Acta Math., 170 (1993), 1–28

Distribution of simple zeros of polynomials

by

VILMOS TOTIK[1]

Bolyai Institute
Szeged, Hungary
and
University of South Florida
Tampa, FL, U.S.A.

1. Introduction and main results

It is well known that if $P_n(x)=x^n+\ldots$ is a monic polynomial of degree n, then its supremum norm on $[-1,1]$ is at least as large as 2^{1-n}:

$$\|P_n\|_{[-1,1]} \geqslant \frac{1}{2^{n-1}},$$

and here the equality sign holds only for the Chebyshev polynomials

$$T_n(x)=2^{1-n}\cos(n \arccos x).$$

It is also known that if $\{P_n\}$ is a sequence of monic polynomials with the property

$$\lim_{n\to\infty} \|P_n\|_{[-1,1]}^{1/n} = \tfrac{1}{2},$$

then the zeros of the P_n's are distributed according to the arcsine distribution.

More precisely, we associate with P_n the normalized zero counting measure

$$\nu_{P_n}(A)=\frac{\text{number of zeros of } P_n \text{ on } A}{n}$$

where A is any point set in $\mathbf{C}$. Let ω be the arcsine distribution, i.e.

$$\omega([a,b])=\frac{1}{\pi}\int_a^b \frac{dx}{\sqrt{1-x^2}}$$

for any subinterval $[a,b]$ of $[-1,1]$. Then the above statement about the zeros means that

$$\lim_{n\to\infty} \nu_{P_n}=\omega$$

[1]This work was supported by NSF Grant No. DMS 9002794

in the weak* topology on measures on $\overline{\mathbf{C}}$. In particular, if all the zeros of the P_n's are real, then

$$\lim_{n\to\infty} |(\nu_{P_n} - \omega)([a,b])| = 0 \tag{1.1}$$

uniformly for $[a,b] \subseteq [-1,1]$. The supremum of the left hand side for all interval $[a,b] \subseteq [-1,1]$ is called the *discrepancy* of the zeros of P_n.

From now on we shall assume that all the zeros $x_{i,n}$ of the P_n's are real and lie in $[-1,1]$.

In [6] Erdős and Turán gave a quantitative version of the convergence in (1.1) in the form

$$|(\nu_{P_n} - \omega)([a,b])| \leqslant \frac{8}{\log 3}\sqrt{\frac{\log A_n}{n}}$$

for any interval $[a,b] \subseteq [-1,1]$, where

$$\|P_n\|_{[-1,1]} \leqslant A_n \frac{1}{2^n}. \tag{1.2}$$

This result is sharp up to the constant $8/\log 3$.

In the literature this basic estimate has been widely used in various discrepancy theorems. Erdős [4] proved a sharper estimate under the assumption that the maximum modulus of the polynomial on each interval determined by consecutive zeros is comparable to its maximum on the whole interval $[-1,1]$. Later Erdős and Turán [5] proved the analogue of the above result for the case when the norm is considered on the unit circle. (In such situations one gets discrepancy for the distribution of the arguments of the zeros.)

Returning to the real case, in a recent breaktrough H.-P. Blatt [3] noticed that if we know that all zeros of P_n are simple, then the Erdős–Turán estimate may be strengthened. He assumed a lower bound for the derivative $|P_n'(x_{i,n})|$ at the zeros of P_n, namely

$$|P_n'(x_{i,n})| \geqslant \frac{1}{B_n}\frac{1}{2^n}, \quad 1 \leqslant i \leqslant n. \tag{1.3}$$

He proved

THEOREM A. *Let P_n be monic polynomials with zeros in $[-1,1]$ satisfying the conditions* (1.2) *and* (1.3). *Then there exists a constant C (independent of n) such that*

$$|(\nu_{P_n} - \omega)([a,b])| \leqslant C\frac{\log C_n}{n}\log n \tag{1.4}$$

for any interval $[a,b] \subset [-1,1]$, where

$$C_n = \max(A_n, B_n, n).$$

For two remarkable applications of this theorem concerning zeros of orthogonal polynomials and Kadec-type distribution of extremal points of best polynomial approximation see [3]. The point is that in these applications (and probably in many other ones) the additional assumption (1.3) is automatically satisfied, thus with no additional work we can get a remarkable improvement on the Erdős–Turán estimate.

H.-P. Blatt has also given an example which shows that in some cases his estimate is not very far from the best possible one, although in that example the additional log term on the right is missing. Let us also note that in certain cases this log factor places Blatt's result into a different category than the Erdős–Turán one, namely we do not get (1.1) from it. For example, if we know that $C_n \leqslant \exp(\varepsilon_n n)$, then (1.4) gives for the discrepancy only the estimate $O(\varepsilon_n \log n)$, although we know from (1.1) that this discrepancy tends to zero together with ε_n. However, we shall also see that in other ranges of C_n (namely if $C_n = O(\exp(O(n^\alpha)))$ with $\alpha < 1$) Blatt's estimate is sharp.

In this paper our aim is to determine the best possible estimate for the discrepancy under the conditions of Theorem A. Since our estimate will be best possible, it will be continuous in the sense that it gives back (1.1) (as well as (1.4)). The methods reach beyond the theorems presented here, but we shall not pursue the most general form of our results.

THEOREM 1.1. *With the assumptions and notations of Theorem* A *we have*

$$|(\nu_{P_n} - \omega)([a,b])| \leqslant C \frac{\log C_n}{n} \log \frac{n}{\log C_n}. \tag{1.5}$$

for any interval $[a,b] \subset [-1,1]$.

Here C is an absolute constant. Note that in the case $C_n \leqslant \exp(\varepsilon_n n)$ discussed above this gives the rate $\varepsilon_n \log 1/\varepsilon_n$ for the discrepancy, which tends to zero together with ε_n.

Of course, in Theorem 1.1 one has to restrict C_n to, say, $C_n \leqslant e^{n/2}$, for otherwise nothing can be said about the distribution of the zeros. Actually, only the case $C_n = e^{o(n)}$ is interesting. Note also, that $C_n \geqslant n$ is always satisfied.

Theorem 1.1 is best possible.

THEOREM 1.2. *Let* $\{C_n\}$ *be an arbitrary sequence with the property that* $n \leqslant C_n \leqslant e^{n/2}$. *Then there are monic polynomials* P_n *of corresponding degree* $n = 1, 2, \ldots$ *such that*

$$\|P_n\|_{[-1,1]} \leqslant C_n \frac{1}{2^n}, \tag{1.6}$$

for every zero $x_{i,n}$ *of* P_n

$$|P_n'(x_{i,n})| \geqslant \frac{1}{C_n} \frac{1}{2^n}, \quad 1 \leqslant i \leqslant n, \tag{1.7}$$

and such that for some intervals $[a_n, b_n]$ *of* $[-1,1]$ *the estimate*

$$|(\nu_{P_n}-\omega)([a_n,b_n])| \geqslant c\frac{\log C_n}{n}\log\frac{n}{\log C_n} \tag{1.8}$$

holds with some positive c *independent of* n.

Exactly as in [3] we shall get Theorem 1.1 by reformulating it in terms of logarithmic potentials, and then prove a discrepancy theorem for potentials. Since this reformulation is an integral part of the proof and it is not long, for the sake of completeness we copy the argument here from [3].

Let $G(z)$ denote Green's function of $\overline{\mathbf{C}}\setminus[-1,1]$ with pole at infinity, i.e. $G(z)=\log|z+\sqrt{z^2-1}|$, where we take that branch of $\sqrt{z}$ which is positive for positive z. Bernstein's inequality together with (1.2) yields

$$\frac{1}{n}\log|P_n(z)|-G(z)-\log\frac{1}{2}\leqslant\frac{\log A_n}{n}\quad\text{for all } z\in\mathbf{C}. \tag{1.9}$$

We also need a matching lower estimate on the left hand side. Lagrange's interpolation formula shows that

$$1=\sum_{i=1}^{n}\frac{P_n(z)}{P_n'(x_{i,n})(z-x_{i,n})}.$$

For $z\notin[-1,1]$ let $d(z)$ denote the distance from the point z to the interval $[-1,1]$. Then, the preceding inequality yields

$$1\leqslant n\frac{|P_n(z)|}{d(z)}B_n2^n,$$

i.e.

$$|P_n(z)|\geqslant\frac{1}{n}\frac{d(z)}{B_n}\frac{1}{2^n}.$$

Let $\Gamma_\varkappa=\{z\in\mathbf{C}\,|\,G(z)=\log\varkappa\}$, $\varkappa>1$ be a level curve of the Green's function $G(z)$. Then $\Gamma_\varkappa$ is an ellipse with foci at ±1 and major axis $\varkappa+1/\varkappa$. Hence,

$$\inf_{z\in\Gamma_\varkappa} d(z)=\frac{1}{2}\left(\varkappa+\frac{1}{\varkappa}\right)-1.$$

Choosing

$$\varkappa=\varkappa_n:=1+n^{-12}$$

in the last inequality leads to

$$\frac{1}{n}\log|P_n(z)|-G(z)-\log\frac{1}{2}\geqslant-d\frac{\log C_n}{n} \tag{1.10}$$

for $z\in\Gamma_{\varkappa_n}$, where $d>0$ is an absolute constant independent of n. The minimum principle for harmonic functions shows that (1.10) is actually satisfied for all z with $G(z)\geqslant\log\varkappa_n$. (1.9) and (1.10) together show that

$$\left|\frac{1}{n}\log|P_n(z)|-G(z)-\log\frac{1}{2}\right|\leqslant D\frac{\log C_n}{n} \tag{1.11}$$

for all z where $G(z)\geqslant\log\varkappa_n$.

Now we are going to rewrite this inequality in potential theoretical form. If μ is a Borel measure of compact support on $\mathbf{C}$, then its *logarithmic potential* is defined as

$$U^\mu(z)=\int\log\frac{1}{|z-t|}\,d\mu(t).$$

Since $-(1/n)\log|P_n(z)|$ is the logarithmic potential $U^{\nu_{P_n}}$ of the measure ν_{P_n}, and $-G(z)-\log\frac{1}{2}$ is the logarithmic potential $U^\omega(z)$ of the arcsine distribution ω, (1.11) can be written as

$$|U^{\nu_{P_n}}(z)-U^\omega(z)|\leqslant D\frac{\log C_n}{n}$$

for all z with $G(z)\geqslant\log\varkappa_n$. Now Theorem 1.1 follows from the last estimate and from the next theorem if we set $\sigma=\nu_{P_n}-\omega$, $\varepsilon=(\log C_n)/n$, and $\Delta=\frac{1}{2}$ in it.

THEOREM 1.3. *Let $\sigma=\sigma_+-\sigma_-$ be a signed measure such that $\sigma_\pm$ are probability measures on $[-1,1]$ with the property that for some $0<\Delta\leqslant1$ the estimate*

$$\sigma_-(E)\leqslant C_0 m(E)^\Delta \tag{1.12}$$

holds for every interval E, where m denotes the linear Lebesgue measure. Then if with $L=5/\Delta+2$ we have

$$|U^\sigma(z)|\leqslant C_1\varepsilon$$

for every z with

$$\operatorname{dist}(z,[-1,1])\geqslant\varepsilon^L,$$

then

$$|\sigma([a,b])|\leqslant C_2\varepsilon\log\frac{1}{\varepsilon}$$

holds for every interval $[a,b]$, where the constant C_2 depends exclusively on C_0, C_1 and Δ.

The outline of the paper is as follows. In the next section we prove Theorem 1.3 with the help of a theorem on condenser potentials, which in turn will be proven in Section 4. The proof of Theorem 1.2 will be given in Section 3. The proof is distinctly different in the cases when $C_n\geqslant n^4$ and $C_n<n^4$. In the former case we can use weighted potentials with a discretization technique. This will be done in subsection 3.1. In the second case the theorem is proved by moving certain zeros of the Chebyshev polynomials, the details of which will be given in subsection 3.2.

2. Proof of Theorem 1.3

The main idea of the proof can be explained as follows. Let $[a,b]\subseteq[-1,1]$ be arbitrary. Suppose we had a signed measure μ of compact support lying at a distance $\geqslant\varepsilon^L$ from $[-1,1]$ such that $\|\mu\|\leqslant 2$, and if

$$\chi_{[a,b]}:=\begin{cases}1 & \text{if } x\in[a,b]\\ 0 & \text{otherwise}\end{cases}$$

is the characteristic function of the interval $[a,b]$, then with

$$\tau_\varepsilon=\frac{1}{\log 1/\varepsilon}$$

and some constant c we have

$$U^\mu(x)=c+\tau_\varepsilon\chi_{[a,b]}$$

for all $x\in[-1,1]$. Then, using Fubini's theorem, we could write

$$2C_1\varepsilon\geqslant\left|\int U^\sigma\,d\mu\right|=\left|\int U^\mu\,d\sigma\right|=\left|\int_a^b\tau_\varepsilon\,d\sigma\right|=\tau_\varepsilon|\sigma([a,b])|,$$

from which

$$|\sigma([a,b])|\leqslant 2C_1\varepsilon\log\frac{1}{\varepsilon}$$

follows immediately, and this is what we need to prove.

Unfortunately, the signed measure μ with the above properties does not exist. We can, however, get a measure, the properties of which will be close to the above ones; hence this measure can serve as a substitute. The rest of this section is devoted to the construction of that measure and to showing that the weaker properties it will possess are still sufficient for our purposes.

More precisely, we will construct a μ with the following properties.

LEMMA 2.1. *Let L and Δ be the numbers from Theorem 1.3, and let $[a,b]\subseteq[-1,1]$, and $0<\varepsilon<\frac{1}{2}$ be arbitrary with $b-a\geqslant 2\varepsilon^{1/\Delta}$. Then there is a signed measure $\mu=\mu_{\varepsilon,a,b}$ and two numbers $c=c_{\varepsilon,a,b}$ and $\tau=\tau_{\varepsilon,a,b}$ with the following properties:*

(1) $\operatorname{supp}(\mu)$ *is compact and is at distance* $\geqslant\varepsilon^L$ *from* $[-1,1]$,

(2) $\|\mu\|\leqslant 2$,

(3) $c\leqslant U^\mu(x)\leqslant c+\tau$ *for every* $x\in[-1,1]$,

(4) $|U^\mu(x)-\tau\chi_{[a,b]}(x)-c|\leqslant C_3\varepsilon$ *for* $x\in[a,b]$ *and* $x\in[-1,1]\setminus(a-\varepsilon^{2/\Delta},b+\varepsilon^{2/\Delta})$,

(5) $1/C_3\leqslant\tau\log 1/\varepsilon\leqslant C_3$.

Furthermore, here C_3 is an absolute constant.

The proof of Lemma 2.1 is quite long and involves explicit construction of some extremal measures, but before we set out to prove it we show how Theorem 1.3 can be obtained from it.

Proof of Theorem 1.3. First of all we simplify the problem, namely it is enough to prove the inequality

$$\sigma([a,b]) \leqslant C_2\varepsilon \log \frac{1}{\varepsilon} \tag{2.1}$$

with some constant C_2 for all $[a,b]\subseteq[-1,1]$. In fact, then by applying (2.1) to the intervals $[-1,a]$ and $[b,1]$ instead of $[a,b]$ and using that $\sigma([-1,1])=0$, we obtain the counterpart

$$\sigma([a,b]) \geqslant -2C_2\varepsilon \log \frac{1}{\varepsilon}$$

of (2.1), and with (2.1) this proves the claim.

Next we we observe that we may assume without loss of generality that $b-a\geqslant 2\varepsilon^{1/\Delta}$. In fact, suppose (2.1) has been verified in this case. Then if a and b are closer than $2\varepsilon^{1/\Delta}$, then we can enlarge $[a,b]$ to have length $2\varepsilon^{1/\Delta}$. If the enlarged interval is $[a',b']$, then we can apply (2.1) to $[a',b']$ instead of $[a,b]$ to get

$$\sigma_+([a,b]) \leqslant \sigma_+([a',b']) \leqslant \sigma_-([a',b'])+C_2\varepsilon\log\frac{1}{\varepsilon} \leqslant 2C_0\varepsilon+C_2\varepsilon\log\frac{1}{\varepsilon},$$

where, in the last step we applied (1.12). This proves (2.1) for all $[a,b]$ (with a possibly bigger constant).

Now we can apply Lemma 2.1. With the signed measure μ obtained there and with $\delta:=\varepsilon^{2/\Delta}$ we get exactly as in the sketch above

$$\begin{aligned} 2C_1\varepsilon &\geqslant \int U^\sigma\,d\mu = \int U^\mu\,d\sigma = \int (U^\mu - c)\,d\sigma \\ &= \int_{[a,b]} + \int_{[-1,1]\setminus[a-\delta,b+\delta]} + \int_{(a-\delta,a)\cup(b,b+\delta)}, \end{aligned} \tag{2.2}$$

where the domain of the last integral has to be appropriately adjusted if $a-\delta<-1$ or $b+\delta>1$. Using properties of U^μ we can continue this inequality as

$$2C_1\varepsilon \geqslant \int_{[a,b]} \tau\,d\sigma - \int_{(a-\delta,a)\cup(b,b+\delta)} \tau\,d\sigma_- - 2C_3\varepsilon \geqslant \tau\sigma[a,b] - 2\tau C_0\delta^\Delta - 2C_3\varepsilon,$$

where we have used (1.12) again. Since $\delta^\Delta=\varepsilon^2$, and by property (5) of the measure μ

$$\tau \sim \frac{1}{\log 1/\varepsilon},$$

we immediately arrive at (2.1) from this estimate. □

2.1. Proof of Lemma 2.1

In the proof of Lemma 2.1 we shall use the so-called condenser potentials.

Let Σ_1 and Σ_2 be disjoint compact sets on $\mathbf{C}$ of positive capacity (the conductors) such that with $\Sigma=\Sigma_1\cup\Sigma_2$ the complement $\mathbf{C}\setminus\Sigma$ is connected. Such a pair (Σ_1,Σ_2) is called a *condenser*. To each $j=1,2$ we assign a sign $\varepsilon_j=\pm1$ (the sign of the charge), and let us agree that Σ_1 is the positive 'plate', i.e. $\varepsilon_1=+1$ and $\varepsilon_2=-1$. We want to minimize the energy

$$\iint \log\frac{1}{|z-t|}\,d\mu(z)\,d\mu(t)$$

for all signed measures of the form $\mu=\mu_1-\mu_2$, where μ_j is a positive measure on Σ_j of total mass 1.

There is a unique extremal signed measure $\mu=\mu^*$ for which the infimum is attained. We call μ^* *the equilibrium measure* for the condenser (Σ_1,Σ_2). The logarithmic potential of this extremal measure has the properties that there exist two constants F_1 and F_2 such that

$$-F_2\leqslant U^{\mu^*}(z)\leqslant F_1$$

for every $z\in\mathbf{C}$,

$$U^{\mu^*}(z)=F_1 \quad \text{for every } z\in\Sigma_1 \tag{2.3}$$

with the exception of a set of zero capacity (see Section 4; in what follows we shall abbreviate this fact as 'for quasi-every $z\in\Sigma_1$'), and

$$U^{\mu^*}(z)=-F_2 \quad \text{for quasi-every } z\in\Sigma_2. \tag{2.4}$$

Furthermore, if $\mathbf{C}\setminus\Sigma$ is regular with respect to the Dirichlet problem, then the last two equalities hold true for *every* $z\in\Sigma_1$ respectively $z\in\Sigma_2$. For all these results see [1] and [7].

Now we shall need to explicitly determine the extremal measure and the constant F_1+F_2 when Σ_1 and Σ_2 consist of finitely many intervals on the real line. Thus, let $\Sigma=\Sigma_1\cup\Sigma_2$ be the union of some intervals $[a_j,b_j]$, $b_j<a_{j+1}$, $j=1,\dots,m$.

The following theorem is of independent interest, and will be proved in Section 4.

THEOREM 2.2. *Let Σ_1 and Σ_2 consist of intervals on the real line, $\Sigma=\Sigma_1\cup\Sigma_2=\bigcup_{j=1}^m[a_j,b_j]$. Then*

$$F_1+F_2=\left|\int_{b_{j_0}}^{a_{j_0+1}}\frac{P_{m-2}(t)}{\sqrt{R(t)}}\,dt\right|, \tag{2.5}$$

and

$$d\mu^*(t)=\frac{P_{m-2}(t)}{-\pi i\sqrt{R(t)}}\,dt,$$

where

$$R(z)=\prod_{k=1}^{m}(z-a_k)(z-b_k),$$

j_0 is an index such that b_{j_0} and a_{j_0+1} belong to different sets Σ_1 and Σ_2, and where the coefficients of the polynomial

$$P_{m-2}(t)=c_{m-2}t^{m-2}+\ldots+c_0$$

are the solutions of the linear system of equations

$$\left(\int_{b_j}^{a_{j+1}}+\int_{b_{l(j)}}^{a_{l(j)+1}}\right)\frac{P_{m-2}(t)}{\sqrt{R(t)}}\,dt=0,\quad j\neq j_1,j_2,$$

$$\int_{\Sigma_1}\frac{P_{m-2}(t)}{-\pi i\sqrt{R(t)}}\,dt=1.$$

In this system for $1\leqslant j\leqslant m$ the number $l(j)\geqslant j$ denotes the smallest index for which $[a_j,b_j]$ and $[a_{l(j)+1},b_{l(j)+1}]$ belong to the same set Σ_1 or Σ_2, and j_1 and j_2 denote those two j's for which such an $l(j)$ does not exist. This system of equations has a unique solution.

Above we used that branch of the square root that is positive for positive z. We also note that the system of equations in the theorem is a real system for the coefficients of P_{m-2} hence, P_{m-2} is a real polynomial.

We shall need the following corollary of Theorem 2.2.

COROLLARY 2.3. *In Theorem 2.2 let*

$$\Sigma_1=[-\alpha,\alpha]\quad\textit{and}\quad\Sigma_2=[-2-\alpha,-\alpha-\eta]\cup[\alpha+\eta,2+\alpha]$$

with some $0<\eta\leqslant\alpha^2\leqslant\frac{1}{4}$. Then

$$F_1+F_2\sim\frac{1}{\log 1/\eta},\tag{2.6}$$

where $\sim$ means that the ratio of the two sides is bounded away from 0 and ∞ by two absolute constants. Furthermore, the signed measure μ^* is absolutely continuous with respect to Lebesgue measure, and if we set

$$d\mu^*(t)=v(t)\,dt,$$

then for $j=1,2,3$ and $t\in[a_j,b_j]$

$$|v(t)|\leqslant\frac{1}{\eta}v_j(t)=:\frac{1}{\eta}\frac{1}{\pi}\frac{1}{\sqrt{(t-a_j)(b_j-t)}}.\tag{2.7}$$

Recall that here $[a_j, b_j]$ denote the intervals of $\Sigma = \Sigma_1 \cup \Sigma_2$.

Proof of Corollary 2.3. According to Theorem 2.2 we have to solve the system of equations

$$\left(\int_{-\alpha-\eta}^{-\alpha} + \int_{\alpha}^{\alpha+\eta}\right) \frac{c_1 t + c_0}{\sqrt{(t^2-(2+\alpha)^2)(t^2-(\alpha+\eta)^2)(t^2-\alpha^2)}}\, dt = 0 \tag{2.8}$$

$$\int_{-\alpha}^{\alpha} \frac{1}{\pi i} \frac{c_1 t + c_0}{\sqrt{(t^2-(2+\alpha)^2)(t^2-(\alpha+\eta)^2)(t^2-\alpha^2)}}\, dt = 1 \tag{2.9}$$

(we have incorporated the $-$ sign from $-i\pi$ in (2.9) into c_1 and c_0 in order to get a positive c_0 below). Since the denominator in (2.8) takes opposite sign on $[-\alpha-\eta, -\alpha]$ and $[\alpha, \alpha+\eta]$, we get that c_1 must be zero. Then c_0 is obtained from the second equation:

$$c_0 = 1 \Big/ \int_{-\alpha}^{\alpha} \frac{1}{\pi} \frac{1}{\sqrt{((2+\alpha)^2-t^2)((\alpha+\eta)^2-t^2)(\alpha^2-t^2)}}\, dt.$$

This easily yields

$$c_0 \sim \frac{\alpha}{\log \alpha/\eta} \sim \frac{\alpha}{\log 1/\eta}.$$

But

$$\begin{aligned} F_1 + F_2 &= \left| \int_{\alpha}^{\alpha+\eta} \frac{c_1 t + c_0}{\sqrt{(t^2-(2+\alpha)^2)(t^2-(\alpha+\eta)^2)(t^2-\alpha^2)}}\, dt \right| \\ &\sim \frac{\alpha}{\log 1/\eta} \int_{\alpha}^{\alpha+\eta} \frac{1}{\sqrt{((2+\alpha)^2-t^2)((\alpha+\eta)^2-t^2)(t^2-\alpha^2)}}\, dt, \end{aligned}$$

and if we use that

$$\int_{\alpha}^{\alpha+\eta} \frac{1}{\pi} \frac{1}{\sqrt{(t-\alpha)(\alpha+\eta-t)}}\, dt = 1,$$

we get (2.6).

Since

$$v(t) = \frac{1}{\pi i} \frac{c_0}{\sqrt{(t^2-(2+\alpha)^2)(t^2-(\alpha+\eta)^2)(t^2-\alpha^2)}}$$

if $t \in \Sigma = [-2-\alpha, -\alpha-\eta] \cup [\alpha+\eta, \alpha+2] \cup [-\alpha, \alpha]$, while

$$v_j(t) = \frac{1}{\pi} \frac{1}{\sqrt{(t-a_j)(b_j-t)}},$$

(2.7) also easily follows. $\square$

LEMMA 2.4. *With the assumptions and notations of Corollary* 2.3 *we have for the potential of* μ^* *the estimate*

$$|U^{\mu^*}(x)-U^{\mu^*}(x\pm i\xi)|\leqslant \frac{6}{\eta}\left(\frac{\xi}{\alpha}\right)^{1/2} \tag{2.10}$$

for every $x\in\mathbf{R}$ *and* $0<\xi\leqslant\alpha/2$.

Proof of Lemma 2.4. Using the second part of Corollary 2.3 we can write

$$\begin{aligned}|U^{\mu^*}(x)-U^{\mu^*}(x\pm i\xi)| &= \left|\int \log\left|\frac{x-t\pm i\xi}{x-t}\right| d\mu^*(t)\right|\\ &\leqslant \int \log\left|\frac{x-t\pm i\xi}{x-t}\right| d|\mu^*|(t)\\ &\leqslant \frac{1}{\eta}\int \log\left|\frac{x-t\pm i\xi}{x-t}\right|(v_1(t)+v_2(t)+v_3(t))\,dt\\ &= \frac{1}{\eta}\sum_{j=1}^{3}|U^{v_j}(x)-U^{v_j}(x\pm i\xi)|,\end{aligned}$$

where we have used the self explanatory notation for the potential of a measure given by its density function. But with

$$v(t)=\frac{1}{\pi}\frac{1}{\sqrt{1-t^2}},$$

we have

$$U^{v_j}(z)=U^{v}(y)+\log\frac{2}{b_j-a_j},$$

where z and y are connected by the formula

$$y=\left(z-\frac{a_j+b_j}{2}\right)\frac{2}{b_j-a_j},$$

hence the last sum is at most as large as

$$\frac{3}{\eta}\max_{y\in\mathbf{R}}\left|U^{v}(y)-U^{v}\left(y\pm 2i\frac{\xi}{b_j-a_j}\right)\right|. \tag{2.11}$$

Here

$$\left|\frac{2\xi}{b_j-a_j}\right|\leqslant\frac{2\xi}{2\alpha}\leqslant\frac{1}{2} \tag{2.12}$$

and

$$U^{v}(z)=-\log\left|z+\sqrt{z^2-1}\right|+\log 2.$$

One can easily prove that for fixed real ζ, $|\zeta|\leqslant\frac{1}{2}$ the function

$$|U^{v}(y)-U^{v}(y\pm i\zeta)|$$

attains its maximum at $y=\pm 1$ and this maximum is at most $2\sqrt{|\zeta|}$.

Substituting this into (2.11) we arrive at (2.10) (cf. also (2.12)). □

Finally we can turn to the

Proof of Lemma 2.1. Suppose first that $[a,b]$ is symmetric on the origin, say $[a,b]=[-\alpha,\alpha]$. Then we set $\eta=\varepsilon^{2/\Delta}$ and choose μ to be equal to the translation of the measure μ^* from the previous two lemmas by $i\varepsilon^L$. With $c=-F_2$ and $\tau=F_1+F_2=F_1-(-F_2)$ the first two properties in Lemma 2.1 follow from the construction, the third one follows from the fact that for every z the potential U^{μ^*} lies in between F_1 and $-F_2$ (see the discussion before Theorem 2.2). Property (4) is a consequence of the properties of U^{μ^*} (see the discussion before Theorem 2.2) and Lemma 2.4 if we also use that by the choice of the parameters we have $\alpha\geqslant\varepsilon^{1/\Delta}$, and so

$$\frac{1}{\eta}\left(\frac{\varepsilon^L}{\alpha}\right)^{1/2}\leqslant\varepsilon.$$

Note that this property (property (4)) actually holds in a wider range, namely for all

$$x\in\Sigma=[-2-\alpha,-\alpha-\eta]\cup[\alpha+\eta,\alpha+2]\cup[-\alpha,\alpha]. \tag{2.13}$$

Finally, the last property was proved in (2.6). These prove Lemma 2.1 in the symmetric case.

If $[a,b]\subseteq[-1,1]$ is arbitrary, then let $[a',b']=[-(b-a)/2,(b-a)/2]$, and let the just constructed signed measure for $[a',b']$ be μ'. Now we choose μ as the translation of the measure μ' by $(a+b)/2$. Since we have verified property (4) in the larger range (2.13), the translation of which (by $(a+b)/2$) certainly covers the set appearing in (4), the signed measure μ satisfies all the requirements. □

3. Proof of Theorem 1.2

The proof is distinctively different in the ranges $C_n\geqslant n^4$ and $C_n<n^4$. We shall separate these two cases below. Of course, the sequence $\{C_n\}$ need not satisfy either $C_n\geqslant n^4$ or $C_n<n^4$ for all n, in which case one has to separate the terms with these two properties, respectively, and apply the two methods below to the appropriate terms.

3.1. Proof of Theorem 1.2 in the case when $C_n\geqslant n^4$

First we need a lemma.

LEMMA 3.1. *For any $x \in \mathbf{R}$ and $0<\eta<\theta$*

$$\left| \int_\eta^\theta \log \left| \frac{x+t}{x-t} \right| \frac{1}{t} \, dt \right| \leqslant 10. \tag{3.1}$$

Proof. By the homogeneity of the integral we can assume without loss of generality that $\eta=1$ and $x \geqslant 0$. Furthermore, the ratio

$$\left| \frac{x+t}{x-t} \right| \tag{3.2}$$

is increasing as x increases on $(0,1)$ for every fixed $t \geqslant 1$, hence we may even assume $x \geqslant 1$. Now we divide the domain of integration in (3.1) into three parts: $(1, x/2)$, $(x/2, 2x)$ and $(2x, \theta)$ with the obvious modifications if $x \leqslant 2$ or $x \geqslant \theta/2$. On the first part we use that (3.2) is at most

$$1+\frac{2t}{x-t} \leqslant 1+\frac{4t}{x},$$

and so the integrand is at most $4/x$, from which the contribution of this part to the left side of (3.1) is at most 2. In a similar manner, on $(2x, \theta)$ we have for (3.2) the upper estimate

$$1+\frac{2x}{t-x} \leqslant 1+\frac{4x}{t},$$

so the contribution of the third integral is also at most 2.

Finally,

$$\left| \int_{x/2}^{2x} \log \left| \frac{x+t}{x-t} \right| \frac{1}{t} \, dt \right| \leqslant \frac{2}{x} \int_{x/2}^{2x} \log \left| \frac{x+t}{x-t} \right| dt = \frac{2}{x} \frac{x}{2} (2.5 \log 2 + 3 \log 3) < 6. \qquad \square$$

With this technical lemma at our hand we can now prove Theorem 1.2 in the case when $C_n \geqslant n^4$.

Consider for an $0<\varepsilon \leqslant e^{-13}$ the function

$$v_\varepsilon(t) := \begin{cases} (t-(1-\varepsilon))^{-1} & \text{if } \varepsilon^{3/2} \leqslant |t-(1-\varepsilon)| \leqslant \varepsilon \\ 0 & \text{otherwise,} \end{cases}$$

and the signed measure ν_ε that it defines:

$$d\nu_\varepsilon(t) = c_\varepsilon v_\varepsilon(t) \, dt,$$

where the normalizing constant c_ε is chosen so that the total variation of ν_ε be 2, i.e.

$$c_\varepsilon = \frac{2}{\log 1/\varepsilon}. \tag{3.3}$$

For $\varepsilon^{3/2}\leqslant|t-(1-\varepsilon)|\leqslant\varepsilon$ we have

$$|\varepsilon c_\varepsilon v_\varepsilon(t)|\leqslant\frac{2\varepsilon}{\log 1/\varepsilon}\frac{1}{\varepsilon^{3/2}}\leqslant\frac{13}{\log 1/\varepsilon}\frac{1}{\pi\sqrt{1-t^2}}\leqslant\frac{1}{\pi\sqrt{1-t^2}},$$

hence the signed measure

$$\mu:=\omega+\varepsilon\nu_\varepsilon$$

is a positive measure of total mass 1 which has density

$$\leqslant\frac{2}{\pi\sqrt{1-t^2}} \tag{3.4}$$

on $[-1,1]$ (recall that ω denotes the arcsine measure). Furthermore, we can immediately get from Lemma 3.1 and the equality $U^\omega(x)=\log 2$ for $x\in[-1,1]$ that for such x

$$|U^\mu(x)-U^\omega(x)|=|U^\omega(x)||\varepsilon U^{\nu_\varepsilon}(x)|=(\log 2)\varepsilon c_\varepsilon\left|\int_{\varepsilon^{3/2}}^{\varepsilon}\log\left|\frac{x+t}{x-t}\right|\frac{1}{t}\,dt\right|\leqslant\frac{20\varepsilon}{\log 1/\varepsilon}. \tag{3.5}$$

Now we shall utilize an idea of E. A. Rahmanov on how to distribute the zeros of a polynomial if we want to get a discretized version of a potential. We need the following quantitative version (see [10, Lemma 6.1]).

For an integer n let

$$-1=y_{0,n}<y_{1,n}<\ldots<y_{n,n}=1$$

be that partition of $[-1,1]$ for which $\mu([y_{j,n},y_{j+1,n}])=1/n$ for all $0\leqslant j\leqslant n-1$. Consider the polynomials

$$P_n(x)=\prod_{j=1}^{n-1}(x-y_{j,n}).$$

Using the monotonicity of the logarithmic function it is not too hard to see (see [10, pp. 40–43]) that if for some constants α and β the inequality

$$\int_{|x-t|\leqslant n^{-\alpha}}|\log|x-t||\,d\mu(t)\leqslant\beta\frac{\log\, n}{n} \tag{3.6}$$

holds, then

$$|P_n(x)|\leqslant n^{\alpha+\beta}\exp(-nU^\mu(x)),\qquad x\in\mathbf{R},$$

and

$$|P_n(x)|\geqslant\tfrac14\exp(-nU^\mu(x))|x-y_{n_x,n}|,$$

where $y_{n_x,n}$ denotes the zero of P_n closest to x.

If the potential U^μ is continuous on $[-1,1]$, then the latter inequality immediately implies

$$|P_n'(y_{j,n})| \geqslant \tfrac{1}{4}\exp(-nU^\mu(y_{j,n}))$$

for every zero $y_{j,n}$ of P_n (actually this is true without the continuity assumption).

In our case the potential U^μ is obviously continuous, furthermore (3.6) holds with $\alpha=\frac{5}{2}$ and $\beta=\frac{1}{2}$ for large n (cf. the estimate (3.4) for the density of μ). On applying (3.5) we can thus write

$$|P_n(x)| \leqslant \exp\left(\frac{20\varepsilon}{\log 1/\varepsilon}n+3\log n\right)\frac{1}{2^n},$$

and for each $j=1,\dots,n$

$$|P_n'(y_{j,n})| \geqslant \exp\left(-\frac{20\varepsilon}{\log 1/\varepsilon}n-2\right)\frac{1}{2^n}.$$

Now if $C_n \geqslant n^4$ is given, then we define $\varepsilon=\varepsilon_n$ by the equality

$$\log C_n = \frac{20\varepsilon}{\log 1/\varepsilon}n+3\log n. \tag{3.7}$$

Since $\log C_n - 3\log n \geqslant \frac{1}{4}\log C_n$, we can deduce that

$$\varepsilon_n \sim \frac{\log C_n}{n}\log\frac{n}{\log C_n}. \tag{3.8}$$

Now if we assume that this ε satisfies $\varepsilon \leqslant e^{-13}$, then we can apply all of our estimates so far to deduce

$$\|P_n\|_{[-1,1]} \leqslant \frac{C_n}{2^n}$$

and

$$|P_n'(y_{j,n})| \geqslant \frac{1}{C_n}\frac{1}{2^n}, \quad j=1,\dots,n.$$

But the polynomial P_n has $[n\mu([1-2\varepsilon,1-\varepsilon])]$ plus minus one zeros on the interval $[1-2\varepsilon,1-\varepsilon]$, hence for the discrepancy of its zeros we have

$$\begin{aligned}|(\nu_{P_n}-\omega)([1-2\varepsilon,1-\varepsilon])| &\geqslant |\varepsilon\nu_\varepsilon([1-2\varepsilon,1-\varepsilon])|-\frac{1}{n}\\ &= \frac{\varepsilon\|\nu_\varepsilon\|}{2}-\frac{1}{n} = \varepsilon-\frac{1}{n} \geqslant c\frac{\log C_n}{n}\log\frac{n}{\log C_n},\end{aligned} \tag{3.9}$$

with some absolute constant $c>0$, where at the last step we used (3.8).

These inequalities prove Theorem 1.2 in the case when $C_n \geqslant n^4$ and the $\varepsilon=\varepsilon_n$ from (3.7) satisfies $\varepsilon \leqslant e^{-13}$. If the latter condition is not satisfied, then all we have to do to copy the above argument is to choose $\varepsilon=e^{-13}$, for which the last inequality in (3.9) is still valid with some positive c. □

3.2. Proof of Theorem 1.2 in the case $C_n < n^4$

Let
$$T_n(x) = \frac{1}{2^{n-1}} \cos(n \arccos x)$$
be the monic Chebyshev polynomials. T_n has the zeros $\cos((2k-1)\pi/2n)$, $k=1,\dots,n$ which are the projections onto $[-1,1]$ of the equidistant points $\exp((2k-1)\pi i/2n)$, $k=1,2,\dots,2n$ lying on the unit circumference. This easily implies that the discrepancy of T_n is at most $1/n$. We shall construct our P_n by moving some zeros of T_n.

For an n define
$$\varepsilon = \frac{\log^2 n}{n},$$
and $a=1-2\varepsilon$. The point a will be the center of the zero movements, we shall, roughly speaking, reflect some zeros of T_n, distributed according to a logarithmic scale, onto a.

To this end we choose a large constant C that will be specified later (we shall see that actually any $C>80$ will do the job), and with it we define some numbers $\xi_0, \dots, \xi_J$ as follows: we set $\xi_0 = \varepsilon^{4/3}$, and for other j's we define ξ_{j+1} in terms of ξ_j via the formula
$$\int_{\xi_j}^{\xi_{j+1}} \frac{1}{t}\, dt = \frac{C}{\log n}, \tag{3.10}$$
and let J be the largest number for which $\xi_{J+1} \leqslant \varepsilon$. Then
$$J \sim \frac{\log^2 n}{C}.$$

Now let x_j be the nearest zero of T_n to $a-\xi_j$ and y_j the nearest zero of T_n' (note the prime!) to $a+\xi_j$, and form the rational function
$$r_J(t) = \prod_{j=0}^{J} \frac{t-y_j}{t-x_j}.$$
We transform the zeros of T_n with the help of r_J, namely we set
$$P_n(t) := T_n(t) r_J(t).$$
We claim that for large enough C (to be chosen below) and large n the following estimates hold: for every $t \in [-1,1]$
$$|P_n(t)|_{[-1,1]} \leqslant \frac{n^{3/4}}{2^n}, \tag{3.11}$$
and for every zero θ of P_n
$$|P_n'(\theta)| \geqslant \frac{1}{n^{3/4}} \frac{1}{2^n}. \tag{3.12}$$

From these Theorem 1.2 immediately follows in the case $n \leqslant C_n \leqslant n^4$. In fact, by our construction we have removed $J+1$ zeros of T_n from the interval $[1-3\varepsilon, 1-2\varepsilon]$, hence the discrepancy of P_n is at least as large as

$$\frac{J}{n} \geqslant c \frac{\log^2 n}{n},$$

which is

$$\geqslant c \frac{\log C_n}{n} \log \frac{n}{\log C_n}$$

in the present case.

Thus, it remains to prove (3.11) and (3.12). We start with (3.11). In the proof below D will denote absolute constants that may vary from line to line, but C is one and the same throughout the proof.

Proof of (3.11). First of all, the definition of the ξ_j's gives

$$\xi_{j+1} - \xi_j = \xi_j (e^{C/\log n} - 1) = \xi_j \frac{C}{\log n} + O\left(\frac{\xi_j}{\log^2 n}\right). \tag{3.13}$$

This is much larger than the largest distance between consecutive zeros of T_n and T_n' on $[1-4\varepsilon, 1-\varepsilon/2]$ which is

$$\sim \frac{\sqrt{\varepsilon}}{n} \leqslant D \frac{\log n}{n^{3/2}}.$$

Thus, we immediately get the estimates

$$|y_j - (a+\xi_j)| \leqslant D \frac{\log n}{n^{3/2}}$$

and

$$|x_j - (a-\xi_j)| \leqslant D \frac{\log n}{n^{3/2}}.$$

Since every ratio

$$\left|\frac{t-y_j}{t-x_j}\right|, \quad j = 0, 1, \ldots, J, \tag{3.14}$$

is increasing on the interval $[-1, x_J]$, and the polynomial T_n attains its maximum on $[x_{J+1}, x_J]$, we can restrict our attention to $t \in [x_{J+1}, 1]$. It is also immediate that for $t \in [a+\varepsilon^{4/3}/2, 1]$ the rational function $r_J(t)$ is at most 1 in absolute value, so this leaves us to consider the case $t \in [x_{J+1}, a+\varepsilon^{4/3}/2]$. We shall prove (3.11) for $t \in [x_J, x_0]$ because the consideration is the same (actually somewhat simpler) for $t \in [x_{J+1}, x_J]$ or $t \in [x_0, a+\varepsilon^{4/3}/2]$.

Thus, let $x_{j_0+1} \leqslant t \leqslant x_{j_0}$ for some $j=0,\dots,J-1$. We separate the j_0th and (j_0+1)st terms in r_J, and first estimate the products of the terms with index smaller than j_0 and then with index greater than j_0, respectively.

We write

$$\prod_{j=0}^{j_0-1} \frac{t-y_j}{t-x_j} = \prod_{j=0}^{j_0-1} \frac{1-(y_j-(a+\xi_j))/(t-(a+\xi_j))}{1-(x_j-(a-\xi_j))/(t-(a-\xi_j))} \cdot \frac{t-(a+\xi_j)}{t-(a-\xi_j)} =: \Pi_1 \Pi_2.$$

Here we have for the denominators

$$|t-(a\pm\xi_j)| \geqslant \tfrac{1}{2}|\xi_{j_0}-\xi_{j_0-1}| \geqslant \xi_{j_0-1}\frac{C}{3\log n} \geqslant \frac{C}{3}\frac{\log^{5/3} n}{n^{4/3}}, \tag{3.15}$$

and so

$$\left|\frac{y_j-(a+\xi_j)}{t-(a+\xi_j)}\right| \leqslant \frac{Dn^{-3/2}\log n}{(C/3)n^{-4/3}\log^{5/3} n} \leqslant \frac{D}{C} n^{-1/6}$$

and

$$\left|\frac{x_j-(a-\xi_j)}{t-(a-\xi_j)}\right| \leqslant \frac{Dn^{-3/2}\log n}{(C/3)n^{-4/3}\log^{5/3} n} \leqslant \frac{D}{C} n^{-1/6}.$$

These yield

$$|\Pi_1| \leqslant \left(\frac{1+(D/C)n^{-1/6}}{1-(D/C)n^{-1/6}}\right)^J \leqslant \exp\left(\frac{D}{C^2}\frac{\log^2 n}{n^{1/6}}\right). \tag{3.16}$$

In the estimate of Π_2 we shall make use of Lemma 3.1. Using the monotonicity of the ratios (3.14) we can write with $\tau := t-a < 0$

$$\begin{aligned}\left|\log \prod_{j=0}^{j_0-1}\left|\frac{\tau-\xi_j}{\tau+\xi_j}\right|\right| &= \frac{\log n}{C}\sum_{j=0}^{j_0-1} \log\left|\frac{\tau-\xi_j}{\tau+\xi_j}\right| \int_{\xi_j}^{\xi_{j+1}} \frac{1}{u}\,du \\ &\leqslant \frac{\log n}{C}\int_{\xi_0}^{\xi_{j_0}} \log\left|\frac{\tau-u}{\tau+u}\right| \frac{1}{u}\,du \leqslant \frac{10\log n}{C},\end{aligned} \tag{3.17}$$

where, at the last step we used Lemma 3.1. From (3.16) and (3.17) we finally arrive at

$$\prod_{j=0}^{j_0-1}\left|\frac{t-y_j}{t-x_j}\right| = |\Pi_1||\Pi_2| \leqslant \exp\left(\frac{D}{C^2}\frac{\log^2 n}{n^{1/6}} + \frac{10\log n}{C}\right). \tag{3.18}$$

Since

$$\begin{aligned}\left|\log \prod_{j=j_0+2}^{J}\left|\frac{\tau-\xi_j}{\tau+\xi_j}\right|\right| &= \frac{\log n}{C}\sum_{j=j_0+2}^{J} \log\left|\frac{\tau-\xi_j}{\tau+\xi_j}\right| \int_{\xi_{j-1}}^{\xi_j} \frac{1}{u}\,du \\ &\leqslant \frac{\log n}{C}\int_{\xi_{j_0+1}}^{\xi_J} \log\left|\frac{\tau-u}{\tau+u}\right| \frac{1}{u}\,du \leqslant \frac{10\log n}{C},\end{aligned}$$

we get similarly

$$\prod_{j=j_0+2}^{J} \left|\frac{t-y_j}{t-x_j}\right| \leqslant \exp\left(\frac{D}{C^2}\frac{\log^2 n}{n^{1/6}}+\frac{10\log n}{C}\right). \tag{3.19}$$

As for the remaining two factors

$$\frac{t-y_j}{t-x_j}$$

in r_J with $j=j_0$ and $j=j_0+1$, we note that only one of them can be really large. In fact, t lies either closer to x_{j_0} or closer to x_{j_0+1}. Consider the first case, the other one is similar. Then for the second term we get from (3.13)

$$\left|\frac{t-y_j}{t-x_j}\right| \leqslant \frac{3\xi_{j_0}}{\xi_{j_0}C/(3\log n)} \leqslant \frac{9}{C}\log n. \tag{3.20}$$

Finally, for the other term with $j=j_0$ we get by the mean value theorem

$$\left|T_n(t)\frac{t-y_j}{t-x_j}\right| = \left|\frac{T_n(t)-T_n(x_{j_0})}{t-x_{j_0}}\right| |t-y_{j_0}| = |T_n'(\theta)||t-y_{j_0}|$$

with some $\theta\in[1-3\varepsilon, 1-\varepsilon]$. We can explicitly calculate the derivative of T_n, and with the inequality $|t-y_{j_0}|\leqslant 2\varepsilon$ we finally arrive at

$$\left|T_n(t)\frac{t-y_j}{t-x_j}\right| \leqslant \frac{n}{2^{n-1}}\frac{1}{\sqrt{1-(1-\varepsilon)^2}}2\varepsilon \leqslant \frac{4\sqrt{n}\log n}{2^n}. \tag{3.21}$$

From (3.18)–(3.21) it follows that

$$|P_n(t)| = |T_n(t)r_n(t)| \leqslant \frac{4\sqrt{n}\log n}{2^n}\frac{9\log n}{C}\exp\left(\frac{D}{C^2}\frac{\log^2 n}{n^{1/6}}+\frac{20\log n}{C}\right) \leqslant \frac{n^{3/4}}{2^n}$$

if we choose C larger than, say 80, and n is sufficiently large. This proves (3.11). □

Proof of (3.12). Let θ be a zero of P_n. Then θ is either a y_j, or a zero of T_n different from every x_j. Let us consider first the case when $\theta=y_{j_0}$ for some $j_0\in\{0, ..., J\}$. Then

$$|P_n'(\theta)| = |T_n'(\theta)r_J(\theta)+T_n(\theta)r_J'(\theta)| = \frac{1}{2^{n-1}}|r_J'(y_{j_0})|$$

because θ is a zero of T_n' by the choice of the numbers y_j, and at every zero of T_n' the value of T_n equals 2^{-n+1}. The derivative of r_J at y_{j_0} equals

$$\prod_{j\neq j_0}\frac{y_{j_0}-y_j}{y_{j_0}-x_j}\cdot\frac{1}{y_{j_0}-x_{j_0}}.$$

In the proof of (3.11) we have verified that

$$\left|\prod_{j\neq j_0}\frac{y_{j_0}-x_j}{y_{j_0}-y_j}\right|\leqslant n^{3/4},$$

more precisely we have proved in (3.18)–(3.21) a similar inequality in which the role of the x_j's and y_j's have been switched. Thus, taking reciprocal, we finally get

$$|P_n'(\theta)|\geqslant\frac{1}{n^{3/4}}\frac{1}{2^n},$$

which is exactly (3.12).

If θ is one of the zeros of T_n, then

$$P_n'(\theta)=T_n'(\theta)r_J(\theta)+T_n(\theta)r_J'(\theta)=T_n'(\theta)r_J(\theta).$$

Here

$$|T_n'(\theta)|\geqslant\frac{n}{2^{n-1}},$$

while exactly as above

$$|r_J(\theta)|\geqslant\frac{1}{n^{3/4}},$$

by which (3.12) has been verified. □

4. Proof of Theorem 2.2

Let $\mu^*=\mu_1^*-\mu_2^*$, $\|\mu_i^*\|=1$, $\operatorname{supp}(\mu_i^*)\subset\Sigma_i$, $i=1,2$ be the equilibrium measure from the energy problem discussed in the beginning of Section 2.1. We know that U^{μ^*} equals some constant F_1 on Σ_1 and another one $-F_2$ on Σ_2. Using these facts, first we determine the signed measure μ^*.

In the proof we need the concept of equilibrium measure associated with a compact set on the plane, and the concept of balayage measure.

The *logarithmic energy* of a measure ν of compact support is defined as

$$I(\nu):=\int U^\nu(z)\,d\nu(z)=\iint\log\frac{1}{|z-t|}\,d\nu(t)\,d\nu(z).$$

If K is a compact set, then its *logarithmic capacity* $\operatorname{cap}(K)$ is defined by the formula

$$\log\frac{1}{\operatorname{cap}(K)}:=\inf\{I(\nu)\mid\operatorname{supp}(\nu)\subset K,\ \nu\geqslant 0,\ \|\nu\|=1\},$$

where $\|\nu\|$ denotes the total variation (total mass) of ν.

If K is of positive capacity, then there exists a unique probability measure $\nu=\nu_K$ on K for which the infimum on the right is attained, that is, ν_K is the unique measure that minimizes the energy integral $I(\nu)$ among all probability measures defined on K (see [12, Chapter II]).

This so-called *equilibrium measure* ν_K possesses the following properties:

(i) $U^{\nu_K}(z) \leqslant \log 1/\mathrm{cap}(K)$ for $z \in \mathbf{C}$,

(ii) $U^{\nu_K}(z) = \log 1/\mathrm{cap}(K)$ for quasi-every $z \in K$.

These properties can also be used to define ν_K. Furthermore, the equilibrium measure ν_K is supported on the outer boundary of K, which is defined as the boundary of the unbounded component of $\mathbf{C} \setminus K$. For example, the equilibrium measure of the interval $[-1,1]$ is the arcsine measure ω, while that of a disk or circle is the normalized Lebesgue measure on the circumference.

Consider in $\mathbf{C}$ an open set G with compact boundary ∂G, and let μ be a measure with $\mathrm{supp}(\mu) \subseteq \overline{G}$. The problem of *balayage* (or 'sweeping out') consists of finding a new measure μ', $\|\mu'\| = \|\mu\|$ *supported on* ∂G such that

$$U^{\mu}(z) = U^{\mu'}(z) \quad \text{for quasi-every } z \notin G. \tag{4.1}$$

For bounded G such a measure always exists ([9, Chapter IV, §2, Section 2]), but for unbounded G we must replace (4.1) by

$$U^{\mu}(z) = U^{\mu'}(z) + c \quad \text{for quasi-every } z \notin G. \tag{4.2}$$

Here the constant c turns out to be equal to

$$-\int_{\Omega} G_{\infty}(z)\, d\mu(y),$$

where Ω is the component of G that contains the point infinity and $G_{\infty}(z)$ is the Green function of that component with pole at infinity ([9, (4.2.6)]). Besides (4.1)–(4.2) we also know ([9, (4.210)]) that

$$U^{\mu'}(z) \leqslant U^{\mu}(z), \tag{4.3}$$

respectively

$$U^{\mu'}(z) \leqslant U^{\mu}(z) + \int_{\Omega} G_{\infty}(y)\, d\mu(y) \tag{4.4}$$

hold for all $z \in \mathbf{C}$.

Furthermore, if G is connected and regular with respect to the Dirichlet problem (i.e. every Dirichlet problem with continuous boundary function has a continuous solution up to the boundary), then in (4.1)–(4.2) we have equality for all $z \notin G$ ([9, Theorem 4.5]). The equality for $z \notin \overline{G}$ occurs automatically.

The balayage measure μ' has the additional property (see [9, Chapter IV, §1]), that if h is a continuous function on $\overline{G}$ which is harmonic in G, then

$$\int h\,d\mu = \int h\,d\mu'. \tag{4.5}$$

After these preparations we return to the equilibrium measure $\mu^* = \mu_1^* - \mu_2^*$ associated with the condenser (Σ_1, Σ_2).

Let us compare the measures μ_1^* and $(\mu_2^*)'$, where the latter one is the measure that we get when we sweep μ_2^* out of $\mathbf{C}\setminus\Sigma_1$ onto Σ_1. Both of these measures are probability measures on Σ_1 and their difference is constant on Σ_1 by the properties of μ^* and the balayage measures. Thus, it follows from the principle of domination ([9, Theorem 1.27]) that the potential $U^{\mu_1^* - (\mu_2^*)'}$ of $\mu_1^* - (\mu_2^*)'$ is identically constant, and since this constant must be zero (consider the potential around infinity), we get that the two potentials $U^{\mu_1^*}$ and $U^{(\mu_2^*)'}$ coincide, which implies that the measures μ_1^* and $(\mu_2^*)'$ are the same ([9, Theorem 1.12′]).

Since the same can be said when we sweep out the measure μ_1^* from $\mathbf{C}\setminus\Sigma_2$ onto Σ_2, we get that the measures μ_1^* and μ_2^* are each other's balayage measures.

We need one more thing before we can proceed with the proof of Theorem 2.2. Let us consider e.g. Σ_1, and let G be a disk containing Σ_1. Since the equilibrium measure of $\overline{G}$ is the normalized Lebesgue measure $m_{\partial G}$ on the boundary of G, and the equilibrium potential of $\overline{G}$ is constant on Σ_1, it follows exactly as in the preceding paragraph that if we take the balayage of $m_{\partial G}$ out of $\mathbf{C}\setminus\Sigma_1$ onto Σ_1, then we obtain ν_{Σ_1}. Let us now apply (4.5) and let the radius of the disk G tend to infinity. Then we arrive at the formula

$$h(\infty) = \int h\,d\nu_{\Sigma_1} \tag{4.6}$$

for every h that is continuous on $\overline{\mathbf{C}}$ and harmonic on $\overline{\mathbf{C}}\setminus\Sigma_1$.

After these preparations we set out to prove Theorem 2.2.

First we show that there are constants c, C such that

$$c\nu_{\Sigma_j} \leqslant \mu_j^* \leqslant C\nu_{\Sigma_j}, \quad j = 1, 2. \tag{4.7}$$

Let h be an arbitrary nonnegative continuous function on Σ_1. Since Σ_1 is regular with respect to the solution of the Dirichlet problem in $\overline{\mathbf{C}}\setminus\Sigma_1$, h can be extended to a nonnegative harmonic function to $\overline{\mathbf{C}}\setminus\Sigma_1$, which we continue to denote by h, so that h is continuous on the whole Riemann sphere. Using that μ_1^* is the balayage of μ_2^* onto Σ_2 we have

$$\int h\,d\mu_2^* = \int h\,d\mu_1^*.$$

As we have seen in (4.6), we also have

$$h(\infty) = \int h \, d\nu_{\Sigma_1}$$

Now Harnack's inequality for nonnegative harmonic functions implies that there are constants c, C independent of h such that

$$ch(\infty) \leqslant h(t) \leqslant Ch(\infty)$$

for $t \in \operatorname{supp}(\mu_2^*) \subseteq \Sigma_2$. On integrating this inequality with respect to μ_2^* and taking into account the preceding relations, we arrive at

$$\int h \, d(\mu_1^* - c\nu_{\Sigma_1}) \geqslant 0, \qquad \int h \, d(C\nu_{\Sigma_1} - \mu_1^*) \geqslant 0.$$

The signed measures with respect to which the integrals are taken are supported on Σ_1, and since these inequalities hold for all nonnegative continuous function h on Σ_1, we can conclude that the signed measures $\mu_1^* - c\nu_{\Sigma_1}$ and $C\nu_{\Sigma_1} - \mu_1^*$ are actually positive measures and this is the inequality (4.7) for $j=1$. When $j=2$, the proof is similar.

Next we need an estimate on the equilibrium measures ν_{Σ_j}. Namely we need that they are absolutely continuous with respect to Lebesgue measure on Σ_j, and if

$$\Sigma_j = \bigcup_{k=1}^{m_j} [a_k^{(j)}, b_k^{(j)}], \quad b_k^{(j)} < a_{k+1}^{(j)}, \quad k = 1, ..., m_j - 1,$$

then there are numbers $y_k^{(j)} \in (b_k^{(j)}, a_{k+1}^{(j)})$, $k=1, ..., m_j - 1$ such that

$$d\mu_j^*(t) = \frac{S_j(t)}{\pi\sqrt{|R_j(t)|}} \, dt$$

where

$$R_j(t) = \prod_{k=1}^{m_j} (t - a_k^{(j)})(t - b_k^{(j)})$$

and

$$S_j(t) = \prod_{k=1}^{m_k - 1} |t - y_k^{(j)}|$$

([11, Lemma 4.4.1]).

From this representation of the equilibrium measures ν_{Σ_j} and from (4.7), it easily follows that the function

$$H(z) = \left(\int \frac{d\mu^*(t)}{z - t} \right)^2$$

has a simple pole at each a_j, b_j. We claim that elsewhere H is analytic. This is obvious in $\overline{\mathbf{C}}\setminus\Sigma$, and the analyticity on each of (a_j, b_j) can be proved as follows. If we cut $\mathbf{C}$ along Σ, then

$$\int \frac{d\mu^*(t)}{z-t} \tag{4.8}$$

is purely imaginary on the cut, because the real part of

$$\int \log(z-t)\, d\mu^*(t) \tag{4.9}$$

is the potential $U^{\mu^*}(z)$ and so it is constant on each interval of Σ; hence the real part of the derivative of (4.9) vanishes on Σ. Furthermore, (4.8) takes conjugate values for conjugate arguments; therefore, (4.8) takes opposite values on the upper and lower parts of the cut. Squaring these opposite values as in H we get that H is real on the cut and takes conjugate values for conjugate arguments on the upper and lower parts of the cut; hence the analyticity of H on $\bigcup(a_j, b_j)$ follows from the continuation principle for analytic functions. Of course, to do all these deductions, we need that H, which on $\bigcup_{j=1}^{m}(a_j, b_j)$ must be understood in principal value sense, is continuous on the cut. Seeing however that e.g. on Σ_1 the measure μ_1^* is given as the balayage of μ_2^* onto Σ_1, the density function of μ^* is analytic on $\bigcup(a_j, b_j)$ (cf. [9, (4.1.6)]), from which the claimed continuity easily follows.

In summary, the function H is a rational function. Obviously, H has a zero at infinity with multiplicity 4 (recall that μ^* is orthogonal to constants) and each of its zeros is of even multiplicity; hence H is of the form

$$H(z) = \frac{(P_{m-2}(z))^2}{R(z)},$$

where

$$R(z) = \prod_{k=1}^{m} (z-a_k)(z-b_k)$$

and

$$P_{m-2}(z) = c_{m-2}z^{m-2} + \ldots + c_0$$

is a polynomial of degree at most $m-2$. Thus, by multiplying P_{m-2} by -1 if necessary we can conclude that

$$\int \frac{d\mu^*(t)}{z-t} = \frac{P_{m-2}(z)}{\sqrt{R(z)}}, \quad z \in \mathbf{C}\setminus\Sigma.$$

Here, and in what follows, we take that branch of the square root that is positive on the positive part of the real line. From Cauchy's formula applied to $\overline{\mathbf{C}}\setminus\Sigma$ we can see that

$$\frac{P_{m-2}(z)}{\sqrt{R(z)}} = \frac{1}{2\pi i}\oint_{\Sigma} \frac{P_{m-2}(\xi)}{\sqrt{R(\xi)}}\,\frac{1}{\xi - z}\, d\xi = \int_{\Sigma} \frac{P_{m-2}(t)}{\pi i\sqrt{R(t)}}\,\frac{1}{t-z}\, dt,$$

where the first integral is taken on the cut in the clockwise direction and the second integral is an ordinary Lebesgue integral and the values of $\sqrt{R(t)}$ in it are taken on the upper part of the cut. Since Cauchy transforms determine their generating signed (or even complex) measures if these measures have support of zero two dimensional Lebesgue measure (see [2]), it follows from the preceding two formulae that

$$d\mu^*(t) = \frac{P_{m-2}(t)}{-\pi i\sqrt{R(t)}}\,dt. \tag{4.10}$$

Since $i\sqrt{R(t)}$ is real on the upper part of the cut, we can also conclude that P_{m-2} has real coefficients.

Let now x and y belong to the same interval $[a_j, b_j]$. Then the function

$$\frac{P_{m-2}(z)}{-\pi i\sqrt{R(z)}}\log\frac{x-z}{y-z}$$

is analytic on $\overline{\mathbf{C}}\setminus\Sigma$ and has at least double zero at infinity; hence

$$\oint_\Sigma \frac{P_{m-2}(\xi)}{-\pi i\sqrt{R(\xi)}}\log\frac{x-\xi}{y-\xi}\,d\xi = 0. \tag{4.11}$$

Taking real parts, we can see that whatever the real polynomial P_{m-2} of degree at most $m-2$ is, the potential of the (signed) measure

$$d\sigma(t) = \frac{P_{m-2}(t)}{-\pi i\sqrt{R(t)}}\,dt, \quad t\in\Sigma$$

is constant on each interval $[a_j, b_j]$, in particular,

$$U^\sigma(a_j) = U^\sigma(b_j). \tag{4.12}$$

Next we compute $U^\sigma(b_j) - U^\sigma(a_{j+1})$. If $L = \Sigma\cup[b_j, a_{j+1}]$, then (4.11) holds again if the integration on Σ is replaced by integration around L, and for the same reason. Taking again real parts we can see from the facts that $\sqrt{R(t)}$ is real on (b_j, a_{j+1}) and

$$\log\frac{a_{j+1}-t}{b_j-t} = \log\left|\frac{a_{j+1}-t}{b_j-t}\right| \pm i\pi$$

there, that

$$\mathrm{Re}\left(\log\frac{a_{j+1}-t}{b_j-t}\,\frac{P_{m-2}(t)}{-\pi i\sqrt{R(t)}}\right) = -\frac{P_{m-2}(t)}{\sqrt{R(t)}}$$

on the upper part of the cut along L on (b_j, a_{j+1}); therefore,

$$\int_\Sigma \log\left|\frac{a_{j+1}-t}{b_j-t}\right|\frac{P_{m-2}(t)}{-\pi i\sqrt{R(t)}}\,dt = \int_{b_j}^{a_{j+1}}\frac{P_{m-2}(t)}{\sqrt{R(t)}}\,dt. \tag{4.13}$$

It follows from (4.12) and (4.13) that for any $l\geqslant j$

$$\int_\Sigma \log\left|\frac{a_{l+1}-t}{b_j-t}\right| \frac{P_{m-2}(t)}{-\pi i\sqrt{R(t)}}\,dt=\left(\int_{b_j}^{a_{j+1}}+\int_{b_{j+1}}^{a_{j+2}}+\ldots+\int_{b_l}^{a_{l+1}}\right)\frac{P_{m-2}(t)}{\sqrt{R(t)}}\,dt. \tag{4.14}$$

From these formulae we can easily derive necessary and sufficient conditions for the fact that the potential U^σ be constant on Σ_1 and on Σ_2. In fact, let j_1 and j_2 be the indices of the last (more precisely, rightmost) intervals of Σ_1 and Σ_2, respectively, and set

$$\mathcal{I}=\{j\mid j\neq j_1,j_2,\ [a_j,b_j]\subset\Sigma_1 \text{ and } [a_{j+1},b_{j+1}]\subset\Sigma_2 \text{ or } [a_j,b_j]\subset\Sigma_2 \text{ and } [a_{j+1},b_{j+1}]\subset\Sigma_1\}$$

and

$$\mathcal{J}=\{j\mid j\neq j_1,j_2,\ [a_j,b_j]\subset\Sigma_1 \text{ and } [a_{j+1},b_{j+1}]\subset\Sigma_1 \text{ or } [a_j,b_j]\subset\Sigma_2 \text{ and } [a_{j+1},b_{j+1}]\subset\Sigma_2\}.$$

Then $\mathcal{I}\cup\mathcal{J}$ has $m-2$ elements because the indices of the last intervals of Σ_1 and Σ_2 do not appear in $\mathcal{I}\cup\mathcal{J}$. If $j\in\mathcal{J}$ and U^σ is constant on Σ_1 and on Σ_2, then U^σ must take the same value on $[a_j,b_j]$ and $[a_{j+1},b_{j+1}]$; hence by (4.13)

$$\int_{b_j}^{a_{j+1}}\frac{P_{m-2}(t)}{\sqrt{R(t)}}\,dt=0,\quad j\in\mathcal{J} \tag{4.15}$$

(note that the left hand side in (4.14) is nothing else than $U^\sigma(b_j)-U^\sigma(a_{j+1})$). Let now $j\in\mathcal{I}$, and let $l(j)\geqslant j$ be the smallest index such that $[a_j,b_j]$ and $[a_{l(j)+1},b_{l(j)+1}]$ belong to the same set Σ_1 or Σ_2. $j\in\mathcal{I}$ means that $l(j)>j$. If U^σ is constant on Σ_1 and on Σ_2, then U^σ must take the same value on $[a_j,b_j]$ and $[a_{l(j)+1},b_{l(j)+1}]$; hence by (4.14)

$$\left(\int_{b_j}^{a_{j+1}}+\int_{b_{j+1}}^{a_{j+2}}+\ldots+\int_{b_{l(j)}}^{a_{l(j)+1}}\right)\frac{P_{m-2}(t)}{\sqrt{R(t)}}\,dt=0.$$

But the indices $j+1,j+2,\ldots,l(j)-1$ belong then to $\mathcal{J}$; hence in view of (4.15) we can see that the last sum is the same as

$$\left(\int_{b_j}^{a_{j+1}}+\int_{b_{l(j)}}^{a_{l(j)+1}}\right)\frac{P_{m-2}(t)}{\sqrt{R(t)}}\,dt=0. \tag{4.16}$$

(4.15) and (4.16) give $m-2$ equations on the $m-1$ coefficients of P_{m-2}. The $(m-1)$st condition

$$\int_{\Sigma_1}\frac{P_{m-2}(t)}{-\pi i\sqrt{R(t)}}\,dt=1 \tag{4.17}$$

comes from (4.10) because we only consider signed measures that have total mass 1 on Σ_1. From Cauchy's formula it then follows from (4.17) that

$$\int_{\Sigma_2} \frac{P_{m-2}(t)}{-\pi i\sqrt{R(t)}}\,dt = -1$$

as is required for our measures.

From our considerations it is clear that if the coefficients of P_{m-2} are chosen to satisfy (4.15)–(4.17), then

$$d\sigma(t) = \frac{P_{m-2}(t)}{-\pi i\sqrt{R(t)}}\,dt$$

is a signed measure on Σ such that $\sigma(\Sigma_1)=1$, $\sigma(\Sigma_2)=-1$, and U^σ is constant on each of Σ_1 and Σ_2. We claim that then σ must be μ^*, i.e. the equilibrium measure for the signed energy problem (cf. [8]). In fact, since $\mu^*=\mu_1^*-\mu_2^*$ also has these properties, it follows that there are constants α and β such that the potential of the signed measure $\sigma-\alpha\mu^*$ is identically equal to β on Σ. Thus, if $\sigma=\sigma_1-\sigma_2$ where $(-1)^{j-1}\sigma_j$ denotes the restriction of σ to Σ_j, and if $\nu_\pm$ denotes the positive and negative parts of a measure ν, then we have for all $z\in\Sigma$

$$U^{\sigma_{1+}+\sigma_{2-}+\alpha\mu_2^*}(z) = U^{\sigma_{2+}+\sigma_{1-}+\alpha\mu_1^*}(z)+\beta. \tag{4.18}$$

Here for the positive measures $\sigma_{1+}+\sigma_{2-}+\alpha\mu_2^*$ and $\sigma_{2+}+\sigma_{1-}+\alpha\mu_1^*$ we have

$$\|\sigma_{1+}+\sigma_{2-}+\alpha\mu_2^*\| = \|\sigma_{2+}+\sigma_{1-}+\alpha\mu_1^*\|$$

because $\|\sigma_{1+}\|-\|\sigma_{1-}\|=\sigma(\Sigma_1)=1$, $\|\sigma_{2+}\|-\|\sigma_{2-}\|=\sigma(\Sigma_1)=1$ and $\|\mu_1^*\|=\|\mu_2^*\|$. Furthermore they have finite logarithmic energy; hence it follows from the principle of domination (see [9, Theorem 1.27]) that (4.18) is true for all z. Then for $z\to\infty$ we get $\beta=0$ and so

$$U^{\sigma_{1+}+\sigma_{2-}+\alpha\mu_2^*}(z) \equiv U^{\sigma_{2+}+\sigma_{1-}+\alpha\mu_1^*}(z)$$

everywhere. Hence $\sigma_{1+}+\sigma_{2-}+\alpha\mu_2^*=\sigma_{2+}+\sigma_{1-}+\alpha\mu_1^*$, i.e. $\sigma=\alpha\mu^*$, and since $\sigma(\Sigma_1)=1=\mu^*(\Sigma_1)$, we get $\sigma=\mu^*$ as we claimed above.

Finally we compute F_1+F_2. Since this is the difference of the potential values taken on Σ_1 and on Σ_2, the above formulae (see e.g. (4.13)) yield

$$|F_1+F_2| = \left|\int_{b_j}^{a_{j+1}} \frac{P_{m-2}(t)}{\sqrt{R(t)}}\,dt\right|,$$

where j is an index such that b_j and a_{j+1} belong to different sets Σ_1 and Σ_2. But F_1+F_2 is nonnegative. In fact, U^{μ^*} coincides with F_1 on Σ_1 and with $-F_2$ on Σ_2, hence

$$F_1+F_2 = \int U^{\mu^*}\,d\mu^* = I(\mu^*) \geqslant 0,$$

because the logarithmic energy of any compactly supported signed measure μ^* with the property $\mu^*(\mathbf{C})=0$ is nonegative (see [9, Theorem 1.16]). This gives (2.5). $\square$

The author thanks J. Szabados for valuable discussion and for providing the references [4] and [5].

References

[1] BAGBY, TH., The modulus of a plane condenser. *J. Math. Mech.*, 17 (1967), 315–329.

[2] BISHOP, E., A minimal boundary for function algebras. *Pacific J. Math.*, 9 (1959), 629–642.

[3] BLATT, H.-P., On the distribution of simple zeros of polynomials. *J. Approx. Theory*, 69 (1992), 250–268.

[4] ERDŐS, P., On the uniform distribution of the roots of certain polynomials. *Ann. of Math.*, 43 (1942), 59–64.

[5] ERDŐS, P. & TURÁN, P., On the distribution of roots of polynomials. *Ann. of Math.*, 51 (1950), 105–119.

[6] — On the uniformly dense distribution of certain sequences of points. *Ann. of Math.*, 41 (1940), 162–173.

[7] GONCHAR, A. A., On the speed of rational approximation of some analytic functions. *Mat. Sb.*, 105 (147) (1978), 147–163; English translation in *Math. USSR-Sb.*, 34 (1978), 131–145.

[8] GONCHAR, A. A. & RAHMANOV, E. A., Equilibrium measure and distribution of zeros of extremal polynomials. *Mat. Sb.*, 125 (167) (1984), 117–127.

[9] LANDKOF, N. S., *Foundations of Modern Potential Theory.* Grundlehren der Mathematischen Wissenschaften, 190. Springer-Verlag, New York, 1972.

[10] LUBINSKY, D. S. & SAFF, E. B., *Strong Asymptotics for Extremal Polynomials Associated with Weights on* **R**. Lecture Notes in Mathematics, 1305. Springer-Verlag, New York, 1988.

[11] STAHL, H. & TOTIK, V., *General Orthogonal Polynomials.* Encyclopedia of Mathematics, 43. Cambridge University Press, New York, 1992.

[12] TSUJI, M., *Potential Theory in Modern Function Theory.* Maruzen, Tokyo, 1959.

VILMOS TOTIK
Bolyai Institute
Aradi vertanuk tere 1
6720 Szeged
Hungary
E-mail address: h871tot@ella.hu

Department of Mathematics
University of South Florida
Tampa, FL 33620
U.S.A.
E-mail address: totik@gauss.math.usf.edu

Received August 5, 1991