# Connes' bicentralizer problem and uniqueness of the injective factor of type $\mathrm{III}_1$

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## Introduction

In Connes' fundamental work "Classification of injective factors" [7], it is proved that injective factors of type $\mathrm{III}_\lambda$, $\lambda \neq 1$ on a separable Hilbert space are completely classified by their "smooth flow of weights". Since the flow of weights of factors of type $\mathrm{III}_1$ is trivial, one would expect that there is only one isomorphism class of injective factors of type $\mathrm{III}_1$. During the years 1976–78, Connes spent much effort to prove that there is only one injective factor of type $\mathrm{III}_1$, and found a number of conditions for an injective factor of type $\mathrm{III}_1$ to be isomorphic to the Araki-Woods' factor $R_\infty$. One of these conditions is the following:

Let $\varphi$ be a normal faithful state on a von Neumann algebra $M$, and let the *bicentralizer* of $\varphi$ be the set $B_\varphi$ of operators $a$ in $M$ for which

$$x_n a - a x_n \to 0 \quad (\sigma\text{-strongly})$$

whenever $(x_n)$ is a bounded sequence in $M$ satisfying $\lim_{n\to\infty} \|x_n \varphi - \varphi x_n\| = 0$. Connes proved that if an injective factor of type $\mathrm{III}_1$ with separable predual has a normal faithful state $\varphi$ for which $B_\varphi = \mathbf{C}1$, then $M$ is isomorphic to the Araki-Woods factor $R_\infty$. In particular, if $M$ has a normal faithful state $\varphi$, such that $M'_\varphi \cap M = \mathbf{C}1$, then $M \cong R_\infty$.

In this paper we provide the last step in the proof of uniqueness of the injective factor of type $\mathrm{III}_1$ by proving that *every* injective factor of type $\mathrm{III}_1$ has a normal faithful state $\varphi$, such that $B_\varphi = \mathbf{C}1$.

The starting point in our proof is the Connes-Takesaki relative commutant theorem for dominant weights (cf. [13, Section 2]): For every dominant weight $\psi$ on a $\mathrm{III}_1$-factor with separable predual

$$M'_\psi \cap M = \mathbf{C}1.$$

If $M$ is an injective factor of type $\mathrm{III}_1$, then the centralizer $M_\psi$ is the hyperfinite $\mathrm{II}_\infty$-factor. In particular, $M_\psi$ has Schwartz' property $P$, so in this case $M'_\psi \cap M = \mathbf{C}1$ implies that for every $x \in M$:

$$\overline{\mathrm{conv}}\,\{uxu^* \mid u \in U(M_\psi)\} \cap \mathbf{C}1 \neq \emptyset \tag{*}$$

where the closure is taken in the $\sigma$-weak topology. Let now $\varphi$ be a normal faithful state on $M$, and let $\mathscr{H}$ be an infinite dimensional separable Hilbert space. By approximating the weight $\varphi \otimes \mathrm{Tr}$ on $M \bar{\otimes} B(\mathscr{H})$ with dominant weights, we obtain from (*) that for every $x \in M \setminus \{0\}$ with $\varphi(x)=0$ and for every $\delta>0$, there exists a sequence $(a_i)_{i=1}^\infty$ of operators in $M$ such that

(i) $\mathrm{sp}_{\sigma^\varphi}(a_i) \subseteq [-\delta, \delta]$ for all $i \in \mathbf{N}$

(ii) $\sum_{i=1}^\infty a_i^* a_i = 1$

(iii) $\sum_{i=1}^\infty \|a_i x - x a_i\|_\varphi^2 \geq \frac{1}{2}\|x\|_\varphi^2$

(cf. Lemma 2.7). These three conditions imply intuitively that "$x \notin B_\varphi$", because the $a_i$'s almost commute with $\varphi$, while some of the $a_i$'s must be far from commuting with $x$. However, we have only little control over the operator norm of the $a_i$'s relative to the size of $\|a_i x - x a_i\|_\varphi$, and it is actually necessary to make a very long detour in order to prove that $x \notin B_\varphi$. This detour occupies the main part of Section 2 and it is strongly inspired by the techniques in Connes' and Størmer's proof of the homogeneity of the state space of $\mathrm{III}_1$-factors (cf. [12]). Once we know that ($\varphi(x)=0$ and $x \neq 0$) $\Rightarrow x \notin B_\varphi$, it follows immediately that $B_\varphi = \mathbf{C}1$. The details in Connes' proof of

$$[M \text{ injective } \mathrm{III}_1\text{-factor and } B_\varphi = \mathbf{C}1] \Rightarrow [M \cong R_\infty]$$

has appeared very recently in [10]. We have checked independently that the above implication can also be proved using the ideas of [16, Sections 3, 4 and 5]. Our proof is quite long and will be presented elsewhere.

In the last section (Section 3) of this paper we prove that for a general $\mathrm{III}_1$-factor $M$ with separable predual, the following three conditions are equivalent:

(1) For every (faithful) dominant weight $\psi$ on $M$ and every $x \in M$

$$\overline{\mathrm{conv}}\,\{uxu^*|u\in U(M_\psi)\}\cap \mathbf{C}1\neq\emptyset$$

($\sigma$-weak closure).

(2) For every normal faithful state $\varphi$ on $M$, $B_\varphi=\mathbf{C}1$.

(3) The set of normal faithful states $\varphi$ on $M$, for which $M'_\varphi\cap M=\mathbf{C}1$ is norm dense in the set of normal states on $M$.

We have not been able to decide whether these conditions are fulfilled for all $\mathrm{III}_1$-factors with separable predual.

The rest of the paper is organized in the following way:



## 1. Preliminaries on Connes' bicentralizer problem

The material presented in this section has been known to Connes since 1976–78. I learned about it during a number of conversations with Connes in May 1978 and November 1978. As mentioned in the introduction, Connes defined the bicentralizer of a normal faithful state $\varphi$ on a von Neumann algebra $M$ to be the set of operators $a\in M$ for which

$$x_n a-ax_n\to 0 \quad (\sigma\text{-strongly})$$

whenever $(x_n)_{n\in\mathbf{N}}$ is a bounded sequence in $M$ for which $\lim_{n\to\infty}\|x_n\varphi-\varphi x_n\|=0$. Connes proved that if $M$ is a $\mathrm{III}_1$-factor, and $B_\varphi=\mathbf{C}1$ for one n.f. (normal faithful) state $\varphi$ on $M$, then $B_\varphi=\mathbf{C}1$ for all normal faithful states on $M$. From this it follows that $B_\varphi=\mathbf{C}1$ for all n.f. states on the Araki-Woods factor $R_\infty$ (cf. Corollary 1.5 and Example 1.6 below). He conjectured that $B_\varphi=\mathbf{C}1$ for some (and hence for every) n.f. state $\varphi$ on any $\mathrm{III}_1$-factor $M$. Connes' interest in this problem lies in the fact that he was able to prove:

THEOREM 1.1 (Connes [8], [10]). *Let $M$ be an injective $\mathrm{III}_1$-factor with separable predual. If $M$ admits a normal faithful state $\varphi$ for which $B_\varphi=\mathbf{C}1$, then $M$ is isomorphic to the Araki-Woods factor $R_\infty$.*

The above theorem was announced in the end of Connes' survey paper [9] in a slightly different formulation. A detailed proof appeared very recently in [10]. In the rest of this section we present some basic properties of the bicentralizer $B_\varphi$, which will be needed in the following sections.

For any unital $C^*$-algebra we let $U(A)$ denote the unitary group of $A$.

LEMMA 1.2. *Let $M$ be a von Neumann algebra with a normal faithful state $\varphi$. For $A \in M$, put*

$$C_\varphi(a, \delta) = \overline{\operatorname{conv}}\{u^*au \mid u \in U(M),\ \|u\varphi - \varphi u\| \leq \delta\}$$

*where $\overline{\operatorname{conv}}\{\cdot\}$ is the closure of the convex hull in the $\sigma$-weak topology. Then*

$$a \in B_\varphi \Leftrightarrow \bigcap_{\delta>o} C_\varphi(a, \delta) = \{a\}.$$

*Proof.* For $x \in M$, put $\|x\|_\varphi = \varphi(x^*x)^{1/2}$. Then $\|\ \|_\varphi$ is a norm on $M$ and it generates the $\sigma$-strong topology on bounded sets of $M$. Put

$$\mathscr{A} = \{(x_n) \in l^\infty(\mathrm{N}, M) \mid \lim_{n\to\infty} \|x_n\varphi - \varphi x_n\| = 0\}.$$

Then $\mathscr{A}$ is a unital $C^*$-algebra. Therefore $\mathscr{A}$ is spanned by $U(\mathscr{A})$. Note that $U(\mathscr{A})$ consists of those sequences $(u_n)_{n\in\mathrm{N}}$ of unitaries in $M$ for which $\|u_n\varphi - \varphi u_n\| \to 0$ for $n \to \infty$.

Thus

$$B_\varphi = \{a \in M \mid \lim_{n\to\infty} \|u_n a - a u_n\|_\varphi = 0 \text{ for all } (u_n) \in U(\mathscr{A})\}.$$

Equivalently

$$B_\varphi = \{a \in M \mid \lim_{n\to\infty} \|a - u_n^* a u_n\|_\varphi = 0 \text{ for all } (u_n) \in U(\mathscr{A})\}. \tag{*}$$

The last equality (*) follows, because the $\varphi$-norm is invariant under multiplication from left with unitary operators from $M$. For $a \in M$, and $\delta > 0$ put

$$\varepsilon(a, \delta) = \sup\{\|u^*au - a\|_\varphi \mid u \in U(M),\ \|u\varphi - \varphi u\| \leq \delta\}.$$

Since $\|x\|_\varphi = \sup\{\varphi(y^*x) \mid y \in M, \|y\|_\varphi \leq 1\}$, the $\varphi$-norm is lower semi continuous in the $\sigma$-weak topology on $M$. Therefore

$$\|x - a\|_\varphi \leq \varepsilon(a, \delta) \quad \text{for every } x \in C_\varphi(a, \delta).$$

By (*) we have

$$a \in B_\varphi \Leftrightarrow \lim_{\delta\to 0} \varepsilon(a,\delta) = 0$$

Hence $\bigcap_{\delta>0} C_\varphi(a,\delta) = \{a\}$ for all $a \in B_\varphi$.

Conversely, if $a \notin B_\varphi$, then we can choose a sequence $(u_n)$ of unitaries in $M$ such that

$$\lim_{n\to\infty} \|u_n \varphi - \varphi u_n\| = 0$$

while

$$\limsup_{n\to\infty} \|u_n^* a u_n - a\|_\varphi > 0.$$

By passing to a subsequence, we can even obtain that there exists an $\varepsilon>0$, such that

$$\|u_n^* a u_n - a\|_\varphi \geqslant \varepsilon \quad \text{for all } n.$$

Let $b$ be a cluster point for the sequence $\{u_n^* a u_n | n \in \mathbf{N}\}$ in the $\sigma$-weak topology. Clearly $b \in \bigcap_{\delta>0} C_\varphi(a,\delta)$. We will prove that $b \neq a$. Note first that

$$\lim_{n\to\infty} \|u_n^* a u_n\|_\varphi^2 = \lim_{n\to\infty} \varphi(u_n^* a^* a u_n) = \varphi(a^* a) = \|a\|_\varphi^2$$

because $\|u_n \varphi u_n^* - \varphi\| \to 0$ for $n \to \infty$. Using

$$2\,\mathrm{Re}\,\varphi(a^* u_n^* a u_n) = \|a\|_\varphi^2 + \|u_n^* a u_n\|_\varphi^2 - \|a - u_n^* a u_n\|_\varphi^2$$

we get in the limit $n \to \infty$

$$\varphi(a^* b) \leqslant \|a\|_\varphi^2 - \tfrac{1}{2}\varepsilon^2 = \varphi(a^* a) - \tfrac{1}{2}\varepsilon^2$$

Hence $b \neq a$. This completes the proof of Lemma 1.2.

PROPOSITION 1.3 [8]. *Let $M$ be a von Neumann algebra with a normal faithful state $\varphi$. Then*

(1) *$B_\varphi$ is a von Neumann subalgebra of $M$.*

(2) *The following two conditions are equivalent:*

(a) $B_\varphi = \mathbf{C}1$

(b) *For every $a \in M$ and every $\delta > 0$*

$$\overline{\text{conv}}\,\{u^* a u \mid u \in U(M), \|u\varphi - \varphi u\| \leq \delta\} \cap \mathbf{C}1 \neq \emptyset$$

(closure in $\sigma$-weak topology).

*Proof.* (1) It is clear that $B_\varphi$ is a unital subalgebra of $M$. Moreover, by Lemma 1.2,

$$a \in B_\varphi \quad \Leftrightarrow \bigcap_{\delta>0} C_\varphi(a, \delta) = \{a\} \quad \Leftrightarrow \bigcap_{\delta>0} C_\varphi(a^*, \delta) = \{a^*\} \quad \Leftrightarrow \quad a^* \in B_\varphi.$$

It remains to be proved that $B_\varphi$ is $\sigma$-strongly closed. Let $a \in \bar{B}_\varphi^{\sigma-s}$, and let $u_n$ be a sequence of unitaries in $M$, such that

$$\|u_n \varphi - \varphi u_n\| \to 0 \quad \text{for } n \to \infty.$$

For every $\varepsilon > 0$, we can choose $b \in B_\varphi$, such that $\|a-b\|_\varphi < \varepsilon$. Then

$$\|u_n^* (a-b) u_n\|_\varphi^2 = \varphi(u_n^* (a-b)^* (a-b) u_n) \to \|a-b\|_\varphi^2 \quad \text{for } n \to \infty$$

because $\|u_n \varphi u_n^* - \varphi\| \to 0$ for $n \to \infty$. Using

$$\|u_n^* a u_n - a\|_\varphi \leq \|u_n^* (a-b) u_n\|_\varphi + \|u_n^* b u_n - b\|_\varphi + \|b-a\|_\varphi$$

we get

$$\limsup_{n\to\infty} \|u_n^* a u_n - a\|_\varphi \leq 2\|a-b\|_\varphi < 2\varepsilon.$$

Since $\varepsilon$ was arbitrary, it follows that $a \in B_\varphi$.

(2) (a) $\Rightarrow$ (b): Let $a \in M$. The set

$$C_\varphi(a) = \bigcap_{\delta>0} C_\varphi(a, \delta)$$

is a $\sigma$-weakly compact convex subset of $M$, and it is non-empty because it contains $a$. Let $\mathcal{H}_\varphi$ be the completion of $M$ with respect to the $\varphi$-norm. Then $C_\varphi(a)$ is a norm closed convex subset of $\mathcal{H}_\varphi$. Since $\mathcal{H}_\varphi$ is a Hilbert space, there exists $b \in C_\varphi(a)$, such that

$$\|x\|_\varphi > \|b\|_\varphi \quad \text{for all } x \in C_\varphi(a) \setminus \{b\}.$$

We will show that $b \in B_\varphi$. If $u$, $v$ are two unitary operators in $M$, then

$$\begin{aligned} \|(uv)\varphi - \varphi(uv)\| &\leq \|(u\varphi - \varphi u) v\| + \|u(v\varphi - \varphi v)\| \\ &= \|u\varphi - \varphi u\| + \|v\varphi - \varphi v\|. \end{aligned}$$

From this it follows easily that if $a' \in C_\varphi(a, \delta)$ then

$$C_\varphi(a', \delta) \subseteq C_\varphi(a, 2\delta), \quad \delta < 0.$$

Since $b \in \bigcap_{\delta>0} C_\varphi(a, \delta)$ it follows that for all $\delta>0$

$$C_\varphi(b) = \bigcap_{\delta>0} C_\varphi(b, \delta) \subseteq \bigcap_{\delta>0} C_\varphi(a, 2\delta) = C_\varphi(a).$$

If $u \in U(M)$, and $\|u\varphi-\varphi u\| \leq \delta$, then

$$\begin{aligned} \|u^*bu\|_\varphi^2 &= \varphi(u^*b^*bu) \\ &= \varphi(b^*b)+(u\varphi u^*-\varphi)(b^*b) \\ &\leq \|b\|_\varphi^2+\|u\varphi u^*-\varphi\|\,\|b\|^2 \\ &\leq \|b\|_\varphi^2+\delta\|b\|^2. \end{aligned}$$

Using the lower semi continuity of $\|\ \|_\varphi$ in the $\sigma$-weak topology we get

$$\|x\|_\varphi^2 \leq \|b\|_\varphi^2+\delta\|b\|^2 \quad \text{for all } x \in C_\varphi(b, \delta)$$

and consequently

$$\|x\|_\varphi \leq \|b\|_\varphi \quad \text{for all } x \in C_\varphi(b).$$

Since $C_\varphi(b) \subseteq C_\varphi(a)$, this inequality implies that $x=b$. Hence $C_\varphi(b)=\{b\}$ so by Lemma 1.2, $b \in B_\varphi$. Therefore (a) implies that

$$b \in C_\varphi(a, \delta) \cap \mathbf{C}1$$

for all $\delta>0$. Thus (a)$\Rightarrow$(b).

(b)$\Rightarrow$(a): Assume (b) and let $a \in M$. Since the sets

$$C_\varphi(a, \delta) \cap \mathbf{C}1$$

form a decreasing family of non-empty $\sigma$-weakly compact sets, they have a non-empty intersection. Hence there exists $\lambda \in \mathbf{C}$, such that

$$\lambda 1 \in \bigcap_{\delta>0} C_\varphi(a, \delta).$$

If $u \in U(M)$, and $\|u\varphi-\varphi u\| \leq \delta$, then

$$|\varphi(u^*au)-\varphi(a)|\leq \|u\varphi u^*-\varphi\|\,\|a\|$$
$$\leq \delta\|a\|.$$

Hence $|\varphi(x)-\varphi(a)|\leq\delta\|a\|$ for all $x\in C_\varphi(a, \delta)$ and all $\delta>0$. Therefore $\lambda=\varphi(a)$, i.e.

$$\varphi(a)\,1\in\overline{\mathrm{conv}}\,\{u^*au|u\in U(M),\ \|u\varphi-\varphi u\|\leq\delta\}$$

for all $\delta>0$. Equivalently

$$a-\varphi(a)1\in\overline{\mathrm{conv}}\{a-u^*au|u\in U(M),\ \|u\varphi-\varphi u\|\leq\delta\}.$$

Using that the $\|\ \|_\varphi$-norm is lower semi continuous in the $\sigma$-weak topology, we get

$$\|a-\varphi(a)\,1\|_\varphi\leq\sup\{\|a-u^*au\|_\varphi|u\in U(M),\ \|u\varphi-\varphi u\|\leq\delta\}.$$

If $a\in B_\varphi$, the supremum goes to zero for $\delta\to 0$. Hence $a=\varphi(a)\,1$. This proves (a).

*Remark* 1.4. By the proof of (b) $\Rightarrow$ (a) it follows that $B_\varphi=\mathbf{C}1$ is also equivalent to

(c) For all $a\in M$

$$\varphi(a)1\in\bigcap_{\delta>0}\overline{\mathrm{conv}}\,\{u^*au|u\in U(M),\ \|u\varphi-\varphi u\|\leq\delta\}$$

(closure in $\sigma$-weak topology). Moreover, a simple duality argument shows that this condition is again equivalent to

(d) For all $\psi\in M_*$,

$$\psi(1)\varphi\in\bigcap_{\delta>0}\overline{\mathrm{conv}}\,\{u\psi u^*|u\in U(M),\ \|u\varphi-\varphi u\|\leq\delta\}$$

(closure in norm topology).

COROLLARY 1.5 [8]. *Let $M$ be a $\sigma$-finite factor of type* $\mathrm{III}_1$. *If $B_\varphi=\mathbf{C}1$ for some* n.f. *state $\varphi$ on $M$, then $B_\omega=\mathbf{C}1$ for all* n.f. *states $\omega$ on $M$.*

*Proof.* Assume that $B_\varphi=\mathbf{C}1$, for some n.f. state $\varphi$ on $M$. Then, by the Connes-Størmer transitivity theorem [12], the set of n.f. states $\omega$ on $M$ for which $B_\omega=\mathbf{C}1$ is norm dense in the set of normal states on $M$. Let $\omega$ be a n.f. state on $M$, and let $\delta>0$. Choose a normal state $\omega_\delta$ on $M$, such that $B_{\omega_\delta}=\mathbf{C}1$ and $\|\omega-\omega_\delta\|\leq\delta$. By Proposition 1.3 (2) we have

$$C_{\omega_\delta}(a,\delta)\cap\mathbf{C}1\neq\emptyset$$

for all $a \in M$. However, for $u \in U(M)$, $\|u\omega_\delta - \omega_\delta u\| \leq \delta$ implies that $\|u\omega - \omega u\| \leq 3\delta$. Therefore $C_\omega(a, 3\delta) \cap \mathbf{C}1 \neq \emptyset$. Using again Proposition 1.3 (2) we get $B_\omega = \mathbf{C}1$.

*Example* 1.6 [8]. In [2] Araki and Woods proved that there is up to isomorphism only one ITPFI-factor with asymptotic ratio set $r_\infty(M)$ equal to $[0, \infty[$. This factor is called the Araki-Woods factor and is denoted $R_\infty$. We shall see that $B_\varphi = \mathbf{C}1$ for all n.f. states on $M$, but that $M_\omega = \mathbf{C}1$ (and hence $M'_\omega \cap M = M$) for some state $\omega$ on $M$.

Note first that $R_\infty$ can be written as the tensor product

$$R_\infty = R_{\lambda_1} \hat{\otimes} R_{\lambda_2}$$

of two Powers factors $R_{\lambda_1}$ and $R_{\lambda_2}$ where $\log \lambda_1 / \log \lambda_2$ is irrational, because $M = R_{\lambda_1} \hat{\otimes} R_{\lambda_2}$ is clearly an ITPFI-factor and both $\lambda_1$ and $\lambda_2$ are contained in $r_\infty(M)$, so that $r_\infty(M) = [0, \infty[$ ($r_\infty(M) \cap \mathbf{R}_+$ is always a closed subgroup of $\mathbf{R}_+$). Let $\varphi_1$ and $\varphi_2$ be the usual tensor product states on $R_{\lambda_1}$ and $R_{\lambda_2}$ (cf. [20]). Then by [5, section 4]

$$M'_{\varphi_i} \cap R_{\lambda_i} = \mathbf{C}1, \quad i = 1, 2.$$

Therefore $\varphi = \varphi_1 \otimes \varphi_2$ satisfies

$$M'_\varphi \cap M = \mathbf{C}1.$$

In particular $B_\varphi = \mathbf{C}1$. Since $R_\infty$ is of type $\mathrm{III}_1$ we have $B_\omega = \mathbf{C}1$ for all n.f. states $\omega$ on $R_\infty$ by Corollary 1.5.

On the other hand Hermann and Takesaki gave in [17] an example of a n.f. state $\omega$ on a factor $M$, such that $M_\omega = \mathbf{C}1$. The factor in question is of type $\mathrm{III}_1$, because if $M$ was not of type $\mathrm{III}_1$, then by [5, Section 3–4] $M_\omega$ would contain a maximal abelian subalgebra of $M$. The factor in [17] comes from the G.N.S.-representation of the CAR-algebra given by a quasi free state. By [20] quasi free states on the CAR-algebra induce ITPFI-factor representations, and by [5, Section 3] $R_\infty$ is the only ITPFI-factor of type $\mathrm{III}_1$. Therefore the factor $M$ in Hermann and Takesaki's example is isomorphic to $R_\infty$.

## 2. Uniqueness of the injective factor of type $\mathrm{III}_1$

In [13], Connes and Takesaki introduced the notion of dominant weights on a factor of type III. The weights considered in [13] are not necessarily faithful, but for simplicity we shall here only consider faithful weights.

Let $M$ be a von Neumann algebra with separable predual. A normal faithful semifinite (n.f.s.) weight $\psi$ on $M$ is called *dominant* if

(i) $\psi$ has infinite multiplicity

and

(ii) $\lambda\psi \sim \psi$ for all $\lambda \in \mathbf{R}_+$.

The first condition means that the centralizer $M_\psi$ of $\psi$ is properly infinite, and $\lambda\psi \sim \psi$ means that $\lambda\psi = \psi\,(u \cdot u^*)$ for some unitary operator $u \in M$. Connes and Takesaki proved that every properly infinite von Neumann algebra has a dominant weight, and that two dominant weights are unitarily equivalent ([13, pp. 496–497]).

By [24], every properly infinite von Neumann algebra $M$ can be written as a crossed product

$$M = N \rtimes_\theta \mathbf{R}$$

where $N$ is a von Neumann algebra with a n.f.s. trace $\tau$, and $(\theta_s)_{s \in \mathbf{R}}$ is a continuous one parameter group of automorphisms for which

$$\tau \circ \theta_s = e^{-s}\tau, \quad s \in \mathbf{R}.$$

By [13, p. 497] the dual weight $\psi$ of $\tau$ is a dominant weight on $M = N \rtimes_\theta \mathbf{R}$, and the centralizer $M_\psi$ of $\psi$ is equal to $\pi_\theta(N)$ (the usual imbedding of $N$ in the crossed product $N \rtimes_\theta \mathbf{R}$ [24]). If $M$ is a factor of type $\mathrm{III}_1$, then $N$ is a factor of type $\mathrm{II}_\infty$ [24, Corollary 9.7]. Since dominant weights are unitarily equivalent, it follows that $M_\psi$ is a $\mathrm{II}_\infty$-factor for every dominant weight $\psi$ on a factor $M$ of type $\mathrm{III}_1$ (with separable predual).

By Connes' and Takesaki's relative commutant theorem [13, p. 513],

$$M'_\psi \cap M = Z(M_\psi)$$

for every integrable (in particular for every dominant) weight on $M$. Hence

THEOREM 2.1 (Connes, Takesaki [13]). *Let $M$ be a factor of type* $\mathrm{III}_1$ *with separable predual, and let $\psi$ be a dominant weight on $M$. Then*

$$M'_\psi \cap M = \mathbf{C}1.$$

COROLLARY 2.2. *Let $M$ be an injective factor of type* $\mathrm{III}_1$ *with separable predual, and let $\psi$ be a dominant weight on $M$. Then for every $x \in M$,*

$$\overline{\operatorname{conv}}\,\{uxu^* \mid u \in U(M_\psi)\} \cap \mathbf{C}1 \neq \emptyset$$

*where the closure is in the $\sigma$-weak topology on $M$.*

*Proof.* Let $m$ be an invariant mean on $\mathbf{R}$. Then

$$x \to \int_{-\infty}^{\infty} \sigma_t^{\psi}(x)\, dm(t)$$

defines a projection of norm 1 from $M$ to $M_\psi$. Hence, when $M$ is injective, so is $M_\psi$. In particular $M_\psi$ satisfies property $P$ of Schwartz (cf. [7]). Hence, for all $x \in M$,

$$\overline{\mathrm{conv}}\,\{uxu^* | u \in U(M_\psi)\} \cap M'_\psi \neq \emptyset.$$

This proves Corollary 2.2 because the above intersection is clearly contained in $M \cap M'_\psi = \mathbf{C}1$.

We are now able to state the main results of this section:

THEOREM 2.3. *Let $M$ be a factor of type $\mathrm{III}_1$ with separable predual. If $M$ satisfies the property:*

(1) *For every (faithful) dominant weight $\psi$ on $M$ and every $x \in M$,*

$$\overline{\mathrm{conv}}\,\{uxu^* | u \in U(M_\psi)\} \cap \mathbf{C}1 \neq \emptyset$$

*($\sigma$-weak closure),*

*then*

(2) *For every normal faithful state $\varphi$ on $M$, $B_\varphi = \mathbf{C}1$.*

*Particularly, $B_\varphi = \mathbf{C}1$ for any normal faithful state $\varphi$ on an injective factor of type $\mathrm{III}_1$ with separable predual.*

The above theorem combined with Connes' result cited in section 1 (Theorem 1.1) gives immediately:

COROLLARY 2.4. *Every injective factor of type $\mathrm{III}_1$ on a separable Hilbert space is isomorphic to the Araki-Woods factor $R_\infty$.*

In Section 3 we will prove that the two conditions (1) and (2) in Theorem 2.3 are actually equivalent (cf. Theorem 3.1). The rest of this section will be used to prove Theorem 2.3, i.e. to prove that (1) $\Rightarrow$ (2). We shall need some definitions from the spectral theory of automorphism groups (cf. [1] and [5, Section 3]): Let $(\alpha_t)_{t \in \mathbf{R}}$ be a $\sigma$-weakly continuous one-parameter group of automorphisms on a von Neumann algebra $M$. For $f \in L^2(\mathbf{R})$ and $x \in M$, one puts

$$\alpha_f(x) = \int_{-\infty}^{\infty} f(t)\,\alpha_t(x)\,dt.$$

The $\alpha$-spectrum $\mathrm{sp}_\alpha(x)$ of an operator $x \in M$ is the set of characters $\gamma \in \hat{\mathbf{R}}$, for which $\hat{f}(\gamma)=0$ for all $f \in L^1(\mathbf{R})$ satisfying $\alpha_f(x)=0$. We will identify $\hat{\mathbf{R}}$ with $\mathbf{R}$ in the usual way, such that

$$\hat{f}(\gamma) = \int_{-\infty}^{\infty} f(x) e^{i\gamma x} dx, \quad \gamma \in \mathbf{R}, f \in L^1(\mathbf{R}).$$

LEMMA 2.5. *Let $M$ and $\alpha_t$ be as above. Let $x \in M$ and let $\delta>0$. If the function $s \to \alpha_s(x)$ can be extended to an entire (analytic) $M$-valued function, such that*

$$\|\alpha_s(x)\| \leq K e^{\delta|\mathrm{Im}\, s|}, \quad s \in \mathbf{C}$$

*for some constant $K>0$, then* $\mathrm{sp}_\alpha(x) \subseteq [-\delta, \delta]$.

*Proof.* For every $\varphi \in M_*$, there exists a constant $K'>0$, such that

$$|\varphi(\alpha_s(x))| \leq K' e^{\delta|\mathrm{Im}\, s|}, \quad s \in \mathbf{C}.$$

Hence, by the Paley-Wiener theorem the function $t \to \varphi(\alpha_t(x))$, $t \in \mathbf{R}$ is the Fourier transformed of a tempered distribution with support in the interval $[-\delta, \delta]$. Thus, if $f$ is any Schwartz function, such that $\hat{f}$ has support in $\mathbf{R} \setminus [-\delta, \delta]$, then

$$\int_{-\infty}^{\infty} \varphi(\alpha_s(x)) f(x)\,dx = 0.$$

Hence $\alpha_f(x)=0$ for every Schwartz function $f$ for which $\mathrm{supp}(\hat{f}) \subseteq \mathbf{R} \setminus [-\delta, \delta]$. This proves Lemma 2.5.

LEMMA 2.6. *Let $M$ be a factor of type* $\mathrm{III}_1$ *with separable predual, and let $\psi$ be a weight on $M$ of infinite multiplicity (i.e. $M_\psi$ is properly infinite). If $M$ satisfies* (1) *in Theorem* 2.3, *then for all $x \in M$ and all $\delta>0$,*

$$\overline{\mathrm{conv}}\,\{uxu^* \mid u \in U(M),\ \mathrm{sp}_{\sigma^\psi}(u) \subseteq [-\delta, \delta]\} \cap \mathbf{C}1 \neq \emptyset$$

($\sigma$-*weak closure*).

*Proof.* Let $\delta>0$ and put $\alpha=\delta/2$. By [13, Chapter II, Theorem 4.7 and Corollary 3.2], there exists a dominant weight $\psi'$ on $M$, such that

$$e^{-\alpha}\psi' \leq \psi \leq e^{\alpha}\psi' \quad (\infty). \tag{*}$$

By definition of the ordering "$\leq$ $(\infty)$" (*) is equivalent to that the cocycle Radon Nikodym derivative $t \to (D_\psi : D_{\psi}')_t$ can be extended to an entire $M$-valued function satisfying

$$\|(D_\psi : D_{\psi'})_s\| \leq e^{\alpha|\mathrm{Im}\, s|}, \quad s \in \mathbf{C},$$

(cf. [13, pp. 508–509]). If $x \in M_{\psi'}$, then for $t \in \mathbf{R}$,

$$\begin{aligned}\sigma_t^{\psi}(x) &= (D_\psi : D_{\psi'})_t \sigma_t^{\psi'}(x)(D_\psi : D_{\psi'})_t^* \\ &= (D_\psi : D_{\psi'})_t x (D_\psi : D_{\psi'})_t^*.\end{aligned}$$

Hence $t \to \sigma_t^{\psi}(x)$ can be extended to an entire $M$-valued function, namely

$$s \to (D_\psi : D_{\psi'})_s x((D_\psi : D_{\psi'})_{\bar{s}})^*, \quad s \in \mathbf{C}$$

and

$$\|\sigma_s^{\psi}(x)\| \leq e^{2\alpha|\mathrm{Im}\, s|}\|x\|, \quad s \in \mathbf{C}.$$

Thus, by Lemma 2.5

$$\mathrm{sp}_{\sigma^\psi}(x) \subseteqq [-2\alpha, 2\alpha] = [-\delta, \delta].$$

Therefore,

$$U(M_{\psi'}) \subseteqq \{u \in U(M) \,|\, \mathrm{sp}_{\sigma^\psi}(u) \subseteqq [-\delta, \delta]\}.$$

This, together with the assumption (1) in Theorem 2.3, proves Lemma 2.6.

LEMMA 2.7. *Assume that $M$ satisfies* (1) *in Theorem* 2.3. *Let $\varphi$ be a normal faithful state on $M$, and let $x$ be an operator in $M$ for which $\varphi(x)=0$. Then for every $\delta>0$ there exists a sequence $(a_i)_{i\in \mathbf{N}}$ of operators in $M$, such that*

(i) $\mathrm{sp}_{\sigma^\varphi}(a_i) \subseteqq [-\delta, \delta]$ for all $i \in \mathbf{N}$

(ii) $\sum_{i=1}^{\infty} a_i^* a_i = 1$

(iii) $\sum_{i=1}^{\infty} \|xa_i - a_i x\|_\varphi^2 \geq \frac{1}{2}\|x\|_\varphi^2$,

*where as usual* $\|x\|_\varphi = \varphi(x^*x)^{1/2}$.

*Proof.* We can assume that $M$ acts on a Hilbert space $\mathcal{H}$, such that $\varphi$ is the vector state given by a vector $\xi_0 \in \mathcal{H}$.

Let $\mathcal{K}$ be an infinite dimensional Hilbert space with orthonormal basis $(e_i)_{i=1}^{\infty}$, and let $\psi$ be the weight on $M \bar{\otimes} B(\mathcal{K})$ given by $\psi = \varphi \otimes \mathrm{Tr}$, where Tr is the trace on $B(\mathcal{K})$. Then $\psi$ has infinite multiplicity. Since $M \bar{\otimes} B(\mathcal{K})$ is isomorphic to $M$ we get by Lemma 2.6 that there exists $\lambda \in \mathbf{C}1$, such that

$$\lambda(1 \otimes 1) \in \overline{\mathrm{conv}}\,\{u(x \otimes 1)u^* | u \in U(M \bar{\otimes} B(\mathcal{K})),\ \mathrm{sp}_{\psi}(u) \subseteqq [-\delta, \delta]\}.$$

Hence,

$$(x - \lambda 1) \otimes 1 \in \overline{\mathrm{conv}}\,\{x \otimes 1 - u(x \otimes 1)u^* | u \in U(M \bar{\otimes} B(\mathcal{K})),\ \mathrm{sp}_{\psi}(u) \subseteqq [-\delta, \delta]\}.$$

Since convex sets in $M \bar{\otimes} B(\mathcal{K})$ has the same closure in $\sigma$-weak and $\sigma$-strong topology, we have for every $\zeta \in \mathcal{H} \otimes \mathcal{K}$, that

$$\begin{aligned}\|((x-\lambda 1) \otimes 1)\zeta\| &\leq \sup\{\|(x \otimes 1 - u(x \otimes 1)u^*)\zeta\| \,|\, u \in U(M \bar{\otimes} B(\mathcal{K})),\ \mathrm{sp}_{\sigma^{\psi}}(u) \subseteqq [-\delta, \delta]\} \\ &= \sup\{\|(u^*(x \otimes 1) - (x \otimes 1)u^*)\zeta\| \,|\, u \in U(M \bar{\otimes} B(\mathcal{K})),\ \mathrm{sp}_{\sigma^{\psi}}(u) \subseteqq [-\delta, \delta]\}.\end{aligned}$$

By applying the above inequality to the vector $\zeta = \zeta_0 \otimes e_1$, we find that there exists $u \in U(M \bar{\otimes} B(\mathcal{K}))$, such that $\mathrm{sp}_{\sigma^{\psi}}(u) \subseteqq [-\delta, \delta]$ and

$$\|(x - \lambda 1)\xi_0\| \leq \sqrt{2}\,\|(u^*(x \otimes 1) - (x \otimes 1)u^*)(\xi_0 \otimes e_1)\|. \qquad (*)$$

The operator $u^*$ can be represented as an infinite matrix $(a_{ij})_{i,j=1}^{\infty}$ with elements in $M$ where $a_{ij}$ is characterized by

$$(u^*\xi, \eta) = (a_{ij}\xi \otimes e_j, \eta \otimes e_i), \quad \xi, \eta \in \mathcal{H}.$$

Since $\mathrm{sp}_{\sigma^{\psi}}(u) \subseteqq [-\delta, \delta]$ also $\mathrm{sp}_{\sigma^{\psi}}(u^*) \subseteqq [-\delta, \delta]$, and since $\sigma_t^{\psi} = \sigma_t^{\varphi} \otimes \mathrm{id}_{B(\mathcal{K})}$, we have

$$\mathrm{sp}_{\sigma^{\varphi}}(a_{ij}) \subseteqq [-\delta, \delta] \quad \text{for all } i, j \in \mathbf{N}.$$

The inequality (*) can now be expressed as

$$\|(x - \lambda 1)\xi_0\|^2 \leq 2 \sum_{i=1}^{\infty} \|(a_{i1}x - xa_{i1})\xi_0\|^2$$

because the set of vectors $(a_{i1}x - xa_{i1})(\xi_0 \otimes e_1)$ are pairwise orthogonal. Since

$$(x\xi_0, \lambda\xi_0) = \bar{\lambda}\varphi(x) = 0$$

we have

$$\|(x-\lambda 1)\,\xi_0\|^2=\|x\xi_0\|^2+|\lambda|^2\geqslant\|x\xi_0\|^2.$$

Hence also

$$\|x\xi_0\|^2\leqslant 2\sum_{i=1}^{\infty}\|(a_{i1}x-xa_{i1})\,\xi_0\|^2.$$

Since $u^*$ is unitary, we have $\Sigma_{i=1}^{\infty} a_{i1}^* a_{i1}=1$.

This proves Lemma 2.7.

The remaining part of the proof of Theorem 2.3 is strongly inspired by the techniques from Connes' and Størmer's paper [12]. As in [12] we shall consider $M$ in its standard representation (cf. [1], [6], [15]). Following the notation of [15], we can to every von Neumann algebra $M$ associate a unique quadruple $(M, \mathcal{H}, J, P)$, where $\mathcal{H}$ is a Hilbert space on which $M$ acts, $J$ is an isometric involution in $\mathcal{H}$, such that

(i) $JMJ=M'$,
(ii) $JcJ=c^*$, $c\in Z(M)$,

and $P^\natural$ is a selfdual cone in $\mathcal{H}$, such that

(iii) $J\xi=\xi$, $\xi\in P^\natural$,
(iv) $xJxJ(P^\natural)\subseteq P^\natural$, $x\in M$.

We put

$$\mathcal{H}_{\text{s.a.}}=\{\xi\in\mathcal{H}\,|\,J\xi=\xi\}.$$

Moreover, we will consider $\mathcal{H}$ as a two-sided $M$-module, where the right multiplication is given by

$$\eta x=Jx^*J\eta,\quad x\in M,\ \eta\in\mathcal{H}.$$

Recall that every positive normal functional $\varphi$ on $M$ is implemented by a unique vector $\xi_\varphi\in P^\natural$. By Araki's generalization of the Powers-Størmer inequality, one has for $\varphi$, $\psi\in M_*^+$:

$$\|\xi_\varphi-\xi_\psi\|^2\leqslant\|\varphi-\psi\|\leqslant\|\xi_\varphi-\xi_\psi\|\,\|\xi_\varphi+\xi_\psi\|$$

(cf. [1, Theorem 4(8)], [15, Lemma 2.9], [21]). Note that in the above notation the quantity $I(\varphi, x)$ used in [12] is simply given by

$$I(\varphi, x)=\tfrac{1}{2}\|x\xi_\varphi-\xi_\varphi x\|^2, \quad x\in M,\ \varphi\in M_*^+.$$

For later reference we prove:

LEMMA 2.8. *Let $M$ be a von Neumann algebra with standard form $(M, \mathcal{H}, J, P^\natural)$ and let $\varphi$ be a normal faithful state on $M$. Then:*

(a) *For every unitary operator $u$ in $M$*

$$\|u\xi_\varphi-\xi_\varphi u\|^2\leq\|u\varphi-\varphi u\|\leq 2\|u\xi_\varphi-\xi_\varphi u\|.$$

(b) *For every bounded sequence $(x_n)_{n\in\mathbf{N}}$ in $M$,*

$$\lim_{n\to\infty}\|x_n\varphi-\varphi x_n\|=0 \Leftrightarrow \lim_{n\to\infty}\|x_n\xi_\varphi-\xi_\varphi x_n\|=0.$$

*Proof.* (a) It is elementary to check that $u\xi_\varphi u^*\in P^\natural$, and that the vector functional on $M$ given by $u\xi_\varphi u^*$ is equal to $u\varphi u^*$. Hence by the Araki-Powers-Størmer unequality cited above

$$\|u\xi_\varphi u^*-\xi_\varphi\|^2\leq\|u\varphi u^*-\varphi\|\leq 2\|u\xi_\varphi u^*-\xi_\varphi\|$$

which is equivalent to the stated inequality.

(b) Let $\mathcal{A}$ (resp. $\mathcal{B}$) denote the set of bounded sequences in $M$ for which $\lim_{n\to\infty}\|x_n\varphi-\varphi x_n\|=0$ (resp. $\lim_{n\to\infty}\|x_n\xi_\varphi-\xi_\varphi x_n\|=0$). Then $\mathcal{A}$ and $\mathcal{B}$ are unital $C^*$-subalgebras of $l^\infty(\mathbf{N}, M)$. Moreover, by (a) their unitary groups $U(\mathcal{A})$ and $U(\mathcal{B})$ coincide. Since any unital $C^*$-algebra is the linear span of its unitaries, we have $\mathcal{A}=\mathcal{B}$.

Throughout the rest of this section, $M$ is a $\mathrm{III}_1$-factor with separable predual, and with standard form $(M, \mathcal{H}, J, P^\natural)$.

LEMMA 2.9. *Assume that $M$ satisfies* (1) *in Theorem* 2.3. *Let $\xi\in P^\natural$ be a cyclic and separating unit vector, and let $\eta\in\mathcal{H}_{\mathrm{s.a.}}$ (i.e. $J\eta=\eta$) be a unit vector orthogonal to $\xi$. For every $\delta>0$, there exists $a\in M$, $a\neq 0$, such that*

$$\|a\xi\|^2+\|a\eta\|^2<8\|a\eta-\eta a\|^2$$

*and*

$$\|a\xi-\xi a\|^2<\delta\|a\eta-\eta a\|^2.$$

*Proof.* We may assume that $0<\delta<1$. Define normal states $\varphi$, $\psi$ on $M$ by $\varphi=(\cdot\xi, \xi)$ and $\psi=(\cdot\eta, \eta)$. We treat first the case where $\psi$ is dominated by some scalar multiple of $\varphi$; i.e. $\psi\leq K\varphi$ for some $K\in\mathbf{R}_+$. Then the operator $x\xi\to x\eta$, $x\in M$ extends by continuity to a bounded operator $x'\in M'$, such that $\|x'\|\leq K^{1/2}$ and $\eta=x'\xi$. Put $x=Jx'J\in M$. Since $J\xi=\xi$ and $J\eta=\eta$, we have $\eta=x\xi$. Note that $\varphi(x)=0$ because $\eta\perp\xi$. Put

$$\delta_1=\min\left\{\left(\frac{\delta}{8}\right)^{1/2}, (2^7K)^{-1/2}\right\}.$$

By Lemma 2.8 we can choose $(a_j)_{j=1}^{\infty}$ in $M$, such that

$$\mathrm{sp}_{\sigma^\varphi}(a_i)\subseteq[-\delta_1, \delta_1], \quad i\in\mathbf{N},$$
$$\sum_{i=1}^{\infty} a_i^* a_i=1,$$
$$\sum_{i=1}^{\infty}\|(a_i x-xa_i)\xi\|^2\geq\tfrac{1}{2}\|x\xi\|^2$$
$$=\tfrac{1}{2}\|\eta\|^2.$$

Let $\Delta_\varphi$ be the modular operator associated with $\xi$ via Tomita-Takesaki theory [22]. For every $f\in L^1(\mathbf{R})$ for which the Fourier transformed $\hat{f}$ vanishes on $[-\delta_1, \delta_1]$ we have for every $j\in\mathbf{N}$

$$\hat{f}(\log\Delta_\varphi)a_j\xi=\int_{-\infty}^{\infty}f(t)\,\Delta_\varphi^{it}a_j\xi\,dt$$
$$=\int_{-\infty}^{\infty}f(t)\,\sigma_t^\varphi(a_j)\,\xi\,dt$$
$$=0$$

because $\mathrm{sp}_{\sigma^\varphi}(a_j)\subseteq[-\delta_1, \delta_1]$. Hence $a_j\xi$ is contained in the spectral subspace of $\log\Delta_\varphi$ corresponding to the interval $[-\delta_1, \delta_1]$.

Since $\xi\in P^\natural$, the isometry $J$ in the quadruple $(M, \mathcal{H}, J, P^\natural)$ coincides with the isometry $J_\varphi$ obtained in the polar decomposition of the modular conjugation $S_\varphi$ associated with $\xi$, i.e.

$$S_\varphi=J_\varphi\Delta_\varphi^{1/2}=J\Delta_\varphi^{1/2},$$

(cf. [6], [15, Lemma 2.9]). Since $S_\varphi(x\xi)=x^*\xi$, $x\in M$, we have

$$\xi a_i=Ja_i^*J\xi=Ja_i^*\xi=\Delta_\varphi^{1/2}a_i\xi.$$

Clearly

$$\sup\{|1-e^{s/2}|\,|s\in[-\delta_1,\delta_1]\}=e^{\delta_1/2}-1.$$

Therefore,

$$\begin{aligned}\|a_i\xi-\xi a_i\| &= \|(1-\Delta_\varphi^{1/2})\,a_i\xi\|\\ &\leqslant (e^{\delta_1/2}-1)\,\|a_i\xi\|\\ &\leqslant \delta_1\,\|a_i\xi\|.\end{aligned}$$

For the last inequality we have used that $\delta_1\leqslant(\delta/8)^{1/2}<1$. Using $\Sigma_{i=1}^\infty a_i^* a_i=1$, we have

$$\sum_{i=1}^\infty \|a_i\xi-\xi a_i\|^2\leqslant\delta_1^2\sum_{i=1}^\infty\|a_i\xi\|^2=\delta_1^2\leqslant\delta/8.$$

Clearly,

$$a_i\eta-\eta a_i=(a_ix-xa_i)\,\xi+x(a_i\xi-\xi a_i).$$

Using the triangle inequality in the Hilbert space $\otimes_{i=1}^\infty \mathscr{H}$ we get

$$\begin{aligned}\left(\sum_{i=1}^\infty\|a_i\eta-\eta a_i\|^2\right)^{1/2} &\geqslant \left(\sum_{i=1}^\infty\|(a_ix-xa_i)\,\xi\|^2\right)^{1/2}-\|x\|\left(\sum_{i=1}^\infty\|a_i\xi-\xi a_i\|^2\right)^{1/2}\\ &\geqslant \frac{1}{\sqrt{2}}-K^{1/2}\cdot\delta_1\\ &\geqslant \frac{1}{\sqrt{2}}-2^{-7/2}\\ &=\frac{7}{8\sqrt{2}}.\end{aligned}$$

Hence $8\,\Sigma_{i=1}^\infty\|a_i\eta-\eta a_i\|^2\geqslant 49/16>3$, while

$$\sum_{i=1}^\infty(\|a_i\xi\|^2+\|a_i\eta\|^2+8\delta^{-1}\|a_i\xi-\xi a_i\|^2)\leqslant 1+1+1=3.$$

Hence, for at least one $i\in\mathbf{N}$,

$$8\|a_i\eta-\eta a_i\|^2>\|a_i\xi\|^2+\|a_i\eta\|^2+8\delta^{-1}\|a_i\xi-\xi a_i\|^2.$$

In particular, $a_i\neq 0$ and

$$\|a_i \xi\|^2+\|a_i \eta\|^2<\|a_i \eta-\eta a_i\|^2$$
$$\|a_i \xi-\xi a_i\|^2<\delta\|a_i \eta-\eta a_i\|^2.$$

This proves the lemma in the case $\psi \leq K\varphi$ for some $K$.

Assume next that $\psi$ is not dominated by a multiple of $\varphi$. In this case we can choose a non zero projection $p\in M$, such that

$$\psi(p)>\frac{16}{\delta}\varphi(p).$$

Moreover, since the reduced algebra $pMp$ has no minimal projections, we can choose a projection $q\in M$, $0\leq q\leq p$ such that

$$\psi(q)=\frac{1}{16}\psi(p).$$

Note that $q\neq 0$ *and* $p-q\neq 0$. Since $M$ is of type III, any two non zero projections in $M$ are equivalent, so we can choose $v\in M$, such that

$$v^*v=p-q \quad \text{and} \quad vv^*=q.$$

Then

$$\|v\eta\|^2=\psi(p-q)=\frac{15}{16}\psi(p)$$

and

$$\|\eta v\|^2=\|J(v^*\eta)\|^2=\|v^*\eta\|^2=\psi(q)=\frac{1}{16}\psi(p).$$

Hence,

$$\|v\eta-\eta v\|\geq\|v\eta\|-\|\eta v\|\geq\frac{\sqrt{15}-1}{4}\psi(p)^{1/2}\geq\tfrac{1}{2}\psi(p)^{1/2}.$$

Therefore,

$$\begin{aligned}\|v\xi\|^2+\|v\eta\|^2&\leq\varphi(p)+\psi(p)\\&\leq\left(\frac{\delta}{16}+1\right)\psi(p)\\&<2\psi(p)\leq 8\|v\eta-\eta v\|^2\end{aligned}$$

and by the parallelogram identity

$$\begin{aligned}\|v\xi-\xi v\|^2 &\leq 2(\|v\xi\|^2+\|\xi v\|^2)\\ &= 2(\|v\xi\|^2+\|v^*\xi\|^2)\\ &\leq 4\varphi(p)\\ &< \frac{1}{4}\delta\psi(p)\\ &\leq \delta\|v\eta-\eta v\|^2.\end{aligned}$$

This finishes the proof of Lemma 2.9.

LEMMA 2.10. *Let $M$, $\xi$, $\eta$ be as in Lemma 2.9, and let $\delta>0$. There exists $b\in M_{\text{s.a.}}$, $b\neq 0$ such that*

$$\|b\xi\|^2+\|b\eta\|^2 < 32\|b\eta-\eta b\|^2$$
$$\|b\xi-\xi b\|^2 < \delta\|b\eta-\eta b\|^2.$$

*Proof.* It is sufficient to consider $0<\delta<1$. By Lemma 2.9 we can choose $a\in M$, such that

$$\|a\xi\|^2+\|a\eta\|^2 < 8\|a\eta-\eta a\|^2$$
$$\|a\xi-\xi a\|^2 < \frac{\delta}{4}\|a\eta-\eta a\|^2.$$

Put $b_1=(a+a^*)/2$ and $b_2=(a-a^*)/2i$. We will show that either $b_1$ or $b_2$ satisfies the conditions of the lemma. If $b_1=0$ then $b_2=-ia$ clearly satisfies the conditions. Also if $b_2=0$ then $b_1=a$ satisfies the conditions. Hence we can assume that $b_1\neq 0$ and $b_2\neq 0$.

First, note that

$$\begin{aligned}\|a^*\xi\| = \|J(\xi a)\| = \|\xi a\| &\leq \|a\xi\|+\|a\xi-\xi a\|\\ &\leq \left(8^{1/2}+\left(\frac{\delta}{4}\right)^{1/2}\right)\|a\eta-\eta a\|\\ &< 4\|a\eta-\eta a\|\end{aligned}$$

and

$$\begin{aligned}\|a^*\eta\| = \|\eta a\| &\leq \|a\eta\|+\|a\eta-\eta a\|\\ &\leq (8^{1/2}+1)\,\|a\eta-\eta a\|\\ &< 4\|a\eta-\eta a\|.\end{aligned}$$

Moreover, since $J\xi=\xi$,

$$\|a^*\xi-\xi a^*\| = \|J(\xi a-a\xi)\| = \|a\xi-\xi a\|$$

and similarly

$$\|a^*\eta-\eta a^*\| = \|a\eta-\eta a\|.$$

Hence,

$$\|a^*\xi\|^2+\|a^*\eta\|^2+32\delta^{-1}\|a^*\xi-\xi a^*\|^2 < \left(2\cdot 16+\tfrac{1}{4}\cdot 32\right)\|a\eta-\eta a\|^2$$
$$= 40\|a\eta-\eta a\|^2.$$

Clearly,

$$\|a\xi\|^2+\|a\eta\|^2+32\delta^{-1}\|a\xi-\xi a\|^2 < \left(8+\frac{1}{4}\cdot 32\right)\|a\eta-\eta a\|^2$$
$$\leq 24\|a\eta-\eta a\|^2.$$

Hence,

$$(\|a\xi\|^2+\|a^*\xi\|^2)+(\|a\eta\|^2+\|a^*\eta\|^2)+32\delta^{-1}(\|a\xi-\xi a\|^2+\|a^*\xi-\xi a^*\|)^2$$
$$< 64\|a\eta-\eta a\|^2$$
$$= 32(\|a\eta-\eta a\|^2+\|a^*\eta-\eta a^*\|^2).$$

Using $a=b_1+ib_2$ and $a^*=b_1-ib_2$ we get now by the parallelogram identity that

$$\|b_1\xi\|^2+\|b_2\xi\|^2+\|b_1\eta\|^2+\|b_2\eta\|^2+32\delta^{-1}(\|b_1\xi-\xi b_1\|^2+\|b_2\xi-\xi b_2\|^2)$$
$$< 32(\|b_1\eta-\eta b_1\|^2+\|b_2\eta-\eta b_2\|^2).$$

Hence, for either $b=b_1$ or $b=b_2$ we have

$$\|b\xi\|^2+\|b\eta\|^2+32\delta^{-1}\|b\xi-\xi b\|^2 < 32\|b\eta-\eta b\|^2.$$

Thus, $b$ satisfies the conditions of the lemma.

The following lemma is very similar to [7, Proposition I.1].

LEMMA 2.11. *Let $\zeta\in\mathcal{H}$, and let $b\in M$ be selfadjoint. Then there exists a positive bounded measure $\nu$ on $\mathbf{R}^2$ with support in* $\mathrm{sp}(b)\times\mathrm{sp}(b)$, *such that for any two bounded Borel functions $f,g$* on $\mathbf{R}$

$$\|f(b)\zeta-\zeta g(b)\|^2 = \int_{\mathbf{R}^2} |f(s)-g(t)|^2 d\nu(s,t).$$

*Proof.* Since left and right multiplication with $b$ on $\mathscr{H}$ commute, there exists a representation $\pi$ of the abelian $C^*$-algebra $C(\mathrm{sp}(b)\times\mathrm{sp}(b))$ on $\mathscr{H}$ such that

$$\pi(f\otimes g)\xi = f(b)\xi g(b)$$

for $\xi\in\mathscr{H}$ and $f, g\in C(\mathrm{sp}(b))$. Let $\nu$ be the positive measure $\nu$ on $\mathrm{sp}(b)\times\mathrm{sp}(b)$ defined by

$$\langle \nu, h\rangle = (\pi(h)\zeta, \zeta), \quad h\in C(\mathrm{sp}(b)\times\mathrm{sp}(b)).$$

For $f, g\in C(\mathrm{sp}(b))$,

$$(f(b)\zeta g(b), \zeta) = (\pi(f\otimes g)\zeta, \zeta)$$
$$\iint_{\mathrm{sp}(b)\times\mathrm{sp}(b)} f(s)\,g(t)\,d\nu(s,t).$$

By standard arguments the above equality can be extended to all bounded Borel functions $f,g$ on $\mathrm{sp}(b)$. Hence, for any pair of bounded Borel functions $f,g$ on $\mathrm{sp}(b)$

$$\begin{aligned}\|f(b)\zeta-\zeta g(b)\|^2 &= \|f(b)\zeta\|^2+\|\zeta g(b)\|^2-2\,\mathrm{Re}\,(f(b)\zeta,\zeta g(b))\\ &= (|f|^2(b)\zeta,\zeta)+(\zeta|g|^2(b),\zeta)-2\,\mathrm{Re}\,(f(b)\zeta\bar{g}(b),\zeta)\\ &= \iint_{\mathrm{sp}(b)\times\mathrm{sp}(b)} (|f|^2(s)+|g|^2(t)-2\,\mathrm{Re}\,(f(s)\,\overline{g(t)}\,))\,d\nu(s,t)\\ &= \iint_{\mathrm{sp}(b)\times\mathrm{sp}(b)} |f(s)-g(t)|^2\,d\nu(s,t).\end{aligned}$$

We can extend $\nu$ to a measure on $\mathbf{R}^2$ by putting

$$\nu(\mathbf{R}^2\setminus \mathrm{sp}(b)\times\mathrm{sp}(b)) = 0.$$

This finishes the proof of Lemma 2.11.

LEMMA 2.12. *Let $\zeta\in\mathscr{H}$, and let $b\in M$ be selfadjoint. If*

$$b=\int_{-\infty}^{\infty}\lambda\,de_\lambda$$

*is the spectral resolution of $b$ (i.e. $e_\lambda=\chi_{]-\infty,\lambda]}(b)$), then*

(a) $\displaystyle\int_{-\infty}^{\infty}\|e_\lambda\zeta-\zeta e_\lambda\|^2|\lambda|\,d\lambda\le\|b\zeta\|\,\|b\zeta-\zeta b\|$

*and*

$$\text{(b)} \int_{-\infty}^{\infty} \|e_\lambda \zeta - \zeta e_\lambda\|^2 |\lambda| d\lambda \geq \frac{1}{4} \|b\zeta - \zeta b\|^2.$$

*Proof.* Let $\nu$ be as in Lemma 2.11. Put

$$h(s,t,\lambda) = \begin{cases} 1 & s \leq \lambda < t \text{ or } t \leq \lambda < s \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$\|e_\lambda \zeta - \zeta e_\lambda\|^2 = \iint_{\mathbf{R}^2} |\chi_{]-\infty,\lambda]}(s) - \chi_{]-\infty,\lambda]}(t)|^2 d\nu(s,t)$$
$$= \iint_{\mathbf{R}^2} h(s,t,\lambda)\, d\nu(s,t).$$

By Fubini's theorem,

$$\int_{-\infty}^{\infty} \|e_\lambda \zeta - \zeta e_\lambda\|^2 |\lambda| d\lambda = \iint_{\mathbf{R}^2} \left( \int_{-\infty}^{\infty} h(s,t,\lambda) |\lambda| d\lambda \right) d\nu(s,t).$$

If $s \leq t$,

$$\int_{-\infty}^{\infty} h(s,t,\lambda)|\lambda| d\lambda = \int_s^t |\lambda| d\lambda = \tfrac{1}{2}(t^2 \operatorname{sign} t - s^2 \operatorname{sign} s).$$

Using $h(s,t,\lambda) = h(t,s,\lambda)$, we get for all $s,t \in \mathbf{R}$,

$$\int_{-\infty}^{\infty} h(s,t,\lambda)\, |\lambda| d\lambda = \tfrac{1}{2}|t^2 \operatorname{sign} t - s^2 \operatorname{sign} s|.$$

A simple computation shows that for $s \cdot t \geq 0$

$$|t^2 \operatorname{sign} t - s^2 \operatorname{sign} s| = |s-t|(|s|+|t|)$$

and for $s \cdot t < 0$

$$|t^2 \operatorname{sign} t - s^2 \operatorname{sign} s| = s^2 + t^2 \leq |s-t|\,(|s|+|t|)$$

so in all cases

$$\int_{-\infty}^{\infty} h(s,t,\lambda)\, |\lambda| d\lambda \leq \tfrac{1}{2}|t-s|(|s|+|t|).$$

Therefore,

$$\int_{-\infty}^{\infty} \|e_\lambda \zeta - \zeta e_\lambda\|^2 |\lambda| d\lambda = \tfrac{1}{2} \iint_{\mathbf{R}^2} |t-s|(|s|+|t|)\, d\nu(s,t)$$

$$\leq \tfrac{1}{2} \left( \iint_{\mathbf{R}^2} |t-s|^2 d\nu(\mathrm{s},\mathrm{t}) \right)^{1/2} \left( \iint_{\mathbf{R}^2} (|s|+|t|)^2 d\nu(s,t) \right)^{1/2}.$$

By Lemma 2.11,

$$\iint_{\mathbf{R}^2} |t-s|^2 d\nu(s,t) = \|b\zeta - \zeta b\|^2.$$

Since $(|s|+|t|)^2 \leq 2|s|^2+2|t|^2$, we get by Lemma 2.11

$$\iint_{\mathbf{R}^2} (|s|+|t|)^2 d\nu(s,t) \leq 2(\|b\zeta\|^2 + \|\zeta b\|^2).$$

But since $J\zeta = \zeta$,

$$\|\zeta b\| = \|Jb\zeta\| = \|b\zeta\|.$$

Hence

$$\iint_{\mathbf{R}^2} (|s|+|t|)^2 d\nu(s,t) \leq 4\|b\zeta\|^2.$$

This proves (a). To prove (b), observe that for $s \cdot t \geq 0$,

$$|t^2 \operatorname{sign} t - s^2 \operatorname{sign} s| \geq (t-s)^2.$$

Moreover, for $s \cdot t < 0$

$$|t^2 \operatorname{sign} - s^2 \operatorname{sign} s| = t^2 + s^2 \geq \tfrac{1}{2}(t-s)^2.$$

Hence,

$$\int_{-\infty}^{\infty} h(s,t,\lambda)\, |\lambda| d\lambda \geq \tfrac{1}{4}(t-s)^2.$$

Therefore,

$$\int_{-\infty}^{\infty} \|e_\lambda \zeta - \zeta e_\lambda\|^2 |\lambda| d\lambda \geq \tfrac{1}{4} \iint_{\mathbf{R}^2} |t-s|^2 d\nu(s,t)$$

$$= \tfrac{1}{4} \|b\zeta - \zeta b\|^2.$$

**Lemma 2.13.** *Let $M$, $\xi$, $\eta$ be as in Lemma 2.9. For any $\delta>0$ there exists a projection $p\neq 0$ in $M$, such that*

$$\|p\xi\|^2+\|p\eta\|^2<2^7\|p\eta-\eta p\|^2$$
$$\|p\xi-\xi p\|^2<\delta\|p\eta-\eta p\|^2.$$

*Proof.* Let $\delta>0$, and put $\delta_1=(2^{-7}\cdot\delta)^2$. Assume that $b\in M_{\text{s.a.}}$ satisfies the conditions of Lemma 2.10 with respect to $\delta_1$. Let

$$b=\int_{-\infty}^{\infty}\lambda de_\lambda$$

be the spectral resolution of $b$. Put

$$f_\lambda=\begin{cases}e_\lambda, & -\infty<\lambda<0\\ 1-e_\lambda, & 0\leq\lambda<\infty.\end{cases}$$

Using that $e_\lambda=0$ for $\lambda<-\|b\|$, and $e_\lambda=1$ for $\lambda>\|b\|$ we get by partial integration

$$\int_{-\infty}^{0}\|f_\lambda\xi\|^2|\lambda|\delta\lambda=-\int_{-\infty}^{0}(e_\lambda\xi,\xi)\,d\left(\frac{\lambda^2}{2}\right)$$
$$=\int_{-\infty}^{0}\frac{\lambda^2}{2}d(e_\lambda\xi,\xi)$$

and

$$\int_{0}^{\infty}\|f_\lambda\xi\|^2|\lambda|d\lambda=\int_{0}^{\infty}(1-(e_\lambda\xi,\xi))\,d\left(\frac{\lambda^2}{2}\right)$$
$$=\int_{0}^{\infty}\frac{\lambda^2}{2}d(e_\lambda\xi,\xi).$$

Thus

$$\int_{-\infty}^{\infty}\|f_\lambda\xi\|^2|\lambda|d\lambda=\tfrac{1}{2}\int_{-\infty}^{\infty}\lambda^2d(e_\lambda\xi,\xi)=\tfrac{1}{2}\|b\xi\|^2.$$

Similarly

$$\int_{-\infty}^{\infty}\|f_\lambda\eta\|^2|\lambda|d\lambda=\tfrac{1}{2}\|b\eta\|^2.$$

Since for all $\zeta \in \mathscr{H}_{\text{s.a.}}$ we have

$$f_\lambda \zeta - \zeta f_\lambda = \pm(e_\lambda \zeta - \zeta e_\lambda)$$

we get from Lemma 2.12, that

$$\int_{-\infty}^{\infty} \|f_\lambda \eta - \eta f_\lambda\|^2 |\lambda| d\lambda \geq \tfrac{1}{4}\|b\eta - \eta b\|^2$$

and

$$\int_{-\infty}^{\infty} \|f_\lambda \xi - \xi f_\lambda\|^2 |\lambda| d\lambda \leq \|b\xi\|\,\|b\xi - \xi b\|.$$

Using, $\|b\xi\|^2 < 32\|b\eta - \eta b\|^2$ and $\|b\xi - \xi b\|^2 < \delta_1 \|b\eta - \eta b\|^2$, we have

$$\int_{-\infty}^{\infty} \|f_\lambda \xi - \xi f_\lambda\|^2 |\lambda| d\lambda < (32\delta_1)^{1/2}\|b\eta - \eta b\|^2$$
$$\leq 6\delta_1^{1/2}\|b\eta - \eta b\|^2.$$

Hence

$$\begin{aligned}\int_{-\infty}^{\infty} (\|f_\lambda \xi\|^2 + \|f_\lambda \eta\|^2 + \delta_1^{-1/2}\|f_\lambda \xi - \xi f_\lambda\|^2)|\lambda| d\lambda \\ &< \tfrac{1}{2}(\|b\xi\|^2 + \|b\eta\|^2) + 6\|b\eta - \eta b\|^2 \\ &\leq \frac{32}{2}\|b\eta - \eta b\|^2 + 6\|b\eta - \eta b\|^2 \\ &\leq 32\|b\eta - \eta b\|^2 \\ &\leq 2^7 \int_{-\infty}^{\infty} \|f_\lambda \eta - \eta f_\lambda\|^2 |\lambda| d\lambda.\end{aligned}$$

Thus for some $\lambda \in \mathbf{R}$, one has

$$\|f_\lambda \xi\|^2 + \|f_\lambda \eta\|^2 + \delta_1^{-1/2}\|f_\lambda \xi - \xi f_\lambda\|^2 < 2^7 \|f_\lambda \eta - \eta f_\lambda\|^2.$$

In particular, for this $\lambda$, $f_\lambda \neq 0$ and

$$\|f_\lambda \xi\|^2 + \|f_\lambda \eta\| < 2^7 \|f_\lambda \eta - \eta f_\lambda\|^2$$
$$\|f_\lambda \xi - \xi f_\lambda\|^2 < 2^7 \delta_1^{1/2} \|f_\lambda \eta - \eta f_\lambda\|^2$$
$$= \delta \|f_\lambda \eta - \eta f_\lambda\|^2.$$

This proves Lemma 2.13.

LEMMA 2.14. *Assume that $M$ satisfies (1) in Theorem 2.3, and let $\xi \in P^\natural$ be a cyclic and separating unit vector. Let $\eta \in \mathscr{H}_{\mathrm{s.a.}}$ be a unit vector, $\eta \neq \xi$, and $\eta \neq -\xi$, and let $\theta$ be the angle between $\xi$ and $\eta$, i.e.*

$$\theta = \arc\cos(\xi, \eta).$$

*Then for every $\delta > 0$ there exists a projection $p \neq 0$ in $M$, such that*

$$\|p\xi\|^2 + \|p\eta\|^2 < \frac{2^{10}}{\sin^2\theta} \|p\eta - \eta p\|^2$$

$$\|p\xi - \xi p\| < \delta \|p\eta - \eta p\|^2.$$

*Proof.* Note first that the angle $\theta$ is well defined because $\mathscr{H}_{\mathrm{s.a.}} = \{\zeta \in \mathscr{H} | J\zeta = \zeta\}$ is a real Hilbertspace. Moreover, $0 < \theta < \pi$. It is sufficient to consider the case $\delta < 1$. The vector $\eta$ can be written in the form

$$\eta = \cos\theta\xi + \sin\theta\eta'$$

where $\eta' \in \mathscr{H}_{\mathrm{s.a.}}$ is a unit vector orthogonal to $\xi$. Put $\delta_1 = \frac{1}{4}\delta\sin^2\theta$. By Lemma 2.13 there exists a non-zero projection $p \in M$, such that

$$\|p\xi\|^2 + \|p\eta'\|^2 < 2^7 \|p\eta' - \eta' p\|^2$$

$$\|p\xi - \xi p\|^2 < \delta_1 \|p\eta' - \eta' p\|^2.$$

Since

$$\sin\theta\eta' = \eta - \cos\theta\xi$$

we have

$$\begin{aligned} \sin\theta \|p\eta' - \eta' p\| &\leq \|p\eta - \eta p\| + \|p\xi - \xi p\| \\ &\leq \|p\eta - \eta p\| + \delta_1^{1/2} \|p\eta' - \eta' p\|. \end{aligned}$$

Thus

$$\begin{aligned} \|p\eta - \eta p\| &\geq (\sin\theta - \delta_1^{1/2}) \|p\eta' - \eta' p\| \\ &= \sin\theta(1 - \tfrac{1}{2}\delta^{1/2}) \|p\eta' - \eta' p\| \\ &\geq \tfrac{1}{2}\sin\theta \|p\eta' - \eta' p\| \end{aligned}$$

which implies that

$$\|p\xi-\xi p\|^2<\frac{4\delta_1}{\sin^2\theta}\|p\eta-\eta p\|^2$$
$$=\delta\|p\eta-\eta p\|^2.$$

Moreover,

$$\|p\eta\|\leq\cos\theta\|p\xi\|+\sin\theta\|p\eta'\|$$
$$\leq(\|p\xi\|^2+\|p\eta'\|^2)^{1/2}.$$

Thus

$$\|p\xi\|^2+\|p\eta\|^2\leq 2\|p\xi\|^2+2\|p\eta'\|^2$$
$$<2^8\|p\eta'-\eta'p\|^2$$
$$\leq\frac{2^{10}}{\sin^2\theta}\|p\eta-\eta p\|^2.$$

This proves Lemma 2.14.

LEMMA 2.15. *Assume that $M$ satisfies* (1) *in Theorem* 2.3. *Let $\xi\in P^\natural$ be a cyclic and separating unit vector, and let $\eta\in\mathscr{H}_{\text{s.a.}}$ be a unit vector such that $\xi\perp\eta$. Then for every $\delta>0$ there exists a family $(e_i)_{i\in I}$ of orthogonal projections in $M$ with sum* 1, *such that*

$$\|\xi-\sum_{i\in I}e_i\xi e_i\|^2\leq\delta$$

$$\|\eta-\sum_{i\in I}e_i\eta e_i\|^2\geq 2^{-18}.$$

*Proof.* Let $\mathscr{F}$ be the collection of all sets of projections $\{p_i\}_{i\in I}$ in $M$ for which

(1) $p_i\neq 0$ for all $i$ and $p_i\perp p_j$ for $i\neq j$.

(2) With $p=1-\sum_{i\in I}p_i$,

$$\|\xi-p\xi p\|^2+\|\eta-p\eta p\|^2\leq 2^{14}\|\eta-p\eta p-\sum_{i\in I}p_i\eta p_i\|^2$$

and

$$\|\xi-p\xi p-\sum_{i\in I}p_i\xi p_i\|^2\leq\delta\|\eta-p\eta p-\sum_{i\in I}p_i\eta p_i\|^2.$$

The collection $\mathcal{F}$ is a partially ordered set with respect to inclusion. $\mathcal{F}$ is non empty, because $\emptyset \in \mathcal{F}$. Moreover, it is easy to check that $\mathcal{F}$ is inductively ordered, i.e. every totally ordered subset of $\mathcal{F}$ has a least upper bound in $\mathcal{F}$. Hence by Zorn's lemma $\mathcal{F}$ has a maximal element $\{q_i\}_{i\in I}$. Put $q=1-\Sigma_{i\in I}q_i$. We will show that the family of projections:

$$\{q_i\}_{i\in I}\cup\{q\}$$

satisfies the inequalities stated in the lemma. Since $\{q_i\}_{i\in I}\cup\{q\}$ is a family of pairwise orthogonal projections, the family

$$\{q_iJq_iJ\}_{i\in I}\cup\{qJqJ\}$$

consists also of orthogonal projections. Therefore

$$\begin{aligned}\|\eta-q\eta q-\sum_{i\in I}q_i\eta q_i\|^2 &= \|\left(1-qJqJ-\sum_{i\in I}q_iJq_iJ\right)\eta\|^2\\ &\leq \|\eta\|^2.\end{aligned}$$

Thus since $\{q_i\}_{i\in I}\in\mathcal{F}$, we have

$$\|\xi-q\xi q-\sum_{i\in I}q_i\xi q_i\|^2\leq\delta\|\eta\|^2=\delta$$

so to complete the proof of Lemma 2.15 we have to show that

$$\|\eta-q\eta q-\sum_{i\in I}q_i\eta q_i\|^2\geq 2^{-18}.$$

Assume that $\|\eta-q\eta q-\Sigma_{i\in I}q_i\eta q_i\|^2<2^{-18}$. Then by the definition of $\mathcal{F}$,

$$\|\xi-q\xi q\|^2+\|\eta-q\eta q\|^2<2^{14}\cdot 2^{-18}=\tfrac{1}{16}.$$

Put $\xi'=q\xi q$ and $\eta'=q\eta q$. Then

$$\|\xi-\xi'\|\leq\tfrac{1}{4}\quad\text{and}\quad\|\eta-\eta'\|\leq\tfrac{1}{4}.$$

In particular $q\neq 0$, $\|\xi'\|\geq\frac{3}{4}$ and $\|\eta'\|\geq\frac{3}{4}$. Moreover,

$$\begin{aligned}(\xi',\eta')=(q\xi q,q\eta q)&=(q\xi q,\eta)\\ &=(\xi,\eta)-(\xi-q\xi q,\eta).\end{aligned}$$

Thus

$$|(\xi', \eta')| \leq 0 + \|\xi - \xi'\| \, \|\eta\| \leq \tfrac{1}{4}.$$

Let $\theta$ be the angle between $\xi'$ and $\eta'$. Since

$$|\cos\theta| = \frac{|(\xi', \eta')|}{\|\xi'\| \, \|\eta'\|} \leq \frac{1}{4}\left(\frac{4}{3}\right)^2 < \frac{1}{2}$$

we have $\sin^2\theta > \frac{3}{4}$.

Let $J_q$ denote the restriction of $J$ to $q\mathcal{H}q$.

By [15, Lemma 2.6], $(qMq, q\mathcal{H}q, J_q, qP^\natural q)$ is a standard form of the reduced algebra. It is clear that $\xi' \in qP^\natural q$ and $\eta' \in (q\mathcal{H}q)_{\text{s.a.}}$

Since $\xi$ is cyclic and separating for $M$, the face in $P^\natural$ generated by $\xi$ is dense in $P^\natural$. Hence the face in $qP^\natural q$ generated by $\xi' = q\xi q$ is dense in $qP^\natural q$, which by [6, Lemma 4.3] implies that $\xi'$ is cyclic and separating for $qMq$. Since $M$ is of type III and $q \neq 0$, $qMq$ is isomorphic to $M$. Therefore we can apply Lemma 2.14 to $qMq$ and the vectors

$$\xi'' = \xi'/\|\xi'\| \quad \text{and} \quad \eta'' = \eta'/\|\eta'\|.$$

Hence, there exists a projection $r \in M$, $r \leq q$, $r \neq 0$, such that

$$\begin{aligned} \|r\xi''\|^2 + \|r\eta''\|^2 &\leq \frac{2^{10}}{\sin^2\theta} \|r\eta'' - \eta'' r\|^2 \\ &\leq \frac{4}{3} 2^{10} \|r\eta'' - \eta'' r\|^2 \end{aligned}$$

and

$$\|r\xi'' - \xi'' r\|^2 \leq \frac{\delta}{2} \|r\eta'' - \eta'' r\|^2.$$

Hence, using $\frac{3}{4} \leq \|\xi'\| \leq 1$ and $\frac{3}{4} \leq \|\eta'\| \leq 1$, we get

$$\begin{aligned} \|r\xi'\|^2 + \|r\eta'\|^2 &\leq \left(\frac{4}{3}\right)^3 2^{10} \|r\eta'' - \eta'' r\|^2 \\ &\leq 2^{13} \|r\eta' - \eta' r\|^2 \end{aligned}$$

and

$$\begin{aligned} \|r\xi' - \xi' r\|^2 &\leq \left(\frac{4}{3}\right)^2 \frac{\delta}{2} \|r\eta' - \eta' r\|^2 \\ &\leq \delta \|r\eta' - \eta' r\|^2. \end{aligned}$$

We will show next that $\{q_i\}_{i\in I}\cup\{r\}$ is contained in $\mathscr{F}$, i.e. we will check that

$$\|\xi-(q-r)\,\xi(q-r)\|^2+\|\eta-(q-r)\,\eta(q-r)\|^2\leqslant 2^{14}\|\eta-(q-r)\eta(q-r)-r\eta r-\sum_{i\in I}q_i\eta q_i\|^2 \quad (*)$$

and that

$$\|\xi-(q-r)\,\xi(q-r)-r\xi r-\sum_{i\in I}q_i\xi q_i\|^2\leqslant\delta\|\eta-(q-r)\,\eta(q-r)-r\eta r-\sum_{i\in I}q_i\xi q_i\|^2. \quad (**)$$

To prove (**), observe that

$$1-(q-r)\,J(q-r)\,J-rJrJ-\sum_{i\in I}q_iJq_iJ=(1-qJqJ-\sum_{i\in I}q_iJq_iJ)+rJ(q-r)\,J+(q-r)\,JrJ),$$

where the right side of the equality is the sum of three orthogonal projections. Therefore

$$\|\xi-(q-r)\,\xi(q-r)-r\xi r-\sum_{i\in I}q_i\xi q_i\|^2=\|\xi-q\xi q-\sum_{i\in I}q_i\xi q_i\|^2+\|r\xi(q-r)\|^2+\|(q-r)\,\xi r\|^2.$$

Since

$$r\xi'-\xi'r=r\xi q-q\xi r=r\xi(q-r)-(q-r)\,\xi r$$

and since

$$rJ(q-r)\,J\perp(q-r)\,JrJ$$

we have

$$\|r\xi'-\xi'r\|^2=\|r\xi(q-r)\|^2+\|(q-r)\,\xi r\|^2.$$

Thus

$$\|\xi-(q-r)\,\xi(q-r)-r\xi r-\sum_{i\in I}q_i\xi q_i\|^2=\|\xi-q\xi q-\sum_{i\in I}q_i\xi q_i\|^2+\|r\xi'-\xi'r\|^2.$$

Similarly,

$$\|\eta-(q-r)\,\eta(q-r)-r\eta r-\sum_{i\in I}q_i\eta q_i\|^2=\|\eta-q\eta q-\sum_{i\in I}q_i\eta q_i\|^2+\|r\eta'-\eta'r\|^2.$$

Since $\{q_j\}_{j\in J}\in\mathscr{F}$ and since $\|r\xi'-\xi'r\|^2\leq\delta\|r\eta'-\eta'r\|^2$ we have proved (**). To prove (*), we use that

$$1-(q-r)J(q-r)J=(1-qJqJ)+qJrJ+rJ(q-r)J$$

where the right side is a sum of three orthogonal projections. Hence

$$\begin{aligned}\|\xi-(q-r)\xi(q-r)\|^2&=\|\xi-q\xi q\|^2+\|q\xi r\|^2+\|r\xi(q-r)\|^2\\&\leq\|\xi-q\xi q\|^2+\|q\xi r\|^2+\|r\xi q\|^2.\end{aligned}$$

Since $J\xi=\xi$ we have $\|q\xi r\|^2=\|J(r\xi q)\|^2=\|r\xi q\|^2$.

Moreover, $r\xi q=r\xi'$. Therefore

$$\|\xi-(q-r)\xi(q-r)\|^2\leq\|\xi-q\xi q\|^2+2\|r\xi'\|^2.$$

Similarly

$$\|\eta-(q-r)\eta(q-r)\|^2\leq\|\eta-q\eta q\|^2+2\|r\eta'\|^2.$$

Since $\{q_i\}_{i\in I}$ is in $\mathscr{F}$

$$\|\xi-q\xi q\|^2+\|\eta-q\eta q\|^2\leq 2^{14}\|\eta-q\eta q-\sum_{i\in I}q_i\eta q_i\|^2.$$

Moreover we have proved that

$$\|r\xi'\|^2+\|r\eta'\|^2\leq 2^{13}\|r\eta'-\eta'r\|^2.$$

Hence

$$\begin{aligned}\|\xi-(q-r)\xi(q-r)\|^2+\|\eta-(q-r)\eta(q-r)\|^2&\leq 2^{14}(\|\eta-q\eta q-\sum_{i\in I}q_i\eta q_i\|^2+\|r\eta'-\eta'r\|^2)\\&=2^{14}\|\eta-(q-r)\eta(q-r)-r\eta r-\sum_{i\in I}q_i\eta q_i\|^2.\end{aligned}$$

This proves (*). Hence we have proved that $\{q_i\}_{i\in I}\cup\{r\}$ is contained in $\mathscr{F}$, which contradicts the maximality of $\{q_i\}_{i\in I}$. Therefore

$$\|\eta-q\eta q-\sum_{i\in I}q_i\eta q_i\|^2\geq 2^{-18},$$

while

$$\|\xi-q\xi q-\sum_{i\in I} q_i\xi q_i\|^2\leq\delta.$$

Since $\{q_i\}_{i\in I}\cup\{q\}$ is a set of pairwise orthogonal projections with sum 1 we have proved Lemma 2.15.

LEMMA 2.16. *Assume that $M$ satisfies* (1) *in Theorem* 2.3. *Let $\xi\in P^\natural$ be a cyclic and separating unit vector, and let $\eta\in\mathcal{H}$ be a vector orthogonal to $\xi$. For every $\delta>0$, there exists a projection $p\in M$, such that*

$$\|p\xi-\xi p\|^2\leq\delta$$
$$\|p\eta-\eta p\|^2\geq 2^{-21}\|\eta\|^2.$$

*Proof.* Assume first that $\eta\in\mathcal{H}_{\text{s.a.}}$. By Lemma 2.15 there exists a set of pairwise orthogonal non-zero projections $\{e_i\}_{i\in I}$ with sum 1, such that

$$\|\xi-\sum_{i\in I} e_i\xi e_i\|^2\leq\delta$$

and

$$\|\eta-\sum_{i\in I} e_i\eta e_i\|^2\geq 2^{-18}\|\eta\|^2.$$

Since $M$ is $\sigma$-finite, the index set $I$ is countable. Let $G$ be the compact abelian group

$$G=\{-1,1\}^I.$$

For $g\in G$, $g=(g_i)_{i\in I}$ we put

$$u_g=\sum_{i\in I} g_i e_i.$$

Clearly $u_g$ is a selfadjoint unitary operator for all $g$. Moreover

$$g\to u_g$$

is a strongly continuous representation of $G$ on $\mathcal{H}$. Therefore

$$g\to u_g J u_g J$$

is also a strongly continuous unitary representation of $G$. Let $dg$ be the normalized Haar measure on $G$. Then

$$\int_G u_g(Ju_gJ)\,dg = \sum_{i,j} \int_G g_i g_j (e_i J e_j J)\,dg.$$

Since $dg = \Pi_{i\in I}\, dg_i$, where $dg_i$ has mass $\frac{1}{2}$ at both 1 and $-1$, it is clear that

$$\int_G g_i g_j\,dg = \begin{cases} 1 & i=j \\ 0 & i\neq j \end{cases}.$$

Hence

$$\int_G u_g(Ju_gJ)\,dg = \sum_i e_i J e_i J.$$

Therefore,

$$\int_G (\eta - u_g \eta u_g)\,dg = \eta - \sum_{i\in I} e_i \eta e_i.$$

In particular

$$\begin{aligned}\int_G \|\eta - u_g \eta u_g\|\,dg &\geq \|\eta - \sum_{i\in I} e_i \eta e_i\| \\ &\geq 2^{-9}\|\eta\|,\end{aligned}$$

so for at least one $g\in G$,

$$\|\eta - u_g \eta u_g\| \geq 2^{-9}\|\eta\|.$$

Equivalently

$$\|u_g \eta - \eta u_g\|^2 \geq 2^{-18}\|\eta\|^2.$$

Put $\xi' = \Sigma_{i\in I}\, e_i \xi e_i$. Then $u_h \xi' = \xi' u_h$ for all $h\in G$. Therefore

$$\begin{aligned}\|u_g \xi - \xi u_g\| &\leq 2\|\xi - \xi'\| \\ &\leq 2\delta^{1/2}.\end{aligned}$$

Let now $p$ be the projection $p = \frac{1}{2}(1+u_g)$.

Then clearly

$$\|p\xi-\xi p\|^2\leqslant\delta \quad\text{and}\quad \|p\eta-\eta p\|^2\geqslant 2^{-20}\|\eta\|^2.$$

Let finally $\eta\in\mathcal{H}$ be a general vector orthogonal to $\xi$. Put

$$\eta_1=\tfrac{1}{2}(\eta+J\eta),\quad \eta_2=\frac{1}{2i}(\eta-J\eta).$$

Then $\eta_1,\eta_2\in\mathcal{H}_{\mathrm{s.a.}}$, $\eta_i\perp\xi$, $i=1,2$, $\eta=\eta_1+i\eta_2$ and $\|\eta\|^2=\|\eta_1\|^2+\|\eta_2\|^2$. Therefore we can choose $j\in\{1,2\}$ such that $\|\eta_j\|^2\geqslant\frac{1}{2}\|\eta\|^2$. By the above arguments, there exists a projection $p\in M$, such that

$$\|p\xi-\xi p\|^2\leqslant\delta \quad\text{and}\quad \|p\eta_j-\eta_j p\|^2\geqslant 2^{-20}\|\eta_j\|^2.$$

Clearly

$$p\eta-\eta p=(p\eta_1-\eta_1 p)+i(p\eta_2-\eta_2 p).$$

Moreover, one checks easily that

$$\begin{aligned} p\eta_1-\eta_1 p&\in i\mathcal{H}_{\mathrm{s.a.}}\\ i(p\eta_2-\eta_2 p)&\in\mathcal{H}_{\mathrm{s.a.}}.\end{aligned}$$

Therefore

$$\begin{aligned}\|p\eta-\eta p\|^2&=\|p\eta_1-\eta_1 p\|^2+\|p\eta_2-\eta_2 p\|^2\\ &\geqslant 2^{-20}\|\eta_j\|^2\\ &\geqslant 2^{-21}\|\eta\|^2.\end{aligned}$$

This proves Lemma 2.16.

*End of proof of Theorem* 2.3. Assume that $M$ satisfies (1) in Theorem 2.3, and let $\varphi$ be a normal faithful state on $M$. We shall show that $B_\varphi=\mathbf{C}1$. Let $a\in B_\varphi$, and put $a'=a-\varphi(a)1$.

Let $\xi_\varphi\in P^\natural$ be the unique vector in $P^\natural$ that implements $\varphi$. Then $\xi_\varphi$ is a cyclic and separating unit vector. The vector $\eta=a'\xi_\varphi$ is orthogonal to $\xi_\varphi$, because $\varphi(a')=0$. Thus by Lemma 2.16 we can choose a sequence $(p_n)_{n\in\mathbf{N}}$ of projections in $M$, such that for all $n\in\mathbf{N}$,

$$\|p_n\xi_\varphi-\xi_\varphi p_n\|\leqslant\frac{1}{n}\quad\text{and}\quad \|p_n\eta-\eta p_n\|\geqslant 2^{-11}\|\eta\|.$$

By Lemma 2.8 (b) the first inequality implies that $\lim_{n\to\infty}\|p_n\varphi-\varphi p_n\|=0$ and since $a\in B_\varphi$, it now follows that

$$\lim_{n\to\infty}\|p_n a-ap_n\|_\varphi=0.$$

On the other hand

$$\begin{aligned}\|p_n a-ap_n\|_\varphi &= \|(p_n a'-a'p_n)\xi_\varphi\|\\ &\geq \|p_n a'\xi_\varphi-a'\xi_\varphi p_n\|-\|a'\xi_\varphi p_n-a'p_n\xi_\varphi\|\\ &\geq \|p_n\eta-\eta p_n\|-\|a'\|\,\|p_n\xi_\varphi-\xi_\varphi p_n\|.\end{aligned}$$

Thus

$$\liminf_{n\to\infty}\|p_n a-ap_n\|_\varphi\geq\|p_n\eta-\eta p_n\|\geq 2^{-11}\|\eta\|.$$

Therefore $\eta=0$, which implies that $a'=0$. This proves that $B_\varphi=\mathbf{C}1$.

## 3. Characterization of $\mathrm{III}_1$-factors for which $B_\varphi=\mathbf{C}1$

In this section we will prove the following extension of Theorem 2.3:

THEOREM 3.1. *Let $M$ be a factor of type* $\mathrm{III}_1$ *with separable predual. Then the following three conditions are equivalent:*

(1) *For every (faithful) dominant weight $\psi$ on $M$ and every $x\in M$*

$$\overline{\mathrm{conv}}\{uxu^*\,|\,u\in U(M_\psi)\}\cap\mathbf{C}1\neq\emptyset$$

*($\sigma$-weak closure).*

(2) *For every normal faithful state $\varphi$ on $M$, $B_\varphi=\mathbf{C}1$.*

(3) *The set of normal faithful states on $M$ for which $M'_\varphi\cap M=\mathbf{C}1$ is norm dense in the set of all normal states on $M$.*

It is very likely that all $\mathrm{III}_1$-factors on a separable Hilbert space satisfy the above conditions (see Remark 3.9). The implication (1)$\Rightarrow$(2) was proved in Section 2. It remains to be proved that (2)$\Rightarrow$(3) and (3)$\Rightarrow$(1). The first three lemmas of this section is used to come from $B_\varphi=\mathbf{C}1$ back to the situation we had in Lemma 2.16. The rest of the proof of (2)$\Rightarrow$(3) is inspired by Popa's techniques from [19].

Throughout this section $M$ is a factor of type $\mathrm{III}_1$ with separable predual and with standard form $(M,\mathcal{H},J,P^\natural)$. As usual we define right multiplication of $M$ on $\mathcal{H}$ by

$$\eta a = Ja^*J\eta, \quad a \in M, \ \eta \in \mathscr{H}.$$

LEMMA 3.2. *Assume that* $B_\varphi = \mathbf{C}1$ *for all* n.f. *states on* $M$. *Let* $\xi \in P^\natural$ *be a cyclic and separating unit vector and let* $\eta \in \mathscr{H}$ *be orthogonal to* $\xi$. *For every* $\delta > 0$, *there exists a unitary operator* $u \in M$, *such that*

$$\|u\xi - \xi u\|^2 \leq \delta \quad \text{and} \quad \|u\eta - \eta u\|^2 \geq \tfrac{1}{2}\|\eta\|^2.$$

*Proof.* Let $\varphi$ be the vector state on $M$ given by $\xi$. By Lemma 2.8

$$\|u\xi - \xi u\|^2 \leq \|u\varphi - \varphi u\|, \quad u \in U(M). \tag{*}$$

It is sufficient to consider the case $\eta \neq 0$. Assume first that $\eta$ can be written in the form $\eta = a\xi$ for some $a \in M$. Since $n \perp \xi$ we have $\varphi(a) = 0$. Let $\delta > 0$ and put

$$\delta_1 = \min\{\delta, (\|\eta\|/8\|a\|)^2\}.$$

By Proposition 1.3(2), $(a) \Rightarrow (b)$, there exists $\lambda \in \mathbf{C}$, such that

$$a - \lambda I \in \overline{\mathrm{conv}}\{a - u^*au \mid u \in U(M), \ \|u\varphi - \varphi u\| \leq \delta_1\}.$$

Since the norm $\| \ \|_\varphi$ is $\sigma$-weakly lower semi-continuous, we get

$$\|a - \lambda I\|_\varphi \leq \sup\{\|a - u^*au\|_\varphi \mid u \in U(M), \ \|u\varphi - \varphi u\| \leq \delta_1\}$$

and since $\varphi(a) = 0$, $\|a - \lambda I\|_\varphi^2 = \|a\|_\varphi^2 + |\lambda|^2 \geq \|a\|_\varphi^2$.
Hence we can choose a unitary operator $u \in M$, such that $\|u\varphi - \varphi u\| \leq \delta_1$ and

$$\|a - u^*au\|_\varphi \geq \tfrac{7}{8}\|a\|_\varphi$$

or equivalently

$$\|ua - au\|_\varphi \geq \tfrac{7}{8}\|a\|_\varphi.$$

Thus

$$\begin{aligned} \|u\eta - \eta u\| &= \|ua\xi - a\xi u\| \\ &\geq \|(ua - au)\xi\| - \|a\|\,\|u\xi - \xi u\| \\ &\geq \tfrac{7}{8}\|a\|_\varphi - \|a\|\,\|u\xi - \xi u\|. \end{aligned}$$

By the inequality (*) we get

$$\|u\xi_\varphi - \xi_\varphi u\| \leq \delta_1^{1/2}.$$

Since $\|a\|_\varphi=\|\eta\|$ it follows that

$$\|u\eta-\eta u\|\geq \tfrac{7}{8}\|\eta\|-\delta_1^{1/2}\|a\|$$
$$\geq \tfrac{6}{8}\|\eta\|.$$

Hence

$$\|u\xi_\varphi-\xi_\varphi u\|^2\leq\delta$$

and

$$\|u\eta-\eta u\|^2\geq(\tfrac{3}{4})^2\|\eta\|^2>\tfrac{1}{2}\|\eta\|^2.$$

Finally, let $\eta\in\mathscr{H}$ be an arbitrary vector orthogonal to $\xi$. For every $\varepsilon>0$, there exists $\eta'\in M\xi$, such that $\|\eta-\eta'\|<\varepsilon$. Moreover, $\eta'$ can be chosen orthogonal to $\xi$, because the projection of $\eta'$ onto the orthogonal complement of $\mathbf{C}\xi$ also belongs to $M\xi$. It is clear that the distance between the two numbers,

$$\sup\{\|u\eta-\eta u\|\mid u\in U(M),\ \|u\xi-\xi u\|^2\leq\delta\}$$

and

$$\sup\{\|u\eta'-\eta' u\|\mid u\in U(M),\ \|u\xi-\xi u\|^2\leq\delta\}$$

is at most $2\varepsilon$. Hence, by letting $\varepsilon\to0$, we get by the first part of the proof that

$$\sup\{\|u\eta-\eta u\|\mid u\in U(M),\ \|u\xi-\xi u\|^2\leq\delta\}\geq\tfrac{3}{4}\|\eta\|.$$

Since $(\tfrac{3}{4})^2>\tfrac{1}{2}$ we have proved Lemma 3.2.

LEMMA 3.3 *Let $u\in M$ be a unitary operator, and let for $0<\theta\leq2\pi$, $p_\theta$ denote the spectral projection of $u$ corresponding to the semi circle $\{e^{it}|\theta\leq t<\theta+\pi\}$. For every $\zeta\in\mathscr{H}$*

(i) $$\int_0^{2\pi}\|p_\theta\zeta-\zeta p_\theta\|^2d\theta\leq\pi\|\zeta\|\,\|u\zeta-\zeta u\|$$

*and*

(ii) $$\int_0^{2\pi}\|p_\theta\zeta-\zeta p_\theta\|^2d\theta\geq\|u\zeta-\zeta u\|^2.$$

*Proof.* Let $\mathbf{T}$ be the unit circle in $\mathbf{C}$. Arguing as in the proof of Lemma 2.11, one can find a positive measure $\mu$ on $\mathbf{T}^2$ such that

$$\|f(u)\,\xi-\xi g(v)\|^2=\int\int_{\mathbf{T}^2}|f(s)-g(t)|^2 d\mu(s,t)$$

for all bounded Borel functions $f$, $g$ on $\mathbf{T}$. (See also [11, proof of Lemma 3.3].) Define a function $h$ on $\mathbf{T}\times\mathbf{T}\times]0,2\pi]$ by

$$h(s,t,\theta)=\begin{cases}1 & \text{if } \theta\leqslant\arg s<\theta+\pi \text{ and } \theta-\pi\leqslant\arg t<\theta\\ 1 & \text{if } \theta-\pi\leqslant\arg s<\theta \text{ and } \theta\leqslant\arg t<\theta+\pi\\ 0 & \text{otherwise.}\end{cases} \tag{*}$$

Then it follows that for all $0<\theta\leqslant 2\pi$

$$\|p_\theta\,\xi-\xi p_\theta\|^2=\int\int_{\mathbf{T}^2} h(s,t,\theta)\,d\mu(s,t).$$

Hence, by Fubini's theorem

$$\int_0^{2\pi}\|p_\theta\,\xi-\xi p_\theta\|^2 d\theta=\int\int_{\mathbf{T}^2}\left(\int_0^{2\pi} h(s,t,\theta)\,d\theta\right)d\mu(s,t).$$

Let $\beta\in[0,\pi]$. Then

$$h(1,e^{i\beta},\theta)=\begin{cases}1 & \theta\in\,]0,\beta]\cup\,]\pi,\beta+\pi]\\ 0 & \text{otherwise.}\end{cases}$$

Similarly, for $\beta\in[-\pi,0]$,

$$h(1,e^{i\beta},\theta)=\begin{cases}1 & \theta\in\,]\pi+\beta,\pi]\cup\,]2\pi+\beta,2\pi]\\ 0 & \text{otherwise.}\end{cases}$$

Hence,

$$\int_0^{2\pi} h(1,e^{i\beta},\theta)\,d\theta=2|\beta|\quad\text{for } -\pi\leqslant\beta\leqslant\pi.$$

Assume now that $h(s,t,\theta)$ is extended to a function on $\mathbf{T}\times\mathbf{T}\times\mathbf{R}$ periodic in $\theta$ with period $2\pi$. Then, for $\alpha$, $\beta\in\mathbf{R}$

$$h(e^{i\alpha},e^{i\beta},\theta)=h(1,e^{i(\beta-\alpha)},\theta-\alpha).$$

Therefore, if $|\alpha-\beta|\leqslant\pi$, we get

$$\int_0^{2\pi} h(e^{i\alpha}, e^{i\beta}, \theta)\, d\theta = \int_0^{2\pi} h(1, e^{i(\beta-\alpha)}, \theta)\, d\theta$$
$$= 2|\alpha-\beta|.$$

It is elementary to check that for $|\alpha-\beta|\leq\pi$ one has

$$\frac{2}{\pi}|\alpha-\beta| \leq |e^{i\alpha}-e^{i\beta}| \leq |\alpha-\beta|.$$

Since, for every pair $(s, t)\in\mathbf{T}^2$, one can choose $\alpha, \beta\in\mathbf{R}$, such that $e^{i\alpha}=s$, $e^{i\beta}=t$ and $|\alpha-\beta|\leq\pi$, it follows that

$$2|s-t| \leq \int_0^{2\pi} h(s, t, \theta)\, d\theta \leq \pi|s-t|$$

for all $(s, t)\in\mathbf{T}^2$. Hence,

$$\int_0^{2\pi} \|p_\theta\zeta-\zeta p_\theta\|^2 d\theta \leq \pi \int_{\mathbf{T}^2} |s-t|\, d\mu(s, t)$$
$$\leq \pi\left(\int_{\mathbf{T}^2} |s-t|^2 d\mu(s, t)\right)^{1/2}\left(\int_{\mathbf{T}^2} d\mu\right)^{1/2}$$
$$= \pi\|u\zeta-\zeta u\|\,\|\zeta\|,$$

and

$$\int_0^{2\pi} \|p_\theta\zeta-\zeta p_\theta\|^2 d\theta \geq 2\int_{\mathbf{T}^2} |s-t|\, d\mu(s, t)$$
$$\geq \int_{\mathbf{T}^2} |s-t|^2 d\mu(s, t)$$
$$= \|u\zeta-\zeta u\|^2.$$

This completes the proof of Lemma 3.3

LEMMA 3.4. *Assume that $M$ satisifes* (2) *in Theorem* 3.1. *Let $\xi\in P^\natural$ be a cyclic and separating unit vector and let $\eta\in\mathcal{H}$, $\eta\perp\xi$. Then, for every $\delta>0$, there exists a projection $p\in M$ such that*

$$\|p\xi-\xi p\|^2 \leq \delta$$
$$\|p\eta-\eta p\|^2 \geq \tfrac{1}{32}\|\eta\|^2.$$

*Proof.* We can assume that $\|\eta\|=1$. By Lemma 3.2 there exists $u\in U(M)$, such that

$$\|u\xi-\xi u\|^2\leqslant(\delta/16)^2$$

and

$$\|u\eta-\eta u\|^2\geqslant\tfrac{1}{2}.$$

Let $p_\theta$, $0<\theta\leqslant 2\pi$ be as in Lemma 3.3. Then

$$\int_0^{2\pi}\|p_\theta\xi-\xi p_\theta\|^2 d\theta\leqslant\delta/16$$

and

$$\int_0^{2\pi}\|p_\theta\eta-\eta p_\theta\|^2 d\theta\geqslant\tfrac{1}{2}.$$

Therefore,

$$\int_0^{2\pi}\left(\frac{1}{2\pi}+\frac{16}{\delta}\|p_\theta\xi-\xi p_\theta\|^2\right)d\theta\leqslant 2$$
$$\leqslant 4\int_0^{2\pi}\|p_\theta\eta-\eta p_\theta\|^2 d\theta.$$

Hence, for some $\theta\in\,]0,2\pi]$ we must have

$$\frac{1}{2\pi}+\frac{16}{\delta}\|p_\theta\xi-\xi p_\theta\|^2\leqslant 4\|p_\theta\eta-\eta p_\theta\|^2.$$

In particular, for this $\theta$,

$$\begin{aligned}\|p_\theta\xi-\xi p_\theta\|^2&\leqslant\frac{\delta}{4}\|p_\theta\eta-\eta p_\theta\|^2\\&\leqslant\frac{\delta}{4}(\|p_\theta\eta\|+\|\eta p_\theta\|)^2\\&\leqslant\delta\end{aligned}$$

and

$$\|p_\theta\eta-\eta p_\theta\|^2\geqslant\frac{1}{8\pi}>\frac{1}{32}.$$

This completes the proof.

For any von Neumann subalgebra $N$ of $M$ we put

$$\mathscr{H}_N = \{\eta \in \mathscr{H} \mid a\eta = \eta a,\ a \in N\},$$

and we let $Q_N$ be the projection of $\mathscr{H}$ onto $\mathscr{H}_N$. It is clear that $\mathscr{H}_N$ is invariant under $J$. We let $J_N$ denote the restriction of $J$ to $\mathscr{H}_N$.

LEMMA 3.5. *Let $N$ be a finite dimensional subfactor of $M$. Then*
(a) *$\mathscr{H}_N$ is invariant under $N' \cap M$, and*

$$(N' \cap M, \mathscr{H}_N, J_N, P^\natural \cap \mathscr{H}_N)$$

*is a standard form for $N' \cap M$.*
(b) *If $\xi \in P^\natural$ then $Q_N(\xi) \in P^\natural \cap \mathscr{H}_N$. If, moreover, $\xi$ is cyclic and separating for $M$, then $Q_N(\xi)$ is cyclic and separating for $N' \cap M$ on $\mathscr{H}_N$.*

*Proof.* (a) It is clear that $\mathscr{H}_N$ is $N' \cap M$-invariant. Let $(e_{ij})_{i,j=1}^n$ be a set of matrix units for $N$, and put $e = e_{11}$. By [15, Lemma 2.6]

$$(eMe, e\mathscr{H}e, J_e, eP^\natural e)$$

is a standard form for $eNe$. ($J_e$ is the restriction of $J$ to $e\mathscr{H}e$.) We will establish an explicit isomorphism between this quadruple and $(N' \cap M, \mathscr{H}_N, J_N, P^\natural \cap \mathscr{H}_N)$.

Since $N$ is a finite factor, $M$ can be identified with $(N' \cap M) \otimes N$. From this it follows that the map

$$x \to xe, \quad x \in N' \cap M$$

is a $*$-isomorphism of $N' \cap M$ onto $eMe$.

It is easy to check that the orthogonal projection $Q_N$ of $\mathscr{H}$ into $\mathscr{H}_N$ is given by

$$Q_N = \frac{1}{n} \sum_{i,j=1}^n e_{ij} J e_{ij} J.$$

Put

$$w = \frac{1}{\sqrt{n}} \sum_{i=1}^n e_{i1} J e_{i1} J.$$

Then $w^* w = eJeJ$ and $ww^* = Q_N$. Thus $w$ is an isometry of $e\mathscr{H}e$ onto $\mathscr{H}_N$. Since $w$ commutes with every $x \in N' \cap M$, we have for $x \in N' \cap M$ and $\xi \in e\mathscr{H}e$,

$$w^*xw\xi = xw^*w\xi = x\xi = (xe)\xi.$$

Hence $w$ implements a spatial isomorphism of $(eMe, e\mathcal{H}e)$ onto $(N'\cap M, \mathcal{H}_N)$. Since $aJaJ(P^\natural)\subseteq P^\natural$ for all $a\in M$,

$$w(eP^\natural e)\subseteq w(P^\natural)\subseteq P^\natural\cap\mathcal{H}_N$$

and

$$w^*(P^\natural\cap\mathcal{H}_N)\subseteq w^*(P^\natural)\subseteq eP^\natural e.$$

Since also $wJ=Jw$, one gets that $w$ implements an isomorphism between $(eNe, e\mathcal{H}e, Je, eP^\natural e)$ and $(N'\cap M, \mathcal{H}_N, J_N, P^\natural\cap\mathcal{H}_N)$. This proves (a).

(b) It is clear from the computations above that

$$Q_N(P^\natural)\subseteq P^\natural\cap\mathcal{H}_N.$$

Let $\xi\in P^\natural$ be cyclic and separating. Put

$$\zeta = w^*\xi = \frac{1}{\sqrt{n}}\sum_{i=1}^{n} e_{1i}\xi e_{1i}^*.$$

Then $\zeta\in e\mathcal{H}e$ and $w(\xi)=Q_N(\xi)$. By [6, Lemma 4.3], $e\xi e\in eP^\natural e$ is cyclic and separating for $eMe$ acting on $e\mathcal{H}e$. Since

$$\zeta\geqslant\frac{1}{\sqrt{n}}e\xi e$$

in ordering on $e\mathcal{H}e$ given by the cone $eP^\natural e$, $\zeta$ is also cyclic and separating for $eMe$. Therefore $Q_N(\xi)=w\zeta$ is cyclic and separating for $N'\cap M$ on $\mathcal{H}_N$.

LEMMA 3.6. *Assume that $M$ satisfies* (2) *in Theorem* 3.1, *and let $\xi\in P^\natural$ be a cyclic and separating unit vector. Let $\eta\in\mathcal{H}$, $\eta\perp\xi$. For every $\delta>0$ there exists a finite dimensional subfactor $N$ of $M$, such that*

$$\|\xi-Q_N(\xi)\|^2\leqslant\delta$$

and

$$\|Q_N(\eta)\|^2\leqslant\tfrac{31}{32}\|\eta\|^2.$$

*Proof.* We may assume that $\delta<1$. By Lemma 3.4 there exists a projection $p\in M$, such that

$$\|p\xi-\xi p\|^2 \leq \delta^2/36$$

and

$$\|p\eta-\eta p\|^2 \geq \tfrac{1}{32}\|\eta\|^2.$$

Clearly $p\neq 0$ and $(1-p)\neq 0$. Choose a rational number $\varrho \in ]0,1[$, such that

$$\varrho-\delta/6 < \|p\xi\|^2 < \varrho+\delta/6.$$

Write $\varrho=k/d$, where $d\in \mathbf{N}$, and $k$ is an integer, $0<k<d$. Put $\varphi=\omega_\xi$ on $M$ and put $\varphi'=p\varphi p+(1-p)\varphi(1-p)$. Let $u=2p-1$. Then $u$ is a selfadjoint unitary, and

$$\varphi' = \tfrac{1}{2}(\varphi+u\varphi u^*).$$

The state $u\varphi u^*$ is implemented by the vector $u\xi u^* \in P^\natural$. A simple computation shows that any two vector states $\omega_\eta$ and $\omega_\zeta$, one has $\|\omega_\eta-\omega_\zeta\| \leq \|\eta-\zeta\|\,\|\eta+\zeta\|$. Hence

$$\begin{aligned}\|\varphi'-\varphi\| &= \tfrac{1}{2}\|\varphi-u\varphi u^*\| \\ &\leq \tfrac{1}{2}\|\xi-u\xi u^*\|\,\|\xi+u\xi u^*\| \\ &\leq \|u\xi-\xi u\| \\ &= 2\|p\xi-\xi p\| \\ &\leq \delta/3.\end{aligned}$$

Choose next a normal faithful state $\psi$ on $M$, such that the centralizer $M_\psi$ of $\psi$ contains a subfactor $F$ isomorphic to the $d\times d$-matrices $M_d$. This is possible because $M\cong M\otimes M_d$ and because the centralizer of $\varphi\otimes \mathrm{tr}$ contains $1\otimes M_d$ (tr is here the normalized trace on $M_d$). Let $q\in F$ be a projection of dimension $k$ (relative to $F$). Then $\psi(q)=k/d$. Since $M$ is of type III, we have $p\sim q$ and $(1-p)\sim(1-q)$ as projections in $M$, so we can choose a unitary operator $v\in M$, such that $vqv^*=p$. Now, put

$$\psi' = v\psi v^* \quad \text{and} \quad F' = vFv^*.$$

Then $\psi'$ is a faithful normal state on $M$, and

$$p\in F' \subseteqq M_{\psi'}.$$

Note that by the definition of $\varphi'$ also $p\in M_{\varphi'}$. Moreover,

$$\varphi'(p) = \varphi(p) = \|p\xi\|^2.$$

Put

$$\varphi_1' = \frac{1}{\varphi'(p)} p\varphi', \quad \varphi_2' = \frac{1}{\varphi'(p^\perp)} p^\perp \varphi'$$
$$\psi_1' = \frac{1}{\psi'(p)} p\psi', \quad \psi_2' = \frac{1}{\psi'(p^\perp)} p^\perp \psi'.$$

Then $\varphi_1'$, $\psi_1'$ are faithful states on $pMp$ and $\varphi_2'$, $\psi_2'$ are faithful states on $p^\perp M p^\perp$. Since $pMp \cong p^\perp M p^\perp \cong M$ and since $M$ is of type $\mathrm{III}_1$, we can by the Connes-Størmer transitivity theorem [12] find unitaries $w_1 \in pMp$ and $w_2 \in p^\perp M p^\perp$, such that

$$\|\varphi_i' - w_i \psi_i' w_i^*\| \leq \delta/3, \quad i = 1, 2.$$

Then $w = w_1 + w_2$ is a unitary in $M$.

Since $\psi'(p) = \psi(q) = k/d$ we have

$$\psi' = \frac{k}{d}\psi_1' + \frac{d-k}{d}\psi_2'.$$

Therefore

$$\left\| \left( \frac{k}{d}\varphi_1' + \frac{d-k}{d}\varphi_2' \right) - w\psi' w^* \right\| \leq \delta/3.$$

Using

$$\varphi' = \varphi'(p)\,\varphi_1' + \varphi'(p^\perp)\,\varphi_2'$$

and that

$$\frac{k}{d} - \frac{\delta}{6} \leq \varphi'(p) \leq \frac{k}{d} + \frac{\delta}{6},$$

we have

$$\frac{d-k}{d} - \frac{\delta}{6} \leq \varphi'(p^\perp) \leq \frac{d-k}{d} + \frac{\delta}{6}.$$

Thus

$$\left\| \varphi' - \left( \frac{k}{d}\varphi_1' + \frac{d-k}{d}\varphi_2' \right) \right\| \leq \frac{\delta}{3}.$$

Since $\|\varphi - \varphi'\| \leq \delta/3$ we have altogether

$$\|\varphi - w\psi' w^*\| \leq \delta.$$

Put $\omega = w\psi' w^*$. Then $\omega$ is a faithful state and $M_\omega$ contains the finite dimensional factor $N = wF'w^*$. Moreover, $p \in N$ because $wpw^* = p$. Let $\xi_\omega$ be the unique vector in $P^\natural$ that implements $\omega$. Then, by Araki's generalization of the Powers-Størmer inequality [1, Theorem 4(8)],

$$\|\xi - \xi_\omega\|^2 \leq \|\varphi - \omega\| \leq \delta.$$

Since $u\omega u^* = \omega$ for all $u \in U(N)$, we have $u\xi_\omega u^* = \xi_\omega$ for all $u \in U(N)$. Hence,

$$\xi_\omega \in \mathcal{H}_N = \{\zeta \in \mathcal{H} \mid a\zeta = \zeta a,\ a \in N\}.$$

Therefore,

$$\|\xi - Q_N(\xi)\|^2 = \operatorname{dist}(\xi, \mathcal{H}_N)^2 \leq \|\xi - \xi_\omega\|^2 \leq \delta.$$

Put

$$\mathcal{K} = \{\eta \in \mathcal{H} \mid p\eta = \eta p\}.$$

Then $\mathcal{K}$ is a closed subspace of $\mathcal{H}$, and the orthogonal projection $Q$ of $\mathcal{H}$ onto $\mathcal{K}$ is given by

$$Q(\zeta) = p\zeta p + (1-p)\,\zeta(1-p).$$

Since $p\zeta p$, $(1-p)\,\zeta(1-p)$, $p\zeta(1-p)$, $(1-p)\,\zeta p$ are orthogonal vectors in $\mathcal{H}$ with sum $\zeta$, we have for all $\zeta \in \mathcal{H}$,

$$\begin{aligned}\|\zeta\|^2 &= \|Q(\zeta)\|^2 + \|p\zeta(1-p) - (1-p)\,\zeta p\|^2 \\ &= \|Q(\zeta)\|^2 + \|p\zeta - \zeta p\|^2\end{aligned}$$

Using that $\mathcal{H}_N \subseteq \mathcal{K}$, and that $\|p\eta - \eta p\|^2 \geq \frac{1}{32}\|\eta\|^2$, we get

$$\begin{aligned}\|Q_N(\eta)\|^2 \leq \|Q(\eta)\|^2 &= \|\eta\|^2 - \|p\eta - \eta p\|^2 \\ &\leq \tfrac{31}{32}\|\eta\|^2.\end{aligned}$$

This completes the proof of Lemma 3.6.

LEMMA 3.7. *Assume that $M$ satisfies* (2) *in Theorem* 3.1. *Let $\xi \in P^\natural$ be a cyclic and separating unit vector and let $\eta \in \mathcal{H}$. For every $\delta > 0$ and $\varepsilon > 0$ there exists a finite dimensional subfactor $N$ of $M$ such that*

$$\|\xi - Q_N(\xi)\| \leq \delta$$
$$\operatorname{dist}(Q_N(\eta), \mathbf{C}Q_N(\xi)) \leq \varepsilon.$$

*Proof.* We prove by induction that for every $r \in \mathbf{N}$ there exists a finite dimensional subfactor $N_r$ of $M$, such that

$$\|\xi - Q_{N_r}(\xi)\| \leq (1-2^{-r})\,\delta \tag{*}$$

and

$$\operatorname{dist}(Q_{N_r}(\eta), \mathbf{C}Q_{N_r}(\xi)) \leq (\tfrac{31}{32})^{r/2} \operatorname{dist}(\eta, \mathbf{C}\xi). \tag{**}$$

First, let $r=1$. Put $c=(\eta, \xi)$. Then

$$\eta = c\xi + \eta'$$

where $\eta' \perp \xi$, and $\operatorname{dist}(\eta, \mathbf{C}\xi) = \|\eta'\|$.

By Lemma 3.6 there exists a finite dimensional subfactor $N_1$ of $M$, such that

$$\|\xi - Q_{N_1}(\xi)\| \leq \delta/2$$

and

$$\|Q_{N_1}(\eta')\|^2 \leq \tfrac{31}{32}\|\eta'\|^2.$$

Hence,

$$\begin{aligned}\operatorname{dist}(Q_{N_1}(\eta), \mathbf{C}Q_N(\xi)) &= \operatorname{dist}(Q_N(\eta'), \mathbf{C}Q_N(\xi))\\ &\leq (\tfrac{31}{32})^{1/2}\|\eta'\|\\ &= (\tfrac{31}{32})^{1/2} \operatorname{dist}(\eta, \mathbf{C}\xi).\end{aligned}$$

This proves (*) and (**) for $r=1$. Assume next that we have found $N_r$ satisfying (*) and (**). We proceed to construct $N_{r+1}$. Put

$$\xi' = Q_{N_r}(\xi) \quad \text{and} \quad \eta' = Q_{N_r}(\eta).$$

By Lemma 3.5,

$$(N_r' \cap M, \mathcal{H}_{N_r}, J_{N_r}, P^\natural \cap \mathcal{H}_{N_r})$$

is a standard form for $N_r' \cap M$ and $N_r' \cap M$ is isomorphic to $M$. Moreover, $\xi'$ is cyclic and separating for $N_r' \cap M$ on $\mathcal{H}_{N_r}$. Using the above argument for $r=1$ to the two vectors $\xi'' = \xi'/\|\xi'\|$ and $\eta'$ we can find a finite dimensional subfactor $F$ of $N_r' \cap M$, such that

$$\|\xi' - Q_F'(\xi')\| \leq 2^{-r-1}\delta$$

and

$$\operatorname{dist}(Q_F'(\eta'), \mathbf{C}Q_F'(\xi')) \leq (\tfrac{31}{32})^{1/2} \operatorname{dist}(\eta', \mathbf{C}\xi')$$

where $Q_F'$ is the projection of $\mathcal{H}_{N_r}$ onto

$$\{\eta \in \mathcal{H}_{N_r} |\ a\eta = \eta a,\ a \in F\}.$$

Put $N_{r+1} = \operatorname{span}\{ab | a \in N_r, b \in F\}$. Since $N_r$ and $F$ are commuting finite dimensional factors, $N_{r+1}$ is also a finite dimensional factor. Moreover,

$$\begin{aligned} \mathcal{H}_{N_{r+1}} &= \{\eta \in \mathcal{H} |\ a\eta = \eta a,\ a \in N_{r+1}\} \\ &= \{\eta \in \mathcal{H}_{N_r} |\ b\eta = \eta b,\ b \in F\}. \end{aligned}$$

Therefore $Q_{N_{r+1}} = Q_F' Q_{N_r}$. Hence,

$$\begin{aligned} \|\xi - Q_{N_{r+1}}(\xi)\| &\leq \|\xi - Q_{N_r}(\xi)\| + \|\xi' - Q_F'(\xi')\| \\ &\leq (1 - 2^{-r-1})\delta \end{aligned}$$

and

$$\begin{aligned} \operatorname{dist}(Q_{N_{r+1}}(\eta), \mathbf{C}Q_{N_{r+1}}(\xi)) &\leq (\tfrac{31}{32})^{1/2} \operatorname{dist}(\eta', \mathbf{C}\xi') \\ &\leq (\tfrac{31}{32})^{(r+1)/2} \operatorname{dist}(\eta, \mathbf{C}\xi), \end{aligned}$$

which proves (*) and (**) for $r+1$. Thus we can find $N_r$ satisfying (*) and (**) for all $r$. Choose now $r$ such that

$$(\tfrac{31}{32})^{r/2} \operatorname{dist}(\eta, \mathbf{C}\xi) \leq \varepsilon,$$

then Lemma 3.7 holds with $N = N_r$.

LEMMA 3.8. *Assume that $M$ satisfies* (2) *in Theorem* 3.1, *and let $\xi \in P^\natural$ be a cyclic and separating unit vector. Let $0 < \delta < 1$. There exists an increasing sequence of finite* *dimensional subfactors $(N_n)_{n\in\mathbf{N}}$ of $M$, such that when $N$ is the von Neumann algebra generated by $\bigcup_{n=1}^{\infty} N_n$, then*

$$\|\xi - Q_N(\xi)\| \leq \delta$$
$$Q_N(\mathcal{H}) = \mathbf{C}Q_N(\xi).$$

*Proof.* Since $M$ has separable predual, the Hilbert space $\mathcal{H}$ in the standard form of $M$ is also separable. Let $(\eta_n)_{n=1}^{\infty}$ be a dense sequence in $\mathcal{H}$. We will construct an increasing sequence $(N_n)_{n\in\mathbf{N}}$ of finite dimensional subfactors of $M$, such that

$$\|\xi - Q_{N_n}(\xi)\| \leq (1-2^{-n})\,\delta \tag{*}$$

$$\operatorname{dist}(Q_{N_n}(\eta_n), \mathbf{C}Q_{N_n}(\xi)) \leq 2^{-n} \tag{**}$$

for all $n \in \mathbf{N}$. Lemma 3.7 shows that we can choose $N_1$, such that (*) and (**) are fulfilled for $n=1$. Assume next that we have found

$$N_1 \subseteq N_2 \subseteq \ldots \subseteq N_r$$

satisfying the conditions up to $n=r$, and let us proceed to construct $N_{r+1}$:

Put $\xi' = Q_{N_r}(\xi)$ and $\eta' = Q_{N_r}(\eta_{r+1})$. By applying Lemma 3.7 to the standard form

$$(N_r' \cap M, \mathcal{H}_{N_r}, J_{N_r}, P^{\natural} \cap \mathcal{H}_{N_r})$$

and the two vectors $\xi'' = \xi'/\|\xi'\|$ and $\eta'$, one can find a finite dimensional subfactor $F$ of $N_r' \cap M$, such that

$$\|\xi' - Q_{N_r}(\xi')\| \leq 2^{-r-1}\delta$$
$$\operatorname{dist}(Q_F'(\eta'), \mathbf{C}Q_F'(\xi')) \leq 2^{-r}$$

where $Q_F'$ is the projection of $\mathcal{H}_{N_r}$ onto the elements in $\mathcal{H}_{N_r}$, that commutes with $F$.

As in the proof of Lemma 3.6, one sees that

$$N_{r+1} = \operatorname{span}\{ab \mid a \in N_r,\ b \in F\}$$

is a finite dimensional subfactor of $M$, and that

$$\|\xi - Q_{N_{r+1}}(\xi)\| \leq (1-2^{-r-1})\,\delta$$

$$\operatorname{dist}(Q_{N_{r+1}}(\eta_{r+1}), \mathbf{C}Q_{N_{r+1}}(\xi)) \leq 2^{-r}.$$

Moreover, $N_r \subseteq N_{r+1}$. Hence $N_1 \subseteq \dots \subseteq N_r \subseteq N_{r+1}$ satisfy the conditions (*) and (**) up to $n=r+1$. By induction we get an increasing sequence $(N_n)_{n\in\mathbf{N}}$ of subfactors satisfying (*) and (**). Put now $N=\bigcup_{n=1}^{\infty} N_n$.

Since $\mathcal{H}_{N_n}$ is a decreasing sequence of Hilbert spaces, and since

$$\mathcal{H}_N = \bigcap_{n=1}^{\infty} \mathcal{H}_{N_n}$$

we have $Q_N=\lim_{n\to\infty} Q_{N_n}$ (strongly).

Therefore $\|\xi-Q_N(\xi)\|\leq\delta$. For each $n\in\mathbf{N}$ we can choose $c_n\in\mathbf{C}$ such that

$$\|Q_{N_n}(\eta_n)-c_n Q_{N_n}(\xi)\| \leq 2^{-n}.$$

Since $Q_N Q_{N_n}=Q_N$ it follows that

$$\|Q_N(\eta_n)-c_n Q_N(\xi)\| \leq 2^{-n}, \quad n\in\mathbf{N}.$$

Hence

$$\operatorname{dist}(Q_N(\eta_n), \mathbf{C}Q_N(\xi)) \leq 2^{-n}.$$

For each $n\in\mathbf{N}$, the sequence $(\eta_m)_{m\geq n}$ is also dense in $\mathcal{H}$. Therefore

$$\operatorname{dist}(Q_N(\mathcal{H}), \mathbf{C}Q_N(\xi)) \leq 2^{-n}.$$

Hence

$$Q_N(\mathcal{H}) \subseteq \overline{\mathbf{C}Q_N(\xi)} = \mathbf{C}Q_N(\xi).$$

This proves Lemma 3.8.

*End of proof of* (2)⇒(3). Assume that $M$ satisfies condition (2) in Theorem 3.1, and let $\xi\in P^\natural$ be a cyclic and separating unit vector. Let $N_n$ and $N$ be as in Lemma 3.8 with $\delta=\frac{1}{2}$, and put $\xi'=Q_N(\xi)$. Then $\xi'\neq 0$ and

$$\mathcal{H}_N = \mathbf{C}\xi'.$$

Since $\xi'=\lim_{n\to\infty} Q_{N_n}(\xi)$, it follows from Lemma 3.5 that $\xi'\in P^\natural$. Let $e\in M$ be the projection of the vector functional $\varphi'$ on $M$ given by $\xi'$. Then $e\xi'=\xi'$, and since $J\xi'=\xi'$ also $\xi' e=\xi'$. Hence $\xi'\in e\mathcal{H}e$. By [15, Lemma 2.6]

$$(eMe, e\mathcal{H}e, J_e, eP^\natural e)$$

is a standard form for $eMe$. (Here $J_e$ is the restriction of $J$ to $e\mathcal{H}e$.) Moreover, $\xi'$ is cyclic and separating for $eMe$ acting on $e\mathcal{H}e$. Since $\xi' \in Q_N(\mathcal{H})$, we have

$$u\xi' u^* = \xi', \quad u \in U(N).$$

Hence also

$$u\varphi' u^* = \varphi', \quad u \in U(N)$$

and

$$ueu^* = e, \quad u \in U(N).$$

Thus $e \in N' \cap M$. Let $\psi$ be the restriction of $\varphi'/\|\varphi'\|$ *to* $eMe$. Then $\psi$ is a normal faithful state on $eMe$, and

$$eN \subseteqq M_\psi.$$

We will show that $(eN)' \cap eMe = \mathbf{C}I_e$, where $I_e = e$ is the identity in $eMe$. Let

$$x \in (eN)' \cap eMe$$

regarded as an operator on $\mathcal{H}$ and

$$\eta = x\xi' \in \mathcal{H}.$$

Since $e \in N'$ we have for all $a \in N$ that $ax = xa$. Thus, for $a \in N$,

$$a\eta = ax\xi' = xa\xi' = x\xi' a = \eta a.$$

Hence $\eta \in \mathcal{H}_N = \mathbf{C}\xi'$. Since $\xi'$ is separating for $eMe$, it follows that $x = I_e$. Thus

$$(eN)' \cap eMe = \mathbf{C}I_e$$

and since $eN \subseteq M_\psi$ we have also

$$M'_\psi \cap eMe = \mathbf{C}I_e.$$

Since $eMe \cong M$, we have proved that $M$ has at least one normal faithful state $\omega$, such that $M'_\omega \cap M = \mathbf{C}I$. The density of such states in the set of normal states follows now from the Connes-Størmer transitivity theorem [12].

*Proof of* (3)⇒(1) *in Theorem* 3.1. Assume that $M$ is a type $\mathrm{III}_1$-factor with separable predual, and that $\varphi$ is a n.f. state on $M$, such that $M'_\varphi \cap M = \mathbf{C}1$. By ([4] or [23]) there exists a normal faithful conditional expectation of $M$ onto $M_\varphi$. Since $M_\varphi$ is a finite factor, we get by Popa's result [19, Theorem 3.2] that $M_\varphi$ contains a maximal abelian $*$-subalgebra $A$ of $M$. Let $\omega$ be the n.f.s. weight on $B(L^2(\mathbf{R}))$ for which

$$(D\omega : D(\mathrm{Tr}))_t = u_t$$

where

$$(u_t f)(s) = f(s-t), \quad s, t \in \mathbf{R}, f \in L^2(\mathbf{R}).$$

By [13, p. 497] $\psi = \varphi \otimes \omega$ is a dominant weight on $M \bar{\otimes} B(L^2(\mathbf{R}))$. It is clear that $M_\omega$ contains a maximal abelian subalgebra of $B(L^2(\mathbf{R}))$, namely the von Neumann algebra $B$ generated by $\{u_t | t \in \mathbf{R}\}$. Thus $C = A \bar{\otimes} B$ is a maximal abelian von Neumann subalgebra of $M \bar{\otimes} B(L^2(\mathbf{R}))$. Moreover, $C$ is contained in $M_\psi$. Since $M \bar{\otimes} B(L^2(\mathbf{R})) \cong M$ it follows that $M$ has a dominant weight $\psi$, such that $M_\psi$ contains a maximal abelian $*$-subalgebra $C$ of $M$.

Since the unitary group $U(C)$ of $C$ is abelian, it has an invariant mean $m$. For every $x \in M$, the integral

$$y = \int_{U(C)} uxu^* dm(u)$$

defines an element in $C' \cap M = C \subseteq M_\psi$. Moreover,

$$y \in \overline{\mathrm{conv}}\,\{uxu^* | u \in U(M_\psi)\}, \quad (\sigma\text{-weak closure}). \tag{*}$$

Since $M_\psi$ is a factor, we get by "the Diximier averaging process" (cf. [14, Part III, Chapter 5, Lemma 4], that

$$\overline{\mathrm{conv}}\,\{uyu^* | u \in U(M_\psi)\} \cap \mathbf{C}1 \neq \emptyset.$$

By (*) it now follows that

$$\overline{\mathrm{conv}}\,\{uxu^* | u \in U(M_\psi)\} \cap \mathbf{C}1 \neq \emptyset.$$

Since any two dominant weights on $M$ are unitary equivalent, we have proved (1).

*Remark* 3.9. The problem whether the conditions (1), (2) and (3) in Theorem 3.1 holds in all $\mathrm{III}_1$-factors with separable predual is related to the following problem of Kadison (cf. [18], [19]): Let $N$ be a subfactor of a factor $M$, such that $N' \cap M = \mathbf{C}1$. Does $N$ contain a maximal abelian $*$-subalgebra which is also maximal abelian in $M$? Indeed, if Kadison's problem has an affirmative solution for factors on a separable Hilbert space, then for any dominant weight $\psi$ on a $\mathrm{III}_1$-factor $M$ with separable predual, $M_\psi$ contains a maximal abelian $*$-subalgebra $C$ of $M$, and hence by the above proof of (3)$\Rightarrow$(1) it follows that condition (1) in Theorem 3.1 holds.

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*Received January 21, 1985*