A NON-LINEAR HODGE–DE RHAM THEOREM

BY

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Introduction

The classical Hodge–de Rham theorem for Riemannian manifolds establishes an isomorphism between the de Rham cohomology groups and the groups of harmonic forms living on the manifold. This may be restated in the following way: if γ is a given closed form on the manifold, then there exists a unique harmonic form which differs from γ by an exact form.

Since a harmonic 1-form describes an incompressible potential fluid flow, the Hodge–de Rham theorem, in this case, can be expressed in the following form for compact manifolds: there exists a unique incompressible potential flow having prescribed periods (that is, prescribed circulation around the handles).

The stationary flow of a *compressible* fluid is described by a quasi-linear second order partial differential equation of divergence type. The equation is elliptic or hyperbolic depending upon whether the flow is subsonic or supersonic. Bers has conjectured the existence of a compressible subsonic flow on a Riemannian manifold having prescribed periods.

Our main theorem establishes existence and uniqueness for a certain non-linear, non-regular global problem on a Riemannian n-manifold. The problem is suggested by the classical framework of gas dynamics and its solution gives an affirmative answer to Bers' conjecture.

In the Hilbert space $L^2(M)$ of 1-forms with square integrable coefficients, the collection of harmonic 1-forms (locally, $\omega = d\phi$ where $\Delta\phi = 0$ for the Laplace–Beltrami operator Δ) spans a b_1 dimensional subspace H ($b_1 = \dim H^1(M, R)$ = dimension of first cohomology group of M over the reals = first Betti number of M). Roughly speaking, the content of our main theorem can be described as follows: the collection of 1-forms ω such that locally

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$\omega = d\phi$, $\Delta_\varrho \phi = 0$ for a (non linear) operator Δ_ϱ of "gas dynamics type", spans a b_1-dimensional manifold $G \subset L^2(M)$. There is a natural projection map π: $G \to H$. If the equation is *regular* (cf. § 1.2) then π is bijective. If it is *admissible* but *not* regular (cf. § 1.2—this is the case for the classical gas dynamics equation in Euclidean space) then π is injective but *not* surjective. In fact the image $\pi(G)$ is a bounded star shaped subdomain of H.

In this way a non-linear analogue of the *Hodge decomposition* theorem for 1-forms is established. A "weak" decomposition theorem is also obtained for p-forms in the regular case.

In section 1, we formulate the problem and state the main theorem. Section 2 contains motivation from classical gas dynamics. In section 3, these ideas are formulated for an arbitrary manifold, and the "duality" between a certain conjugate problem and the non-linear Hodge–de Rham theorem for 1-forms is established. In section 4, we prove the conjugate theorem using variational techniques first introduced in the classical case by Shiffman [7]. The appendix contains a new proof of Hodge's theorem for harmonic p-forms.

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I. Preliminaries

1.1. We consider an orientable Riemannian manifold M of dimension n. Let $\tilde{\Omega}^p$ be the space of p-forms on M with coefficients of class C^1. The inner product on the tangent space at each point $x \in M$ induces a (pointwise) inner product on $\tilde{\Omega}^p$ which we denote by $Q^p(\omega, \nu)$; we write $Q^p(\omega)$ for $Q^p(\omega, \omega)$.

Let d: $\tilde{\Omega}^p \to \tilde{\Omega}^{p+1}$ be exterior differentiation and $*: \tilde{\Omega}^p \to \tilde{\Omega}^{n-p}$ the canonical isomorphism between these spaces satisfying $(*)^2 = (-1)^{np+p}$. Set $\delta = (-1)^{np+n+1} * d *$ so that δ is a map of $\tilde{\Omega}^p$ into $\tilde{\Omega}^{p-1}$. It can be verified that $\omega \wedge * \nu = Q^p(\omega, \nu)\, dV$ where we have written dV for the n-form $*1$. For $\omega \in \tilde{\Omega}^1$ we can write $\omega = \omega_i dx^i$ in terms of local coordinates $x^1, \ldots, x^n$. If g_{ij} are the components of the metric tensor in these coordinates one can show that $Q^1(\omega) = g^{ij} \omega_i \omega_j$. (We are observing the standard summation convention.)

Let $L^2_{(p)}$ be the completion with respect to the norm $\|\omega\|_p = (\int_M \omega \wedge * \omega)^{1/2}$ of the subspace of $\tilde{\Omega}^p$ consisting of p-forms having compact carrier. The associated inner product is

$$(\omega, \nu) = \int_M \omega \wedge * \nu = \int_M Q^p(\omega, \nu)\, dV.$$

Let $\Omega^p = \tilde{\Omega}^p \cap L^2_{(p)}$. A p-form $\omega \in \Omega^p$ is closed if $d\omega = 0$ and coclosed if $\delta\omega = 0$. If E_p is the closure in $L^2_{(p)}$ of exact forms, $\omega = d\nu$ with $\nu \in \Omega^{p-1}$ and having compact carrier, and E_p^* is the closure of coexact forms, $\omega = \delta\nu$ with $\nu \in \Omega^{p+1}$ and having compact carrier, then it is easily verified using Stoke's theorem that $E_p^{*\perp}$, the orthogonal complement in L_p^2 of E_p^*, is the L_p^2 closure of closed forms.

1.2. *Definitions.* A map $\sigma: M \times R \to R$ is said to be *admissible* if there exist constants $0 < a_\sigma < \infty$ and $0 < k < \infty$ such that, on the interval $0 \leqslant a < a_\sigma$

(i) $\sigma(x, a) \in C^{2+\alpha}$ in x

$\sigma(x, a) \in C^{1+\alpha}$ in a

(ii) $\dfrac{1}{k} < \sigma < k$

(iii) $\dfrac{\partial}{\partial a}(a\sigma^2(x, a)) > 0.$

Definition. Q_σ, the least upper bound over all a_σ for which conditions (i), (ii), and (iii) hold, is called the *sonic value* of σ.

Definition. A map $\sigma: M \times R \to R$ is said to be *regular* if it is admissible with $Q_\sigma = \infty$ and, in addition, there exists a constant $0 < k_1 < \infty$ such that for $0 \leqslant a < \infty$

(iv) $\dfrac{1}{k_1} < \dfrac{\partial}{\partial a}(a\sigma^2(x, a)) < k_1.$

Definition. $\omega \in \Omega^p$ is said to be *σ-subsonic* if for any compact subdomain $D \subset M$

$$\sup_D Q^p(\omega) < Q_\sigma,$$

where Q_σ is the sonic value.

Remark. If σ is regular then every $\omega \in \Omega^p$ is σ-subsonic.

1.3. We now state the main result.

NON-REGULAR THEOREM. *Let M be compact, $\gamma \in E_1^{*\perp}$ be given, and ϱ admissible. Then there is a constant t_ϱ such that for each t, $0 \leqslant t < t_\varrho$, there exists a unique $\omega_t \in \Omega^1$ satisfying*

(i) $d\omega_t = 0$

(ii) $\delta\varrho\omega_t = 0$ $(\varrho = \varrho(x, Q^1(\omega_t)))$

(iii) $\omega_t - t\gamma \in E_1$ *and is exact if* $\gamma \in \Omega^1$

(iv) ω_t *is ϱ-subsonic*

(v) $\lim_{t \to t_\varrho} (\max_{x \in M} Q^1(\omega_t)) = Q_\varrho.$

REGULAR THEOREM. *If ϱ is regular, then the above results hold for $t_\varrho = \infty$ (i.e., for arbitrary $t\gamma$) and without the assumption that M is compact.*

Remark. The Regular Theorem includes the linear case where ϱ is a strictly positive function of x alone; $\varrho = 1$ corresponds to the classical Hodge theorem.

1.4. For $p > 1$, our methods give the following

THEOREM. *Let $\gamma \in E_p^{*\perp}$ be given and suppose that ϱ is regular. Then there exists a unique $\omega \in L^2_{(p)}$ such that ω is a weak solution of*

(i) $d\omega = 0$

(ii) $\delta\varrho\omega = 0$

(iii) $\omega - \gamma \in E_p$

To show that ω is differentiable and to obtain the estimate necessary to extend the theorem to admissible ϱ requires a DeGiorgi type theorem for elliptic systems (compare § 4.3). Such a result is not known for the above system and is not true for arbitrary elliptic systems.

II. Gas dynamics in R^n

2.1. In R^n, the stationary potential flow of a compressible fluid is described by an equation for the *velocity potential* $\phi(x)$ of the flow. The *velocity vector* of the flow is $\nabla\phi$ and the *speed* of the flow is given by $q = |\nabla\phi|^{\frac{1}{2}}$.

The *density* $\varrho > 0$ of the flow is assumed to be a given function of $Q = q^2$. It follows from the equations of motion and continuity, and from Bernoulli's law that ϕ is a solution of the non-linear equation

$$\sum_i \frac{\partial}{\partial x^i}\left(\varrho \frac{\partial \phi}{\partial x^i}\right) = 0, \qquad \varrho = \varrho(Q). \tag{1}$$

The character of the flow depends on the Mach number M which is defined by

$$M^2 = \frac{-2Q}{\varrho}\,\varrho'(Q).$$

The flow is called *subsonic* if $M < 1$, *sonic* if $M = 1$, and *supersonic* if $M > 1$. Since $\varrho > 0$ and

$$1 - M^2 = \frac{1}{\varrho}(\varrho + 2Q\varrho'),$$

the flow is subsonic if and only if

$$\frac{d}{dQ}(Q\varrho^2) > 0. \tag{2}$$

On the other hand, equation (1) will be elliptic if and only if the matrix

$$\left(\varrho\delta_{ij} + 2\varrho' \frac{\partial\phi}{\partial x^i}\frac{\partial\phi}{\partial x^j}\right)$$

is positive definite. A calculation shows that there are two distinct eigenvalues, namely

$$\lambda_1 = \varrho + 2\varrho' Q$$
$$\lambda_2 = \ldots = \lambda_n = \varrho.$$

This, together with the fact that ϱ is strictly positive, shows that (1) is elliptic if and only if (2) holds, or the flow is subsonic.

2.2. A particular example is furnished by *polytropic flow* for which

$$\varrho = \left(1 - \frac{\gamma-1}{2}Q\right)^{1/(\gamma-1)},$$

where $\gamma > 1$ is the adiabatic constant.

In this case,

$$M^2 = \frac{Q}{1-\left(\frac{\gamma-1}{2}\right)Q}$$

and hence, the flow is subsonic whenever the flow speed Q is less than the *sonic value*

$$Q_\varrho = \left(\frac{2}{\gamma+1}\right),$$

In the classical theory of gas dynamics, it is generally assumed that ϱ behaves like the density of a polytropic gas; namely, there are constants k and Q_ϱ such that

(i) $0 < \frac{1}{k} < \varrho(Q) < k < \infty$ for $Q \leqslant Q_\varrho$

(ii) $\frac{d}{dQ}(Q\varrho^2) > 0$ for $Q < Q_\varrho$

(iii) $\frac{d}{dQ}(Q\varrho^2) \to 0$ as $Q \to Q_\varrho$.

The classical problem in the plane is to show the existence of smooth subsonic flows past an obstacle for a range $0 \leqslant Q < Q_\varrho$ of prescribed speeds Q at ∞. If Q is too large at infinity, then no subsonic flow exists.

This problem was solved independently by Bers [2] and Shiffman [7]. A more detailed description of the physical situation may be found in Bers' book [1].

III. Gas dynamics on a manifold

3.1. Let M be a Riemannian manifold of dimension n and ϱ an admissible map ϱ: $M \times R \to R$. A form $\omega \in \Omega^p$ is closed if $d\omega = 0$ and coclosed if $\delta\omega = 0$. We will say that ω is *ϱ-coclosed* if $\delta\varrho\omega = 0$ $(\varrho = \varrho(x, Q^p(\omega)))$.

Definition. A form $\omega \in \Omega^p$ is said to be *ϱ-harmonic* if it is closed and ϱ-coclosed for some admissible ϱ. If, in addition, it is ϱ-subsonic (with the same ϱ) then it is said to be *subsonic ϱ-harmonic.*

If M is compact and ϱ is regular then, by the remark in § 1.2, any ϱ-harmonic form is necessarily subsonic ϱ-harmonic. Observe also that a 1-harmonic form is a harmonic form in the sense of Hodge [4]. Letting $\delta_\varrho = \varrho^{-1}\delta\varrho$ one can define a generalized Laplace–Beltrami operator

$$\Delta_\varrho = d\delta_\varrho + \delta_\varrho d \tag{1}$$

and, in the same way as for harmonic forms, one obtains the

PROPOSITION. *If $\omega \in \Omega^p$ is ϱ-harmonic then it satisfies $\Delta_\varrho\omega = 0$. If M is compact then a solution of $\Delta_\varrho\omega = 0$ is ϱ-harmonic.*

Proof. The second statement is obtained by observing that $(\varrho\omega, \Delta_\varrho\omega) = \frac{1}{\varrho}\|\delta\varrho\omega\|^2 + \varrho\|d\omega\|^2$ and recalling that $\varrho > 0$.

The connection between ϱ-harmonic 1-forms and solutions to the gas dynamics equation is given by the following:

PROPOSITION. *If ω is ϱ-harmonic then, locally, $\omega = d\phi$ where $\phi \in \Omega^0$ satisfies*

$$\delta\varrho\, d\phi = 0. \tag{2}$$

In local coordinates $x^1, ..., x^n$ equation (2) has the form

$$\frac{1}{\sqrt{g}}\frac{\partial}{\partial x^i}\left(\sqrt{g}\, g^{ij}\varrho\frac{\partial\phi}{\partial x^j}\right) = 0, \tag{3}$$

where g_{ij} is the given metric tensor on M and $g = \det(g_{ij})$. If M is flat we can choose orthonormal coordinates and (3) reduces to the classical gas dynamics equation

$$\sum_i \frac{\partial}{\partial x^i}\left(\varrho \frac{\partial \phi}{\partial x^i}\right) = 0.$$

3.2. We can now give a simple statement of the regular and non-regular theorems of § 1.3 under the assumption that γ is closed: *Let M be a Riemannian manifold (not necessarily compact) and ϱ regular. Given a closed 1-form γ there exists a ϱ-harmonic $\omega \in \Omega^1$ such that $\omega - \gamma$ is exact.* If ϱ is not assumed to be regular, one cannot expect to find *subsonic* solutions for arbitrary γ (i.e., with arbitrary periods). In fact max $Q(\omega)$ on a curve must be large if the corresponding period is assumed large. Therefore, in analogy with the classical (plane) case, one obtains subsonic solutions for a *range* of closed differentials $t\gamma$, $0 \leqslant t < t_\varrho$. *Let γ be a closed 1-form on a compact Riemannian manifold M and let ϱ be an admissible map of $M \times R \to R$. There exists a number $t_\varrho > 0$ and a family of subsonic ϱ-harmonic 1-forms ω_t such that $\omega_t - t\gamma$ is exact for $0 \leqslant t < t_\varrho$ and* $\sup_{x \in M} Q^1(\omega_t) \to Q_\varrho$ as $t \to t_\varrho$. *(Q_ϱ is the sonic value, defined in § 1.2).*

3.3. Rather than consider the regular and non-regular problems directly we turn to "conjugate" problems (which we formulate in § 3.4).

PROPOSITION. *Let ϱ: $M \times R \to R$ be admissible. There exists a map μ: $M \times R \to R$, called the conjugate of ϱ, such that, at each point x,*

(i) $\mu(Q^{n-p}(*\varrho\omega))\varrho(Q^p(\omega)) = 1, \quad \omega \in \Omega^p$

(ii) $\mu(Q^{n-p}(\nu))\varrho(Q^p(*\mu\nu)) = 1, \quad \nu \in \Omega^{n-p}$

(iii) *ϱ is admissible with sonic value Q_ϱ if and only if μ is admissible with sonic value $Q_\mu = Q_\varrho \varrho^2(Q_\varrho)$.*

(iv) *ϱ is regular if and only if μ is regular.*

Proof. Let $b = f(a) = a\varrho^2(a)$. The admissibility condition $db/da > 0$ on the interval $0 \leqslant a < Q_\varrho$ ensures that the function f has a single valued inverse f^{-1}. Let

$$\mu(b) = 1/\varrho \circ f^{-1}(b) = 1/\varrho(a). \tag{4}$$

Since $a = b/\varrho^2(a)$ we can write

$$a = f^{-1}(b) = b\mu^2(b). \tag{5}$$

Conclusion (i) follows by setting $b = Q^{n-p}(*\varrho\omega)$ in (4). Since $Q^{n-p}(*\varrho\omega) = Q^p(\varrho\omega) = Q^p(\omega)\varrho^2(Q^p(\omega)) = f(Q^p(\omega))$ we obtain $a = Q^p(\omega)$. Conclusion (ii) follows by setting $a = Q^p(*\mu\nu)$

in (4) and using (5) to obtain $b = Q^{n-p}(\nu)$. To obtain (iii) we observe that for all x, $k_1^{-1} < \varrho(a) < k_1 < \infty$ on the interval $0 \leq a < a_\varrho$ if and only if $k_1^{-1} < \mu(b) < k_1 < \infty$ on the interval $0 \leq b < b_\mu = f(a_\varrho)$. The smoothness conditions on these corresponding intervals are easily verified. Moreover, $(d/da)\,(a\varrho^2(a)) = (db/da) > 0$ if and only if $(d/db)\,(b\mu^2(b)) = (da/db) > 0$ on corresponding intervals. The results of (iii) for $Q_\varrho = Q_\mu = \infty$ together with $k_2^{-1} < (db/da) < k_2 < \infty$ if and only if $k_2^{-1} < (da/db) < k_2 < \infty$ give statement (iv).

COROLLARY. *The map* $_*\varrho$: $\omega \to * \varrho\omega$ *establishes a* $1-1$ *correspondence between the* ϱ*-subsonic forms in* Ω^p *with the* μ*-subsonic forms in* Ω^{n-p}. *Moreover, the map* $_*\mu: \Omega^{n-p} \to \Omega^p$ *defined by* $\nu \to (-1)^{np+p} * \mu\nu$ *is the inverse of* $_*\varrho$:

$$({}_*\varrho) \circ ({}_*\mu) = ({}_*\mu) \circ ({}_*\varrho) = \text{identity}.$$

Remark. We will see that the above correspondence is, in fact, between subsonic ϱ-harmonic forms and subsonic μ-harmonic forms.

3.4. The results of § 3.3 enable us to obtain solutions of the "ϱ-harmonic" problems of § 3.2 from solutions of the following conjugate problems:

NON-REGULAR CONJUGATE THEOREM. *Let* M *be compact,* $\gamma \in E_1^{*\perp}$ *be given, and* μ *admissible. Then there is a constant* t_μ *such that for each* t, $0 \leq t < t_\mu$, *there exists a unique* $\nu_t \in \Omega^{n-1}$ *such that*

(i) $d\nu_t = 0$

(ii) $\delta\mu\nu_t = 0 \quad (\mu = \mu(x, Q^{n-1}(\nu)))$

(iii) $_*\mu\nu_t + (-1)^n t\gamma \in E_1$

(iv) ν_t *is* μ*-subsonic*

(v) $\lim_{t \to t_\mu} (\max_{x \in M} Q^{n-1}(\nu_t)) = Q_\mu$.

(The essential difference from the theorems in § 1.3 is in condition (iii).)

REGULAR CONJUGATE THEOREM. *If* μ *is regular, then the above results hold for* $t_\mu = \infty$ *(i.e., for arbitrary* $t\gamma$*) and without the assumption that* M *is compact.*

These theorems will be proved in section IV.

Let, now, ϱ be regular and μ the conjugate of ϱ. Let ν be the $n-1$ form obtained from the regular conjugate theorem with this μ and a given $\gamma \in E_1^{*\perp}$. Then it is easily verified that $\omega = {}_*\mu\nu = (-1)^{n+1} * \mu\nu$ is closed, ϱ-coclosed and satisfies $\omega - \gamma \in E_1$, so that ω is the 1-form satisfying the conclusions of the regular theorem (§ 1.3) for ϱ and γ. Only the statement that ω is ϱ-coclosed is not immediately clear. But by the corollary in § 3.3, since ω is the image of ν under the map ${}_*\mu$, we have

$$*\varrho\omega = ({}_*\varrho) \circ ({}_*\mu)\nu = \nu$$

and $d\nu = 0$.

If now, ϱ is assumed only to be admissible, let ν_t, $0 \leq t < t_\mu$ be the family of μ-subsonic elements of Ω^{n-1} obtained from the non-regular conjugate theorem. Letting $\omega_t = (-1)^{n+1} {}_*\mu\nu_t$ we obtain a family of ϱ-subsonic elements of Ω^1 satisfying conclusions (i)–(v) of the non-regular theorem (cf. § 1.3).

Corresponding statements hold with ϱ and μ interchanged. Thus we obtain

PROPOSITION. *There exists a one to one correspondence between the solutions of the regular (non-regular) problem and the solutions of the regular conjugate (non-regular conjugate) problem.*

IV. Solution of the conjugate problem

The variational method of proof in the plane case is due to Max Shiffman [7]. We shall follow his technique of first solving the regular conjugate problem and then obtaining the solution of the non-regular problem from the estimates in the regular case.

4.1. *The regular case.* For $\nu = \sum_{i_k \neq i} \nu_i \, dx^{i_1} \dots dx^{i_{n-1}}$, $Q^{n-1}(\nu) = Q^1(*\nu) = g^{ij}\nu_i\nu_j$. For brevity, we shall henceforth write $Q(\nu)$ for the "pointwise norm" $Q^{n-1}(\nu)$ of an $n-1$ form ν. Let $F(x,\nu) = \int_0^{Q(\nu)} \mu(x,\xi)\, d\xi$, where $x \in M$. Then $(\partial/\partial\nu_i) F(x,\nu) = \mu(x,Q)\,\partial Q/\partial\nu_i$, $i = 1, \dots, n$ and $F(x,0) = 0$. At every point $x \in M$, F is a convex function of $\nu_1, \dots, \nu_n$ if and only if the matrix

$$(F_{\nu_i\nu_j}) = \left(\mu \frac{\partial^2 Q}{\partial\nu_i\,\partial\nu_j} + \mu' \frac{\partial Q}{\partial\nu_i}\frac{\partial Q}{\partial\nu_j}\right) = (\mu Q_{ij} + \mu' Q_i Q_j)$$

is positive definite (where we have written μ' for $\partial\mu/\partial Q$, Q_i for $\partial Q/\partial\nu_i$ and Q_{ij} for $\partial^2 Q/\partial\nu_i\partial\nu_j = 2g^{ij}$). Since the eigenvalues of a symmetric matrix are invariant under an orthogonal coordinate transformation, and since *at a fixed point* $x \in M$, the matrix g_{ij} can be diagonalized by such a transformation, we may assume that the coordinates $x^1, \dots, x^n$ are such that *at the point* x

$$Q(\nu) = \sum_k g^{kk}\nu_k^2$$

so that $Q_i(\nu) = 2g^{ii}\nu_i$ (no summation) and $Q_{ij} = 2g^{ij}$ is zero if $i \neq j$.

LEMMA 4.1. *A symmetric matrix A with characteristic polynomial $P(\lambda)$ always satisfies $P(-\infty) = +\infty$. A is positive definite if and only if $P(0) > 0$ and $dP(\lambda)/d\lambda < 0$ for $\lambda \leq 0$.*

Proof. Trivial. The conclusion of the lemma implies, of course, that $P(\lambda) > 0$ for $\lambda \leqslant 0$.

Let A be an $n \times n$ matrix and r an integer $\leqslant n$. An $r \times r$ submatrix of A formed by the rows $i_1 < i_2 < \ldots < i_r$ and the columns $i_1 < \ldots < i_r$ will be called a *distinguished submatrix* of order r.

THEOREM 4.1. *If $\mu > 0$, $(d/dQ)Q\mu^2(Q) > 0$ and $Q_{ij} = 0$ for $i \neq j$ then the matrix $A = (\mu Q_{ij} + \mu' Q_i Q_j)$ is positive definite. In particular, the hypotheses are satisfied if μ is regular.*

Proof. We will show that any distinguished submatrix A_r of order r is positive definite if all distinguished submatrices of order $r-1$ are positive definite. Let $P_r(\lambda) = \det(A_r - \lambda I)$. Then for $\lambda < 0$,

$$\frac{d}{d\lambda} P_r(\lambda) = -\sum_{j=1}^{r} P_{r-1}^{(j)}(\lambda) < 0$$

since the $P_{r-1}^{(j)}(\lambda)$ are determinants of distinguished submatrices of order $r-1$, hence all positive for $\lambda \leqslant 0$ by the induction hypothesis. Moreover, by elementary matrix manipulation, we can obtain the following recursion relation:

$$P_r(0) = \det A_r = \mu Q_{rr} P_{r-1}(0) + \mu' \mu^{r-1} Q_r^2 \prod_{k=1}^{r-1} Q_{kk}, \quad P_1(0) = Q_{11} + \mu' Q_1^2,$$

where $P_{r-1} = \det A_{r-1}$ and A_{r-1} is the matrix obtained from A_r by omitting the rth row and the rth column. This recursion relation is satisfied by

$$P_r(0) = \mu^{r-1}(\mu + 2\mu' \sum_{j=1}^{r} g^{jj} \nu_j^2) \prod_{k=1}^{r} Q_{kk}.$$

If $\mu' > 0$ then $P_r(0) > 0$. If $\mu' < 0$, then since

$$\left| \sum_{j=1}^{r} g^{jj} \nu_j^2 \right| \leqslant \left| \sum_{j=1}^{n} g^{jj} \nu_j^2 \right|$$

and recalling that $\mu + \mu' \sum_{j=1}^{n} g^{jj} \nu_j^2 = \mu + 2\mu' Q > 0$, we again obtain $P_r(0) > 0$.

This last argument also shows that all distinguished submatrices of order 1 are positive definite and the proof of the theorem is complete.

4.4. *The extremal*. We assume in this section that μ is regular. Then the function F defined in 4.1 satisfies the following set of conditions:

$F(x^1, \ldots, x^n, \nu_1, \ldots, \nu_n) \in C_\alpha^{2+\alpha}$, $0 < \alpha < 1$, in each of its arguments

$$m_0(\nu, \nu) \leqslant \int_M F(x, \nu)\, dV \leqslant M_0(\nu, \nu)$$

$$m_1 |\xi|^2 \leqslant F_{\nu_i \nu_j} \cdot \xi_i \xi_j \leqslant M_1 |\xi|^2, \tag{A}$$

where m_0, m_1, M_0 and M_1 are positive constants.

Let
$$I(\nu)=\int_M F(x,\nu)\,dV-2\,(\nu,*\gamma), \tag{1}$$
where $dV=*1=\sqrt{g}\,dx^1\dots dx^n$ and $\gamma\in E_1^{*\perp}$ is given.

LEMMA 4.2. *Let F be any function satisfying conditions* (A). *Then there is a ν_0 which minimizes $I(\nu)$ over all $\nu\in E_{n-1}^{*\perp}$.*

To prove the lemma, we observe that because of (A), $I(\nu)$ is bounded below. Moreover the class of admissible forms is all of $E_{n-1}^{*\perp}$ and hence is non-empty.

We next show that $I(\nu)$ is lower-semi-continuous with respect to weak convergence in $L_{n-1}^2(M)$. Since F is convex, for smooth ν and τ we have
$$F(x,\tau)\geqslant F(x,\nu)+F_{\nu_i}(x,\nu)\,(\tau_i-\nu_i).$$
Let $\nu=(\nu_1,\dots,\nu_n)$ be any element of $L_{n-1}^2(M)$ and let $\nu^j=(\nu_1^j,\dots,\nu_n^j)$ be a sequence of smooth forms converging weakly to ν. Then, by Fatou's lemma
$$\int_M F(x,\nu^j)\,dV\geqslant\int_M F(x,\nu)\,dV+2\int_M \mu(x,Q^{n-1}(\nu))\,g^{ik}\nu_k(\nu_i^j-\nu_i)\,dV.$$

Since the second integral on the right is the inner product $(\mu\nu,\nu^j-\nu)$ which tends to zero by weak convergence,
$$\varliminf_{j\to\infty}\int_M F(x,\nu^j)\,dV\geqslant\int_M F(x,\nu)\,dV.$$
Because $E_{n-1}^{*\perp}$ is weakly closed, we obtain the result of the lemma.

The hypotheses of Lemma 4.2 are satisfied by the function $F(x,\nu)=\int_0^Q\mu(x,\xi)\,d\xi$. Moreover, in this case we may state

LEMMA 4.3. *The extremal ν_0 satisfies the Euler equation*
$$\mu\nu_0-*\gamma\in E_{n-1}^{*} \tag{2}$$
and ν_0 is the unique solution of (2) *in $E_{n-1}^{*\perp}$.*

Proof. For fixed $\tau\in E_{n-1}^{*\perp}$, let $\phi(\varepsilon)=I(\nu_0+\varepsilon\tau)$. Expand ϕ by Taylor's formula:
$$\phi(\varepsilon)=I(\nu_0)+\varepsilon\phi'(0)+\varepsilon^2R,$$
where
$$\phi'(0)=(\mu\nu_0-*\gamma,\tau)$$

and
$$R = \int_0^1 \left(\int_M F_{\nu_i \nu_j}(x, \nu + \varepsilon t \tau) \tau_i \tau_j \, dV \right) (1-t)\, dt.$$

By conditions (A),
$$c \| \tau \|^2_{n-1} \leqslant R \leqslant C \| \tau \|^2_{n-1},$$

where the constants are independent of ε.

Since ϕ has a relative minimum at $\varepsilon = 0$, $(\mu\nu_0 - *\gamma, \tau) = 0$, and since τ was arbitrary, we obtain (2).

To prove uniqueness, set $\tau = \nu - \nu_0$ and $\varepsilon = 1$. From the positive definiteness of R, we obtain
$$I(\nu) = I(\nu_0) + R > I(\nu_0)$$

which proves that the minimizing ν_0 is unique.

On the other hand, if $\nu_0 \in E^{*\perp}_{n-1}$ and satisfies (2), then by the above inequality ν_0 gives the (unique) minimum of $I(\nu)$ over all forms in $E^{*\perp}_{n-1}$.

THEOREM 4.2. *Let μ be regular and let $\gamma \in E^{*\perp}_1$ be given. Then there is a unique ν satisfying*

(i) $\nu \in E^{*\perp}_{n-1}$

(ii) $*\mu\nu \in E^{*\perp}_1$

(iii) $*\mu\nu + (-1)^n \gamma \in E_1$.

Proof. We shall show that the extremal ν_0 satisfies the conditions of the theorem. By definition, $\nu_0 \in E^{*\perp}_{n-1}$. By (2), $*\mu\nu_0 + (-1)^n \gamma \in E_1$ and since $\gamma \in E^{*\perp}_1$ and $E_1 \subset E^{*\perp}_1$, (ii) follows. On the other hand, any solution ν of (i), (ii), and (iii) is a solution of (2). By the preceding remarks, a solution of (2) provides a unique minimum for $I(\nu)$.

4.3. *Smoothness.*

THEOREM 4.3. *The extremal ν_0 is Hölder continuous on every compact subdomain of M with modulus and exponent of Hölder continuity depending only on $\|\nu_0\|_{n-1}$, the regularity constant for μ, and bounds on the g_{ij}. Moreover, ν_0 has Hölder continuous first derivatives on every compact subdomain. If $\mu \in C^\infty$ then $\nu_0 \in C^\infty$. If μ is analytic then ν_0 is analytic.*

Proof. Let $\omega_0 = *\mu\nu_0$. We have, from Theorem 4.2 and the results of § 3.4, that locally $\omega_0 = d\phi$ where $\phi \in L^2$ is a weak solution of the divergence equation
$$\delta\varrho\, d\phi = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^\alpha} \left(\sqrt{g}\, g^{\alpha\beta} \varrho \frac{\partial \phi}{\partial x^\beta} \right) = 0,$$

where ϱ (the conjugate of μ) is regular.

Let $p_\beta = \partial\phi/\partial x^\beta$ (so that $d\phi = p_\beta dx^\beta$) and $A^\alpha = \sqrt{g}\, g^{\alpha\beta}\varrho(x, Q)p_\beta$ where we regard $Q = Q(x, d\phi)$ as a function of $2n$ variables $Q(x^1, \ldots, x^n, p_1, \ldots, p_n)$.

Since ϱ is regular, the function $Q\partial\varrho/\partial Q$ is bounded for all Q at every $x \in D$. Since Q is a positive definite quadratic form in the p_α,

$$\left|\frac{\partial \varrho}{\partial Q} p_\alpha p_\beta\right| \leqslant C_1$$

and

$$\left|\frac{\partial \varrho}{\partial Q}\frac{\partial Q}{\partial x^i}\right| \leqslant C_2.$$

It follows by elementary computations that the A^α satisfy

$$\sum_\alpha |A^\alpha|^2 + \sum_{\alpha, i}\left|\frac{\partial A^\alpha}{\partial x^i}\right|^2 \leqslant K^2(1 + \sum_\beta p_\beta^2) \qquad \sum_{\alpha,\beta}\left|\frac{\partial A^\alpha}{\partial p_\beta}\right|^2 \leqslant k^2$$

$$m_1|\xi|^2 \leqslant \frac{\partial A^\alpha}{\partial p_\beta}\xi_\alpha\xi_\beta \leqslant M_1|\xi|^2. \tag{B}$$

Since the coefficients of ν_0 are smooth linear combinations of $\partial\phi/\partial x^\beta$, Theorem 4.3 now follows from

THEOREM (De Giorgi–Moser–Morrey). *Let $\phi \in L^2$, $d\phi \in L^2_{(1)}$ be a weak solution of $\partial A^\alpha/\partial x^\alpha = 0$, where the $A^\alpha = A^\alpha(x, \nabla\phi)$ satisfy* (B). *Then, locally, the first derivatives of ϕ satisfy a Hölder condition depending only on $\|d\phi\|_1$, m_1, M_1, k and K. Furthermore, $\phi \in C^{2+\alpha}$. If $A^\alpha \in C^\infty$ then $\phi \in C^\infty$; if A^α is analytic then so is ϕ.*

The results stated in the above theorem are proved in Morrey's book (see [6], §§ 1:10, 1:11, and Chapter 5).

The regular conjugate theorem (§ 3.4) follows from Theorems 4.2 and 4.3. In the next two sections we prove the non-regular conjugate theorem.

4.4. *Continuous dependence.* From now on we assume that M is compact. Let F be defined as in § 4.1 and

$$I(\nu, t) = \int_M F(x, \nu)\, dV - 2t(\nu, *\gamma).$$

Denote by $\nu(t)$ the extremal for each t.

LEMMA 4.4. *$\nu(t)$ is a continuous function of t.*

Proof. Let t_k converge to t. Then $\nu_k = \nu(t_k)$ is bounded in L^2 independently of k. By Theorem 4.3, $\{\nu_k\}$ is equicontinuous. Since it is uniformly bounded in L_2, it is also uniformly bounded. By Arzela's Theorem, a subsequence converges uniformly to ν. It remains to show that ν is the extremal of $I(\nu, t)$, so that the selection of a subsequence was unnecessary.

Let $\tau \in E^{*\perp}_{n-1}$. For every k, $I(\nu_k, t_k) \leqslant I(\tau, t_k)$. Both integrals converge uniformly from which it follows that $I(\nu, t) \leqslant I(\tau, t)$.

4.5. *The Shiffman regularization.* Given a regular map σ and $\gamma \in E_1^{*\perp}$, the problem of finding an $\nu \in \Omega^{n-1}$ satisfying

(i) $\nu \in E^{*\perp}_{n-1}$

(ii) $*\sigma\nu \in E_1^{*\perp}$

(iii) $*\sigma\nu + (-1)^n t\gamma \in E_1$

will be called the "(σ, t) regular conjugate problem" and its solution will be denoted by $\nu(\sigma, t)$. To simplify notation we write

$$|\nu(\sigma, t)| = \max_{x \in M} Q(\nu(\sigma, t)).$$

In view of Lemma 4.4 we may state

Remark 1. If σ is a regular map, then for any constant C there exists a $t_\sigma(C)$ such that

$$|\nu(\sigma, t)| < C \quad \text{for } t < t_\sigma(C)$$

and

$$|\nu(\sigma, t)| \to C \quad \text{as } t \to t_\sigma(C).$$

We state also

Remark 2. If $C_1 \leqslant C_2$ then $t_\sigma(C_1) \leqslant t_\sigma(C_2)$.

This follows from the observation that the interval $0 \leqslant t < t_\sigma(C_1)$ is the largest interval in which $|\nu(\sigma, t)| < C_1$. If $C_2 \geqslant C_1$, then the interval on which $|\nu(\sigma, t)| < C_2$ is at least as large as the interval on which $|\nu(\sigma, t)| < C_1$.

LEMMA 4.5. *If σ and τ are regular maps and $\sigma = \tau$ on the interval $0 \leqslant Q \leqslant C$ then the solutions $\nu(\sigma, t)$ and $\nu(\tau, t)$, of the (σ, t) and of the (τ, t) regular conjugate problems respectively, agree for $t < t_\sigma(C)$. Moreover, $t_\sigma(C) = t_\tau(C)$.*

Proof. For $t < t_\sigma(C)$ we have $|\nu(\sigma, t)| < C$ and since $\sigma = \tau$ for $Q < C$, $\nu(\sigma, t)$ is a solution of the (τ, t) regular conjugate problem. Hence, by the uniqueness part of Lemma 4.2, $\nu(\sigma, t) = \nu(\tau, t)$. Then $|\nu(\sigma, t)| = |\nu(\tau, t)|$ for every $t < t_\sigma(C)$ and $t_\tau(C) = t_\sigma(C)$.

Consider now an admissible map μ with sonic value Q_μ. Let $\{Q_\mu^n\}$ be an increasing sequence $Q_\mu^n \to Q_\mu$, and let $\{\mu_n\}$ be a sequence of *regular* maps with $\mu_n(x, Q) = \mu(x, Q)$ for $Q \leqslant Q_\mu^n < Q_\mu$. Denote by $\nu_n(t)$ the solution $\nu(\mu_n, t)$ of the (μ_n, t) regular conjugate problem and write $t_n(C) = t_{\mu_n}(C)$. Thus

Remark 3. $|\nu_n(t)| < Q_\mu^n$ for $t < t_n(Q_\mu^n)$ and $|\nu_n(t_n(Q_\mu^n))| = Q_\mu^n$.

LEMMA 4.6. *For $m > n$ we have $\nu_m(t) = \nu_n(t)$ in the interval $0 \leqslant t \leqslant t_n(Q_\mu^n)$.*

Proof. If $t < t_n(Q_\mu^n)$ then $|\nu_n(t)| < Q_\mu^n$. But for $Q < Q_\mu^n$ and $m > n$ we have $\mu_m = \mu_n$. Then by Lemma 4.5, $\nu_n(t) = \nu_m(t)$.

Combining the final assertion of Lemma 4.5 with Remark 2 we obtain,

$$t_n(Q_\mu^n) = t_m(Q_\mu^n) \leqslant t_m(Q_\mu^m)$$

so that $\{t_n(Q_\mu^n)\}$ is a non-decreasing sequence. Let $t_\mu = \lim t_n(Q_\mu^n)$ as $n \to \infty$. For each t, the forms $\nu_n(t)$ are defined and *coincide* for all n for which $t_n(Q_\mu^n) > t$. In this way they define a form $\nu(t)$ for $t < t_\mu$. Clearly $|\nu(t)| < Q_\mu$ so that $\nu(t)$ is μ-subsonic.

Remark 4. Suppose now that t is such that $|\nu_n(t)| \leqslant Q_\mu^n$ (this is certainly true for $t \leqslant t_n(Q_\mu^n)$ but we do not explicitly assume this). Then $\nu_n(t)$ is μ-subsonic and solves the (μ_m, t) regular conjugate problem for $m > n$, so that $\nu_n(t) = \nu(t)$.

For each $t < t_\mu$ the form $\nu(t)$ coincides with the solution $\nu_n(t)$ of a (μ_n, t) regular conjugate problem (we need only choose n so that $t < t_n(Q_\mu^n) < t_\mu$). Thus

(i) $\nu \in E_{n-1}^{*\perp}$

(ii) $*\mu_n \nu \in E_1^{*\perp}$

(iii) $*\mu_n \nu + (-1)^n t\gamma \in E_1$

Since $t < t_n(Q_\mu^n)$, however, we have in addition that $\mu = \mu_n$ so that (i), (ii) and (iii) above hold with μ_n replaced by μ. We have already observed that for $t < t_\mu$

(iv) $\nu(t)$ is μ-subsonic.

To complete the proof of the non-regular conjugate theorem stated in § 3.4 it remains only to show

(v) $|\nu(t)| \to Q_\mu$ as $t \to t_\mu$.

If (v) were not valid there would exist an $\varepsilon > 0$ and an increasing sequence $t_j \to t_\mu$ such that

$$|\nu(t_j)| < Q_\mu - \varepsilon. \tag{3}$$

Recall that $Q_\mu^n \to Q_\mu$ and choose n such that $Q_\mu - \varepsilon < Q_\mu^n < Q_\mu$. Then by definition, $\nu(t_j) = \nu_n(t_j)$, for all j such that $t_j < t_n(Q_\mu^n)$. But (3) ensures that this holds for all t_j. By continuity (Lemma 4.4) $\nu(t_j) = \nu_n(t_j) \to \nu_n(t_\mu)$ as $t_j \to t_\mu$ so that $|\nu(t_j)| \to |\nu_n(t_\mu)|$. Consequently

$$|\nu_n(t_\mu)| \leqslant Q_\mu - \varepsilon < Q_\mu^n$$

so that $|\nu_n(t)| \leqslant Q_\mu^n$ for t sufficiently close to t_μ. By Remark 4, we have

$$|\nu(t)| \leqslant Q_\mu^n \quad \text{for } t \text{ sufficiently close to } t_\mu.$$

However $t_m(Q_\mu^m) \to t_\mu$ as $m \to \infty$ and using Remark 3,

$$|\nu(t_m(Q_\mu^m))| = |\nu_m(t_m(Q_\mu^m))| = Q_\mu^m \to Q_\mu$$

which contradicts (3).

This concludes the proof of the non-regular conjugate theorem and hence, by the results of § 3.4, the proof of the non-regular theorem (§ 1.3).

Appendix — The Hodge–de Rham theorem for harmonic p-forms

If, in the previous work, we take $\varrho \equiv 1$, then $\mu \equiv 1$ and $F = Q$ (cf. § 4.1). Then

$$I(\nu) = \int_M Q(\nu)dV - 2(\nu, *\gamma) = \|\nu - *\gamma\|^2 - \|*\gamma\|^2$$

which is minimized for $\nu_0 \in E_{n-p}^{*\perp}$ minimizing $\|\nu - *\gamma\|$, that is, for the projection ν_0 of $*\gamma$ on $E_{n-p}^{*\perp}$.

Thus, our proof leads to a (slight) modification of existing proofs of the classical Hodge theorem for harmonic p-forms.

Given $\gamma \in E_p^{*\perp}$ consider $*\gamma \in L^2_{(n-p)}$. Let ν_0 be the projection of $*\gamma$ on the subspace $E_{n-p}^{*\perp}$. Then $\nu_0 - *\gamma \in E_{n-p}^*$ so that $*\nu_0 - (-1)^{np+p}\gamma \in E_p$ (and hence, since $\gamma \in E_p^{*\perp}$ and $E_p \subset E_p^{*\perp}$, we have $*\nu_0 \in E_p^{*\perp}$). Then letting $\omega_0 = (-1)^{np+p} * \nu_0$ we obtain

(i) $\omega_0 \in E_p^{*\perp}$

(ii) $*\omega_0 \in E_{n-p}^{*\perp}$

(iii) $\omega_0 - \gamma \in E_p$

By Weyl's lemma [5] conditions (i) and (ii) guarantee that ω has C^1 coefficients and is therefore harmonic.

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