BOUNDARY BEHAVIOR OF A CONFORMAL MAPPING

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1. Suppose given in the complex w-plane a simply connected domain $\mathcal{D}$, which is not the whole plane, and let $w=f(z)$ be a function mapping the open unit disc D in the z-plane one-to-one and conformally onto $\mathcal{D}$. As is well known, for almost every θ $(0\leq\theta<2\pi)$, $f(z)$ has a finite *angular limit* $f(e^{i\theta})$ at $e^{i\theta}$, that is, for any open triangle Δ contained in D and having one vertex at $e^{i\theta}$, $f(z)\to f(e^{i\theta})$ as $z\to e^{i\theta}$, $z\in\Delta$. An *arc at* $e^{i\theta}$ is a curve $A\subset D$ such that $A\cup\{e^{i\theta}\}$ is a Jordan arc. As a preliminary form of our main result (Theorem 2), we state

THEOREM 1. *For almost every θ either*

$$\frac{f(z)-f(e^{i\theta})}{z-e^{i\theta}} \text{ and } f'(z) \text{ have the same finite, nonzero angular limit at } e^{i\theta}, \tag{1.1}$$

or $\arg(f(z)-f(e^{i\theta}))$, *defined and continuous in D, is unbounded above and below on each arc at $e^{i\theta}$.* (1.2)

Note that if (1.1) holds, the mapping is *isogonal* at $e^{i\theta}$ in the sense that

$$\arg(f(z)-f(e^{i\theta}))-\arg(z-e^{i\theta}),$$

where both argument functions are defined and continuous in D, has a finite angular limit at $e^{i\theta}$.

If $f(z)$ has a finite angular limit at $e^{i\theta}$, then the image under $f(z)$ of the radius at $e^{i\theta}$ determines an (ideal) accessible boundary point $\mathfrak{a}_\theta$ of $\mathcal{D}$ whose complex coordinate $w(\mathfrak{a}_\theta)=f(e^{i\theta})$ is finite. The set of all such $\mathfrak{a}_\theta$ is denoted by $\mathfrak{A}$. On $\mathcal{D}\cup\mathfrak{A}$ we use the *relative metric*, the relative distance between two points of $\mathcal{D}\cup\mathfrak{A}$ being defined as the infimum of the Euclidean diameters of the open Jordan arcs that lie in $\mathcal{D}$ and join these two points. Any limits involving accessible boundary points are taken in this relative metric.

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We shall see (Lemma 1) that (1.2) is equivalent to

$$\liminf_{w\to\mathfrak{a}} \arg (w-w(\mathfrak{a})) = -\infty \quad \text{and} \quad \limsup_{w\to\mathfrak{a}} \arg (w-w(\mathfrak{a})) = +\infty, \tag{1.3}$$

where $\mathfrak{a}=\mathfrak{a}_\theta$ and $\arg (w-w(\mathfrak{a}))$ is defined and continuous in $\mathcal{D}$. The condition (1.3) says, roughly speaking, that $\mathcal{D}$ and consequently also its boundary $\partial\mathcal{D}$ twist around $w(\mathfrak{a})$ infinitely often in both directions, arbitrarily near $\mathfrak{a}$.

2. We proceed to state Theorem 2. We say that *the* (unique) *inner tangent* to $\partial\mathcal{D}$ exists at an accessible boundary point $\mathfrak{a}\in\mathfrak{A}$ provided there exists one and only one number φ_0 $(0\leqslant\varphi_0<2\pi)$ with the property that for each positive number $\varepsilon\,(\varepsilon<\pi/2)$ there exists a positive number δ such that the sector

$$\mathcal{A} = \{w(\mathfrak{a})+\varrho e^{i\varphi}\colon 0<\varrho<\delta,\ |\varphi-\varphi_0|<\pi/2-\varepsilon\}$$

is contained in $\mathcal{D}$, and is such that $w\to\mathfrak{a}$ (relative metric) as $w\to w(\mathfrak{a})$, $w\in\mathcal{A}$ (our terminology is slightly different from that of Lavrentieff [5]). For convenience we call these sectors the angles at $\mathfrak{a}$. Set

$$\mathfrak{A}_1 = \{\mathfrak{a}\colon \mathfrak{a}\in\mathfrak{A}, \text{ the inner tangent to } \partial\mathcal{D} \text{ exists at } \mathfrak{a}\};$$
$$\mathfrak{A}_2 = \{\mathfrak{a}\colon \mathfrak{a}\in\mathfrak{A}, \text{ (1.3) holds}\}.$$

We say that a subset $\mathfrak{N}$ of $\mathfrak{A}$ is a *$\mathcal{D}$-conformal null-set* provided $\{\theta\colon \mathfrak{a}_\theta\in\mathfrak{N}\}$ is a set of measure zero. Note that this definition is independent of f.

Let $z=g(w)$ be a function mapping $\mathcal{D}$ one-to-one and conformally onto D. Then for each $\mathfrak{a}\in\mathfrak{A}$ the limit

$$\lim_{w\to\mathfrak{a}} g(w) = g(\mathfrak{a})$$

exists. We say that $g(w)$ has a nonzero *angular derivative* at a point $\mathfrak{a}\in\mathfrak{A}_1$ provided there exists a finite, nonzero complex number $g'(\mathfrak{a})$ such that for each angle $\mathcal{A}$ at $\mathfrak{a}$,

$$\lim_{\substack{w\to\mathfrak{a}\\ w\in\mathcal{A}}} \frac{g(w)-g(\mathfrak{a})}{w-w(\mathfrak{a})} = g'(\mathfrak{a}) \quad \text{and} \quad \lim_{\substack{w\to\mathfrak{a}\\ w\in\mathcal{A}}} g'(w) = g'(\mathfrak{a}). \tag{2.1}$$

THEOREM 2. (i) $\mathfrak{A}=\mathfrak{A}_1\cup\mathfrak{A}_2\cup\mathfrak{N}$, *where $\mathfrak{N}$ is a $\mathcal{D}$-conformal null-set.*

(ii) *$g(w)$ has a nonzero angular derivative at each point of $\mathfrak{A}_1$, with the possible exception of those points in a $\mathcal{D}$-conformal null-set.*

(iii) *A subset of $\mathfrak{A}_1$ is a $\mathcal{D}$-conformal null-set if and only if the set of complex coordinates of its points has linear measure zero.* (1)

(1) A subset of the plane is said to have linear measure zero provided for each $\varepsilon>0$ it can be covered by a countable collection of open discs the sum of whose diameters is less than ε.

We are indebted to M. A. Lavrentieff for an earlier theorem [5, Theorem 1], which is contained in Theorem 2.

3. We return now to the notation of Section 1. It is convenient to use the special notation $\operatorname{Arg}(w-f(e^{i\theta}))$ for the continuous branch of $\arg(w-f(e^{i\theta}))$ which is defined in $\mathcal{D}$ and satisfies

$$0 \leqslant \operatorname{Arg}(f(0)-f(e^{i\theta}))<2\pi.$$

LEMMA 1. (a) *If there exists an arc at $e^{i\theta}$ on which $\operatorname{Arg}(f(z)-f(e^{i\theta}))$ is bounded above, then*

$$\limsup_{w\to a_\theta} \operatorname{Arg}(w-f(e^{i\theta}))<\infty.$$

(b) *If there exists an arc at $e^{i\theta}$ on which $\operatorname{Arg}(f(z)-f(e^{i\theta}))$ is bounded below, then*

$$\liminf_{w\to a_\theta} \operatorname{Arg}(w-f(e^{i\theta}))>-\infty.$$

Remark. The reader who is only interested in the proof of Theorem 2 can skip to Section 4. For the proof of Theorem 2 we only need to know that for almost every θ, either (1.1) or (1.3) holds.

Proof of Lemma 1. We give the proof of (a); the proof of (b) is analogous. Suppose there exists an arc A^z at $e^{i\theta}$ on which $\operatorname{Arg}(f(z)-f(e^{i\theta}))$ is bounded above. Let z_0 be the initial point of A^z (that is, the endpoint of A^z in D), and let ϱ_n $(n=0, 1, \ldots)$ be numbers such that

$$0<\varrho_n<\varrho_0<|f(z_0)-f(e^{i\theta})| \quad (n=1, 2, \ldots) \tag{3.1}$$

and such that $\varrho_n\to 0$. Set

$$C_n=\{|w-f(e^{i\theta})|=\varrho_n\} \quad (n=0, 1, \ldots).$$

By standard theorems, for each n each component of the preimage $f^{-1}(C_n)$ is a crosscut of D neither endpoint of which is $e^{i\theta}$. Let V_n^z $(n=0, 1, \ldots)$ be the component of $D-f^{-1}(C_n)$ such that $re^{i\theta}\in V_n^z$ for all r sufficiently near 1 $(r<1)$, and set $\gamma_n^z=D\cap\partial V_n^z$. Note that for each $n=0, 1, \ldots$, $A^z\cap V_n^z\neq\emptyset$, for otherwise some component of $f^{-1}(C_n)$ would have $e^{i\theta}$ as an endpoint. Thus $A^z\cap\gamma_n^z\neq\emptyset$ $(n=0, 1, \ldots)$, because $z_0\notin\bar V_n^z$ by (3.1) (the bar denotes closure). Also by (3.1), $D\cap\bar V_n^z\subset V_0^z$ $(n=1, 2, \ldots)$ and in particular $\gamma_n^z\subset V_0^z$ $(n=1, 2, \ldots)$. Set

$$\Gamma_n^z=V_0^z\cap f^{-1}(C_n) \quad (n=1, 2, \ldots).$$

Then $\gamma_n^z\subset\Gamma_n^z$, and consequently, since $A^z\cap\gamma_n^z\neq\emptyset$, $A^z\cap\Gamma_n^z\neq\emptyset$. Thus for each $n=1, 2, \ldots$, A^z contains a Jordan arc that joins γ_0^z to Γ_n^z. We note that only finitely many components of $f^{-1}(C_n)$ intersect this Jordan arc. It follows readily that there exist open Jordan arcs

$\alpha_n^z \subset A^z$ $(n=1, 2, \ldots)$ such that α_n^z joins γ_0^z to Γ_n^z and does not intersect $\gamma_0^z \cup \Gamma_n^z$. Since one endpoint of α_n^z is in V_0^z, $\alpha_n^z \subset V_0^z$.

Let A, V_0, γ_0, and α_n $(n=1, 2, \ldots)$ denote the images under $f(z)$ of A^z, V_0^z, γ_0^z, and α_n^z, respectively. Clearly $\gamma_0 \subset C_0$. Also, α_n lies in the open annulus U_n whose boundary is $C_0 \cup C_n$, and α_n joins a point $w_n \in \gamma_0$ to a point of C_n. For each n let

$$\varphi_n(w) = \arg (w - f(e^{i\theta}))$$

be defined and continuous on $\bar{U}_n - \bar{\alpha}_n$. Let $\varphi_n^-(w)$ and $\varphi_n^+(w)$ $(w \in \bar{\alpha}_n)$ be the boundary values of φ_n from the two sides of $\bar{\alpha}_n$, defined so that φ_n^- and φ_n^+ are continuous functions on $\bar{\alpha}_n$. Then each of the functions φ_n^- and φ_n^+ differs from $\operatorname{Arg} (w - f(e^{i\theta}))$ $(w \in \bar{\alpha}_n)$ by a constant, and $\varphi_n^+(w_n) = \varphi_n^-(w_n) \pm 2\pi$. Thus

$$\varphi_n(w) - \varphi_n^-(w_n) \leqslant 2\pi + \sup_{w \in \alpha_n} (\operatorname{Arg} (w - f(e^{i\theta})) - \operatorname{Arg} (w_n - f(e^{i\theta}))), \tag{3.2}$$

because it is readily seen that all boundary values of the function on the left are less than or equal to the number on the right.

We now note that $\operatorname{Arg} (w - f(e^{i\theta}))$ is bounded on γ_0. To see this let w' and w'' be any two points of γ_0, and let J be an open Jordan arc lying in V_0 and joining w' and w''. Consider the bounded component of the complement of $C_0 \cup J$ that does not contain $f(e^{i\theta})$. We define $\arg (w - f(e^{i\theta}))$ as a continuous function on the closure of this component so that it agrees with $\operatorname{Arg} (w - f(e^{i\theta}))$ on J, and we see that

$$|\operatorname{Arg} (w'' - f(e^{i\theta})) - \operatorname{Arg} (w' - f(e^{i\theta}))| \leqslant 2\pi.$$

Thus $\operatorname{Arg} (w - f(e^{i\theta}))$ is bounded on γ_0.

Hence by (3.2) the functions $\varphi_n(w) - \varphi_n^-(w_n)$ are uniformly bounded above, because $\operatorname{Arg} (w - f(e^{i\theta}))$ is bounded above on A and $w_n \in \gamma_0$.

Now consider any point $w^* \in V_0 - A$, and let β be an open Jordan arc lying in V_0 and joining w^* to a point of γ_0. Choose n sufficiently large so that $\beta \cup \{w^*\} \subset U_n$. Then w^* is in a component of $V_0 \cap U_n$ whose boundary contains a component of γ_0. We readily see that this component of $V_0 \cap U_n$ contains an open Jordan arc that joins w^* to a point $w' \in \gamma_0$ $(w' \neq w_n)$ and does not intersect α_n. Thus

$$\operatorname{Arg} (w^* - f(e^{i\theta})) - \operatorname{Arg} (w' - f(e^{i\theta})) = \varphi_n(w^*) - \varphi_n(w') \leqslant \varphi_n(w^*) - \varphi_n^-(w_n) + 2\pi.$$

Since the functions $\varphi_n(w) - \varphi_n^-(w_n)$ are uniformly bounded above, and since $\operatorname{Arg} (w - f(e^{i\theta}))$ is bounded on γ_0, we see that $\operatorname{Arg} (w - f(e^{i\theta}))$ is bounded above on $V_0 - A$, and thus also on V_0. The proof of Lemma 1 is complete.

4. *Proof of Theorem 1. Part I.* The proof of Theorem 1 will be given in the next five sections.

Let $\arg f'(z)$ be defined and continuous in D, and set

$$\log f'(z) = \log |f'(z)| + i \arg f'(z).$$

A routine argument shows that if $f'(z)$ has a finite, nonzero angular limit at $e^{i\theta}$, then the difference quotient in (1.1) has the same angular limit at $e^{i\theta}$. Thus (1.1) holds if

$$\log f'(z) \text{ has a finite angular limit at } e^{i\theta}. \tag{4.1}$$

By Lemma 1 it is sufficient, in order to prove Theorem 1, to prove that for almost every θ either (4.1) holds or both of the following hold:

$$\limsup_{w \to \alpha_\theta} \operatorname{Arg}(w - f(e^{i\theta})) = +\infty; \tag{4.2}$$

$$\liminf_{w \to \alpha_\theta} \operatorname{Arg}(w - f(e^{i\theta})) = -\infty. \tag{4.3}$$

We prove that for almost every θ either (4.1) or (4.2) holds. A completely analogous argument (which we omit) shows that for almost every θ either (4.1) or (4.3) holds; and these two facts combined yield the desired result.

Suppose contrary to the assertion that there exists a subset $E_z^{(1)}$ of ∂D of positive outer measure (that is, $\{\theta\colon e^{i\theta} \in E_z^{(1)}\}$ has positive outer measure) such that neither (4.1) nor (4.2) holds if $e^{i\theta} \in E_z^{(1)}$. We suppose without loss of generality that $f(z)$ has a finite angular limit at each point of $E_z^{(1)}$. For each $e^{i\theta} \in E_z^{(1)}$, let Δ_θ be the open equilateral triangle of side length $\frac{1}{2}$ that is contained in D, has one vertex at $e^{i\theta}$, and is symmetric with respect to the radius at $e^{i\theta}$.

Suppose for the moment that for almost every $e^{i\theta} \in E_z^{(1)}$ (that is, for almost every θ in $\{\theta\colon e^{i\theta} \in E_z^{(1)}\}$), $\arg f'(z)$ is bounded above in Δ_θ. Then by Plessner's extension of Fatou's theorem [12], $\log f'(z)$ has an angular limit at almost every point of $E_z^{(1)}$. By assumption, $\log f'(z)$ does not have a finite angular limit at any point of $E_z^{(1)}$, and consequently it has the angular limit ∞ at almost every point of $E_z^{(1)}$. It is easy to see that the set of points $e^{i\theta}$ at which a continuous function in D has the angular limit ∞ is an $F_{\sigma\delta}$-set (for the type of argument involved, see [4, p. 308]), and is therefore measurable. Hence $\log f'(z)$ has the angular limit ∞ at each point of a set of positive measure, and by a theorem of Lusin and Priwalow [8], we have a contradiction. We conclude that $E_z^{(1)}$ contains a set $E_z^{(2)}$ of positive outer measure such that for each $e^{i\theta} \in E_z^{(2)}$, $\arg f'(z)$ is unbounded above in Δ_θ.

Consider a fixed $e^{i\theta} \in E_z^{(2)}$, and let C be a rational circle (that is, C is a circumference whose radius is rational and whose center has rational real and imaginary parts) which satisfies the following conditions:

$$f(e^{i\theta}) \in \operatorname{int} C, \tag{4.4}$$

where int C denotes the disc of interior points of C;

$$f(z_\theta^{(j)}) \notin \operatorname{int} C \quad (j=1, 2), \tag{4.5}$$

where $z_\theta^{(1)}$ and $z_\theta^{(2)}$ are the vertices of Δ_θ in D; and finally, if $\mathcal{D}_C = \mathcal{D}_C(e^{i\theta})$ denotes the component of $\mathcal{D} \cap \operatorname{int} C$ such that

$$f(z) \in \mathcal{D}_C \text{ if } z \in \bar{\Delta}_\theta - \{e^{i\theta}\} \text{ and } z \text{ is sufficiently near } e^{i\theta} \tag{4.6}$$

(the bar denotes closure), then

$$\operatorname{Arg}(w - f(e^{i\theta})) \text{ is bounded above in } \mathcal{D}_C. \tag{4.7}$$

The existence of C satisfying (4.7) is assured, because (4.2) fails to hold at $e^{i\theta}$.

Note that $\mathcal{D} \cap \partial\mathcal{D}_C$ is a relatively open subset of C, each component of which is a free boundary arc of $\mathcal{D}_C$. We prove (as in the proof of Lemma 1) that all values of Arg $(w - f(e^{i\theta}))$ on $\mathcal{D} \cap \partial\mathcal{D}_C$ lie in an interval of length 2π. To this end let w' and w'' be any two points of $\mathcal{D} \cap \partial\mathcal{D}_C$, and let J be an open Jordan arc lying in $\mathcal{D}_C$ and joining w' and w''. Consider the bounded component of the complement of $C \cup J$ that does not contain $f(e^{i\theta})$. We define arg $(w - f(e^{i\theta}))$ on the closure of this component so that it agrees with Arg $(w - f(e^{i\theta}))$ on J, and we see that

$$|\operatorname{Arg}(w'' - f(e^{i\theta})) - \operatorname{Arg}(w' - f(e^{i\theta}))| \leqslant 2\pi. \tag{4.8}$$

Thus all values of Arg $(w - f(e^{i\theta}))$ on $\mathcal{D} \cap \partial\mathcal{D}_C$ lie in an interval of length 2π.

Hence (4.7) is equivalent to the existence of a positive integer M such that

$$\operatorname{Arg}(w - f(e^{i\theta})) - \operatorname{Arg}(w_0 - f(e^{i\theta})) \leqslant M \quad \text{if } w \in \mathcal{D}_C \text{ and } w_0 \in \mathcal{D} \cap \partial\mathcal{D}_C. \tag{4.9}$$

Here M is independent of w and w_0.

Define $\mathfrak{C}(e^{i\theta})$ to be the collection of all triples $(C, \mathcal{D}_C, M)$ satisfying the above conditions, that is, satisfying (4.4), (4.5) and (4.9), where C is a rational circle, $\mathcal{D}_C$ is the component of $\mathcal{D} \cap \operatorname{int} C$ satisfying (4.6), and M is a positive integer. Since for each C there are at most countably many components of $\mathcal{D} \cap \operatorname{int} C$, the union $\bigcup \mathfrak{C}(e^{i\theta})$, taken over all $e^{i\theta} \in E_z^{(2)}$, is a countable set. Thus there exists in this union a particular triple $(C, \mathcal{D}_C, M)$, *which is fixed throughout the rest of the proof of Theorem 1*, such that the set

$$E_z^{(3)} = \{e^{i\theta}: e^{i\theta} \in E_z^{(2)}, (C, \mathcal{D}_C, M) \in \mathfrak{C}(e^{i\theta})\}$$

has positive outer measure.

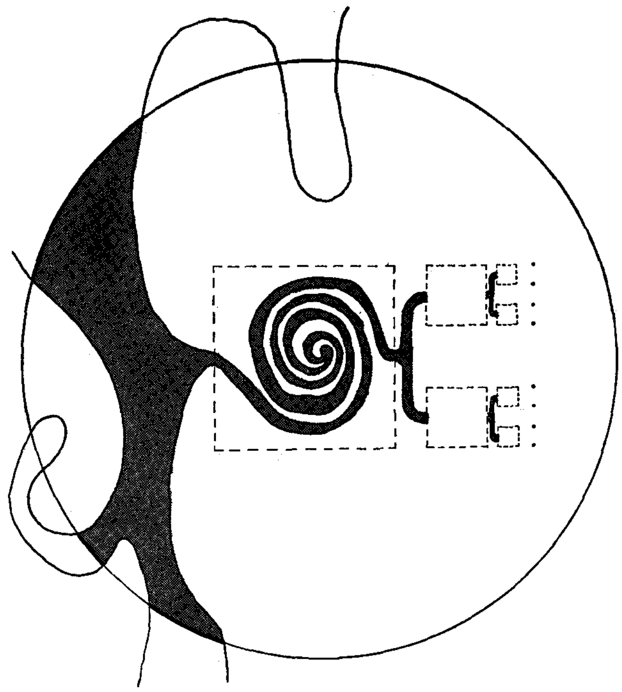

Fig. 1.

Before proceeding to prove some lemmas, we summarize the pertinent facts that will yield the desired contradiction.

(a) $E_z^{(3)}$ has positive outer measure.

(b) (4.5) and (4.6) hold for each $e^{i\theta} \in E_z^{(3)}$.

(c) $\arg f'(z)$ is unbounded above in Δ_θ for each $e^{i\theta} \in E_z^{(3)}$.

(d) The upper bound (4.9) holds uniformly for $e^{i\theta} \in E_z^{(3)}$.

Our method of proof will be to use (b), (c), and (d) to prove that $E_z^{(3)}$ is a set of measure zero, and thereby contradict (a).

An example for which (b), (c), and (d) can hold is suggested by Fig. 1. In this figure $\mathcal{D}_C$ is represented by the shaded area, except that the portion of $\mathcal{D}_C$ inside the smaller dotted squares is not shown. In each of these smaller squares $\mathcal{D}_C$ twists around some point in the positive direction a certain number of times and then twists back, as it does in the largest dotted square; and this number of times tends to ∞ as the diameter of the square tends to zero. The Cantor set on the vertical segment represents $\{f(e^{i\theta}): e^{i\theta} \in E_z^{(3)}\}$. The heavily drawn arcs on C represent $\mathcal{D} \cap \partial\mathcal{D}_C$. In this example there is at least some doubt whether $E_z^{(3)}$ is a set of measure zero or not.

5. The main result of this section is Lemma 3, the proof of which uses the following lemma.

LEMMA 2. *$f'(z)$ and $\log f'(z)$ are normal holomorphic functions.*

Proof. Clearly

$$\frac{|f''(z)|}{1+|f'(z)|^2} \leqslant \left|\frac{f''(z)}{f'(z)}\right| \leqslant \frac{k}{1-|z|}$$

(k is independent of z), the second inequality being well known [3, p. 395], and it follows that $f'(z)$ is a normal holomorphic function (see [6] or [11]). Similarly, if we set $h(z) = \log f'(z)$, then

$$\frac{|h'(z)|}{1+|h(z)|^2} \leqslant |h'(z)| = \left|\frac{f''(z)}{f'(z)}\right| \leqslant \frac{k}{1-|z|},$$

and we see that $\log f'(z)$ is a normal holomorphic function. The proof of Lemma 2 is complete.

LEMMA 3. *There exists a countable subset N of ∂D such that for each $e^{i\theta} \notin N$ the following holds: If $\arg f'(z)$ is unbounded above in Δ_θ, then there exists a sequence $\{A_n\}$ such that*

each A_n is an arc at some point of N, and A_n intersects the closure $\bar{\Delta}_\theta$ of Δ_θ in exactly one point z_n, which is the initial point of A_n (that is, the endpoint of A_n in D), (5.1)

$$\arg f'(z_n) \to +\infty, \tag{5.2}$$

and

$f(A_n)$ is contained in some closed half-plane whose boundary contains $f(z_n)$. (5.3)

Proof. Set $h(z) = \log f'(z)$. Let $\{\lambda_\nu\}$ be a sequence of real numbers that is dense on the real line, and is such that if we let L_ν denote the horizontal line through $i\lambda_\nu$, then $h(z) \notin L_\nu$ if $h'(z) = 0$ ($\nu = 1, 2, \ldots$). Then for each ν each component of the set

$$\{z: \arg f'(z) = \lambda_\nu\} = \{z: h(z) \in L_\nu\}$$

is a simple level curve (that is, a level curve without multiple points) of $\arg f'(z)$, and there are at most countably many such components. Note that $h(z)$ maps each such component one-to-one onto an open connected subset of L_ν. We shall need the following two facts concerning these level curves:

for each ν each component of $\{z: \arg f'(z) = \lambda_\nu\}$ tends at each end to a point of ∂D; (5.4)

if $\{\lambda_{\nu_n}\}$ is a subsequence of $\{\lambda_\nu\}$ such that $\lambda_{\nu_n} \to +\infty$ (or $-\infty$), and if for each n, Λ_n is a component of $\{z: \arg f'(z) = \lambda_{\nu_n}\}$, then diam $\Lambda_n \to 0$, where diam Λ_n denotes the Euclidean diameter of Λ_n. (5.5)

Since by Lemma 2, $h(z)$ is a normal holomorphic function, (5.4) and (5.5) follow from a theorem of Bagemihl and Seidel [2], which says, roughly speaking, that a nonconstant normal meromorphic function in D cannot tend to a limit along a sequence of Jordan arcs that tend to an arc of ∂D. We define a countable subset N of ∂D as follows: $e^{i\theta} \in N$ if and only if there exists a ν such that $e^{i\theta}$ is an endpoint of a component of $\{z\colon \arg f'(z) = \lambda_\nu\}$.

Consider a fixed $e^{i\theta}$ such that $e^{i\theta} \notin N$ and $\arg f'(z)$ is unbounded above in Δ_θ. Let $\{\lambda_{\nu_n}\}$ be a subsequence of $\{\lambda_\nu\}$ such that $\lambda_{\nu_n} \to +\infty$ and such that for each n some component Λ_n of $\{z\colon \arg f'(z) = \lambda_{\nu_n}\}$ intersects Δ_θ. By (5.5), $\operatorname{diam} \Lambda_n \to 0$, and consequently we can suppose without loss of generality that

$$\Lambda_n \subset \{|z - e^{i\theta}| < \tfrac{1}{2}\} \quad (n = 1, 2, \ldots). \tag{5.6}$$

For each n, since $e^{i\theta} \notin N$, $e^{i\theta}$ is not an endpoint of Λ_n. Thus since $h(z)$ is one-to-one on Λ_n, there exists $A_n \subset \Lambda_n$ satisfying (5.1) and

$$h(A_n) \subset \{h(z_n) + t\colon t \leqslant 0\}. \tag{5.7}$$

We note that since the side length of Δ_θ is $\frac{1}{2}$, (5.6) implies that A_n is contained in a closed half-plane H_n whose boundary contains $e^{i\theta}$ and z_n.

Let A_n be parametrized by a continuously differentiable function $z_n(t)$, $0 \leqslant t < 1$, with $z_n(0) = z_n$. By (5.7), $\log |f'(z_n(t))|$ is a decreasing function of t $(0 \leqslant t < 1)$. Thus $\arg f'(z_n(t))$ is constant and $|f'(z_n(t))|$ is decreasing for $0 \leqslant t < 1$. It is now intuitively obvious that (5.3) follows from the inclusion $A_n \subset H_n$. We prove this fact as follows.

Fix n. Let $\zeta = az + b$ $(|a| = 1)$ be a linear transformation taking H_n to the upper half-plane and z_n to 0. Set

$$F(\zeta) = f\left(\frac{\zeta - b}{a}\right), \quad \zeta(t) = \sigma(t) + i\tau(t) = az_n(t) + b.$$

Then $\arg F'(\zeta(t))$ has a constant value λ' and $|F'(\zeta(t))|$ is decreasing for $0 \leqslant t < 1$. Clearly

$$F(\zeta(t)) - F(0) = e^{i\lambda'} \left(\int_0^t |F'(\zeta(t))| \, d\sigma(t) + i \int_0^t |F'(\zeta(t))| \, d\tau(t) \right).$$

Since $\tau(t) \geqslant 0$, and since $|F'(\zeta(t))|$ is a decreasing function, integration by parts yields

$$\int_0^t |F'(\zeta(t))| \, d\tau(t) = |F'(\zeta(t))| \, \tau(t) - \int_0^t \tau(t) \, d|F'(\zeta(t))| \geqslant 0.$$

Thus (5.3) holds.

The proof of Lemma 3 is complete.

6. In this section we prove a lemma which is stated in terms of the triple $(C, \mathfrak{D}_C, M)$, defined in Section 4. Let N be the countable subset of ∂D whose existence is asserted by Lemma 3, and set

$$E_z^{(4)} = E_z^{(3)} - N, \quad \mathfrak{E}^{(4)} = \{\mathfrak{a}_\theta : e^{i\theta} \in E_z^{(4)}\}.$$

LEMMA 4. *Let z_0 and z^* be points of D satisfying*

$$\arg f'(z^*) - \arg f'(z_0) > M + 23\pi, \tag{6.1}$$

and set $z(t) = z_0(1-t) + z^* t$ $(0 \leqslant t \leqslant 1)$ and $w(t) = f(z(t))$. Suppose

$$w(0) \in \mathfrak{D} \cap \partial \mathfrak{D}_C,\ w(t) \in \mathfrak{D}_C \quad (0 < t \leqslant 1).$$

Let A^* be an arc at some point of N such that z^* is the initial point of A^* and $z(t) \notin A^*$ $(0 \leqslant t < 1)$; and suppose that $f(A^*)$ is contained in some closed half-plane whose boundary contains $f(z^*)$. Then

$$\operatorname{dist}_{\mathfrak{D}}(f(A^*), \mathfrak{E}^{(4)}) \geqslant \operatorname{diam} f(A^*),$$

where $\operatorname{diam} f(A^*)$ *and* $\operatorname{dist}_{\mathfrak{D}}(f(A^*), \mathfrak{E}^{(4)})$ *denote, respectively, the Euclidean diameter of $f(A^*)$ and the relative distance between $f(A^*)$* and $\mathfrak{E}^{(4)}$.

Proof. Set $w_0 = f(z_0)$, $w^* = f(z^*)$, and

$$\sigma = \{w(t) : 0 \leqslant t < 1\}.$$

We first obtain a lower bound in terms of $\arg f'(z^*) - \arg f'(z_0)$ for the twisting of σ around w^*. It is possible to do this because σ does not twist around w_0.

On the set $T = \{(\tau, t) : 0 < t \leqslant 1,\ 0 \leqslant \tau < t\}$ the function $w(t) - w(\tau)$ is continuous and nowhere zero. Thus by applying the monodromy theorem in the w-plane, we can define $\log(w(t) - w(\tau))$ as a continuous function of $(\tau, t) \in T$. The imaginary part of this function is denoted by

$$\varphi(\tau, t) = \arg(w(t) - w(\tau)).$$

Since $w(0) \in C$ and $w(t) \in \mathfrak{D}_C$ $(0 < t \leqslant 1)$, all values of $w(t) - w(0)$ lie on the same side of a certain straight line through the origin, and consequently we can require that

$$-\pi \leqslant \varphi(0, t) \leqslant 3\pi \quad (0 < t \leqslant 1). \tag{6.2}$$

Since $w'(t)$ is continuous and $w'(t) \neq 0$ $(0 \leqslant t \leqslant 1)$, we easily see that for each t_0 $(0 \leqslant t_0 \leqslant 1)$ the limit

$$\varphi(t_0) = \lim_{\substack{(\tau, t) \to (t_0, t_0) \\ (\tau, t) \in T}} \varphi(\tau, t) \tag{6.3}$$

exists. It follows that $\varphi(t)$ $(0 \leqslant t \leqslant 1)$ is continuous. Thus since $\varphi(t)$ is the angle (mod 2π) from the positive horizontal direction to the direction of the forward pointing tangent to σ at $w(t)$, $\varphi(t) - \arg f'(z(t))$ is constant; and in particular

$$\varphi(1) - \varphi(0) = \arg f'(z^*) - \arg f'(z_0). \tag{6.4}$$

By (6.2), $-\pi \leqslant \varphi(0, 1) \leqslant 3\pi$; and by (6.2) and (6.3), $-\pi \leqslant \varphi(0) \leqslant 3\pi$. Thus by (6.4)

$$\varphi(1) - \varphi(0, 1) \geqslant \arg f'(z^*) - \arg f'(z_0) - 4\pi. \tag{6.5}$$

Note that by (6.3), $\varphi(1) = \lim\limits_{\tau \to 1^-} \varphi(\tau, 1)$; and consequently $\varphi(1) - \varphi(0, 1)$ is the change in $\varphi(\tau, 1)$ as τ increases from 0 to 1.

Suppose now that the conclusion of Lemma 4 is false. Set $\alpha^* = f(A^*)$. Then there exists an open Jordan arc $\gamma \subset \mathcal{D}$ such that γ joins a point of α^* to a point $\mathfrak{a}_\theta \in \mathfrak{E}^{(4)}$ and diam $\gamma <$ diam α^*. Since A^* is an arc at a point of N, A^* and the preimage $f^{-1}(\gamma)$ have different endpoints on ∂D, and consequently γ contains an open subarc that joins a point of α^* to $\mathfrak{a}_\theta$ and does not intersect α^*. By replacing γ by this subarc, we can suppose without loss of generality that $\gamma \cap \alpha^* = \emptyset$. The endpoint of γ on α^* is denoted by w_γ. Since diam $\gamma <$ diam α^*, there exists an open half-plane H satisfying $\alpha^* \cap H = \emptyset$ and $\alpha^* \cap \partial H \neq \emptyset$ such that $\bar{\gamma} \cap \bar{H} = \emptyset$. By hypothesis there exists an open half-line L^* such that w^* is the finite endpoint of L^* and $L^* \cap \alpha^* = \emptyset$. Let $L^{(1)}$ be an open half-line such that $L^{(1)} \subset H - L^*$ and the finite endpoint of $L^{(1)}$ is a point $w^{(1)} \in \alpha^* \cap \partial H$. We note that $w^{(1)} \neq w_\gamma$ $(\bar{\gamma} \cap \bar{H} = \emptyset)$ and that

$$(\alpha^* \cup \gamma) \cap L^{(1)} = \emptyset. \tag{6.6}$$

Concerning Figure 2, we note that α^* may or may not tend at one end to a point of $\mathfrak{A}$.

We wish to establish the existence of a point $w_0' \in \mathcal{D} \cap \partial \mathcal{D}_C$ and a point $w_1' \in \mathcal{D}_C$ such that

$$\operatorname{Arg}(w_1' - f(e^{i\theta})) - \operatorname{Arg}(w_0' - f(e^{i\theta})) > M,$$

and thereby contradict (4.9).

We must now make a trivial observation, namely, that $\alpha^* \subset \mathcal{D}_C$. Suppose contrary to this assertion that $\alpha^* \not\subset \mathcal{D}_C$. Then since $w^* \in \mathcal{D}_C$, $\alpha^* \cap C \neq \emptyset$, and α^* contains a Jordan arc α' that joins w^* to a point of C and intersects C only at this one point. We can define $\arg(w^* - w)$ as a continuous function in (int $C) - \alpha'$; and since $\alpha' \cap L^* = \emptyset$, all values of this function lie in some interval of length 4π. Thus since $\sigma \cap \alpha' = \emptyset$, all values of $\varphi(\tau, 1)$ $(0 < \tau < 1)$ lie in some interval of length 4π, contrary to (6.1) and (6.5). Thus $\alpha^* \subset \mathcal{D}_C$.

We do not prove that $\gamma \subset \mathcal{D}_C$, although this is true.

Since $\alpha^* \cap L^* = \emptyset$, it is rather obvious that σ twists around $w^{(1)}$ almost as much as it twists around w^*. We now make this statement precise. Since $\sigma \cap \alpha^* = \emptyset$, we can easily define

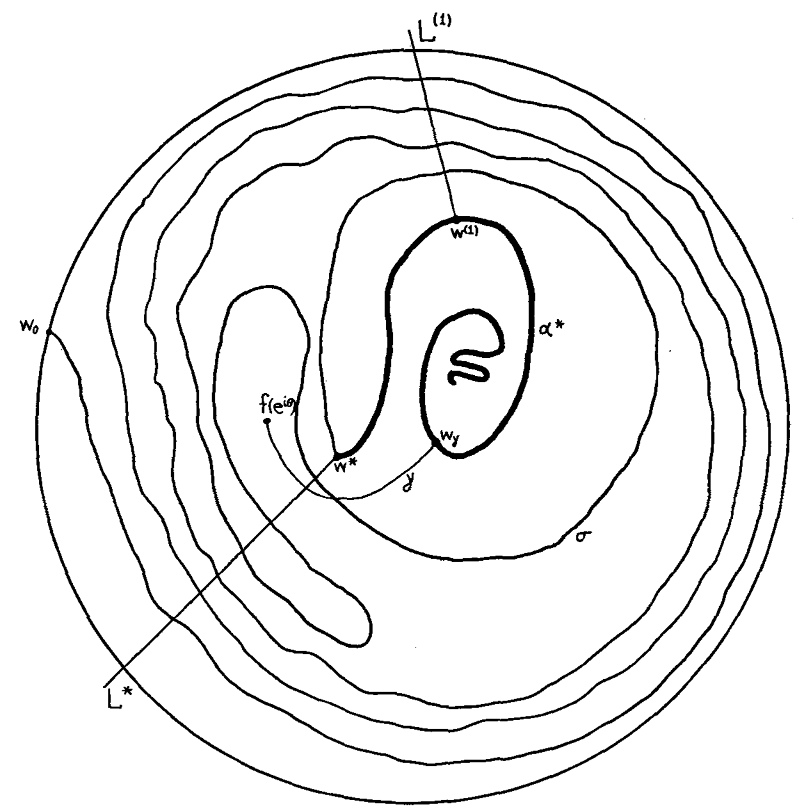

Fig. 2.

$$\psi(w, w') = \arg(w' - w)$$

as a continuous function of two variables for $w \in \sigma$ and $w' \in \alpha^*$. Then $\psi(w(\tau), w^*)$ differs from $\varphi(\tau, 1)$ $(0 \leqslant \tau < 1)$ by a constant, and we can suppose without loss of generality that this constant is zero:

$$\psi(w(\tau), w^*) = \varphi(\tau, 1) \quad (0 \leqslant \tau < 1). \tag{6.7}$$

Consider a particular τ satisfying $w(\tau) \notin L^* \cup L^{(1)}$ $(0 < \tau < 1)$. The union $\alpha^* \cup L^* \cup L^{(1)} \cup C$ contains a unique Jordan curve whose interior domain does not contain $w(\tau)$. By considering this Jordan curve we readily see that

$$|\psi(w(\tau), w^*) - \psi(w(\tau), w^{(1)})| \leqslant 4\pi. \tag{6.8}$$

Thus by continuity (6.8) holds for each τ $(0 \leqslant \tau < 1)$. Upon setting $\tau = 0$ in (6.8) and using (6.7), we obtain

$$|\varphi(0, 1) - \psi(w_0, w^{(1)})| \leqslant 4\pi. \tag{6.9}$$

Again using (6.7) and (6.8), we obtain by taking the limit as $\tau \to 1$ of the left-hand side of (6.8),

$$|\varphi(1) - \lim_{\tau\to 1^-} \psi(w(\tau), w^{(1)})| \leqslant 4\pi. \tag{6.10}$$

Combining (6.5), (6.9) and (6.10), we obtain

$$\lim_{\tau\to 1^-} \psi(w(\tau), w^{(1)}) - \psi(w_0, w^{(1)}) \geqslant \arg f'(z^*) - \arg f'(z_0) - 12\pi. \tag{6.11}$$

Define $\Psi(w) = \arg(w^{(1)} - w)$ as a continuous function on the simply connected domain $\mathcal{D}^* = \mathcal{D} - \alpha^*$. By (6.11)

$$\lim_{\tau\to 1^-} \Psi(w(\tau)) - \Psi(w_0) \geqslant \arg f'(z^*) - \arg f'(z_0) - 12\pi. \tag{6.12}$$

Since $\alpha^* \cap L^{(1)} = \emptyset$, we readily see that

$$|\lim_{\tau\to 1^-} \Psi(w(\tau)) - \lim_{\substack{w\to w_\gamma \\ w\in\gamma}} \Psi(w)| \leqslant 4\pi. \tag{6.13}$$

Combining (6.12) and (6.13), we obtain

$$\lim_{\substack{w\to w_\gamma \\ w\in\gamma}} \Psi(w) - \Psi(w_0) \geqslant \arg f'(z^*) - \arg f'(z_0) - 16\pi. \tag{6.14}$$

The curve σ will be of no further use. Note that γ is a crosscut of $\mathcal{D}^*$, which divides $\mathcal{D}^*$ into two domains. One of these domains intersects C and consequently contains an open Jordan arc β such that $\beta \cap C = \emptyset$, β joins a point $w_0' \in \mathcal{D} \cap C$ to w_γ, and such that β and γ determine the same accessible boundary point of $\mathcal{D}^*$ having the complex coordinate w_γ. This last property of β implies that

$$\lim_{\substack{w\to w_\gamma \\ w\in\gamma}} \Psi(w) = \lim_{\substack{w\to w_\gamma \\ w\in\beta}} \Psi(w). \tag{6.15}$$

Since $w_\gamma \in \mathcal{D}_C$, $\beta \subset \mathcal{D}_C$ and $w_0' \in \mathcal{D} \cap \partial\mathcal{D}_C$. Since also $w_0 \in \mathcal{D} \cap \partial\mathcal{D}_C$, we can join w_0' to w_0 by an open Jordan arc lying in the domain $\mathcal{D}_C - \alpha^*$, and consequently we see as we saw (4.8) that

$$|\Psi(w_0') - \Psi(w_0)| \leqslant 2\pi. \tag{6.16}$$

The restriction of $\Psi(w)$ to β has a continuous extension, which we denote by $\Psi_\beta(w)$, to the closure $\bar{\beta}$ of β. With this notation we obtain using (6.14), (6.15) and (6.16),

$$\Psi_\beta(w_\gamma) - \Psi_\beta(w_0') \geqslant \arg f'(z^*) - \arg f'(z_0) - 18\pi. \tag{6.17}$$

Note that (6.1) and (6.17) imply in particular that $\beta \cap L^{(1)} \neq \emptyset$. Let w_1' be the point of $\beta \cap L^{(1)}$ such that the open subarc of β joining w_γ and w_1' does not intersect $L^{(1)}$. Then

$$|\Psi_\beta(w_\gamma) - \Psi_\beta(w_1')| \leqslant 2\pi,$$

and by combining this inequality and (6.17), we obtain

$$\Psi_\beta(w_1') - \Psi_\beta(w_0') \geqslant \arg f'(z^*) - \arg f'(z_0) - 20\pi. \tag{6.18}$$

Let β' be the open subarc of β joining w_0' and w_1', and let $\gamma^{(1)}$ be the union of γ and the Jordan arc on α^* joining w_γ and $w^{(1)}$. Since $\bar{\beta}' \cap \bar{\gamma}^{(1)} = \emptyset$, we can define

$$\Phi(w', w) = \arg (w - w')$$

as a continuous function of two variables for $w' \in \bar{\beta}'$ and $w \in \bar{\gamma}^{(1)}$. By (6.18).

$$\Phi(w_1', w^{(1)}) - \Phi(w_0', w^{(1)}) \geqslant \arg f'(z^*) - \arg f'(z_0) - 20\pi. \tag{6.19}$$

We have $\gamma^{(1)} \cap L^{(1)} = \emptyset$ by (6.6), and consequently

$$|\Phi(w_1', w^{(1)}) - \Phi(w_1', f(e^{i\theta}))| \leqslant 2\pi. \tag{6.20}$$

Since $\gamma^{(1)}$ is contained in a half-plane whose boundary contains w_0',

$$|\Phi(w_0', w^{(1)}) - \Phi(w_0', f(e^{i\theta}))| \leqslant \pi. \tag{6.21}$$

Combining (6.19), (6.20) and (6.21), we obtain

$$\Phi(w_1', f(e^{i\theta})) - \Phi(w_0', f(e^{i\theta})) \geqslant \arg f'(z^*) - \arg f'(z_0) - 23\pi. \tag{6.22}$$

Thus by (6.1) and (6.22), we have

$$\operatorname{Arg} (w_1' - f(e^{i\theta})) - \operatorname{Arg} (w_0' - f(e^{i\theta})) > M. \tag{6.23}$$

Since $w_0' \in \mathcal{D} \cap \partial \mathcal{D}_C$ and $w_1' \in \mathcal{D}_C$, (6.23) contradicts (4.9). The proof of Lemma 4 is complete.

7. This section depends only on the notation of Section 1. Its main result is Lemma 6, which is of independent interest. The proof of Lemma 6 is based on extremal length, and uses the following simple lemma.

LEMMA 5. *Let R be a subset of the open interval $(0, \delta)$ $(\delta > 0)$, and let $m^*(R)$ denote the outer measure of R. For any $r > 0$, set*

$$\gamma_r = \{z: y > 0,\ |z| = r\} \quad (z = x + iy),$$

and set $\Gamma = \{\gamma_r: r \in R\}$. Then the extremal length $\lambda(\Gamma)$ of the family Γ satisfies

$$\lambda(\Gamma) \leqslant \frac{\pi}{\log \dfrac{1}{1-\varkappa}}, \quad \textit{where } \varkappa = \frac{1}{\delta} m^*(R).$$

Proof. Let $\varrho(z)$ be any measurable function defined in the whole plane such that $\varrho(z) \geqslant 0$ and the integral

$$A(\varrho)=\iint \varrho^2\,dx\,dy,$$

taken over the whole plane, is finite and nonzero. Set

$$L(\varrho)=\inf_{\gamma\in\Gamma}\int_\gamma \varrho\,|dz|,$$

where the integral is taken to be infinite if ϱ is not measurable on γ and may be infinite in any case. Then by definition [1]

$$\lambda(\Gamma)=\sup_\varrho \frac{L(\varrho)^2}{A(\varrho)}.$$

For almost every $r\in R$ both of the following integrals are finite, and by Schwarz's inequality

$$L(\varrho)^2\leqslant\left(\int_{\gamma_r}\varrho\,|dz|\right)^2\leqslant \pi r\int_{\gamma_r}\varrho^2\,|dz|.$$

Hence the inequality
$$\frac{L(\varrho)^2}{\pi r}\leqslant\int_{\gamma_r}\varrho^2\,|dz|$$

holds for each r in a measurable subset R_0 of $(0,\delta)$ that contains R, and we have

$$\frac{L(\varrho)^2}{\pi}\int_{R_0}\frac{dr}{r}\leqslant\int_{R_0}\left(\int_{\gamma_r}\varrho^2\,|dz|\right)dr\leqslant A(\varrho).$$

We readily see that

$$\int_{R_0}\frac{dr}{r}\geqslant\int_{\delta-m(R_0)}^{\delta}\frac{dr}{r}\geqslant\int_{\delta-m^*(R)}^{\delta}\frac{dr}{r}=\log\frac{1}{1-\varkappa},$$

where $m(R_0)$ denotes the measure of R_0. Thus

$$\frac{L(\varrho)^2}{A(\varrho)}\leqslant\frac{\pi}{\log\dfrac{1}{1-\varkappa}},$$

and the proof of Lemma 5 is complete.

LEMMA 6. *Let E_z be a subset of ∂D (which is not assumed to be measurable) at each point of which $f(z)$ has a finite angular limit, and set*

$$\mathfrak{E}=\{\mathfrak{a}_\theta\colon e^{i\theta}\in E_z\}.$$

Suppose that for each $e^{i\theta}\in E_z$ there exists a sequence $\{A_n\}$ with the following properties:

for each n, A_n is an arc at some point of ∂D whose endpoint in D is denoted by z_n; (7.1)

$$z_n \to e^{i\theta}, \textit{ and some open triangle contained in } D \textit{ contains all } z_n; \tag{7.2}$$

and
$$\sup_n \frac{\operatorname{diam} \alpha_n}{\operatorname{dist}_{\mathcal{D}}(\alpha_n, \mathfrak{E})} < \infty, \quad \textit{where } \alpha_n = f(A_n), \tag{7.3}$$

and where diam α_n *and* $\operatorname{dist}_{\mathcal{D}}(\alpha_n, \mathfrak{E})$ *denote, respectively, the Euclidean diameter of* α_n *and the relative distance between* α_n *and* $\mathfrak{E}$. *Then* E_z *is a set of measure zero.*

Proof. Consider any fixed $e^{i\theta} \in E_z$, and let $\{A_n\}$ be a sequence satisfying (7.1), (7.2) and (7.3). Set $w_n = f(z_n)$, and note that by (7.2), $w_n \to \mathfrak{a}_\theta$. Thus $\operatorname{dist}_{\mathcal{D}}(\alpha_n, \mathfrak{E}) \to 0$, and (7.3) implies that

$$\operatorname{diam} \alpha_n \to 0. \tag{7.4}$$

For any curve $\beta \subset \mathcal{D}$, we define a family $\Gamma(\beta)$ as follows: $\gamma \in \Gamma(\beta)$ if and only if γ is an open Jordan arc lying in $\mathcal{D}$, each compact subarc of which is rectifiable, and γ joins a point of β to a point of $\mathfrak{E}$. We define another notion of distance from β to $\mathfrak{E}$ as follows:

$$\delta(\beta, \mathfrak{E}) = \sup \{\delta : \delta > 0, \gamma \not\subset \{w : \operatorname{dist}(w, \beta) < \delta\} \text{ if } \gamma \in \Gamma(\beta)\},$$

where dist (w, β) denotes the Euclidean distance from w to β. If no such δ exists, set $\delta(\beta, \mathfrak{E}) = 0$.

We construct a sequence of open Jordan arcs $\beta_n \subset \mathcal{D}$ such that β_n joins w_n to a point of $\mathfrak{A}$, diam $\beta_n \to 0$, and

$$\inf_n \lambda(\Gamma(\beta_n)) > 0. \tag{7.5}$$

Actually, we construct the sequence $\{\beta_n\}$ so that

$$\sup_n \frac{\operatorname{diam} \beta_n}{\delta(\beta_n, \mathfrak{E})} < \infty, \tag{7.6}$$

and then prove that (7.6) implies (7.5).

By (7.3) there exists an h $(0 < h < 1)$ independent of n such that

$$\operatorname{dist}_{\mathcal{D}}(\alpha_n, \mathfrak{E}) > 4\delta_n, \text{ where } \delta_n = h \operatorname{diam} \alpha_n. \tag{7.7}$$

Let α_n be parametrized by $w_n(t)$, $0 \leq t < 1$, with $w_n(0) = w_n$. Set

$$t_n = \sup \{\tau : 0 \leq \tau < 1, w \in \mathcal{D} \text{ if } 0 \leq t \leq \tau \text{ and } |w - w_n(t)| \leq \delta_n\}. \tag{7.8}$$

If no such τ exists, set $t_n = 0$. Clearly $t_n < 1$, because otherwise α_n would be relatively compact in $\mathcal{D}$. Let s_n be an open rectilinear segment whose length is at most δ_n such that s_n lies in $\mathcal{D}$ and joins $w_n(t_n)$ to a point of $\mathfrak{A}$. We readily see that $w_n(t) \notin s_n$ if $0 \leq t \leq t_n$. Thus the set

$$\beta_n = \{w_n(t) : 0 < t \leq t_n\} \cup s_n$$

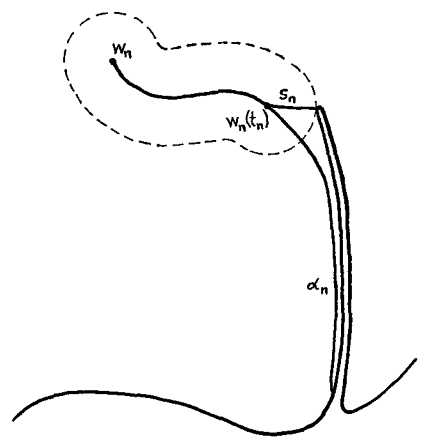

Fig. 3.

is an open Jordan arc that lies in $\mathcal{D}$ and joins w_n to a point of $\mathfrak{A}$. Note that since $h<1$,

$$\operatorname{diam} \beta_n \leqslant 2 \operatorname{diam} \alpha_n. \tag{7.9}$$

Concerning Fig. 3, we note that α_n may or may not tend at one end to an accessible boundary point of $\mathcal{D}$.

We now establish (7.6). To this end we prove that $\delta(\beta_n, \mathfrak{E}) \geqslant \delta_n$ for each n. Suppose to the contrary that for some n, $\delta(\beta_n, \mathfrak{E}) < \delta_n$. Then there exists a $\gamma \in \Gamma(\beta_n)$ such that

$$\gamma \cup \{w(\mathfrak{a})\} \subset \{w\colon \operatorname{dist}(w, \beta_n) < \delta_n\}, \tag{7.10}$$

where $\mathfrak{a}$ is the endpoint of γ in $\mathfrak{E}$. Set

$$V_n = \{w\colon \operatorname{dist}(w, s_n) < \delta_n\},$$

and note that diam $V_n \leqslant 3\delta_n$. By (7.8) and (7.10), $w(\mathfrak{a}) \in V_n$. If $\gamma \subset V_n$, then by considering the two cases $\gamma \cap s_n = \emptyset$ and $\gamma \cap s_n \neq \emptyset$ separately, we readily see that $\gamma \cup s_n$ contains an open Jordan arc that joins $\mathfrak{a}$ to some $w_n(t)$ $(0 \leqslant t \leqslant t_n)$ and lies in V_n, contrary to (7.7). On the other hand, if $\gamma \not\subset V_n$, then an open subarc γ' of γ lies in V_n and joins $\mathfrak{a}$ to a point $w_n' \in \partial V_n$ $(w_n' \in \gamma)$. By (7.10), $|w_n' - w_n(t)| < \delta_n$ for some t $(0 \leqslant t \leqslant t_n)$; and (7.8) implies that the closed rectilinear segment joining this $w_n(t)$ and w_n' lies in $\mathcal{D}$. Since the union of γ' and this rectilinear segment is in diameter at most $4\delta_n$, and since this union contains an open Jordan arc joining this $w_n(t)$ to $\mathfrak{a}$, we again have a contradiction of (7.7). We conclude that $\delta(\beta_n, \mathfrak{E}) \geqslant \delta_n$ for each n. Combining this inequality, (7.7) and (7.9) we obtain

$$\frac{\operatorname{diam} \beta_n}{\delta(\beta_n, \mathfrak{E})} \leqslant \frac{2 \operatorname{diam} \alpha_n}{\delta_n} = \frac{2}{h} \quad (n = 1, 2, \ldots).$$

This proves (7.6).

We now prove that (7.6) implies (7.5). By (7.6) there exists a positive integer k independent of n such that

$$\delta(\beta_n, \mathfrak{E}) > \frac{1}{k} \operatorname{diam} \beta_n \quad (n=1, 2, \ldots). \tag{7.11}$$

Consider on the square

$$Q_n = \{w: |\operatorname{Re} w - \operatorname{Re} w_n| \leqslant 2 \operatorname{diam} \beta_n, \ |\operatorname{Im} w - \operatorname{Im} w_n| \leqslant 2 \operatorname{diam} \beta_n\}$$

a mesh of horizontal and vertical line segments that subdivides Q_n into $(16\,k)^2$ nonoverlapping closed squares Q_{nm}, each of side length $(1/4k) \operatorname{diam} \beta_n$. Let K_n be the union of all Q_{nm} that intersect the closure of β_n, and let G_n be the interior of the union of all Q_{nm} that intersect K_n. Then $K_n \subset G_n$. For each n define a family Γ_n as follows: $\gamma \in \Gamma_n$ if and only if γ is an open Jordan arc, each compact subarc of which is rectifiable, that lies in $G_n - K_n$ and joins a point of K_n to a point of ∂G_n. We note that

$$G_n \subset \{w: \operatorname{dist}(w, \beta_n) \leqslant 2 \operatorname{diam} Q_{nm} < (1/k) \operatorname{diam} \beta_n\}.$$

Thus (7.11) implies that $\gamma \not\subset G_n$ if $\gamma \in \Gamma(\beta_n)$. It follows that each $\gamma \in \Gamma(\beta_n)$ contains some $\gamma' \in \Gamma_n$, and we conclude that $\lambda(\Gamma(\beta_n)) \geqslant \lambda(\Gamma_n)$ (see [1]). We observe that for each n there are only finitely many possible values of $\lambda(\Gamma_n)$, and each of these values is positive. Moreover, since k is independent of n and the extremal length is invariant under translation and change of scale, the set of possible values of $\lambda(\Gamma_n)$ is independent of n. This proves (7.5).

Let $z = T(\zeta)$ be a linear transformation taking the open upper half-plane H onto D and ∞ to 1. We continue to consider the same $e^{i\theta}$, although we suppose $e^{i\theta} \neq 1$. Define ξ and E_ζ by requiring

$$T(\xi) = e^{i\theta}, \quad T(E_\zeta) = E_z - \{1\}.$$

Set $F(\zeta) = f(T(\zeta))$ $(\zeta \in H)$, and define β_n^ζ $(n=1, 2, \ldots)$ by requiring $F(\beta_n^\zeta) = \beta_n$. By (7.4) and (7.9), $\operatorname{diam} \beta_n \to 0$; and consequently, since $\xi \neq \infty$, it follows readily from Koebe's lemma that $\operatorname{diam} \beta_n^\zeta \to 0$. Also using Koebe's lemma, we see that each β_n^ζ has an endpoint $\xi_n \in \partial H$, and since $\operatorname{diam} \beta_n^\zeta \to 0$, we can suppose without loss of generality that $\xi_n \neq \infty$ $(n=1, 2, \ldots)$. Also, $\xi_n \to \xi$. By (7.6), $\xi_n \notin E_\zeta$ $(\delta(\beta_n, \mathfrak{E}) = 0$ if $\xi_n \in E_\zeta)$, and in particular $\xi_n \neq \xi$. Infinitely many ξ_n lie on the same side of ξ, and by replacing $\{\xi_n\}$ by a certain subsequence, we can suppose without loss of generality that all ξ_n lie on the same side of ξ. We consider the case where $\xi_n > \xi$ $(n=1, 2, \ldots)$; the other case is completely analogous.

Define $\varrho_n e^{i\varphi_n}$ $(0 < \varphi_n < \pi)$ by $T(\varrho_n e^{i\varphi_n}) = z_n$. By (7.2) there exists a number η independent of n such that $0 < \eta < \pi/4$ and $\eta < \varphi_n < \pi - \eta$ $(n=1, 2, \ldots)$. Set

$$r_n = (\xi_n - \xi) \sin \eta \quad (n=1, 2, \ldots),$$

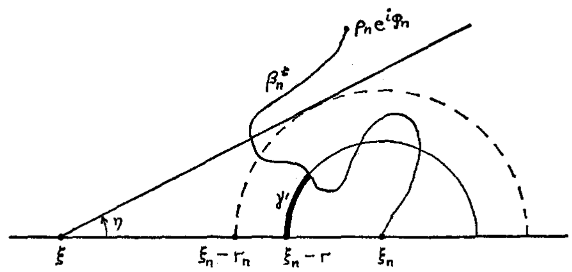

Fig. 4.

and let Γ_n'' be the family of all semicircles $H \cap \{|\zeta - \xi_n| = r\}$, where $0 < r < r_n$ and $\xi_n - r \in E_\zeta$ (see Fig. 4). We readily see that each $\gamma'' \in \Gamma_n''$ contains some curve γ' in the family

$$\Gamma_n' = \{\gamma' : \gamma' \subset H,\ F(\gamma') \in \Gamma(\beta_n)\}.$$

Thus

$$\lambda(\Gamma_n'') \geqslant \lambda(\Gamma_n'). \tag{7.12}$$

Since extremal length is a conformal invariant [1], (7.5) implies

$$\inf_n \lambda(\Gamma_n') > 0. \tag{7.13}$$

By Lemma 5

$$\lambda(\Gamma_n'') \leqslant \frac{\pi}{\log \dfrac{1}{1 - \varkappa_n}}, \quad \text{where } \varkappa_n = \frac{1}{r_n} m^*(E_\zeta \cap (\xi_n - r_n, \xi_n)).$$

Thus by (7.12) and (7.13), $\sup_n \varkappa_n < 1$; and since the ratio $r_n/(\xi_n - \xi) = \sin \eta$ is independent of n, we see that

$$\sup_n \frac{m^*(E_\zeta \cap (\xi, \xi_n))}{\xi_n - \xi} < 1.$$

This implies that no point of E_ζ is a point of outer density for E_ζ, and we conclude that E_ζ is a set of measure zero [14, p. 129]. Thus E_z is a set of measure zero, and the proof of Lemma 6 is complete.

Remark. An immediate consequence of Lemma 6 is the following result: Let $\mathfrak{E}$ be a subset of $\mathfrak{A}$, and suppose that for each $\mathfrak{a} \in \mathfrak{E}$ there exists a sequence $\{c_n\}$ of crosscuts of $\mathcal{D}$, each of which separates $\mathfrak{a}$ from a fixed point $w_0 \in \mathcal{D}$, such that diam $c_n \to 0$ and

$$\sup_n \frac{\operatorname{diam} c_n}{\operatorname{dist}_{\mathcal{D}}(c_n, \mathfrak{E})} < \infty.$$

Then $\mathfrak{E}$ is a $\mathcal{D}$-conformal null-set. (The condition that c_n have endpoints in $\mathfrak{A}$ can be relaxed to require that c_n tend at each end to a prime end of $\mathcal{D}$.) This result is applied in [9] to prove the following theorem: If for each $\mathfrak{a} \in \mathfrak{A}$ and each sufficiently small $r > 0$, $L(\mathfrak{a}, r)$ denotes the length of the component of $\mathcal{D} \cap \{|w - w(\mathfrak{a})| = r\}$ nearest $\mathfrak{a}$ that separates $\mathfrak{a}$ from w_0, and if

$$A(\mathfrak{a}, r) = \int_0^r L(\mathfrak{a}, r)\, dr$$

(which exists as a Lebesgue integral), then

$$\limsup_{r \to 0} \frac{A(\mathfrak{a}, r)}{\pi r^2} \geqslant \frac{1}{2} \left(\text{which implies } \limsup_{r \to 0} \frac{L(\mathfrak{a}, r)}{2\pi r} \geqslant \frac{1}{2}\right),$$

with the possible exception of those $\mathfrak{a}$ in a $\mathcal{D}$-conformal null-set.

8. *Proof of Theorem 1. Conclusion.* The sets $E_z^{(4)}$ and $\mathfrak{E}^{(4)}$ are defined in Section 6. Consider a fixed $e^{i\theta} \in E_z^{(4)}$. We recall from Section 4 that since $e^{i\theta} \in E_z^{(3)}$, $\arg f'(z)$ is unbounded above in Δ_θ. Thus since $e^{i\theta} \notin N$, Lemma 3 states the existence of a sequence $\{A_n\}$ satisfying (5.1), (5.2) and (5.3). Since for each n the initial point z_n of A_n is on $\partial\Delta_\theta$, and since $z_n \to e^{i\theta}$ by (5.2), one side of Δ_θ contains a subsequence of $\{z_n\}$, which of course converges to $e^{i\theta}$. By using (4.5) and (4.6), we see that this side of Δ_θ contains an open rectilinear segment S joining a point z_0 to $e^{i\theta}$ such that $f(z_0) \in \mathcal{D} \cap \partial\mathcal{D}_C$ and $f(S) \subset \mathcal{D}_C$. By replacing the sequence $\{A_n\}$ by a certain subsequence, we can suppose without loss of generality that S contains all z_n; and since $\arg f'(z_n) \to +\infty$ by (5.2), we can also suppose without loss of generality that

$$\arg f'(z_n) - \arg f'(z_0) > M + 23\pi \quad (n = 1, 2, \ldots),$$

where M is the number defined in Section 4. We now fix n and apply Lemma 4 with $z^* = z_n$ and $A^* = A_n$. Note that by (5.1), $A_n \cap S = \{z_n\}$. Thus using (5.3), we see that all hypotheses of Lemma 4 are fulfilled, and we conclude that

$$\operatorname{dist}_{\mathcal{D}}(f(A_n), \mathfrak{E}^{(4)}) \geqslant \operatorname{diam} f(A_n).$$

Since such a sequence $\{A_n\}$ exists for each $e^{i\theta} \in E_z^{(4)}$, Lemma 6 implies that $E_z^{(4)}$ is a set of measure zero. Thus since N is countable, $E_z^{(3)}$ is a set of measure zero, and this is the desired contradiction. The proof of Theorem 1 is complete.

9. In this section we prove Theorem 2. The proof uses the following simple lemma.

LEMMA 7. *If $f(z)$ is isogonal at $e^{i\theta}$, then $\mathfrak{a}_\theta \in \mathfrak{A}_1$.*

Proof. Suppose to the contrary that for some $e^{i\theta}$, $f(z)$ is isogonal at $e^{i\theta}$ and $\mathfrak{a}_\theta \notin \mathfrak{A}_1$. Then there exists a Jordan domain U_w (that is, ∂U_w is a single Jordan curve) contained in $\mathcal{D}$ and having the following three properties:

(a) $f(e^{i\theta}) \in \bar{U}_w \subset \mathcal{D} \cup \{f(e^{i\theta})\}$;

(b) for any open triangle Δ contained in D and having one vertex at $e^{i\theta}$, $f(z) \in U_w$ if $z \in \Delta$ and z is sufficiently near $e^{i\theta}$; and

(c) for some α satisfying $0<\alpha<1$, the function $(w-f(e^{i\theta}))^\alpha$, which is defined and continuous in $\mathcal{D}$, maps U_w onto a Jordan domain whose boundary has a tangent at the origin.

By (a) and (b) the preimage $U_z = f^{-1}(U_w)$ under $f(z)$ is a Jordan domain satisfying

$$e^{i\theta} \in \bar{U}_z \subset D \cup \{e^{i\theta}\};$$

and ∂U_z is tangent to ∂D at $e^{i\theta}$. By (c) the function $(f(z)-f(e^{i\theta}))^\alpha$ maps U_z onto a Jordan domain whose boundary has a tangent at the origin. Thus a well-known theorem of Lindelöf implies that $(f(z)-f(e^{i\theta}))^\alpha$ is isogonal at $e^{i\theta}$, contrary to the assumption that $f(z)$ is isogonal at $e^{i\theta}$. The proof of Lemma 7 is complete.

Proof of Theorem 2. Part (i) is an immediate consequence of Theorem 1 and Lemma 7.

It follows from a routine argument that $g(w)$ has a nonzero angular derivative at a point $\mathfrak{a} \in \mathfrak{A}_1$ if there exists a finite, nonzero complex number $g'(\mathfrak{a})$ such that for each angle $\mathcal{A}$ at $\mathfrak{a}$,

$$\lim_{\substack{w\to\mathfrak{a}\\ w\in\mathcal{A}}} g'(w) = g'(\mathfrak{a}); \tag{9.1}$$

that is, the first equality of (2.1) is a consequence of the second. If we let $w=f(z)$ denote the inverse function of $z=g(w)$, then we see that if $f'(z)$ has a finite, nonzero angular limit $f'(e^{i\theta})$ at $e^{i\theta}$, then (9.1), where $\mathfrak{a}=\mathfrak{a}_\theta$ and $g'(\mathfrak{a})=1/f'(e^{i\theta})$, holds for each angle $\mathcal{A}$ at $\mathfrak{a}_\theta$. Thus (ii) is an immediate consequence of Theorem 1.

We now prove (iii). Let $\mathfrak{E}$ be a subset of $\mathfrak{A}_1$, and take $w=f(z)$ to be the inverse function of $z=g(w)$. Set

$$E_z = \{e^{i\theta}\colon \mathfrak{a}_\theta \in \mathfrak{E}\}, \quad E_w = \{f(e^{i\theta})\colon e^{i\theta} \in E_z\}.$$

Then E_w is the set of complex coordinates of the points of $\mathfrak{E}$.

We first suppose that E_z has measure zero and that E_w does not have linear measure zero, and we derive a contradiction. We shall define subsets $E_z^{(j)}$ $(j=1, 2, 3)$ of E_z, and for each j it shall be understood that

$$E_w^{(j)} = \{f(e^{i\theta})\colon e^{i\theta} \in E_z^{(j)}\}.$$

Associate with each $\mathfrak{a}_\theta \in \mathfrak{E}$ rational numbers $\varphi(\theta)$ and $\alpha(\theta)$ $(0<\alpha(\theta)<\pi/2)$ such that for some angle $\mathcal{A}$ at $\mathfrak{a}_\theta$, all points of the set

$$\Delta(\theta) = \{f(e^{i\theta}) + \varrho e^{i\varphi}\colon \varrho>0,\ |\varphi-\varphi(\theta)|<\alpha(\theta)\}$$

that are sufficiently near $f(e^{i\theta})$ are in $\mathcal{A}$. There exist φ_0, α_0 and a subset $E_z^{(1)}$ of E_z such that $E_w^{(1)}$ does not have linear measure zero, and such that $\varphi(\theta)=\varphi_0$ and $\alpha(\theta)=\alpha_0$ for each

$e^{i\theta} \in E_z^{(1)}$. Associate with each $e^{i\theta} \in E_z^{(1)}$ a straight line $L(\theta)$ in the w-plane with the following properties:

$$L(\theta) \text{ intersects the half-line } \{f(e^{i\theta}) + \varrho e^{i\varphi_\theta}: \varrho > 0\} \text{ at right angles;} \tag{9.2}$$

$$\text{the Euclidean distance from the origin to } L(\theta) \text{ is a rational number;} \tag{9.3}$$

$$\Delta'(\theta) \subset \mathfrak{D}, \text{ where } \Delta'(\theta) \text{ is the bounded component of } \Delta(\theta) - L(\theta). \tag{9.4}$$

By (9.2) and (9.3), the family $\{L(\theta): e^{i\theta} \in E_z^{(1)}\}$ is at most countable. Thus there exist L_0 and a subset $E_z^{(2)}$ of $E_z^{(1)}$ such that $E_w^{(2)}$ does not have linear measure zero, and such that $L(\theta) = L_0$ for each $e^{i\theta} \in E_z^{(2)}$. There are at most countably many components of $\bigcup \Delta'(\theta)$, where the union is taken over all $e^{i\theta} \in E_z^{(2)}$. Thus one of these components, which we denote by G, is of the form

$$G = \bigcup_{e^{i\theta} \in E_z^{(3)}} \Delta'(\theta),$$

where $E_z^{(3)} \subset E_z^{(2)}$ and $E_w^{(3)}$ does not have linear measure zero. Note that $G \subset \mathfrak{D}$ by (9.4). It is readily seen that ∂G is a rectifiable Jordan curve and that $E_w^{(3)}$ has positive outer measure with respect to length on ∂G. Thus under one-to-one conformal mapping $w = w(\zeta)$ of $\{|\zeta| < 1\}$ onto G, $E_w^{(3)}$ corresponds to a set $E_\zeta^{(3)}$ on $\{|\zeta| = 1\}$ of positive outer measure [13, p. 127]. Set $F(\zeta) = g(w(\zeta))$, and let E_z^* be a G_δ-set on $\{|z| = 1\}$ of measure zero such that $E_z^{(3)} \subset E_z^*$. Since the angular-limit function $F(e^{i\theta})$ is a function of the first Baire class defined on an $F_{\sigma\delta}$-set [4, p. 311], the set

$$E_\zeta^* = \{e^{i\theta}: F(e^{i\theta}) \in E_z^*\}$$

is a Borel set [4, p. 303]. Since $E_\zeta^{(3)} \subset E_\zeta^*$, E_ζ^* has positive measure, and we have a contradiction of an extension of Löwner's lemma [11, p. 34]. We conclude that E_w has linear measure zero if E_z has measure zero.

We now suppose that E_w has linear measure zero and that E_z has positive outer measure, and we again derive a contradiction. We define G as above, except that for each $j = 1, 2, 3$, we replace the requirement "$E_w^{(j)}$ does not have linear measure zero" by the requirement "$E_z^{(j)}$ has positive outer measure". By part (ii) of Theorem 2 we can suppose without loss of generality that $g(w)$ has a nonzero angular derivative at each point of $\mathfrak{E}$. Thus $g(w)$ is "isogonal" at each point of $\mathfrak{E}$, and consequently we can associate with each $e^{i\theta} \in E_z^{(3)}$ rational numbers $\psi(\theta)$ and $\beta(\theta)$ $(0 < \beta(\theta) < \pi/2)$ such that all points of the set

$$\{e^{i\theta} + \sigma e^{i\psi}: \sigma > 0. \ |\psi - \psi(\theta)| < \beta(\theta)\}$$

that are sufficiently near $e^{i\theta}$ are in $g(\Delta'(\theta))$. For each $e^{i\theta} \in E_z^{(3)}$ let $\mathfrak{b}_\theta$ denote the accessible boundary point of $g(G)$ that is determined by the segment

$$\{e^{i\theta}+\sigma e^{i\varphi(\theta)}\colon 0<\sigma\leqslant\sigma_0\},$$

where σ_0 is sufficiently small to make this segment lie in $g(G)$. Let $z=z(\zeta)$ be a function mapping $\{|\zeta|<1\}$ one-to-one and conformally onto $g(G)$, and let $E_\zeta^{(3)}$ be the subset of $\{|\zeta|=1\}$ that corresponds under this mapping to $\{\mathfrak{b}_\theta\colon e^{i\theta}\in E_z^{(3)}\}$. Since $E_z^{(3)}$ has positive outer measure, it does not have linear measure zero; and we see, by using the argument in the first part of this proof of part (iii), that $E_\zeta^{(3)}$ has positive outer measure. On the other hand, $f(z(\zeta))$ maps $\{|\zeta|<1\}$ onto G with $E_\zeta^{(3)}$ corresponding to $E_w^{(3)}$; and it follows easily from the special nature of ∂G that $E_w^{(3)}$ has measure zero with respect to length on ∂G. This is the desired contradiction. We conclude that E_z has measure zero if E_w has linear measure zero.

The proof of Theorem 2 is complete.

Remark. Let $\mathfrak{a}\in\mathfrak{A}$, and suppose there exists a curve $A_w\subset\mathcal{D}$ such that $A_w\cup\{\mathfrak{a}\}$ is a Jordan arc in the metric space $\mathcal{D}\cup\mathfrak{A}$, and such that $g'(w)$ has a finite, nonzero limit $g'(\mathfrak{a})$ on A_w at $\mathfrak{a}$. Then $\mathfrak{a}\in\mathfrak{A}_1$ and $g(w)$ has a nonzero angular derivative at $\mathfrak{a}$. We see this as follows. Take $w=f(z)$ to be the inverse function of $z=g(w)$, and let θ be such that $\mathfrak{a}=\mathfrak{a}_\theta$. Then the curve $A_z=g(A_w)$ is an arc at $e^{i\theta}$, and $f'(z)$ has the limit $1/g'(\mathfrak{a})$ on A_z at $e^{i\theta}$. By Lemma 2, $f'(z)$ is a normal holomorphic function, and consequently the theorem of Lehto and Virtanen [6] implies that $f(z)$ has the angular limit $1/g'(\mathfrak{a})$ at $e^{i\theta}$. Thus by Lemma 7, $\mathfrak{a}\in\mathfrak{A}_1$; and as we saw in the proof of part (ii) of Theorem 2, $g(w)$ has a nonzero angular derivative at $\mathfrak{a}_\theta$ (whose value is $g'(\mathfrak{a})$).

10. In this section we give two counterexamples.

EXAMPLE 1. There exists a Jordan domain $\mathcal{D}$ such that $\mathfrak{A}=\mathfrak{A}_2\cup\mathfrak{N}$ for some $\mathcal{D}$-conformal null-set $\mathfrak{N}$. By Theorem 2, parts (i) and (iii), $\mathcal{D}$ will have this property provided $\mathfrak{A}_1$ has linear measure zero (for a Jordan domain we make no distinction between $\mathfrak{a}$ and $w(\mathfrak{a})$). We easily construct a $\mathcal{D}$ with this property, as follows.

By the middle third of a closed rectilinear segment S we mean the closed segment on S whose length is one third that of S and which is equidistant from the endpoints of S. Let Δ_1 be a closed equilateral triangle of side length 1. Let $\Delta_{1,k}$ $(k=1, 2, 3)$ be closed equilateral triangles of side length $\frac{1}{3}$ such that $\Delta_1\cap\Delta_{1,k}$ $(k=1, 2, 3)$ are the middle thirds of the sides of Δ_1. Set

$$\Delta_2=\Delta_1\cup(\bigcup\Delta_{1,k}).$$

Let $\Delta_{2,k}$ $(k=1, \ldots, 12)$ be closed equilateral triangles of side length $(\frac{1}{3})^2$ such that $\Delta_2\cap\Delta_{2,k}$ $(k=1, \ldots, 12)$ are the middle thirds of the rectilinear segments (whose endpoints are corners of $\partial\Delta_2$) on $\partial\Delta_2$. Set

$$\Delta_3=\Delta_2\cup(\bigcup\Delta_{2,k}).$$

Continuing in this way, we define Δ_n $(n=1, 2, \ldots)$. Let $\mathcal{D}$ be the interior of $\bigcup \Delta_n$. Then $\partial\mathcal{D}$ is a Jordan curve, since it could have been defined by means of Knopp's triangle construction [4, p. 233]. It is easy to see that $\mathfrak{A}_1$ is contained in a countable union of "middle-third" Cantor sets, and consequently that $\mathfrak{A}_1$ has linear measure zero.

It was previously known that there exists a Jordan domain $\mathcal{D}$ such that for almost every θ, $f(e^{i\theta})$ is not an endpoint of an open rectilinear segment lying in $\mathcal{D}$ (see Lavrentieff [5] and Lohwater and Piranian [7]).

Remark. Theorem 2 has the following geometrical consequence: If $\mathfrak{A}_2$ is at most countable, then the set of complex coordinates of points of $\mathfrak{A}_1$ does not have linear measure zero (this set is a Borel set, and is therefore linearly measurable; but we do not prove this). Also the local analogue in terms of intervals of prime ends is true.

EXAMPLE 2. The set of points $e^{i\theta}$ at which neither (1.1) nor (1.2) holds can be a compact set of positive logarithmic capacity.

Let $\{v_n\}$ be a sequence of distinct real numbers, and let $\{u_n\}$ be a sequence of positive numbers having the limit zero such that if we set

$$\mathcal{D} = \{w\colon \operatorname{Re} w > 0\} - \bigcup_{n=1}^{\infty} \{u + iv_n\colon 0 < u \leqslant u_n\},$$

then the inner tangent to $\partial\mathcal{D}$ does not exist at any point of the imaginary axis. Let $w=f(z)$ be a function mapping D one-to-one and conformally onto $\mathcal{D}$, and let $f(z)$ also denote the continuous extension of this function to $\bar{D}$. Define E_z to be the set of all $e^{i\theta}$ satisfying one of the following conditions: $f(e^{i\theta})=\infty$, $\operatorname{Re} f(e^{i\theta})=0$, or $f(e^{i\theta})=u_n+iv_n$ for some n. Clearly (1.2) does not hold for any $e^{i\theta}$; and since $\mathfrak{a}_\theta \in \mathfrak{A}_1$ if (1.1) holds at $e^{i\theta}$, we see that (1.1) holds if and only if $e^{i\theta} \notin E_z$. Also, E_z is a compact, totally disconnected set, and each component of $(\partial D) - E_z$ is mapped by $f(z)$ onto a horizontal segment. By reflection the real part of $f(z)$ is extended to a *single-valued* (nonconstant) positive harmonic function in the complement of E_z, and consequently E_z has positive logarithmic capacity [10, p. 140].

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