 2^{\frac{n-4}{2}}\, \Gamma_*(\frac{3}{2}\pm m)}{\Gamma(\frac{1}{2}-\frac{1}{2}n)} {}_2F_1\left[\begin{matrix} \frac{3}{2}\pm m \\ 2 \end{matrix} ; -\frac{1}{4s}\right],$$

$R(\frac{3}{2}\pm m) > 0$, $R(s) > 0$ and $|s| > \frac{1}{2}$.

But

$$\tfrac{1}{4} e^{-\frac{1}{4}t} \left\{ D_n(t^{\frac{1}{2}}) - \frac{\Gamma(\frac{1}{2}) 2^{\frac{1}{2}n}}{\Gamma(\frac{1}{2}-\frac{1}{2}n)} \right\}$$

$$\text{(3.14)} \qquad \underset{m}{\overset{0}{\rightarrow}} \frac{1}{4} \frac{(-2)^{\frac{n}{2}} \Gamma_*(\frac{3}{2}\pm m)\, \Gamma\left(\frac{n+1}{2}\right)}{\Gamma(\frac{1}{2})} {}_3F_2\left[\begin{matrix} \frac{3}{2}\pm m, \frac{n+1}{2} \\ \frac{1}{2}, 2 \end{matrix} ; -\frac{1}{2s}\right] -$$

$$- \frac{\Gamma(\frac{1}{2})\, 2^{\frac{n}{2}}\, \Gamma_*(\frac{3}{2}\pm m)}{4\,\Gamma(\frac{1}{2}-\frac{1}{2}n)} {}_2F_1\left[\begin{matrix} \frac{3}{2}\pm m \\ 2 \end{matrix} ; -\frac{1}{4s}\right],$$

$R(\frac{3}{2}\pm m) > 0$, $R(s) > 0$ and $|s| > \frac{1}{2}$.

Hence, from (3.13) and (3.14), we get

$$t^{-\frac{1}{2}} e^{-\frac{1}{4}t} D_n'(t^{\frac{1}{2}}) \underset{m}{\overset{0}{\rightarrow}} \frac{(-)^{\frac{n}{2}} 2^{\frac{n-2}{2}} \Gamma_*(\frac{3}{2}\pm m)\, \Gamma\left(\frac{n+1}{2}\right)}{\Gamma(\frac{1}{2})} \left\{ {}_3F_2\left[\begin{matrix} \frac{3}{2}\pm m, \frac{n+1}{2} \\ \frac{1}{2}, 2 \end{matrix} ; -\frac{1}{2s}\right] - \right.$$

$$\left. - 2(n+1)\, {}_3F_2\left[\begin{matrix} \frac{3}{2}\pm m, \frac{n+3}{2} \\ \frac{3}{2}, 2 \end{matrix} ; -\frac{1}{2s}\right] \right\},$$

$R(\frac{3}{2}\pm m) > 0$, $R(s) > 0$ and $|s| > \frac{1}{2}$.

(iv). We know

$$\text{(3.15)} \qquad \frac{1}{z} D_n'(z) + \tfrac{1}{2} D_n(z) = \frac{n}{2} D_{n-1}(z).$$

Using (2.24), we get

$$\text{(3.16)} \qquad \left\{ \begin{aligned} & \frac{n\, e^{-\frac{1}{4}t} D_{n-1}(t^{\frac{1}{2}})}{t^{\frac{1}{2}}} = e^{-\frac{1}{4}t} t^{-\frac{1}{2}} D_n'(t^{\frac{1}{2}}) + \tfrac{1}{2} e^{-\frac{1}{4}t} D_n(t^{\frac{1}{2}}) \\ & \underset{m}{\overset{0}{\rightarrow}} \frac{(-)^{\frac{n}{2}} 2^{\frac{n-2}{2}} \Gamma_*(\frac{3}{2}\pm m)\, \Gamma\left(\frac{n+1}{2}\right)}{\Gamma(\frac{1}{2})} \left\{ {}_3F_2\left[\begin{matrix} \frac{3}{2}\pm m, \frac{n+1}{2} \\ \frac{1}{2}, 2 \end{matrix} ; -\frac{1}{2s}\right] - \right. \\ & \qquad \left. - 2(n+1)\, {}_3F_1\left[\begin{matrix} \frac{3}{2}\pm m, \frac{n+3}{2} \\ \frac{3}{2}, 2 \end{matrix} ; -\frac{1}{2s}\right] \right\} + \end{aligned} \right.$$

$$
\left|\begin{array}{l}
+\dfrac{(-)^{\frac{n}{2}}\,2^{\frac{n}{2}-1}\,\Gamma_*\left(\frac{3}{2}\pm m\right)\Gamma\left(\dfrac{n+1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)}\;{}_3F_2\left[\begin{matrix}\frac{3}{2}\pm m,\ \dfrac{n+1}{2}\\ \frac{1}{2},\ 2\end{matrix}\;;\;-\dfrac{1}{2s}\right]\\[2ex]
=\dfrac{(-)^{\frac{n}{2}}\,2^{\frac{n}{2}}\,\Gamma_*\left(\frac{3}{2}\pm m\right)\Gamma\left(\dfrac{n+1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)}\left\{{}_3F_2\left[\begin{matrix}\frac{3}{2}+m,\ \dfrac{n+1}{2}\\ \frac{1}{2},\ 2\end{matrix}\;;\;-\dfrac{1}{2s}\right]-\right.\\[2ex]
\qquad\qquad\qquad\left.-(n+1)\,{}_3F_2\left[\begin{matrix}\frac{3}{2}\pm m,\ \dfrac{n+3}{2}\\ \frac{3}{2},\ 2\end{matrix}\;;\;-\dfrac{1}{2s}\right]\right\},
\end{array}\right.
$$

$$R\left(\tfrac{3}{2}\pm m\right)>0,\ R(s)>0 \text{ and } |s|>\tfrac{1}{2}.$$

In the same manner we can obtain a result for n positive odd integer.

(v). We have ([2], p. 389)

$$
\begin{aligned}
t^n J_\nu(t)\ \underset{m}{\overset{k+\frac{1}{2}}{\rightarrow}}\ &\frac{\left(\frac{1}{2}\right)^\nu\,\Gamma_*(n+\nu-k+1\pm m)}{s^{n+\nu}\,\Gamma(n+\nu-2k+1)\,\Gamma(\nu+1)}\times\\
(3.17)\qquad &\times{}_4F_3\left[\begin{matrix}\dfrac{n+\nu-k+1\pm m}{2},\ \dfrac{n+\nu-k+2\pm m}{2}\\ \nu+1,\ \dfrac{n+\nu-2k+1}{2},\ \dfrac{n+\nu-2k+2}{2}\end{matrix}\;;\;-\dfrac{1}{s^2}\right]\\
&=\varphi_{n,k+\frac{1}{2},m},
\end{aligned}
$$

$$R(n+\nu-k+1\pm m)>0,\ R(s)>0 \text{ and } |s|>1.$$

Now applying Rule 3, we get

$$
\begin{aligned}
&s\frac{d}{ds}\left\{\frac{1}{s^{n+\nu}}\,{}_4F_3\left[\begin{matrix}\dfrac{n+\nu+\frac{3}{2}\pm m}{2},\ \dfrac{n+\nu+\frac{5}{2}\pm m}{2}\\ \nu+1,\ \dfrac{n+\nu+2}{2},\ \dfrac{n+\nu+3}{2}\end{matrix}\;;\;-\dfrac{1}{s^2}\right]\right.\\
(3.18)\qquad &=\frac{1}{s^{n+\nu}}\,{}_4F_3\left[\begin{matrix}\dfrac{n+\nu+\frac{3}{2}\pm m}{2},\ \dfrac{n+\nu+\frac{5}{2}\pm m}{2}\\ \nu+1,\ \dfrac{n+\nu+2}{2},\ \dfrac{n+\nu+3}{2}\end{matrix}\;;\;-\dfrac{1}{s^2}\right]-\\
&\qquad-\frac{(n+\nu+1)}{s^{n+\nu}}\,{}_4F_3\left[\begin{matrix}\dfrac{n+\nu+\frac{3}{2}\pm m}{2},\ \dfrac{n+\nu+\frac{5}{2}\pm m}{2}\\ \nu+1,\ \dfrac{n+\nu+1}{2},\ \dfrac{n+\nu+2}{2}\end{matrix}\;;\;-\dfrac{1}{s^2}\right],
\end{aligned}
$$

$$R\left(n+\nu+\tfrac{3}{2}\pm m\right)>0,\ R(s)>0 \text{ and } |s|>1.$$

Section III

4. Jaiswal ([2], p. 386) has proved the following:

$$\text{If}\quad f_1(t) \xrightarrow[m]{k+\frac{1}{2}} \varphi_1(s),$$

$$f_2(t) \xrightarrow[m]{k+\frac{1}{2}} \varphi_2(s),$$

then

$$\int_0^\infty f_1(t)\,\varphi_2(t)\,\frac{dt}{t} = \int_0^\infty f_2(t)\,\varphi_1(t)\,\frac{dt}{t}. \tag{4.1}$$

If in (4.1), we take $k = -m$, then we get the result of Goldstein ([7], p. 106):

$$\text{If}\quad f_1(x) \doteqdot \varphi_1(p),$$

$$f_2(x) \doteqdot \varphi_2(p),$$

then

$$\int_0^\infty f_1(x)\,\varphi_2(x)\,\frac{dx}{x} = \int_0^\infty f_2(x)\,\varphi_1(x)\,\frac{dx}{x}. \tag{4.2}$$

In this section we shall make use of this result and obtain certain properties of Meijer and Laplace Transforms, which enables us to find new Meijer and Laplace transforms and evaluate integrals.

4.1. (i). **Result 1.** If in (4.1), we take $f(t)$ and $\varphi(s)$ for $f_1(t)$ and $\varphi_1(s)$, and the right hand side of (2.5) as $f_2(t)$ and l.h.s. as $\varphi_2(s)$, then

$$\int_0^\infty t^{\lambda-1} e^{-a/t} f(t)\,dt = \frac{\Gamma(1-\lambda-2k)}{\Gamma_*(1-\lambda-k\pm m)} \int_0^\infty t^{-\lambda-1}\,{}_1F_2\left[\begin{matrix}1-\lambda-2k\\ 1-\lambda-k\pm m\end{matrix};\ -at\right]\varphi(t)\,dt, \tag{4.3}$$

provided $R(1-\lambda-k\pm m)>0$, and the integrals converge.

Example 1. If

$$f(t) = e^{-t}, \quad \text{then}$$

$$\varphi(s) = \frac{\Gamma(1-k\pm m)}{\Gamma(1-2k)}\,{}_2F_1\left[\begin{matrix}1-k\pm m\\ 1-2k\end{matrix};\ -\frac{1}{s}\right],$$

$$R(1-k\pm m)>0 \quad\text{and}\quad R(s+1)>0.$$

Substituting in (4.3) and taking $\lambda = -\nu$, we get

$$\int_0^\infty t^{\nu-1}{}_1F_2\left[\begin{matrix}1+\nu-2k\\1+\nu-k\pm m\end{matrix};\ -\frac{z^2t}{4}\right]{}_2F_1\left[\begin{matrix}1-k\pm m\\1-2k\end{matrix};\ -\frac{1}{t}\right]dt$$
$$=\frac{\Gamma_*(1-k+\nu\pm m)\,\Gamma(1-2k)}{\Gamma(1+\nu-2k)\,\Gamma_*(1-k\pm m)}\int_0^\infty t^{-\nu-1}\,e^{-t-\frac{z^2}{4t}}dt.$$

Now, using the integral ([9], p. 183)

$$\text{(A)}\qquad K_\nu(z)=\tfrac{1}{2}\left(\tfrac{1}{2}z\right)^\nu\int_0^\infty e^{-x-\frac{z^2}{4x}}\frac{dx}{x^{\nu+1}},\qquad R(z^2)>0,$$

we obtain

$$\int_0^\infty t^{\nu-1}{}_1F_2\left[\begin{matrix}1+\nu-2k\\1+\nu-k\pm m\end{matrix};\ -\frac{z^2t}{4}\right]{}_2F_1\left[\begin{matrix}1-k\pm m\\1-2k\end{matrix};\ -\frac{1}{t}\right]dt$$
$$=\frac{2^{\nu+1}\,\Gamma_*(1+\nu-k\pm m)\,\Gamma(1-2k)}{z^\nu\,\Gamma(1+\nu-2k)\,\Gamma_*(1-k\pm m)}K_\nu(z),$$

$R(1-2k)>0,\ R(\nu+1-k\pm m)>0,\ R(1-k\pm m)>0$ and $R(z^2)>0$

(ii). **Result 2.** If (4.1), if we take the two transforms to be

$$f(t)\ \frac{k+\frac{1}{2}}{m}\ \varphi(s),$$

and (2.2), then

$$\text{(4.4)}\qquad \int_0^\infty t^{\nu-\mu-1}{}_2F_2\left[\begin{matrix}\nu,\ \nu-\mu-2k+1\\\nu-\mu-k+1\pm m\end{matrix};\ -at\right]\varphi(t)\,dt=\frac{\Gamma_*(\nu-\mu-k+1\pm m)}{\Gamma(\nu-\mu-2k+1)}\int_0^\infty\frac{t^{\mu-1}}{(t+a)^\nu}f(t)\,dt,$$

provided $R(\nu-\mu-k+1\pm m)>0$ and the integrals converge.

Example 1. Let $f(t)=1$ and $a=1$, then

$$\varphi(s)=\frac{\Gamma_*(1-k\pm m)}{\Gamma(1-2k)},$$

$R(1-k\pm m)>0$ and $R(s)>0$,

and (4.4) reduces to

$$\int_0^\infty t^{\nu-\mu-1}{}_2F_2\left[\begin{matrix}\nu,\ \nu-\mu-2k+1\\\nu-\mu-k+1\pm m\end{matrix};\ -t\right]dt=\frac{\Gamma_*(\nu-\mu-k+1\pm m)\,\Gamma(1-2k)}{\Gamma(\nu-\mu-2k+1)\,\Gamma_*(1-k\pm m)}\int_0^\infty\frac{t^{\mu-1}}{(1+t)^\nu}dt$$
$$=\frac{\Gamma_*(\nu-\mu-k+1\pm m)\,\Gamma(1-2k)\,\Gamma(\mu)\,\Gamma(\nu-\mu)}{\Gamma(\nu-\mu-2k+1)\,\Gamma_*(1-k\pm m)\,\Gamma(\nu)},$$

$R(1-2k)>0,\ R(\nu)>R(\mu)>0$ and $R(1-k\pm m)>0.$

Example 2. If $f(t)=J_\lambda(bt)$, then

$$\varphi(s)=\frac{b_\lambda\Gamma_*(\lambda-k+1\pm m)}{(2s)^\lambda\Gamma(\lambda-2k+1)\Gamma(\lambda+1)}\,{}_4F_3\left[\begin{matrix}\frac{\lambda-k+1\pm m}{2},\ \frac{\lambda-k+2\pm m}{2}\\ \lambda+1,\ \frac{\lambda-2k+1}{2},\ \frac{\lambda-2k+2}{2}\end{matrix};\ -\frac{b^2}{s^2}\right],$$

$$R(\lambda-k+1\pm m)>0 \text{ and } R(s)>0.$$

Substituting in (4.4), and evaluating the integral ([9], p. 436), we get

$$\left(\frac{b}{2}\right)^\lambda\int_0^\infty t^{\nu-\mu-\lambda-1}\,{}_2F_2\left[\begin{matrix}\nu,\ \nu-\mu-2k+1\\ \nu-\mu-k+1\pm m\end{matrix};\ -at\right]{}_4F_3\left[\begin{matrix}\frac{\lambda-k+1\pm m}{2},\ \frac{\lambda-k+2\pm m}{2}\\ \lambda+1,\ \frac{\lambda-2k+1}{2},\ \frac{\lambda-2k+2}{2}\end{matrix};\ -\frac{b^2}{t^2}\right]dt$$

$$=\frac{\Gamma_*(\nu-\mu-k+1\pm m)\Gamma(\lambda-2k+1)\Gamma(\lambda+1)}{\Gamma_*(\lambda-k+1\pm m)\Gamma(\nu-\mu-2k+1)}\int_0^\infty\frac{t^{\mu-1}}{(t+a)^\nu}J_\lambda(bt)\,dt$$

$$=\frac{\pi a^{\mu-\nu}\Gamma_*(\nu-\mu-k+1\pm m)\Gamma(\lambda-2k+1)\Gamma(\lambda+1)}{\Gamma_*(\lambda-k+1\pm m)\Gamma(\nu-\mu-2k+1)\Gamma(\nu)\sin(\mu+\lambda-\nu+1)\pi}$$

$$\times\left\{\frac{(\frac{1}{2}ab)^\lambda\Gamma(\mu+\lambda)}{\Gamma(\lambda+1)\Gamma(\mu+\lambda-\nu+1)}\,{}_2F_3\left[\begin{matrix}\frac{\mu+\lambda}{2},\ \frac{\mu+\lambda+1}{2}\\ \lambda+1,\ \frac{\mu+\lambda-\nu+1}{2},\ \frac{\mu+\lambda-\nu+2}{2}\end{matrix};\ -\frac{a^2b^2}{4}\right]-\right.$$

$$\left.-(\tfrac{1}{2}ab)^{\nu-\mu}\sum_{m'=0}^\infty\frac{(\frac{1}{2}ab)^{m'}\Gamma(\nu+m')\sin\frac{1}{2}(\mu+\lambda-\nu-m'+1)\pi}{\Gamma(\frac{1}{2}\nu+\frac{1}{2}\lambda-\frac{1}{2}\mu+\frac{1}{2}m'+1)\Gamma(\frac{1}{2}\nu-\frac{1}{2}\lambda-\frac{1}{2}\mu+\frac{1}{2}m')}\right\},$$

$b>0$, $|\arg a|<\pi$, $R(\lambda)>0$, $R(\lambda-2k+1)>0$, $R(\nu-\mu-k+1\pm m)>0$, $R(\nu)>R(\mu+\lambda)>$ >0 and $R(\nu-\mu+\frac{3}{2})>0$.

(iii). **Result 3.** If we take $a=\dfrac{1}{p}$ in (4.4) and interpret with (2.1) after replacing k and m by k' and m' respectively, we obtain

$$p^{1-\nu}\int_0^\infty t^{\nu-\mu-1}\,{}_2F_2\left[\begin{matrix}\nu,\ \nu-\mu-2k+1\\ \nu-\mu-k+1\pm m\end{matrix};\ -\frac{t}{p}\right]\varphi(t)\,dt$$

$$(4.5)\quad \xleftarrow[m']{k'+\frac{1}{2}}\ \frac{x^{\nu-1}\Gamma_*(\nu-\mu-k+1\pm m)}{\Gamma(\nu)\Gamma(\nu-\mu-2k+1)}\int_0^\infty t^{\mu-\nu-1}f(t)\sum_{r=0}^\infty\frac{\Gamma(\nu+r)\Gamma(\nu-2k'+r)}{r!\,\Gamma_*(\nu-k'+r\pm m')}\left(-\frac{x}{t}\right)^r dt$$

$$=\frac{x^{\nu-1}\Gamma_*(\nu-\mu-k+1\pm m)\Gamma(\nu-2k')}{\Gamma(\nu-\mu-2k+1)\Gamma_*(\nu-k'\pm m')}\int_0^\infty t^{\mu-\nu-1}f(t)\,{}_2F_2\left[\begin{matrix}\nu,\ \nu-2k'\\ \nu-k'\pm m'\end{matrix};\ -\frac{x}{t}\right]dt,$$

provided $R(\nu-\mu-k+1\pm m)>0$, $R(\nu-k'\pm m')>0$, $R(p)>0$ and the integrals converge.

Putting $k' = -m'$ in (4.5), we get

$$p^{1-\nu}\int_0^\infty t^{\nu-\mu-1}\,{}_2F_2\left[\begin{matrix}\nu,\ \nu-\mu-2k\\ \nu-\mu-k+1\pm m\end{matrix};\ -\frac{t}{p}\right]\varphi(t)\,dt$$

(4.6)

$$\doteqdot \frac{x^{\nu-1}\,\Gamma_*(\nu-\mu-k+1\pm m)}{\Gamma(\nu)\,\Gamma(\nu-\mu-2k+1)}\int_0^\infty e^{-x/t}\,t^{\mu-\nu-1} f(t)\,dt,$$

provided $R(\nu)>0$, $R(\nu-\mu+1-k\pm m)>0$, $R(p)>0$ and the integrals converge.

Example 1. If we take $f(t)=e^{-t}$, then

$$\varphi(s)=\frac{\Gamma_*(1-k\pm m)}{\Gamma(1-2k)}\,{}_2F_1\left[\begin{matrix}1-k\pm m\\ 1-2k\end{matrix};\ -\frac{1}{s}\right].$$

Substituting these values in (4.5) and using (A), we get

$$p^{1-\nu}\int_0^\infty t^{\nu-\mu-1}\,{}_2F_2\left[\begin{matrix}\nu,\ \nu-\mu-2k+1\\ \nu-\mu-k+1\pm m\end{matrix};\ -\frac{t}{p}\right]{}_2F_1\left[\begin{matrix}1-k\pm m\\ 1-2k\end{matrix};\ -\frac{1}{t}\right]dt$$

$$\doteqdot \frac{x^{\nu-1}\,\Gamma_*(\nu-\mu-k+1\pm m)\,\Gamma(1-2k)}{\Gamma(\nu)\,\Gamma_*(1-k\pm m)\,\Gamma(\nu-\mu-2k+1)}\int_0^\infty e^{-t-\frac{x}{t}}\,t^{\mu-\nu-1}\,dt$$

$$=\frac{2\,\Gamma_*(\nu-\mu-k+1\pm m)\,\Gamma(1-2k)}{\Gamma_*(1-k\pm m)\,\Gamma(\nu-\mu-2k+1)}\,x^{\frac{\mu+\nu}{2}-1}\,K_{\nu-\mu}(2\sqrt{x}),$$

$R(1-2k)>0$, $R(\mu)>R(\nu)>0$, $R(\nu-\mu+1-k\pm m)>0$ and $R(p)>0$.

(iv) **Result 4.** If we take one of the Meijer transforms in (4.1) to be (2.3), then

$$\int_0^\infty t^{2\nu-\mu-1}\,{}_3F_4\left[\begin{matrix}\nu,\ \tfrac12(2\nu-\mu-2k+1),\ \tfrac12(2\nu-\mu-2k+2)\\ \tfrac12(2\nu-\mu-k+1\pm m),\ \tfrac12(2\nu-\mu-k+2\pm m)\end{matrix};\ -\frac{a^2t^2}{4}\right]\varphi_1(t)\,dt$$

(4.7)

$$=\frac{\Gamma_*(2\nu-\mu-k+1\pm m)}{\Gamma(2\nu-\mu-2k+1)}\int_0^\infty \frac{t^{\mu-1}}{(t^2+a^2)^{\nu}}\,f_1(t)\,dt,$$

provided $R(2\nu-\mu-k+1\pm m)>0$ and the integrals converge.

If in (4.7), we put $a^2=\dfrac{1}{p}$ and interpret, then

(4.8)
$$p^{1-\nu}\int_0^\infty t^{2\nu-\mu-1}\,{}_3F_4\left[\begin{matrix}\nu,\ \tfrac12(2\nu-\mu-2k+1),\ \tfrac12(2\nu-\mu-2k+2)\\ \tfrac12(2\nu-\mu-k+1\pm m),\ \tfrac12(2\nu-\mu-k+2\pm m)\end{matrix};\ \frac{t^2}{4p}\right]\varphi_1(t)\,dt$$
$$\underset{m'}{\overset{k'+\frac12}{\longleftarrow}}\ \frac{x^{\nu-1}\Gamma_*(2\nu-\mu-k+1\pm m)\,\Gamma(\nu-2k')}{\Gamma(2\nu-\mu-2k+1)\,\Gamma_*(\nu-k'\pm m')}\int_0^\infty t^{\mu-2\nu-1}\,f_1(t)\,{}_2F_2\left[\begin{matrix}\nu,\ \nu-2k'\\ \nu-k'\pm m'\end{matrix};\ -\frac{x}{t^2}\right]dt,$$

$R(2\nu-\mu-k+1\pm m)>0$, $R(\nu-k'\pm m')>0$, $R(p)>0$ and the integrals converge.

Now, putting $k'=-m'$, we get

(4.9)
$$p^{1-\nu}\int_0^\infty t^{2\nu-\mu-1}\,\varphi_1(t)\,{}_3F_4\left[\begin{matrix}\nu,\ \tfrac12(2\nu-\mu-2k+1),\ \tfrac12(2\nu-\mu-2k+2)\\ \tfrac12(2\nu-\mu-k+1\pm m),\ \tfrac12(2\nu-\mu-k+2\pm m)\end{matrix};\ -\frac{t^2}{4p}\right]dt$$
$$\doteqdot \frac{x^{\nu-1}\Gamma_*(2\nu-\mu-k+1\pm m)}{\Gamma(2\nu-\mu-2k+1)\,\Gamma(\nu)}\int_0^\infty e^{-x/t^2}\,t^{\mu-2\nu-1}f_1(t)\,dt,$$

provided $R(2\nu-\mu-k+1\pm m)>0$, $R(\nu)>0$, $R(p)>0$ and the integrals converge.

Example 1. Let $f_1(t)=1$, then

$$\varphi_1(s)=\frac{\Gamma_*(1-k\pm m)}{\Gamma(1-2k)},$$
$$R(1-k\pm m)>0,\ R(s)>0.$$

Substituting in (4.7) with $a=1$, we get

$$\int_0^\infty t^{2\nu-\mu-1}\,{}_3F_4\left[\begin{matrix}\nu,\ \tfrac12(2\nu-\mu-2k+1),\ \tfrac12(2\nu-\mu-2k+2)\\ \tfrac12(2\nu-\mu-k+1\pm m),\ \tfrac12(2\nu-\mu-k+2\pm m)\end{matrix};\ -\frac{t^2}{4}\right]dt$$
$$=\frac12\frac{\Gamma_*(2\nu-\mu-k+1\pm m)\,\Gamma(1-2k)}{\Gamma(2\nu-\mu-2k+1)\,\Gamma_*(1-k\pm m)}\int_0^\infty \frac{t^{\mu/2-1}}{(1+t)^\nu}\,dt$$
$$=\frac{\Gamma_*(2\nu-\mu-k+1\pm m)\,\Gamma(1-2k)\,\Gamma(\mu/2)\,\Gamma(\nu-\mu/2)}{2\,\Gamma(2\nu-\mu-2k+1)\,\Gamma_*(1-k\pm m)\,\Gamma(\nu)},$$
$$R(1-2k)>0,\ R(2\nu)>R(\mu)>0\ \text{and}\ R(1-k\pm m)>0.$$

Example 2. We know

$$J_\lambda(bt)\ \underset{m}{\overset{k+\frac12}{\longrightarrow}}\ \frac{b^\lambda\Gamma_*(\lambda-k+1\pm m)}{(2s)^\lambda\,\Gamma(\lambda-2k+1)\,\Gamma(\lambda+1)}\,{}_4F_3\left[\begin{matrix}\dfrac{\lambda-k+1\pm m}{2},\ \dfrac{\lambda-k+2\pm m}{2}\\ \lambda+1,\ \dfrac{\lambda-2k+1}{2},\ \dfrac{\lambda-2k+2}{2}\end{matrix};\ -\frac{b^2}{s^2}\right],$$
$$R(\lambda-k+1\pm m)>0,\ R(s)>0.$$

Substituting in (4.7) and evaluating one of the integrals ([9], p. 434), we obtain

$$\left(\frac{b}{2}\right)^{\lambda}\int_0^{\infty} t^{2\nu-\mu-\lambda-1}\,{}_3F_4\left[\begin{matrix}\nu,\ \dfrac{2\nu-\mu-2k+1}{2},\ \dfrac{2\nu-\mu-2k+2}{2}\\ \dfrac{2\nu-\mu-k+1\pm m}{2},\ \dfrac{2\nu-\mu-k+2\pm m}{2}\end{matrix};\ -\frac{a^2t^2}{4}\right]\times$$

$$\times\,{}_4F_3\left[\begin{matrix}\dfrac{\lambda-k+1\pm m}{2},\ \dfrac{\lambda-k+2\pm m}{2}\\ \lambda+1,\ \dfrac{\lambda-2k+1}{2},\ \dfrac{\lambda-2k+2}{2}\end{matrix};\ -\frac{b^2}{t^2}\right]dt$$

$$=\frac{\Gamma_*(2\nu-\mu-k+1\pm m)\,\Gamma(\lambda-2k+1)\,\Gamma(\lambda+1)}{\Gamma(2\nu-\mu-2k+1)\,\Gamma_*(\lambda-k+1\pm m)}\times$$

$$\times\left\{\frac{b^{\lambda}a^{\mu+\lambda-2\nu}\,\Gamma(\tfrac{1}{2}\mu+\tfrac{1}{2}\lambda)\,\Gamma(\nu-\tfrac{1}{2}\mu-\tfrac{1}{2}\lambda)}{2^{\lambda+1}\,\Gamma(\nu)\,\Gamma(\lambda+1)}\,{}_1F_2\left[\begin{matrix}\dfrac{\mu+\lambda}{2}\\ \dfrac{\mu+\lambda-2\nu+2}{2},\ \lambda+1\end{matrix};\ \frac{a^2b^2}{4}\right]+\right.$$

$$\left.+\frac{b^{2\nu-\mu}\,\Gamma(\tfrac{1}{2}\lambda+\tfrac{1}{2}\mu-\nu)}{2^{2\nu-\mu+1}\,\Gamma(\nu+\tfrac{1}{2}\lambda-\tfrac{1}{2}\mu+1)}\,{}_1F_2\left[\begin{matrix}\nu\\ \nu+1+\dfrac{\lambda-\mu}{2},\ \nu+1-\dfrac{\lambda+\mu}{2}\end{matrix};\ \frac{a^2b^2}{4}\right]\right\},$$

$R(\lambda-k+1\pm m)>0$, $R(\lambda)>0$, $R(2\nu)>R(\lambda+\mu)>0$, and $R(\lambda-2k+1)>0$; b is a real positive number.

(v) **Result 5.** Using (2.7) in (4.1), we get

$$\left(\frac{a}{2}\right)^{\lambda}\int_0^{\infty} t^{\nu-\mu-1}\,{}_2F_5\left[\begin{matrix}\dfrac{\nu-\mu-2k+1}{2},\ \dfrac{\nu-\mu-2k+2}{2}\\ \nu+1,\ \dfrac{\nu-\mu-k+1\pm m}{2},\ \dfrac{\nu-\mu-k+2\pm m}{2}\end{matrix};\ -\frac{a^2t^2}{4}\right]\varphi_1(t)\,dt$$

$$=\frac{\Gamma_*(\nu-\mu-k+1\pm m)\,\Gamma(\nu+1)}{\Gamma(\nu-\mu-2k-1)}\int_0^{\infty} t^{\mu-1}J_{\nu}(a/t)\,f_1(t)\,dt, \tag{4.10}$$

provided $R(\nu-\mu-k+1\pm m)>0$, $R(s)>0$ and the integrals converge.

Again, putting $a=2\sqrt{x}$ in (4.10), we have

$$x^{\nu}\int_0^{\infty} t^{\nu-\mu-1}\,{}_2F_5\left[\begin{matrix}\dfrac{\nu-\mu-2k+1}{2},\ \dfrac{\nu-\mu-2k+2}{2}\\ \nu+1,\ \dfrac{\nu-\mu-k+1\pm m}{2},\ \dfrac{\nu-\mu-k+2\pm m}{2}\end{matrix};\ -xt^2\right]\varphi_1(t)\,dt$$

$$=\frac{\Gamma_*(\nu-\mu-k+1\pm m)\,\Gamma(\nu+1)}{\Gamma(\nu-\mu-2k+1)}\int_0^{\infty} t^{\mu-\nu-1}f_1(t)\sum_{\nu=0}^{\infty}\frac{(-)^r x^{r+\nu}}{r!\,\Gamma(\nu+r+1)\,t^{2r}}\,dt \tag{4.11}$$

$$\frac{k'+\frac{1}{2}}{m'} \frac{\Gamma_*(\nu-\mu-k+1\pm m)\,\Gamma_*(\nu-k'+1\pm m')}{\Gamma(\nu-\mu-2k+1)\,\Gamma(\nu-2k'+1)\,p^\nu}\times$$

$$\times\int_0^\infty t^{\mu-\nu-1} f_1(t)\,{}_2F_2\left[\begin{matrix}\nu-k'+1\pm m'\\ \nu-2k'+1,\ \nu+1\end{matrix};\ -\frac{1}{pt^2}\right]dt,$$

provided $f_1(t)$ is continuous for $t>0$, $R(\nu-\mu-k+1\pm m)>0$, $R(\nu-k'+1\pm m)>0$, $R(p)>0$ and the integrals converge absolutely.

Taking $k'=-m'$ in (4.11), we get

$$x^\nu\int_0^\infty t^{\nu-\mu-1}\varphi_1(t)\,{}_2F_5\left[\begin{matrix}\frac{\nu-\mu-2k+1}{2},\ \frac{\nu-\mu-2k+2}{2}\\ \nu+1,\ \frac{\nu-\mu-k+1\pm m}{2},\ \frac{\nu-\mu-k+2\pm m}{2}\end{matrix};\ -xt^2\right]dt$$

(4.12)

$$\doteqdot\frac{\Gamma_*(\nu-\mu-k+1\pm m)\,\Gamma(\nu+1)}{\Gamma(\nu-\mu-2k+1)\,p^\nu}\int_0^\infty e^{-\frac{1}{pt^2}}t^{\mu-\nu-1}f_1(t)\,dt,$$

provided $f_1(t)$ is continuous for $t>0$, $R(\nu-\mu-k+1\pm m)>0$, $R(\nu+1)>0$, $R(p)>0$ and the integrals converge absolutely.

Example 1. Let $f_1(t)=J_\lambda(bt)$, then

$$\varphi_1(s)=\frac{b^\lambda\,\Gamma_*(\lambda-k+1\pm m)}{(2s)^\lambda\,\Gamma(\lambda-2k+1)\,\Gamma(\lambda+1)}\,{}_4F_3\left[\begin{matrix}\frac{\lambda-k+1\pm m}{2},\ \frac{\lambda-k+2\pm m}{2}\\ \lambda+1,\ \frac{\lambda-2k+1}{2},\ \frac{\lambda-2k+2}{2}\end{matrix};\ -\frac{b^2}{s^2}\right],$$

$R(\lambda+1-k\pm m)>0$, $R(s)>0$ and $|s|>|b|$.

Substituting in (4.10) and evaluating the integrals on the r.h.s. ([9], p. 437), we get

$$\frac{a^\nu b^\lambda}{2^{\nu+\lambda}}\int_0^\infty t^{\nu-\mu-\lambda-1}\,{}_4F_3\left[\begin{matrix}\frac{\lambda-k+1\pm m}{2},\ \frac{\lambda-k+2\pm m}{2}\\ \lambda+1,\ \frac{\lambda-2k+1}{2},\ \frac{\lambda-2k+2}{2}\end{matrix};\ -\frac{b^2}{t^2}\right]\times$$

$$\times\,{}_2F_5\left[\begin{matrix}\frac{\nu-\mu-2k+1}{2},\ \frac{\nu-\mu-2k+2}{2}\\ \nu+1,\ \frac{\nu-\mu-k+1\pm m}{2},\ \frac{\nu-\mu-k+2\pm m}{2}\end{matrix};\ -\frac{a^2t^2}{4}\right]dt$$

$$=\frac{\Gamma_*(\nu-\mu-k+1\pm m)\,\Gamma(\nu+1)\,\Gamma(\lambda-2k+1)\,\Gamma(\lambda+1)}{\Gamma(\nu-\mu-2k+1)\,\Gamma_*(\lambda-k+1\pm m)}\times$$

$$\times\left\{\frac{b^{\nu-\mu}a^{\nu}\Gamma(\frac{1}{2}\lambda+\frac{1}{2}\mu-\frac{1}{2}\nu)}{2^{2\nu-\mu+1}\Gamma(\nu+1)\Gamma(\frac{1}{2}\lambda+\frac{1}{2}\nu-\frac{1}{2}\mu+1)}{}_0F_3\left[\nu+1,\frac{\nu-\lambda-\mu+2}{2},\frac{\nu+\lambda-\mu+2}{2};\frac{a^2b^2}{16}\right]\right.$$
$$\left.+\frac{b^{\nu}a^{\lambda+\mu}\Gamma(\frac{1}{2}\nu-\frac{1}{2}\lambda-\frac{1}{2}\mu)}{2^{2\lambda+\mu+1}\Gamma(\lambda+1)\Gamma\left(\frac{\lambda+\mu+\nu+2}{2}\right)}{}_0F_3\left[\lambda+1,\frac{\lambda-\nu+\mu+2}{2},\frac{\lambda+\mu+\nu+2}{2};\frac{a^2b^2}{16}\right]\right\},$$

provided a and b are positive, $R(\nu-\mu-k+1\pm m)>0$, $-R(\lambda-\frac{1}{2})<R(\mu)<R(\nu+3/2)$, $R(\lambda-k+1\pm m)>0$ and $R(\lambda-2k+1)>0$.

(vi) **Result 6.** Using (2.9) in (4.1), we get

(4.13)
$$\left(\frac{a}{2}\right)^{\nu}\int_0^{\infty}t^{\nu/2-\mu-1}\varphi_1(t)\,{}_1F_3\left[\begin{matrix}\nu/2-\mu-2k+1\\ \nu+1,\ \nu/2-\mu-k+1\pm m\end{matrix};-\frac{a^2t}{4}\right]dt$$
$$=\frac{\Gamma_*(\nu/2-\mu-k+1\pm m)\Gamma(\nu+1)}{\Gamma(\nu/2-\mu-2k+1)}\int_0^{\infty}t^{\mu-1}J_{\nu}\left(\frac{a}{\sqrt{t}}\right)f_1(t)\,dt,$$

provided $R(\nu/2-\mu-k+1\pm m)>0$, and the integrals converge.

Putting, $a=2\sqrt{x}$ and interpreting, we get

(4.14)
$$x^{\nu}\int_0^{\infty}t^{\nu/2-\mu-1}\varphi_1(t)\,{}_1F_3\left[\begin{matrix}\nu/2-\mu-2k+1\\ \nu+1,\ \nu/2-\mu-k+1\pm m\end{matrix};-xt\right]dt$$
$$\frac{k'+\frac{1}{2}}{m'}\ \frac{\Gamma_*(\nu/2-\mu-k+1\pm m)\Gamma_*(\nu-k'+1\pm m')p^{-\nu}}{\Gamma(\nu/2-\mu-2k+1)\Gamma(\nu-2k'+1)}\times$$
$$\times\int_0^{\infty}t^{\mu-\nu/2-1}f_1(t)\,{}_2F_2\left[\begin{matrix}\nu-k'+1\pm m'\\ \nu+1,\ \nu-2k'+1\end{matrix};-\frac{1}{pt}\right]dt,$$

provided $f_1(t)$ is continuous for $t>0$,

$$R(\nu-k'+1\pm m)>0,\ R(p)>0,\ R(\nu/2-\mu-k+1\pm m)>0,$$

and the integrals converge absolutely.

Taking $k'=-m'$ in (4.14), we get

(4.15)
$$x^{\nu}\int_0^{\infty}t^{\nu/2-\mu-1}\varphi_1(t)\,{}_1F_3\left[\begin{matrix}\nu/2-\mu-2k+1\\ \nu+1,\ \nu/2-\mu-k+1\pm m\end{matrix};-xt\right]dt$$
$$\doteq\frac{\Gamma_*(\nu/2-\mu-k+1\pm m)\Gamma(\nu+1)}{\Gamma(\nu/2-\mu-2k+1)p^{\nu}}\int_0^{\infty}e^{-\frac{1}{pt}}t^{\mu-\nu/2-1}f_1(t)\,dt$$

provided $f_1(t)$ is continuous for $t>0$, $R(\nu+1)>0$, $R(p)>0$, $R(\nu/2-\mu-k+1\pm m)>0$ and the integrals converge absolutely.

Example 1. Let $f_1(t)=e^{-t}$, then

$$\varphi_1(s)=\frac{\Gamma_*(1-k\pm m)}{\Gamma(1-2k)}\,{}_2F_1\left[\begin{matrix}1-k\pm m\\ 1-2k\end{matrix};\ -\frac{1}{s}\right],$$

$R(1-k\pm m)>0$ and $R(s)>0$.

Substituting in (4.15), we get

$$x^{\nu}\int_0^{\infty} t^{\nu/2-\mu-1}\,{}_1F_3\left[\begin{matrix}\nu/2-\mu-2k+1\\ \nu+1,\ \nu/2-\mu-k+1\pm m\end{matrix};\ -xt\right]{}_2F_1\left[\begin{matrix}1-k\pm m\\ 1-2k\end{matrix};\ -\frac{1}{t}\right]dt$$

$$\doteqdot\frac{\Gamma_*(\nu/2-\mu-k+1\pm m)\,\Gamma(\nu+1)\,\Gamma(1-2k)}{\Gamma(\nu/2-\mu-2k+1)\,\Gamma_*(1-k\pm m)\,p^{\nu}}\int_0^{\infty} e^{-t-\frac{1}{pt}}\frac{dt}{t^{\nu/2-\mu+1}}$$

$$=\frac{2\,\Gamma_*(\nu/2-\mu-k+1\pm m)\,\Gamma(\nu+1)\,\Gamma(1-2k)}{\Gamma(\nu/2-\mu-2k+1)\,\Gamma_*(1-k\pm m)\,p^{\frac{1}{4}(3\nu+2\mu)}}K_{\nu/2-\mu}\left(\frac{2}{\sqrt{p}}\right),$$

$R(1-2k)>0$, $R(1-k\pm m)>0$, $R(\nu/2-\mu-k+1\pm m)>0$, $R(\nu+1)>0$ and $R(p)>0$.

(vii) **Result 7.** If in (4.10) we replace a by $\dfrac{2}{p}$ and interpret, then

$$p^{-\nu}\int_0^{\infty} t^{\nu-\mu-1}\varphi_1(t)\,{}_2F_5\left[\begin{matrix}\dfrac{\nu-\mu-2k+1}{2},\ \dfrac{\nu-\mu-2k+2}{2}\\ \nu+1,\ \dfrac{\nu-\mu-k+1\pm m}{2},\ \dfrac{\nu-\mu-k+2\pm m}{2}\end{matrix};\ -\frac{t^2}{4p^2}\right]dt$$

$$(4.16)\qquad \underset{m'}{\overset{k'+\frac{1}{2}}{\leftarrow}}\ \frac{\Gamma_*(\nu-\mu-k+1\pm m)\,\Gamma(\nu-2k'+1)\,x^{\nu}}{\Gamma(\nu-\mu-2k+1)\,\Gamma_*(\nu-k'+1\pm m')}\times$$

$$\times\int_0^{\infty} t^{\mu-\nu-1}f_1(t)\,{}_2F_5\left[\begin{matrix}\dfrac{\nu-2k'+1}{2},\ \dfrac{\nu-2k'+2}{2}\\ \nu+1,\ \dfrac{\nu-k'+1\pm m'}{2},\ \dfrac{\nu-k'+2\pm m'}{2}\end{matrix};\ -\frac{x^2}{t^2}\right]dt,$$

provided $R(\nu-\mu-k+1\pm m)>0$, $R(\nu-k'+1\pm m')>0$, $R(p)>0$ and the integrals converge.

Taking $k'=-m'$,

$$p^{-\nu}\int_0^\infty t^{\nu-\mu-1}{}_2F_5\left[\begin{matrix}\dfrac{\nu-\mu-2k+1}{2},\dfrac{\nu-\mu-2k+2}{2}\\ \nu+1,\dfrac{\nu-\mu-k+1\pm m}{2},\dfrac{\nu-\mu-k+2\pm m}{2}\end{matrix};-\frac{t^2}{4p^2}\right]\varphi_1(t)\,dt$$

(4.17)
$$\doteqdot\frac{\Gamma_*(\nu-\mu-k+1\pm m)\,x^\nu}{\Gamma(\nu-\mu-2k+1)\,\Gamma(\nu+1)}\int_0^\infty t^{\mu-\nu-1}{}_0F_3\left[\nu+1,\frac{\nu+1}{2},\nu/2+1;\frac{i^2x^2}{t^2}\right]f_1(t)\,dt$$

$$=\frac{2^{-\nu}\Gamma_*(\nu-\mu-k+1\pm m)\,\Gamma(\nu+1)}{\Gamma(\nu-\mu-2k+1)\,i^\nu}\int_0^\infty t^{\mu-1}\chi_\nu^b\left\{2\sqrt{\frac{2ix}{t}}\right\}f_1(t)\,dt,$$

provided $R(\nu+1)>0$, $R(\nu-\mu-k+1\pm m)>0$, $R(p)>0$ and the integrals converge.

Again, putting $a=\dfrac{2}{\sqrt{p}}$ in (4.10) and interpreting, we get

$$p^{-\nu/2}\int_0^\infty t^{\nu-\mu-1}{}_2F_5\left[\begin{matrix}\dfrac{\nu-\mu-2k+1}{2},\dfrac{\nu-\mu-2k+2}{2}\\ \nu+1,\dfrac{\nu-\mu-k+1\pm m}{2},\dfrac{\nu-\mu-k+2\pm m}{2}\end{matrix};-\frac{t^2}{4p}\right]\varphi_1(t)\,dt$$

(4.18)
$$\underset{m'}{\overset{k'+\frac12}{\longleftarrow}}\ \frac{\Gamma_*(\nu-\mu-k+1\pm m)\,\Gamma(\nu/2-2k'+1)\,x^{\nu/2}}{\Gamma(\nu-\mu-2k+1)\,\Gamma(\nu/2-k'+1\pm m')}\times$$

$$\times\int_0^\infty t^{\mu-\nu-1}f_1(t)\,{}_1F_3\left[\begin{matrix}\nu/2-2k'+1\\ \nu+1,\nu/2-k'+1\pm m'\end{matrix};-\frac{x}{t^2}\right]dt,$$

provided $R(\nu-\mu-k+1\pm m)>0$, $R(\nu/2-k'+1\pm m'>0$, $R(p)>0$ and the integrals converge.

Taking $k'=-m'$ in (4.18), we get

$$p^{-\nu/2}\int_0^\infty t^{\nu-\mu-1}\varphi_1(t)\,{}_2F_5\left[\begin{matrix}\dfrac{\nu-\mu-2k+1}{2},\dfrac{\nu-\mu-2k+2}{2}\\ \nu+1,\dfrac{\nu-\mu-k+1\pm m}{2},\dfrac{\nu-\mu-k+2\pm m}{2}\end{matrix};-\frac{t^2}{4p}\right]dt$$

(4.19)
$$=\frac{\Gamma_*(\nu-\mu-k+1\pm m)\,x^{\nu/2}}{\Gamma(\nu-\mu-2k+1)\,\Gamma(\nu/2+1)}\int_0^\infty t^{\mu-\nu-1}{}_0F_2\left[\nu+1,\nu/2+1,-\frac{x}{t^2}\right]f_1(t)\,dt$$

$$=\frac{\Gamma_*(\nu-\mu-k+1\pm m)\,\Gamma(\nu+1)}{\Gamma(\nu-\mu-2k+1)}\int_0^\infty t^{\mu-1}J_{\nu,\nu/2}\{3(x/t^2)^{1/3}\}f_1(t)\,dt,$$

provided $R(\nu+2)>0$, $R(\nu-\mu-k+1\pm m)>0$ and the integrals converge.

(viii) **Results 8.** Using (2.11) in (4.1), we get

$$
\begin{aligned}
&\frac{\Gamma(\frac{1}{2})\,\Gamma(-\frac{1}{2}n)\,\Gamma(1-\lambda-2k)}{\Gamma_*(1-\lambda-k\pm m)}\times\\
&\qquad\times\int_0^\infty t^{-\lambda-1}\,{}_3F_5\left[\begin{matrix}-\frac{1}{2}n,\ \dfrac{1-\lambda-2k}{2},\ \dfrac{2-\lambda-2k}{2}\\ \frac{1}{2},\ \dfrac{1-\lambda-k\pm m}{2},\ \dfrac{2-\lambda-k\pm m}{2}\end{matrix};\ \frac{a^2t^2}{8}\right]\varphi_1(t)\,dt+\\
&\qquad+\frac{a\,\Gamma(-\frac{1}{2})\,\Gamma(\frac{1}{2}-\frac{1}{2}n)\,\Gamma(2-\lambda-2k)}{2^{\frac{1}{2}}\,\Gamma_*(2-\lambda-k\pm m)}\times\\
&\qquad\times\int_0^\infty t^{-\lambda}\,{}_3F_5\left[\begin{matrix}\frac{1}{2}-\frac{1}{2}n,\ \dfrac{2-\lambda-2k}{2},\ \dfrac{3-\lambda-2k}{2}\\ \frac{3}{2},\ \dfrac{2-\lambda-k\pm m}{2},\ \dfrac{3-\lambda-k\pm m}{2}\end{matrix};\ \frac{a^2t^2}{8}\right]\varphi_1(t)\,dt\\
&\qquad=\frac{\Gamma(-\frac{1}{2}n)\,\Gamma(\frac{1}{2}-\frac{1}{2}n)}{2^{\frac{1}{2}n}}\int_0^\infty t^{\lambda-1}\,e^{\frac{1}{4}\frac{a^2}{t^2}}\,D_n\left(\frac{a}{t}\right)f_1(t)\,dt,
\end{aligned}
\tag{4.20}
$$

provided $R(1-k-\lambda\pm m)>0$ and the integrals converge.

Putting $a^2=\dfrac{1}{p}$ in (4.20) and interpreting, we have

$$
\begin{aligned}
&\frac{p^{\frac{1}{2}n+1}\,\Gamma(\frac{1}{2})\,\Gamma(-\frac{1}{2}n)\,\Gamma(1-\lambda-2k)}{\Gamma_*(1-\lambda-k\pm m)}\times\\
&\qquad\times\int_0^\infty t^{-\lambda-1}\,{}_3F_5\left[\begin{matrix}-\frac{1}{2}n,\ \dfrac{1-\lambda-2k}{2},\ \dfrac{2-\lambda-2k}{2}\\ \frac{1}{2},\ \dfrac{1-\lambda-k\pm m}{2},\ \dfrac{2-\lambda-k\pm m}{2}\end{matrix};\ \frac{t^2}{8p}\right]\varphi_1(t)\,dt+\\
&\qquad+\frac{p^{\frac{1}{2}n+\frac{1}{2}}\,\Gamma(-\frac{1}{2})\,\Gamma(\frac{1}{2}-\frac{1}{2}n)\,\Gamma(2-\lambda-2k)}{2^{\frac{1}{2}}\,\Gamma_*(2-\lambda-k\pm m)}\times\\
&\qquad\times\int_0^\infty t^{-\lambda}\,{}_3F_5\left[\begin{matrix}\frac{1}{2}-\frac{1}{2}n,\ \dfrac{2-\lambda-2k}{2},\ \dfrac{3-\lambda-2k}{2}\\ \frac{3}{2},\ \dfrac{2-\lambda-k\pm m}{2},\ \dfrac{3-\lambda-k\pm m}{2}\end{matrix};\ \frac{t^2}{8p}\right]\varphi_1(t)\,dt
\end{aligned}
\tag{4.21}
$$

$$\underset{m'}{\overleftarrow{k'+\tfrac{1}{2}}}\ \Gamma(\tfrac{1}{2})\left\{\frac{\Gamma(-\tfrac{1}{2}n)\,\Gamma(-\tfrac{1}{2}n-2k')}{\Gamma_*(-\tfrac{1}{2}n-k'\pm m')}x^{-\frac{1}{2}n-1}\times\right.$$

$$\times\int_0^\infty t^{\lambda-1}\,{}_2F_3\left[\begin{matrix}-\tfrac{1}{2}n, & -\tfrac{1}{2}n-2k' \\ \tfrac{1}{2}, & -\tfrac{1}{2}n-k'\pm m'\end{matrix};\frac{x}{2t^2}\right]f_1(t)\,dt$$

$$-\frac{\Gamma(\tfrac{1}{2}-\tfrac{1}{2}n)\,\Gamma(\tfrac{1}{2}-\tfrac{1}{2}n-2k')\sqrt{2}\,x^{-\frac{1}{2}n-\frac{1}{2}}}{\Gamma_*(\tfrac{1}{2}-\tfrac{1}{2}n-k'\pm m')}\times$$

$$\left.\times\int_0^\infty t^{\lambda-2}\,{}_2F_3\left[\begin{matrix}\tfrac{1}{2}-\tfrac{1}{2}n, & \tfrac{1}{2}-\tfrac{1}{2}n-2k' \\ \tfrac{3}{2}, & \tfrac{1}{2}-\tfrac{1}{2}n-k'\pm m'\end{matrix};\frac{x}{2t^2}\right]f_1(t)\,dt\right\},$$

provided $R(1-k-\lambda\pm m)>0$, $R(p)>0$, $R(\tfrac{1}{2}n-k'\pm m')>0$ and the integrals converge.

Taking $k'=-m'$, we get

$$\frac{p^{\frac{1}{2}n+1}\,\Gamma(-\tfrac{1}{2}n)\,\Gamma(1-\lambda-2k)}{\Gamma_*(1-\lambda-k\pm m)}\times$$

$$\times\int_0^\infty t^{-\lambda-1}\,{}_3F_5\left[\begin{matrix}-\tfrac{1}{2}n, & \dfrac{1-\lambda-2k}{2}, & \dfrac{2-\lambda-2k}{2} \\ \tfrac{1}{2}, & \dfrac{1-\lambda-k\pm m}{2}, & \dfrac{2-\lambda-k\pm m}{2}\end{matrix};\frac{t^2}{8p}\right]\varphi_1(t)\,dt-$$

$$-\frac{p^{\frac{1}{2}n+3/2}\sqrt{2}\,\Gamma(\tfrac{1}{2}-\tfrac{1}{2}n)\,\Gamma(2-\lambda-2k)}{\Gamma_*(2-\lambda-k\pm m)}\times$$

$$\times\int_0^\infty t^{-\lambda}\,{}_3F_5\left[\begin{matrix}\tfrac{1}{2}-\tfrac{1}{2}n, & \dfrac{2-\lambda-2k}{2}, & \dfrac{3-\lambda-2k}{2} \\ \tfrac{3}{2}, & \dfrac{2-\lambda-k\pm m}{2}, & \dfrac{3-\lambda-k\pm m}{2}\end{matrix};\frac{t^2}{8p}\right]\varphi_1(t)\,dt \tag{4.22}$$

$$\doteqdot x^{-\frac{1}{2}n-1}\int_0^\infty t^{\lambda-1}\,{}_0F_1\left(\tfrac{1}{2};\frac{x}{2t^2}\right)f_1(t)\,dt-\sqrt{2}\,x^{-\frac{1}{2}n-\frac{1}{2}}\int_0^\infty t^{\lambda-2}\,{}_0F_1\left[\tfrac{3}{2};\frac{x}{2t^2}\right]f_1(t)\,dt$$

$$=x^{-\frac{1}{2}n-1}\int_0^\infty t^{\lambda-1}f_1(t)\cosh\left(\frac{\sqrt{2x}}{t}\right)dt-x^{-\frac{1}{2}n-1}\int_0^\infty t^{\lambda-1}\sinh\left(\frac{\sqrt{2x}}{t}\right)f_1(t)\,dt$$

$$=x^{-\frac{1}{2}n-1}\int_0^\infty t^{\lambda-1}e^{-\frac{\sqrt{2x}}{t}}f_1(t)\,dt,$$

provided $R(1-k-\lambda\pm m)>0$, $R(n)<0$ and the integrals converge.

(ix) **Result 9.** Using (2.24) in (4.1), we get

$$\int_0^\infty t^{-1} {}_3F_2\left[\begin{matrix}1-k\pm m,\ \frac{1}{2}(n+1)\\ \frac{1}{2},\ 1-2k\end{matrix};\ -\frac{1}{2t}\right] f_1(t)\,dt=$$

$$(4.23)\qquad = \frac{\Gamma(\frac{1}{2})\,\Gamma(1-2k)}{(-2)^{n/2}\,\Gamma_*(1-k\pm m)\,\Gamma\left(\frac{n+1}{2}\right)}\int_0^\infty t^{-1} e^{-\frac{t}{4}} D_n\left(\sqrt{t}\right)\varphi_1(t)\,dt,$$

for n positive even integer, $R(1-k\pm m)>0$ and the integrals converge.

Example 1. Let

$$f_1(t)=\frac{t^{\nu-\frac{1}{2}}\,\Gamma(\nu+\frac{1}{2}-2k)}{\Gamma_*(\nu+\frac{1}{2}-k\pm m)}\,{}_2F_2\left[\begin{matrix}\nu+\frac{1}{2}-2k,\ \nu\\ \nu+\frac{1}{2}-k\pm m\end{matrix};\ -bt\right] \xrightarrow[m]{k+\frac{1}{2}} \frac{s^{\frac{1}{2}}}{(b+s)^{\nu}}=\varphi_1(s),$$

$R(\nu-k+\frac{1}{2}\pm m)>0,\ b+s\neq 0$ and $R(s)>0$.

Substituting in (4.23), we get

$$\int_0^\infty t^{\nu-3/2}\,{}_3F_2\left[\begin{matrix}1-k\pm m,\ \frac{n+1}{2}\\ \frac{1}{2},\ 1-2k\end{matrix};\ -\frac{1}{2t}\right]{}_2F_2\left[\begin{matrix}\nu+\frac{1}{2}-2k,\ \nu\\ \nu+\frac{1}{2}-k\pm m\end{matrix};\ -bt\right]dt$$

$$=\frac{\Gamma(\frac{1}{2})\,\Gamma(1-2k)\,\Gamma_*(\nu+\frac{1}{2}-k\pm m)}{(-2)^{n/2}\,\Gamma_*(1-k\pm m)\,\Gamma\left(\frac{n+1}{2}\right)\Gamma(\nu+\frac{1}{2}-2k)}\int_0^\infty \frac{e^{-\frac{t}{4}}\,D_n\left(\sqrt{t}\right)t^{-\frac{1}{2}}}{(b+t)^{\nu}}\,dt.$$

Evaluating the right hand side ([5], p. 13, with $a^2=b$ and n even positive integer), we get

$$\int_0^\infty t^{\nu-3/2}\,{}_3F_2\left[\begin{matrix}1-k\pm m,\ \frac{n+1}{2}\\ \frac{1}{2},\ 1-2k\end{matrix};\ -\frac{1}{2t}\right]{}_2F_2\left[\begin{matrix}\nu+\frac{1}{2}-2k,\ \nu\\ \nu+\frac{1}{2}-k\pm m\end{matrix};\ -bt\right]dt$$

$$=\frac{\Gamma(\frac{1}{2})\,\Gamma(1-2k)\,\Gamma_*(\nu+\frac{1}{2}-k\pm m)\,(-)^{\frac{1}{2}n}\,(\pi/2)^{1/2}}{(-2)^{n/2}\,\Gamma_*(1-k\pm m)\,\Gamma\left(\frac{n+1}{2}\right)\Gamma(\nu+\frac{1}{2}-2k)\,2^{\nu-2}\,\Gamma(\nu)}\times$$

$$\times\sum_{r=1}^{r=\nu}\frac{e^{\frac{1}{4}b}\,b^{-\left(\frac{\nu+r-1}{2}\right)}\,\Gamma(\nu+r-1)\,\Gamma(n+\nu-r+1)}{2^{r-1}\,\Gamma(r)\,\Gamma(\nu-r+1)}\,D_{-(n+\nu-r+1)}^{(b^{\frac{1}{2}})},$$

$R(b)>0,\ R(1-2k)>0,\ R(\nu-k+\frac{1}{2}\pm m)>0$ and $R(1-k\pm m)>0$.

In particular when $\nu = 1$

$$\int_0^\infty t^{-1/2} {}_3F_2\left[\begin{matrix} 1-k \pm m, \frac{n+1}{2} \\ \frac{1}{2};\, 1-2k \end{matrix}; -\frac{1}{2t}\right] {}_2F_2\left[\begin{matrix} \frac{3}{2}-2k,\, 1 \\ \frac{3}{2}-k \pm m \end{matrix}; -bt\right] dt$$

$$= \frac{\Gamma(\frac{1}{2})\,\Gamma(1-2k)\,\Gamma_*(\frac{3}{2}-k \pm m)\,\Gamma(n+1)\,\pi^{\frac{1}{2}}\,e^{b/4}}{2^{\frac{1}{2}(n+1)}\,\Gamma_*(1-k \pm m)\,\Gamma\left(\frac{n+1}{2}\right)\Gamma(\frac{3}{2}-2k)\,b^{\frac{1}{2}}} D^{(b^{\frac{1}{2}})}_{-(n+1)}.$$

(x) **Result 10.** Using (2.25) in (4.1), we get

$$\int_0^\infty t^{-3/2} {}_3F_2\left[\begin{matrix} \frac{3}{2}-k \pm m,\, \frac{1}{2}(n+2) \\ \frac{3}{2},\, \frac{3}{2}-2k \end{matrix}; -\frac{1}{2t}\right] f_1(t)\,dt$$

(4.24)
$$= \frac{\Gamma(\frac{3}{2})\,\Gamma(\frac{3}{2}-2k)}{(-2)^{\frac{1}{2}(n-1)}\,\Gamma_*(\frac{3}{2}-k \pm m)\,\Gamma\left(\frac{n}{2}+1\right)} \int_0^\infty t^{-1} e^{-\frac{1}{4}t} D_n(t^{\frac{1}{2}})\,\varphi_1(t)\,dt,$$

provided n is a positive odd integer, $R(\frac{3}{2}-k \pm m) > 0$ and the integrals converge.

Example 1. Let $\varphi_1(s) = \dfrac{s}{(s+b)^\nu}$, then

$$f_1(t) = \frac{t^{\nu-1}\,\Gamma(\nu-2k)}{\Gamma_*(\nu-k \pm m)} {}_2F_2\left[\begin{matrix} \nu,\, \nu-2k \\ \nu-k \pm m \end{matrix}; -bt\right],$$

$R(\nu-k \pm m) > 0$ and $R(s) > 0$.

Substituting in (4.24), we get

$$\int_0^\infty t^{\nu-5/2} {}_3F_2\left[\begin{matrix} \frac{3}{2}-k \pm m,\, \frac{1}{2}(n+2) \\ \frac{3}{2},\, \frac{3}{2}-2k \end{matrix}; -\frac{1}{2t}\right] {}_2F_2\left[\begin{matrix} \nu,\, \nu-2k \\ \nu-k \pm m \end{matrix}; -bt\right] dt$$

$$= \frac{\Gamma(\frac{3}{2})\,\Gamma(\frac{3}{2}-2k)\,\Gamma_*(\nu-k \pm m)}{(-2)^{\frac{1}{2}(n-1)}\,\Gamma_*(\frac{3}{2}-k \pm m)\,\Gamma(n/2+1)\,\Gamma(\nu-2k)} \int_0^\infty \frac{e^{-t/4} D_n(t^{\frac{1}{2}})}{(b+t)^\nu} dt.$$

Evaluating the right hand side ([5], p. 13), we get

$$\int_0^\infty t^{\nu-5/2} {}_3F_2\left[\begin{matrix} \frac{3}{2}-k \pm m,\, \frac{1}{2}(n+2) \\ \frac{3}{2},\, \frac{3}{2}-2k \end{matrix}; -\frac{1}{2t}\right] {}_2F_2\left[\begin{matrix} \nu,\, \nu-2k \\ \nu-k \pm m \end{matrix}; -bt\right] dt$$

$$= \frac{\Gamma(\frac{3}{2})\,\Gamma(\frac{3}{2}-2k)\,\Gamma_*(\nu-k \pm m)\,(\pi/2)^{\frac{1}{2}}\,e^{\frac{b}{4}}}{\Gamma_*(\frac{3}{2}-k \pm m)\,\Gamma(n/2+1)\,\Gamma(\nu-2k)\,(\nu-1)!\,(\nu-2)!\,2^{n+\nu-\frac{5}{2}}}$$

$$\sum_{r=2}^{r=\nu} \frac{b^{-\frac{(\nu+r-3)}{2}}\,(\nu+r-4)!\,\Gamma(n+\nu-r+2)}{2^{r-2}\,(r-2)!\,(\nu-r)!} D^{(b^{\frac{1}{2}})}_{-(n+\nu-r+2)}.$$

In particular when $\nu = 1$

$$\int_0^\infty t^{-3/2}\, {}_3F_2\left[\begin{matrix}\frac{3}{2} - k \pm m,\ \frac{1}{2}(n+2)\\ \frac{3}{2},\ \frac{3}{2} - 2k\end{matrix};\ -\frac{1}{2t}\right] {}_2F_2\left[\begin{matrix}1,\ 1-2k\\ 1-k\pm m\end{matrix};\ -bt\right] dt$$

$$= \frac{\Gamma(\frac{1}{2})\Gamma(\frac{3}{2}-2k)\,\Gamma_*(1-k\pm m)\,\Gamma(n+1)\, e^{\frac{1}{2}b}}{2^{n/2}\,\Gamma_*(\frac{3}{2}-k\pm m)\,\Gamma(n/2+1)\,\Gamma(1-2k)} D_{-(n+1)}(b^{\frac{1}{2}});\ b > 0.$$

(xi) **Result 11.** Using (2.14) in (4.1), gives

$$a^{\nu+1}\int_0^\infty t^{\nu-\lambda}\, {}_3F_6\left[\begin{matrix}1,\ \frac{1}{2}(\nu-\lambda-2k+2),\ \frac{1}{2}(\nu-\lambda-2k+3)\\ \frac{3}{2},\ \nu+\frac{3}{2},\ \frac{1}{2}(\nu-\lambda-k+2\pm m),\ \frac{1}{2}(\nu-\lambda-k+3\pm m)\end{matrix};\ -\frac{a^2t^2}{4}\right]\varphi_1(t)\,dt$$

$$(4.25)\qquad = \frac{\Gamma(\frac{3}{2})\,\Gamma(\nu+\frac{3}{2})\,\Gamma_*(\nu-\lambda-k+2\pm m)}{\Gamma(\nu-\lambda-2k+2)}\int_0^\infty t^{\lambda-1} H_\nu\left(\frac{2a}{t}\right) f_1(t)\,dt,$$

provided $R(\nu-\lambda+2-k\pm m) > 0$ and the integrals converge.

Putting $a^2 = x$, expanding $H_\nu(z)$, and interpreting, we get

$$x^{\frac{2\nu+1}{2}}\int_0^\infty t^{\nu-\lambda}\, {}_3F_6\left[\begin{matrix}1,\ \dfrac{\nu-\lambda-2k+2}{2},\ \dfrac{\nu-\lambda-2k+3}{2}\\ \frac{3}{2},\ \nu+\frac{3}{2},\ \dfrac{\nu-\lambda-k+2\pm m}{2},\ \dfrac{\nu-\lambda-k+3\pm m}{2}\end{matrix};\ -\frac{t^2x}{4}\right]\varphi_1(t)\,dt$$

$$(4.26)\qquad \xrightarrow[m']{k'+\frac{1}{2}} \frac{\Gamma_*(\nu-\lambda-k+2\pm m)\,\Gamma_*(\nu+\frac{3}{2}-k'\pm m')}{\Gamma(\nu-\lambda-2k+2)\,\Gamma(\nu+\frac{3}{2}-2k')\,p^{\nu+\frac{1}{2}}}\times$$

$$\times\int_0^\infty t^{\lambda-\nu-2}\, {}_3F_3\left[\begin{matrix}(\nu+\frac{3}{2}-k'\pm m',\ 1\\ \nu+\frac{3}{2}-2k',\ \frac{3}{2},\ \nu+\frac{3}{2}\end{matrix};\ -\frac{1}{pt^2}\right] f_1(t)\,dt,$$

provided $R(\nu-\lambda-k+2\pm m) > 0$, $R(\nu+\frac{3}{2}-k'\pm m') > 0$, $R(p) > 0$, the integrals converge and $f_1(t)$ is continuous function of t for $t > 0$.

Taking $k' = -m'$, (4.26) reduces to

$$x^{\nu+\frac{1}{2}}\int_0^\infty t^{\nu-\lambda}\, {}_3F_6\left[\begin{matrix}1,\ \dfrac{\nu-\lambda-2k+2}{2},\ \dfrac{\nu-\lambda-2k+3}{2}\\ \frac{3}{2},\ \nu+\frac{3}{2},\ \dfrac{\nu-\lambda-k+2\pm m}{2},\ \dfrac{\nu-\lambda-k+3\pm m}{2}\end{matrix};\ -\frac{xt^2}{4}\right]\varphi_1(t)\,dt$$

$$\div \frac{\Gamma(\nu+\frac{3}{2})\,\Gamma_*(\nu-\lambda-k+2\pm m)}{\Gamma(\nu-\lambda-2k+2)\,p^{\nu+\frac{1}{2}}}\int_0^\infty t^{\lambda-\nu-2}\, {}_1F_1\left[1,\ \tfrac{3}{2};\ -\frac{1}{pt^2}\right] f_1(t)\,dt,$$

provided $R(\nu+\frac{3}{2}) > 0$, $R(\nu-\lambda+2-k\pm m) > 0$, $R(p) > 0$ and the integrals converge.

Again, putting $a^2 = 1/p$ and interpreting, we get

$$\int_0^\infty t^{\nu-\lambda}\, {}_3F_6\left[\begin{matrix} 1, \dfrac{\nu+2-\lambda-2k}{2}, \dfrac{\nu-\lambda-2k+3}{2} \\ \frac{3}{2}, \nu+\frac{3}{2}, \dfrac{\nu-\lambda-k+2\pm m}{2}, \dfrac{\nu-\lambda-k+3\pm m}{2} \end{matrix}; -\frac{t^2}{4p}\right] \varphi_1(t)\, dt$$

$$(4.27)\qquad \overset{k'+\frac{1}{2}}{\underset{m'}{\rightarrow}} \frac{\Gamma_*(\nu-\lambda-k+2\pm m)\,\Gamma(1-2k')}{\Gamma(\nu-\lambda-2k+2)\,\Gamma^*(1-k'\pm m')} \int_0^\infty t^{\nu-\lambda-2} \times$$

$$\times\, {}_2F_4\left[\begin{matrix} 1-2k', 1 \\ \frac{3}{2}, \nu+\frac{3}{2}, 1-k'\pm m' \end{matrix}; -\frac{x}{t^2}\right] f_1(t)\, dt,$$

provided $R(\nu-\lambda-k+2\pm m)>0$, $R(1-k'\pm m')>0$, $R(p)>0$ and the integrals converge.

Taking $k' = -m'$, we get

$$\int_0^\infty t^{\nu-\lambda}\, {}_3F_6\left[\begin{matrix} 1, \dfrac{\nu-\lambda-2k+2}{2}, \dfrac{\nu-\lambda-2k+3}{2} \\ \frac{3}{2}, \nu+\frac{3}{2}, \dfrac{\nu-\lambda-k+2\pm m}{2}, \dfrac{\nu-\lambda-k+3\pm m}{2} \end{matrix}; -\frac{t^2}{4p}\right] \varphi_1(t)\, dt$$

$$\doteqdot \frac{\Gamma_*(\nu-\lambda-k+2\pm m)}{\Gamma(\nu-\lambda-2k+2)} \int_0^\infty t^{\lambda-\nu-2}\, {}_0F_2\left[\tfrac{3}{2}, \nu+\tfrac{3}{2}; -\frac{x}{t^2}\right] f_1(t)\, dt$$

$$= \frac{\Gamma(\frac{3}{2})\,\Gamma(\nu+\frac{3}{2})\,\Gamma_*(\nu-\lambda-k+2\pm m)}{\Gamma(\nu-\lambda-2k+2)\, x^{\frac{\nu+1}{3}}} \int_0^\infty t^{\lambda-\frac{\nu}{3}-\frac{4}{3}} J_{\frac{1}{2},\, \nu+\frac{1}{2}}\left\{3\left(\frac{x}{t^2}\right)^{\frac{1}{3}}\right\} f_1(t)\, dt,$$

provided $R(\nu-\lambda-k+2\pm m)>0$, $R(p)>0$ and the integrals converge.

(xii) **Result 12.** We have ([2′], p. 132)

$$t^\lambda H_\nu(2at) \overset{k+\frac{1}{2}}{\underset{m}{\rightarrow}} \frac{a^{\nu+1}\,\Gamma_*(\nu+\lambda+2-k\pm m)}{s^{\nu+\lambda+1}\,\Gamma(\frac{3}{2})\,\Gamma(\nu+\frac{3}{2})\,\Gamma(\nu+\lambda+2-2k)} \times$$

$$\times\, {}_5F_4\left[\begin{matrix} \frac{1}{2}(\nu+2+\lambda-k\pm m), \frac{1}{2}(\nu+3+\lambda-k\pm m), 1 \\ \frac{3}{2}, \nu+\frac{3}{2}, \frac{1}{2}(\nu+\lambda+2-2k), \frac{1}{2}(\nu+\lambda+3-2k) \end{matrix}; -\frac{4a^2}{s^2}\right],$$

$R(\nu+\lambda+2-k\pm m)>0$ and $R(s)>0$.

Using this in (4.1), we get

$$(4.28)\qquad a^{\nu+1}\int_0^\infty t^{-\nu-\lambda-2}\, {}_5F_4\left[\begin{matrix} \dfrac{\nu+2+\lambda-k\pm m}{2}, \dfrac{\nu+3+\lambda-k\pm m}{2}, 1 \\ \frac{3}{2}, \nu+\frac{3}{2}, \dfrac{\nu+\lambda+2-2k}{2}, \dfrac{\nu+\lambda+3-2k}{2} \end{matrix}; -\frac{4a^2}{t^2}\right] f_1(t)\, dt$$

$$= \frac{\Gamma(\frac{3}{2})\,\Gamma(\nu+\frac{3}{2})\,\Gamma(\nu+\lambda+2-2k)}{\Gamma_*(\nu+\lambda+2-k\pm m)} \int_0^\infty t^{\lambda-1} H_\nu(2at)\, \varphi_1(t)\, dt,$$

provided $R(\nu+\lambda+2-k\pm m)>0$ and the integrals converge.

Putting $a=1/p$ and interpreting, we get

$$\int_0^\infty t^{-\nu-\lambda-2}\,{}_5F_4\left[\begin{matrix}1,\ \dfrac{\nu+2+\lambda-k\pm m}{2},\ \dfrac{\nu+3+\lambda-k\pm m}{2} \\ \frac{3}{2},\ \nu+\frac{3}{2},\ \dfrac{\nu+\lambda+2-2k}{2},\ \dfrac{\nu+\lambda+3-2k}{2}\end{matrix};\ -\frac{4}{p^2t^2}\right] f_1(t)\,dt$$

(4.29)
$$\overleftarrow{\frac{k'+\frac{1}{2}}{m'}}\ \frac{\Gamma(\nu+\lambda+2-2k)\,\Gamma(1-2k)}{\Gamma_*(\nu+\lambda+2-k\pm m)\,\Gamma_*(1-k'\pm m')}\times$$

$$\times\int_0^\infty t^{\lambda+\nu}\,{}_3F_6\left[\begin{matrix}\dfrac{1-2k'}{2},\ 1-k',\ 1 \\ \frac{3}{2},\ \nu+\frac{3}{2},\ \dfrac{1-k'\pm m'}{2},\ \dfrac{2-k'\pm m'}{2}\end{matrix};\ -\frac{x^2t^2}{4}\right]\varphi_1(t)\,dt,$$

provided $R(\nu+\lambda+2-k\pm m)>0$, $R(1-k'\pm m')>0$, $R(p)>0$ and the integrals converge.

Taking $k'=-m'$, we get

$$\int_0^\infty t^{-\nu-\lambda-2}\,{}_5F_4\left[\begin{matrix}\dfrac{\nu+2+\lambda-k\pm m}{2},\ \dfrac{\nu+3+\lambda-k\pm m}{2},\ 1 \\ \frac{3}{2},\ \nu+\frac{3}{2},\ \dfrac{\nu+\lambda+2-2k}{2},\ \dfrac{\nu+\lambda+3-2k}{2}\end{matrix};\ -\frac{4}{p^2t^2}\right] f_1(t)\,dt$$

$$\doteqdot\frac{\Gamma(\nu+\lambda+2-2k)}{\Gamma_*(\nu+\lambda+2-k\pm m)}\int_0^\infty t^{\lambda+\nu}\,{}_0F_3\left[\tfrac{3}{2},\nu+\tfrac{3}{2},\tfrac{1}{2};\ -\frac{x^2t^2}{4}\right]\varphi_1(t)\,dt,$$

provided $R(\nu+\lambda+2-k\pm m)>0$, $R(p)>0$ and the integrals converge.

(xiii) **Result 13.** Using (2.13) in (4.1), we get

$$a^{\mu+\nu}\int_0^\infty t^{\mu+\nu-\lambda-1}\times$$

$$\times\,{}_3F_8\left[\begin{matrix}\dfrac{\mu+\nu-\lambda-2k+1}{3},\ \dfrac{\mu+\nu-\lambda-2k+2}{3},\ \dfrac{\mu+\nu-\lambda-2k+3}{3} \\ \mu+1,\ \nu+1,\ \dfrac{\mu+\nu-\lambda-k+1\pm m}{3},\ \dfrac{\mu+\nu-\lambda-k+2\pm m}{3},\ \dfrac{\mu+\nu-\lambda-k+3\pm m}{3}\end{matrix};\ -\frac{t^3a^3}{3^3}\right]\varphi_1(t)\,dt$$

(4.30)
$$=\frac{\Gamma(\mu+1)\,\Gamma(\nu+1)\,\Gamma_*(\mu+\nu-\lambda-k+1\pm m)}{\Gamma(\mu+\nu-\lambda-2k+1)}\int_0^\infty t^{\lambda-1}J_{\mu,\nu}\left(\frac{3a}{t}\right)f_1(t)\,dt,$$

provided $R(\mu+\nu-\lambda-k+1\pm m)>0$ and the integrals converge.

Putting $a^3 = \frac{1}{p}$ and interpreting, we get

$$p^{-\frac{1}{2}(\nu-1)} \int_0^\infty p^{\mu+\nu-\lambda-1} \times$$

$$\times\ {}_3F_8 \left[\begin{matrix} \frac{\mu+\nu-\lambda-2k+1}{3}, \frac{\mu+\nu-\lambda-2k+2}{2}, \frac{\mu+\nu-\lambda-2k+3}{3} \\ \mu+1,\ \nu+1,\ \frac{\mu+\nu-\lambda-k+1\pm m}{3}, \frac{\mu+\nu-\lambda-k+2\pm m}{3}, \frac{\mu+\nu-\lambda-k+3\pm m}{3} \end{matrix} ;\ -\frac{t^3}{3^3 p} \right] \varphi_1(t)\, d$$

(4.31)

$$\overset{k'+\frac{1}{2}}{\underset{m'}{\leftarrow}} \frac{\Gamma_*(\mu+\nu-\lambda-k+1\pm m)\, \Gamma(\frac{1}{2}\nu+\frac{1}{2}-2k')\, x^{\frac{1}{2}(\nu-1)}}{\Gamma(\mu+\nu-\lambda-2k+1)\, \Gamma_*(\frac{1}{2}\nu+\frac{1}{2}-k'\pm m')} \times$$

$$\times \int_0^\infty t^{\lambda-\mu-\nu-1}\ {}_1F_4 \left[\begin{matrix} \frac{1}{2}\nu+\frac{1}{2}-2k' \\ \mu+1,\ \nu+1,\ +\frac{1}{2}\nu+\frac{1}{2}-k'\pm m' \end{matrix} ;\ -\frac{x}{t^3} \right] f_1(t)\, dt,$$

provided $R(\frac{1}{2}\nu+\frac{1}{2}-k'\pm m') > 0$, $R(\mu+\nu-\lambda-k+1\pm m) > 0$, $R(p) > 0$ and the integrals converge.

Taking $k' = -m'$, we get

$$p^{-\frac{1}{2}(\nu-1)} \int_0^\infty t^{\mu+\nu-\lambda-1} \times$$

$$\times\ {}_3F_8 \left[\begin{matrix} \frac{\mu+\nu-\lambda-2k+1}{3}, \frac{\mu+\nu-\lambda-2k+2}{2}, \frac{\mu+\nu-\lambda-2k+3}{3} \\ \mu+1,\ \nu+1,\ \frac{\mu+\nu-\lambda-k+1\pm m}{3}, \frac{\mu+\nu-\lambda-k+2\pm m}{3}; \frac{\mu+\nu-\lambda-k+3\pm m}{3} \end{matrix} ;\ -\frac{t^3}{3^3 p} \right] \varphi_1(t)\, dt$$

$$\doteqdot \frac{\Gamma_*(\mu+\nu-\lambda-k+1\pm m)\, x^{\frac{1}{2}(\nu-1)}}{\Gamma(\mu+\nu-\lambda-2k+1)\, \Gamma(\frac{1}{2}\nu+\frac{1}{2})} \times$$

$$\times \int_0^\infty t^{\lambda-\mu-\nu-1}\ {}_0F_3 \left[\mu+1,\ \nu+1,\ \frac{\nu+1}{2};\ -\frac{x}{t^2} \right] f_1(t)\, dt,$$

provided $R(\nu+1) > 0$, $R(\mu+\nu-\lambda-k+1\pm m) > 0$ and the integrals converge.

Now, putting $\mu = \frac{1}{2}\nu$, we get

$$p^{-\frac{1}{2}(\nu-1)} \int_0^\infty t^{\frac{3}{2}\nu-\lambda-1} \times$$

$$\times\ {}_3F_8\left[\begin{matrix} \frac{\frac{3}{2}\nu-\lambda-2k+1}{3}, \frac{\frac{3}{2}\nu-\lambda-2k+2}{3}, \frac{\frac{3}{2}\nu-\lambda-2k+3}{3} \\ \frac{\nu}{2}+1,\ \nu+1,\ \frac{\nu+1+\nu/2-\lambda-k\pm m}{3}, \frac{\frac{3}{2}\nu-\lambda-k+2\pm m}{3}, \frac{\frac{3}{2}\nu-\lambda-k+3\pm m}{3} \end{matrix};\ -\frac{t^3}{3^3 p}\right] \varphi_1(t)\,dt$$

$$\doteqdot \frac{\Gamma_*\left(\frac{3\nu}{2}-\lambda-k+1\pm m\right)\Gamma(\nu+1)\,\Gamma(\nu+1)\,x^{-\frac{1}{2}}}{\Gamma(\frac{3}{2}\nu-\lambda-2k+1)\,\Gamma(\frac{1}{2}\nu+\frac{1}{2})\,2^\nu i^\nu} \int_0^\infty t^{\lambda-1} \chi_\nu^b \left\{2\sqrt{\frac{2i}{t}\left(\frac{x}{t}\right)^{\frac{1}{2}}}\right\} f_1(t)\,dt.$$

(xiv) **Result 14.** Using (2.4) in (4.1), we get

$$\int_0^\infty t^{4\nu-\mu-1} \times \tag{4.32}$$

$$\times\ {}_5F_8\left[\begin{matrix} \nu, \frac{4\nu-\mu-2k+1}{4}, \frac{4\nu-\mu-2k+2}{4}, \frac{4\nu-\mu-2k+3}{4}, \frac{4\nu-\mu-2k+4}{4} \\ \frac{4\nu-\mu-k+1\pm m}{4}, \frac{4\nu-\mu-k+2\pm m}{4}, \frac{4\nu-\mu-k+3\pm m}{4}, \frac{4\nu-\mu-k+4\pm m}{4} \end{matrix};\ -\frac{a^4 t^4}{4^4}\right] \varphi_1(t)\,dt$$

$$= \frac{\Gamma_*(4\nu-\mu-k+1\pm m)}{\Gamma(4\nu-\mu-2k+1)} \int_0^\infty \frac{t^{\mu-1}}{(t^4+a^4)^\nu} f_1(t)\,dt,$$

provided $R(4\nu-\mu-k+1\pm m)>0$ and the integrals converge.

Example 1. Let $f_1(t) = J_\lambda(bt)$, then

$$\varphi_1(s) = \frac{\left(\frac{b}{2}\right)^\lambda \Gamma_*(\lambda-k+1\pm m)}{s^\lambda\,\Gamma(\lambda-2k+1)\,\Gamma(\lambda+1)}\,{}_4F_3\left[\begin{matrix} \frac{\lambda-k+1\pm m}{2}, \frac{\lambda-k+2\pm m}{2} \\ \lambda+1, \frac{\lambda-2k+1}{2}, \frac{\lambda-2k+2}{2} \end{matrix};\ -\frac{b^2}{s^2}\right],$$

$R(\lambda+1-k\pm m)>0$, $R(s)>0$ and $|s|>|a|$.

Substituting in (4.32), and evaluating the r.h.s. [(9], p. 434), we get

$$\left(\frac{b}{2}\right)^\lambda \int_0^\infty t^{4\nu-\mu-\lambda-1} \times$$

$$\times\ {}_5F_8\left[\begin{matrix} \nu, \frac{4\nu-\mu-2k+1}{4}, \frac{4\nu-\mu-2k+2}{4}, \frac{4\nu-\mu-2k+3}{4}, \frac{4\nu-\mu-2k+4}{4} \\ \frac{4\nu-\mu-k+1\pm m}{4}, \frac{4\nu-\mu-k+2\pm m}{4}, \frac{4\nu-\mu-k+3\pm m}{4}, \frac{4\nu-\mu-k+4\pm m}{4} \end{matrix};\ -\frac{a^4 t^4}{4^4}\right] \times$$

$$\times {}_4F_3\left[\begin{matrix}\frac{\lambda-k+1\pm m}{2}, \frac{\lambda-k+2\pm m}{2} \\ \lambda+1, \frac{\lambda-2k+1}{2}, \frac{\lambda-2k+2}{2}\end{matrix}; -\frac{b^2}{t^2}\right] dt$$

$$=\frac{\Gamma_*(4\nu-\mu-k+1\pm m)\,\Gamma(\lambda-2k+1)\,\Gamma(\lambda+1)}{\Gamma(4\nu-\mu-2k+1)\,\Gamma_*(\lambda-k+1\pm m)}\int_0^\infty \frac{t^{\mu-1}}{(t^4+a^4)^\nu} J_\lambda(bt)\,dt$$

$$(4.33)\quad =\frac{\Gamma_*(4\nu-\mu-k+1\pm m)\,\Gamma(\lambda-2k+1)\,\Gamma(\lambda+1)\,\frac{1}{2}\pi\, a^{\mu-4\nu}}{\Gamma(4\nu-\mu-2k+1)\,\Gamma_*(\lambda-k+1\pm m)\,\sin\left(\frac{1}{2}\mu+\frac{1}{2}\lambda-2\nu+2\right)\pi\,\Gamma(\nu)}\times$$

$$\times\left[\sum_{r=0}^\infty \frac{(ab)^{r+2\nu}\,\Gamma\left(\frac{\mu}{4}+\frac{\lambda}{4}+\frac{r}{2}\right)}{2^{\lambda+2r}\, r!\,\Gamma(\lambda+r+1)\,\Gamma\left(\frac{\mu}{4}+\frac{\lambda}{4}-\nu+1+\frac{r}{2}\right)}\cos\left(\frac{\mu}{4}+\frac{\lambda}{4}-\nu+1+\frac{r}{2}\right)\pi+\right.$$

$$\left.+\sum_{r=0}^\infty \frac{(-)^r\left(\frac{ab}{2}\right)^{4\nu-\mu+4r}\Gamma(\nu+r)}{r!\,\Gamma\left(2\nu-\frac{1}{2}\mu+\frac{1}{2}\lambda+2r+1\right)\Gamma\left(2\nu-\frac{1}{2}\mu-\frac{1}{2}\lambda+2r+1\right)}\right],$$

$R(\lambda-2k+1)>0, 4R(\nu-1)+\frac{11}{2}>R(\mu)>-R(\lambda)$, $R(\lambda+1-k\pm m)>0$ and $R(4\nu-\mu-k+$ $+1\pm m)>0$.

Particular cases

(i) If $\mu=\lambda+2$, $\nu=\lambda+\frac{1}{2}$, then (4.33) reduces to

$$\left(\frac{b}{2}\right)^\lambda\int_0^\infty t^{2\lambda-1}\,{}_5F_8\left[\begin{matrix}\lambda+\frac{1}{2}, \frac{3\lambda-2k+1}{4}, \frac{3\lambda-2k+2}{4}, \frac{3\lambda-2k+3}{4}, \frac{3\lambda-2k+4}{4} \\ \frac{3\lambda-k+1\pm m}{4}, \frac{3\lambda-k+2\pm m}{4}, \frac{3\lambda-k+3\pm m}{4}, \frac{3\lambda-k+4\pm m}{4}\end{matrix}; -\frac{a^4t^4}{4^4}\right]\times$$

$$\times {}_4F_3\left[\begin{matrix}\frac{\lambda-k+1\pm m}{2}, \frac{\lambda-k+2\pm m}{2} \\ \lambda+1, \frac{\lambda-2k+1}{2}, \frac{\lambda-2k+2}{2}\end{matrix}; -\frac{b^2}{t^2}\right] dt$$

$$=\frac{\Gamma_*(3\lambda-k+1\pm m)\,\Gamma(\lambda-2k+1)\,\Gamma(\lambda+1)\left(\frac{1}{2}b\right)^\lambda\sqrt{\pi}}{\Gamma(3\lambda-2k+1)\,\Gamma_*(\lambda-k+1\pm m)\,(a\sqrt{2})^{2\lambda}\,\Gamma\left(\lambda+\frac{1}{2}\right)} J_\lambda\left(\frac{ab}{\sqrt{2}}\right) K_\lambda\left(\frac{ab}{\sqrt{2}}\right),$$

$R(\lambda-2k+1)>0$, $R(\lambda-k+1\pm m)>0$, $R(\lambda)>-\frac{1}{2}$, $b>0$ and $|\arg a|<\pi/4$.

(ii) If $\mu=\lambda+4$, $\nu=\lambda+\frac{1}{2}$, then (4.33) reduces to

$$\left(\frac{b}{2}\right)^{\lambda}\int_0^{\infty} t^{2\lambda-3}\,{}_5F_8\left[\begin{matrix}\lambda+\frac{1}{2}, \frac{3\lambda-1-2k}{4}, \frac{3\lambda-2k}{4}, \frac{3\lambda-2k+1}{4}, \frac{3\lambda-2k+2}{4}\\ \frac{3\lambda-1-k\pm m}{4}, \frac{3\lambda-k\pm m}{4}; \frac{3\lambda-k+1\pm m}{4}, \frac{3\lambda-k+2\pm m}{4}\end{matrix}; -\frac{a^4t^4}{4^4}\right]\times$$

$$\times\,{}_4F_3\left[\begin{matrix}\frac{\lambda-k+1\pm m}{2}, \frac{\lambda-k+2\pm m}{2}\\ \lambda+1, \frac{\lambda-2k+1}{2}, \frac{\lambda-2k+2}{2}\end{matrix}; -\frac{b^2}{t^2}\right]dt$$

$$=\frac{\Gamma_*(3\lambda-k-1\pm m)\,\Gamma(\lambda-2k+1)\,\Gamma(\lambda+1)\,(\frac{1}{2}b)^{\lambda}\sqrt{\pi}}{\Gamma(3\lambda-2k-1)\,\Gamma_*(\lambda-k+1\pm m)\,\Gamma(\lambda+\frac{1}{2})\,2\,(a\sqrt{2})^{2(\lambda-1)}}\,J_{\lambda-1}\left(\frac{ab}{\sqrt{2}}\right)K_{\lambda-1}\left(\frac{ab}{\sqrt{2}}\right),$$

$R(\lambda-2k+1)>0$, $R(\lambda-k+1\pm m)>0$, $R(\lambda)>\frac{1}{6}$, $b>0$ and $|\arg a|<\pi/4$.

Section IV

5. In this section we have obtained some properties of Meijer Transform involving self-reciprocal functions in the Hankel transform.

5.1. Theorem 1. *If* $f(t) \underset{m}{\overset{k+\frac{1}{2}}{\longrightarrow}} \varphi_1(s)$ *and* $t^{-\mu-3/2}f\left(\frac{1}{t}\right)$ *is self-reciprocal in the Hankel Transform of order* ν, *then*

$$\int_0^{\infty} t^{\nu-\mu-1}\,{}_2F_5\left[\begin{matrix}\frac{1}{2}(\nu-\mu-2k+1), \frac{1}{2}(\nu-\mu-2k+2)\\ \nu+1, \frac{1}{2}(\nu-\mu-k+1\pm m), \frac{1}{2}(\nu-\mu-k+2\pm m)\end{matrix}; -\frac{a^2t^2}{4^2}\right]\varphi(t)\,dt$$

(5.1)

$$=\frac{2^{\nu}\,\Gamma_*(\nu-\mu-k+1\pm m)\,\Gamma(\nu+1)}{\Gamma(\nu-\mu-2k+1)}\,a^{-\mu-\nu-2}f\left(\frac{1}{a}\right),$$

provided $R(\nu-\mu-k+1\pm m)>0$ *and the integrals converge.*

Proof. We know, (2.7)

$$\frac{\left(\frac{a}{2}\right)^{\nu}\Gamma(\nu-\mu-2k+1)\,t^{\nu-\mu}}{\Gamma_*(\nu-\mu-k+1\pm m)\,\Gamma(\nu+1)}\times$$

$$\times\,{}_2F_5\left[\begin{matrix}\frac{\nu-\mu-2k+1}{2}, \frac{\nu-\mu-2k+2}{2}\\ \nu+1, \frac{\nu-\mu-k+1\pm m}{2}, \frac{\nu-\mu-k+2\pm m}{2}\end{matrix}; -\frac{a^2t^2}{4^2}\right] \underset{m}{\overset{k+\frac{1}{2}}{\longrightarrow}} s^{\mu}J_{\nu}(a/s),$$

$R(\nu-\mu-k+1\pm m)>0$ and $R(s)>0$,

also

$$f(t) \xrightarrow[m]{k+\frac{1}{2}} \varphi(s).$$

Now, applying these in (4.1), we get

$$\int_0^\infty t^{\nu-\mu-1}\,{}_2F_5\left[\begin{matrix} \frac{\nu-\mu-2k+1}{2}, \frac{\nu-\mu-2k+2}{2} \\ \nu+1, \frac{\nu-\mu-k+1\pm m}{2}, \frac{\nu-\mu-k+2\pm m}{2}\end{matrix}; -\frac{a^2t^2}{4^2}\right]\varphi(t)\,dt$$

$$= \frac{2^\nu \Gamma_*(\nu-\mu-k+1\pm m)\,\Gamma(\nu+1)}{a^\nu\,\Gamma(\nu-\mu-2k+1)}\int_0^\infty t^{\mu-1} J_\nu\left(\frac{a}{t}\right) f(t)\,dt$$

$$= \frac{2^\nu \Gamma_*(\nu-\mu-k+1\pm m)\,\Gamma(\nu+1)}{a^{\nu+\frac{1}{2}}\,\Gamma(\nu-\mu-2k+1)}\int_0^\infty \sqrt{au}\, J_\nu(au)\, u^{-\mu-3/2} f\left(\frac{1}{u}\right) du$$

$$= \frac{2^\nu \Gamma_*(\nu-\mu-k+1\pm m)\,\Gamma(\nu+1)}{\Gamma(\nu-\mu-2k+1)\,a^{\mu+\nu+2}} f\left(\frac{1}{a}\right),$$

$R(\nu-\mu-k+1\pm m)>0$ and the integrals converge.

5.2. Theorem 2. *If* $f(t) \xrightarrow[m]{k+\frac{1}{2}} \varphi(s)$, *then*

$$x^{n-\frac{1}{2}}\int_0^\infty e^{-xt^2} f(t)\,dt \doteqdot \frac{\Gamma(n+\frac{1}{2})\,\Gamma(2n-2k+1)\,p}{\Gamma_*(n-k+\frac{1}{2}\pm m)} \times \tag{5.2}$$

$$\times \int_0^\infty t^{n-3/2}\varphi(t)\,{}_3F_4\left[\begin{matrix} n+\frac{1}{2}, \frac{2n-2k+1}{2}, \frac{n-2k+2}{2} \\ \frac{2n-k+1\pm m}{2}, \frac{2n-k+2\pm m}{2}\end{matrix}; -\frac{pt^2}{4}\right]dt,$$

provided $R(2n-k+1\pm m)>0$, $R(p)>0$ *and the integrals converge. Also, if* $t^{-n-\frac{1}{2}} f(t)$ *is self-reciprocal in the Hankel Transform of order* n, *then*

$$x^{-1} f(x) \doteqdot \frac{2^n\,\Gamma(2n-2k+1)\,\Gamma(n+\frac{1}{2})\,p}{\pi^{\frac{1}{2}}\,\Gamma_*(2n-k+1\pm m)} \times \tag{5.3}$$

$$\times \int_0^\infty t^{2n-1}\varphi(t)\,{}_3F_4\left[\begin{matrix} n+\frac{1}{2}, \frac{2n-2k+1}{2}, n-k+1 \\ \frac{2n-k+1\pm m}{2}, \frac{2n-k+2\pm m}{2}\end{matrix}; -\frac{p^2t^2}{4}\right]dt,$$

provided $R(2n-k+1\pm m)>0$, $R(2n+1)>0$, $R(p)>0$ *and the integrals converge.*

Proof. We know, (4.7),

$$\int_0^\infty \frac{f(t)}{(t^2+a^2)^{n+\frac{1}{2}}}\,dt = \frac{\Gamma(2n-2k+1)}{\Gamma_*(2n-k+1\pm m)}\times$$

$$\times \int_0^\infty t^{2n-1}\,{}_3F_4\left[\begin{matrix} n+\frac{1}{2},\ \dfrac{2n-2k+1}{2},\ n-k+1 \\ \dfrac{2n-k+1\pm m}{2},\ \dfrac{2n-k+2\pm m}{2} \end{matrix};\ -\frac{a^2 t^2}{4}\right]\varphi(t)\,dt,$$

provided $R(2n-k+1\pm m)>0$ and the integrals converge.

If we put $a^2=p$ and interpret, assuming that $p^{-n} \doteqdot \dfrac{x^n}{\Gamma(n+1)}$, we get

$$x^{n-\frac{1}{2}}\int_0^\infty e^{-xt^2} f(t)\,dt \doteqdot \frac{\Gamma(2n-2k+1)\,\Gamma(n+\frac{1}{2})\,p}{\Gamma_*(2n-k+1\pm m)}\int_0^\infty t^{2n-1}\varphi(t)\times$$

$$\times\,{}_3F_4\left[\begin{matrix} n+\frac{1}{2},\ \dfrac{2n-2k+1}{2},\ n-k+1 \\ \dfrac{2n-k+1\pm m}{2},\ \dfrac{2n-k+2\pm m}{2} \end{matrix};\ -\frac{pt^2}{4}\right]dt,$$

provided $R(2n-k+1\pm m)>0$, $R(p)>0$, $R(2n+1)>0$ and the integrals converge.

Again, if we put $a=p$ and interpret, we get

$$\frac{p\,\Gamma(n+\frac{1}{2})\,\Gamma(2n-2k+1)\,2^n}{\Gamma(2n-k+1\pm m)\,\pi^{\frac{1}{2}}}\int_0^\infty t^{2n-1}\varphi(t)\,{}_3F_4\left[\begin{matrix} n+\frac{1}{2},\ \dfrac{2n-2k+1}{2},\ n-k+1 \\ \dfrac{2n-k+1\pm m}{2},\ \dfrac{2n-k+2\pm m}{2} \end{matrix};\ -\frac{p^2t^2}{4}\right]$$

(5.4)
$$\doteqdot x^{n-\frac{1}{2}}\int_0^\infty \sqrt{xt}\,J_\nu(xt)\,t^{-n-\frac{1}{2}} f(t)\,dt$$

$$= x^{-1} f(x),$$

provided $R(2n+1)>0$, $R(2n-k+1\pm m)>0$, $R(p)>0$ and the integrals converge.

Exemple. Let $x^{-n-\frac{1}{2}} f(x) = x^{n+\frac{1}{2}} e^{-\frac{1}{4}x^2}$, then

$$f(x) = x^{2n+1} e^{-\frac{1}{4}x^2}$$

$$\doteqdot \Gamma(2n+2)\,p\,e^{\frac{1}{4}p^2} D_{-(2n+2)}(p)$$

$R(2n+1)>0$ and $R(p)>0$.

Putting $k = -m$ in (5.4) and substituting the values of $f(t)$ and $\varphi(s)$, we get

$$\frac{2^n p(2n+1)}{\pi^{\frac{1}{2}}} \int_0^\infty t^{2n} e^{\frac{1}{4}t^2} D_{-(2n+2)}^{(t)} {}_1F_2\left[\begin{matrix} n+\frac{1}{2} \\ n+\frac{1}{2}, n+1 \end{matrix}; -\frac{p^2 t^2}{4}\right] dt$$
$$= x^{2n} e^{-\frac{1}{2}x^4},$$

$R(n+\frac{1}{2}) > 0$ and $R(p) > 0$.

5.3. Theorem 3. *If* $f(t) \overset{k+\frac{1}{2}}{\underset{m}{\longrightarrow}} \varphi(s)$ *then*

$$\text{(5.5)}\quad \int_0^\infty t^{2n-1} {}_4F_4\left[\begin{matrix} n, \frac{2n-2k+1}{2}, n-k+1, n+\frac{1}{2} \\ \frac{2n-k+1\pm m}{2}, \frac{2n-k+2\pm m}{2} \end{matrix}; -\frac{t^2}{2}\right] \varphi(t)\, dt$$
$$= \frac{\Gamma_*(2n-k+1\pm m}{2^n \Gamma(2n-2k+1)} \int_0^\infty e^{\frac{1}{4}t^2} D_{-2n}^{(t)} \frac{f(t)}{t} dt,$$

provided $R(2n-k+1\pm m) > 0$ *and the integrals converge.*

Proof. We know, (2.12),

$$\frac{t^{2n}\Gamma(2n-2k+1)}{\Gamma_*(2n-k+1\pm m)} {}_4F_4\left[\begin{matrix} n, \frac{2n-2k+1}{2}, n-k+1, n+\frac{1}{2} \\ \frac{2n-k+1\pm m}{2}, \frac{2n-k+2\pm m}{2} \end{matrix}; -\frac{t^2}{2}\right] \overset{k+\frac{1}{2}}{\underset{m}{\longrightarrow}} e^{\frac{1}{4}s^2} D_{-2n}^{(s)},$$

$R(n+\frac{1}{2}) > 0$, $R(2n-k+1\pm m) > 0$ and $R(s) > 0$,

also $f(t) \overset{k+\frac{1}{2}}{\underset{m}{\longrightarrow}} \varphi(s)$.

Applying these in (4.1), we get

$$\int_0^\infty t^{2n-1} {}_4F_4\left[\begin{matrix} n, \frac{2n-2k+1}{2}, n-k+1, n+\frac{1}{2} \\ \frac{2n-k+1\pm m}{2}, \frac{2n-k+2\pm m}{2} \end{matrix}; -\frac{t^2}{2}\right] \varphi(t)\, dt$$
$$= \frac{\Gamma_*(2n-k+1\pm m)}{2^n \Gamma(2n-2k+1)} \int_0^\infty e^{\frac{1}{4}t^2} D_{-2n}^{(t)} \frac{f(t)}{t} dt,$$

provided $R(n+\frac{1}{2}) > 0$, $R(2n-k+1\pm m) > 0$ and the integrals converge.

Example. If $f(t) = t^{n+1} J_n(tx)$, then

$$\varphi(s) = \frac{x^n \Gamma_*(2n-k+2\pm m)}{2^n s^{2n+1}\Gamma(2n-2k+2)\Gamma(n+1)}\, {}_4F_3\left[\begin{matrix}\frac{2n-k+2\pm m}{2}, \frac{2n-k+3\pm m}{2} \\ n+1, n-k+1, \frac{2n-2k+3}{2}\end{matrix}; -\frac{x^2}{s^2}\right],$$

$$R(2n-k+2\pm m) > 0 \text{ and } R(s) > 0.$$

Substituting in (5.5) and knowing that $t^{n-\frac{1}{2}} e^{\frac{1}{4}t^2} D_{-2n}(t)$ is self-reciprocal in the Hankel transform of order n, ([6], p. 12), we get

$$\int_0^\infty t^{-2}\, {}_4F_4\left[\begin{matrix} n, \frac{2n-2k+1}{2}, n-k+1, n+\frac{1}{2} \\ \frac{2n-k+1\pm m}{2}, \frac{2n-k+2\pm m}{2}\end{matrix}; -\frac{t^2}{2}\right]\times$$

$$\times\, {}_4F_3\left[\begin{matrix}\frac{2n-k+2\pm m}{2}, \frac{2n-k+3\pm m}{2} \\ n+1, n-k+1, \frac{2n-2k+3}{2}\end{matrix}; -\frac{x^2}{t^2}\right] dt$$

$$= \frac{2^n \Gamma_*(2n-k+1\pm m)\Gamma(2n-2k+2)\Gamma(n+1)}{x^{n+\frac{1}{2}}\Gamma(2n-2k+1)\Gamma_*(2n-k+2\pm m)} \int_0^\infty \sqrt{tx}\, J_n(tx)\, t^{n-\frac{1}{2}} e^{\frac{1}{4}t^2} D^{(t)}_{-2n}\, dt$$

$$= \frac{2^n \Gamma_*(2n-k+1\pm m)\Gamma(2n+2-2k)\Gamma(n+1)}{\Gamma(2n-2k+1)\Gamma_*(2n+2-k\pm m)} x^{-1} e^{\frac{1}{4}x^2} D^{(x)}_{-2n},$$

$$R(2n+1-k\pm m) > 0, \text{ and } R(n+\tfrac{1}{2}) > 0.$$

5.4. Theorem 4. *If* $f(t) \underset{m}{\overset{k+\frac{1}{2}}{\doteqdot}} \varphi(s)$ *and* $t^{-\frac{3}{2}(\lambda+\mu/2-\nu+9/4)} f(t^{-\frac{2}{3}})$ *is self-reciprocal in the Hankel transform of order* μ, *then*

$$\frac{3\Gamma(\mu+\nu-\lambda-2k+1)\, 2^{\frac{2}{3}(\lambda+\mu-\nu+3/2)} x^\nu}{\Gamma(\mu+1)\Gamma(\nu+1)\Gamma_*(\mu+\nu-\lambda-k+1\pm m)} \int_0^\infty t^{\mu+\nu-\lambda-1} \varphi(t)\times$$

$$\times\, {}_3F_8\left[\begin{matrix}\frac{\mu+\nu-\lambda-2k+1}{3}, \frac{\mu+\nu-\lambda-2k+2}{3}, \frac{\mu+\nu-\lambda-2k+3}{3} \\ \mu+1, \nu+1, \frac{\mu+\nu-\lambda-k+1\pm m}{3}, \frac{\mu+\nu-\lambda-k+2\pm m}{3}, \frac{\mu+\nu-\lambda-k+3\pm m}{3}\end{matrix}; -\frac{xt^3}{3^3}\right] dt$$

$$\doteqdot p^{\frac{1}{3}(\lambda+2\mu-4\nu+3)} f(p^{1/3}/2^{2/3}), \tag{5.7}$$

provided $R(\mu+\nu-\lambda-k+1\pm m) > 0$, $R(\mu+1) > 0$, $R(p) > 0$ *and the integrals converge.*

Proof. We have, (2.13),

$$\frac{\Gamma(\mu+\nu-\lambda-2k+1)\, t^{\mu+\nu-\lambda}\, a^{\mu+\nu}}{\Gamma(\mu+1)\,\Gamma(\nu+1)\,\Gamma_*(\mu+\nu-\lambda-k+1\pm m)}\times$$

$$\times\, {}_3F_8\left[\begin{matrix} \frac{\mu+\nu-\lambda-2k+1}{3}, \frac{\mu+\nu-\lambda-2k+2}{3}, \frac{\mu+\nu-\lambda-2k+3}{3} \\ \mu+1,\ \nu+1,\ \frac{\mu+\nu-\lambda-k+1\pm m}{3}, \frac{\mu+\nu-\lambda-k+2\pm m}{3}, \frac{\mu+\nu-\lambda-k+3\pm m}{3} \end{matrix};\ -\frac{a^3 t^3}{3^3}\right]$$

$$\xrightarrow[m]{k+\frac{1}{2}}\ s^\lambda J_{\mu,\nu}\left(\frac{3a}{s}\right),$$

$$R(\mu+\nu-\lambda-k+1\pm m)>0 \text{ and } R(s)>0,$$

also $f(t) \xrightarrow[m]{k+\frac{1}{2}} \varphi(s)$.

Using these in (4.1), we get

$$\frac{\Gamma(\mu+\nu-\lambda-2k+1)\, a^{\mu+\nu}}{\Gamma(\mu+1)\,\Gamma(\nu+1)\,\Gamma_*(\mu+\nu-\lambda-k+1\pm m)}\int_0^\infty t^{\mu+\nu-\lambda-1}\varphi(t)\times$$

$$\times\, {}_3F_8\left[\begin{matrix} \frac{\mu+\nu-\lambda-2k+1}{3}, \frac{\mu+\nu-\lambda-2k+2}{3}, \frac{\mu+\nu-\lambda-2k+3}{3} \\ \mu+1,\ \nu+1,\ \frac{\mu+\nu-\lambda-k+1\pm m}{3}, \frac{\mu+\nu-\lambda-k+2\pm m}{3}, \frac{\mu+\nu-\lambda-k+3\pm m}{3} \end{matrix};\ -\frac{a^3 t^3}{3^3}\right] dt$$

$$=\int_0^\infty t^{\lambda-1} J_{\mu,\nu}\left(\frac{3a}{t}\right) f(t)\, dt,$$

provided $R(\mu+\nu-\lambda-k+1\pm m)>0$ and the integrals converge.

Putting $a^3=x$ and interpreting, assuming that $x^n \doteqdot p^{-n}\, n!$, we get

$$\frac{\Gamma(\mu+\nu-\lambda-2k+1)\, x^{\nu}}{\Gamma(\mu+1)\,\Gamma(\nu+1)\,\Gamma_*(\mu+\nu-\lambda-k+1\pm m)}\int_0^\infty t^{\mu+\nu-\lambda-1}\varphi(t)\times$$

$$\times\, {}_3F_8\left[\begin{matrix} \frac{\mu+\nu-\lambda-2k+1}{3}, \frac{\mu+\nu-\lambda-2k+2}{3}, \frac{\mu+\nu-\lambda-2k+3}{3} \\ \mu+1,\ \nu+1,\ \frac{\mu+\nu-\lambda-k+1\pm m}{3}; \frac{\mu+\nu-\lambda-k+2\pm m}{3}, \frac{\mu+\nu-\lambda-k+3\pm m}{3} \end{matrix};\ -\frac{x t^3}{3^3}\right] dt$$

$$\doteqdot p^{-\nu}\int_0^\infty t^{\lambda-\mu-\nu-1} f(t)\, J_\mu\{2(pt^3)^{-\frac{1}{2}}\}\,(pt^3)^{\mu/2}\, dt$$

$$= \frac{\sqrt{2}}{3} p^{\mu/2-\nu+\frac{1}{4}} \int_0^\infty \sqrt{2\, v\, p^{-\frac{1}{2}}}\, J_\mu (2\, p^{-\frac{1}{2}}\, v)\, v^{-2/3\,(\lambda+\mu/2-\nu+9/4)} f\,(v^{-2/3})\, d\,v$$

$$= 2^{-2/3\,(\lambda+\mu/2-\nu+3/2)}\; 3^{-1}\; p^{\frac{1}{3}\,(\lambda+2\,\mu-4\,\nu+3)} f\left(\frac{p^{\frac{1}{3}}}{2^{\frac{2}{3}}}\right);$$

provided $R\,(\mu+\nu-\lambda-k+1\pm m)>0,\ R\,(\mu+1)>0,\ R\,(p)>0$ and the integrals converge.

5.5. Theorem 5. *If* $f\,(x) \doteqdot \varphi\,(p)$ *and* $\sqrt{\frac{\pi}{2}}\, e^{-a x} \frac{\varphi\,(x)}{x}$ *is self-reciprocal in the Sine-transform, then*

$$\frac{\Gamma\,(2-2\,k)}{\Gamma_*\,(2-k\pm m)}\, t \int_0^\infty {}_3F_4 \left[\begin{matrix} 1-k,\ \frac{3-2\,k}{2},\ 1 \\ \frac{2-k\pm m}{2},\ \frac{3-k\pm m}{2} \end{matrix};\ -\frac{t^2\,(x+a)^2}{4}\right] f\,(x)\, d\,x$$

(5.8)

$$= \frac{k+\frac{1}{2}}{m} \sqrt{\frac{\pi}{2}}\, e^{-a\,s} \frac{\varphi\,(s)}{s},$$

provided $R\,(a)>0,\ R\,(s)>0,\ R\,(2-k\pm m)>0$ *and the integrals converge.*

Proof. We know ([10], p. 19)

$$e^{-a\,n} \sin b\,x \doteqdot \frac{p\,b}{(p+a)^2+b^2},\quad R\,(a+p)>0 \text{ and } R\,(p)>0,$$

and

$$f\,(x) \doteqdot \varphi\,(p).$$

Now, applying (4.2), we get

$$\int_0^\infty e^{-a\,x} \sin b\,x \frac{\varphi\,(x)}{x}\, d\,x = \int_0^\infty \frac{b}{(x+a)^2+b^2} f\,(x)\, d\,x,$$

provided $R\,(a)>0$ and the integrals converge.

Putting $b=s$ and interpreting (using (2.3) with $\mu=1,\ \nu=1$), we get

$$\frac{\Gamma\,(2-2\,k)}{\Gamma_*\,(2-k\pm m)}\, t \int_0^\infty {}_3F_4 \left[\begin{matrix} 1,\ 1-k,\ \frac{3-2\,k}{2} \\ \frac{2-k\pm m}{2},\ \frac{3-k\pm m}{2} \end{matrix};\ -\frac{t^2}{4}(x+a)^2\right] f\,(x)\, d\,x$$

$$= \frac{k+\frac{1}{2}}{m} \int_0^\infty e^{-a\,n}\, x^{-1}\, \varphi\,(x) \sin s\,x\, d\,x = \sqrt{\frac{\pi}{2}}\, e^{-a\,s} \frac{\varphi\,(s)}{s},$$

provided $R\,(a)>0,\ R\,(s)>0,\ R\,(2-k\pm m)>0$ and the integrals converge.

5.6. Theorem 6. *If* $f(x) \doteqdot \varphi(p)$ *and* $\sqrt{\frac{\pi}{2}}\, e^{-ax}\, \frac{\varphi(x)}{x}$ *is self-reciprocal in the Cosine-transform, then*

$$\frac{\Gamma(3-2k)\,t^2}{\Gamma_*(3-k\pm m)} \int_0^\infty (x+a)\,{}_3F_4\left[\begin{matrix} 1, \frac{3-2k}{2}, 2-k \\ \frac{3-k\pm m}{2}, \frac{4-k\pm m}{2} \end{matrix}; -\frac{t^2}{4}(x+a)^2\right] f(x)\,dx \tag{5.9}$$

$$\xrightarrow[m]{k+\frac{1}{2}} \sqrt{\frac{\pi}{2}}\, e^{-as}\, \frac{\varphi(s)}{s},$$

provided $R(a) > 0$, $R(s) > 0$, $R(3-k\pm m) > 0$ *and the integrals converge.*

Proof. We know ([10], p. 19)

$$e^{-ax} \cos bx \doteqdot \frac{p(p+a)}{(p+a)^2+b^2}, \qquad R(p+a) > 0,$$

and

$$f(x) \doteqdot \varphi(p).$$

Applying (4.2), we get

$$\int_0^\infty e^{-ax} \cos bx\, \frac{\varphi(x)}{x}\, dx = \int_0^\infty \frac{x+a}{(x+a)^2+b^2}\, f(x)\, dx.$$

Putting $b = s$ and interpreting (using (2.3) with $\mu = 0$ and $\nu = 1$), we get

$$\frac{\Gamma(3-2k)\,t^2}{\Gamma_*(3-k\pm m)} \int_0^\infty (x+a)\,{}_3F_4\left[\begin{matrix} 1, \frac{1}{2}(3-2k), 2-k \\ \frac{1}{2}(3-k\pm m), \frac{1}{2}(4-k\pm m) \end{matrix}; -\frac{t^2}{2}(x+a)^2\right] f(x)\,dx$$

$$\xrightarrow[m]{k+\frac{1}{2}} \int_0^\infty e^{-ax}\, \frac{\varphi(x)}{x} \cos sx\, dx = \sqrt{\frac{\pi}{2}}\, e^{-as}\, \frac{\varphi(s)}{s},$$

provided $R(a) > 0$, $R(s) > 0$, $R(3-k\pm m) > 0$ and the integrals converge.

5.7. Theorem 7. *If* $f(x) \doteqdot \varphi(p)$ *and* $x^{\mu-3/2}\,\varphi(x)$ *is self-reciprocal transform of order* ν, *then*

$$\frac{\Gamma(\mu+\nu+1)\,\Gamma(\nu-2k+1)\,t^\nu}{2^\mu\,\Gamma(\nu+1)\,\Gamma_*(\nu-k+1\pm m)} \int_0^\infty x^{-\mu-\nu-1} \times$$

$$\times\, {}_4F_5\left[\begin{matrix} \frac{\mu+\nu+1}{2}, \frac{\mu+\nu+2}{2}, \frac{\nu+2k+1}{2}, \frac{\nu-2k+2}{2} \\ \nu+1, \frac{\nu-k+1\pm m}{2}, \frac{\nu-k+2\pm m}{2} \end{matrix}; -\frac{t^2}{x^2}\right] f(x)\,dx \xrightarrow[m]{k+\frac{1}{2}} s^{-\mu+2}\,\varphi\left(\frac{1}{s}\right), \tag{5.10}$$

provided $R(\nu-k+1\pm m) > 0$, $R(s) > 0$ *and the integrals converge.*

Proof. We know ([10], p. 28)

$$a^{\mu} x^{\mu} J_{\nu}(a x) \doteqdot \frac{\Gamma(\mu+\nu+1)}{2^{\mu}\Gamma(\nu+1)} a^{\mu+\nu} p^{-\mu-\nu} {}_2F_1\left[\begin{matrix}\frac{\mu+\nu+1}{2}, \frac{\mu+\nu+2}{2} \\ \nu+1\end{matrix} ; -\frac{a^2}{p^2}\right],$$

$$R(\mu+\nu+1)>0,\ R(p)>0 \text{ and } |p|>|a|,$$

and

$$f(x) \doteqdot \varphi(p).$$

Applying (4.2), we get

$$\int_0^{\infty} x^{\mu-1} J_{\nu}(a x)\varphi(x)\,dx = \frac{a^{\nu}}{2^{\mu}}\int_0^{\infty} x^{-\mu-\nu-1}\sum_{r=0}^{\infty}\frac{\Gamma(\mu+\nu+2r+1)(-)^r a^{2r}}{r!\,\Gamma(\nu+r+1)\,2^{2r} x^{2r}} f(x)\,dx.$$

Putting $a=\frac{1}{s}$ and interpreting, we get

$$\frac{\Gamma(\mu+\nu+1)\Gamma(\nu-2k+1)t^{\nu}}{2^{\mu}\Gamma(\nu+1)\Gamma_*(\nu-k+1\pm m)}\times$$

$$\times\int_0^{\infty} x^{-\mu-\nu-1} f(x)\,{}_4F_5\left[\begin{matrix}\frac{\mu+\nu+1}{2}, \frac{\mu+\nu+2}{2}, \frac{\nu-2k+1}{2}, \frac{\nu-2k+2}{2} \\ \nu+1, \frac{\nu-k+1\pm m}{2}, \frac{\nu-k+2\pm m}{2}\end{matrix} ; -\frac{t^2}{4x^2}\right] f(x)\,dx$$

$$\frac{k+\frac{1}{2}}{m}\ \sqrt{s}\int_0^{\infty}\sqrt{\frac{x}{s}}\,J_{\nu}\left(\frac{x}{s}\right) x^{\mu-3/2}\varphi(x)\,dx$$

$$= s^{-\mu+2}\varphi\left(\frac{1}{s}\right),$$

provided $R(\nu-k+1\pm m)>0$, $R(s)>0$ and the integrals converge.

Putting $k=\pm m$, we get

$$\frac{\Gamma(\mu+\nu+1)t^{\nu}}{2^{\mu}\Gamma(\nu+1)\Gamma(\nu+1)}\int_0^{\infty} x^{-\mu-\nu-1}\,{}_2F_3\left[\begin{matrix}\frac{\mu+\nu+1}{2}, \frac{\mu+\nu+2}{2} \\ \nu+1, \frac{\nu+1}{2}, \frac{\nu+2}{2}\end{matrix} ; -\frac{t^2}{4x^2}\right] f(x)\,dx$$

$$:\ s^{2-\mu}\varphi\left(\frac{1}{s}\right),$$

$R(\nu+1)>0$, $R(s)>0$ and the integrals converge.

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