THE INVERSE PROBLEM OF THE NEVANLINNA THEORY

BY

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1. Introduction

1.1. Statement of Theorem 1. Our main result is

THEOREM 1. *Let sequences* $\{\delta_i\}$, $\{\theta_i\}$ $(1 \leqslant i < N \leqslant \infty)$ *of non-negative numbers be assigned such that*

$$0 < \delta_i + \theta_i \leqslant 1 \qquad (1 \leqslant i < N),$$

$$\sum_i \{\delta_i + \theta_i\} \leqslant 2,$$

together with a sequence $\{a_i\}$ $(1 \leqslant i < N)$ *of distinct complex numbers. Then there exists a meromorphic function* $f(z)$ *having*

$$\delta(a_i, f) = \delta_i,\ \theta(a_i, f) = \theta_i \qquad (1 \leqslant i < N), \tag{1.1}$$

$$\delta(a, f) = \theta(a, f) = 0 \qquad (a \notin \{a_i\}). \tag{1.2}$$

Further, if $\phi(r)$ *is a positive increasing function with*

$$\phi(r) \to \infty \qquad (r \to \infty), \tag{1.3}$$

the function $f(z)$ *may be chosen so that its Nevanlinna characteristic satisfies*

$$T(r, f) < r^{\phi(r)} \tag{1.4}$$

for all large r.

Here we use the standard notations of R. Nevanlinna's theory (cf. Nevanlinna [13], [15], A. A. Goldberg and I. V. Ostrovskii [8] and W. K. Hayman [9]); for example

$$\delta(a, f) = \liminf_{r \to \infty} \left\{ 1 - \frac{N(r, a, f)}{T(r, f)} \right\}, \tag{1.5}$$

$$\theta(a, f) = \liminf_{r \to \infty} \left\{ \frac{N(r, a, f) - \bar{N}(r, a, f)}{T(r, f)} \right\}. \tag{1.6}$$

The function $f(z)$ thus provides a complete solution to the inverse problem of the theory of meromorphic functions (for a discussion of this problem see [8], Ch. 7 and H. Wittich [19], Ch. 8).

The problem of constructing a function whose deficiencies and ramifications are arbitrarily chosen consistent with the first and second fundamental theorems has a long history. It is proposed in Nevanlinna's first book ([13], p. 90) but solved only in very special cases. Nevanlinna achieved a major advance in 1932 [14] when, in introducing the class of Riemann surfaces with finitely many logarithmic branch points, he proved that the restricted inverse problem

$$\delta(a_i, f) = \delta_i,\ 0 < \delta_i \leqslant 1 \quad (i = 1, \ldots, N < \infty),$$

$$\delta(a, f) = 0 \quad (a \notin \{a_i\}),$$

$$\Sigma\, \delta_i = 2, \quad \delta_i \text{ rational},$$

with $\{a_1, \ldots, a_N\}$ any preassigned set of distinct complex numbers, could be solved by choosing an appropriate surface from this class and taking $f(z)$ to be the meromorphic function which maps the plane conformally onto (uniformizes) this surface. A sketch of this procedure is in [15], Ch. 11, and an excellent exposition with some extensions is given in [7].

Later, F. E. Ullrich [17] introduced a more general class of surfaces and conjectured that (1.1), (1.2), (with now all δ_i, θ_i rational, $N < \infty$ and $\Sigma\, \delta_i + \theta_i = 2$) could be solved by uniformizing a suitable surface of this type. This was confirmed by Le-Van Thiem [11] for most cases, in a paper also notable for being the first to apply a general principal of Teichmüller [16] to the inverse problem. Teichmüller had come to these discoveries also while studying Ullrich's surfaces, and a modified form is the starting point for this investigation (chapter 2).

More recently, Goldberg applied Teichmüller's principle to a more general class of surfaces to solve the problem $\Sigma^N \delta_i \leq 2$ $(N < \infty)$ without the δ_i being rational, and also gave a complete solution to the restricted problem $\Sigma\, \theta_i \leqslant 2$. A useful account of Goldberg's successes appears in chapter 7 of [8].

Finally, we recall the well-known example of W. H. J. Fuchs and Hayman (cf. [9], chapter 4) which solves the restricted problem $\Sigma\, \delta_i \leqslant 2$ for entire functions.

The solution to the inverse problem cannot in general be of finite order. Indeed, A. Weitsman [18] has shown $\Sigma\, \delta(a_i)^{1/3} < \infty$ whenever

$$\liminf_{r \to \infty} \frac{T(2r, f)}{T(r, f)} < \infty. \tag{1.7}$$

Assertion (1.4) implies that our solution $f(z)$ may be chosen of as 'small' infinite order as desired, and the construction also shows that $T(2r, f)/T(r, f)$ may tend arbitrarily slowly to infinity, complementing (1.7).

It is a pleasure to make several acknowledgments. The viewpint of chapter 2, which replaces all notions of Riemann surfaces and uniformization by properties of solutions to the Beltrami equation, was shown me by my colleague K. V. Rajeswara Rao. This approach uses notions now standard in the study of quasi-conformal mappings, and leads to a more transparent and essentially self-contained exposition. It was with another colleague, Allen Weitsman, that I discovered the literature on this problem, and in our earlier paper [6] we made a major step in properly adapting Teichmüller's principle; [6] also showed the relevance of the Lindelöf functions. Professors W. H. J. Fuchs and Seppo Rickman caught several substantial errors in the first version of this paper. A suggestion from Professor Fuchs has simplified my proof of Theorem 4.

Finally, I thank Nancy Eberle for the excellent typing she has given to the many versions of this manuscript.

1.2. Principle of construction. Consider the restricted inverse problem $\Sigma\, \delta_i \leqslant 2$. Given a positive integer n, choose $2n$ extended complex numbers $b_{-(n-1)}, \ldots, b_0, \ldots, b_n$ with $b_j \neq b_{j+1}$, $b_n \neq b_{-(n-1)}$. The method of Nevanlinna [14] produces a meromorphic function f_n, of order n, such that

$$\delta_n(a) \equiv \delta(a, f_n) = n^{-1}[\operatorname{card}\{j;\, -(n-1) \leqslant j \leqslant n,\, b_j = a\}]. \tag{1.8}$$

Hence, if $\{b_j\}(-\infty < j < \infty)$ is a sequence chosen so that the numbers $\delta_n(a)$ defined by (1.8) tend to δ_i when $a = a_i$ and 0 otherwise, it is natural to try to construct the solution to this deficiency problem as a limit of the corresponding functions $\{f_n\}$. We achieve this in the following manner: there will be a very rapidly increasing sequence $\{r_n\}(1 \leqslant n < \infty)$ with the property that near $\{|z| = r_n\}$ $f(z)$ has the same value-distribution as does $f_n(z)$. Further, the definition of f in the intermediate regions $\{r_n < |z| < r_{n+1}\}$ will ensure that $\delta(a, f) = \lim_n \delta_n(a)$ for all a.

The solution to the full inverse problem (1.1), (1.2) is made in a similar manner, but based on a family modelled after that introduced in [6].

The function $f(z)$ of Theorem 1 is obtained by indirect methods. The inverse problem is solved formally by an explicit function $g(\zeta)$; although g is not meromorphic, it may be 'factored' as $g = f \circ \psi$ where f is a meromorphic function and ψ a (quasi-conformal) homeomorphism of the plane. In chapter 2, we derive conditions to ensure that the Nevanlinna data of g transfer to f (i.e. that g be *Nevanlinna admissible*) so that in addition (1.4) holds. Much of the material in this chapter is implicit in other sources, but the importance of the parameters in Theorem 2 and Lemma 4 warrants a complete exposition.

The definition of $g(\zeta)$ is based on a family of auxilliary functions $g_j(\zeta)$ $(|j| < \infty)$ and $g_j^*(\zeta)$ $(j \geqslant 0)$. These functions are introduced in § 3.1, where their important properties are listed in Theorem 3. Assuming Theorem 3, the proof of Theorem 1 is completed in § 3.2.

The proof of Theorem 3 itself depends on Theorem 4. Theorem 4 is stated in § 4.1, and additional preliminaries to the proof of Theorem 3 are given in § 4.3–4.5. This makes it easy to obtain Theorem 3, in chapter 5. Finally, Theorem 4 is proved in chapter 6.

The methods of this paper may be used to solve other problems. For example, it is easy to modify the approach to construct a function $f(z)$ order $\varepsilon > 0$ which solves the restricted problem $\Sigma\,\theta_i \leqslant 2$, and only a little harder to show that f may be chosen of order zero.

2. Nevanlinna admissibility

2.1. Nevanlinna theory and quasi-meromorphic functions. To keep a distinction between meromorphic and not-necessarily-meromorphic functions, we usually reserve the complex variable $z(=re^{i\theta})$ to be the domain of a meromorphic function, while functions of the complex variables $w(=se^{it}=u+iv)$ and $\zeta=(\varrho e^{i\phi}=\xi+i\eta)$ need not be meromorphic.

Let $g(\zeta)$ be a continuous map from the finite complex plane C into the extended complex plane $\hat{C}$ which has partial derivatives a.e. and such that each ζ_0 has a neighborhood $N(\zeta_0)$ in which either

$$g_\xi(\zeta),\ g_\eta(\zeta) \in L^2(N(\zeta_0)) \tag{2.1}$$

or

$$(1/g)_\xi(\zeta),\ (1/g)_\eta(\zeta) \in L^2(N(\zeta_0)) \tag{2.2}$$

((2.2) is preferred when $g(\zeta_0)=\infty$). In terms of the formal derivatives

$$g_\zeta = \tfrac{1}{2}(g_\xi - ig_\eta),\ g_{\bar\zeta} = \tfrac{1}{2}(g_\xi + ig_\eta) \tag{2.3}$$

we introduce the fundamental assumption that there is a fixed number k_0, $0 \leqslant k_0 < 1$ such that either

$$|g_{\bar\zeta}(\zeta)| < k_0 |g_\zeta(\zeta)| \qquad \text{a.e. in } N(\zeta_0) \tag{2.4}$$

or

$$|(1/g)_{\bar\zeta}(\zeta)| < k_0 |(1/g)_\zeta(\zeta)| \qquad \text{a.e. in } N(\zeta_0). \tag{2.5}$$

A continuous function $g\colon C \to \hat{C}$ such that either or both (2.1) and (2.4) or (2.2) and (2.5) hold in a neighborhood of each $\zeta \in C$ is called *quasi-meromorphic;* if D is open and $g\colon D \to \hat{C}$ satisfies the analogous conditions, then g is quasi-meromorphic on D. Finally, if D is a set whose boundary has planar measure zero, a continuous function $g\colon D \to \hat{C}$ is quasi-meromorphic in D if g is quasi-meromorphic in the interior of D.

The measurable function μ defined locally by an appropriate choice of the formulae

$$\mu_g(\zeta) = g_{\bar\zeta}(\zeta)/g_\zeta(\zeta), \tag{2.6}$$

$$\mu_g(\zeta) = (1/g)_{\bar\zeta}(\zeta)/(1/g)_\zeta(\zeta) \tag{2.7}$$

gauges the deviation of g from a meromorphic function: $\mu \equiv 0$ if g is meromorphic, and $\|\mu_g\|_\infty \leqslant k_0$.

Much of the theory of quasi-conformal mappings depends on the fact that the partial differential equation (Beltrami equation)

$$\psi_{\bar{\zeta}}(\zeta) = \mu_g(\zeta)\psi_\zeta(\zeta) \quad (\|\mu_g\|_\infty \leqslant k_0 < 1) \tag{2.8}$$

has a solution $z = \psi(\zeta)$ which is a homeomorphic self-map of the finite plane, and the normalizations

$$\psi(0) = 0, \qquad \psi(1) = 1 \tag{2.9}$$

render ψ unique (cf. [1], Ch. 5; for history cf. [2]).

The importance of this 'fundamental solution' of (2.8) is that the function $f(z)$ defined by

$$f(z) = g \circ \psi^{-1}(z) \tag{2.10}$$

is meromorphic in the complex plane. Indeed the question is purely local, and g and ψ are both solutions of the same Beltrami equation in the sense of Bers [4] (this is why regularity conditions (2.1) and (2.2) are required). Thus the analyticity of f follows from [4], p. 94.

The *factorization* (2.10) permits a natural extension of the standard value-distribution functional to g. For example, if Γ_ϱ is the curve in the z-plane which is the image of $\{|\zeta| = \varrho\}$ by ψ, then

$$n(\varrho, a, g) \qquad (\text{resp. } \bar{n}(\varrho, a, g))$$

is the number of solutions inside Γ_ϱ of the equation $f(z) = a$ with (resp. without) due account of multiplicity. Further,

$$N(\varrho, a, g) = \int_0^\varrho \{n(u, a, g) - n(0, a, g)\} \frac{du}{u} + n(0, a, g) \log \varrho, \tag{2.11}$$

$$\bar{N}(\varrho, a, g) = \int_0^\varrho \{\bar{n}(u, a, g) - \bar{n}(0, a, g)\} \frac{du}{u} + \bar{n}(0, a, g) \log \varrho, \tag{2.12}$$

$$T(\varrho, g) = \frac{1}{2\pi} \int_0^{2\pi} N(\varrho, e^{i\theta}, g)\, d\theta, \tag{2.13}$$

and, finally, $\delta(a, g)$ and $\theta(a, g)$ are defined by (1.5) and (1.6). When g is meromorphic, these reduce to standard (or equivalent) definitions.

2.2. Nevanlinna admissibility

Definition. Let g be quasi-meromorphic and ψ a homeomorphism of the plane which satisfies (2.8) and (2.9). Then g is *Nevanlinna admissible* if the function $f(z)$ determined in (2.10) satisfies

$$\delta(a, f) = \delta(a, g); \quad \theta(a, f) = \theta(a, g) \tag{2.14}$$

for all $a \in \hat{C}$.

Each set of conditions sufficient for Nevanlinna admissibility presents new possibilities to construct meromorphic functions. The classical criteria are due to Le-Van [11] and are based on Teichmüller's discoveries in [16]: for some λ, $0<\lambda<\infty$,

$$T(\varrho, g) \sim \varrho^{\lambda} \quad (\varrho \to \infty), \tag{2.15}$$

all limits

$$\lim_{\varrho\to\infty} \varrho^{-\lambda} n(\varrho, a, g), \qquad \lim_{\varrho\to\infty} \varrho^{-\lambda} \bar{n}(\varrho, a, g) \qquad (a \in \hat{C}) \tag{2.16}$$

exist and

$$\iint_{|\zeta|>1} |\mu_g(\zeta)|\,|\zeta|^{-2} d\xi\, d\eta \equiv \iint_{|\zeta|>1} |\mu_\psi(\zeta)|\,|\zeta|^{-2} d\xi\, d\eta < \infty. \tag{2.17}$$

Indeed, not only does (2.14) hold, but in addition

$$T(r, f) \sim \alpha r^{\lambda} \qquad (r \to \infty) \tag{2.18}$$

for some $\alpha > 0$.

Since our solution $f(z)$ in Theorem 1 will have infinite order, (2.18) cannot hold, and our construction will almost always violate (2.17). Thus more flexible conditions are needed: in terms of the representation (2.10), they balance the growth of the characteristic of $g(\zeta)$ with the rate at which ψ becomes conformal at ∞. In this section we obtain a substitute for (2.17); modifications of (2.15) and (2.16) will be given in § 2.3.

Hence, consider the mapping of the plane given by $\psi(\zeta)$. For $r>0$ let Γ_r be the Jordan curve which surrounds $\zeta=0$ and is the image of $\{|z|=r\}$ under ψ^{-1}, and define

$$\begin{aligned} \varrho_2(r) &= \sup\{|\zeta|;\, \zeta \in \Gamma_r\}, \\ \varrho_1(r) &= \inf\{|\zeta|;\, \zeta \in \Gamma_r\}. \end{aligned} \tag{2.19}$$

Assumption (2.9) implies that $\varrho_1(r)$, $\varrho_2(r)$ are increasing functions of r which vanish when $r=0$. The deviation of ψ from conformality at ∞ is measured by the 'distortion'

$$\omega(r)\,(=\omega(r, \psi)) = \log\{\varrho_2(r)/\varrho_1(r)\}. \tag{2.20}$$

LEMMA 1. *If ψ is as above with $|\mu_\psi| \leq k_0 < 1$ a.e., then there is an $M = M(k_0) < \infty$ such that*

$$\varrho_2(2r)/\varrho_1(r) < M \qquad (r > 0). \tag{2.21}$$

Proof. Given z_1, z_2 with $|z_1| = r$, $|z_2| = 2r$ and

let
$$\varrho_2(2r) = |\psi^{-1}(z_2)|, \quad \varrho_1(r) = |\psi^{-1}(z_1)|,$$
$$B' = \{\zeta; \varrho_1(r) < |\zeta| < \varrho_2(2r)\}.$$

Then $B = \psi(B')$ is a doubly-connected region in the z-plane which separates 0 and z_1 from z_2 and ∞. Teichmüller's inequality (cf. [1], Ch. 3; [10], p. 58] for the module of B, $M(B)$, gives

$$M(B) \leqslant 2\nu\left\{\left(\frac{|z_1|}{|z_1|+|z_2|}\right)^{1/2}\right\} = 2\nu(3^{-1/2})$$

where ν may be expressed in terms of elliptic integrals. But ψ is $(1+k_0)(1-k_0)^{-1}$ quasiconformal, so

$$\log\frac{\varrho_2(2r)}{\varrho_1(r)} \equiv M(B') < \frac{1+k_0}{1-k_0}M(B),$$

which yields

$$\frac{\varrho_2(2r)}{\varrho_1(r)} < \exp\{2(1+k_0)(1-k_0)^{-1}\nu(3^{-1/2})\} \equiv M.$$

COROLLARY 1. *The hypotheses of Lemma* 1 *imply*

$$\omega(r) < \log M \qquad (M = M(k_0),\ r > 0), \tag{2.22}$$

where ω is defined in (2.20) *and M is the bound of* (2.21).

COROLLARY 2. *Let $z = \psi(\zeta)$ be a homeomorphism of the plane which satisfies* (2.8) *and* (2.9). *Then there is an $r_0 = r_0(k_0)$ such that if M is as in* (2.21) *and either $r(=|z|) > r_0$ or $\varrho(=|\zeta|) > r_0$, then*

$$|\zeta| = |\psi^{-1}(z)| \leqslant M r^{2\log M} \qquad (r > r_0 \quad or \quad \varrho > r_0), \tag{2.23}$$

and

$$|z| = |\psi(\zeta)| \leqslant M \varrho^{2\log M} \qquad (r > r_0 \quad or \quad \varrho > r_0). \tag{2.24}$$

Proof. By symmetry it suffices to show (2.23). Let $2^n \leqslant |z| < 2^{n+1}$, $n > 1$. Then the normalization $\psi(0) = 0$ with (2.21) yields

$$|\zeta| \leqslant \varrho_2(2)\prod_{j=1}^{n}\frac{\varrho_2(2^{j+1})}{\varrho_1(2^j)} \leqslant \varrho_2(2)\exp\left\{\frac{\log M}{\log 2}\log|z|\right\} = \varrho_2(2)|z|^{\log M/\log 2}.$$

In addition, Lemma 1 and the normalization $\psi(1) = 1$ give

$$\varrho_2(2) \leqslant \frac{\varrho_2(2)}{\varrho_1(1)} < M, \tag{2.25}$$

and hence if $|z| = r > 2$

$$|\zeta| < M r^{\log M/\log 2} \qquad (r = |z| > 2). \tag{2.26}$$

Since $|z| = |\psi(\zeta)| \geq 2$ when $|\zeta| \geq \varrho_2(2)$, (2.23) follows from (2.25) and (2.26) with $r_0 = \max(2, M)$.

We also need an '$o(1)$' form of Corollary 1. The simplest way to do this is to take k_0 in (2.4) and (2.5) small, or require that $\mu_g(\zeta) \to 0$ as $\zeta \to \infty$, but this is not adequate here. Sufficient flexibility is attained by studying the dependence of the expressions $\varrho_2(2r)/\varrho_1(r)$ and $\omega(r)$ as functions of

$$D(\varrho) = D(\varrho, \psi) = \frac{1}{2\pi}\int_0^{2\pi} |\mu_\psi(\varrho e^{i\phi})|\, d\phi. \tag{2.27}$$

LEMMA 2. *Let ψ, k_0, M, r_0 be the constants of Lemma 1 and Corollary 2. Then given $\varepsilon > 0$ there exist $\eta > 0$, $A < \infty$ such that if $\varrho' > r_0$ and*

$$D(\varrho, \psi) < \eta \qquad (\varrho > \varrho'), \tag{2.28}$$

then

$$\omega(r) < \varepsilon \qquad (r > A(\varrho')^{2\log M}) \tag{2.29}$$

and

$$2 - \varepsilon < \frac{\varrho_2(2r)}{\varrho_1(r)} < 2 + \varepsilon \qquad (r > A(\varrho')^{2\log M}). \tag{2.30}$$

Proof. Both (2.29) and (2.30) follow from similar considerations, so we consider only (2.29). If (2.29) were false, there would exist sequences $A_m \to \infty$, $\eta_m \to 0$, $\varrho'_m \geq r_0$ and r_m with

$$r_m \geq A_m(\varrho'_m)^{2\log M} \qquad (m = 1, 2, \ldots), \tag{2.31}$$

and to each m would correspond a normalized solution $z = \psi_m(\zeta)$ of the Beltrami equation (2.8) with

$$D(\varrho, \psi_m) < \eta_m \qquad (\varrho > \varrho'_m), \tag{2.32}$$

and yet

$$\omega(r_m, \psi_m) \geq \varepsilon \qquad (m = 1, 2, \ldots). \tag{2.33}$$

For appropriate real θ_m, ϕ_m let $\varrho_2(r_m)e^{i\phi_m} = \psi_m^{-1}(r_m e^{i\theta_m})$. Then for each m consider the homeomorphism $\Psi_m(\zeta)$,

$$\Psi_m(\zeta) = \frac{\psi_m(\varrho_2(r_m)\, e^{i\phi_m}\zeta)}{r_m e^{i\theta_m}}. \tag{2.34}$$

Clearly $\|\mu_{\Psi_m}\|_\infty = \|\mu_{\psi_m}\|_\infty \leq k_0$, and (2.9) holds for each Ψ_m. In addition, we claim that given $\eta > 0$, $\delta > 0$, then

$$D(\varrho, \Psi_m) < \eta \qquad (\varrho > \delta,\ m > m_0(\eta, \delta)). \tag{2.35}$$

For (2.31) implies that $r_m > r_0$ for large m, and hence if $\varrho = |\zeta| > \delta$, (2.24) and (2.31) yield

$$|\varrho_2(r_m)\zeta| > \varrho_2(A_m(\varrho'_m)^{2\log M})\,\delta > \delta\{M^{-1}A_m(\varrho'_m)^{2\log M}\}^{1/(2\log M)} = \delta(M^{-1}A_m)^{1/(2\log M)}\varrho'_m.$$

But $\{A_m\} \to \infty$ and the ϱ'_m are bounded below, so this computation implies that $|\varrho_2(r_m)\zeta| > \varrho'_m$ for large m. Thus (2.35) follows from this, (2.32) and the fact that $D(\varrho, \Psi_m) = D(\varrho_2(r_m)\varrho, \psi)$.

That the Ψ_m form a normal family is clear from Corollary 2 to Lemma 1 and (2.9) for each Ψ_m (cf. [10], pp. 74–76). By taking subsequences and then relabelling, we obtain a limit function $\Psi(\zeta)$ such that $\Psi_m \to \Psi$, $\Psi_m^{-1} \to \Psi^{-1}$ with convergence uniform on compacta, and (2.35) shows that $\mu_\Psi = 0$ a.e. Thus Ψ is a schlicht self-map of the plane which satisfies (2.9): $\Psi(\zeta) = \zeta$. This with (2.34) contradicts (2.33).

Remark. Conclusions (2.29) and (2.30) follow when (2.28) is weakened to

$$E(\varrho) \equiv \int_\varrho^{2\varrho} u^{-1} D(u, \psi)\, du < \eta \qquad (\eta > 0,\ \ \varrho > \varrho'(\eta)) \tag{2.36}$$

for sufficiently small $\eta > 0$, since the normal family argument again implies $\Psi(\zeta) = \zeta$. That conclusions of the nature (2.24) hold when (2.17) is replaced by (2.28) or (2.36) was first shown by P. Belinskii, and is discussed in his recent book ([3], p. 53). These ideas were also used in [6].

2.3. Sufficient conditions for Nevanlinna admissibility

Here we derive alternatives to (2.15) and (2.16). Let b be a complex number that is to satisfy $\delta(b, f) = 0$ (e.g., in the language of Theorem 1, b is disjoint from the $\{a_i\}$). Then we introduce the hypothesis that all limits

$$\lim_{\varrho\to\infty} \frac{n(\varrho, a, g)}{n(\varrho, b, g)},\ \lim_{\varrho\to\infty} \frac{\bar{n}(\varrho, a, g)}{n(\varrho, b, g)} \qquad (a \in \hat{C}) \tag{2.37}$$

exist. Since (2.10) and (2.19) lead at once to

$$n(\varrho_1(r), a, g) \leqslant n(r, a, f) \leqslant n(\varrho_2(r), a, g) \qquad (a \in \hat{C},\ r > 0), \tag{2.38}$$

$$\bar{n}(\varrho_1(r), a, g) \leqslant \bar{n}(r, a, f) \leqslant \bar{n}(\varrho_2(r), a, g) \qquad (a \in \hat{C},\ r > 0), \tag{2.39}$$

(2.37)–(2.39) and the definitions readily imply

LEMMA 3. *Let g be quasi-meromorphic and assume all limits in* (2.37) *exist, where* $\delta(b, g) = 0$. *Then if*

$$n(\varrho_1(r), b, g) \sim n(\varrho_2(r), b, g) \qquad (r \to \infty), \tag{2.40}$$

the function $g(\zeta)$ *is Nevanlinna admissible. More precisely, if the meromorphic function* $f(z)$ *is defined by* (2.10), *then*

$$\delta(a, f) = 1 - \lim_{r\to\infty} \frac{n(r, a, f)}{n(r, b, f)} = 1 - \lim_{\varrho\to\infty} \frac{n(\varrho, a, g)}{n(\varrho, b, g)} \qquad (a \in \hat{C}), \tag{2.41}$$

$$\theta(a, f) = \lim_{r\to\infty} \frac{n(r, a, f) - \bar{n}(r, a, f)}{n(r, b, f)} = \lim_{\varrho\to\infty} \frac{n(\varrho, a, g) - \bar{n}(\varrho, a, g)}{n(\varrho, b, g)} \qquad (a \in \hat{C}). \tag{2.42}$$

Condition (2.40) is the key to our method. Lemma 2 shows that $\varrho_1(r) \sim \varrho_2(r)$ when $D(\varrho, \psi) \to 0$ $(\varrho \to \infty)$ and (2.40) relates this to the growth of g.

Our function g will be defined in a manner to make it easy to check (2.37) and (2.40). We will introduce an increasing function $\lambda(\varrho)$ $(\varrho \geq 0)$ which is continuously differentiable off a discrete set P such that

$$\lambda(\varrho) \geq 1 \qquad (\varrho \geq 0), \tag{2.43}$$

$$\varrho|\lambda'(\varrho)| < 1 \qquad (\varrho > 0, \varrho \notin P). \tag{2.44}$$

Let

$$S(\varrho) = \exp\left\{\int_1^\varrho \lambda(u)\, u^{-1} du\right\} \qquad (\varrho > 0); \tag{2.45}$$

then we will construct a sequence $\{\varrho_m\} \to \infty$ with

$$\varrho_{m+1} > 2\varrho_m \qquad (m \geq 1), \tag{2.46}$$

such that

$$n(\varrho, b, g) \sim m\pi^{-1} S(\varrho) \qquad (\varrho_{m-1} \leq \varrho \leq \varrho_m,\ m \to \infty) \tag{2.47}$$

for some, and by (2.37) all, b having $\delta(b, g) = 0$. Assumptions (2.37), (2.45) and (2.47) replace (2.15) and (2.16); (2.45) and (2.47) are analogous to the classic proximate order representation but more flexible [5].

THEOREM 2. *For fixed* $k_0 < 1$, *let* g *be quasi-meromorphic with* $\|\mu_g\|_\infty \leq k_0$, *and let* $M = M(k_0)$, $r_0 = r_0(k_0)$ *be the constants determined in* (2.21), (2.23) *and* (2.24). *Let* $\{A_m\}$, $\{\eta_m\}$ *be sequences with the property that whenever*

$$D(\varrho) \leq \eta_m \qquad (\varrho > \varrho' > r_0) \tag{2.48}$$

for any $\varrho' > r_0$, *it follows that*

$$\omega(r) < m^{-2} \qquad (r > A_m M(\varrho')^{2\log M}). \tag{2.49}$$

Suppose $\{\varrho_m\}$ $(m \geqslant 1)$ *is chosen in accord with* (2.46) *such that in addition*

$$\varrho_m > r_0 \qquad (m = 1, 2, \ldots) \tag{2.50}$$

and

$$M^{1+2\log M} A_m^{2\log M} \varrho_m^{(2\log M)^2} \leqslant \varrho_{m+1} \qquad (m \geqslant 1). \tag{2.51}$$

If for some b *with* $\delta(b, g) = 0$ *all limits in* (2.37) *exist and* $n(\varrho, b, g)$ *may be represented as in* (2.43)–(2.47) *with*

$$\lambda(\varrho) \leqslant m+1 \qquad (\varrho \leqslant \varrho_m) \tag{2.52}$$

and

$$D(\varrho) \leqslant \eta_m \qquad (\varrho > \varrho_m), \tag{2.53}$$

then the function $g(\zeta)$ *is Nevanlinna admissible.*

Proof. (The existence of the $\{A_m\}$, $\{\eta_m\}$ is evident from Lemma 2). Since (2.53) shows that $D(\varrho, \psi) \to 0$ $(\varrho \to \infty)$, it follows from (2.46) and (2.47) that (2.40) is equivalent to

$$\int_{\varrho_1(r)}^{\varrho_2(r)} \lambda(u)\, u^{-1} du \to 0 \qquad (r \to \infty). \tag{2.54}$$

Also, (2.20), (2.22) and (2.44) give the estimate

$$\int_{\varrho_1(r)}^{\varrho_2(r)} \lambda(u)\, u^{-1} du \leqslant \{\lambda(\varrho_1(r)) + \log M\} \int_{\varrho_1(r)}^{\varrho_2(r)} u^{-1} du = \omega(r) \{\lambda(\varrho_1(r)) + \log M\}. \tag{2.55}$$

We may suppose that the A_m increase with m. Hence, given sufficiently large r there is a unique m with

$$A_m M \varrho_m^{2\log M} < r \leqslant A_{m+1} M \varrho_{m+1}^{2\log M}. \tag{2.56}$$

That

$$\omega(r) < m^{-2} \tag{2.57}$$

follows at once from the left inequality of (2.56) with (2.49), (2.50) and (2.53). The right inequality of (2.56) with (2.23) and (2.51) shows that if r (i.e. m) is large,

$$\varrho_1(r) \leqslant M r^{2\log M} \leqslant M^{1+2\log M} A_{m+1}^{2\log M} \varrho_{m+1}^{(2\log M)^2} \leqslant \varrho_{m+2}$$

and so, from (2.52),

$$\lambda(\varrho_1(r)) \leqslant m+3. \tag{2.58}$$

When (2.57) and (2.58) are used to estimate the right side of (2.55), we see that (2.54) and (2.40) are proved.

LEMMA 4. *Assume in addition to the hypotheses of Theorem 2 that*

$$\phi\left(\left[\frac{\varrho_m}{M}\right]^{1/(2\log M)}\right) > 3(m+2)\log M \qquad (m \geq 1), \tag{2.59}$$

where ϕ is as in (1.3). *Then the Nevanlinna characteristic of the meromorphic function $f = g\circ\psi^{-1}$ satisfies* (1.4).

Proof. Choose a with $T(r, f) \sim N(r, a, f)$ ([15], p. 280) and recall the number $b \in C$ with $\delta(b, g) = 0$ (cf. (2.37)). Then (2.37) and (2.41) imply that $N(r, b, f) \sim T(r, f)$.

Thus, if r is large with

$$\varrho_{m-1} < Mr^{2\log M} \leq \varrho_m \qquad (m \geq 2) \tag{2.60}$$

we deduce from (2.47), (2.23), (2.45), (2.52), (2.60), (2.46) and (2.59) that

$$T(r, f) < 2N(r, b, f) < 4m\pi^{-1}S(\varrho_2(r))\log r \leq 4m\pi^{-1}S(Mr^{2\log M})\log r$$
$$\leq 4m\pi^{-1}(Mr^{2\log M})^{m+1}\log r \leq r^{3(m+1)\log M} \leq r^{\phi([\varrho_{m-1}/M]^{1/(2\log M)})} < r^{\phi(r)},$$

which is (1.4).

3. Outline of construction

3.1. Functions g_j, g_j^*. The basic goals of the construction are easy to describe, but their realization requires much attention.

To include the possibility that $\Sigma(\delta_i + \theta_i) < 2$ (in particular that the $\{a_i\}$ be an empty set), let a_0, a_N be complex numbers disjoint from the $\{a_i\}$ $(1 \leq i < N)$, set

$$\mathcal{A} = \{a_i\} \qquad (0 \leq i \leq N), \tag{3.1}$$

$$\mathcal{A}^* = \{a_i\} \qquad (1 \leq i < N), \tag{3.2}$$

and assume, with no loss of generality, that $\infty \notin \mathcal{A}$.

Next let $\mathcal{B} = \{b_j\}$ $(-\infty < j < \infty)$ be a sequence all of whose elements are in $\mathcal{A}$, with

$$b_j = b_{-j} \qquad (-\infty < j < \infty), \tag{3.3}$$

$$b_j \neq b_{j+1} \qquad (-\infty < j < \infty), \tag{3.4}$$

(compare with § 1.2). In the enumeration of $\mathcal{B}$, each element of $\mathcal{A}$ is repeated sufficiently often to ensure that if

$$E(a) = \{j;\ b_j = a\},\ E_m(a) = E(a) \cap [-m, m] \qquad (a \in \mathcal{A},\ m = 1, 2, \ldots), \tag{3.5}$$

and

$$\Delta_m(a) = m^{-1}\,\mathrm{card}[E_m(a)], \tag{3.6}$$

then

$$\Delta_m(a_i) \to \Delta_i \equiv \delta_i + \theta_i \qquad (m \to \infty,\ 1 \leqslant i < N), \tag{3.7}$$

$$\Delta_m(a_0) \to \Delta_0 \equiv 1 - \tfrac{1}{2} \sum_{1 \leqslant i < N} \{\delta_i + \theta_i\}, \tag{3.8}$$

$$\Delta_m(a_N) \to \Delta_N \equiv 1 - \tfrac{1}{2} \sum_{1 \leqslant i < N} \{\delta_i + \theta_i\}. \tag{3.9}$$

Thus, $0 \leqslant \Delta_i \leqslant 1$, $\sum_A \Delta_i = 2$. The set $\mathcal{B}$ may be constructed, for example, by adjusting the procedure of [9], Lemma 4.4.

Let the $\{\delta_i\}$, $\{\theta_i\}$ be as in the statement of Theorem 1 and the $E(a_i)$ as in (3.5). Then for $-\infty < j < \infty$ choose Λ_j with

$$\Lambda_0 = 2, \tag{3.10}$$

$$\Lambda_{-j} = \Lambda_j \qquad (-\infty < j < \infty), \tag{3.11}$$

$$\tfrac{3}{2} < \Lambda_j \leqslant 2 \qquad (-\infty < j < \infty), \tag{3.12}$$

$$|\sin \pi \Delta_j| \to \frac{\theta_i}{\delta_i + \theta_i} \qquad (j \to \infty,\ j \in E(a_i),\ 1 \leqslant i \leqslant N), \tag{3.13}$$

$$|\sin \pi \Lambda_j| \to 1 \qquad (j \in E(a_0) \cup E(a_N)) \tag{3.14}$$

(when $\delta_i > 0$, (3.13) may be simplified to $|\sin \pi \Lambda_j| = \theta_i(\delta_i + \theta_i)^{-1}(j \in E(a_i))$, but it is convenient that $|\sin \pi \Lambda_j| < 1$ for all j, as guaranteed by (3.12)).

Now, once and for all, choose

$$k_0 = 2^{-4} \tag{3.15}$$

in (2.4) and (2.5); this choice yields r_0, M, $\{A_m\}$, $\{\eta_m\}$ as in Theorem 2. We recall from the programme of § 2.3 that the value-distribution of $g(\zeta)$ is to be compared with a function $S(\varrho)$ as evinced by (2.37) and (2.47). The representation (2.45) shows that $S(\varrho)$ is determined in turn by an increasing function $\lambda(\varrho)$. At that time, $\lambda(\varrho)$ was to satisfy (2.43) and (2.44).

We now impose more specific conditions on λ:

$$\lambda(\varrho) = 1 \qquad (\varrho \leqslant \varrho_0 = 1), \tag{3.16}$$

$$\lambda(\varrho_1) = 2, \tag{3.17}$$

$$\lambda(\varrho_m) = 1 + 2 \sum_{0}^{m-1} (\Lambda_k - \tfrac{3}{2}) \qquad (m \geqslant 2). \tag{3.18}$$

Finally, in § 5.2 we will determine a positive sequence $\{\tau'_m\}$ and require that

$$(0 \leqslant) \varrho \lambda'(\varrho) < \tau'_m \qquad (\varrho_m < \varrho < \varrho_{m+1}) \tag{3.19}$$

where $\tau'_m \to 0$ $(m \to \infty)$. Note that (3.16)–(3.19) are compatible if ϱ_1 and the ratios $\{\varrho_{m+1}/\varrho_m\}$ $(m \geq 1)$ are sufficiently large, and that (3.16) and (3.18) imply (2.52).

The precise sequence $\{\varrho_m\}$, which is to satisfy (2.46), (2.50), (2.51), (2.53) and (2.59) as well as to interact with $\lambda(\varrho)$ as required in (2.52) and (3.16)–(3.19), will be constructed in § 5.2.

Next, the ζ-plane is divided into disjoint regions D_j $(-\infty < j < \infty)$, D_j^* $(j \geq 0)$ with

$$\sum_{-\infty}^{\infty} \operatorname{meas} \partial D_j + \sum_{0}^{\infty} \operatorname{meas} \partial D_j^* = 0 \tag{3.20}$$

((3.20) refers to planar measure),

$$D_0^* = \{|\zeta| < 1\}, \tag{3.21}$$

$$D_j \subset \{|\zeta| \geq \varrho_{|j|}\} \qquad (-\infty < j < \infty), \tag{3.22}$$

$$D_j^* \subset \{\varrho_{j-1} \leq |\zeta| \leq \varrho_j\} \qquad (1 \leq j < \infty). \tag{3.23}$$

For appropriate functions $\alpha_j(\varrho)$, $\beta_j(\varrho)$, we will have

$$D_j \cap \{|\zeta| = \varrho\} = \{\varrho e^{i\phi};\ \varrho \geq \varrho_{|j|},\ \alpha_j(\varrho) \leq \phi \leq \beta_j(\varrho)\} \qquad (-\infty < j < \infty), \tag{3.24}$$

$$D_j^* \cap \{|\zeta| = \varrho\} = \{\varrho e^{i\phi};\ \varrho_{j-1} \leq \varrho \leq \varrho_j,\ |\phi| \leq \alpha_{j-1}(\varrho)\} \qquad (j \geq 1), \tag{3.25}$$

where

$$\beta_{-j}(\varrho) = 2\pi - \alpha_j(\varrho),\ \alpha_{-j}(\varrho) = 2\pi - \beta_j(\varrho) \qquad (j \geq 0,\ \varrho \geq \varrho_j), \tag{3.26}$$

$$\alpha_j(\varrho) = \beta_{j+1}(\varrho) \qquad (-\infty < j < \infty,\ \varrho \geq \max(\varrho_{|j|}, \varrho_{|j+1|})). \tag{3.27}$$

Thus the interiors of these sets are mutually disjoint, and $\bigcup_j \bar{D}_j \cup \bigcup_j \bar{D}_j^*$ is the full ζ-plane (see Figure 1, p. 98).

The function $g(\zeta)$ which solves the inverse problem for the data $\{\delta_i\}$, $\{\theta_i\}$ is defined by

$$g(\zeta) = \begin{cases} g^*(\zeta) = T_0(e^{-\zeta}) & (\zeta \in D_0^*) \\ g_j(\zeta) = T_j \circ H_j^{\#} \circ \psi_j(\zeta) & (\zeta \in D_j) \\ g_j^*(\zeta) = T_j \circ H_j^* \circ \psi_j^*(\zeta) & (\zeta \in D_j^*,\ j \geq 1). \end{cases} \tag{3.28}$$

Here the g_j, g_j^* are continuous in the closures of their respective domains, the Möbius transformation T_j is

$$T_j(W) = \frac{b_j W + b_{|j|+1}}{W+1} \qquad (-\infty < j < \infty) \tag{3.29}$$

(where the $\{b_j\}$ are determined by (3.3), (3.4), (3.7)–(3.9)) and the $H_j^{\#}$, ψ_j, H_j^*, ψ_j^* are to be specified in §§ 4.4 and 4.5.

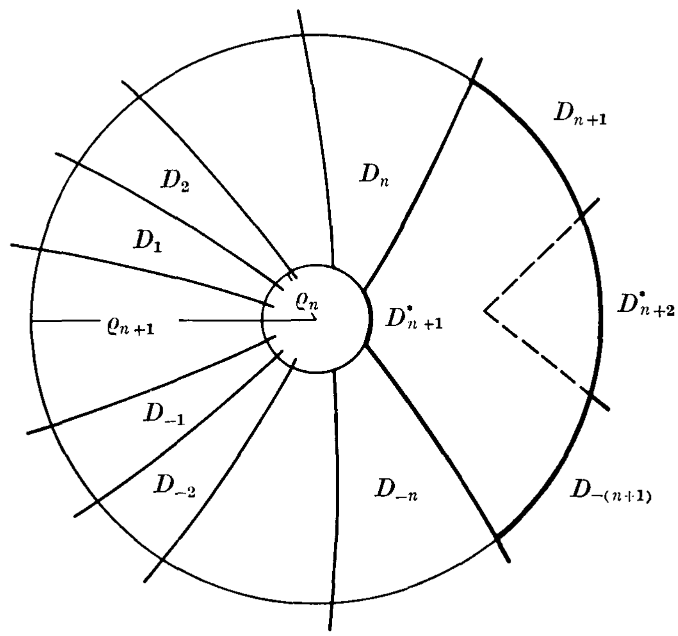

Fig. 1

We summarize below, in Theorem 3, the properties of g and the g_j, g_j^* which are needed for the proof of Theorem 1; the proof of Theorem 3 is deferred to chapters 4 and 5, although an important component, Theorem 4, is considered separately in chapter 6.

THEOREM 3. *It is possible to choose the D_j, D_j^*, H_j, $H_j^\#$, ψ_j, ψ_j^* so that if $g(\zeta)$ is defined as in* (3.28), *then the following conditions hold:*

g is continuous in the finite plane, quasi-meromorphic and Nevanlinna admissible; (3.30)

the meromorphic function $f(z)$, defined by (2.10), *satisfies* (1.4) *and* (2.48). (3.31)

Further there is an absolute constant[1] *A such that if*

$$n(\varrho, a, g, D) \qquad (\bar{n}(\varrho, a, g, D)) \tag{3.32}$$

is the number of solutions to the equation $g(\zeta) = a$ with (*without*) *account of multiplicity with ζ in $D^\circ \cap \{|\zeta| < \varrho\}$* ($D^\circ$ = *interior of D*), *then*

---

[1] Until chapter 6 A will be used to represent constants which depend at most on the choice of k_0 in (3.15). The conventions in chapter 6 are discussed on p. 129.

$$n(\varrho, a, g, D_j) < A\,S(\varrho) \qquad (a \in \hat{C}, \varrho > 0), \tag{3.33}$$

$$n(\varrho, a, g, D_j^*) < A\,S(\varrho) \qquad (a \in \hat{C}, \varrho > 0). \tag{3.34}$$

Moreover, if a belongs to

$$p_j(a) \tag{3.35}$$

of the regions

$$\left\{w;\, 0 < \left|\frac{w-b_j}{w-b_{j-1}}\right| < 1\right\},\ \left\{w;\, 0 < \left|\frac{w-b_j}{w-b_{j+1}}\right| < 1\right\}$$

then

$$n(\varrho, a, g, D_j) \sim \bar{n}(\varrho, a, g, D_j) \sim (2\pi)^{-1} p_j(a)\, S(\varrho) \tag{3.36}$$

in a manner such that for each $\varepsilon > 0$

$$\begin{aligned} &|n(\varrho, a, g, D_j) - (2\pi)^{-1} p_j(a)\, S(\varrho)| \\ &\quad \leqslant A m^{-1} S(\varrho) \qquad (m \geqslant M(\sigma),\ |j| \leqslant m,\ \varrho_m \leqslant \varrho \leqslant \varrho_{m+1}) \end{aligned} \tag{3.37}$$

in each region

$$|a - b_j| > \sigma > 0. \tag{3.38}$$

Finally, let the sets $E(a_i)$ *be as in* (3.5), *the* $\{\Lambda_j\}$ *as in* (3.10)–(3.14), *and*

$$\alpha_j = \begin{cases} 1 & j \in E(a_0) \cup E(a_N) \\ 0 & j \in E(a_i),\ 1 \leqslant i < N. \end{cases} \tag{3.39}$$

Then

$$|n(\varrho, b_j, g, D_j) - \pi^{-1}|\sin \pi\Lambda_j|\, S(\varrho)| < |j|^{-1} S(\varrho) \qquad (|k| < |j|,\ \varrho \geqslant \varrho_{|j|}), \tag{3.40}$$

$$|\bar{n}(\varrho, b_j, g, D_j) - \alpha_j n(\varrho, b_j, g, D_j)| < |j|^{-1} S(\varrho) \qquad (|k| < |j|,\ \varrho \geqslant \varrho_{|j|}). \tag{3.41}$$

3.2. Proof of Theorem 1. We now assume the assertions of Theorem 3. Since (1.4) is contained in (3.31), and g is Nevanlinna admissible, it suffices to establish those equalities in (2.41) and (2.42) which involve g. We recall that $\mathcal{A} = \{a_i\}\ 0 \leqslant i \leqslant N$ in (3.1).

LEMMA 5. *Let* $S(\varrho)$ *be as in* (2.45). *Then*

$$n(\varrho, a, g) \sim m\pi^{-1} S(\varrho) \qquad (a \notin \mathcal{A},\ \varrho_m \leqslant \varrho \leqslant \varrho_{m+1},\ m \to \infty) \tag{3.42}$$

for each $a \notin \mathcal{A}$.

Proof. For the moment, suppose

$$g(\zeta) \neq a \qquad (\zeta \in \{\bigcup \partial D_j\} \cup \{\bigcup \partial D_j^*\}); \tag{3.43}$$

according to (3.20) and elementary properties of quasi-conformal mappings ([1], p. 33) this means that only a set of a's having measure 0 is excluded.

Let $\varepsilon > 0$ be given and choose $M < \infty$ so that if

$$\mathcal{A}_\varepsilon = \{a_0, a_N, a_1, \ldots, a_M\} \qquad (\subset \mathcal{A}), \tag{3.44}$$

then

$$\sum_{\mathcal{A}_\varepsilon} \Delta_i > 2 - \varepsilon \qquad (= \sum_{\mathcal{A}} \Delta_i - \varepsilon), \tag{3.45}$$

where the $\{\Delta_i\}$ are described in (3.6)–(3.9). Thus if

$$F(\varepsilon) = \{j;\, b_j \in \mathcal{A}_\varepsilon\}, \tag{3.46}$$

it follows from (3.45) and (3.46) that

$$\operatorname{card}\,[F(\varepsilon) \cap (-m, m)] > (2-\varepsilon)m \qquad (m > m_0(\varepsilon)). \tag{3.47}$$

Define D_ε and D'_ε by

$$D_\varepsilon = \bigcup_{j \in F(\varepsilon)} D_j,\; D'_\varepsilon = \{D_\varepsilon\}' \tag{3.48}$$

(where $\{D_\varepsilon\}'$ is the complement of D_ε). Then assumption (3.43) yields that

$$n(\varrho, a, g) = n(\varrho, a, g, D_\varepsilon) + n(\varrho, a, g, D'_\varepsilon) = \sum_{j \in F(\varepsilon)} n(\varrho, a, g, D_j) + n(\varrho, a, g, D'_\varepsilon) \qquad (\varrho > 0). \tag{3.49}$$

If $\varrho_m \leqslant \varrho \leqslant \varrho_{m+1}$, (3.21)–(3.23), (3.33), (3.34) and (3.47) lead to

$$\begin{aligned} n(\varrho, a\, g, D'_\varepsilon) &\leqslant \sum_{\substack{j \notin F_\varepsilon \\ |j| \leqslant m}} n(\varrho, a, g, D_j) + \sum_{|j| \leqslant m} n(\varrho_j, a, g, D_j^*) + n(\varrho, a, g, D_{m+1}^*) \\ &\leqslant A S(\varrho)(\varepsilon m + 2) + A \sum_{|j| \leqslant m} S(\varrho_j) \qquad (\varrho_m \leqslant \varrho \leqslant \varrho_{m+1}). \end{aligned} \tag{3.50}$$

According to (2.43), (2.45) and (2.46)

$$S(\varrho_j) \geqslant (\varrho_j/\varrho_{j-1})^{\lambda(\varrho_{j-1})} S(\varrho_{j-1}) \geqslant 2S(\varrho_{j-1}) \qquad (j \geqslant 1). \tag{3.51}$$

so (3.50) becomes

$$n(\varrho, a, g, D'_\varepsilon) \leqslant A(\varepsilon m + 1)\, S(\varrho) \qquad (a \in \hat{C},\; \varrho_m \leqslant \varrho \leqslant \varrho_{m+1}). \tag{3.52}$$

Next, we note from assumption (3.43) and the definition of $p_j(a)$ in (3.35) that

$$\Big| \sum_{|j| < m} p_j(a) - 2m \Big| \leqslant 2, \tag{3.53}$$

and, since $0 \leqslant p_j(a) \leqslant 2$ for all j, this with (3.47) yields that

$$\Big| \sum_{\substack{|j| \leqslant m \\ j \in F(\varepsilon)}} p_j(a) - 2m \Big| \leqslant A(1 + \varepsilon m).$$

Thus, if σ is chosen so small that $|a-b_j|>\sigma$ $(j\in F(\varepsilon))$ it follows from (3.37) that

$$\Big|\sum_{j\in F(\varepsilon)} n(\varrho, a, g, D_j) - m\pi^{-1}S(\varrho)\Big| \leqslant A(1+\varepsilon m)\,S(\varrho) \qquad (m>M(\sigma),\ \varrho_m\leqslant\varrho\leqslant\varrho_{m+1}). \tag{3.54}$$

(This is possible since $\mathcal{A}_\varepsilon$ in (3.44) is a finite set.) However, as ε tends to zero (3.52) and (3.54), with (3.49), yield (3.42).

Finally, we remove restriction (3.43). Given $\varepsilon>0$ and $a\notin\mathcal{A}$, choose $\sigma>0$ so small that

$$\{|w-a|\leqslant 2\sigma\}\cap\mathcal{A}_\varepsilon=\phi. \tag{3.55}$$

To compute $n(\varrho, a, g)$, we may suppose that $g(\varrho e^{i\phi})\neq a$ $(0\leqslant\phi\leqslant 2\pi)$, and choose $a'\notin\mathcal{A}$ such that $|a-a'|<\sigma$, (3.43) holds for a', and

$$n(\varrho, a', g) = n(\varrho, a, g). \tag{3.56}$$

Let $F(\varepsilon)$ be as in (3.46). Then (3.49) and (3.52) show that

$$\Big|n(\varrho, a', g) - \sum_{j\in F(\varepsilon)} n(\varrho, a', g, D_j)\Big| \leqslant A(1+\varepsilon m)\,S(\varrho) \qquad (\varrho_m\leqslant\varrho\leqslant\varrho_{m+1}), \tag{3.57}$$

and since (3.55) implies that $|a'-a_i|>\sigma$ $(a_i\in\mathcal{A}_\varepsilon)$, the argument which gave (3.54) leads to

$$\Big|\sum_{j\in F(\varepsilon)} n(\varrho, a', g, D_j) - m\pi^{-1}S(\varrho)\Big| \leqslant A(1+\varepsilon m)\,S(\varrho) \qquad (\varrho_m\leqslant\varrho\leqslant\varrho_{m+1}), \tag{3.58}$$

at least when $m\geqslant M(\sigma)$. Now (3.42) is an obvious consequence of (3.56)–(3.58).

The proof of Theorem 1 is completed by

LEMMA 6. *The value-distribution of g satisfies*

$$n(\varrho, a, g) \sim (1-\delta_i)\,m\pi^{-1}S(\varrho) \qquad (m\to\infty,\ \varrho_m\leqslant\varrho\leqslant\varrho_{m+1},\ a=a_i\in\mathcal{A}^*), \tag{3.59}$$

$$n(\varrho, a_0, g) \sim n(\varrho, a_N, g) \sim m\pi^{-1}S(\varrho) \qquad (m\to\infty,\ \varrho_m\leqslant\varrho\leqslant\varrho_{m+1}), \tag{3.60}$$

$$\bar n(\varrho, a_i, g) \sim (1-\delta_i-\theta_i)\,m\pi^{-1}S(\varrho) \qquad (m\to\infty,\ \varrho_m\leqslant\varrho\leqslant\varrho_{m+1},\ a=a_i\in\mathcal{A}^*), \tag{3.61}$$

$$n(\varrho, a, g) \sim \bar n(\varrho, a, g) \qquad (\varrho\to\infty,\ a\notin\mathcal{A}^*). \tag{3.62}$$

Proof. We suppose a satisfies (3.43) since otherwise the procedure used to eliminate this restriction in Lemma 5 may again be applied.

First, consider (3.59) and (3.60). Fiven $a\in\hat{C}$, $\varepsilon>0$, let $F(\varepsilon)$, $E(a)$ be as in (3.46) and (3.5). Let $F(\varepsilon)$ be partitioned into $F(a,\varepsilon)$ and $F'(a,\varepsilon)$ where

$$F(a,\varepsilon) = F(\varepsilon)\cap E(a), \tag{3.63}$$

$$F'(a,\varepsilon) = F(\varepsilon) - F(a,\varepsilon). \tag{3.64}$$

Then if D_ε, D'_ε are as in (3.48) we recall as in (3.49) and (3.52) that

$$|n(\varrho, a, g) - n(\varrho, a, g, D_\varepsilon)| \leqslant A(\varepsilon m + 1) S(\varrho) \qquad (\varrho_m \leqslant \varrho \leqslant \varrho_{m+1}), \tag{3.65}$$

and since ε may be taken arbitrarily small, it suffices to estimate $n(\varrho, a, g, D_\varepsilon)$; this is made by an analysis of the identity

$$n(\varrho, a, g, D_\varepsilon) = \sum_{F(a,\varepsilon)} n(\varrho, a, g, D_j) + \sum_{F'(a,\varepsilon)} n(\varrho, a, g, D_j). \tag{3.66}$$

Now (3.13) shows that

$$(\delta_i + \theta_i)|\sin \pi\Lambda_j| \to \theta_i \qquad (j \to \infty,\ j \in E(a_i),\ a_i \in \mathcal{A}^*), \tag{3.67}$$

and it is easy to see from (3.6)–(3.9), (3.47), (3.63), (3.64) that

$$|\operatorname{card}\{F(a, \varepsilon) \cap [-m, m]\} - m(\delta_i + \theta_i)| \leqslant A\varepsilon m \qquad (a = a_i \in \mathcal{A}^*,\ m > m_0(\varepsilon, a)), \tag{3.68}$$

and

$$|\operatorname{card}\{F'(a, \varepsilon) \cap [-m, m]\} - m\{2 - (\delta_i + \theta_i)\} \leqslant A\varepsilon m \qquad (a = a_i \in \mathcal{A}^*,\ m > m_0(\varepsilon, a)). \tag{3.69}$$

Hence (3.33), (3.40), (3.67) and (3.68) yield

$$\Big|\sum_{F(a,\varepsilon)} n(\varrho, a, g, D_j) - \theta_i m\pi^{-1} S(\varrho)\Big| \leqslant A\varepsilon m S(\varrho) \qquad (a = a_i \in \mathcal{A}^*,\ \varrho_m \leqslant \varrho \leqslant \varrho_{m+1},\ m > m_0(\varepsilon, a)). \tag{3.70}$$

It is clear from definition (3.35) that $p_j(a) = 2$ ($j \in F(a, \varepsilon)$) so (3.53) and (3.68) give

$$\Big|\sum_{F'(a,\varepsilon)} p_j(a) - 2m\{1 - (\delta_i + \theta_i)\}\Big| \leqslant A\varepsilon m \qquad (a = a_i \in \mathcal{A}^*,\ m > m_0(\varepsilon, a)). \tag{3.71}$$

Thus (3.7), (3.37) and (3.71) readily yield

$$\Big|\sum_{F'(a,\varepsilon)} n(\varrho, a, g, D_j) - m\pi^{-1}\{1 - (\delta_i + \theta_i)\} S(\varrho)\Big| \leqslant A\varepsilon m S(\varrho) \qquad (a = a_i \in \mathcal{A}^*,\ \varrho_m \leqslant \varrho \leqslant \varrho_{m+1}).$$

which with (3.65), (3.66) and (3.70) implies (3.59). The same reasoning with (3.14) in place of (3.13) gives (3.60).

Next, (3.39) and (3.41) yield

$$\sum_{F(a,\varepsilon)} \bar{n}(\varrho, a, g, D_j) = o(1)\Big\{\sum_{F(a,\varepsilon)} n(\varrho, a, g, D_j)\Big\} \qquad (a \in \mathcal{A}^*,\ \varrho \to \infty), \tag{3.72}$$

$$\sum_{F(a,\varepsilon)} \bar{n}(\varrho, a, g, D_j) \sim \sum_{F(a,\varepsilon)} n(\varrho, a, g, D_j) \qquad (a = a_0, a_N, \varrho \to \infty), \tag{3.73}$$

and according to (3.33) and (3.37)

$$\sum_{F'(a,\varepsilon)} \{n(\varrho, a, g, D_j) - \tilde{n}(\varrho, a, g, D_j)\} \leqslant A\varepsilon m S(\varrho) \qquad (\varrho_m \leqslant \varrho \leqslant \varrho_{m+1},\ m \to \infty). \tag{3.74}$$

Thus (3.61) is an easy consequence of (3.48), (3.65), (3.72) and (3.74), and (3.62) follows from (3.48), (3.65), (3.73) and (3.74).

4. Auxiliary functions

In this chapter we develop the necessary material to prove Theorem 4.

4.1. The fundamental auxiliary function. Let $f_1(z) = e^z$, $f_2(z) = e^{-z^2}$ and write these formulae as

$$\log f_1(re^{i\theta}) = -r\,e^{i(\theta-\pi)} \qquad (r > 0,\ 0 \leqslant \theta < 2\pi), \tag{4.1}$$

$$\log f_2(re^{i\theta}) = -r^2\,e^{i2(\theta-\pi)} \qquad (r > 0,\ 0 \leqslant \theta < 2\pi). \tag{4.2}$$

The functions which generalize (4.1) and (4.2) to arbitrary Λ, $1 < \Lambda < 2$, are the classical Lindelöf functions of order Λ (cf. [12], Ch. 1, § 17). Indeed, if f_Λ is a canonical product with positive zeros and zero-counting function $n(r, 0, f_\Lambda) \sim \pi^{-1}|\sin \pi\Lambda|\, r^\Lambda$, an appropriate branch of $\log f_\Lambda$ satisfies

$$\log f_\Lambda(re^{i\theta}) = -r^\Lambda e^{i\Lambda(\theta-\pi)}\{1 + k(z)\} \qquad (r > 0,\ 0 < \theta < 2\pi), \tag{4.3}$$

where $k(z)$ tends to zero uniformly in any sector $\{|\theta - \pi| < \pi - \delta\}$ $(\delta > 0)$ as $r \to \infty$.

We will construct a quasi-meromorphic function $H(w)$ $(w = se^{it})$ which 'interpolates' the family $f_\Lambda(1 \leqslant \Lambda \leqslant 2)$. Thus on each circle $\{|w| = s\}$ an equation of the nature (4.1), (4.2) or (4.3) will hold for some Λ, but Λ will vary with s. The relevance of H to our construction is discussed in § 4.2.

Let $\Lambda(s)$ $(s > 0)$ be a continuous function which has continuous derivatives off some discrete set P having no finite accumulation point, with

$$1 \leqslant \Lambda(s) \leqslant 2 \qquad (s > 0), \tag{4.4}$$

$$|s\Lambda'(s)| < (2\pi)^{-1} \qquad (s > 0,\ s \notin P), \tag{4.5}$$

$$s\Lambda'(s) \to 0 \qquad (s \to \infty,\ s \notin P), \tag{4.6}$$

and define

$$S(s) = \exp\left\{\int_1^s \Lambda(u)\,u^{-1}du\right\} \qquad (s > 0); \tag{4.7}$$

note the similarity between (4.4)–(4.7) and (2.43)–(2.45). We have the obvious (and useful) consequences of (4.4) and (4.7):

$$s \leqslant S(s) \leqslant s^2 \qquad (s > 1), \tag{4.8}$$

$$S(1) = 1; \quad S(s)(s'/s) \leqslant S(s') \leqslant S(s)(s'/s)^2 \qquad (s' > s). \tag{4.9}$$

The relation between $\Lambda(r)$ (subject to (4.6)) and $S(r)$ is analogous to that between a proximate order $\varrho(r)$ (subject to $r\varrho'(r)\log r \to 0$) and the classical comparison function $r^{\varrho(r)}$, but permits more flexibility ([5]).

THEOREM 4. *Let* $(50)^{-1} > \eta > 0$ *and* $0 \leqslant \alpha \leqslant 1$ *be given. Then there exist* $M^\infty < \infty$, $\tau_0 > 0$ *such that if* $\Lambda(s)$ *is a differentiable function off a discrete set* P *(where* P *has no finite accumulation point) which satisfies* (4.4)–(4.6),

$$\sin \pi\Lambda(s) = 0 \qquad (0 < s < M^\infty) \tag{4.10}$$

(so that $\Lambda(s) \equiv 1$ *or* $\equiv 2$ *for* $s \leqslant M^\infty$*), and*

$$s\,|\Lambda'(s)| < \tau_0(<(2\pi)^{-1}) \qquad (s > 0,\ s \notin P), \tag{4.11}$$

then a quasi-meromorphic function $H(w)$ *may be associated to* $\Lambda(s)$ *with the following properties. The dilatation of* H *satisfies*

$$\|\mu_H\|_\infty < \eta, \tag{4.12}$$

$$\mu_H(w) \to 0 \qquad (w \to \infty) \tag{4.13}$$

and, if $S(s)$ *is as in* (4.7)*, then*

$$\log H(w) = -S(s)e^{i\Lambda(s)(t-\pi)} \qquad (\eta < t < 2\pi - \eta) \tag{4.14}$$

for a proper choice of branch. Moreover, whenever

$$\Lambda(s) = m \qquad (m = 1, 2), \tag{4.15}$$

(4.14) *may be improved to*

$$\log H(w) = -S(s)e^{im(t-\pi)} \qquad (0 \leqslant t \leqslant 2\pi). \tag{4.16}$$

The value-distribution of H *satisfies*

$$n(s, a, H) < AS(s) \qquad (a \in \hat{C},\ s > 1), \tag{4.17}$$

where A *is an absolute constant (independent of* η *and* τ_0*). Also*

$$n(s, \infty, H) = o(1)S(s) \qquad (w \to \infty), \tag{4.18}$$

the zeros of H *are on the positive axis with*

$$|n(s, 0, H) + \pi^{-1}\sin \pi\Lambda(s)S(s)| = o(1)S(s) \qquad (s \to \infty) \tag{4.19}$$

and

$$|\tilde{n}(s, 0, H) - \alpha n(s, 0, H)| = o(1)S(s) \qquad (s \to \infty). \tag{4.20}$$

Finally, if in addition $\Lambda^{\#}$ *satisfies*

$$\tfrac{3}{2} < \Lambda^{\#} \leqslant 2 \qquad (s > 0) \tag{4.21}$$

and for all large $s (s > s_{\#})$

$$\Lambda(s) = \Lambda^{\#} \qquad (s > s_{\#}), \tag{4.22}$$

then

$$\log |H(w)| \leqslant -A \sin(\Lambda^{\#} - \tfrac{3}{2})\, S(s), \qquad \left(s > s(s_{\#}, \Lambda^{\#}),\ |\arg w| < \frac{\pi}{2}\left(1 - \frac{3}{2\Lambda^{\#}}\right)\right) \tag{4.23}$$

again with A *an absolute constant, and*

$$n(s, a, H) - \tilde{n}(s, a, H) \leqslant o(1) S(s) \quad (0 < |a| < \infty;\ s \to \infty), \tag{4.24}$$

where the $o(1)$ *of* (4.24) *is uniform for* a *in each region*

$$|\log |a|| \leqslant A_1. \tag{4.25}$$

Remarks. With some care, it may be shown that the asymptotics in (4.18)–(4.20), (4.23) and (4.24) are attained at a rate which depends only on τ_0, $\Lambda^{\#}$, $s_{\#}$ and the rate at which (4.6) is achieved. This, and precise asymptotic computations for $n(s, a, H)$ $(0 < |a| < \infty)$, is not needed here.

If $\Lambda(s) \equiv \Lambda_0$, $1 < \Lambda_0 < 2$, then Theorem 3 (with no references to (4.20), (4.23) and (4.24)) is implicit in [6].

4.2. On the role of Theorem 4. Let us consider a function $\Lambda(s)$ as in (4.4)–(4.6), (4.10) and (4.11) with

$$\Lambda(s) = 1 \qquad (s < M^{\infty}),$$

$$\Lambda(s) = 2 \qquad (s > M'),$$

where M'/M^{∞} is sufficiently large (to be compatible with (4.11)) and let H be associated to Λ as in Theorem 4. We consider the subsets of the plane

$$\mathcal{D}_0^* = \{w;\ \eta < |t| < \pi - \pi/2\Lambda(s)\},$$

$$\mathcal{D}_0 = \{w;\ |t - \pi| < \pi/2\Lambda(s)\}$$

(compare with Figure 1, p. 98). It is immediate from (4.14) that $|H(w)| < 1$ in $\mathcal{D}_0$. Further, while $|H(w)| > 1$ on $\{|w| = s\} \cap \mathcal{D}_0^*$ when $\Lambda(s) < \tfrac{3}{2}$, we observe that $\{|w| = s\} \cap \mathcal{D}_0^*$ contains two subarcs on which $|H| < 1$ when $\Lambda(s) > \tfrac{3}{2}$. In particular, when $s > M'$, $|H(se^{it})| < 1$ for $|t| < \pi/4$, since (4.16) now applies.

Recall the sequence $\mathcal{B}$ of (3.3)–(3.9), let $T(w)$ be the Möbius transformation $(b_0 W + b_1)(W + 1)^{-1}$ (cf. (3.29)), and consider the behavior of $g(w) = T \circ (H(w))^{-1}$ (note the similarity to (3.28)). Clearly $|g(w)-b_0|<|g(w)-b_1|$ on $\{|w|=s\}\cap \mathcal{D}_0$. We find also that $|g(w)-b_1|<|g(w)-b_0|$ on $\{|w|=s\}\cap \mathcal{D}_0^*$ when $\Lambda(s)\pm\frac{3}{2}$, but when $s>M'$ the set on $\mathcal{D}_0^*\cap\{|w|=s\}$ on which $|g(w)-b_1|<|g(w)-b_0|$ has divided into two intervals, separated by an interval on which $|g(w)-b_0|<|g(w)-b_1|$.

In order to introduce b_2 we need $|g-b_2|<|g-b_1|$ in this middle interval. However, we cannot assume $b_0=b_2$.

To achieve adequate flexibility, slight non-analytic changes of variables will be made; this requires §§ 4.3–4.4. In particular, the functions H_j^*, $H_j^\#$ required in (3.28) will be defined in § 4.4. Next, the ψ_j, ψ_j^*, also needed in (3.28), are given in § 4.5. Modulo the proof of Theorem 4, the verification of Theorem 3 is performed in chapter 5. Finally Theorem 4 itself is proved in chapter 6.

4.3. A quasi-conformal homeomorphism. To facilitate computations, we state a Lemma to which appeal will frequently be made; the proof is immediate from the definitions (2.3), (2.6) and (2.7) (let $\xi=\log\varrho$, $\eta=\phi$).

LEMMA 7. *Let $G(\zeta)$ $(\zeta=\varrho e^{i\phi})$ be C^1 in a neighborhood N of $\zeta_0\neq 0$ with $G(\zeta_0)\neq 0$. Assume there are positive numbers $c>1$, $\eta<(50)^{-1}$, such that*

$$\left|\frac{\partial \log G(\zeta)}{\partial \log \varrho}-c\right|<2\eta \qquad (\zeta\in N), \tag{4.26}$$

$$\left|\frac{\partial \log G(\zeta)}{\partial \phi}-ic\right|<2\eta \qquad (\zeta\in N). \tag{4.27}$$

Then

$$|\mu_G(\zeta)|=|\mu_{\log G}(\zeta)|<3\eta \qquad (\zeta\in N). \tag{4.28}$$

LEMMA 8. *Given complex numbers γ, $\sigma(\sigma\neq 0)$ and $0<\eta<(50)^{-1}$, $M'\geq 1$, choose M so large that*

$$\eta\log(M/M')>4\max(|\gamma|,\log|\sigma|+\pi). \tag{4.29}$$

Then there exists a quasi-conformal homeomorphism $\omega(W)$ of the W-plane $(W=Se^{iT})$ with

$$\|\mu_\omega\|_\infty\leq 3\eta, \tag{4.30}$$

such that

$$\omega(W)=\gamma+\sigma W \qquad (|W|\leq M'), \tag{4.31}$$

$$\omega(W)=W \qquad (|W|\geq M). \tag{4.32}$$

Proof. Let $a(S)$, $b(S)$ be complex valued continuously differentiable functions with

$$a(S)=\log\sigma \qquad (0\leq S\leq M') \tag{4.33}$$

(here $|\mathfrak{J}(\log \sigma)| \leqslant \pi$),

$$a(S) = 0 \qquad (M \leqslant S), \tag{4.34}$$

$$|a'(S)| < \tfrac{1}{4}\eta S^{-1} \qquad (0 < S < \infty); \tag{4.35}$$

$$b(S) = 0 \qquad (0 \leqslant S \leqslant M'), \tag{4.36}$$

$$b(S) = \gamma \qquad (M \leqslant S), \tag{4.37}$$

$$|b'(S)| < \tfrac{1}{4}\eta \qquad (0 < S < \infty). \tag{4.38}$$

(That (4.35) and (4.38) are compatible with the other conditions follows from the choice of M in (4.29) and the inequality $M - M' > \log(M/M')$).

Then if ω is defined by

$$\omega(W) = \gamma + e^{a(S)}(W - b(S)) \qquad (S = |W|) \tag{4.39}$$

it is clear from (4.33), (4.34), (4.36) and (4.37) that (4.31) and (4.32) hold, and

$$|\mu_\omega(W)| = 0 \qquad (|W| \leqslant M',\ |W| \geqslant M). \tag{4.40}$$

If $M' < |W_0| < M$, we rewrite (4.39) as

$$\log(\omega(W) - \gamma) = a(S) + \log W + \log(1 - b(S)\, W^{-1}) \tag{4.41}$$

in a neighborhood of W_0. Thus (4.36) and (4.38) yield that

$$|b(S)\, W^{-1}| < \tfrac{1}{4}\eta \qquad (S > 0) \tag{4.42}$$

so Lemma 7 may be applied with $c = 1$. We obtain that $|\mu_\omega(W)| \leqslant 3\eta$ near W_0, and this with (4.40) gives (4.30).

That ω is a homeomorphism depends on the argument principle (that the argument principle applies to quasi-meromorphic functions is immediate from (2.10)). Indeed, ω is a local homeomorphism ([10], p. 250) and the explicit formula (4.32) shows that for fixed ω_0 and large S, the image of $\{|W| = S\}$ winds once about ω_0. Thus ω is a global homeomorphism and Lemma 8 is proved.

4.4. Functions H^*, $H^\#$. This class of functions is an important component of definition (3.28).

Construction of H^*. As starting point, we take $\eta>0$ and complex numbers γ, $\sigma(\sigma\neq 0)$ and a function $\omega(W)$ as in Lemma 8. Recall also that numbers $M'>1$ and $M>M'$ are assigned to ω as in (4.29). We use this ω to modify the boundary values of the fundamental auxilliary function $H(w)$ of Theorem 4.

Thus, choose M^∞, $\tau_0>0$ such that to any function $\Lambda(s)$ which satisfies (4.4)–(4.6), (4.10), (4.11) may be associated a function $H(w)$ in accord with (4.14), (4.16) (when (4.15) holds) and (4.17) (the more refined conclusions (4.18)–(4.25) are not required). Let M' and M be as in (4.29), (4.31), (4.32) and let M^* satisfy

$$M^* \geqslant \max(4 \log M,\ M^\infty). \tag{4.43}$$

Then let $\Lambda(s)$ satisfy the additional constraints

$$\Lambda(s) = 1 \qquad (s \leqslant M^*), \tag{4.44}$$

$$\Lambda(s) = 2 \qquad (s \geqslant S^*), \tag{4.45}$$

where S^* is sufficiently large to be compatible with (4.11) and (4.44).

It is easy to see that as t increases

$$-S(s)\cos\Lambda(s)(t-\pi) \qquad (\pi-\pi/\Lambda(s)\leqslant t\leqslant \pi-\pi/2\Lambda(s))$$

decreases from $S(s)$ to 0. According to (4.8), $S(s)>\log M$ when $s>\log M$, so (4.4) implies that there is a unique function $t=t_0(s)$ such that

$$\pi-\pi/\Lambda(s)\leqslant t_0(s)\leqslant \pi-\pi/2\Lambda(s) \qquad (s>\log M) \tag{4.46}$$

and

$$-S(s)\cos\Lambda(s)(t_0(s)-\pi) = [S(s)\log M]^{1/2} \qquad (s>\log M). \tag{4.47}$$

The definition of $t_0(s)$ in (4.46) and (4.47) is augmented by

$$t_0(s) = 0 \qquad (s\leqslant \log M); \tag{4.48}$$

according to (4.43) and (4.44), this means that t_0 is continuous for $s\geqslant 0$.

Next, let

$$l_0(s) = \pi-\pi/2\Lambda(s) \qquad (s>0) \tag{4.49}$$

and

$$\mathcal{D}^* = \{w;\ s>0,\ |t|\leqslant l_0(s)\}. \tag{4.50}$$

It follows from (4.4) that $\pi/2 \leqslant l_0(s) \leqslant 3\pi/4$,

$$0 \leqslant t_0(s) \leqslant l_0(s) \qquad (s > 0), \tag{4.51}$$

and from (4.8) and (4.47) that

$$t_0(s) \to l_0(s) \qquad (s \to \infty). \tag{4.52}$$

Define $H^*(w)$ in $\mathcal{D}^*$ by

$$H^*(w) = \begin{cases} \omega(H(w)) & s > 0,\ t_0(s) \leqslant t \leqslant l_0(s), \\ H(w) & s > \log M,\ |t| \leqslant t_0(s), \\ \omega(H(w)) & s > 0,\ -l_0(s) \leqslant t \leqslant -t_0(s). \end{cases} \tag{4.53}$$

LEMMA 9. *H^* is continuous and quasi-meromorphic in $\mathcal{D}^*$ with*

$$|\mu_{H^*}(w)| < 8\eta \qquad (w \in \mathcal{D}^*). \tag{4.54}$$

Further, if M', γ and σ are associated to ω as in (4.31), *then*

$$H^*(se^{il_0(s)}) = \gamma + \sigma e^{iS(s)} \qquad (s > 0), \tag{4.55}$$

$$H^*(se^{-il_0(s)}) = \gamma + \sigma e^{-iS(s)} \qquad (s > 0), \tag{4.56}$$

$$H^*(se^{it}) = \gamma + \sigma e^{S(s)e^{it}} \qquad (s \leqslant \log M',\ |t| \leqslant l_0(s)). \tag{4.57}$$

Finally,

$$n(s, a, H^*, \mathcal{D}^*) < AS(s) \qquad (s > 1,\ a \in \hat{C}) \tag{4.58}$$

for an absolute constant A.

Remark. Since (4.52) holds, we see from (4.53) that $H^* = H$ on most of $\mathcal{D}^*$; however the boundary values (4.55)–(4.57) have been modified by ω. This, together with Lemma 10 (cf. (4.81)–(4.84)) resolves the difficulty which we discussed in § 4.2.

Proof. It is clear from (4.53) that H^* is continuous in $\mathcal{D}^*$ save perhaps on the curves $se^{\pm it_0(s)}$ $(s > 0)$. When $s \leqslant \log M$ this continuity is evident from (4.16), (4.43), (4.44), (4.48) and (4.53) since $H^*(s) = e^{S(s)} = e^s = H^*(se^{2\pi i})$.

Now let $\log M \leqslant s$, $t_0(s) > 0$; it is necessary to investigate both curves $se^{\pm it_0(s)}$. First let $\log M \leqslant s \leqslant M^*$. Then (4.44) shows that $\Lambda(s) = 1$, and it follows from (4.8), (4.16) and (4.47) that

$$|H(se^{it_0(s)})| = \exp\{[S(s) \log M]^{1/2}\} \geqslant M; \tag{4.59}$$

thus property (4.32) of ω implies that the two determinations in (4.53) for $H^*(se^{it_0(s)})$

agree. When $s > M^*$, we obtain from (4.4), (4.8) and (4.43) that $\frac{1}{2}\{S(s)\}^{1/2} \geq \frac{1}{2}(M^*)^{1/2} \geq [\log M]^{1/2}$ and consequently

$$-S(s)\cos(7\pi/8) \geq \tfrac{1}{2}S(s) \geq [S(s)\log M]^{1/2}.$$

According to the defining property (4.47) of $t_0(s)$, this means that $t_0(s) \geq \pi/8$ and hence, from (4.14),

$$\log H(se^{it}) = -S(s)e^{i\Lambda(s)(t-\pi)} \qquad (t_0(s) \leq t \leq 2\pi - t_0(s)). \tag{4.60}$$

Once more, (4.59) holds and so (4.32) implies that H^* is continuous on the full curve $se^{it_0(s)}$. The analysis for $se^{-it_0(s)}$ is similar, using (4.60). However, to use (4.60) in (4.53), we must compute with $H(we^{2\pi i})$ to reconcile the branches of arg w.

The estimate of μ_{H^*} is an immediate consequence of (4.12), (4.30) and the inequality

$$|\mu_{f\circ g}(\zeta)| \leq 2|\mu_f(g(\zeta))| + 2|\mu_g(\zeta)|, \tag{4.61}$$

which holds when $\|\mu f\|_\infty < \frac{1}{2}$, $\|\mu_g\|_\infty \leq \frac{1}{2}$ (cf. [1], pp. 9, 10).

The proofs of (4.55)–(4.58) follow at once from (4.14), (4.16), (4.31), (4.43), (4.44), (4.47), (4.49) with (4.53). For example since $\Lambda(s) = 1$ ($s \leq \log M'$) we have that $|H(se^{it})| \leq M'$ ($s \leq \log M'$), so (4.57) is a restatement of (4.31) and (4.53). When computing (4.56), (4.14) is used with arg $w = 2\pi - l_0(s)$. In both (4.55) and (4.56), the bound $M' > 1$ is needed to apply (4.31) in (4.53). Finally, (4.58) follows from (4.53) and (4.17) since ω is a homeomorphism.

Construction of $H^\#$. Again we use a function $\omega(W)$ from Lemma 8 to modify one of the functions $H(w)$ produced by Theorem 4.

Choose $\Lambda^\#$ as in (4.21), $0 \leq \alpha \leq 1$ and

$$(0<) \qquad \eta^\# = \frac{\pi}{2}\left(1 - \frac{3}{2\Lambda^\#}\right) \qquad \left(\leq \frac{\pi}{8}\right). \tag{4.62}$$

According to Theorem 4, there exist $M^\#$, τ such that if

$$s|\Lambda'(s)| < \tau \qquad (s > 0), \tag{4.63}$$

$$\Lambda(s) = 2 \qquad (0 < s < M^\#) \tag{4.64}$$

and

$$\Lambda^\# \leq \Lambda(s) \leq 2 \qquad (s > 0), \tag{4.65}$$

then there exists a function H which satisfies Theorem 4; in particular (4.14) is to hold for $\eta = \eta^\#$. It is consistent with (4.63)–(4.65) to assume $M^\#$ so large that

$$M^{\#} \sin \frac{\pi}{2} (\Lambda^{\#} - \tfrac{3}{2}) > \log M \tag{4.66}$$

(where M has been associated with ω in (4.29)), and that (4.22) holds for sufficiently large $s^{\#} > M^{\#}$.

It follows from (4.21), (4.62) and (4.65) that

$$-2\pi \leqslant \Lambda(s)(\eta^{\#} - \pi) \leqslant -\frac{\pi}{2}\{3 + (\Lambda^{\#} - \tfrac{3}{2})\}. \tag{4.67}$$

Thus if $s > M^{\#}$ and

$$l(s) = \pi - 3\pi/2\Lambda(s), \tag{4.68}$$

it is easy to check that $\cos \Lambda(s)(t-\pi)$ decreases from $\cos \Lambda(s)(\eta^{\#}-\pi)$ to 0 as t increases from $\eta^{\#}$ to $l(s)$. Thus using (4.8), (4.66) and (4.67) we may construct a unique function $t(s)$ such that

$$\eta^{\#} < t(s) < l(s) \qquad (s > M^{\#}) \tag{4.69}$$

and

$$S(s) \cos \Lambda(s)(t(s) - \pi) = \{S(s) \log M\}^{1/2} \qquad (s > M^{\#}); \tag{4.70}$$

further (compare with (4.52))

$$t(s) \to l(s) \qquad (s \to \infty). \tag{4.71}$$

Then with

$$\mathcal{D}_{M^{\#}} = \{w;\, s \geqslant M^{\#},\ |t| \leqslant l(s)\}, \tag{4.72}$$

we define $H^{\#}$ on $D_{M^{\#}}$ by either of the formulas

$$H^{\#}(w) = \begin{cases} \omega\{(1/H)(w)\} & s > M^{\#},\ l(s) \geqslant t \geqslant t(s), \\ (1/H)(w) & s > M^{\#},\ -l(s) \leqslant t \leqslant t(s), \end{cases} \tag{4.73}$$

or

$$H^{\#}(w) = \begin{cases} (1/H)(w) & s > M^{\#},\ -t(s) \leqslant t \leqslant l(s), \\ \omega\{(1/H)(w)\} & s > M^{\#},\ -l(s) \leqslant t \leqslant -t(s). \end{cases} \tag{4.74}$$

LEMMA 10. *Given $0 < \eta < 50^{-1}$, $0 \leqslant \alpha \leqslant 1$ and sufficiently large $M^{\#}$, let $H^{\#}$ be defined in $\mathcal{D}_{M^{\#}}$ by (4.73) or (4.74). Then $H^{\#}$ is continuous and quasi-meromorphic in $\mathcal{D}_{M^{\#}}$ with*

$$|\mu_{H^{\#}}(w)| < 8\eta \qquad (w \in \mathcal{D}_{M^{\#}}), \tag{4.75}$$

$$\int_{-l(s)}^{l(s)} |\mu_{H^{\#}}(se^{it})|\, dt = o(1) \qquad (s \to \infty). \tag{4.76}$$

The value-distribution of the poles of H is governed by

$$n(s, \infty, H^{\#}, \mathcal{D}_{M^{\#}}) = \{\pi^{-1}|\sin \pi\Lambda^{\#}| + o(1)\}S(s) \qquad (s \to \infty), \tag{4.77}$$

$$\tilde{n}(s, \infty, H^{\#}, \mathcal{D}_{M^{\#}}) = \{\alpha + o(1)\}n(s, \infty, H^{\#}, \mathcal{D}_{M^{\#}}) \qquad (s \to \infty). \tag{4.78}$$

Further, if $a \in C$ *belongs to* $p(a)$ *of the punctured discs* $\{1 < |a| < \infty\}$, $\{|\sigma| < |a-\gamma| < \infty\}$ *(where* σ *and* γ *are associated with* ω *by Lemma* 8) *then*

$$n(s, a, H^{\#}, \mathcal{D}_{M^{\#}}) \sim \tilde{n}(s, a, H^{\#}, \mathcal{D}_{M^{\#}}) = \{(2\pi)^{-1} + o(1)\}p(a)S(s) \qquad (s \to \infty), \tag{4.79}$$

and the asymptotics in (4.78) *are uniform for a in each region* $\log|a| < A_1$.

For all $a \in \hat{C}$,

$$n(s, a, H^{\#}, \mathcal{D}_{M^{\#}}) < AS(s) \qquad (s > 1,\ a \in \hat{C}) \tag{4.80}$$

holds for an absolute constant A.

Finally, if $H^{\#}$ *is given by* (4.73) *we have*

$$H^{\#}(se^{it(s)}) = \gamma + \sigma e^{iS(s)} \qquad (s \geq M^{\#}), \tag{4.81}$$

$$H^{\#}(se^{-it(s)}) = e^{-iS(s)} \qquad (s \geq M^{\#}), \tag{4.82}$$

and if $H^{\#}$ *is given by* (4.74) *then*

$$H^{\#}(se^{it(s)}) = e^{iS(s)} \qquad (s \geq M^{\#}), \tag{4.83}$$

$$H^{\#}(se^{-it(s)}) = \gamma + \sigma e^{-iS(s)} \qquad (s \geq M^{\#}). \tag{4.84}$$

Proof. For simplicity, only the case that $H^{\#}$ is defined by (4.73) will be studied. Conclusions (4.75) and (4.80)–(4.82) follow by straightforward modification of the steps used to achieve (4.54)–(4.56) and (4.58) in Lemma 9.

It is easy to see that $H^{\#}$ is continuous. Indeed we may use (4.14) when $s > M^{\#}$ and $\eta^{\#} \leq t \leq 2\pi - \eta^{\#}$, where $\eta^{\#}$ is defined by (4.62). Since $t(s)$ satisfies (4.69), we have from (4.64) and (4.66) that $\{S(s) \log M\}^{1/2} \geq \log M (s \geq M^{\#})$. Thus $|(1/H)(se^{it(s)})| = \exp\{S(s)\log M\}^{1/2} \geq M (s \geq M^{\#})$; hence (4.32) guarantees that $\omega\{(1/H)(se^{it(s)})\} = (1/H)(se^{it(s)})$. This means $H^{\#}$ is continuous in $\mathcal{D}_{M^{\#}}$.

That (4.76) holds is a simple consequence of (4.12), (4.13), (4.30), (4.61) and (4.71), since then

$$\int_{-t(s)}^{t(s)} |\mu_{H^{\#}}(se^{it})|\,dt = \int_{-t(s)}^{t(s)} |\mu_H(se^{it})|\,dt = o(1) \qquad (s \to \infty)$$

and

$$\int_{t(s)}^{t(s)} |\mu_{H^{\#}}(se^{it})|\,dt \leq \int_{t(s)}^{t(s)} dt = o(1) \qquad (s \to \infty).$$

In addition, the poles of $H^{\#}$ arise from the zeros of H, so (4.77) and (4.78) follow from (4.19) and (4.20).

We turn to the proof of (4.79) and use the decomposition $\mathcal{D}_{M^{\#}} = \mathcal{D}_0 \cup \mathcal{D}_+ \cup \mathcal{D}_-$ where

$$\mathcal{D}_0 = \mathcal{D}_{M^{\#}} \cap \{|t| < \eta^{\#}\},$$

$$\mathcal{D}_+ = \mathcal{D}_{M^{\#}} \cap \{\eta^{\#} < t < l(s)\},$$

$$\mathcal{D}_- = \mathcal{D}_{M^{\#}} \cap \{-l(s) < t < -\eta^{\#}\}.$$

Property (4.23) of H with (4.62) and (4.73) shows that $H^{\#} \to \infty$ as $w \to \infty$ in $\bar{\mathcal{D}}_0$ and thus

$$\bar{n}(s, a, H^{\#}, \mathcal{D}_0) \leqslant n(s, a, H^{\#}, \mathcal{D}_0) = 0(1) \qquad (s > 0, \log|a| > -A_1), \tag{4.85}$$

where the $O(1)$ in (4.85) depends on A_1 and $H^{\#}$.

Next, consider the value-distribution of $H^{\#}$ in $\mathcal{D}_+$. It is easy to check from (4.14), (4.65) and (4.67) that the image of $\mathcal{D}_+ \cap \{|w| < s\}$ under $W = \log(1/H) = Se^{iT}$ is contained in

$$\Delta^*(s) = \{W; S(M^{\#}) \leqslant S \leqslant S(s), -2\pi \leqslant T \leqslant -\tfrac{3}{2}\pi\} \tag{4.86}$$

and contains

$$\Delta_*(s) = \left\{W; S(M^{\#}) \leqslant S \leqslant S(s), -\frac{\pi}{2}[3 + (\Lambda^{\#} - \tfrac{3}{2})] \leqslant T \leqslant -\tfrac{3}{2}\pi\right\}. \tag{4.87}$$

Then (4.86) and the argument principle yield that

$$n(s, a, 1/H, \mathcal{D}_+) \leqslant \frac{S(s)}{2\pi} + 1 \qquad (s > M^{\#}, 1 < |a| < \infty), \tag{4.88}$$

$$n(s, a, 1/H, \mathcal{D}_+) = 0 \qquad (s > M^{\#}, |a| < 1), \tag{4.89}$$

and the usual properties of the exponential function with (4.8) and (4.87) imply that to each $\varepsilon > 0$, $A_1 > 0$ corresponds $s(\varepsilon, A_1)$ with the property that

$$n(s, a, 1/H, \mathcal{D}_+) \geqslant (1 - \varepsilon) \frac{S(s)}{2\pi} \qquad (s > s(\varepsilon, A_1),\ 0 < \log|a| < A_1).$$

Thus, since $M' \geqslant 1$ in Lemma 8, we achieve from the properties (4.31), (4.32) of ω and (4.73), (4.88), (4.89) that

$$(1+\varepsilon)\frac{S(s)}{2\pi} \geqslant n(s, a, H^{\#}, \mathcal{D}_+) \geqslant (1-\varepsilon)\frac{S(s)}{2\pi} \qquad (s > s(\varepsilon, A_1), \log|\sigma| < \log|a - \gamma| < A_1), \tag{4.90}$$

$$n(s, a, H^{\#}, \mathcal{D}_+) = 0 \qquad (s > M^{\#}, \log|a - \gamma| < \log \sigma). \tag{4.91}$$

Similarly,

$$(1+\varepsilon)\frac{S(s)}{2\pi} \geqslant n(s, a, H^{\#}, \mathcal{D}_-) \geqslant (1-\varepsilon)\frac{S(s)}{2\pi} \qquad (s < s(\varepsilon, A_1),\ 0 < \log|a| < A_1), \tag{4.92}$$

$$n(s, a, H^{\#}, \mathcal{D}_-) = 0 \qquad (s > M,\ |a| \leqslant 1). \tag{4.93}$$

It is now clear from (4.85) and (4.90)–(4.93) that

$$n(s, a, H^{\#}, \mathcal{D}_{M^{\#}}) = \{(2\pi)^{-1}p(a) + o(1)\}S(s) \qquad (s \to \infty) \tag{4.94}$$

with asymptotics as claimed in the statement of Lemma 10. Moreover,

$$\bar{n}(s, a, H^{\#}, \mathcal{D}_+ \cup \mathcal{D}_-) = 0 \qquad (s > 0,\ a \in \hat{C}), \tag{4.95}$$

since, in $\mathcal{D}_+ \cup \mathcal{D}_-$, $H^{\#}$ is the composition of local homeomorphisms. Thus (4.85) and (4.94) show that $\bar{n}(s, a, H^{\#}, \mathcal{D}_{M^{\#}}) = n(s, a, H^{\#}, \mathcal{D}_{M^{\#}}) + O(1)$, with the $O(1)$ uniform in each region $|\log|a|| < A_1$. This with (4.95) completes the proof of (4.79) and Lemma 10.

4.5. Mappings ψ_j^*, ψ_j. We describe the remaining ingredients of (3.28). The need for the $\{\psi_j^*\}$, $\{\psi_j\}$ arises from the fact that the functions H^*, $H^{\#}$ of § 4.4 are defined in normalized regions $\mathcal{D}^*$ (in (4.50)) and $\mathcal{D}_{M^{\#}}$ (in (4.72)) of the w-plane. However an inspection of Figure 1 p. 98 shows that the annulus $\{\varrho_m < |\zeta| < \varrho_{m+1}\}$ contains $2(m+1)$ regions D_j, D_j^*, whose angular measure tends to zero as $m \to \infty$. Thus $z = \psi_j^*(\zeta)\ (\zeta \in D_j^*)$ or $z = \psi_j(\zeta)\ (\zeta \in D_j)$ is a quasi-conformal homeomorphism from a D_j^* to a $\mathcal{D}^*$ or from a D_j to a $\mathcal{D}_{M^{\#}}$ which is "locally" a power of ζ.

Definition of ψ_j^* $(j \geqslant 1)$. Assume ϱ_{j-1} and $\lambda(\varrho)$ for $\varrho \leqslant \varrho_{j-1}$ are determined and set

$$s_j^* = \log S(\varrho_{j-1}) = \int_1^{\varrho_{j-1}} \lambda(u)\, u^{-1} du \tag{4.96}$$

with S as in (2.45). We shall construct a function $\Lambda_j^*(s)$ of the nature considered in Theorem 4. In addition to (4.4), we require

$$\Lambda_j^*(s) = 1 \qquad (s \leqslant s_j^*) \tag{4.97}$$

with s_j^* defined in (4.96). We also assume s_j^* so large that if H_j is associated to $\Lambda_j^*(s)$ by Theorem 4, we may achieve

$$\|\mu_{H_j}\|_\infty < 2^{-8}\eta_{j-1} \tag{4.98}$$

by (4.97) and taking τ_0 sufficiently small in (4.11); in (4.11) and (4.98) η_j is determined by the normalization (3.15) and Theorem 2, p. 93.

The definition of $\Lambda_j^*(s)$ is balanced with that of $\lambda(\varrho)$ on $\varrho_{j-1} \leqslant \varrho \leqslant \varrho_j$ by an increasing function $s_j^*(\varrho)$ so that if

$$\psi_j^*(\zeta) = s_j^*(\varrho)\, e^{it_j^*(\zeta)} \tag{4.99}$$

(with t_j^* defined below in (4.105)) then

$$\int_1^{s_j^*(\varrho)} \Lambda_j^*(u)\, u^{-1} du = \int_1^{\varrho} \lambda(u)\, u^{-1} du \qquad (\varrho_{j-1} \leqslant \varrho \leqslant \varrho_j). \tag{4.100}$$

To achieve this, construct a continuous increasing function $L_j^*(\varrho)$ with

$$\begin{cases} L_j^*(\varrho) = 1 & (\varrho \leqslant \varrho_{j-1}) \\ (L_j^*)'(\varrho) = \dfrac{\lambda'(\varrho)}{\lambda(\varrho_j) - \lambda(\varrho_{j-1})} & (\varrho_{j-1} < \varrho < \varrho_j) \\ L_j^*(\varrho) = 2 & (\varrho \geqslant \varrho_j), \end{cases} \tag{4.101}$$

and then define an increasing function $s_j^*(\varrho)$ subject to

$$s_j^*(\varrho_{j-1}) = s_j^*, \quad \frac{d \log s_j^*(\varrho)}{d \log \varrho} = \frac{\lambda(\varrho)}{L_j^*(s)}. \tag{4.102}$$

Then if $\varrho(s)$ is inverse to $s_j^*(\varrho)$, we complement (4.97) by

$$\Lambda_j^*(s) = \begin{cases} L_j^*(\varrho(s)) & (s_j^* \leqslant s \leqslant s_j^*(\varrho_j)) \\ 2 & (s \geqslant s_j^*(\varrho_j)), \end{cases} \tag{4.103}$$

and verify that Λ_j^* is continuous, and differentiable for all s save perhaps s_j^* and $s^*(\varrho_j)$.

Note from (3.16)–(3.18) and (4.101)–(4.103) that

$$s(\Lambda_j^*)'(s) = \frac{\varrho\lambda'(\varrho)}{\lambda(\varrho_j) - \lambda(\varrho_{j-1})} \left\{ \frac{d \log \varrho}{d \log s} \right\} \qquad (s = s_j^*(\varrho),\ \varrho_{j-1} < \varrho < \varrho_j) \tag{4.104}$$

and that $\{d(\log \varrho)/d(\log s)\}$ is bounded above and below by positive constants; this means then any bounds of the nature (4.11) can be achieved by restricting τ_{j-1}' in (3.19). In addition (4.102) and (4.103) yield (4.100).

Now definition (4.99), with $s_j^*(\varrho)$ given in (4.102), is complemented by

$$t_j^*(\zeta) = \frac{\lambda(\varrho)}{\Lambda_j^*(s)} \phi \qquad (s = s_j^*(\varrho),\ |\phi| < \pi). \tag{4.105}$$

It is easy to check that ψ_j^* maps

$$D_j^* = \left\{ \zeta;\ \varrho_{j-1} \leqslant |\zeta| \leqslant \varrho_j,\ |\phi| \leqslant \frac{\pi}{2\lambda(\varrho)} (2\Lambda_j^*(s) - 1) \right\} \qquad (s = s_j^*(\varrho)) \tag{4.106}$$

topologically onto

$$\mathcal{D}_j^* = \left\{ w;\ s_j^*(\varrho_{j-1}) \leqslant s \leqslant s_j^*(\varrho_j),\ |t| \leqslant \frac{\pi}{2\Lambda_j^*(s)}(2\Lambda_j^*(s)-1) \right\} \qquad (s = s_j^*(\varrho)) \tag{4.107}$$

(we show (4.106) compatible with (3.75) on p. 118).

A straightforward computation using (3.19), (4.100) and (4.105) shows that

$$\frac{\partial \log s_j^*}{\partial \log \varrho} = \frac{\lambda(\varrho)}{\Lambda_j^*(s)} = \frac{\partial t_j^*}{\partial \phi} \qquad (s = s_j^*(\varrho)),$$

$$\frac{\partial \log s_j^*}{\partial \phi} = 0,$$

$$\left| \frac{\partial t_j^*}{\partial \log \varrho} \right| \leqslant \frac{|\phi|}{(\Lambda_j^*(s))^2} \left\{ \Lambda_j^*(s)\, \tau_{j-1}' + |\lambda(\varrho)| \frac{d\Lambda_j^*(s)}{d \log \varrho} \right\} \qquad (\zeta \in D_j^*).$$

Thus Lemma 7 and the discussion following (4.104) show that if τ_{j-1}' is sufficiently small we may assume

$$|\mu_{\varphi_j^*}(\zeta)| \leqslant 2^{-5} \eta_{j-1} \qquad (\zeta \in D_j^*), \tag{4.108}$$

where $\eta_0 = 2^{-4}$ in (3.15), and the η_j are determined in Theorem 2.

Definition of $\psi_j(-\infty < j < \infty)$. This parallels ideas already introduced.

First, let γ_j, σ_j ($\sigma_j \neq 0$) be given and $\omega_j(W)$ as in Lemma 8, subject to

$$\omega_j(W) = \omega_{-j}(W) \qquad (-\infty < j < \infty), \tag{4.109}$$

$$\|\mu_{\omega_j}\|_\infty \leqslant 2^{-(|j|+8)} \eta_{|j|}, \tag{4.110}$$

with the $\{\eta_j\}$ determined by Theorem 2 and (3.15), and let α_j be chosen in accord with (3.39). We then have from Lemma 10 that if

$$s\,|\Lambda_j'(s)| < \tau_j'' \qquad (s > 0) \tag{4.111}$$

for sufficiently small τ_j'' and $M_j^\#(=M_{-j}^\#)$ is sufficiently large (cf. (4.66)), we may construct $H_j^\#$ in accord with (4.73) if $j \geqslant 0$ and (4.74) if $j \leqslant -1$. Then $H_j^\#$ is quasi-meromorphic in

$$\mathcal{D}_{M_j^\#} = \{ s \geqslant M_j^\#,\ |t| \leqslant \pi - 3\pi/2\Lambda_j(s) \} = \mathcal{D}_{M_{-j}^\#} \tag{4.112}$$

with

$$\|\mu_{H_j^\#}\|_\infty < 2^{-(|j|+5)} \eta_{|j|}, \tag{4.113}$$

(cf. (4.75) and (4.110)).

We will construct an increasing function $s = s_j(\varrho)$ and $\Lambda_j(s)$ so that

$$s_j(\varrho) = s_{-j}(\varrho), \tag{4.114}$$

$$\Lambda_j(s) = \Lambda_{-j}(s) \qquad (s > 0), \tag{4.115}$$

and both

$$\int_1^{s_j(\varrho)} \Lambda_j(u)\, u^{-1} du = \int_1^{\varrho} \lambda(u)\, u^{-1} du, \tag{4.116}$$

and (4.10), with $M_j^{\#}$ in place of M^{∞}, hold.

In general, if $\varrho_{|j|}$ is so large that

$$S(\varrho_{|j|}) \geq (M_j{\#})^2 \tag{4.117}$$

((2.43) and (2.45) show this is certainly possible if $\varrho_{|j|} > (M_j{\#})^2$) we introduce functions $L_j(\varrho)$, $L_{-j}(\varrho)$ with

$$L_j(\varrho) = L_{-j}(\varrho) \qquad (-\infty < j < \infty), \tag{4.118}$$

by the formulas

$$\begin{cases} L_j(\varrho) = 2 & (\varrho \leq \varrho_{|j|}) \\ L_j'(\varrho) = \dfrac{\Lambda_j - 2}{\lambda(\varrho_{|j|+1}) - \lambda(\varrho_{|j|})}\lambda'(\varrho) & (\varrho_{|j|} < \varrho < \varrho_{|j|+1}) \\ L_j(\varrho) = \Lambda_j & (\varrho \geq \varrho_{|j|+1}), \end{cases} \tag{4.119}$$

where the Λ_j are as in (3.10)–(3.14).

Now let $s_j(\varrho)$ $(-\infty < |j| < \infty)$ satisfy (4.114),

$$s_j \equiv s_j(\varrho_{|j|}) = \{S(\varrho_{|j|})\}^{1/2} (> M_j{\#}), \tag{4.120}$$

$$\frac{d \log s_j(\varrho)}{d \log \varrho} = \frac{\lambda(\varrho)}{L_j(\varrho)} \qquad (\varrho_{|j|} \leq \varrho \leq \varrho_{|j|+1}). \tag{4.121}$$

Then if $\Lambda_j(s)$ is given by

$$\Lambda_j(s) = L_j(\varrho(s)), \tag{4.122}$$

($\varrho(s)$ the inverse function to $s_j(\varrho)$) it is easy to check that $\Lambda_j(s)$ satisfies (4.64), (4.65) (with Λ_j in place of $\Lambda^{\#}$) and (4.116). Note also that

$$\Lambda_j(s) = 2 \qquad (s < s_j(\varrho_{|j|})), \tag{4.123}$$

$$\Lambda_j(s) = \Lambda_j \qquad (s > s_j(\varrho_{|j|+1})). \tag{4.124}$$

We can now describe the D_k, D_k^* (cf. p. 97). Let

$$\beta_0(\varrho) = \pi + \frac{\pi}{2\lambda(\varrho)} \qquad (\varrho \geq \varrho_0 = 1), \tag{4.125}$$

$$\beta_j(\varrho)=\pi-\frac{\pi}{\lambda(\varrho)}\left\{-\tfrac{1}{2}+2\sum_{0}^{j-1}(\Lambda_k(s)-\tfrac{3}{2})\right\}\qquad(\varrho\geqslant\varrho_j,\ j\geqslant 1),\tag{4.126}$$

$$\beta_j(\varrho)-\alpha_j(\varrho)=\frac{2\pi}{\lambda(\varrho)}\{\Lambda_j(s)-\tfrac{3}{2}\}\qquad(s=s_j(\varrho),\ \varrho\geqslant\varrho_j,\ j\geqslant 0),\tag{4.127}$$

and define the $\alpha_{-j}(\varrho)$, $\beta_{-j}(\varrho)\,(j<0)$ by (3.26), (3.27). Note that if D_j, D_j^* are given by (3.21)–(3.25), then (3.20) holds, the D_j, D_j^* are disjoint, and $\bigcup\bar{D}_j\bigcup\bar{D}_j^*$ is the full ζ-plane.

We remark that the representations of D_j^* in (3.25) and (4.106) agree. For example, if $j\geqslant 1$ it is easy to obtain from (4.126), (4.127) that

$$\alpha_{j-1}(\varrho)=\beta_{j-1}(\varrho)-\frac{2\pi}{\lambda(\varrho)}(\Lambda_{j-1}(s)-\tfrac{3}{2})=\pi-\frac{\pi}{\lambda(\varrho)}\left\{-\tfrac{1}{2}+2\sum_{0}^{j-2}(\Lambda_k-\tfrac{3}{2})\right\}-\frac{2\pi}{\lambda(\varrho)}\{\Lambda_{j-1}(s)-\tfrac{3}{2}\}$$
$$(s=s_{j-1}(\varrho)).\tag{4.128}$$

We then see from (4.96), (4.97) and (4.123) that $\Lambda_{j-1}(s_{j-1}(\varrho_{j-1}))=2$, $\Lambda_j^*(s_j^*(\varrho_{j-1}))=1$ and so (4.128) and (3.18) yield

$$\alpha_{j-1}(\varrho)=\frac{\pi}{2\lambda(\varrho)}(2\Lambda_j^*(s)-1)\qquad(s=s_j^*(\varrho),\ \varrho_{j-1}\leqslant\varrho\leqslant\varrho_j)\tag{4.129}$$

when $\varrho=\varrho_{j-1}$. Also both sides of (4.129) have the same derivative with respect to ϱ for $\varrho_{j-1}\leqslant\varrho\leqslant\varrho_j$ (consider the derivative of $2\pi^{-1}\lambda(\varrho)\alpha_{j-1}(\varrho)$ with respect to $\log\varrho$; according to (4.128), (4.119), (4.121), (4.122) and (3.18) this is $2(\lambda(\varrho_j)-\lambda(\varrho_{j-1}))^{-1}\varrho\lambda'(\varrho)$, so (4.129) follows from (4.104)). Hence (3.25) agrees with (4.106).

It is easy to check using (4.127) that the function

$$\psi_j(\zeta)=s_j(\varrho)\,e^{it_j(\zeta)},\tag{4.130}$$

where $s_j(\varrho)$ is determined in (4.120), (4.121) and

$$t_j(\zeta)=\frac{\lambda(\varrho)}{\Lambda_j(s)}\{\phi-\alpha_j(\varrho)\}-(\pi-3\pi/2\Lambda_j(s))\qquad(s=s_j(\varrho)),\tag{4.131}$$

maps D_j (cf. (3.24)) topologically onto

$$\mathcal{D}_j=\{s\geqslant s_j,\ |t|\leqslant\pi-3\pi/2\Lambda_j(s)\}=\mathcal{D}_{-j};\tag{4.132}$$

note from (4.120) and (4.117) that $\mathcal{D}_j$ is a subset of $\mathcal{D}_{M_j^\#}$ (cf. (4.112)). It is easy to compute from Lemma 7, (4.130), (4.131), (4.121), and (4.116) that if τ_j' in (3.19) and τ_j'' in (4.111) are sufficiently small we may arrange

$$|\mu_{\psi_j}(\zeta)| \leqslant 2^{-(|j|+5)}\eta_{|j|} \qquad (\zeta \in D_j); \tag{4.133}$$

$$|\mu_{\psi_j}(\zeta)| \to 0 \qquad (\zeta \to \infty,\ \zeta \in D_j). \tag{4.134}$$

Finally, as in the analysis of (4.104), we note that such restrictions on τ_j'' are guaranteed by sufficiently restricting τ_j' in (3.19).

5. Proof of Theorem 3

Recall that Theorem 3 is stated in § 3.1. We continue to assume Theorem 4 (§ 4.1); Theorem 4 is proved in chapter 6.

5.1. Sequences $\{\gamma_j\}$, $\{\sigma_j\}$. We now determine the $\{\gamma_j\}, \sigma_j\}$ used to construct $\omega_j(W)$ in (4.109), (4.110). Let the $\{b_j\}$ be the fundamental sequence associated in (3.3)–(3.9) to the given data $\{a_j\}(1 \leqslant i < N)$, $\{\delta_i\}$, $\{\theta_i\}$ which appear in the statement of Theorem 1, and let the Möbius transformations $\{T_j\}$ be as in (3.29). Let

$$\gamma_0 = 0,\ \sigma_0 = 1 \tag{5.1}$$

and for $|j| \geqslant 1$ determine γ_j, σ_j by

$$T_j^{-1} \circ T_{|j|-1}(W) = \gamma_j + \sigma_j W^{-1} \qquad (\sigma_j \neq 0); \tag{5.2}$$

that this is possible depends on (3.4) and the assumption that $\infty \notin \{b_j\}$. Thus $T_j^{-1} \circ T_{|j|-1}(\infty) = T_j^{-1}(b_{|j|-1}) \neq \infty$ (for $T_j^{-1}(b_j) = \infty$ and $b_j \neq b_{|j|-1}$). Consequently there are (finite) complex numbers γ_j, σ_j, p_j with $\sigma_j \neq 0$ such that

$$T_j^{-1} \circ T_{|j|-1}(W) = \gamma_j + \frac{\sigma_j}{W - p_j}.$$

However, $p_j = (T_j^{-1} \circ T_{|j|-1})^{-1}(\infty) = T_{|j|-1}^{-1} \circ T_j(\infty) = T_{|j|-1}^{-1}(b_j) = 0$, and this yields (5.2).

5.2. Determination of the $\{\varrho_m\}$ and $\lambda(\varrho)$. In (3.15) we set the *a priori* bound $\|\mu_g\|_\infty < 2^{-4}$. which, according to § 2.3 (cf. Theorem 2) induced constants M, r_0, $\{A_m\}$, $\{\eta_m\}$. These constants and the need to ensure (1.4) (cf. (2.59)) already yield lower bounds for the numbers $\{\varrho_m\}$ and $\{\varrho_m/\varrho_{m-1}\}$ (cf. (2.46), (2.50), (2.51)). Of course, any restrictions on τ_m' in (3.19) also increase the ratios ϱ_{m+1}/ϱ_m, as in clear from (3.16)–(3.18).

Note from (3.16) that $\lambda(\varrho)$ has been defined for $\varrho \leqslant \varrho_0 = 1$, and that in (3.28) g is defined for $\{|\zeta| \leqslant \varrho_0\}$.

In general, suppose $\lambda(\varrho)$ has been defined appropriately for $\varrho \leqslant \varrho_m$. Explicitly, this means we have selected functions $\omega_0(W), \omega_{\pm 1}(W) \dots \omega_{\pm m}(W)$ as in Lemma 8 with data

$\sigma=\sigma_j$, $\gamma=\gamma_j$ from (5.2) so that (4.109), (4.110) hold for $|j|\leqslant m$, and have associated $1\leqslant M'_j\leqslant M_j$ to ω_j, $\omega_{-j}(|j|\leqslant m)$ as in (4.29). Then if Λ_j is as in (3.10)–(3.14) and $M^{\#}_j$ satisfies

$$M^{\#}_j \sin\frac{\pi}{2}(\Lambda_j-\tfrac{3}{2})>\log M_j \qquad (|j|\leqslant m) \tag{5.3}$$

(cf. (4.66)), we require ϱ_m be so large that

$$S(\varrho_m)\geqslant (M^{\#}_m)^2. \tag{5.4}$$

Note that (5.4) may be achieved when $m=0$ since, according to (5.1), we may take $\omega_0(W)=W$ and $M^{\#}_0=\varrho_0=1$.

We now determine τ'_m in (3.18); then $\lambda(\varrho)$ is defined arbitrarily for $\varrho_m\leqslant\varrho\leqslant\varrho_{m+1}$ to be compatible with (3.18) and (3.19) and (5.5), (5.8)–(5.13) below. All the definitions below are in turn based on $\lambda(\varrho)$ for $\varrho_m\leqslant\varrho\leqslant\varrho_{m+1}$.

Since (5.4) holds, we are in the situation (4.117), and so may define $\psi_m(\zeta)$, $\psi_{-m}(\zeta)$ according to (4.130), (4.131) and (4.116). Using the choices of $M^{\#}_m$, $\omega_m(W)$, and $\Lambda_m(s)$ we construct a function $H^{\#}_m$ of the class (4.73) when $m\geqslant 0$ and (4.74) otherwise. Note that if τ'_m in (3.19) is sufficiently small (cf. discussion of (4.104) and (4.108), (4.133)) we have from (3.28), (4.61), (4.113) and (4.133) that

$$|\mu_{g_{\pm m}}(\zeta)=|\mu_{H^{\#}_{\pm m}}\circ\psi_{\pm m}(\zeta)|\leqslant 2|\mu_{H^{\#}_{\pm m}}(\psi_{\pm m}(\zeta))|+2|\mu_{\psi_{\pm m}}(\zeta)|\leqslant 2^{-3}\eta_m \qquad (\zeta\in D_{\pm m}). \tag{5.5}$$

Thus g_m and g_{-m} have been introduced for $\{|\zeta|\geqslant\varrho_m\}$, and it remains to describe g^*_{m+1}. With σ_{m+1} and γ_{m+1} as determined by (5.2), let $M'_{m+1}(=M'_{-(m+1)})$ satisfy

$$\log M'_{m+1}=S(\varrho_m) \tag{5.6}$$

and choose $M_{m+1}(=M_{-(m+1)})$ with

$$M_{m+1}=M_{-(m+1)}>e^{M'_{m+1}} \tag{5.7}$$

so large that $\omega_{m+1}(W)(=\omega_{-(m+1)}(W))$ may be introduced from Lemma 8 with data σ_{m+1}, γ_{m+1} so that (4.109) and (4.110) hold for $j=m+1$. Now that (5.6) implies that (4.96) holds with $j=m+1$, we construct $\Lambda^*_{m+1}(s)$ in accord with (4.100) and $H_{m+1}(w)$ as in Theorem 4 and (4.98). With the data $\omega_{m+1}(W)$, $H_{m+1}(w)$, let $H^*_{m+1}(w)$ be obtained according to (4.53), and next determine $\psi^*_{m+1}(\zeta)$ as in (4.99), (4.100), (4.105) and (4.108) in terms of $\Lambda^*_{m+1}(s)$. Then the estimates (4.54), (4.98), (4.108) and (4.110) with (3.28) and (4.61) yield

$$\begin{aligned}|\mu_{g^*_{m+1}}(\zeta)|&=|\mu_{H^*_{m+1}}\circ\psi^*_{m+1}(\zeta)|\\&\leqslant 2|\mu_{H^*_{m+1}}(\psi^*_{m+1}(\zeta))|+2|\mu_{\psi^*_{m+1}}(\zeta)|\leqslant 2^{-4}\eta_m+2^{-4}\eta_m\leqslant 2^{-3}\eta_m \qquad (\zeta\in D^*_m);\end{aligned} \tag{5.8}$$

again, these estimates can be assured if τ'_m is small enough.

In order to achieve (5.5) and (5.8), we have given lower bounds on τ'_m or, what is the same, lower bounds on ϱ_{m+1}/ϱ_m. If necessary ϱ_{m+1} is increased so that in addition

$$(2\pi)^{-1}\int_{D_j\cap\{|\zeta|=\varrho\}}|\mu_{g_j}(\varrho e^{i\phi})|\,d\phi\leqslant 2^{-(|j|+3)}\eta_{m+1}\qquad(\varrho\geqslant\varrho_{m+1},\ |j|\leqslant m),\tag{5.9}$$

$$|n(s_j(\varrho),\infty,H_j^\#,\mathcal{D}_{M_j^\#})-\pi^{-1}|\sin\pi\Lambda_j|S(\varrho)|\leqslant(m+1)^{-1}S(\varrho)\qquad(|j|\leqslant m,\ \varrho\geqslant\varrho_{m+1}),\tag{5.10}$$

$$|\bar{n}(s_j(\varrho),\infty,H_j^\#,\mathcal{D}_{M_j^\#})-\alpha_j n(s_j(\varrho),\infty,H_j^\#,\mathcal{D}_{M_j})|\leqslant(m+1)^{-1}S(\varrho)\qquad(|j|\leqslant m,\ \varrho\geqslant\varrho_{m+1})\tag{5.11}$$

and

$$|n(s_j(\varrho),a,H_j^\#,\mathcal{D}_{M_j^\#})-(2\pi)^{-1}p_j(a)\,S(\varrho)|\leqslant(m+1)^{-1}S(\varrho)$$
$$(\log|a|\leqslant(m+1),\ |j|\leqslant m,\ \varrho\geqslant\varrho_{m+1});\tag{5.12}$$

recall that α_j is defined in (3.39) and, from (3.29) and (5.2), that $p_j(a)$ of (3.35) is the number of punctured discs $\{1<|a|<\infty\}$ and $\{|\sigma_j|<|a-\gamma_j|<\infty\}$ to which a belongs. This is all possible from the corresponding statements in Lemma 10.

Finally, we introduce one more constraint on ϱ_{m+1}. Recall from (4.103) that $\Lambda^*_{m+1}(s^*_{m+1}(\varrho_{m+1}))=2$. Then ϱ_{m+1} is taken so large that $l_0(s^*_m(\varrho_m))-t_0(s^*_m(\varrho_m))<\pi/8$, where t_0 and l_0 are defined in (4.46), (4.47) and (4.49) with $\Lambda=\Lambda^*_{m+1}(s)$; such ϱ_{m+1} exist according to (4.52). When this holds, we see from (4.53), (4.16), (4.100) and (2.45) that

$$\log H^*_{m+1}(s^*_{m+1}(\varrho_{m+1})\,e^{it})=-S(\varrho_{m+1})\,e^{2it}\qquad(|t|\leqslant\pi/2).\tag{5.13}$$

5.3. Continuity of g. We have seen in §5.2 how to construct $\lambda(\varrho)$ and the $\{\varrho_m\}$ so that the programme suggested by (3.28) may be carried out. We now begin the proof of Theorem 3.

It is obvious that the g_j, g_j^* are continuous at each interior $\zeta_0\in D_j$, D_j^*, but it is more troublesome to check continuity at points ζ_0 common to more than one of these regions. There are eight cases to consider:

$$\zeta_0\in D_j\cap D_{j-1}\qquad(-\infty<j<\infty),\tag{5.14}$$

$$\zeta_0\in D_j^*\cap D_{j-1}\qquad(j\geqslant 1),\tag{5.15}$$

$$\zeta_0\in D_j^*\cap D_{-(j-1)}\qquad(j\geqslant 2),\tag{5.16}$$

$$\zeta_0\in D_j\cap D_j^*\qquad(j\geqslant 1),\tag{5.17}$$

$$\zeta_0\in D_{-j}\cap D_j^*\qquad(j\geqslant 1),\tag{5.18}$$

$$\zeta_0\in D_{j-1}^*\cap D_j^*\qquad(j\geqslant 1),\tag{5.19}$$

$$\zeta_0 \in D_j^* \cap \{|\zeta| = \varrho_0 = 1\}, \tag{5.20}$$

$$\zeta_0 \in D_0 \cap \{|\zeta| = \varrho_0 = 1\}. \tag{5.21}$$

The techniques needed to verify continuity in these cases will be apparent from studying (5.14), (5.17) and (5.19); the remaining situations are left to the reader.

In analysing (5.14), suppose for concreteness that $j \geq 1$. According to (4.123), $\Lambda_j(s) = 2$ for $s \leq s_j(\varrho_j)$; thus (2.45), (4.120) and (5.4) imply that $s_j(\varrho) \geq S(\varrho_j)^{1/2} \geq M_j^\#(\varrho \geq \varrho_j)$. Further, it is easy to see from (4.127), (4.130) and (4.131) that $\psi_j(\varrho e^{i\beta_j(\varrho)}) = se^{il(s)}(s = s_j(\varrho))$, where $l(s)$ is defined in (4.68) with $\Lambda(s) = \Lambda_j(s)$. We thus obtain from (3.28) and (4.81) that

$$g(\varrho e^{i\beta_j(\varrho)}) = T_j \circ H_j^\#(s e^{it_j(\varrho e^{i\beta_j(\varrho)})}) = T_j \circ H_j^\#(se^{il(s)}) = T_j\{\gamma_j + \sigma_j e^{iS(s)}\} \qquad (s = s_j(\varrho),\ \varrho \geq \varrho_j),$$

where $S(s) = \exp \int_1^s \Lambda_j(u) u^{-1} du$. An application of (4.116) and (2.45) shows that $S(s) = S(\varrho)$, and thus

$$g(\varrho e^{i\beta_j(\varrho)}) = T_j\{\gamma_j + \sigma_j \exp iS(\varrho)\} \qquad (\varrho \geq \varrho_j). \tag{5.22}$$

Next, let $s = s_{j-1}(\varrho)$; then it is easy to see from (4.131) and (4.68) that

$$t_{j-1}(\varrho e^{i\alpha_{j-1}(\varrho)}) = s e^{-il(s)} \qquad (s = s_{j-1}(\varrho),\ \varrho \geq \varrho_{j-1}),$$

and hence (2.45), (4.82), (4.116) and (4.130) show that

$$H_{j-1}^\#(\psi_{j-1}(\varrho e^{i\alpha_{j-1}(\varrho)})) = \exp - iS(\varrho) \qquad (\varrho \geq \varrho_{j-1}). \tag{5.23}$$

We apply the defining property (5.2) of γ_j, σ_j and see from (3.28), (5.22) and (5.23) that g is continuous at points ζ_0 which satisfy (5.14).

Suppose next that ζ_0 satisfies (5.17). We readily see using (3.18), (3.24), (4.123), (4.126) and (4.127) that

$$D_j \cap \{|\zeta| = \varrho_j\} = \left\{\varrho_j e^{i\phi};\ \frac{\pi}{2\lambda(\varrho_j)} \leq \phi \leq \frac{3\pi}{2\lambda(\varrho_j)}\right\},$$

and (4.123) and (4.131) yield that

$$t_j(\varrho_j e^{i\phi}) = \tfrac{1}{2}[\lambda(\varrho_j)\phi - \pi] \qquad \left(\frac{\pi}{2\lambda(\varrho_j)} \leq \phi \leq \frac{3\pi}{2\lambda(\varrho_j)}\right);$$

thus substitution in (3.28) shows

$$g_j(\varrho_j e^{i\phi}) = T_j \circ H_j^\#(s_j(\varrho_j) e^{(1/2)i[\lambda(\varrho_j)\phi - \pi]}) \qquad \left(\frac{\pi}{2\lambda(\varrho_j)} \leq \phi \leq \frac{3\pi}{2\lambda(\varrho_j)}\right). \tag{5.24}$$

On the other hand, (4.103) asserts that $\Lambda_j^*(s_j^*(\varrho_j))=2$, so (3.28), (4.99), (4.105) and (4.106) yield

$$g_j^*(\varrho_j e^{i\phi}) = T_j \circ H_j^*(s_j^*(\varrho_j)\, e^{it_j^*(\varrho e^{i\phi})}) = T_j \circ H_j^*(s_j^*(\varrho_j)\, e^{(1/2)i\lambda(\varrho_j)\phi}) \qquad \left(|\phi| < \frac{3\pi}{2\lambda(\varrho_j)}\right). \tag{5.25}$$

In order to reconcile (5.24) with (5.25), we consider the definitions (4.53) and (4.73) of the H_j^*, $H_j^\#$. The same $\omega = \omega_j(W)$ is used in these definitions, but we have $H(w) = H_{(j)}(w)$ in (4.53) and $H(w) = H_j(w)$ in (4.73), since these are different functions. Thus it must be shown that

$$(1/H_j)(s_j(\varrho_j)\, e^{(1/2)i(t-\pi)}) = H_{(j)}(s_j^*(\varrho_j)\, e^{(1/2)it}) \qquad \left(\frac{\pi}{2} \leqslant t \leqslant \frac{3\pi}{2}\right). \tag{5.26}$$

According to (4.123), $\Lambda_j(s_j(\varrho_j))=2$, so we may apply (4.16) to all t. A now-standard computation with (4.16) yields

$$(1/H_j)(s_j(\varrho_j)\, e^{(1/2)i(t-\pi)}) = \exp\{S(\varrho_j)\, e^{2i[(1/2)(t-\pi)-\pi]}\} = \exp\{-S(\varrho_j)\, e^{it}\} \qquad \left(\frac{\pi}{2} \leqslant t \leqslant \frac{3\pi}{2}\right).$$

Since $\Lambda_j^*(s_j^*(\varrho_j))=2$, (4.16) may also be applied to $H_{(j)}$ and we obtain in a similar manner that

$$H_{(j)}(s_j^*(\varrho_j)\, e^{(1/2)it}) = \exp\{-S(\varrho_j)\, e^{2i((1/2)t-\pi)}\} = \exp\{-S(\varrho_j)\, e^{it}\} \qquad \left(|t| < \frac{3\pi}{2}\right); \tag{5.27}$$

thus (5.26) is proved.

Finally we study the possibility (5.19). Then $D_j^* \cap D_{j-1}^* = \{\zeta;\, \varrho = \varrho_j,\ |\phi| \leqslant \pi/(2\lambda(\varrho_j))\}$ and $\Lambda_j^*(s_j^*(\varrho_j))=2$, $\Lambda_{j+1}^*(s_{j+1}^*(\varrho_j))=1$. In particular, (3.28), (4.105), (5.13) and the steps used to obtain (5.27) imply that

$$g_j^*(\varrho_j e^{i\phi}) = T_j(\exp[-S(\varrho_j)\, e^{i\lambda(\varrho_j)\phi}]) \qquad \left(|\phi| \leqslant \frac{\pi}{2\lambda(\varrho_j)}\right). \tag{5.28}$$

Also, (4.31) and assumption (5.6) imply that $\omega_{j+1}(W) = \gamma_{j+1} + \sigma_{j+1} W$ if $|W| \leqslant \exp S(\varrho_j)$, and this and (4.57) yield

$$H_{j+1}^*(s_{j+1}^*(\varrho_j)\, e^{it}) = \omega_{j+1}(H_{(j+1)}(s_{j+1}^*(\varrho_j)\, e^{it})) \qquad \left(|t| < \frac{\pi}{2}\right).$$

Since $\Lambda_{j+1}^*(s_{j+1}^*(\varrho_j))=1$, we see from (3.28), (4.99), (4.109) and (4.105) that

$$\begin{aligned} g_{j+1}^*(\varrho_j e^{i\phi}) &= T_{j+1} \circ \omega_{j+1}\{\exp[S(\varrho_j)\, e^{i\lambda(\varrho_j)\phi}]\} \\ &= T_{j+1}\{\gamma_{j+1} + \sigma_{j+1} \exp[S(\varrho_j)\, e^{i\lambda(\varrho_j)\phi}]\} \qquad \left(|\phi| \leqslant \frac{\pi}{2\lambda(\varrho_j)}\right); \end{aligned}$$

a final appeal to (5.2) shows this expression agrees with (5.28).

5.4. Completion of proof. The remaining properties of Theorem 3 are less obnoxious to verify. Note, from (3.18), that $\lambda(\varrho)$ already satisfies (2.52). Thus to check that g is Nevanlinna admissible, it must be checked that the dilatation of g is so small that (2.53) holes, where $D(\varrho)$ is defined in (2.27). According to (3.28), $\mu_g \equiv 0$ for $|\zeta| \leqslant \varrho_0 = 1$. In general, we see from (3.28), (5.5), (5.8) and (5.9) that

$$\int_0^{2\pi} |\mu_g(\varrho e^{i\phi})| d\phi$$

$$= \int |\mu^*_{g_{m+1}}(\varrho e^{i\phi})| d\phi + \left\{ \int |\mu_{g_m}(\varrho e^{i\phi})| d\phi + \int |\mu_{g_{-m}}(\varrho e^{i\phi})| d\phi \right\} + \sum_{|j| \leqslant m-1} \int |\mu_{g_j}(\varrho e^{i\phi})| d\phi$$

$$\leqslant 2\pi \left\{ 2^{-3} + 2^{-2} + 2^{-3} \sum_{-(m-1)}^{m-1} 2^{-|j|} \right\} \eta_m < 2\pi\eta_m \qquad (\varrho_m \leqslant \varrho \leqslant \varrho_{m+1}),$$

which is (2.53).

Also, since the $\{\varrho_m\}$ are chosen in accord with (2.59) we have (1.4).

Next, since the ψ_j, ψ_j^* and T_j are homeomorphisms, we readily obtain (3.33) and (3.34) from (3.28) and the corresponding properties (4.80) and (4.58) of the $H_j^\#$, H_j^*.

Similar reasoning yields (3.37) from (4.79) and (5.12), (3.40) from (5.10) and (4.77) and (3.41) from (5.11) and (4.78).

6. Proof of Theorem 4

Theorem 4 is stated in § 4.1.

6.1. A class of functions of genus one. Our goal in this section is to construct an entire function $F(z)$ for which the conclusions of Theorem 4 almost hold, and then to use quasiconformal methods to satisfy (4.14) and (4.16) exactly. This function $F(z)$ is a slight generalization of the Lindelöf functions.

LEMMA 11. *Let $h>0$ and $0<h_1<(10)^{-1}$ be given. Then there exist $0<\tau$, $0<\nu<\frac{1}{2}$ and $K>2^{10}$ in accord with the following assertions. Let $\Lambda(r)$ be a differentiable function with*

$$1+3h \leqslant \Lambda(r) \leqslant 2-3h \qquad (r>0, \tag{6.1}$$

$$r|\Lambda'(r)| < (4\pi)^{-1} \sin 3\pi h \qquad (r>0), \tag{6.2}$$

let $S(r)$ be defined according to (4.7), let

$$n^*(r) = \pi^{-1} |\sin \pi\Lambda(r)| S(r) \qquad (r>0) \tag{6.3}$$

and let $F(z)$ be a canonical product with positive zeros whose zero-counting function $n(r) = n(r, 0, F)$ is bounded by

$$n(r) < 2n^*(r) \qquad (r>0). \tag{6.4}$$

Suppose for some $r_0 > 10\nu^{-2}h_1^{-5}$ *we also have*

$$r|\Lambda'(r)| < \tau \qquad (r > r_0), \tag{6.5}$$

$$|n(r) - n^*(r)| < \nu^2 n^*(r) \qquad (r > r_0). \tag{6.6}$$

Then in the plane slit along the positive axis, that branch of $\log F(z)$ *having* $\log F(0) = 0$ *satisfies*

$$|\log F(z) + S(r)e^{i\Lambda(r)(\theta-\pi)}| < h_1^2 S(r) \qquad (r > Kr_0,\ h_1 < \theta < 2\pi - h_1). \tag{6.7}$$

Remarks. It will be seen in § 6.4 that such an $n(r)$ may be constructed.

We require (6.1) in place of (4.4) until § 6.6.

Proof. Take $K > 2^{10}$ so that

$$\int_0^{K^{-1}} y^{-1+h}dy < \frac{1}{15}h_1^2,\ \int_K^{\infty} y^{-1-h}dy < \frac{1}{15}h_1^2, \tag{6.8}$$

$$2^{24}h^{-1}K^{-h} \leqslant 1, \tag{6.9}$$

and then find $\sigma = \sigma(K, h_1) < \frac{1}{2}$ with

$$\sigma\int_{K^{-1}}^{K} y^{-p}|y - e^{i\theta}|^{-1}dy < \frac{1}{6}h_1^2 \qquad (-2 \leqslant p \leqslant 2,\ h_1 \leqslant \theta \leqslant 2\pi - h_1). \tag{6.10}$$

We now estimate the oscillation of $\Lambda(r)$ and $n^*(r)$. It follows with little effort from (6.5) and (4.7) that

$$|\Lambda(u) - \Lambda(r)| \leqslant \tau\left|\int_u^r y^{-1}dy\right| \leqslant \tau\log K \qquad (r_0 < K^{-1}r < u < Kr) \tag{6.11}$$

and

$$\left(\frac{r}{u}\right)^{\Lambda(r)}\frac{S(u)}{S(r)} - 1 = \exp\left\{\int_r^u [\Lambda(y) - \Lambda(r)]\,y^{-1}dy\right\} - 1;$$

thus

$$\left|\left(\frac{r}{u}\right)^{\Lambda(r)}\frac{S(u)}{S(r)} - 1\right| \leqslant e^{\tau(\log K)^2} - 1 \qquad (r_0 < K^{-1}r < u < Kr). \tag{6.12}$$

Now the definition (4.7) of $S(u)$ with (6.1)–(6.3) shows that

$$\left|\frac{d\log n^*(u)}{d\log u} - \Lambda(u)\right| = \left|\frac{d\log|\sin\pi\Lambda(u)|}{d\log u}\right| \leqslant \pi u\Lambda'(u)\cot\pi h\ (<\tfrac{1}{2}) \qquad (u > 0) \tag{6.13}$$

which implies that n^* is an increasing function. We then obtain from (6.5), (6.12) and (6.13) that

$$\left|\frac{d\log n^*(u)}{d\log u} - \Lambda(r)\right| \leqslant \left|\frac{d\log n^*(u)}{d\log u} - \Lambda(u)\right| + |\Lambda(u) - \Lambda(r)|$$

$$\leqslant \tau(\pi\cot 3\pi h + \log K) \qquad (r_0 < K^{-1}r < u < Kr).$$

Thus, given $\nu > 0$, τ is chosen in (6.5) sufficiently small to ensure that

$$\left|\left(\frac{r}{u}\right)^{\Lambda(r)}\frac{n^*(u)}{n^*(r)}-1\right| = \left|\exp\left\{\int_r^u\left[\frac{d\log n^*(y)}{d\log y}-\Lambda(r)\right]\frac{dy}{y}\right\}-1\right|$$
$$\leqslant |\exp\{\tau\log K(\pi\cot 3\pi h+\log K)\}-1| < \nu^4 \qquad (r_0 < K^{-1}r < u < Kr). \tag{6.14}$$

With (6.14) in mind, we take ν and then τ in (6.6) and (6.5) so that (6.14),

$$\left|n(u)-\left(\frac{u}{r}\right)^{\Lambda(r)}n^*(r)\right| \leqslant |n(u)-n^*(u)| + \left|n^*(u)-\left(\frac{u}{r}\right)^{\Lambda(r)}n^*(r)\right|$$
$$\leqslant \nu^2 n^*(u) + \nu^4\left(\frac{u}{r}\right)^{\Lambda(r)}n^*(r) \leqslant (2+\nu^2)\nu^2\left(\frac{u}{r}\right)^{\Lambda(r)}n^*(r)$$
$$\leqslant \sigma\left(\frac{u}{r}\right)^{\Lambda(r)}n^*(r) \qquad (r > Kr_0,\ K^{-1}r < u < Kr) \tag{6.15}$$

and

$$\left|n(u)-\left(\frac{u}{r}\right)^{\Lambda(r)}n(r)\right| \leqslant \left|n(u)-\left(\frac{u}{r}\right)^{\Lambda(r)}n^*(r)\right| + \left(\frac{u}{r}\right)^{\Lambda(r)}|n^*(r)-n(r)|$$
$$\leqslant 2\nu^2\left(\frac{u}{r}\right)^{\Lambda(r)}n^*(r) \leqslant \sigma\left(\frac{u}{r}\right)^{\Lambda(r)}n^*(r) \qquad (r > Kr_0),\ K^{-1}r < u < Kr) \tag{6.16}$$

hold, where σ has been chosen in (6.10).

For further reference, we observe from (6.10) that $\sigma = O(h_1^2)$, so we further require of ν and τ in (6.14)–(6.16) that

$$\tau < 2\tau\log K < Ah_1^4, \qquad \nu < Ah_1^4 \tag{6.17}$$

where A is an absolute constant.

Only a weaker form of (6.15) is needed when $0 < u < K^{-1}r$ or $Kr < u$. Restriction (6.1) implies that

$$\left(\frac{s}{s'}\right)^{1+3h}S(s') \leqslant S(s) \leqslant \left(\frac{s}{s'}\right)^{2-3h}S(s') \qquad (1 < s' < s), \tag{6.18}$$

which is a sharpening of (4.9). It is then routine to obtain from (6.3), (6.4), (6.9) and (6.18) that

$$n(u) \leqslant 2n^*(u) \leqslant \csc 3\pi h\left(\frac{u}{r}\right)^{1+3h}n^*(r) \leqslant \left(\frac{u}{r}\right)^{1+2h}n^*(r) \qquad (r > Kr_0,\ 0 < u < K^{-1}r) \tag{6.19}$$

and

$$n(u) < \left(\frac{u}{r}\right)^{2-2h}n^*(r) \qquad (r > Kr_0,\ Kr < u). \tag{6.20}$$

That branch of $\log F(z)$ in $0 < \arg z < 2\pi$ for which $\log F(0) = 0$ may be represented by Valiron's formula:

$$\log F(z) = z^2 \int_0^\infty \frac{n(u, F)}{u^2(u-z)} du \qquad (z = re^{i\theta},\ 0 < \theta < 2\pi). \tag{6.21}$$

Since

$$e^{2i\theta} \int_0^\infty y^{\Lambda-2}(y - e^{i\theta})^{-1} dy = -\pi \csc \pi\Lambda\, e^{i\Lambda(\theta-\pi)} \qquad (0 < \theta < 2\pi,\ 1 < \Lambda < 2), \tag{6.22}$$

it follows that

$$\begin{aligned}
|\log F(z) + S(r) e^{i\Lambda(r)(\theta-\pi)}| &= r^2 \left| \int_0^\infty \frac{n(u) - n^*(r)(u/r)^{\Lambda(r)}}{u^2(u-z)} du \right| \\
&\leqslant r^2 \left| \int_{K^{-1}r}^{Kr} u^{-2}(u-z)^{-1} \{ n(u) - n^*(r)(u/r)^{\Lambda(r)} \} du \right| \\
&\quad + r^2 \left\{ \int_0^{K^{-1}r} n(u) u^{-2} |u-z|^{-1} du + \int_{Kr}^\infty n(u) u^{-2} |u-z|^{-1} du \right\} \\
&\quad + S(r) \left\{ \int_0^{K^{-1}r} (u/r)^{\Lambda(r)-2} |u-z|^{-1} du + \int_{Kr}^\infty (u/r)^{\Lambda(r)-2} |u-z|^{-1} du \right\} \\
&\qquad (r > Kr_0,\ 0 < \theta < 2\pi).
\end{aligned} \tag{6.23}$$

However $|u-z|^{-1} \leqslant 3(u+r)^{-1}$ when $|z| = r$ and $u < \frac{1}{2}r$ or $u > 2r$ and, in particular, when $|\log(u/r)| > \log K$. Thus after (6.1), (6.15), (6.19) and (6.20) are applied to (6.23), elementary manipulations yield

$$\begin{aligned}
|\log F(z) + S(r) e^{i\Lambda(r)(\theta-\pi)}| < S(r) \Bigg\{ &\sigma \int_{K^{-1}}^{K} y^{\Lambda(r)-2} |y - e^{i\theta}|^{-1} dy + 4 \int_0^{K^{-1}} y^{-1+2h} dy \\
&+ 4 \int_K^\infty y^{-1-2h} dy + 4 \int_0^{K^{-1}} y^{-1+3h} dy + 4 \int_K^\infty y^{-1-3h} dy \Bigg\}
\end{aligned}$$

so this, (6.8) and (6.10) prove (6.7).

6.2. A modification. Estimate (6.7) degenerates when z is near the positive axis, and this is to be expected since the zeros of F are located there. However, when $\Lambda(r)$ is close to 1 or 2, (6.3) and (6.4) show that $n(r)$ is small when compared to $S(r)$. Since (6.7) establishes $S(r)$ as the natural comparison function for $\log F$, this suggests that for such r the influence of the zeros whose modulus is "close to" r is small.

To exploit this principle, let $\mathfrak{N}$ be a set of positive numbers which satisfies the separation conditions

$$r^* \in \mathfrak{N} \Rightarrow (r^*, 2^{10} r^*) \cap \mathfrak{N} = \varnothing, \tag{6.24}$$

$$\mathfrak{N} \cap (0, 1) = \varnothing. \tag{6.25}$$

Then $\mathcal{N}$ may be written

$$\mathcal{N}=\{r_i^*\},\ r_i^*<r_{i+1}^*, \qquad i\geqslant 1. \tag{6.26}$$

We introduce the intervals

$$J_i=\{2^{-4}r_i^*\leqslant r\leqslant 2^4 r_i^*,\ r_i^*\in\mathcal{N}\}, \tag{6.27}$$

$$I_i=\{2^{-4}r_i^*\leqslant r\leqslant 2^{-4}r_{i+1}^*,\ r_i^*\in\mathcal{N}\} \qquad (i>0), \tag{6.28}$$

$$I_0=\{0<r\leqslant 2^{-4}r_1^*\} \tag{6.29}$$

and, if $\mathcal{N}$ has $n<\infty$ elements,

$$I_n=\{2^{-4}r_n^*\leqslant r\}. \tag{6.30}$$

Let $F(z)$ be a canonical product which satisfies the conditions of Lemma 11. Then construct a canonical product $F_1(z)$ with

$$n(r,0,F_1)=n(2^{-4}r_i^*,0,F_1) \qquad (r\in J_i), \tag{6.31}$$

$$n(r,0,F_1)=n(r,0,F)+\sum_{\substack{r^*<r\\ r^*\in\mathcal{N}}} n(2^{-4}r^*,F) \qquad (r\notin\bigcup J_i) \tag{6.32}$$

and next a canonical product $F_2(z)$ such that

$$n(r,0,F_2)=\begin{cases}0 & (r\in I_0)\\ n(2^{-4}r_i^*,0,F_1) & (r\in I_i,\ i\geqslant 1),\end{cases} \tag{6.33}$$

and consider the meromorphic function

$$F^*(z)=\frac{F_1(z)}{F_2(z)}. \tag{6.34}$$

It is clear from the construction (6.31)–(6.34) that

$$n(r,0,F^*)=n(r,\infty,F^*) \qquad (r\in\bigcup J_i), \tag{6.35}$$

so that single-valued branches of $\log F^*(z)$ may be defined in each annulus $\{2^{-4}r^*\leqslant |z|\leqslant 2^4r^*,\ r^*\in\mathcal{N}\}$.

LEMMA 12. *Let $h,\ h_1>0$ with*

$$(10^{-1}\geqslant)\,h_1=h^{1/4} \tag{6.36}$$

and let $F(z)$, r_0, τ, v and K be as in Lemma 11. If $\mathcal{N}$ is a set of positive numbers such that (6.24), (6.25)

$$\mathcal{N} \cap (0, 2^6 K r_0) = \varnothing, \tag{6.37}$$

and, for all $r^* \in \mathcal{N}$

$$(r^*, 2^{11} K^2 r^*) \cap \mathcal{N} = \varnothing \qquad (r^* \in \mathcal{N}), \tag{6.38}$$

$$|\sin \pi\Lambda(r^*)| \leqslant 10\pi h \qquad (r^* \in \mathcal{N}) \tag{6.39}$$

hold, then the function $F^*(z)$ *associated to* $F(z)$ *by* (6.31)–(6.34) *satisfies*

$$|\log F^*(z) + S(r) e^{i\Lambda(r)(\theta-\pi)}| < A h_1^2 S(r) \qquad (z = re^{i\theta},\ r > Kr_0,\ h_1 \leqslant \theta \leqslant 2\pi - h_1), \tag{6.40}$$

$$|\log F^*(z) + S(r) e^{i\Lambda(r)(\theta-\pi)}| < A h^2 S(r) \qquad (2^{-3} r^* \leqslant r \leqslant 2^3 r^*,\ 0 \leqslant \theta \leqslant 2\pi,\ r^* \in \mathcal{N}) \tag{6.41}$$

for an absolute constant A.

Remarks. Here and in the future, an absolute constant refers to one which does not depend on $\Lambda(r)$, $\mathcal{N}$, h or h_1 so long as (6.1), (6.2), (6.24), (6.25), (6.36)–(6.39) hold.

The value of Lemma 12 over Lemma 11 is that the error term of (6.41) is small for all θ; thus (6.39) is the key assumption.

Proof. Formula (6.35) will be crucial in the proof of Lemma 17. For now, the useful properties of (6.31)–(6.34) are

$$0 < n(r, 0, F) - [n(r, 0, F^*) - n(r, \infty, F^*)] < A h_1^4 S(r) < A n^*(r) \qquad (r > 0) \tag{6.42}$$

(where n^* is defined in (6.3)) and

$$n(r, 0, F^*) - n(r, \infty, F^*) = n(r, 0, F) \qquad (r \notin \bigcup J_i). \tag{6.43}$$

The left inequality of (6.42) follows at once from (6.31)–(6.33). Next, let $r \in I_i (i \geqslant 1)$. Then (6.19), (6.37) and (6.38) yield

$$n(2^{-4} r_j^*, 0, F) < K^{-2} n^*(2^{-4} r_{j+1}^*) \tag{6.44}$$

where $K > 2^{10}$, so if $r \in I_i$ $(i \geqslant 1)$, iteration of (6.44) with (4.9), (6.28), (6.36) and (6.39) shows

$$\sum_{j \leqslant i} n(2^{-4} r_j^*, 0, F) \leqslant A n^*(r_i^*) \leqslant A h_1^4 S(r_i^*) \leqslant A h_1^4 S(r) \qquad (r \in I_i).$$

According to (6.32), this gives

$$|n(r, 0, F) - n(r, 0, F_1)| \leqslant A h_1^4 S(r) \qquad (r > 0) \tag{6.45}$$

and the proof of

$$n(r, 0, F_2) \leqslant A h_1^4 S(r) \qquad (r > 0) \tag{6.46}$$

is similar. This proves (6.42), and (6.43) follows from (6.32) and (6.33).

The computation of $\log F^*(z)$ will be based on the Valiron-type formula (compare with (6.21))

$$\log F^*(z) = z^2 \int_0^\infty \frac{n_0(u)}{u^2(u-z)}\, du \qquad (0 < \arg z < 2\pi), \tag{6.47}$$

with

$$n_0(r) = n(r, 0, F^*) - n(r, \infty, F^*). \tag{6.48}$$

According to (6.19), (6.20), (6.42) and (6.46)

$$n(u, 0, F^*) + n(u, \infty, F^*) \leqslant A\left(\frac{u}{r}\right)^{1+2h} n(r, 0, F) \qquad (r > Kr_0,\ 0 < u < K^{-1}r),$$

$$n(u, 0, F^*) + n(u, \infty, F^*) \leqslant A\left(\frac{u}{r}\right)^{2-2h} n(r, 0, F) \qquad (r > Kr_0,\ u > Kr),$$

so it is easy to see from (6.21) and (6.47) (compare with the manipulations in (6.23)) that

$$|\log F^*(z) - \log F(z)| \leqslant r^2 \left| \int_{K^{-1}r}^{Kr} \frac{n_0(u) - n(u, 0, F)}{u^2(u-z)}\, du \right| + Ah_1^2 S(r) \qquad (r > Kr_0,\ 0 < \theta < 2\pi), \tag{6.49}$$

$$\left| \log F^*(z) - z^2 \int_{K^{-1}r}^{Kr} \frac{n_0(u)}{u^2(u-z)}\, du \right| \leqslant Ah_1^2 S(r) \qquad (r > Kr_0,\ 0 < \theta < 2\pi). \tag{6.50}$$

First, suppose $z = re^{i\theta}$ where

$$r > Kr_0,\ [K^{-1}r, Kr] \cap \{\bigcup J_i\} = \phi,\ h_1 \leqslant \theta \leqslant 2\pi - h_1. \tag{6.51}$$

Then a glance at (6.7), (6.43) and (6.49) leads at once to

$$|\log F^*(z) + S(r)e^{i\Lambda(r)(\theta-\pi)}| \leqslant |\log F^*(z) - \log F(z)| + |\log F(z) + S(r)e^{i\Lambda(r)(\theta-\pi)}| \leqslant Ah_1^2 S(r). \tag{6.52}$$

We next consider the situation

$$r > Kr_0,\ h_1 \leqslant \theta \leqslant 2\pi - h_1,\ J_i \cap [K^{-1}r, Kr) \neq \phi \tag{6.53}$$

for some (and, by (6.38), only one) $r_i^* \in \mathcal{N}$. Since $\int_{J_i} t^{-1}dt < A$, the bound $1 \leqslant \Lambda(r) \leqslant 2$ shows

$$\int_{J_i} \left| \frac{u^{\Lambda(r)-2}}{r^{\Lambda(r)-2}(u - re^{i\theta})} \right| du \leqslant Ah_1^{-1} \qquad (h_1 \leqslant \theta \leqslant 2\pi - h_1);$$

thus when (6.53) holds we obtain from (4.9), (6.15), (6.35), (6.43) and (6.48) that

$$r^2\left|\int_{K^{-1}r}^{Kr}\frac{n_0(u)-n(u,0,F)}{u^2(u-z)}du\right| = r^2\left|\int_{J_i}\frac{n(u,0,F)}{u^2(u-z)}du\right|$$
$$\leqslant An^*(r)\int_{J_i}\left|\frac{u^{\Lambda(r)-2}}{r^{\Lambda(r)-2}(u-z)}\right|du \leqslant Ah_1^{-1}n^*(r) \tag{6.54}$$

(a more refined analysis can replace h_1^{-1} by $\log h_1^{-1}$ in (6.54)).

Since $|\log(r/r_i^*)| < 2\log K$, (6.11), (6.17) and the method used to obtain (6.11) yield

$$|\Lambda(r)-\Lambda(r_i^*)| < 2\tau\log K^2 < Ah_1^4 \tag{6.55}$$

when r satisfies (6.53). Hence the fundamental assumption (6.39) with the convention (6.36) yields that

$$|\sin\pi\Lambda(r)| \leqslant Ah,$$

so we obtain from (6.3) that

$$n^*(r) \leqslant Ah_1^4 S(r) \qquad (r > Kr_0,\ \mathcal{N}\cap[K^{-1}r, Kr] \neq \phi). \tag{6.56}$$

Now (6.7), (6.54) and (6.56) are used in (6.49), leading to

$$|\log F^*(z)+S(r)e^{i\Lambda(r)(\theta-\pi)}| \leqslant |\log F^*(z)-\log F(z)| + |\log F(z)+S(r)e^{i\Lambda(r)(\theta-\pi)}|$$
$$\leqslant AS(r)\{2h_1^2+h_1^4h_1^{-1}\} = Ah_1^2S(r),$$

when (6.53) holds; this and (6.52) complete the proof of (6.40).

The more delicate inequality is (6.41). In this range, $\log F(z)$ is not a good comparison to $\log F^*(z)$, so (6.50) is preferred to (6.49). We write the principal term of (6.50) as

$$z^2\int_{K^{-1}r}^{Kr}\frac{n_0(u)}{u^2(u-z)}du$$
$$= n(r,0,F)\,e^{2i\theta}\int_{K^{-1}}^{K}\frac{y^{\Lambda(r)-2}}{y-e^{i\theta}}dy + z^2\int_{K^{-1}r}^{Kr}\frac{n_0(u)-(u/r)^{\Lambda(r)}n(r,0,F)}{u^2(u-z)}du \quad (0<\theta<2\pi). \tag{6.57}$$

The first term on the right side of (6.57) provides the main contribution; it may be estimated from (6.22) and the bounds (6.1) on $\Lambda(r)$ and (6.8) of K (cf. (6.23)):

$$\left|n(r,0,F)\,e^{2i\theta}\int_{K^{-1}}^{K}\frac{y^{\Lambda(r)-2}}{y-e^{i\theta}}dy + S(r)\,e^{i\Lambda(r)(\theta-\pi)}\right| \leqslant Ah_1^2S(r) \qquad (r>Kr_0,\ 0<\theta<2\pi). \tag{6.58}$$

In order to estimate the second integral on the right side of (6.57), the range is divided into $[K^{-1}r, 2^{-9}r]$, $[2^{-9}r, 2^{-1}r]$, $[2^{-1}r, 2r]$, $[2r, 2^9r]$, $[2^9r, Kr]$.

Suppose $K^{-1}r \leqslant u \leqslant 2^{-9}r$ or $2^9r \leqslant u \leqslant Kr$. Since it is assumed that $2^{-4}r^* \leqslant r \leqslant 2^4r^*$, we see from (6.38), (6.43) and (6.48) that $n_0(u) = n(u,0,F)$. Thus (6.10), (6.16) and (6.56) give

$$\left|r^2\int_{K^{-1}r}^{2^{-9}r}\frac{n_0(u)-(u/r)^{\Lambda(r)}n(r,0,F)}{u^2(u-z)}du\right|$$
$$\leqslant \sigma n^*(r)\int_{K^{-1}}^{2^{-9}}\left|\frac{t^{\Lambda(r)-2}}{t-e^{i\theta}}\right|dt\leqslant Ah_1^6 S(r) \qquad (2^{-4}r^*<r<2^4r^*,\ 0<\theta<2\pi,\ r^*\in\mathcal{N}); \tag{6.59}$$

in the same way,

$$r^2\left|\int_{2^9r}^{Kr}\frac{n_0(u)-(u/r)^{\Lambda(r)}n(r,0,F)}{u^2(u-z)}du\right|\leqslant Ah_1^6 S(r) \qquad (2^{-4}r^*<r<2^4r^*,\ 0<\theta<2\pi,\ r^*\in\mathcal{N}). \tag{6.60}$$

When $2^{-9}r<u<2^{-1}r$, (6.15), (6.42), (6.48) and (6.56) show

$$r^2\left|\int_{2^{-9}r}^{2^{-1}r}\frac{n_0(u)-(u/r)^{\Lambda(r)}n(r,0,F)}{u^2(u-z)}du\right|$$
$$\leqslant An^*(r)\int_{2^{-9}r}^{2^{-1}r}\left|\frac{1}{u-z}\right|du\leqslant Ah_1^4 S(r) \qquad (2^{-4}r^*<r<2^4r^*,\ 0<\theta<2\pi,\ r^*\in\mathcal{N}) \tag{6.61}$$

and similarly

$$r^2\left|\int_{2r}^{2^9r}\frac{n_0(u)-(u/r)^{\Lambda(r)}n(r,0,F)}{u^2(u-z)}du\right|\leqslant Ah_1^4 S(r) \qquad (2^{-4}r^*<r<2^4r^*,\ 0<\theta<2\pi,\ r^*\in\mathcal{N}). \tag{6.62}$$

Finally, since (6.35) and (6.48) show that $n_0(u)=0$ $(\frac{1}{2}r<u<2r)$ when $2^{-3}r^*<r<2^3r^*$ for some $r^*\in\mathcal{N}$, it follows that

$$r^2\int_{(1/2)r}^{2r}\frac{n_0(u)}{u^2(u-z)}du\equiv 0 \qquad (2^{-3}r^*<r<2^3r^*,\ 0<\theta<2\pi,\ r^*\in\mathcal{N}), \tag{6.63}$$

and (6.56) gives

$$r^2\left|\int_{(1/2)r}^{2r}\left(\frac{u}{r}\right)^{\Lambda(r)}\frac{n(r,0,F)}{u^2(u-z)}du\right|$$
$$=n(r,0,F)\left|\int_{(1/2)r}^{2r}\left(\frac{u}{r}\right)^{\Lambda(r)-2}\frac{du}{u-z}\right|\leqslant Ah_1^4 S(r) \qquad (2^{-3}r^*<r<2^3r^*,\ 0<\theta<2\pi,\ r^*\in\mathcal{N}). \tag{6.64}$$

Thus when $2^{-3}r^*<|z|<2^3r^*$ with $0<\arg z<2\pi$, the expression

$$z\int_{K^{-1}r}^{Kr}\frac{n_0(u)}{u^2(u-z)}du,$$

which appears in (6.50), is written as in (6.57) and estimated by (6.58)–(6.64) and (6.41) is proved.

To apply quasi-conformal methods, it is necessary to differentiate estimates (6.40) and (6.41). Thus, the following lemma, though now elementary to obtain, will also be of central importance.

LEMMA 13. *The function $F^*(z)$ defined by* (6.31)–(6.34) *satisfies*

$$\left|\frac{d}{dz}\{\log F^*(z)\} + z^{-1}\Lambda(r)\,S(r)\,e^{i\Lambda(r)(\theta-\pi)}\right| < Ar^{-1}h_1\,S(r) \tag{6.66}$$

if either

$$|z| > 2Kr_0,\ 3h_1 \leqslant \arg z \leqslant 2\pi - 3h_1 \tag{6.67}$$

or

$$2^{-2}r^* < |z| < 2^2 r^*,\ 0 \leqslant \arg z \leqslant 2\pi, \qquad r^* \in \mathcal{N}. \tag{6.68}$$

Proof. Let z_0 satisfy (6.67) or (6.68), and let $h' = h'(z_0) = h_1$ when (6.67) holds and $h' = \frac{1}{2}$ when z_0 satisfies (6.68). We claim that if

$$D(z_0) = \{z;\ |z - z_0| \leqslant h'r_0\} \qquad (r_0 = |z_0|),$$

then

$$\left|\log F^*(z) + S(r_0)\left(\frac{r}{r_0}\right)^{\Lambda(r_0)(\theta-\pi)}\right| \leqslant Ah_1^2\,S(r_0) \qquad (z \in D(z_0)). \tag{6.69}$$

Once (6.69) is established, (6.66) follows from Cauchy's formula:

$$\left|\frac{d}{dz}\{\log F^*(z)\}_{z=z_0} + z_0^{-1}\Lambda_0(r_0)\,S(r_0)\,e^{i\Lambda(r_0)(\theta-\pi)}\right|$$
$$\leqslant Ah_1^2\,S(r_0)\int_{\partial D(z_0)} |z - z_0|^{-2}|dz| < Ar^{-1}h_1\,S(r).$$

We now prove (6.69). When (6.67) holds, (6.69) is a simple consequence of (6.11), (6.12), (6.17) and (6.40) as

$$\left|S(r)\,e^{i\Lambda(r)(\theta-\pi)} - S(r_0)\left(\frac{r}{r_0}\right)^{\Lambda(r_0)} e^{i\Lambda(r_0)(\theta-\pi)}\right|$$
$$\leqslant \left|S(r) - S(r_0)\left(\frac{r}{r_0}\right)^{\Lambda(r_0)}\right| + S(r_0)\left(\frac{r}{r_0}\right)^{\Lambda(r_0)}\left|e^{i\{\Lambda(r)-\Lambda(r_0)\}(\theta-\pi)} - 1\right| \leqslant Ah_1^4\,S(r_0).$$

In the range (6.68) we must be careful since $D(z_0)$ may cross the positive axis. However according to (6.34) and (6.35)

$$\log F^*(re^{-i\theta}) = \log F^*(re^{i(2\pi-\theta)}),$$

and, if $r \in J_i$, we obtain routinely from (6.39) and (6.34) that

$$|S(r)\,e^{i\Lambda(r)(-\theta-\pi)} - S(r)\,e^{i\Lambda(r)\{(2\pi-\theta)-\pi\}}| = S(r)\,|1-e^{2\pi i\Lambda(r)}| \leqslant AS(r)\,h_1^4 \qquad (r \in J_i,\ -\pi \leqslant \theta \leqslant 0),$$

so (6.69) follows as before.

6.3. Value-distribution of F^*. Inequality (6.70) below can be made more exact, but this is not necessary here.

LEMMA 14. *Let F^* be constructed in accord with* (6.31)–(6.34). *Then there is an absolute constant A such that*

$$n(r, a, F^*) \leqslant AS(r) \qquad (a \in \hat{C},\ r > Kr_0). \tag{6.70}$$

Estimate (6.70) complements the bounds in (6.4), (6.6), (6.45), (6.46) for $n(r, 0, F^*)$, $n(r, \infty, F^*)$, since F^* is defined by (6.34).

Proof. As starting point, we show that

$$T(r, F^*) \leqslant T(r, F_1) + T(r, F_2) \leqslant AS(r) \qquad (r > Kr_0) \tag{6.71}$$

(the left inequality of (6.71) is a consequence of Jensen's formula and the normalization $F_2(0)=1$ (cf. (6.75) below)). Since the $F_i (i=1, 2)$ are canonical products, the characteristics are estimated by the standard inequality (cf. Theorem 1.11 of [9]):

$$T(r, F_i) \leqslant \log M(r, F_i) \leqslant 12\left\{ r\int_0^r \frac{n(u, 0, F_i)}{u^2}\,du + r^2\int_r^\infty \frac{n(u, 0, F_i)}{u^3}\,du \right\}$$

for $i=1, 2$. The integrals may be estimated as follows: according to (6.45), (6.46), (6.4), (6.8), (6.19) and (6.20)

$$r\int_0^{K^{-1}r} \frac{n(u, 0, F_i)}{u^2}\,du + r^2\int_{Kr}^\infty \frac{n(u, 0, F_i)}{u^3}\,du < An^*(r) < AS(r),$$

and (6.45), (6.46), (6.4) and (6.15) yield that

$$r\int_{K^{-1}r}^r \frac{n(u, 0, F_i)}{u^2}\,du + r^2\int_r^{Kr} \frac{n(u, 0, F_i)}{u^3}\,du \leqslant A\,\frac{n^*(r)}{\Lambda(r)-1} + A\,\frac{n^*(r)}{2-\Lambda(r)} < AS(r);$$

thus

$$M(r, F_i) \leqslant AS(r) \qquad (i=1, 2,\ r > Kr_0) \tag{6.72}$$

and (6.71) is proved.

According to the first fundamental theorem,

$$N(r, a, F^*) \leqslant T\left(r, \frac{1}{F^*-a}\right) = T(r, F^*-a) - \log|F^*(0)-a|;$$

since

$$|T(r, F^*-a) - \log^+|a|| \leqslant T(r, F^*) + \log 2,$$

(6.71) implies that

$$N(r, a, F^*) \leqslant AS(r) \qquad (r \geqslant Kr_0, \ |a-1| > \tfrac{1}{4}).$$

We use the standard relation between N and n (cf. (2.11)) and deduce by a simple tauberian argument that (6.70) holds if

$$|a-1| \geqslant \tfrac{1}{4}, \tag{6.73}$$

since $F^*(0)=1$.

To remove the restriction (6.73), we argue as follows: in Lemma 11 it was required that $r_0 > 10\nu^{-2}h_1^{-5}$ so it is easy to see from (6.36), the definitions (6.1), (6.3) of $n^*(r)$ and the growth property (4.8) of S that $n^*(8h_1^{-5}) > 2$. Thus if (6.6) holds with $\nu < \frac{1}{2}$, F must vanish at some γ_1, $0 < \gamma_1 < r_0$, and since $\mathfrak{N}$ satisfies (6.37), (6.32) shows that $F^*(\gamma_1)=0$. Choose γ_0, $0 < \gamma_0 < \gamma_1$ with $|F^*(\gamma_0)| = \frac{1}{2}$ and consider

$$F_0(z) = F^*(z-\gamma_0).$$

We claim that

$$T(r, F_0) \leqslant AS(r) \qquad (r > r_0). \tag{6.74}$$

Indeed, $F_0(z) = (F_1(z-\gamma_0))(F_2(z-\gamma_0))^{-1}$, so Jensen's formula, (4.9) and (6.72) give

$$\begin{aligned} T(r, F_0) &\leqslant T(r, F_1(z-\gamma_0)) + T\left(r, \frac{1}{F_2(z-\gamma_0)}\right) \\ &= T(r, F_1(z-\gamma_0)) + T(r, F_2(z-\gamma_0)) - \log|F_2(\gamma_0)| \\ &\leqslant \log M(r, F_1(z_1-\gamma_0)) + \log M(r, F_2(z-\gamma_0)) - \log|F_2(\gamma_0)| \\ &\leqslant AS(r+r_0) - \log|F_2(\gamma_0)| \leqslant AS(r) - \log|F_2(\gamma_0)| \qquad (r > r_0). \end{aligned} \tag{6.75}$$

Since $\mathfrak{N}$ satisfies (6.37), (6.33) shows that $F_2(z)$ does not vanish for $\{|z| < 2\gamma_0\}$. But since $1/F_2$ is holomorphic in $\{|z| \leqslant 2\gamma_0\}$ the standard estimate

$$\log^+ M\left(\gamma_0, \frac{1}{F_2}\right) \leqslant 3T\left(2\gamma_0, \frac{1}{F_2}\right)$$

([9], p. 18), the normalization $F_2(0)=1$ and (6.71) imply that

$$\begin{aligned} -\log|F_2(\gamma_0)| &\leqslant \log^+ M\left(\gamma_0, \frac{1}{F_2}\right) \leqslant AT\left(2\gamma_0, \frac{1}{F_2}\right) \\ &= AT(2\gamma_0, F_2) \leqslant AS(\gamma_0) \leqslant AS(r) \qquad (r > r_0); \end{aligned}$$

this and (6.75) yield (6.74). The usual tauberian argument based on (2.11) now gives

$$n(r, a, F_0) \leqslant AS(r) \qquad (r > Kr_0, |a - F_0(0)| \geqslant \tfrac{1}{4}),$$

or

$$n(r, a, F_0) \leqslant n(r + r_0, a, F_0) \leqslant AS(r) \qquad (r > Kr_0, |a - F_0(0)| \geqslant \tfrac{1}{4});$$

since $|F_0(0)| = \frac{1}{2}$, this proves (6.70) for those a not included in (6.73).

6.4. On the hypotheses of Lemmas 11 and 12. We now show that the hypotheses of Lemmas 11 and 12 are realistic.

LEMMA 15. *Let* $0 \leqslant \alpha \leqslant 1$, h, $h_1 > 0$ *and* $\Lambda(r)$ *be given where* $\Lambda(r)$ *satisfies* (6.1), (6.2) *and* (6.5) *and* h, h_1 *satisfy* (6.36). *Recall that* h_1 *determines* $0 < \nu < \frac{1}{2}$ *in accord with* (6.6), (6.14)–(6.16). *Then functions* $F(z)$, $F^*(z)$ *may be constructed in accord with Lemmas* 11–14 *and such that*

$$|\bar{n}(r, 0, F^*) - \alpha n(r, 0, F^*)| \leqslant Ah_1 S(r) \qquad (r > h_1^{-1} r_0) \tag{6.76}$$

with

$$r_0 \leqslant 10\nu^{-2} h_1^{-5}. \tag{6.77}$$

Further, given $(2 - 3h \geqslant) \Lambda^{\#} > \frac{3}{2}$, *suppose* (4.21) *and* (4.22) *hold, and that the set* $\mathfrak{N}$ *of* (6.24)–(6.26) *and* (6.37)–(6.39) *has finitely many elements. Then if*

$$D(\Lambda^{\#}) = \left\{ z; |\arg z| < \frac{\pi}{2}\left(1 - \frac{3}{2\Lambda^{\#}}\right)\right\}, \tag{6.78}$$

there exists r_1 *such that*

$$\log |F^*(z)| \leqslant -A \sin(\Lambda^{\#} - \tfrac{3}{2}) S(r) \qquad (A > 0, z \in D(\Lambda_0), |z| > r_1), \tag{6.79}$$

so that

$$n(r, a, F^*) - \bar{n}(r, a, F^*) \leqslant O(1); \tag{6.80}$$

the $O(1)$ *in* (6.80) *is uniform in each region*

$$\log |a| \geqslant -A_1. \tag{6.81}$$

Remark. That $\mathfrak{N}$ be bounded is essential for (6.79) since (6.79) fails at the poles of F^*.

Proof. Once h, h_1 are given, Lemma 11 associates τ, K, ν as in (6.5), (6.6), (6.8), (6.9) and (6.17). According to (6.14), (6.1), (6.3), (6.36) and the bound $\nu < \frac{1}{2}$, we have that

$$\begin{aligned} n^*((1+\nu^2)r) - n^*(r) &\geqslant \{(1+\nu^2)(1-\nu^4)\} n^*(r) - n^*(r) \\ &\geqslant \tfrac{1}{2}\nu^2 n^*(r) \geqslant (4\pi)^{-1} 6\nu^2 h_1^4 r. \end{aligned} \tag{6.82}$$

Thus, we may construct a canonical product $F(z)$ of genus 1 with positive zeros and

$$n(r, 0, F) \leqslant n^*(r) \qquad (r > 0) \tag{6.83}$$

such that each zero a_n of F with

$$a_n \geqslant 8\nu^{-2}h_1^{-5} \tag{6.84}$$

occurs with multiplicity p_n where

$$|p_n^{-1} - \alpha| < h_1; \tag{6.85}$$

for example, if $\alpha \leqslant h_1$, let $p_n \geqslant h_1^{-1}$. Inequalities (6.82) and (6.84) show this may be arranged so that

$$n((1+\nu^2)r, 0, F) \geqslant n^*(r) \qquad (r \geqslant 8\nu^{-2}h_1^{-5})$$

and this and (6.14) lead to

$$n(r, 0, F) \geqslant n^*(\{1+\nu^2\}^{-1}r) \geqslant \frac{1-\nu^4}{(1+\nu^2)^{\Lambda(r)}} n^*(r) \geqslant (1-\nu^2)\, n^*(r) \qquad (r \geqslant 10\nu^{-2}h_1^{-5}). \tag{6.86}$$

Thus (6.83) and (6.86) yield (6.6) with $r_0 = 10\nu^{-2}h_1^{-5} (> 8\nu^{-2}h_1^{-5})$ as required in the statement of Lemma 11 and the proof of (6.70).

It readily follows from (4.9) and the bound (6.85) that the $\{p_n^{-1}\}$ may be chosen bounded or tend to infinity so slowly that

$$|\bar{n}(r, 0, F) - \alpha n(r, 0, F)| \leqslant n^*(10\nu^{-2}h_1^{-5}) + 2h_1 n^*(r) + p_n \leqslant Ah_1 S(r) \qquad (r \geqslant 10\nu^{-2}h_1^{-6})$$

holds. Thus if F^* is obtained from F in accord with (6.31)–(6.34), we obtain from this and (6.45) that

$$\begin{aligned} &|\bar{n}(r, 0, F^*) - \alpha n(r, 0, F^*)| \\ &\quad < |\bar{n}(r, 0, F^*) - \bar{n}(r, 0, F)| + |\bar{n}(r, 0, F) - \alpha n(r, 0, F)| + \alpha |n(r, 0, F^*) - n(r, 0, F)| \\ &\quad \leqslant Ah_1 S(r) + n^*(10\nu^{-2}h_1^{-5}) \leqslant Ah_1 S(r) \qquad (r > r_0), \end{aligned}$$

which is (6.76) and (6.77).

Now suppose (4.21) and (4.22) hold. Thus $r\Lambda'(r) \to 0$, so τ in (6.5) may be chosen as small as desired if r_0 is increased. In particular, it may be supposed that

$$n(r, 0, F^*) = n^*(r) + o(1) S(r) \qquad (r \to \infty) \tag{6.87}$$

and

$$\bar{n}(r, 0, F^*) = \alpha n(r, 0, F^*) + o(1) S(r) \qquad (r \to \infty). \tag{6.88}$$

Let A be the largest of the constants introduced in (6.40), (6.41) and (6.66), and take $h_1 < 10^{-1}$ so small that

$$Ah_1 < \tfrac{1}{2} \sin(\Lambda^{\#} - \tfrac{3}{2}). \tag{6.89}$$

Then if h satisfies (6.36) and τ, K and ν are chosen in accord with Lemma 11, we see that (6.5) and (6.6) hold for sufficiently large r_0, and hence so do the conclusions of Lemma 11. Now the discussion which introduced $t(s)$ in (4.70) (cf. (4.62) and (4.67)) shows that

$$\cos \Lambda(r)(\theta-\pi) \geqslant \sin(\Lambda^{\#}-\tfrac{3}{2}) \qquad \left(\theta=\frac{\pi}{2}\left(1-\frac{3}{2\Lambda^{\#}}\right),\ \theta=2\pi-\frac{\pi}{2}\left(1-\frac{3}{2\Lambda^{\#}}\right)\right),$$

and since the choice of h_1 in (6.89) shows that (6.7) and (6.40) hold for large r when

$$\frac{\pi}{2}\left(1-\frac{3}{2\Lambda^{\#}}\right) \leqslant \theta \leqslant 2\pi-\frac{\pi}{2}\left(1-\frac{3}{2\Lambda^{\#}}\right)$$

it follows from (6.40) that with $D(\Lambda^{\#})$ as in (6.78),

$$\log|F^*(z)| \leqslant -\tfrac{1}{2}(\sin(\Lambda^{\#}-\tfrac{3}{2}))S(r) < 0 \qquad (|z|>R_1,\ z\in\partial D(\Lambda^{\#})) \tag{6.90}$$

if R_1 is sufficiently large. We may suppose R_1 so large that F^* is holomorphic outsice $\{|z|\leqslant R_1\}$ (possible since $\mathcal{N}$ is bounded). Estimates (6.71) and (6.18) show that F^* has order $\leqslant 2-3h$, so (6.90) and Phragmen-Lindelöf yield that $|F^*|<1$ in $D(\Lambda^{\#})\cap\{|z|>r_1/8\}$ for some $r_1 \geqslant R_1$.

Now let $z_0(|z_0|=r_0)\in D(\Lambda^{\#})$ with $|z_0|>r_1$ and let

$$D_0 = D(\Lambda^{\#}) \cap \{\tfrac{1}{8}r_0 \leqslant |z| \leqslant 8r_0\}.$$

Partition ∂D_0 into $\alpha\cup\beta$ where $\alpha\subset\{|z|=\tfrac{1}{8}r_0\}\cup\{|z|=8r_0\}$ and $|\arg z|=\pi(1-3/(2\Lambda^{\#}))$ on β. Then (6.36) and standard estimates on harmonic measure (cf. [15], p. 79, Satz 4) show

$$\omega(z_0,\alpha,D_0) \leqslant 2\exp\left\{-\frac{3\log 16}{\pi^2(\Lambda^{\#}-\tfrac{3}{2})}\right\} \leqslant 2\exp\left\{\frac{-24\log 2}{\pi^2}\right\} \leqslant \tfrac{1}{2}.$$

Since $|F^*|<1$ on α, it follows from (4.9), (6.90) and the two-constants theorem that

$$\log|F^*(z_0)| < \tfrac{1}{2}\sup_{\zeta\in\beta}|F^*(\zeta)| \leqslant -4^{-1}\sin(\Lambda^{\#}-\tfrac{3}{2})\,S(\tfrac{1}{8}r_0)$$
$$\leqslant -A\sin(\Lambda^{\#}-\tfrac{3}{2})\,S(r_0) \qquad (z\in D(\Lambda^{\#}),\ |z|>r_1)$$

with $A>0$; this is (6.79).

It follows from (6.66) and (6.89) that all points of ramification of F^* in $\{|z|>r_0\}$ must occur in $D(\Lambda^{\#})$, so (4.8) and (6.79) yield that if $\log|a| \geqslant -A_1$, then

$$n(r,a,F^*)-\bar{n}(r,a,F^*) \leqslant n(\mathcal{R}(A_1),a,F^*) \tag{6.91}$$

where $\mathcal{R}(A_1)$ is so large that $A\sin(\Lambda^{\#}-\tfrac{3}{2})S(r)>A_1$ if $r\geqslant\mathcal{R}(A_1)$ (cf. (4.79), (4.80)). Thus (6.80) is a simple consequence of (6.70) and (6.91).

6.5. A preliminary form of Theorem 4. Recall the constant $(50)^{-1} > \eta > 0$ from the statement of Theorem 4, and choose h, h_1 according to (6.36). This pair h, h_1 in turn determines K, r_0, $\tau > 0$ as in Lemma 11. Also, let $\mathcal{N}$ be as in (6.24)–(6.26) and (6.37)–(6.39).

Now consider a function $\Lambda(r)$ which satisfies (6.1) and (6.5).

Recall that the function $F^*(z)$ of Lemma 12 almost fulfills (4.14) and (4.16) (cf. (6.40) and (6.41)) and is defined in the full plane. Here we introduce a function $\sigma(w)$ which, while not defined in the full plane, satisfies (4.14) precisely and the values $\sigma(r_i^* e^{it})\,(r_i^* \in \mathcal{N})$ are explicitly determined. Our major goal is Lemma 17, where F^* and σ are "welded" together.

To keep control of error terms, we now recall that $\eta > 0$ and h, h_1 are known, and then introduce a $k > 0$ with

$$h = h_1^4 < h_1 \ll k \ll \eta, \tag{6.92}$$

and let $A(k)$ denote a generic positive function such that

$$A(k) \to 0 \qquad (k \to 0). \tag{6.93}$$

Also, let $\mathcal{N}$ be partitioned into $\mathcal{N}_1 \cup \mathcal{N}_2$, characterized by

$$|\Lambda(r^*) - m| < \tfrac{1}{2} \qquad (r^* \in \mathcal{N}_m,\ m = 1, 2); \tag{6.94}$$

it follows from (6.36) and (6.39) that $\mathcal{N}_1 \cup \mathcal{N}_2 = \mathcal{N}$, $\mathcal{N}_1 \cap \mathcal{N}_2 = \phi$.

Definition of σ. The function σ is defined in

$$\{\mathcal{D}_0 \cup \mathcal{D}_1 \cup \mathcal{D}_2\} \cap \{|w| \geq r_1^*\}$$

where r_1^* is the smallest element of $\mathcal{N}$ and

$$\mathcal{D}_0 = \{s > r_1^*,\ \tfrac{1}{2}\eta \leq t \leq 2\pi - \tfrac{1}{2}\eta\}, \tag{6.95}$$

$$\mathcal{D}_1 = \{s \in J_i,\ |t| \leq \tfrac{1}{2}\eta,\ r_i^* \in \mathcal{N}_1\}, \tag{6.96}$$

$$\mathcal{D}_2 = \{s \in J_i,\ |t| \leq \tfrac{1}{2}\eta,\ r_i^* \in \mathcal{N}_2\}; \tag{6.97}$$

recall that the J_i are defined in (6.27).

The definition of σ is

$$\sigma(w) = \exp\{-S(s)\, e^{i\Lambda(s)(t-\pi)}\} \qquad (w \in \mathcal{D}_0), \tag{6.98}$$

$$\sigma(\omega) = \exp\{-S(s)\, e^{i\Lambda(s)(t-\pi) - 2\pi i \eta^{-1}[\Lambda(s)-1](t-(1/2)\eta)}\} \qquad (w \in \mathcal{D}_1), \tag{6.99}$$

$$\sigma(w) = \exp\{-S(s)\, e^{i\Lambda(s)(t-\pi) + 2\pi i \eta^{-1}[2-\Lambda(s)](t-(1/2)\eta)}\} \qquad (w \in \mathcal{D}_2). \tag{6.100}$$

The reader should verify that σ is well-defined and continuous: that (6.98) (for $w = se^{(1/2)i\eta}$,

$se^{i(2\pi-(1/2)\eta)}$) agrees with (6.99) (with $w=se^{\pm(1/2)i\eta}$) when $s\in J_i$ for some $r_i^*\in\mathcal{N}_1$; that (6.98) (for $w=se^{(1/2)i\eta}$, $se^{i(2\pi-(1/2)\eta)}$) agrees with (6.100) (with $w=se^{\pm(1/2)\eta}$) when $s\in J_i$ for some $r_i^*\in\mathcal{N}_2$.

LEMMA 16. *The function $\sigma(w)$ is quasi-meromorphic in $\{|w|\geqslant r_1^*\}\cap\{\mathcal{D}_0\cup\mathcal{D}_1\cup\mathcal{D}_2\}$ with*

$$|\mu_\sigma(w)|<A(k)\qquad (w\in\mathcal{D}_0\cup\mathcal{D}_1\cup\mathcal{D}_2), \tag{6.101}$$

where (6.93) *holds.*

Proof. We compute locally, using a branch of $\log\{\log\sigma(w)\}$. Then

$$\begin{cases}\dfrac{\partial\log\{\log\sigma\}}{\partial\log s}=\Lambda(s)+is\Lambda'(s)(t-\pi) & (w\in\mathcal{D}_0)\\[2ex] \dfrac{\partial\log\{\log\sigma\}}{\partial t}=i\Lambda(s) & (w\in\mathcal{D}_0)\end{cases} \tag{6.102}$$

and (6.101) follows from Lemma 7 since $\Lambda(s)\geqslant 1$ and (using (6.5), (6.17), (6.36) and (6.92)) $|s\Lambda'(s)|\leqslant A(k)$.

The computation is more subtle in $\mathcal{D}_1\cup\mathcal{D}_2$. For example, if $r_i^*\in\mathcal{N}_1$,

$$\begin{cases}\dfrac{\partial\log\{\log\sigma\}}{\partial\log s}=\Lambda(s)+i\{s\Lambda'(s)[(t-\pi)-2\pi\eta^{-1}(t-\tfrac{1}{2}\eta)]\} & (|w|\in J_i,\ w\in\mathcal{D}_1).\\[2ex] \dfrac{\partial\log\{\log\sigma\}}{\partial t}=i\Lambda(s)-2\pi i\eta^{-1}[\Lambda(s)-1] & (|w|\in J_i,\ w\in\mathcal{D}_1).\end{cases} \tag{6.103}$$

Thus

$$\left|\frac{\partial\log\{\log\sigma\}}{\partial\log s}-\Lambda(s)\right|\leqslant A(k)$$

$$\left|\frac{\partial\log\{\log\sigma\}}{\partial t}-i\Lambda(s)\right|\leqslant A(k)$$

since such bounds are satisfied by $s\Lambda'(s)$ and $[\Lambda(s)-1]$ (cf. (6.5), (6.17), (6.36), (6.39) and (6.92)). When $w\in\mathcal{D}_2$ the argument is similar; it now depends on the estimate $|\Lambda(s)-2|<A(k)\,(|w|\in J_i,\ w\in\mathcal{D}_2)$.

Definition of the welding function Ω (see Figure 2). Let $\mathcal{N}$ be as in (6.24)–(6.26), (6.37)–(6.39), and let

$$\mathcal{J}=\bigcup_{r^*\in\mathcal{N}}\{s;\ 2^{-2}r^*\leqslant s\leqslant 2^2r^*\}.$$

We define $\gamma_0(s)\,(s\geqslant r_1^*)$ as follows:

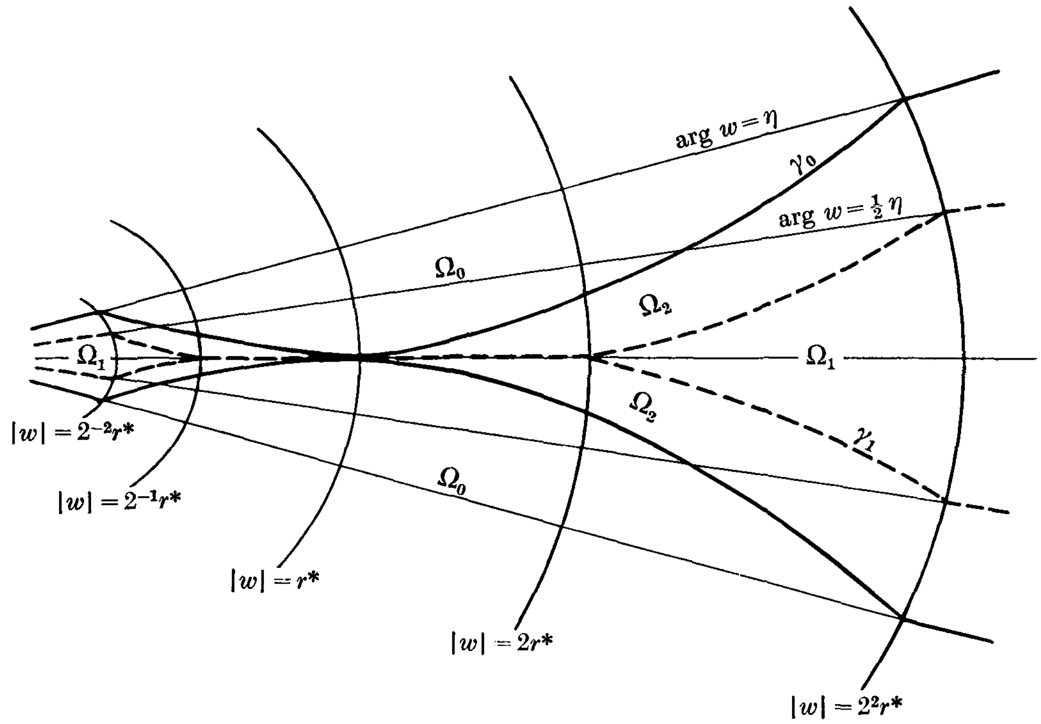

Fig. 2

$$\gamma_0(s) = \begin{cases} \eta & (s \geqslant r_1^*,\ s \notin \mathfrak{J}), \\ \eta \dfrac{|\log(r^*/s)|}{\log(2^2)} & (2^{-2}r^* \leqslant s \leqslant 2^2 r^*,\ r^* \in \mathfrak{N}), \end{cases} \tag{6.104}$$

and

$$\Omega_0 = \{w;\ s \geqslant r_1^*,\ \gamma_0(s) \leqslant t \leqslant 2\pi - \gamma_0(s)\}. \tag{6.105}$$

where r_1^* is the smallest element of $\mathfrak{N}$. Next, let $\gamma_1(s)$ be defined with domain $\{s \geqslant r_1^*\}$ as follows:

$$\gamma_1(s) = \begin{cases} \tfrac{1}{2}\eta & (s \geqslant r_1^*,\ s \notin \mathfrak{J}), \\ \tfrac{1}{2}\eta \dfrac{\log(r^*/2s)}{\log 2} & (2^{-2}r^* \leqslant s \leqslant \tfrac{1}{2}r^*,\ r^* \in \mathfrak{N}), \\ \tfrac{1}{2}\eta \dfrac{\log(s/2r^*)}{\log 2} & (2r^* \leqslant s \leqslant 2^2 r^*,\ r^* \in \mathfrak{N}), \\ 0 & (\tfrac{1}{2}r^* \leqslant s \leqslant 2r^*,\ r^* \in \mathfrak{N}) \end{cases} \tag{6.106}$$

and let

$$\Omega_1 = \left\{ s \geqslant r_1^*,\ \inf_{r^* \in \mathfrak{N}} \left|\log \frac{s}{r}\right| \geqslant \log 2,\ |t| < \gamma_1(s) \right\}, \tag{6.107}$$

$$\Omega_2 = \{\Omega_0 \cup \Omega_1\}' \cap \{|w| \geq r_1^*\} \tag{6.108}$$

($\{\ \}'$ = complement). Note from (6.107) that $\Omega_1 \cap \{\frac{1}{2}r^* < |z| < 2r^*\} = \phi$ if $r^* \in \mathcal{N}$.

We define $\Omega(w)$ for $\{|w| \geq r_1^*\}$ so that Ω is continuous,

$$0 \leq \Omega(w) \leq 1 \qquad (|w| \geq r_1^*), \tag{6.109}$$

$$\Omega(w) = \begin{cases} 0 & (w \in \Omega_0) \\ 1 & (w \in \Omega_1); \end{cases} \tag{6.110}$$

it is easy to see this is possible with

$$\begin{cases} \dfrac{\partial \Omega}{\partial \log s} \leq A\eta^{-1}, \\ \dfrac{\partial \Omega}{\partial t} \leq A\eta^{-1}. \end{cases} \tag{6.111}$$

LEMMA 17. *Let $\Lambda(s)$ be as in* (6.1), (6.95), *let F^* be as in* (6.31)–(6.35), *let $\mathcal{N}$ in* (6.24)–(6.26) *and* (6.37)–(6.39) *be nonempty, and let σ, Ω be as just described. Define $K(w)$ for $\{|w| \geq r_1^*\}$ by*

$$K(w) = \exp\{\Omega(w) \log F^*(w) + [1 - \Omega(w)] \log \sigma(w)\}. \tag{6.112}$$

Then K is continuous and, if h and h_1 in (6.36) *are sufficiently small, quasi-meromorphic in the plane with*

$$|\mu_K(w)| \leq A(k). \tag{6.113}$$

We also have

$$\log K(se^{it}) = -S(s)\, e^{i\Lambda(s)(t-\pi)} = \log \sigma(se^{it}) \qquad (s \geq r_1^*,\ \eta \leq t \leq 2\pi - \eta), \tag{6.114}$$

$$K(r^* e^{it}) = \sigma(r^* e^{it}) \qquad (r^* \in \mathcal{N},\ 0 \leq t \leq 2\pi), \tag{6.115}$$

where $\sigma(r^* e^{it})$ is described in (6.98)–(6.100).

Finally,

$$n(s, a, K) \leq AS(s) \qquad (s > r_1^*) \tag{6.116}$$

for an absolute constant A.

Remarks. 1. In Ω_1, we may write (6.112) more simply as $K(w) = F^*(w)$. Outside Ω_1, the choice of branch of $\log F^*(w)$ is crucial. We take this branch, so that (6.40) holds. This defines a branch of $\log F^*(z)$ in $0 < \arg z < 2\pi$, but according to (6.35), this branch is also single-valued in each annulus $\{2^{-4}r^* \leq |z| \leq 2^4 r^*,\ r^* \in \mathcal{N}\}$.

2. Since K is defined only in $\{|w| > r_1^*\}$, $n(s, a, K)$, $\bar{n}(s, a, K)$ refer to value-distribution in $\{r_1^* < |w| < s\}$.

Proof. Equations (6.114) and (6.115) are obvious since (6.110) and (6.112) assert that $K(w)=\sigma(w)$ on the relevant domains.

To prove (6.113), observe that $K=\sigma$ in Ω_0, so that (6.113) in Ω_0 follows from (6.101). In Ω_1, $K(w)=F^*(w)$, so $\mu_K\equiv 0$ in Ω_1.

Next, suppose $w=se^{it}\in\Omega_2$. Then it is easy to see from (6.40), (6.41), (6.66), (6.98)–(6.100), (6.92) and the computations of Lemma 16 that

$$|\log F^*(w)-\log\sigma|<A(k)S(s),$$
$$|\{\log F^*\}_{\log s}-\{\log\sigma\}_{\log s}|<A(k)S(s),$$
$$|\{\log F^*\}_t-\{\log\sigma\}_t|<A(k)S(s)$$

and from the Cauchy-Riemann conditions that

$$\{\log F^*\}_{\log s}=-i\,\{\log F^*\}_t.$$

Thus if $w\in\Omega_2$,

$$\begin{cases} |\log K(se^{it})+S(s)\,e^{i\Lambda(s)(t-\pi)}|\leqslant A(k)\,S(s),\\ |\{\log K(w)\}_{\log s}+i\,\{\log K(w)\}_t|\leqslant A(k)\,S(s),\\ |\{\log K(w)\}_{\log s}-\Lambda(s)\log K(w)|\leqslant A(k)\,S(s);\end{cases}\tag{6.117}$$

and so (6.109), (6.111), (6.112), (6.117), (6.9) with the definitions (2.3) and (2.6) complete the proof of (6.113). (Remark: the bounds (6.111) show that k must be small in comparison to η, but this is guaranteed by (6.92)). It is obvious that the partials of K satisfy the weak regularity requirements (2.1).

Note from the conventions (6.92) and (6.93) that $A(k)<1$ in (6.113) if h is sufficiently small.

Now consider (6.116). Since $K=F^*$ in Ω_1, (6.70) implies that

$$n(s,a,K,\Omega_1)\leqslant AS(s)\qquad(s\geqslant r_1^*),\tag{6.118}$$

as $\Omega_1\subset\{|w|\geqslant r_1^*\}$ and r_1^* satisfies (6.37). Next let $s_0>r_1^*$ and let

$$\Omega(s_0)=\{\Omega_0\cup\Omega_2\}\cap\{se^{it};\ \tfrac{1}{2}s_0\leqslant s\leqslant s_0,\ \gamma_1(s)<t<2\pi-\gamma_1(s)\}.$$

Note that if $\frac{1}{2}r^*\leqslant s_0\leqslant 2r^*$ for some $r^*\in\mathfrak{N}$, then $\partial\Omega(s_0)$ includes a segment of the positive s-axis, and as w circuits $\partial\Omega(s_0)$ this segment is traversed once in each direction. We apply the first formula of (6.117) on $\partial\Omega(s_0)$ and deduce from (6.46) and the argument principle that

$$n(s_0,a,K,\Omega(s_0))\leqslant AS(s_0)+n(s_0,\infty,\Omega(s_0))\leqslant AS(s_0)\qquad(s_0>r_1^*).\tag{6.119}$$

In general, given $s_0>r_1^*$,

$$\{r_1^* < |w| < s_0\} \subset \bigcup_{n=0}^{N} \left(\Omega\left(\frac{s_0}{2^n}\right)\right),$$

where N is so large that $s_0 \leqslant 2^N r_1^*$. Then (4.9) and (6.119) give

$$n(s_0, a, \Omega_0 \cup \Omega_2) \leqslant \sum_{n=0}^{N} n\left(s, a, K, \Omega\left(\frac{s_0}{2^n}\right)\right) \leqslant A \sum_{n=0}^{N} S\left(\frac{s_0}{2^n}\right) \leqslant As(s_0), \tag{6.120}$$

and (6.118) and (6.120) yield (6.116).

COROLLARY. *Let* $0 \leqslant \alpha \leqslant 1$ *be assigned and suppose the hypotheses of Lemma* 17 *are augmented by*

$$s\Lambda'(s) \to 0 \qquad (s \to \infty) \tag{6.121}$$

and the set $\mathcal{N}$ *of* (6.24)–(6.26) *and* (6.37)–(6.39) *is nonempty and bounded. Then* $K(w)$ *in Lemma* 17 *may be constructed so that in addition*

$$\mu_K(w) \to 0 \qquad (w \to \infty), \tag{6.122}$$

$$|n(s, 0, K) - n^*(s)| \leqslant A(k) S(s) \qquad (s \geqslant r_1^*), \tag{6.123}$$

$$|n(s, 0, K) - n^*(s)| = o(1) S(s) \qquad (s \to \infty), \tag{6.124}$$

$$|\bar{n}(s, 0, K) - \alpha n(s, 0, K)| \leqslant A(k) S(s) \qquad (s \geqslant r_1^*), \tag{6.125}$$

$$|\bar{n}(s, 0, K) - \alpha n(s, 0, K)| = o(1) S(s) \qquad (s \to \infty), \tag{6.126}$$

$$n(s, \infty, K) \leqslant A(k) S(s) \qquad (s \geqslant r_1^*), \tag{6.127}$$

$$n(s, \infty, K) = o(1) S(s) \qquad (s \to \infty). \tag{6.128}$$

If (4.21) *and* (4.22) *also hold with (on account of* (6.1)) $\Lambda^{\#} \leqslant 2\text{–}3h$, *then there exist* $A > 0$, $s' > 0$ *(depending on* $\Lambda(s)$ *and* K*) so that*

$$\log|K(w)| \leqslant -A \sin(\Lambda^{\#} - \tfrac{3}{2}) S(s) \qquad \left(s > s', |t| < \frac{\pi}{2}\left(1 - \frac{3}{2\Lambda^{\#}}\right)\right), \tag{6.129}$$

$$n(s, a, K) - \bar{n}(s, a, K) = O(1) \tag{6.130}$$

with the $O(1)$ *uniform in each region* (6.81).

Proof. The function $\Lambda(s)$ is still assumed to satisfy (6.1) and (6.5).

To compute $\mu_K(w)$ we use the assumption that $\mathcal{N}$ is bounded to see that if M is large, then $K(w) = F^*(w)$ in $\{s > M, |t| < \frac{1}{2}\eta\}$ (cf. (6.106), (6.107), (6.110) and (6.112)). Thus $\mu_K(se^{it}) = 0$ for $s > M$, $|t| < \frac{1}{2}\eta$. For $\{s > M, \frac{1}{2}\eta < t < 2\pi - \frac{1}{2}\eta\}$, formulas (6.98)–(6.100), (6.111) and (6.121) with the computations of (6.102) give (6.122) at once.

The proofs of (6.123)–(6.128) are even easier. In general, $n(s, 0, K) = n(s, 0, F^*, \Omega_1)$, and hence (6.123)–(6.126) follow easily from (6.6), (6.32), (6.34), (6.45), (6.76), (6.87), (6.88) and (6.92). Also, $n(s, \infty, K) = n(s, \infty, F^*, \Omega_1)$ and $\mathcal{N}$ is bounded, so (6.127) and (6.128) are consequences of (6.34), (6.46) and (6.92).

Now let $\Lambda(s)$ satisfy (4.21) and (4.22) with $\Lambda^{\#} \leqslant 2-3h$. It is clear from (6.98)–(6.100) that (6.129) holds in $\{0 < \arg w < 2\pi\} \cap \{\Omega_0 \cup \Omega_2\}$ with σ in place of K, so (6.129) follows from (6.78), (6.79) and definition (6.112). All points of ramification of K are in Ω_1, and so we achieve (6.130) from (6.80) and (6.112).

6.6. Proof of Theorem 4. Let $h(1)$, to be more precisely determined in a moment, satisfy

$$0 < 200\, h(1) < \eta^2, \tag{6.131}$$

where $\eta < (50)^{-1}$ is given in the statement of Theorem 4, and then introduce sequences $h_1(n)$, $h(n)$ with

$$0 < h(n) = 2^{-n} h(1), \; h_1(n) = h(n)^{1/4}, \tag{6.132}$$

(compare with (6.36)). We take $h(1)$ so small that

$$A(k) < \eta, \tag{6.133}$$

which is possible from the conventions (6.92), (6.93). In particular, this means that any function $K(w)$ chosen in accord with Lemma 17 will have

$$|\mu_K(w)| < \eta \qquad (|w| > r_1^*). \tag{6.134}$$

According to Lemma 11, constants $\tau(n)\,(<(2\pi)^{-1})$, $\nu(n)$, $(<\tfrac{1}{2})$ and $K(n)\,(>2^{10})$ may be associated to each pair $\{h(n), h_1(n)\}$ so that (6.7) follows from (6.1)–(6.6). The constants M^{∞} and τ_0 required in the statement of Theorem 4 are then given by

$$M^{\infty} = 10 \cdot 2^4 K(1)\, \nu(1)^{-2} h(1)^{-6} (> 2^{40}) \tag{6.135}$$

and $\tau_0 = \tau_0(1) \leqslant \tau(1)$ so small that

$$h(1)\tau_0^{-1} > 2 \log M^{\infty}. \tag{6.136}$$

In addition choose $\tau_0(n) \leqslant \tau(n)\,(n \geqslant 2)$ so that

$$h(n)\tau_0(n)^{-1} \geqslant 4 \log\{2^6 K(n)\} \tag{6.137}$$

and finally $r_0(n)\,(n \geqslant 1)$ so large that we have

$$|r\Lambda'(r)| < \tau_0(n) \qquad (r > r_0(n)) \tag{6.138}$$

and

$$r_0(n) \geqslant [10\nu(n)^{-2}h_1(n)^{-5}]^2, \tag{6.139}$$

where it is assumed that

$$K(n)r_0(n) < K(n+1)r_0(n+1) \qquad (n \geqslant 2). \tag{6.140}$$

Now let $\varepsilon(s)$ be a decreasing function with

$$2h(1) \geqslant \varepsilon(s) \geqslant h(1), \qquad (0 < s < 2^{22}K(2)r_0(2)^2), \tag{6.141}$$

$$2h(n) \geqslant \varepsilon(s) \geqslant h(n) \qquad (s \geqslant 2^{22}K(n)^2 r_0(n)^2,\ n \geqslant 2) \tag{6.142}$$

(consistent on account of (6.132) and (6.140)) and let

$$\mathfrak{L} = \{s;\ \sin \pi\Lambda(s) = 5\pi\varepsilon(s)\}; \tag{6.143}$$

note from (4.10) that $\mathfrak{L} \subset \{s \geqslant M^\infty\}$. Let

$$I_0 = \bigcup_{i=1}^{M} (\alpha_i, \alpha_i') \qquad (\alpha_i' < \alpha_{i+1},\ \alpha_i,\ \alpha_i' \in \mathfrak{L},\ M \leqslant \infty) \tag{6.144}$$

be a disjoint union of intervals maximal with respect to the property that in each full interval (α_i, α_i')

$$1 + 4\varepsilon(s) \leqslant \Lambda(s) \leqslant 2 - 4\varepsilon(s) \qquad (\alpha_i \leqslant s \leqslant \alpha_i'), \tag{6.145}$$

while for each i there exists a_i such that

$$1 + 6\varepsilon(s) \leqslant \Lambda(a_i) \leqslant 2 - 6\varepsilon(s) \qquad (\alpha_i < a_i < \alpha_i'). \tag{6.146}$$

The complementary intervals $(\alpha_i', \alpha_{i+1})$ are assigned to I_1 or I_2 by the rule

$$(\alpha_i', \alpha_{i+1}) \in I_m \quad \text{if} \quad |\Lambda(s) - m| < 6\varepsilon(s) \qquad (\alpha_i' \leqslant s \leqslant \alpha_{i+1}),\ (m = 1, 2,\ i = 1, 2, \ldots). \tag{6.147}$$

We allow the possibility that some α_i or $\alpha_i' = \infty$; i.e. $M < \infty$ in (6.144).

It is easy to define H for those w having $|w| \in I_1 \cup I_2$:

$$\log H(se^{it}) = -S(s)\, e^{i\Lambda(s)(t-\pi)} \qquad (\tfrac{1}{2}\eta < t < 2\pi - \tfrac{1}{2}\eta,\ s \in I_1 \cup I_2) \tag{6.148}$$

$$\log H(se^{it}) = -S(s)\, e^{i\Lambda(s)(t-\pi) - 2\pi i \eta^{-1}[\Lambda(s)-1](t-(1/2)\eta)} \qquad (|t| < \tfrac{1}{2}\eta,\ s \in I_1), \tag{6.149}$$

$$\log H(se^{it}) = -S(s)\, e^{i\Lambda(s)(t-\pi) + 2\pi i \eta^{-1}[2-\Lambda(s)](t-(1/2)\eta)} \qquad (|t| < \tfrac{1}{2}\eta,\ s \in I_2), \tag{6.150}$$

(as in (6.98)–(6.100) it must be checked that H is well-defined). We observe that H satisfies (2.1) at each interior point of $I_1 \cup I_2$.

That

$$|\mu_H(w)| < \eta \qquad (|w| \in I_1 \cup I_2), \tag{6.151}$$

$$|\mu_H(w)| \to 0 \qquad (w \to \infty,\ |w| \in I_1 \cup I_2), \tag{6.152}$$

follows from (4.6), (6.147)–(6.150) and computations of the nature used in (6.102), (6.103); that $\Lambda(s)-m\to 0$ as $s\to\infty$ in I_m is crucial here as we saw in (6.103).

To define H in the rest of the plane requires Lemmas 15 and 17. Let $\{N(n)\}$ ($n=1, 2, \ldots$) be an increasing sequence such that

$$\alpha_i \geqslant 2^{12} K(n)^2 r_0(n)^2 \qquad (i \geqslant N(n)). \tag{6.153}$$

Then for each $n(\geqslant 1)$ we introduce a differentiable function $\Lambda_n(s)\,(s>0)$ having

$$1+3h(n) \leqslant \Lambda_n(s) \leqslant 2-3h(n), \tag{6.154}$$

$$s\,|\Lambda_n'(s)| < \tau_0(n) \qquad (s>0) \tag{6.155}$$

and sequences $\{\beta_i\}$, $\{\beta_i'\}$ ($N(n)\leqslant i<N(n+1)$) with

$$\ldots\,\beta_i<\beta_i'<\beta_{i+1}<\ldots\;(N(n)\leqslant i<N(n+1)). \tag{6.156}$$

We first require that

$$\beta_i'/\beta_i=\alpha_i'/\alpha_i \qquad (N(n)\leqslant i<N(n+1)) \tag{6.157}$$

and we define $\Lambda_n(s)$ on $\beta_i\leqslant s\leqslant \beta_i'$ by

$$\Lambda_n(s)=\Lambda\left(\frac{\alpha_i}{\beta_i}s\right) \qquad (\beta_i\leqslant s\leqslant\beta_i',\ N(n)\leqslant i<N(n+1)). \tag{6.158}$$

The choice of the $\{\beta_i\}$ and the definition of Λ_n for the remaining s is made so that (6.154), (6.155) and (6.157) hold and in addition

$$\int_1^{\beta_i}\Lambda_n(u)\,u^{-1}du=\int_1^{\alpha_i}\Lambda(u)\,u^{-1}du \qquad (N(n)\leqslant i<N(n+1)). \tag{6.159}$$

Note from (6.154) and the bound $1\leqslant\Lambda(s)\leqslant 2$ in (4.4) that (6.153) and (6.159) give as a lower bound

$$\beta_{N(n)}\geqslant\alpha_{N(n)}^{1/2}\geqslant 2^6K(n)\,r_0(n). \tag{6.160}$$

It is easy to construct such differentiable functions $\Lambda_n(s)$, and (6.156) follows from (6.157), (6.158) and the analogous properties of the $\{\alpha_i\}$, $\{\alpha_i'\}$ in (6.144). It is important to note that (6.155) follows from (6.160), (6.153) and (6.138). In the spirit of (4.7), let

$$S_n(s)=\exp\left\{\int_1^s\Lambda_n(u)\,u^{-1}du\right\} \qquad (s>0) \tag{6.161}$$

and let

$$\mathcal{N}(n)=\{\beta_i,\beta_i';\,N(n)\leqslant i<N(n+1)\}. \tag{6.162}$$

We want to apply Lemma 17 to each pair $\Lambda_n(s)$, $\mathcal{N}(n)$, so it must be be checked that the relevant hypotheses are satisfied. Property (6.154) of $\Lambda_n(s)$ is the exact analogue of (6.1), and (6.155) is (6.5). Clearly (6.2) also follows from (6.155).

Next we check that $\mathcal{N}(n)$, defined in (6.162), satisfies (6.24), (6.25) and (6.37)–(6.39). Of course, (6.24) and (6.25) are now consequences of (6.37)–(6.39), and in obtaining (6.160) we have already checked (6.37).

The construction of I_0 (cf. (6.141)–(6.144)) produces a_i with $\alpha_i < a_i < \alpha_i'$ and

$$|\Lambda(a_i) - \Lambda(\alpha_i)| \geq \tfrac{1}{2}h(n) \qquad (N(n) \leq i < N(n+1)). \tag{6.163}$$

We obtain from (6.163) with (6.155), (6.157)–(6.159) that

$$\tfrac{1}{2}h(n) \leq \left|\int_{\alpha_i}^{a_i} \Lambda'(u)\,du\right| = \left|\int_{\beta_i}^{a_i\beta_i/\alpha_i} \Lambda_n'(u)\,du\right| \leq \tau_0(n)\log\frac{a_i}{\alpha_i}$$
$$\leq \tau_0(n)\log\frac{\alpha_i'}{\alpha_i} \qquad (N(n) \leq i < N(n+1))$$

and these reasons with (6.137) and (6.157) show that

$$\frac{\beta_i'}{\beta_i} = \frac{\alpha_i'}{\alpha_i} > e^{(1/2)h(n)\tau_0(n)^{-1}} \geq 2^{11}K(n)^2 \qquad (N(n) \leq i < N(n+1)), \tag{6.164}$$

which shows that β_i'/β_i satisfies (6.39).

We next consider β_{i+1}/β_i'. According to (6.157)–(6.159),

$$\int_{\beta_i'}^{\beta_{i+1}} \Lambda_n(u)\,u^{-1}du = \int_{\alpha_i'}^{\alpha_{i+1}} \Lambda(u)\,u^{-1}du, \qquad (N(n) \leq i < N(n+1))$$

so reasoning as in (6.160), we deduce that

$$\log\frac{\beta_{i+1}}{\beta_i'} > \tfrac{1}{2}\log\frac{\alpha_{i+1}}{\alpha_i'}.$$

However, the maximality of the (α_i, α_i') in (6.143)–(6.146) shows there must exist $b_i \in (\alpha_i', \alpha_{i+1})$ with $\Lambda(b_i) > 2 - 4\varepsilon(b_i)$ or $\Lambda(b_i) < 1 + 4\varepsilon(b_i)$; thus (6.141) and (6.142) show that $|\Lambda(\alpha_i') - \Lambda(b_i)| \geq \tfrac{1}{2}h(n)$. As in (6.164) we obtain

$$\beta_{i+1}/\beta_i' > 2^{11}K(n)^2 \qquad (N(n) \leq i < N(n+1)) \tag{6.165}$$

and (6.38) follows from (6.164) and (6.165). Finally, since $\alpha_i, \alpha_i' \in \mathcal{L}$ (cf. (6.141), (6.142)), (6.158) shows that

$$|\sin \pi\Lambda_n(\beta_i)| \leq 10\pi h(n); \ |\sin \pi\Lambda_n(\beta_i')| \leq 10\pi h(n) \qquad (N(n) \leq i < N(n+1)),$$

which gives (6.39).

Now (6.135) and (6.153) allow Lemma 15 to be applied. We get $F^*(z) = F_n^*(z)$ in accord with (6.70), (6.76), (6.77), (6.83) and (6.86), and then, with $\mathcal{N} = \mathcal{N}(n)$, of (6.162), Lemma 17 constructs $K_n(w)$ for $\{|w| \geq \beta_{N(n)}\}$. Since $\varepsilon(s) \to 0$, (6.143) and (6.154) show that each $\mathcal{N}(n)$ is bounded. Further, $h(1)$ has been chosen to ensure (6.133) and $h(n)$ satisfies (6.132), so it is clear from (6.113) and (6.122) that

$$|\mu_{K_n}(w)| < \eta \qquad (|w| \geq \beta_{N(n)},\ n \geq 1), \tag{6.166}$$

$$\max_{|w| \geq \beta_{N(n)}} |\mu_{K_n}(w)| = o(1) \qquad (n \to \infty). \tag{6.167}$$

We then complement (6.148)–(6.150) by

$$H(w) = K_n\left(\frac{\beta_i}{\alpha_i} w\right) \qquad (|w| \in I_0,\ \alpha_i \leq |w| \leq \alpha_i',\ N(n) \leq i < N(n+1)). \tag{6.168}$$

It is clear that H satisfies (2.1), but it must be checked that H is continuous. The definitions (4.7) and (6.161) with (6.157) and (6.159) give

$$\begin{aligned} S_n(s) &= S_n(\beta_i) \exp\left\{\int_{\beta_i}^{s} \Lambda_n(u) u^{-1} du\right\} \\ &= S(\alpha_i) \exp\left\{\int_{\alpha_i}^{s\alpha_i/\beta_i} \Lambda(u) u^{-1} du\right\} = S\left(\frac{\alpha_i}{\beta_i} s\right) \qquad (\beta_i \leq s \leq \beta_i',\ n(N) \leq i < N(n+1)). \end{aligned} \tag{6.169}$$

Since $\mathcal{N}(n)$ satisfies (6.24), (6.25) and (6.37)–(6.39), it readily follows from (6.158), (6.169) and a comparison of (6.115) and (6.98)–(6.100) with (6.148)–(6.150) that

$$\log K_n(\alpha_i e^{it}) = \log H(\beta_i e^{it}) \qquad (0 \leq t \leq 2\pi,\ N(n) \leq i < N(n+1))$$

$$\log K_n(\alpha_i' e^{it}) = \log H(\beta_i' e^{it}) \qquad (0 \leq t \leq 2\pi,\ N(n) \leq i < N(n+1)).$$

Thus H is quasi-meromorphic in the plane and (6.151), (6.166) and (6.168) yield (4.12).

It is also clear from the explicit formulas (6.98)–(6.100) and (6.112) (when $|w| \in I_0$) and (6.148) (when $|w| \in I_1 \cup I_2$) that (4.14) holds. Similarly, whenever $\Lambda(s) = m$ $(m = 1, 2)$ in (4.15), our construction ensures that $s \in I_1 \cup I_2$, and (4.16) is a direct consequence of (6.148)–(6.150).

Next, we prove (4.13). The explicit formulas (6.148)–(6.150) show that $\mu_H(w) \to 0$ as $|w| \to \infty$ in $I_1 \cup I_2$. If I_0 is unbounded there are two cases to consider in terms of the decomposition (6.144): $M = \infty$ or $M < \infty$. If $M = \infty$, then (4.13) follows from (6.167) and (6.168); if $M < \infty$, then H is given by (6.168) for all large w with some fixed n, and so then (6.122) gives (4.13).

The proof of (4.17) follows from the argument principle by elementary modifications of the argument used in the proof of (6.116) in Lemma 17.

It is clear that (4.18) holds. Indeed (6.127) and (6.128) apply to each K_n, and (6.148)–(6.150) show that all poles of $H(w)$ occur when $|w| \in I_0$. Thus if $M < \infty$ in (6.144), (6.128) implies (4.18), and if $M = \infty$, (4.18) follows from (6.127). The proofs of (4.19) and (4.20) are of the same nature, since the zeros of $H(w)$ only arise with $|w| \in I_0$. Thus if $s \to \infty$ in $I_1 \cup I_2$, (6.147) gives (4.19), (4.20). When $s \to \infty$ in I_0, the same conclusions follow from (6.124) and (6.126) (when $M < \infty$ in (6.144)) and (6.123) and (6.125) otherwise.

Now suppose (4.21) and (4.22) hold. If $\Lambda^{\#} = 2$, then K is given by (6.148)–(6.150) for all large w, and (4.23) is immediate from these explicit formulas. If $\Lambda^{\#} < 2$, the conditions (6.132), (6.142) show that $\mathcal{C}$ in (6.143) is a finite set. Thus I_0 contains all large s, so $M < \infty$ in (6.144). In this case, (4.23) is a direct consequence of (6.129), (6.168) and (6.169).

The proof of (4.24) subject to (4.25) is similar. If $\Lambda^{\#} = 2$, then H has only a finite number of multiple values. Otherwise, $M < \infty$ in (6.144), and (6.130) subject to (6.81) provides the needed information.

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