SINGULAR LEBESGUE-STIELTJES INTEGRAL EQUATIONS.

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1. Introduction.

The theory of integral equations, broadly outlined, consists, on the one hand, of developments which could be generally classed as of Fredholm type and which may be based on Lebesgue integration; in this connection of particular interest are symmetric kernels, when the characteristic values are real and the characteristic functions form an orthogonal set. On the other hand, there exist developments relating to kernels for which the theory of Fredholm type does not apply and which entail results of form essentially distinct from that involved in the Fredholm theory — prominent in this respect are the names of H. WEYL[1] and

[1] H. WEYL, Singuläre Integralgleichungen mit besonderer Berücksichtigung des Fourierschen Integraltheorems, Göttingen, 1908; pp. 1—86.

T. CARLEMAN[1]; the equations, involved, which may still be expressed with the aid of Lebesgue integrals, contain kernels — one may appropriately term them singular — which are expressible as limits, in one sense or another, of kernels of Fredholm type. Singular kernels have been also studied in a number of papers by the present author[2].

Now, the demands of Mathematical Physics often necessitate consideration of functions of sets instead of points; accordingly a theory has been developed by N. GUNTHER[3] relating to linear equations involving Stieltjes integrals, the integrations being appropriately defined[4]. The kernels considered by Gunther are sufficiently 'regular' so as to secure results resembling those of Fredholm and, in the case of 'symmetry' suitably defined, resembling those of Schmidt. *It is our present purpose to consider kernels more general than those of Gunther and to develop (for the 'symmetric' case) a theory of equations whose kernels are limits in a suitable sense of the 'regular' kernels of* (G); *moreover, our theory will involve Lebesgue Stieltjes (Radon) integration*, which appears to be an appropriate tool for such problems. This explains the aim as well as the title of the present work. The developments given in the following pages will be not of Fredholm type and, in part, will involve use of spectral theory — with regard to the latter aspect the background is given by Carleman's theory.

It is essential to note that in the sequel '*domains*' are *closed* sets of a certain description (cf. section 2).

In section 2 the requisite developments of Gunther are stated. In section 3 we extend the integration methods involved in (G) to Lebesgue-Stieltjes integration. In section 4 the notion of weak convergence is introduced (Definition 4. 1) as a natural extension of the classical concept of such convergence. Theorems 4. 1, 4. 3 allow passage to the limit under the integral sign when certain conditions are satisfied. Theorem 4. 2 gives a condition for weak convergence and in Theorem 4. 4 conditions are given under which change of order of integration for certain integrals is possible. In section 5 singular kernels (T) (Definition 5. 1)

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[1] T. CARLEMAN, Sur les équations intégrales singulières à noyau réel et symétrique, Uppsala, 1923; pp. 1—228; *in the sequel referred to as* (C).

[2] W. J. TRJITZINSKY, General theory of singular integral equations with real kernels, Trans. Amer. Math. Soc. (1939); 202—279.

W. J. TRJITZINSKY, Some problems in the theory of singular integral equations, Annals of Math. (1940); 584—619; *in the sequel referred to as* (T)

[3] N. GUNTHER, Sur les intégrales de Stieltjes et leurs applications aux problèmes de la Physique Mathématique, Leningrad, 1932; *in the sequel referred to as* (G).

[4] The integrals involved in (G) are more general than those of Frechet.

are introduced and two singular integral equation problems are formulated. A detailed 'spectral' theory along lines of (C) is developed for kernels (T) as well as for certain kernels (T^*) (Definition 6. 1), satisfying some continuity conditions. One of the applications of the spectral theory is in the representation of solutions of the various integral equations. In section 7 connections are established in Lemmas 7. 1, 7. 2 between solutions of the *two non homogeneous problems* (*7. 3*), (*7. 3 a*) — in cases when certain conditions are satisfied. In section 8 sets O in the complex plane of the parameter λ (cf. (8. 1 a)) are introduced and problem (7. 3 a) is treated directly; it is established that solutions of the equations (7. 1 a), approximating to (7. 3 a), satisfy certain conditions of compactness and uniform absolute continuity (Lemma 8. 1); in Theorem 8. 1 existence of solutions of (7. 3 a) is established and in Theorem 8. 2 these solutions are represented in terms of spectral functions. Throughout, existence of solutions of the non homogeneous problems is asserted for the parameter λ in O [O contains all the points off the axis of reals and may contain some points on the axis of reals]. Sets $T > O$ are introduced in section 9; a compactness property is established in Lemma 9. 1 for the solutions of the equations approximating to the non homogeneous problem (7. 3) (λ in T); in Theorem 9. 1 spectral representations are given for solutions of (7. 3); two spectral representations for such solutions, of a different type, are given in (9. 12) and (9. 19). Properties of uniqueness for the two homogeneous problems are dealt with in sections 10, 11 (Theorems 10. 1, 10. 2, 11. 1, 11. 2). Kernels which are merely measurable with respect to the measure function u^* are considered in section 12; a 'spectral' theory, analogous to that for similar kernels previously studied by Carleman, is outlined for these kernels. Finally, in section 13 the developments of section 12 are employed to prove existence of solutions for singular integral equations of the first kind (cf. (13. 2) and Theorem 13. 1). In the italics subsequent to (13. 14 b) it is pointed out that formula (6. 31 c) in (T; p. 618) is not justifiable — the developments of section 13 of the present work, however, enable us to assert that the concluding Theorem in (T; p. 619) holds true as formulated in (T).

2. Gunther Integration.

We shall briefly describe the mode of integration and the properties of the integrals, as developed in (G). This will be done in so far as it may be necessary for subsequent developments as well as in order to assist any reader in following the present work.

Intervals are understood to be closed sets of points (square in two dimensions, cube in three, and so on). A set (ω) is a *domain* if it has interior points, if it contains its frontier and if the measure of the latter is zero; the *frontier* of (ω) is the set of limiting points of (ω) which are not interior points of (ω). *Decomposition of* (ω) into two domains (ω_1), (ω_2) is designated by $(\omega) = (\omega_1) + (\omega_2)$. The relation for the corresponding measures will be $\omega = \omega_1 + \omega_2$. Decompositions into any *finite* (but not necessarily infinite) number of domains (ω_1), (ω_2), $\ldots (\omega_p)$ is possible, yielding the relation $(\omega) = (\omega_1) + (\omega_2) + \cdots + (\omega_p)$.

Domains (ω) *and* (τ) *and points* (x) *and* (y) *will be in fixed (bounded) domains* (D_x), (D_y), *respectively,* (D_x) *and* (D_y) *are to be identical except for notation.*

A function $u(\omega)$ of domains (ω) is *mean additive* (*m. a.*) if

$$u(\omega)\,\omega = u(\omega_1)\,\omega_1 + u(\omega_2)\,\omega_2$$

for every decomposition $(\omega) = (\omega_1) + (\omega_2)$. A m. a. $u(\omega)$ is of *bounded variation* (*BV*) if

$$s_n(\omega) = \sum_1^n |u(\omega_j)|\,\omega_j < B < \infty$$

for all finite decompositions of (ω);

$$\text{total variation of } u(\omega) \text{ is } U(\omega)\,\omega = \text{u. b. } s_n(\omega)$$

(u. b. means least upper bound for finite decompositions of (ω)).

It is said that $u(\omega)$ is *absolutely additive* (*AA*) if for all possible infinite decompositions $(\omega) = (\omega_1) + (\omega_2) + \cdots$

$$u(\omega)\,\omega = \sum_1^\infty u(\omega_j)\,\omega_j.$$

Designate by $(\underline{\omega})\,[(\overline{\omega})]$ a domain contained in the interior of (ω) (a domain containing (ω) in its interior); the limits

$$\underline{u}(\omega) = \lim u(\underline{\omega}), \quad \bar{u}(\omega) = \lim u(\overline{\omega})$$

(as $(\underline{\omega})$, $(\overline{\omega}) \to (\omega)$) exist and are unique.

M. a. $u(\omega)$ is *continuous* (*C*) if $\underline{u}(\omega) = u(\omega)$; if $u(\omega) < C$, $\bar{u}(\omega) = u(\omega)$; if $\bar{u}(\omega) = u(\omega)$ then $u(\omega) < C$.

When m. a. $u(\omega) \geqq 0$ one writes $l(\omega)\,\omega = \text{l. b.} \sum_1^n \underline{u}(\omega_j)\,\omega_j$ (l. b. means greatest lower bound for finite decompositions $(\omega) = (\omega_1) + \cdots + (\omega_n)$); $l(\omega)$ is m. a.

$L(\omega)\,\omega$ (saltus function) is u. b. $\sum_1^n (u(\omega_i) - \underline{u}(\omega_i))\,\omega_i$.

For every non negative m. a. $u(\omega)$ one has

$$u(\omega) = l(\omega) + L(\omega).$$

Let $(\omega_1), \ldots (\omega_p)$ (p finite) be possibly non contiguous. It is said that m. a. $u(\omega)$ is *absolutely continuous* AC if, given ε (> 0), there exists η so that

$$\sum_1^p |u(\omega_j)|\,\omega_j < \varepsilon \text{ whenever } \omega_1 + \cdots + \omega_p < \eta.$$

Let $(i_1), (i_2), \ldots$ be a sequence of intervals each containing (x) in the interior and such that $i_\nu \to 0$ (as $\nu \to \infty$). One designates by $\bar{u}_\alpha^{(\nu)}(x)$ the u. b. of $u(\omega)$ for domains (ω), contained in (i_ν), such that $\omega/i_\nu \geqq \alpha\,(> 0)$; furthermore, one lets

$$\bar{u}_\alpha(x) = \overline{\lim_\nu}\ \bar{u}_\alpha^{(\nu)}(x), \quad \bar{u}(x) = \lim_\alpha \bar{u}_\alpha(x) \qquad \text{(as } \alpha \to 0).$$

By taking l. b. and $\underline{\lim}$ in place of u. b. and $\overline{\lim}$, respectively, one obtains the numbers

$$\underline{u}_\alpha^{(\nu)}(x), \quad \underline{u}_\alpha(x), \quad \underline{u}(x).$$

Whenever $\bar{u}(x) = \underline{u}(x)$ it is said that $u(\omega)$ *has a value*

$$u(x) = \bar{u}(x) = \underline{u}(x)$$

at the point (x); thus,

$$u(\omega) = \frac{1}{\omega}\int_{(\omega)} f(x)\,d\omega \qquad (f(x) < L_1 \text{ in } (D_x))$$

has the value $f(x)$ almost everywhere in (D_x).

If m. a. $u(\omega) < BV$, $u(\omega)$ has a value almost everywhere.

The values of $L(\omega)$ are zero almost everywhere in (D_x).

If m. a. $u(\omega) < BV$ then

$$u(\omega) = \frac{1}{\omega}\int_{(\omega)} F(x)\,d\omega + w(\omega) \qquad (F(x) < L_1 \text{ in } (D_x));$$

here the values of $w(\omega)$ are zero almost everywhere.

Let $f(x)$ be a function of points (x), $< (D_x)$, and let m. a. $u(\omega) < BV$. With $(\Omega) < (D_x)$ and $(\Omega) = (\omega_1) + (\omega_2) + \cdots + (\omega_n)$ one forms the sum

$$I_n = \sum_1^n u(\omega_i) f(\xi_i)\, \omega_i;$$

where (ξ_i) is a point in (ω_i). Whenever $\lim I_n$, as the $\omega_i \to 0$ uniformly, exists one writes

$$\int_{(\Omega)} u(\omega) f(x)\, d\,\omega = \lim I_n; \tag{2. 1}$$

this limit is designated in (G) as the *integral of Stieltjes.*

With m. a. $u(\omega) < BV$ and $f(x)$ continuous in (D_x) or, more generally, bounded and Riemann integrable over (D_x), the integral (2. 1) *will exist.* Of the various properties of (2. 1) we shall mention the following:

$$\left|\int_{(\Omega)} u(\omega) f(x)\, d\,\omega\right| \leqq M\, U(\Omega)\, \Omega, \tag{2. 1 a}$$

if $|f(x)| \leqq M$ in (Ω);

$$\int_{(\Omega)} u(\omega) f(x)\, d\,\omega = u(\Omega)\, \Omega f(\xi) \qquad \text{(some } (\xi) \text{ in } (\Omega)), \tag{2. 1 b}$$

if $f(x)$ is continuous in (Ω) and $u \geqq 0$.

$$\left|\int_{(\Omega)} u(\omega) f(x)\, F(x)\, d\,\omega\right|^2 \leqq \int_{(\Omega)} u(\omega) f^2(x)\, d\,\omega \int_{(\Omega)} u(\omega)\, F^2(x)\, d\,\omega; \tag{2. 1 c}$$

$$\left|\int_{(\Omega)} u(\omega) f(x)\, d\,\omega\right| \leqq \int_{(\Omega)} U(\omega)\, |f(x)|\, d\,\omega. \tag{2. 1 d}$$

If the $\varphi_i(x)$ are continuous in (Ω) and

$$f(x) = \sum_1^\infty \varphi_i(x),$$

while $|\varphi_1(x)| + |\varphi_2(x)| + \cdots$ converges uniformly (in (Ω)), one has

$$\int_{(\Omega)} u(\omega) f(x)\, d\,\omega = \sum_1^\infty \int_{(\Omega)} u(\omega)\, \varphi_i(x)\, d\,\omega. \tag{2. 2}$$

A series $u_1(\omega) + u_2(\omega) + \cdots$ (the $u_i(\omega)$ m. a., $< BV$) is said to be *uniformly convergent* if, on writing $r_{n,m}(\omega) = u_n(\omega) + \cdots + u_{n+m}(\omega)$ and on letting $R_{n,m}(\omega)\,\omega$ denote the total variation of $r_{n,m}(\omega)$ one has the following: given $\varepsilon (> 0)$, there exists n_ε independent of (ω) so that $R_{n,m}(\omega)\,\omega < \varepsilon$ for all $n \geqq n_\varepsilon$ and for all $m \geqq 0$[1]. If the series $u_1(\omega) + u_2(\omega) + \cdots$ converges uniformly in (Ω) its sum $u(\omega)$ is m. a. and $< BV$ and one has

$$(2.\ 3) \qquad \int\limits_{(\Omega)} u(\omega) f(x)\, d\omega = \sum_{1}^{\infty} \int\limits_{(\Omega)} u_j(\omega) f(x)\, d\omega,$$

whenever $f(x)$ is continuous in (Ω).

When m. a. $u(\omega) < BV$ and $f(x)$, $F(x)$ are bounded and Riemann integrable in (Ω) one has

$$(2.\ 4) \quad \int\limits_{(\Omega)} u(\omega)\, f(x) F(x)\, d\omega = \int\limits_{(\Omega)} v(\omega) F(x)\, d\omega, \quad v(\omega) = \frac{1}{\omega} \int\limits_{(\omega)} u(\omega) f(x)\, d\omega.$$

If $f(x) < L_1$ in (Ω) and we let $\omega\, u(\omega) = \int\limits_{(\omega)} f(x)\, d\omega$, one has

$$(2.\ 4\,\mathrm{a}) \qquad \int\limits_{(\Omega)} u(\omega) F(x)\, d\omega = \int\limits_{(\Omega)} F(x) f(x)\, d\omega$$

for all bounded Riemann integrable $F(x)$. If $u(\omega) < AC$ (in (Ω)), then $f(x) < L_1$ (in (Ω)) can be found so that $\omega\, u(\omega) = \int\limits_{(\omega)} f(x)\, d\omega$ and, hence, (2. 4 a) holds.

If $L(x, y)$ is continuous in (x) and (y), while $u(\omega)$, $v(\tau)$ are m. a and $< BV$ then

$$(2.\ 5) \quad \int\limits_{(\Omega_x)} u(\omega) \left(\int\limits_{(\Omega_y)} v(\tau) L(x, y)\, d\tau \right) d\omega = \int\limits_{(\Omega_y)} v(\tau) \left(\int\limits_{(\Omega_x)} u(\omega) L(x, y)\, d\omega \right) d\tau.$$

The 'inner' integrals in both members, here, are continuous functions of points.

Let $u(\omega, y) < BV$ for every (y) in (D_y) and suppose $U(D_{x,y})$ is bounded as a function of (y), while

$$\int\limits_{(\Omega_y)} u(\omega, y)\, v(\tau)\, d\tau$$

[1] This formulation of uniform convergence is different from that given in (G).

exists (the latter is the case when $u(\omega, y)$ is continuous in (y) for every $(\omega) < (\Omega_y)$); then for every $\varphi(x)$ continuous in (Ω_x) one has

$$(2.6) \qquad \int_{(\Omega_x)} \varphi(x) \left(\int_{(\Omega_y)} u(\omega, y) v(\tau) d\tau \right) d\omega = \int_{(\Omega_y)} v(\tau) \left(\int_{(\Omega_x)} u(\omega, y) \varphi(x) d\omega \right) d\tau.$$

Let $(\Omega) = (\omega_1) + \cdots + (\omega_n)$ and let (x_i) be a point in (ω_i); form the sum $S_n = u(\omega_1, x_1)\omega_1 + \cdots + u(\omega_n, x_n)\omega_n$. If the limit, as $n \to \infty$, while the $\omega_j \to 0$ uniformly, of S_n exists it is called a generalized Stieltjes integral:

$$(2.7) \qquad \int_{(\Omega)} u(\omega, x) d\omega = \lim S_n.$$

This limit exists and is unique when m. a. $u(\omega, y) < BV$ for every (y), $|u(\omega, y)| < V_1(\omega)$ (with m. a. $V_1(\omega) < BV$) and

$$(2.7\text{ a}) \qquad |u(\omega, y_1) - u(\omega, y_2)| < \varepsilon V_2(\omega) \qquad (\text{m. a. } V_2(\omega) < BV)$$

for all (ω) and for all (y_1), (y_2) in the same sphere (ϱ) of radius $\varrho (= \varrho_\varepsilon)$.

These conditions for existence of (2.7) are satisfied by the function

$$(2.7\text{ b}) \qquad u(\omega, y) = \frac{1}{\omega} \int_{(\omega)} u(\omega) L(x, y) d\omega \qquad (\text{m. a. } u(\omega) < BV);$$

provided L is continuous, $|L| < A$ and $|L(x, y_1) - L(x, y_2)| < \varepsilon$ for (y_1), (y_2) in the same sphere (ϱ).

If $u(\omega, x)$ satisfies the conditions stated subsequent to (2.7) then

$$(2.8) \qquad \int_{(\Omega)} u(\omega, x) \varphi(x) d\omega = \int_{(\Omega)} v(\omega) \varphi(x) d\omega, \quad v(\omega) = \frac{1}{\omega} \int_{(\omega)} u(\omega, x) d\omega$$

for all $\varphi(x)$ continuous in (Ω).

We shall assume throughout that all the m. a. functions of domains under consideration have finite values on every domain. In other words, it will be implied that the m. a. functions, involved, are BV.

3. Extension of Integrations.

Let $\{F\}$ be the class of sets in the 'domain' (D) such that (1°) if $e < \{F\}$ then $(D) - e < \{F\}$, (2°) if $e_1, e_2, \cdots < \{F\}$ then $e_1 + e_2 + \cdots < \{F\}$, (3°) $\{F\}$ contains all intervals (closed) and the empty set. Then $\{F\}$ is a *closed class of sets*, according to the usual terminology. In (2°) the number of sets $e_1, e_2, \ldots$ may be infinite. The class $\{F\}$ may be termed *completely additive*. It will be said that the sets $\{F\}$ are *measurable* $\{F\}$. $\{F\}$ contains the class $\{B\}$ (also a closed class) of *Borel measurable* sets, which will be designated as measurable $\{B\}$. *Domains*, as in section 2, are certain closed sets and, hence, $< \{B\}$.

Henceforth *intervals* will be closed sets of points -- intervals in the ordinary sense in the one dimensional space, rectangles in the two dimensional case and so on. A figure, as usual, will be a sum of a finite number of intervals. A *figure* is a domain (of section 2), if degenerate cases are excluded.

A m. a. function $u(\omega)$ of domains (ω), $< (D)$, gives rise to an additive function of figures and in particular of intervals,

$$\tilde{u}(\omega) = \omega\, u(\omega), \tag{3. 1}$$

where (ω), ω denote figures and measures of figures, respectively. Since m. a. $u(\omega)$ is not necessarily AA, $\tilde{u}(\omega)$ is not necessarily completely additive (i. e., additivity is not extended to an infinity of sets).

We shall translate properties of m. a. functions into those of functions of figures. Inasmuch as we consider only the m. a. $u(\omega) < BV$, our functions $\tilde{u}(\omega)$ are finite on every figure $< (D)$, which agrees with the customarily assumed property of functions of figures. There is a decomposition of $\tilde{u}(\omega)$ into a difference of two non negative additive functions of figures:

$$\tilde{u}(\omega) = \tilde{u}_1(\omega) - \tilde{u}_2(\omega), \tag{3. 2}$$

where $\tilde{u}_i(\omega) = \omega\, u_i(\omega)$ $(i = 1, 2)$, the $u_i(\omega)$ being non negative m. a. functions from the decomposition $u(\omega) = u_1(\omega) - u_2(\omega)$.

If m. a. $u(\omega) \geqq 0$ and $< AC$, in consequence of the definition we shall have in particular

$$\sum_1^p \tilde{u}(\omega_j) < \varepsilon \text{ whenever } \omega_1 + \cdots + \omega_p < \eta_\varepsilon \qquad (\varepsilon > 0), \tag{3. 3}$$

where the (ω_j) are intervals. Thus, the property AC for a m. a. function $u(\omega)$ (of domains) implies the property of *absolute continuity*, in the ordinary sense, of the corresponding functions $\tilde{u}(\omega)$ of figures; that is,

$$\tilde{u}(\omega) \to 0, \tag{3. 3 a}$$

as measure of figure (w) tends to zero. The property (3. 3 a) will hold even when $\tilde{u}(\omega)$ is of variable sign, provided that the non negative components u_1, u_2 in the decomposition $u = u_1 - u_2$ of the corresponding m. a. u belong to AC.

At this point the reader may profitably be referred to a book by C. DE LA VALLÉE POUSSIN[1], *in the sequel referred to as* (P); in particular, to the second part of (P).

According to a theorem in (P), given an additive absolutely continuous function of figures, there exists an absolutely continuous completely additive function of sets $\{B\}$ which coincides with the function of figures on the intervals (and, hence, on the figures). In this connection, $p(e)$ (Borel sets e) is said to be *absolutely continuous* of $p(e) \to 0$ with meas. e — *continuous*, if $p(e) \to 0$ with the diameter of e.

The preceding considerations lead to the following result.

Lemma 3. 1. *Let m. a.* $u(\omega) < AC$ *(then according to* (G) *the two m. a. components of* $u(\omega)$ *belong to* AC*). There exists a completely additive and absolutely continuous function* $u^*(e)$ *of measurable sets* e, $< \{B\}$, *such that*

$$u^* = \tilde{u} \qquad (cf.\ (3.\ 1)) \tag{3. 4}$$

on figures; moreover, if $u = u_1 - u_2 (u_1, u_2 \geqq 0)$ *is the decomposition of m. a.* $u(\omega)$ *then* $u^* = u_1^* - u_2^* \ (u_1^*, u_2^* \geqq 0)$, *where* $u_i^* \ (i = 1, 2)$ *is an absolutely continuous function of sets* $\{B\}$ *such that* $u_i^* = \tilde{u}_i$ *on figures;* u_1^*, u_2^* *are finite for every set* $\{B\}$.

Throughout the paper all sets are contained in the domain (D) [or (D_x), $(D_y) \ldots$, as the case may be]. We also note that according to (P) ordinary continuity of u_1^*, u_2^* would suffice for finiteness on sets $\{B\}$.

If a function of points, $f(x)$, is measurable $\{B\}$ and if $u^*(e)$ is an additive absolutely continuous function of sets $\{B\}$, we have decompositions in the usual manner:

$$f(x) = f^+(x) + f^-(x), \quad u^*(e) = u_1^*(e) - u_2^*(e), \tag{3. 5}$$

where

$$f^+(x) = \begin{cases} f(x) & (\text{when } f(x) \geqq 0), \\ 0 & (\text{when } f(x) < 0); \end{cases} \quad f^-(x) = \begin{cases} f(x) & (\text{when } f(x) < 0), \\ 0 & (\text{when } f(x) \geqq 0); \end{cases} \tag{3. 5 a}$$

[1] C. DE LA VALLÉE POUSSIN. Intégrales de Lebesgue. Fonctions d'ensemble. Classes de Baire. Paris, 1934.

here u_1^*, $u_2^* \geqq 0$ and are absolutely continuous functions of sets $\{B\}$, while f^+, $-f^- \geqq 0$ and are functions measurable $\{B\}$.

A RADON integral (usually termed LEBESGUE-STIELTJES)

$$I = \int_E f(x)\, d\, u^*(e) = I^{+,1} - I^{+,2} - I^{-,1} + I^{-,2}, \tag{3.6}$$

where

$$\begin{aligned} I^{+,1} &= \int_E f^+(x)\, d\, u_1^*(e), \quad I^{+,2} = \int_E f^+(x)\, d\, u_2^*(e), \\ I^{-,1} &= \int_E -f^-(x)\, d\, u_1^*(e), \quad I^{-,2} = \int_E -f^-(x)\, d\, u_2^*(e), \end{aligned} \tag{3.6a}$$

will be said to exist if the four integrals (3. 6 a) have finite values. Any of the integrals (3. 6 a) may be defined either as described in a book of S. SAKS[1], *in the sequal referred to as* (S) — see (S; pp. 19, 20) or according to the classical definition of Lebesgue integrals, except that the Lebesgue measure is replaced by $u_1^*(e)$ (or $u_2^*(e)$), as the case may be.

For our purposes *it will be sufficient to restrict ourselves to Borel measurable sets and to functions of points which are correspondingly measurable.*

It will be said that $f(x)$ *is integrable* $\{u^*\}$ *over a set* E, $< \{B\}$, if the integrals (3. 6 a) all exist. Integrability $\{u^*\}$ of f presupposes corresponding measurability $\{u_1^*\}$ and $\{u_2^*\}$ of f.

Moreover, it is to be noted that the above considerations with respect to Radon integration are applicable even when u_1^*, u_2^* are not necessarily absolutely continuous (in the sense of tending to zero with meas. e, e being in the class $\{B\}$).

Now, by (P), an additive absolutely continuous function F of sets $\{B\}$ has its derivatives — in the sequel denoted by a prime — finite almost everywhere and integrable on every set $\{B\}$; moreover, $F(e) = \int_e F'(x)\, d\, x$. Accordingly, *the function* $u^*(e)$ *of Lemma* 3. 1 *has the representation*[2]

[1] See S. SAKS, Theory of the Integral, Warszawa-Lwow, 1937; in particular Chapters I, II, III.

[2] Derivatives at a point x of a function of sets we always take in reference to families of sets 'regular' (in the sense of Lebesgue; see (P)) with respect to the point x.

$$u^*(e) = \int_e u^{*\prime}(x)\,d\,x \qquad \text{(sets } e, < \{B\}, \text{ in } (D)), \tag{3. 7}$$

where the integral is in the ordinary Lebesgue sense; correspondingly, an integral (3. 6) may be expressed as

$$\int_E f(x)\,d\,u^*(e) = \int_E u^{*\prime}(x) f(x)\,d\,x \quad \text{(all sets } E, < \{B\}, \text{ in } (D)), \tag{3. 7 a}$$

whenever $u^*(e)$ is absolutely continuous. *Thus, when m. a.* $u(\omega) < A\,C$, *we have*

$$\int_{(\omega)} u^{*\prime}(x) f(x)\,d\,x = \int_{(\omega)} f(x)\,d\,u^*(e) = \int_{(\omega)} u(\omega) f(x)\,d\,\omega \tag{3. 8}$$

on all figures (ω), *whenever the last integral exists in the sense of Gunther;* there is on hand a natural extension of Gunther's integration, both with respect to functions $f(x)$ of points and the character of sets.

Suppose now that the m. a. function of domains, $u(\omega)$, does not necessarily belong to $A\,C$.

Let $u(\omega) \geqq 0$ and consider the function of figures $\tilde{u}(\omega)$ (3. 1). We shall define $\tilde{u}(\omega)$ in the whole Euclidean space by the relation

$$\tilde{u}(\omega) = |(\omega)(D)|\, u((\omega)(D)) \qquad \text{(all figures } (\omega)),$$

where $|(\omega)(D)|$ is measure of the domain $(\omega)(D)$ (i. e. of domain consisting of points common to figure (ω) and to domain (D)). Let E be any set in (D); we designate by $u^*(E)$ the lower bound of the sums (finite or infinite)

$$\sum_j \tilde{u}(\omega_j)$$

for sequences of intervals

$$(\omega_1),\ (\omega_2),\ \ldots$$

such that

$$E < \sum_j (\omega_j)^o;$$

here $(\omega_j)^o$ is the interior of (ω_j). In consequence of (S; p. 64) u^* will be an *outer Cartatheodory measure* and will accordingly satisfy the conditions (C_1), (C_2), (C_3) of (S; p. 43). Corresponding to u^* there exists therefore a class $\{L_{u^*}\}$ of sets measurable (in the sense of (S; p. 44)) with respect to u^*. The class $\{L_{u^*}\}$

certainly contains the class of sets $\{B\}$; this is a consequence of (S; pp. 51, 52). Confining ourselves again to sets $\{B\}$, *integration may be defined as in connection with* (3, 6), (3, 6 a), where e, E are sets $\{B\}$ and the outer measures u_1^*, u_2^* are used as measures in the proper sense of the word, inasmuch as sets $\{B\}$, only, are involved (also see (S; p. 65)).

By (S; p. 95) $u^* \geqq 0$ is absolutely continuous or is not absolutely continuous at the same time as the function of figures $\tilde{u} \geqq 0$ has this property. Now we note that if m. a. $u(\omega) \geqq 0$ is not $A\,C$ (section 2) then necessarily $\tilde{u}$ is not absolutely continuous as a function of figures; we then shall have u^* not absolutely continuous. If m. a. $u(\omega)$ is of variable sign and $u(\omega) = u_1(\omega) - u_2(\omega)$ is the decomposition, where the m. a. u_1, u_2 are non negative, the following is observed: if $u(\omega)$ is not $A\,C$ then one at least of the functions u_1, u_2 is not $A\,C$; we denote by u_1^*, u_2^* the corresponding outer Carathéodory measures; one of the functions u_1^*, u_2^* will be not absolutely continuous as a function of sets $\{B\}$. The function $u^* = u_1^* - u_2^*$, corresponding to $u(\omega)$, will be lacking in absolute continuity in the indicated sense.

With the aid of the outer measures u_1^*, u_2^* we define integrals

$$\Phi(E) = \int_E f(x)\, d\, u^*(e) \qquad (u^* = u_1^* - u_2^*)$$

for sets $E < \{B\}$. We shall say that a *measurable function* $f(x)$ *is integrable* $\{B, u^*\}$ if on decomposing $f(x)$ into $f^+ + f^-$ (see (3, 5), (3, 5 a)) the four integrals (3, 6 a) exist; this presupposes, of course, that $f(x)$ is integrable $\{B, u_1^*\}$ and $\{B, u_2^*\}$.

In general u_1^* (and u_2^*) will not coincide on all figures (ω) with $\tilde{u}_1(\omega) = \omega\, u_1(\omega)$ (and $\tilde{u}_2(\omega) = \omega\, u_2(\omega)$).

Let $\tilde{u}(\omega)$ be a function of figures (ω) and let β denote the frontier of some figure (or a hyperplane perpendicular to one of the coordinate axes); we designate by

$$o(\tilde{u};\ \beta) \tag{3.9}$$

the *oscillation of* $\tilde{u}$ at β and define this number in agreement with (S; p. 60); thus the following sequence of relations will define $o(\tilde{u};\ \beta)$:

$$O(\tilde{u};\ e) = \text{u. b. } |\tilde{u}(i)| \qquad (\text{any fixed set } e;\ \text{intervals } (i) < e);$$

$$o_{(\omega)}(\tilde{u};\ \beta) = \text{l. b. } O(\tilde{u};\ (\omega)\cdot g) \qquad (\text{any fixed figure } (\omega);\ \text{open sets } g > \beta);$$

$$o(\tilde{u};\ \beta) = \text{u. b. } o_{(\omega)}(\tilde{u};\ \beta) \qquad (\text{figures } (\omega)).$$

Inasmuch as in this paper we restrict ourselves to m. a. $u(\omega)$ such that $|u(\omega)|\,\omega \leqq A < \infty$ for all 'domains' (ω) (in (D)) it follows that *the corresponding functions of intervals* (figures)

$$\tilde{u}(\omega),\ \tilde{u}_1(\omega),\ \tilde{u}_2(\omega) \qquad [\tilde{u} = \tilde{u}_1 - \tilde{u}_2;\ \tilde{u}_1, \tilde{u}_2 \geqq 0] \tag{3. 10}$$

are of bounded variation in the sense of (S; pp. 61, 62).

Returning now to one of the Caratheodory outer measures, say u_1^* $(\geqq 0)$, *we conclude that* (see (S; p. 68))

$$u_1^*(\omega^o) \leqq \tilde{u}_1(\omega) = \omega\, u_1(\omega) \leqq u_1^*(\omega) \tag{3. 11}$$

for all figures (ω) (in (D)); here $(\omega)^o$ is the interior of (ω). Moreover, in consequence of (S; p. 63), there exists *at most a denumerable infinity of hyperplanes* (perpendicular to the various coordinate axes),

$$h_1, h_2, \ldots, \tag{3. 12}$$

at which the oscillation of $\tilde{u}_1$ *may be positive; for every figure* (ω), *at whose frontier the oscillation of* $\tilde{u}_1$ *is zero*, we have

$$u_1^*(\omega^o) = \tilde{u}_1(\omega) = \omega\, u_1(\omega) = u_1^*(\omega). \tag{3. 13}$$

The figures (ω), at whose frontiers the oscillations of $\tilde{u}_1$ and $\tilde{u}_2$ are zero, are found amongst the figures having the faces (planar portions) of their frontiers not lying in the hyperplanes of discontinuity of $\tilde{u}_1$ (see (3. 12)) nor of $\tilde{u}_2$; for such figures we have

$$u^*(\omega^o) = \tilde{u}(\omega) = \omega\, u(\omega) = u^*(\omega). \tag{3. 13 a}$$

One will have (3. 13 a) for all figures (in (D)) whenever $\tilde{u}_1(i)$, $u_2(i) \to 0$ with i ($i =$ meas. of interval (i)).

For $u_1^*(e) \geqq 0\,(e < \{B\})$ we have the *Lebesgue decomposition*

$$u_1^*(e) = \int_e u_1^{*\prime}(x)\,dx + \zeta_1(e) \qquad (u_1^{*\prime} \geqq 0;\ \zeta_1 \geqq 0) \tag{3. 14}$$

where $u_1^{*\prime}(x)$ is integrable $\{B;$ Lebesgue measure$\}$ (which we express by the designation $u_1^{*\prime}(x) < L_1$) and $\zeta_1(e)$ is additive *singular* $\{B;$ Lebesgue measure$\}$. The latter is to be understood in the sense that there exists a set h_0 of measure zero, so that $\zeta_1(e) = \zeta_1(h_0 e)$ for all $e < \{B\}$. Furthermore, if $f(x)$ is integrable $\{B, u_1^*\}$, we have

$$\int_e f(x)\,d\,u_1^*(e) = \int_e u_1^{*\prime}(x) f(x)\,d\,x + \int_e f(x)\,d\,\zeta_1(e). \tag{3. 14 a}$$

Similarly, if $f(x)$ is integrable $\{B, u_2^*\}$

$$\int_e f(x)\,d\,u_2^*(e) = \int_e u_2^{*\prime}(x) f(x)\,d\,x + \int_e f(x)\,d\,\zeta_2(e),$$

where ζ_2 is additive singular $\{B$; Lebesgue measure$\}$ and $u_2^{*\prime}(x)\,(\geqq 0)$ is integrable $\{B$; Lebesgue measure$\}$.

Thus, *if* $f(x)$ *is integrable* $\{B, u^*\}$ *one has*

$$\Phi(e) = \int_e f(x)\,d\,u^*(e) = \int_e u^{*\prime}(x) f(x)\,d\,x + \beta(e). \tag{3. 15}$$

The last integral, here displayed, is an additive absolutely continuous function $\{B$; Lebesgue meas.$\}$; $u^{*\prime}(x)$ *is integrable* $\{B$; Lebesgue meas.$\}$; *moreover*, $\beta(e)$ *is additive* (of possibly variable sign) *singular* $\{B$; Lebesgue meas.$\}$, *i. e.*

$$\beta(e) = \beta(e\,s_0) \qquad\qquad (\text{all } e < \{B\}),$$

where meas. $s_0 = 0$; (3. 15) is the Lebesgue decomposition of $\varphi(e)$ into sum of an absolutely continuous and singular function.

With the aid of (P; p. 105) the following may be formulated.

Lemma 3. 2. *Let m. a.* $u(\omega)$ *be such that* $\tilde{u}(\omega) = \omega\,u(\omega)$ *is continuous as a function of intervals* (*i. e.*. $\tilde{u}(i) \to 0$ *as the diameter of interval* (i) *tends to zero*). *There exists then a completely additive and continuous function* $u^*(e)$ *of sets* $\{B\}$ (*i. e.* $u^*(e) \to 0$ *with the diameter of* e) *such that*

$$u^* = \tilde{u} \tag{3. 15^1}$$

on figures. Let $u^* = u_1^* - u_2^*\,(u_1^*, u_2^* \geqq 0)$ *be the decomposition of* u^*. *Then* u_1^*, u_2^* *are additive continuous functions of sets* $\{B\}$, *coinciding with the corresponding components of* $\tilde{u}(\omega)$ *on figures; moreover,* u_1^*, u_2^* *are finite for every set* $\{B\}$.

This presents an extension over Lemma 3. 1 inasmuch as continuity of functions of intervals (as stated in Lemma 3. 2) and continuity of functions of sets $\{B\}$ are conditions less stringent than those implied by absolute continuity of such functions.

The Radon integral (3. 6) may be defined with $u^* = u_1^* - u_2^*$ from the above Lemma.

In consequence of (P) every additive function $u^*(e)$ of sets $\{B\}$ is decomposable into a sum

$$u^*(e) = c^*(e) + \delta^*(e), \tag{3. 16}$$

where $c^*(e)$ is additive continuous and $\delta^*(e)$ is additive and of the form

$$\delta^*(e) = \delta^*(e\, e_0),$$

where e_0 is a fixed *denumerable* set; if $u^*(e) \geqq 0$ we have $c^*(e)$, $\delta^*(e) \geqq 0$.

When $\tilde{u}(\omega) = \omega\, u(\omega)$ is continuous as a function of intervals we form the function $u^* = u_1^* - u_2^*$ of (3. 15) and note that

$$\int_{(\omega)} f(x)\, d\, u^*(e) = \int_{(\omega)} u(\omega) f(x)\, d\, \omega \tag{3. 17}$$

on figures (ω), whenever the second integral exists in the sense of Gunther. Moreover, for functions f integrable $\{B, u^*\}$, where $u^* \geqq 0$, the set-function

$$\Phi(e) = \int_e f(x)\, d\, u^*(e) \qquad (e < \{B\})$$

is absolutely continuous $\{u^*\}$; that is, $\varphi(e) \to 0$ as $u^*(e) \to 0$. When u^* is of variable sign, $\varphi(e) \to 0$ whenever both functions (of sets $\{B\}$) $u_1^*(e)$, $u_2^*(e)$ tend to zero.

4. Some Limiting Processes.

We shall establish certain 'compactness' properties and theorems regarding passage to the limit under the integration signs for the general integrals of section 3.

Until stated otherwise *we shall assume $u^*(e)$ to be a non negative completely additive function of sets of the closed class $\{B\}$* (section 2); these sets are to be in the 'domain' (bounded) (D).

Definition 4. 1. *It will be said that a sequence*

$$q_1(x),\ q_2(x),\ \ldots$$

of functions measurable $\{B, u^*\}$ converges to a function $q(x)$ in the weak sense provided the integral

(4. 1) $$\int_E q^2(x)\,d\,u^*(e) \qquad (a \text{ bounded set } E < \{B\})$$

exists,

(4. 1 a) $$\int_E q_m^2(x)\,d\,u^*(e) \leqq M \qquad (m = 1, 2, \ldots)$$

and

(4. 1 b) $$\lim_m \int_e q_m(x)\,d\,u^*(e) = \int_e q(x)\,d\,u^*(e) = \int_e q^+(x)\,d\,u^*(e) + \int_e q^-(x)\,d\,u^*(e)$$

for all $\{B\}$-sets e (in E) such that the set-functions

(4. 1 c) $$\varphi^+(e) = \int_E q^+(x)\,d\,u^*(e), \quad \varphi^-(e) = \int_E -q^-(x)\,d\,u^*(e)$$

vanish on the frontier of e.

A function $\psi(x)$ will be said to be *simple* (see (S)) if it assumes a finite number of values ($\neq \pm\infty$) in a number of sets $\{B\}$.

It is not difficult to see that if $g^2(x)$ is integrable $\{B, u^*\}$ over (D_x) and we assign $\varepsilon(> 0)$, there exists a simple function $\psi_\varepsilon(x)$ such that

(4. 2) $$\int_{(D_x)} (g(x) - \psi_\varepsilon(x))^2\,d\,u^*(e) < \varepsilon^2.$$

Explicitly — there is a decomposition of (D_x) into a finite number of sets $\{B\}$ without common points,

(4. 2 a) $$(D_x) = e_1 + e_2 + \cdots + e_r$$

so that

$$\psi_\varepsilon(x) = c^{\varepsilon, j} \qquad (\text{in } e_j;\ j = 1, \ldots r),$$

the $c^{\varepsilon, j}$ being constants. The e_j may be chosen so that the functions (4. 1 c) vanish on the frontiers of the e_j.

Suppose $q_m(x) \to q(x)$ weakly. One has

$$\sigma_m = \int_{(D_x)} (q(x) - q_m(x))\,\psi_\varepsilon(x)\,d\,u^*(e) = \sigma_{m,1} + \cdots + \sigma_{m,r},$$

where

$$\sigma_{m,j} = \int\limits_{e_j} (q(x) - q_m(x))\, \psi_\varepsilon(x)\, d\,u^*(e) = c^{\varepsilon, j} \int\limits_{e_j} (q(x) - q_m(x))\, d\,u^*(e)$$

By (4. 1 b)

$$\lim_m \sigma_{m,j} = 0.$$

Thus, inasmuch as Φ^+, Φ^- vanish on the frontiers of $e_1, e_2, \ldots e_n$, one has

$$\lim_m \sigma_m = 0.$$

There is on hand a *Schwartzian inequality*,

$$\left| \int\limits_E \alpha(x)\, \beta(x)\, d\,u^*(e) \right|^2 \leqq \int\limits_E \alpha^2(x)\, d\,u^*(e) \int\limits_E \beta^2(x)\, d\,u^*(e), \tag{4. 3}$$

valid for sets $\{B\}$, whenever the two latter integrals exist.

Accordingly, on writing

$$s_m = \int\limits_{(D_x)} (q(x) - q_m(x))\, g(x)\, d\,u^*(e) =$$

$$\int\limits_{(D_x)} (q(x) - q_m(x))\, (g(x) - \psi_\varepsilon(x))\, d\,u^*(e) + \int\limits_{(D_x)} (q(x) - q_m(x))\, \psi_\varepsilon(x)\, d\,u^*(e),$$

with the aid of (4. 2) it is inferred that

$$|s_m| \leqq \left[\int\limits_{(D_x)} (q(x) - q_m(x))^2\, d\,u^*(e) \right]^{\frac{1}{2}} \left[\int\limits_{(D_x)} (g(x) - \psi_\varepsilon(x))^2\, d\,u^*(e) \right]^{\frac{1}{2}} + |\sigma_m|$$

$$< \varepsilon \left[\int\limits_{(D_x)} (q(x) - q_m(x))^2\, d\,u^*(e) \right]^{\frac{1}{2}} + |\sigma_m|.$$

In view of the existence of the integral (4. 1) and in consequence of (4. 1 a) the integral last displayed does not exceed a number independent of m. Thus $\lim\limits_m s_m = 0$.

Theorem 4. 1. *Suppose*

$$q_m(x) \to q(x) \qquad (\textit{in } (D_x);\ \textit{as } m \to \infty)$$

weakly and the integral

(4. 4)
$$\int_{(D_x)} g^2(x)\, d\, u^*(e)$$

exists. Then

(4. 5)
$$\lim_m \int_{(D_x)} g(x)\, q_m(x)\, d\, u^*(e) = \int_{(D_x)} g(x)\, q(x)\, d\, u^*(e).$$

This can be extended as follows. We have

(4. 5 a)
$$\lim_m \int_{(D_x)} g_m(x)\, q_m(x)\, d\, u^*(e_x) = \int_{(D_x)} g(x)\, q(x)\, d\, u^*(e_x)$$

when $g_m(x) \to g(x)$ *(as* $m \to \infty$*) and* $|g_m(x)| \leqq \gamma(x)$ *where* $\gamma^2(x)$ *is integrable* $\{B, u^*\}$, *the conditions for* $q_m(x)$ *being as before.*

The above will hold with (D_x) replaced by a subset $\{B\}$ — the same refers to similar developments in the sequel.

With the aid of Theorem 4. 1 and of (4. 3), following familiar lines of reasoning *it is found that*

(4. 6)
$$\int_{(D_x)} q^2(x)\, d\, u^*(e) \leqq \underline{\lim}_m \int_{(D_x)} q_m^2(x)\, d\, u^*(e)$$

whenever $q_m(x) \to q(x)$ *weakly* (in sense of Definition 4. 1).

Suppose

$$\int_E q_m^2(x)\, d\, u^*(e) \leqq M \qquad (m = 1, 2, \ldots)$$

where E is a fixed set $\{B\}$ (in (D)). Consider the functions

$$h_m(e) = \int_e q_m(x)\, d\, u^*(e),$$

where e, $< E$, are sets $\{B\}$. By (4. 3)

$$|h_m(e)| \leqq \left[\int_e q_m^2(x)\, d\, u^*(e)\right]^{\frac{1}{2}} \left[\int d\, u^*(e)\right]^{\frac{1}{2}}.$$

Hence

$$|h_m(e)| \leqq M^{\frac{1}{2}} [u^*(e)]^{\frac{1}{2}} \qquad (\{B\}\text{-sets } e < E) \tag{4. 7}$$

and, in particular,

$$|h_m(e)| \leqq M^{\frac{1}{2}} [u^*(E)]^{\frac{1}{2}} = M(E) \tag{4. 7 a}$$

for $m = 1, 2, \ldots$ and all $\{B\}$-sets $e < E$.

We recall now a result, which could appropriately be termed *De la V. Poussin-Frostman's* theorem[1], according to which, given a uniformly bounded family $\{\mu\}$ of additive functions of sets $\{B\}$, there exists a sequence $\{\mu_\nu\}$ $(\nu = 1, 2, \ldots)$ of this family and an additive function μ, of sets $\{B\}$, so that

$$\lim_\nu \mu_\nu(e) = \mu(e)$$

on every set $e, < \{B\}$, on whose frontier μ vanishes (frontier of a set e is closure of e minus the set of interior points of e).

Now the $h_m(e)$ are additive functions of sets $\{B\}$, satisfying (4. 7 a); thus, application of the above theorem enables us to assert that there exists a subsequence $\{h_{m_\nu}(e)\}$ $(m_1 < m_2 < \cdots)$ and an additive function of sets e, $h(e)$, so that

$$\lim_\nu h_{m_\nu}(e) = h(e) \qquad (\text{all } \{B\}\text{-sets } e < E), \tag{4. 7 b}$$

except for those sets e on whose frontiers $h(e)$ does not vanish. In view of (4. 7)

$$|h(e)| \leqq M^{\frac{1}{2}} [u^*(e)]^{\frac{1}{2}} \tag{4. 7 c}$$

for *all* $\{B\}$-sets e in E. Hence $h(e) \to 0$, whenever $u^*(e) \to 0$; accordingly $h(e)$ is absolutely continuous $\{B, u^*\}$. Such an additive function is expressible as an 'indefinite' integral

$$h(e) = \int_e q(x)\, d\, u^*(e)$$

for all sets $e, < \{B\}$, in E; here $q(x)$ is integrable $\{B, u^*\}$ over E. This follows by the theorem of *Radon-Nikodym* (S; p. 36). Together with (4. 6) these developments enable assertion of the following result.

[1] C. de la V. Poussin, Les nouvelles méthodes de la Théorie du Potentiel et le problème généralisé de Dirichlet, Actualités scientifiques et industrielles, No 578, Paris, 1937; *referred to as* (*VP*). In particular see p. 9. Also, O. Frostman, Potentiel d'équilibre et capacité des ensembles ..., Meddelanden från Lunds Universitet, 1935, pp. 1—115; in particular see pp. 11—13.

Theorem 4. 2. *Let* $u^*(e)(\geqq 0)$ *be an additive function of sets* $\{B\}$. *Let* E *be a bounded set* $\{B\}$. *If*

$$\int_E q_j^2(x)\,d\,u^*(e) \leqq M < \infty \qquad (j = 1, 2, \ldots), \tag{4. 8}$$

where M *is independent of* j, *there exists a subsequence* $\{q_{\nu_j}(x)\}$ $(\nu_1 < \nu_2 < \cdots)$ *converging weakly on* E (*Definition* 4. 1) *to a function* $q(x)$ *for which*

$$\int_E q^2(x)\,d\,u^*(e) \leqq M. \tag{4. 8 a}$$

In the particular case when $u^*(e)$ is the ordinary Lebesgue measure the definition of weak convergence is simplified in the sense that $q_m(x) \to q(x)$ weakly (over E) if (4. 1) exists, (4. 1 a) holds and if the limiting relation (4. 1 b) takes place for all $\{B\}$-sets $e, < E$, which have frontiers of zero Lebesgue measure; a similar statement may be made when, more generally, additive $u^*(e)$ (of sets $\{B\}$) is absolutely continuous (i. e., $u^*(e) \to 0$ with meas. e).

The following is an extension of Carleman's theorem in (C; p. 20).

Theorem 4. 3. *Let* E *be a fixed bounded set* $< \{B\}$. *Suppose that the integrals*

$$\int_E f_n^2(x)\,d\,u^*(e), \quad \int_E g_n^2(x)\,d\,u^*(e) \qquad (n = 1, 2, \ldots)$$

all exist and the limits

$$\lim f_n(x) = f(x), \quad \lim g_n(x) = g(x) \qquad (\text{as } n \to \infty) \tag{4. 9}$$

exist. Suppose also that

$$|f_n(x)| < h(x), \quad \int_E g_n^2(x)\,d\,u^*(e) < c^2 \qquad (n = 1, 2, \ldots), \tag{4. 9 a}$$

where $h^2(x)$ *is integrable* $\{B, u^*\}$ *over* E. *Then*

$$\lim_n \int_E f_n(x)\,g_n(x)\,d\,u^*(e) = \int_E f(x)\,g(x)\,d\,u^*(e). \tag{4. 10}$$

By a known theorem (S; p. 17) on converging sequences of measurable functions, given $\varepsilon(> 0)$ and $\delta(> 0)$ there is a decomposition of E,

$$E = E_1 + E_2,$$

where $E_1,\ E_2 < \{B\}$ and

$$u^*(E_2) < \delta,$$

so that for some n_0 (independent of (x))

$$|f(x) - f_n(x)|,\ |g(x) - g_n(x)| < \varepsilon \qquad \text{(in } E_1) \tag{4.11}$$

for all $n \geqq n_0$.

We have

$$\sigma_n = \int\limits_E (f(x)\,g(x) - f_n(x)\,g_n(x))\,d\,u^*(e) = \int\limits_{E_2} f(x)\,g(x)\,d\,u^*(e) - \int\limits_{E_2} f_n(x)\,g_n(x)\,d\,u^*(e)$$

$$+ \int\limits_{E_1} (f(x) - f_n(x))\,g(x)\,d\,u^*(e) + \int\limits_{E_1} f_n(x)\,(g(x) - g_n(x))\,d\,u^*(e).$$

Designate the four integrals last displayed by $\alpha_1,\ \alpha_2,\ \alpha_3,\ \alpha_4$, in succession. Since

$$\int\limits_e f_n^2(x)\,d\,u^*(e) \leqq \int\limits_e h^2(x)\,d\,u^*(e) \qquad [\{B\}\text{-sets } e \text{ in } E]$$

application of (4. 9 a) and of Theorem 4. 2 will yield

$$\int\limits_e f^2(x)\,d\,u^*(e) \leqq \int\limits_e h^2(x)\,d\,u^*(e), \quad \int\limits_e g^2(x)\,d\,u^*(e) \leqq c^2.$$

Hency by the Schwartzian inequality

$$|\alpha_1|^2 \leqq \int\limits_{E_2} f^2(x)\,d\,u^*(e) \int\limits_{E_2} g^2(x)\,d\,u^*(e) \leqq c^2 \int\limits_{E_2} h^2(x)\,d\,u^*(e),$$

$$|\alpha_2|^2 \leqq \int\limits_{E_2} f_n^2(x)\,d\,u^*(e) \int\limits_{E_2} g_n^2(x)\,d\,u^*(e) < c^2 \int\limits_{E_2} h^2(x)\,d\,u^*(e);$$

moreover, in view of (4. 11)

$$|\alpha_3|^2 \leqq \varepsilon^2 \left[\int\limits_{E_1} |g(x)|\,d\,u^*(e)\right]^2 \leqq \varepsilon^2 \int\limits_E d\,u^*(e) \int\limits_E g^2(x)\,d\,u^*(e) \leqq \lambda^2 \varepsilon^2$$

$(\lambda^2 = c^2 u^*(E))$ and, similarly,

$$|\alpha_4|^2 \leqq \varepsilon^2 \left[\int\limits_{E_1} |f_n(x)|\,d\,u^*(e)\right]^2 \leqq \varepsilon^2 u^*(E) \int\limits_E h^2(x)\,d\,u^*(e) = \varepsilon^2 \lambda_1^2.$$

Consequently

$$(4.\ 12) \qquad |\sigma_n| \leqq |\alpha_1| + \cdots + |\alpha_4| < 2\,c\left[\int\limits_{E_2} h^2(x)\,d\,u^*(e)\right]^{\frac{1}{2}} + \varepsilon\,\lambda + \varepsilon\,\lambda_1 \qquad (n \geqq n_0).$$

The function $\Phi(e) = \int\limits_e h^2(x)\,d\,u^*(e)$ of sets $\{B\}$ (in E) is absolutely continuous $\{B, u^*\}$ and thus vanishes with $u^*(e)$. Hence on taking δ suitably small, noting that $u^*(E_2) < \delta$ and choosing ε sufficiently small the last member in (4. 12) can be made arbitrarily small for all $n \geqq n_0$ (n_0 suitably great). Thus $\lim\limits_n \sigma_n = 0$, which establishes the theorem.

We shall need an extension of *Carleman's* Theorem I* (C; pp. 8, 9). The extension is as follows.

Theorem 4. 4. *We have*

$$\int\limits_{(\omega_0)} d\,u^*(e_x) \int\limits_{\lambda_0}^{\lambda_1} c(\lambda)\,d_\lambda\,\alpha(\lambda, x) = \int\limits_{\lambda_0}^{\lambda_1} c(\lambda)\,d_\lambda \int\limits_{(\omega_0)} \alpha(\lambda, x)\,d\,u^*(e_x),$$

provided $c(\lambda)$ *is continuous on the finite closed interval* (λ_0, λ_1),

$$\int\limits_{(\omega_0)} \alpha(\lambda, x)\,d\,u^*(e_x)$$

exists for $\lambda_0 \leqq \lambda \leqq \lambda_1$,

$$A(x) = V_{\lambda_0}^{\lambda_1}\,\alpha(\lambda, x) < +\infty \qquad (\textit{variation with respect to } \lambda)$$

for almost all $\{u^*\}(x)$ *in* (ω_0) *and the integral*

$$\int\limits_{(\omega_0)} A(x)\,d\,u^*(e_x)$$

exists.

The proof will be omitted as it may be given following the lines of (C; pp. 8, 9) as well as with the aid of the theorem according to which

$$\lim_n \int\limits_{(\omega_0)} q_n(x)\,d\,u^*(e_x) = \int\limits_{(\omega_0)} q(x)\,d\,u^*(e_x),$$

whenever $q_n(x) \to q(x)$ (as $n \to \infty$), while $q_n(x)$ is integrable $\{B, u^*\}$ and $|q_n(x)| \leqq s(x)$, where $s(x)$ is integrable $\{B, u^*\}$.

5. Formulation of the Integral Equation Problems.

Throughout we let (D_x), (D_y), (D_z) denote bounded domains in the sense of section 2; (x), (y), (z) will denote points and (ω), (τ), (ζ) domains in (D_x), (D_y), (D_z), respectively. Except for the notation of points the domains (D_x), (D_y), ... will be identical; whenever there is no possibility of confusion the subscript will be omitted. *Throughout, e will denote sets $\{B\}$ in (D_x), or (D_y), or ..., as the case may be.*

We consider the Stieltjes integral equations

$$\varphi(x) = \lambda \int_{(D_y)} k(\tau, x)\, \psi(y)\, d\tau + f(x), \tag{5.1}$$

$$\psi(\tau) = \lambda \int_{(D_x)} k(\tau, x)\, \psi(\omega)\, d\omega + F(\tau) \qquad (\text{m. a. } F(\tau)) \tag{5.2}$$

to be satisfied in (D_x) by functions of points $\varphi(x)$ and functions of domains $\psi(\omega)$. Associated with the problem *we have a m. a.* (section 2) *bounded function $u(\omega)$ of domains, which is non negative and for which $\tilde{u} = \omega\, u(\omega)$ is continuous as a function of intervals;* in accordance with Lemma 3.2 *with $u(\omega)$ one may associate a completely additive and continuous (not necessarily absolutely) function $u^*(e)\,(\geqq 0)$ of sets $\{B\}$ such that $u^* = \tilde{u}$ on figures in (D_x). We assume that $rF(r)$ is continuous as a function of intervals* — thus vanishing with the diameter of the latter.

We suppose that the Radon integral

$$\int_{(D_x)} f^2(x)\, d\, u^*(e) \tag{5.3}$$

exists, while for every finite n the function $f_n(x)$, defined by the relations

$$f_n(x) = f(x) \quad (\text{when } |f| \leqq n), \tag{5.3 a}$$

$$f_n(x) = n \quad (\text{when } f > n), \qquad f_n(x) = -n \quad (\text{when } f < -n),$$

is continuous in (D_x). This hypothesis implies that the limit

$$\lim_n \int_e f_n^2(x)\, d\, u^*(e) = \int_e f^2(x)\, d\, u^*(e)$$

exists and that

$$\int_{(\omega)} f_n(x)\, d\, u^*(e) = \int_{(\omega)} u(\omega) f_n(x)\, d\, \omega$$

for figures (ω) in (D_x), the latter integral being in the sense of Gunther.

Definition 5. 1. *It will be said that* $k(\omega, y)$ (*or* $L(x, y)$) *satisfies condition* (T) *if for all figures* (ω) (*in* (D_x))

$$\omega\, k(\omega, y) = \int_{(\omega)} L(x, y)\, d\, u^*(e_x) \qquad (L(x, y) = L(y, x);\ \omega = \text{meas.}\,(\omega)), \tag{5. 4}$$

while the integral

$$L^2(y) = \int_{(D_x)} L^2(x, y)\, d\, u^*(e_x) \tag{5. 4 a}$$

exists for all (y) *interior* (D_y); *moreover, when* (x) *and* (y) *are in domains lying in the interiors of* (D_x), (D_y) $L(x, y)$ *is continuous in* (x) *uniformly with respect to* (y) (*and continuous in* (y) *uniformly with respect to* (x)).

Kernels (T) do not come under the theory of Gunther. Accordingly such kernels may justifiably be termed *singular*.

Since the Radon integral

$$\int_{e} L^2(x, y)\, d\, u^*(e_x)$$

exists the same is true of

$$k^*(e, y) = \int_{e} L(x, y)\, d\, u^*(e_x); \tag{5. 5}$$

moreover,

$$k^*(\omega, y) = \omega\, k(\omega, y)$$

for figures (ω) in (D_x).

We shall establish that if $k(\omega, y)$ satisfies (T) one has

$$k(\omega, y) = \lim_{n} k_n(\omega, y) \qquad \text{(for figures } (\omega)) \tag{5. 6}$$

where $k_n(\omega, y)\,(n = 1, 2, \ldots)$ is a kernel 'symmetric' and *'regular'* in the sense that the theory developed in (G) applies to $k_n(\omega, y)$.

In fact, let

$$L_n(x, y) = \begin{cases} L(x, y) & (\text{when } |L(x, y)| < n), \\ n & (\quad \text{»} \quad L \geqq n), \\ -n & (\quad \text{»} \quad L \leqq -n); \end{cases}$$

$L_n(x, y)$ is symmetric and continuous in (D_x), (D_y); moreover,

$$\lim_n L_n(x, y) = L(x, y).$$

We write

$$(5.7) \qquad \tilde{k}_n(\omega, y) = \omega\, k_n(\omega, y) = \int_{(\omega)} u(\omega)\, L_n(x, y)\, d\,\omega = \int_{(\omega)} L_n(x, y)\, d\,u^*(e_x).$$

The function

$$(5.7\text{a}) \qquad k_n^*(e, y) = \int_e L_n(x, y)\, d\,u^*(e_x) \qquad (\{B\}\text{-sets } e \text{ in } (D_x))$$

is an extension of $\omega\, k_n(\omega, y)$ in the sense that

$$k_n^*(\omega, y) = \omega\, k_n(\omega, y)$$

for figures (ω). Now $|L_n(x, y)| \leqq |L(x, y)|$, while the integral (5.5) exists; the latter fact implies, of course, that the integral

$$\int_e |L(x, y)|\, d\,u^*(e_x)$$

also exists. Thus

$$\lim_n \int_e L_n(x, y)\, d\,u^*(e_x) = \int_e L(x, y)\, d\,u^*(e_x),$$

that is,

$$(5.7\text{b}) \qquad \lim_n k_n^*(e, y) = k^*(e, y)$$

for all sets e, $< \{B\}$, in (D_x); in particular (5.6) will hold.

In considering 'symmetric' kernels $h(\omega, y)$ to which the theory of (G) is applicable one is brought to the consideration of conditions (A), (C) ('symmetry'), (F) ('finiteness'), (D), (D^s) (condition (D) satisfied strictly). These conditions are as follows.

(A). $(1°)$ $h(\omega, y)$ is continuous in (y) for every (ω)

$(2°)$ The total bound $H(D_x, y)D_x$ of $h(\omega, y)$ is a bounded function of (y).

(C) or symmetry with respect to $u(\omega)$:

$$\frac{1}{\omega}\int\limits_{(\omega)} u(\omega)\, h(\tau, x)\, d\omega = \frac{1}{\tau}\int\limits_{(\tau)} u(\tau)\, h(\omega, y)\, d\tau. \tag{5. 8}$$

(F). (α). For every (y) one has

$$|h(\omega, y)| \leqq V_1(\omega) \qquad (\text{m. a. } V_1(\omega) < BV),$$

where $V_1(\omega)$ is independent of (y).

(β). For every $\varepsilon(> 0)$ there exists $\varrho(> 0)$ so that, with (y'), (y'') in the same sphere of radius ϱ, we have

$$|h(\omega, y') - h(\omega, y'')| < \varepsilon V_2(\omega) \qquad (\text{m. a. } V_2(\omega) < BV),$$

where $V_2(\omega)$ is independent of (y'), (y'').

(D).
$$\int\limits_{(D_y)} u(\tau)\, k^2(\omega, y)\, d\tau < C^2 u^2(\omega).$$

(D^s). $k(\omega, y)$ satisfies (F) and $V_1(\omega) < \alpha u(\omega)$.

The main part of the developments of (G) for 'symmetric' kernels $h(\omega, y)$ applies when $h(\omega, y)$ satisfies (A) and (C), while some iterant (i. e. iterated kernel) satisfies (F) and some iterant satisfies (D); we shall establish that $k_n(\omega, y)$ (figures (ω)) of $(5. 7)$ is such a kernel.

By a remark in (G), any kernel $k_n(\omega, y)$ of the form $(5. 7)$ is symmetric with respect to $u(\omega)$ and, thus, satisfies (C). The condition (A), $(1°)$ is satisfied by $k_n(\omega, y)$ in consequence of the statement subsequent to $(2. 5)$. (A), $(2°)$ is satisfied by $k_n(\omega, y)$ with

$$K_n(D_x, y)D_x \leqq n\, u(D_x)D_x.$$

In fact, the first member here is the upper bound of $S_m(y)$, where

$$S_m(y) = \sum_1^m |k_n(\omega_j, y)|\, \omega_j$$

for all finite decompositions $(D_x) = (\omega_1) + \cdots + (\omega_m)$. Now

$$S_m(y) = \sum_1^m \left| \int_{(\omega_j)} u(\omega) L_n(x, y)\, d\omega \right| \leqq \sum_1^m \int_{(\omega_j)} u(\omega) \left| L_n(x, y) \right| d\omega \leqq$$

$$\leqq n \sum_1^m \int_{(\omega_j)} u(\omega)\, d\omega = n \int_{(D_x)} u(\omega)\, d\omega,$$

from which the asserted inequality follows at once.

By (5. 7) and (2. 1 a)

$$| k_n(\omega, y) | \leqq n\, u(\omega);$$

hence $k_n(\omega, y)$ satisfies (F), (α) with

$$V_1(\omega) = n\, u(\omega). \tag{5. 9}$$

In view of Definition (5. 1) it is deduced that $L_n(x, y)$ has the following continuity properties. Given $\varepsilon (> 0)$ there exists $\varrho (> 0)$, independent of (x), so that

$$| L_n(x, y') - L_n(x, y'') | < \varepsilon \tag{5. 10}$$

whenever $\delta(y', y'')$ [= distance between (y') and (y'')] $< \varrho$, this being true for all pairs of points $[(y'), (y'')]$, in (D_y), and for all (x) in (D_x).

By (5. 7) and (2. 1 d)

$$| k_n(\omega, y') - k_n(\omega, y'') | = \frac{1}{\omega} \left| \int_{(\omega)} u(\omega) (L_n(x, y') - L_n(x, y''))\, d\omega \right|$$

$$\leqq \frac{1}{\omega} \int_{(\omega)} u(\omega) \left| L_n(x, y') - L_n(x, y'') \right| d\omega.$$

Whence, by virtue of the property (5. 10) we have

$$| k_n(\omega, y') - k_n(\omega, y'') | < \frac{\varepsilon}{\omega} \int_{(\omega)} u(\omega)\, d\omega = \varepsilon\, u(\omega),$$

for all (ω) and for all (y'), (y'') in (D_y) for which $\delta(y', y'') < \varrho = \varrho_{n,\varepsilon}$. Accordingly it is seen that $k_n(\omega, y)$ satisfies (F), (β) with

$$V_2(\omega) = u(\omega),$$

ϱ being dependent on n, of course.

Inasmuch as $k_n(\omega, y)$ has been shown to satisfy (F) with $V_1 = n\,u(\omega)$ it is noted that $k_n(\omega, y)$ also satisfies (D^s) (with any $a > n$). As observed before, (D^s) implies (D). Thus the following has been established.

Lemma 5. 1. *Every kernel $k(\omega, y)$, satisfying (T) (Definition 5. 1), is the limit, as stated in (5. 1), of approximating kernels $k_n(\omega, y)$ of the form (5. 7); $k_n(\omega, y)$ satisfies (A), (C), (F), (D^s) (and, hence, (D)) and may be appropriately termed 'regular' in the sense of Gunther.*

In consequence of (G), associated with the pair of homogeneous integral equations

$$\varphi(x) = \lambda \int\limits_{(D_y)} k_n(\tau, x)\, \varphi(y)\, d\tau, \tag{5. 11}$$

$$\psi(\tau) = \lambda \int\limits_{(D_x)} k_n(\tau, x)\, \psi(\omega)\, d\omega \tag{5. 12}$$

there is a sequence of real characteristic numbers and characteristic functions

$$(\lambda_{n,1}, \lambda_{n,2}, \ldots), \quad (\varphi_{n,1}(x), \varphi_{n,2}(x), \ldots), \quad (\psi_{n,1}(\tau), \psi_{n,2}(\tau), \ldots)$$

for which there are on hand the following relations

$$\varphi_{n,k}(x) = \lambda_{n,k} \int\limits_{(D_y)} k_n(\tau, x)\, \varphi_{n,k}(y)\, d\tau, \quad \psi_{n,k}(\tau) = \frac{1}{\tau} \int\limits_{(\tau)} u(\tau)\, \varphi_{n,k}(y)\, d\tau, \tag{5. 13}$$

$$\psi_{n,k}(\tau) = \lambda_{n,k} \int\limits_{(D_x)} k_n(\tau, x)\, \psi_{n,k}(\omega)\, d\omega. \tag{5. 13 a}$$

The sequence $\lambda_{n,1}, \lambda_{n,2}, \ldots$ contains at least one member, the $\lambda_{n,k}$ $(k = 1, 2, \ldots)$ are all distinct from zero and the set of points represented by the $\lambda_{n,k}$ $(k = 1, 2, \ldots)$ has no finite limiting points. Moreover, the $\varphi_{n,k}(x)$ may be arranged to form an ortho normal sequence in the sense that

$$\int\limits_{(D_x)} u(\omega)\, \varphi_{n,k}(x)\, \varphi_{n,j}(x)\, d\omega = \begin{cases} 0 & (\text{for } k \neq j), \\ 1 & (\text{for } k = j). \end{cases} \tag{5. 13 b}$$

By (G) there is on hand a 'Bessel's inequality'

$$\sum_k c_{n,k}^2 \leqq \int\limits_{(D_x)} u(\omega) f^2(x)\, d\omega, \quad c_{n,k} = \int\limits_{(D_x)} u(\omega) f(x)\, \varphi_{n,k}(x)\, d\omega, \tag{5. 14}$$

whenever $f(x)$ is continuous in (D_x). However, such an inequality will hold in the more general case when the integral

$$\int_{(D_x)} f^2(x)\, d\, u^*(e)$$

exists; one then has

$$\sum_k c_{n,k}^2 \leqq \int_{(D_x)} f^2(x)\, d\, u^*(e), \quad c_{n,k} = \int_{(D_x)} f(x)\, \varphi_{n,k}(x)\, d\, u^*(e). \tag{5. 14 a}$$

Of importance in the present investigation will be the approximating non homogeneous integral equations

$$\varphi(x) = \lambda \int_{(D_y)} k_n(\tau, x)\, \varphi(y)\, d\,\tau + f_n(x) \qquad \text{(cf. (5. 3 a))}, \tag{5. 15}$$

$$\psi(\tau) = \lambda \int_{(D_x)} k_n(\tau, x)\, \psi(\omega)\, d\,\omega + F(\tau). \tag{5. 15 a}$$

In consequence of (G) one may assert that a solution of (5. 15) may be given in the form

$$\begin{aligned} \varphi_n(x) &= f_n(x) - \sum_{k=1}^{\infty} \frac{\lambda\, c_{n,k}}{\lambda - \lambda_{n,k}} \varphi_{n,k}(x) \\ &= f_n(x) + \lambda \int_{(D_y)} k_n(\tau, x) f_n(y)\, d\,\tau - \sum_{k=1}^{\infty} \frac{\lambda^2}{\lambda - \lambda_{n,k}} \frac{c_{n,k}}{\lambda_{n,k}} \varphi_{n,k}(x) \qquad (c_{n,k} \text{ from (5. 14 a)}), \end{aligned} \tag{5. 16}$$

while a solution of (5. 15 a) is expressible as

$$\psi_n(\tau) = F(\tau) + \lambda \int_{(D_x)} k_n(\tau, x)\, F(\omega)\, d\,\omega - \sum_{k=1}^{\infty} \frac{1}{(\lambda - \lambda_{n,k})\, \lambda_{n,k}} \lambda^2 \sigma_{n,k} \psi_{n,k}(\tau), \tag{5. 17}$$

$$\sigma_{n,k} = \int_{(D_x)} F(\omega)\, \varphi_{n,k}(x)\, d\,\omega; \tag{5. 17 a}$$

this is asserted for λ distinct from the $\lambda_{n,k}$ $(k = 1, 2, \ldots)$. The series involved above converge.

6. **Spectral Theory.**

We shall construct several kinds of spectral functions associated with $k_n(\tau, x)$. Thus $\theta_n(x, y/\lambda)$ is to be defined by the relations

$$\theta_n(x, y/\lambda) = \sum_{0<\lambda_{n,k}<\lambda} \varphi_{n,k}(x)\,\varphi_{n,k}(y) \qquad \text{(for } \lambda > 0\text{)}, \tag{6. 1}$$

$$\theta_n(x, y/\lambda) = -\sum_{\lambda\leqq\lambda_{n,k}<0} \varphi_{n,k}(x)\,\varphi_{n,k}(y) \qquad \text{(for } \lambda < 0\text{)}, \tag{6. 1 a}$$

while $\theta_n(x, y/0) = 0$. On the other hand, $\theta_n(x, \tau/\lambda)$ is to be a function of points (x) and of domains (τ), given as follows:

$$\theta_n(x, \tau/\lambda) = \sum_{0<\lambda_{n,k}<\lambda} \varphi_{n,k}(x)\,\psi_{n,k}(\tau) \qquad \text{(for } \lambda > 0\text{)}, \tag{6. 2}$$

$$\theta_n(x, \tau/\lambda) = -\sum_{\lambda\leqq\lambda_{n,k}<0} \varphi_{n,k}(x)\,\psi_{n,k}(\tau) \qquad \text{(for } \lambda < 0\text{)}, \tag{6. 2 a}$$

the value zero being assigned for $\lambda = 0$. In an analogous way one may define, if necessary, a spectrum $\theta_n(\omega, \tau/\lambda)$.

In view of the second relation (5. 13) and of the definitions, just given,

$$\theta_n(x, \tau/\lambda) = \frac{1}{\tau}\int_{(\tau)} u(\tau)\,\theta_n(x, y/\lambda)\,d\tau. \tag{6. 3}$$

Similarly

$$\theta_n(\omega, \tau/\lambda) = \frac{1}{\tau}\int_{(\tau)} u(\tau)\,\theta_n(y, \omega)\,d\tau = \frac{1}{\omega\,\tau}\int_{(\omega)}\int_{(\tau)} u(\omega)\,u(\tau)\,\theta_n(x, y/\lambda)\,d\omega\,d\tau.$$

Designating a summation as in (6. 1) by a prime and a summation as in (6. 1 a) by a double prime, on using the orthogonality properties of the $\varphi_{n,j}$ we obtain

$$\int_{(D_z)} u(\zeta)\,\theta_n(x, z/\lambda)\,\theta_n(z, y/\lambda)\,d\zeta = \sum_{j,k}{}' \varphi_{n,j}(x)\,\varphi_{n,k}(y)\int_{(D_z)} u(z)\,\varphi_{n,j}(z)\,\varphi_{n,k}(z)\,d\zeta,$$

when $\lambda > 0$, and

$$\int_{(D_z)} \cdots = \sum_{j,k}'' \cdots$$

for $\lambda < 0$ and, finally,

$$\int_{(D_z)} u(\zeta)\,\theta_n(x, z/\lambda)\,\theta_n(z, y/\lambda)\,d\zeta = \pm\,\theta_n(x, y/\lambda) \tag{6. 4}$$

for all real λ; here $+\,(-)$ is for $\lambda > 0$ $(\lambda < 0)$.

We divide the linear interval $(-l, l)$ $(l > 0)$ into a finite number of linear intervals Δ_j $(j = 1, \ldots q)$ as follows:

$$(-l, l) = (\Delta_1) + \cdots + (\Delta_q), \quad (\Delta_j) = (l_{j-1}, l_j),$$

where

$$-l = l_0 < l_1 < \cdots < l_q = l, \quad \Delta_j = l_j - l_{j-1}.$$

In consequence of (6. 1) and (6. 1 a)

$$V_n = \sum_{j=1}^{q} |\theta_n(x, y/l_j) - \theta_n(x, y/l_{j-1})| \leqq \sum_{k}^{(1)} |\varphi_{n,k}(x)\,\varphi_{n,k}(y)|, \tag{6. 5}$$

the summation last displayed being over values k for which $-l \leqq \lambda_{n,k} < l$,

Let $(\tau) < (D_y)$. For finite decompositions $(\tau) = (\tau_1) + \cdots + (\tau_j)$ we form

$$S_j(\tau) = \sum_{\nu=1}^{j} |k_n(\tau_\nu, x)|\,\tau_\nu.$$

By (5. 7)

$$S_j(\tau) = \sum_{1}^{j} \left| \int_{(\tau_\nu)} u(\tau)\,L_n(y, x)\,d\tau \right| \leqq n \sum_{1}^{j} \int_{(\tau_\nu)} u(\tau)\,d\tau = n \int_{(\tau)} u(\tau)\,d\tau.$$

Hence

$$K_n(\tau, x)\,\tau = \text{u. b. } S_j(\tau) \leqq n\,u(\tau)\,\tau \tag{6. 6}$$

and, in view of (2. 1 d),

$$\left| \int_{(D_y)} k_n(\tau, x)\,\varphi_{n,k}(y)\,d\tau \right| \leqq \int_{(D_y)} K_n(\tau, x)\,|\varphi_{n,k}(y)|\,d\tau \leqq n \int_{(D_y)} u(\tau)\,|\varphi_{n,k}(y)|\,d\tau.$$

By (2. 1 c)

$$\left|\int\limits_{(D_y)} k_n(\tau, x)\, \varphi_{n,k}(y)\, d\tau\right|^2 \leqq n^2 \int\limits_{(D_y)} u(\tau)\, d\tau \int\limits_{(D_y)} u(\tau)\, \varphi_{n,k}^2(y)\, d\tau$$

and, by virtue of (5. 13 b),

$$\left|\int\limits_{(D_y)} k_n(\tau, x)\, \varphi_{n,k}(y)\, d\tau\right|^2 \leqq D_y\, n^2\, u(D_y) = a_0^2\, n^2;$$

whence from (5. 13) it follows that

$$|\varphi_{n,k}(x)| \leqq n\, a_0\, |\lambda_{n,k}| \tag{6. 7}$$

for all (x) in (D_x).

The boundedness property, just established, enables application of (24), with $f = L_n$ (so that $v = k_n$) and with $F(x) = \varphi_{n,k}(x)$; thus

$$\int\limits_{(D_x)} k_n(\omega, y)\, \varphi_{n,k}(x)\, d\omega = \int\limits_{(D_x)} u(\omega)\, L_n(x, y)\, \varphi_{n,k}(x)\, d\omega \tag{6. 8}$$

and, accordingly, in consequence of (5. 13) and (2. 1 c) one has

$$|\varphi_{n,k}(y)| \leqq |\lambda_{n,k}| \left|\int\limits_{(D_x)} k_n(\omega, y)\, \varphi_{n,k}(x)\, d\omega\right|$$
$$\leqq |\lambda_{n,k}| \left[\int\limits_{(D_x)} u(\omega)\, L_n^2(x, y)\, d\omega\right]^{\frac{1}{2}} \left[\int\limits_{(D_x)} u(\omega)\, \varphi_{n,k}^2(x)\, d\omega\right]^{\frac{1}{2}};$$

thus by (5. 13 b)

$$|\varphi_{n,k}(y)| \leqq |\lambda_{n,k}| \left[\int\limits_{(D_x)} u(\omega)\, L_n^2(x, y)\, d\omega\right]^{\frac{1}{2}} = |\lambda_{n,k}| \left[\int\limits_{(D_x)} L_n^2(x, y)\, d\, u^*(e_x)\right]^{\frac{1}{2}}.$$

Now $L_n^2(x, y) \leqq L^2(x, y)$, while the integral (5. 4) exists; hence

$$|\varphi_{n,k}(y)| \leqq |\lambda_{n,k}|\, L(y). \tag{6. 9}$$

It is observed that V_n of (6. 5) satisfies

$$V_n^2 \leqq \sum_k^{(1)} \varphi_{n,k}^2(x) \sum_k^{(1)} \varphi_{n,k}^2(y).$$

In consequence of (5. 13) and (6. 8)

$$\sum_k^{(1)} \varphi_{n,k}^2(y) \leqq l^2 \sum_k \left| \int_{(D_x)} u(\omega) L_n(x, y) \varphi_{n,k}(x) \, d\omega \right|^2$$

$$= l^2 \sum_k \left[\int_{(D_x)} L_n(x, y) \varphi_{n,k}(x) \, d u^*(e_x) \right]^2.$$

Applying the Bessel's inequality (5. 14 a) we obtain

$$\sum_k^{(1)} \varphi_{n,k}^2(y) \leqq l^2 \int_{(D_x)} L_n^2(x, y) \, d u^*(e_x) \leqq l^2 \int_{(D_x)} L^2(x, y) \, d u^*(e_x) = l^2 L^2(y).$$

Whence

$$V_n \leqq l^2 L(x) L(y). \tag{6. 10}$$

Taking the upper bound of V_n (n fixed) for all possible finite decompositions of $(-l, l)$, in consequence of (6. 10) it is inferred that

$$V_{-l}^{l} \theta_n(x, y/\lambda) \leqq l^2 L(x) L(y), \tag{6. 11}$$

where $V_{\dots}^{\dots}$ denotes total variation in λ for λ on the interval $(-l, l)$; moreover, since $\theta_n(x, y/0) = 0$,

$$|\theta_n(x, y/\lambda)| \leqq l^2 L(x) L(y) \qquad (\lambda \text{ on } (-l, l)). \tag{6. 11 a}$$

Let $(x), (x') < (\omega_0)$ and $(y), (y') < (\tau_0)$, where $(\omega_0) < (D_x)^0$, $(\tau_0) < (D_y)^0$. With summation extended over certain values k, one has

$$|\theta_n(x', y'/\lambda) - \theta_n(x, y/\lambda)| = \left| \sum_k^{(1)} [\varphi_{n,k}(x') - \varphi_{n,k}(x)] \varphi_{n,k}((y') \right.$$

$$\left. + \sum_k^{(1)} \varphi_{n,k}(x) [\varphi_{n,k}(y') - \varphi_{n,k}(y)] \right| \leqq \left[\sum_k^{(1)} |\varphi_{n,k}(x') - \varphi_{n,k}(x)|^2 \right]^{\frac{1}{2}} \left[\sum_k^{(1)} \varphi_{n,k}^2(y') \right]^{\frac{1}{2}}$$

$$+ \left[\sum_k^{(1)} \varphi_{n,k}^2(x) \right]^{\frac{1}{2}} \left[\sum_k^{(1)} |\varphi_{n,k}(y') - \varphi_{n,k}(y)|^2 \right]^{\frac{1}{2}}$$

(compare with (C)). For values k, here involved, $|\lambda_{n,k}| \leqq |\lambda|$. Now by (5. 13) and (6. 8)

$$\varphi_{n,k}(x') - \varphi_{n,k}(x) = \lambda_{n,k} \int_{(D_y)} u(\tau) (L_n(y, x') - L_n(y, x)) \varphi_{n,k}(y) \, d\tau$$

and, by virtue of (5. 14 a), it is deduced that

$$\sum_k^{(1)} |\varphi_{n,k}(x') - \varphi_{n,k}(x)|^2 \leqq |\lambda|^2 \int_{(D_y)} u(\tau)(L_n(y, x') - L_n(y, x))^2 d\tau$$

$$= |\lambda|^2 \int_{(D_y)} (L_n(y, x') - L_n(y, x))^2 d u^*(e_y) \leqq |\lambda|^2 \int_{(D_y)} (L(y, x') - L(y, x))^2 d u^*(e_y)$$

for all $n > n''$, where n'' may depend on (ω_0). With the aid of the inequalities preceding (6. 10) we finally obtain

$$(6.\ 12) \quad |\theta_n(x', y'/\lambda) - \theta_n(x, y/\lambda)| \leqq l^2 L(y') \left[\int_{(D_y)} (L(x', y) - L(x, y))^2 d u^*(e_y)\right]^{\frac{1}{2}}$$

$$+ l^2 L(x) \left[\int_{(D_x)} (L(x, y') - L(x, y))^2 d u^*(e_x)\right]^{\frac{1}{2}} = c(x', y'; x, y)$$

for $|\lambda| \leqq l$ and for all $n > n'$, where n' may possibly depend on (ω_0), (τ_0).

Definition 6. 1. *It will be said that $k(\omega, y)$ is a kernel (T^*) if it satisfies (T) (Definition 5. 1) and if*

$$\lim \int_{(D_x)} (L(x, y') - L(x, y))^2 d u^*(e_x) = 0 \qquad (as\ (y') \to (y)),$$

the points (y) being in $(D_y)^0$.

If $k(\omega, y)$ is a kernel (T^*), it follows from (6. 12) that for

$$(6.\ 13) \qquad (x) < (\omega_0) < (D_x)^0 \text{ and } (y) < (\tau_0) < (D_y)^0$$

the *continuity of the $\theta_n(x, y/\lambda)$ in $[(x), (y)]$ is uniform with respect to n*; in fact, in this case there is also uniformity of continuity in $[(x), (y)]$ with respect to λ, provided λ is on a fixed interval $(-l, l)$.

The second members in (6. 11) and (6. 11 a) are independent of n and are defined in $(D_x)^0$, $(D_y)^0$; these inequalities enable us to infer that for some subsequence $\{n_j\}$ of $\{n\}$ one has for real values λ, with a possible exception of a denumerable infinity of values λ,

$$(6.\ 14) \qquad \lim \theta_{n_j}(x, y/\lambda) = \theta(x, y/\lambda) \qquad (\text{as } n_j \to \infty),$$

where the limiting function satisfies the inequalities

(6. 14 a) $$V_{-l}^{l}\,\theta(x, y/\lambda) \leqq l^2 L(x) L(y),$$

(6. 14 b) $$|\theta(x, y/\lambda)| \leqq l^2 L(x) L(y)$$

for $|\lambda| \leqq l$ and for $(x) < (D_x)^0$ and $(y) < (D_y)^0$.

When (T^*) is satisfied, the 'Compactness Theorem' of (C; 21, 22) will secure existence of a limit (6. 14) for all real λ; moreover, θ will satisfy the continuity condition (6. 12) in (x) and (y) for $x < (D_x)^0$ and $(y) < (D_y)^0$; furthermore, there will exist a denumerable sequence

(6. 15) $$\mu_1, \mu_2, \ldots,$$

such that $\theta(x, y/\lambda)$ is continuous in λ, for $\lambda \neq \mu_\nu$ $(\nu = 1, 2, \ldots)$, for all $(x) < (D_x)^0$, $(y) < (D_y)^0$, while, for $\lambda = \mu_\nu$, $\theta(x, y/\lambda)$ has a discontinuity, as a function of λ, for some values (x), (y) (in $(D_x)^0$, $(D_y)^0$).

Any function $\theta(x, y/\lambda)$, obtained by the above processes, will be termed spectrum (or spectral function) of the kernel $k(\tau, x)$.

By (6. 4) and (6. 11 a)

(6. 16) $$|\theta_{n_j}(x, x/\lambda)| = \int_{(D_z)} u(\zeta)\,\theta_{n_j}^2(x, z/\lambda)\,d\zeta = \int_{(D_z)} \theta_{n_j}^2(x, z/\lambda)\,d u^*(e_z) \leqq \lambda^2 L^2(x)$$

$(j = 1, 2, \ldots)$. Hence it is possible to apply Theorem (4. 2) (with convergence in the ordinary sense — in view of (6. 14)) so as to infer existence of the integral

$$\int_{(D_z)} \theta^2(x, z/\lambda)\,d u^*(e_z)$$

and the inequality (see (4. 6))

(6. 16 a) $$\int_{(D_z)} \theta^2(x, z/\lambda)\,d u^*(e_z) \leqq |\theta(x, x/\lambda)| \leqq \lambda^2 L^2(x).$$

Accordingly one may form the integrals

(6. 17) $$\psi_n(x, \lambda) = \int_{(D_z)} h(z)\,\theta_n(x, z/\lambda)\,d u^*(e_z), \quad \psi(x, \lambda) = \int_{(D_z)} h(z)\,\theta(x, z/\lambda)\,d u^*(e_z)$$

for all functions $h(z)$ such that $h^2(z)$ is integrable $\{B.\ u^*\}$ over (D_z); moreover, by virtue of (6. 14) and (6. 16) appliction of Theorem 4. 3 will yield

$$\lim_{n_j} \psi_{n_j}(x, \lambda) = \psi(x, \lambda). \tag{6. 17 a}$$

Using the theorems of section 4 and the results of this section, so far established, *a number of other facts can be given which are closely analogous to those of Carleman (see Chapter I of (C)).* We shall state these results in the remainder of this section, giving minimum of details in the proofs. *Unless explicitly said that the kernels are (T^*) it is to be understood that they are (T);* $\lim_n$ is to be understood as the limit as $n = n_j \to \infty$.

By (6. 16 a) and Theorem 4. 3

$$\lim_{(x')\to(x)} [\psi(x, \lambda) - \psi(x', \lambda)] = 0 \qquad [(x), (x') \text{ in } (D_x)^0] \tag{6. 18}$$

in the case (T^*). By (6. 4) and (5. 14 a)

$$\int_{(D_x)} \psi_n^2(x, \lambda)\, du^*(e_x) = \pm \int_{(D_z)} \int_{(D_v)} h(z)\, h(v)\, \theta_n(z, v)\, d u^*(e_z)\, d u^*(e_v) \tag{6. 18 a}$$
$$\leqq \int_{(D_z)} h^2(z)\, d\, u^*(e_z),$$

provided that the latter integral exists. In view of (6. 17 a), (6. 18 a) and Theorem 4. 3

$$\lim_n \int_{(D_x)} g(x)\, \psi_n(x, \lambda)\, d\, u^*(e_x) = \int_{(D_x)} g(x)\, \psi(x, \lambda)\, d\, u^*(e_x) \tag{6. 18 b}$$

when $g^2(x)$ is integrable $\{B, u^*\}$; one has

$$\int_{(D_x)} \int_{(D_z)} g(x)\, h(z)\, \theta(x, z/\lambda)\, d\, u^*(e_x)\, d\, u^*(e_z) \tag{6. 18 c}$$
$$= \lim_n \int_{(D_x)} \int_{(D_z)} g(x)\, h(z)\, \theta_n(x, z/\lambda)\, d\, u^*(e_x)\, d\, u^*(e_z),$$

the order of integration in the integrals involved being immaterial.

By (6. 9), (5. 14 a) we get

$$|\psi_n(x, \lambda)|,\ |\psi(x, \lambda)|,\ V_{-l}^{l}\, \psi_n(x, \lambda),\ V_{-l}^{l}\, \psi(x, \lambda) \leqq l\, L(x) \left[\int_{(D_z)} h^2(z)\, du^*(e_z)\right]^{\frac{1}{2}}; \tag{6. 19}$$

one obtains the same with (D_z) replaced by a subset (ζ_0) if this is done in (6. 17). Also, in view of (5. 14 a)

$$\left|\int_{(D_x)}\int_{(D_y)} g(x)\,h(y)\,\theta_n(x,\,y/\lambda)\,d\,u^*(e_y)\,d\,u^*(e_x)\right|^2 = |\,\psi_n^2(\lambda)\,| \leqq \int_{(D_x)} g^2(x)\,d\,u^*(e_x)\int_{(D_y)} h^2(y)\,d\,u^*(e_y), \tag{6. 19 a}$$

$$V_{-l}^{l}\,\psi_n(\lambda) \leqq \text{last member of (6. 19 a)}; \tag{6. 19 b}$$

the same inequalities are satisfied by

$$\psi(\lambda) = \int_{(D_x)}\int_{(D_y)} g(x)\,h(y)\,\theta(x,\,y/\lambda)\,d\,u^*(e_x)\,d\,u^*(e_y), \tag{6. 19 c}$$

In the case (T^*), using (6. 16 a) and Theorem 4. 3, as well as (6. 18 a), one obtains

$$\lim_{\varepsilon\to 0}\int_{(D_y)} \theta(x,\,y/\lambda\pm\varepsilon)\,h(y)\,d\,u^*(e_y) = \int_{(D_y)} \theta(x,\,y/\lambda\pm 0)\,h(y)\,du^*(e_y), \tag{6. 20}$$

$$\lim_{\varepsilon\to 0}\int_{(D_x)}\int_{(D_y)} \theta(x,\,y/\lambda\pm\varepsilon)\,h(x)\,g(y)\,d\,u^*(e_x)\,d\,u^*(e_y) = \int_{(D_x)}\int_{(D_y)} \theta(x,\,y/\lambda\pm 0)\,h(x)\,g(y)\,d\,u^*(e_x)\,d\,u^*(e_y). \tag{6. 20 a}$$

By (6. 14), (6. 11), (6. 17 a), (6. 19)—(6. 19 b) and a theorem of Helly (C; p. 9)

$$\int_{\lambda_0}^{\lambda_1} c(\lambda)\,d_\lambda\,\theta(x,\,y/\lambda) = \lim_n \int_{\lambda_0}^{\lambda_1} c(\lambda)\,d_\lambda\,\theta_n(x,\,y/\lambda), \tag{6. 21}$$

$$\int_{\lambda_0}^{\lambda_1} c(\lambda)\,d_\lambda\,\psi(x,\,\lambda) = \lim_n \int_{\lambda_0}^{\lambda_1} c(\lambda)\,d_\lambda\,\psi_n(x,\,\lambda), \tag{6. 21 a}$$

$$\int_{\lambda_0}^{\lambda_1} c(\lambda)\,d_\lambda\,\psi(\lambda) = \lim_n \int_{\lambda_0}^{\lambda_1} c(\lambda)\,d_\lambda\,\psi_n(\lambda) \qquad \text{(cf. (6. 19 c))} \tag{6. 21 b}$$

provided $c(\lambda)$ is continuous on the closed finite interval $(\lambda_0,\,\lambda_1)$.

In view of (6. 10) and Theorem 4. 4

$$\int_{(\tau_0)} h(y)\left[\int_{\lambda_0}^{\lambda_1} c(\lambda)\, d_\lambda\, \theta(x,\ y/\lambda)\right] d\, u^*(e_y) = \int_{\lambda_0}^{\lambda_1} c(\lambda)\, d_\lambda \left[\int_{(\tau_0)} h(y)\theta(x,\ y/\lambda)\, d\, u^*(e_y)\right]$$

for domains $(\tau_0) < (D_y)^0$ and (x) in $(D_x)^0$; by (6 19) and Helly's theorem from the above we obtain

$$(6.\ 22)\qquad \int_{(D_y)} h(y)\left[\int_{\lambda_0}^{\lambda_1} c(\lambda)\, d_\lambda\, \theta(x,\ y/\lambda)\right] d\, u^*(e_y) = \int_{\lambda_0}^{\lambda_1} c(\lambda)\left[d_\lambda \int_{(D_y)} h(y)\, \theta(x,\ y/\lambda)\, d\, u^*(e_y)\right].$$

In consequence of Theorem 4. 4

$$\int_{(\omega_0)} \int_{(\tau_0)} g(x)\, h(y)\left[\int_{\lambda_0}^{\lambda_1} c(\lambda)\, d_\lambda\, \theta(x,\ y/\lambda)\right] d\, u^*(e_x)\, d\, u^*(e_y)$$
$$= \int_{\lambda_0}^{\lambda_1} c(\lambda)\, d_\lambda \left[\int_{(\omega_0)} \int_{(\tau_0)} g(x)\, h(y)\, \theta(x,\ y/\lambda)\, d\, u^*(e_x)\, d\, u^*(e_y)\right]$$

for domains (ω_0), (τ_0) in $(D_x)^0$, $(D_y)^0$, respectively; by virtue of Helly's theorem one may let $(\omega_0) \to (D_x)$, $(\tau_0) \to (D_y)$, obtaining

$$(6.\ 22\ \mathrm{a})\qquad \int_{(D_x)} \int_{(D_y)} g(x)\, h(y)\left[\int_{\lambda_0}^{\lambda_1} c(\lambda)\, d_\lambda \theta(x,\ y/\lambda)\right] d\, u^*(e_x)\, d\, u^*(e_y)$$
$$= \int_{\lambda_0}^{\lambda_1} c(\lambda)\, d_\lambda \left[\int_{(D_x)} \int_{(D_y)} g(x)\, h(y)\, \theta(x,\ y/\lambda)\, d\, u^*(e_x)\, d\, u^*(e_y)\right].$$

Multiply the members of (6. 22) by $g(x)\, d\, u^*(e_x)$ and integrate over (D_x); there results an inequality which, in consequence of (6. 22 a), is of the form (cf. (6. 19 c))

$$(6.\ 22\ \mathrm{b})\qquad \int_{(D_x)} g(x)\left[\int_{\lambda_0}^{\lambda_1} c(\lambda)\, d_\lambda \int_{(D_y)} \theta(x,\ y/\lambda)\, h(y)\, d\, u^*(e_y)\right] d\, u^*(e_x) = \int_{\lambda_0}^{\lambda_1} c(\lambda)\, d_\lambda\, \psi(\lambda).$$

By virtue of (5. 13) and (6.8)

$$(6.\ 23)\qquad \varphi_{n,k}(x) = \lambda_{n,k} \int_{(D_z)} L_n(x,\ z)\, \varphi_{n,k}(z)\, d\, u^*(e_z);$$

hence by definition of θ_n we obtain an equality (6. 24, n) which in the limit yields

$$\theta(x, y/\lambda'') - \theta(x, y/\lambda') = \int\limits_{(D_z)} L(x, z) \left[\int\limits_{\lambda'}^{\lambda''} \mu \, d_\mu \theta(z, y/\mu)\right] d u^*(e_z) \tag{6. 24}$$

(in $(D_x)^0$, $(D_y)^0$). To do this we change (by (6. 22)) the order of integration in (6. 24, n), obtaining

$$\theta_n(x, y/\lambda'') - \theta_n(x, y/\lambda') = \int\limits_{\lambda'}^{\lambda''} \mu \, d_\mu \left[\int\limits_{(D_z)} L_n(x, z) \, \theta_n(z, y/\mu) \, d u^*(e_z)\right],$$

and then pass to the limit, on making use of (6. 16 a), Theorem 4. 3, (6. 19) (with $h(z) = L_n(x, z)$) and of Helly's theorem; a change of order of integration in the resulting formula is possible in view of (6. 22), yielding (6. 24).

Consider case (T^*). Let μ_ν be a number (6. 15) and write

$$\theta(x, y \,|\, \mu_\nu + 0) - \theta(x, y \,|\, \mu_\nu - 0) = e_\nu(x, y) \qquad (\nu = 1, 2, \ldots). \tag{6. 25}$$

By definition $e_\nu(x, y) \not\equiv 0$. We have (in $(D_x)^0$, $(D_y)^0$)

$$\lim_{\varepsilon \to 0} \int\limits_{(D_z)} L(x, z) \left[\int\limits_{\mu_\nu - \varepsilon}^{\mu_\nu + \varepsilon} (\mu - \mu_\nu) \, d_\mu \theta(z, y/\mu)\right] d u^*(e_z) = 0,$$

since in consequence of (6. 19) the absolute value of the integral here involved is $\leqq \varepsilon(|\mu_\nu| + \varepsilon) L(y) L(x)$. Now, by (6. 20),

$$\lim_{\varepsilon} \int\limits_{(D_z)} L(x, z) \left[\int\limits_{\mu_\nu - \varepsilon}^{\mu_\nu + \varepsilon} \mu_\nu \, d_\mu \theta(z, y/\mu)\right] d u^*(e_z) = \int\limits_{(D_z)} L(x, z) \, [\mu_\nu e_\nu(z, y)] \, d u^*(e_z);$$

thus (6. 24) (with $\lambda'' = \mu_\nu + \varepsilon$, $\lambda' = \mu_\nu - \varepsilon$) will yield, as $\varepsilon \to 0$,

$$e_\nu(x, y) = \mu_\nu \int\limits_{(D_z)} L(x, z) \, e_\nu(z, y) \, d u^*(e_z) \quad (\mu_\nu \neq 0; \ \nu = 1, 2, \ldots) \tag{6. 26}$$

whenever the kernel is (T^*).

Whence, in the case (T^*), *the functions* $e_\nu(x, y)$ ((y) *fixed*) *are solutions of the homogeneous problem, while the* μ_ν *are 'characteristic values', in a sense*; the μ_ν may be everywhere dense in parts or on the whole of the axis of reals.

In the case (T^*) the theorems (C; 40), (C; 43), (C; 50) apply with suitable changes in formulation.

7. Connection between the two Problems.

The approximating equations (5. 15), (5. 15 a) may be written in the form

$$\varphi(x) = \lambda \int_{(D_y)} L_n(x, y)\, \varphi(y)\, d\, u^*(e_y) + f_n(x), \tag{7. 1}$$

$$\psi^*(e_y) = \lambda \int_{(D_x)} \left[\int_{e_y} L_n(x, y)\, d\, u^*(e_y) \right] d\, \psi^*(e_x) + F^*(e_y). \tag{7. 1 a}$$

In the latter equation $F^*(e_y)$ is an additive function of sets $\{B\}$ which on figures coincides with $\tau F(\tau)$. Inasmuch as $\tau F(\tau)$ is continuous as a function of intervals (i. e. the two non negative components are), $F^*(e_y)$ may be formed so that in the decomposition

$$F^*(e_y) = F_1^*(e_y) - F_2^*(e_y), \quad F_1^*(e) \geqq 0, \quad F_2^*(e) \geqq 0 \tag{7. 2}$$

the components $F_i^*(e)$ are continuous as functions of sets $\{B\}$. We write

$$v^*(e) = F_1^*(e) + F_2^*(e). \tag{7. 2 a}$$

As to the unknown set-function $\psi^*(e_y)$ — this is to be a continuous function of sets which on figures coincides with $\tau\psi(\tau)$, where $\psi(\tau)$ is the unknown function of figures for the problem (5. 15 a). The kernel in (7. 1 a) is justified by (5. 7 b), the relation preceding (5. 7 b) and by (5. 5). The kernel in (7. 1) is justified by (5. 7 a).

The non homogeneous equations for which (7. 1), (7. 1 a) are approximating equations are as follows:

$$\varphi(x) = \lambda \int_{(D_y)} L(x, y)\, \varphi(y)\, d u^*(e_y) + f(x), \tag{7. 3}$$

$$\psi^*(e_y) = \lambda \int_{(D_x)} \left[\int_{e_y} L(x, y)\, d\, u^*(e_y) \right] d\, \psi^*(e_x) + F^*(e_y). \tag{7. 3 a}$$

Whenever dealing with (7. 3 a), *the following Hypothesis will be assumed to hold.*

Hypothesis 7. 4. *With condition (T) or (T^*) satisfied, as the case may be, the integrals*

$$\int_{(D_x)} L(x)\, d\, u^*(e_x), \quad \int_{(D_x)} L(x)\, d\, v^*(e_x) \qquad (cf. \ (7.\ 2\ a),\ (5.\ 4\ a)) \tag{7. 4 a}$$

exist.

In this connection it is to be noted that existence of the first integral (7. 4 a) does not imply existence of

$$\int_{(D_x)} \int_{(D_y)} L^2(x, y)\, d\, u^*(e_x)\, d\, u^*(e_y); \tag{7. 5}$$

this is an essential fact, since it can be shown that whenever (7. 5) exists we are brought back to the main features of Gunther's theory; accordingly, we avoid integrability $\{B, u^*\}$ of $L^2(x, y)$ with respect to $[x, y]$.

We shall now establish a connection between the equations (7. 3), (7. 3 a) in the case when Hypothesis (7. 4) holds. Let $\Phi(y)$, such that $\Phi^2(y)$ is integrable $\{B, u^*\}$, be a solution of

$$\Phi(y) = \lambda \int_{(D_x)} L(x, y)\, \Phi(x)\, d\, u^*(e_x) + q(y), \tag{7. 6}$$

where

$$q(y) = \lambda \int_{(D_x)} L(x, y)\, d\, F^*(e_x). \tag{7. 6 a}$$

Existence of the latter integral follows from that of the second one in (7. 4 a). We have

$$\int_{(D_y)} q^2(y)\, d\, u^*(e_y) = \lambda^2 \int_{(D_x)} \int_{(D_z)} \left[\int_{(D_y)} L(x, y)\, L(z, y)\, d\, u^*(e_y)\right] d\, F^*(e_x)\, d\, F^*(e_z),$$

$$\left|\int_{(D_y)} L(x, y)\, L(z, y)\, d\, u^*(e_y)\right| \leqq \left[\int_{(D_y)} L^2(x, y)\, d\, u^*(e_y)\right]^{\frac{1}{2}} \cdot$$

$$\cdot \left[\int_{(D_y)} L^2(z, y)\, d\, u^*(e_y)\right]^{\frac{1}{2}} = L(x)\, L(z);$$

$|q^2(y)|$ is integrable $\{B, u^*\}$ and

$$\left|\int\limits_{(D_y)} q^2(y)\, d\, u^*(e_y)\right| \leqq |\lambda|^2 \int\limits_{(D_x)} \int\limits_{(D_z)} L(x)\, L(z)\, d\, v^*(e_x)\, d\, v^*(e_z) = |\lambda|^2 \left[\int\limits_{(D_x)} L(x)\, d\, v^*(e_x)\right]^2.$$

Moreover

$$q(y) = \lim_n q_n(y),$$

where

$$q_n(y) = \lambda \int\limits_{(D_x)} L_n(x,\, y)\, d\, F^*(e_x);$$

we have

$$|q_n(y)| \leqq |\lambda|\, n \int\limits_{(D_x)} d\, v^*(e_x) = |\lambda|\, n\, v^*(D_x);$$

furthermore, $q_n(y)$ is continuous in (D_y) in view of the continuity properties of $L_n(x, y)$ (cf. (5. 10)). In the sequel it will be actually proved, that for certain values λ, the equation (7. 6) has solutions $\Phi(y)$, with $\Phi^2(y)$ integrable $\{B, u^*\}$. Forming the set-function

$$\psi^*(e_y) = F^*(e_y) + \int\limits_{e_y} \Phi(y)\, d\, u^*(e_y), \tag{7. 7}$$

the connection sought for is established. To prove this we note that, in consequence of Hypothesis 7. 4 the order of integration in

$$\alpha' = \int\limits_{e_y} \int\limits_{(D_x)} L(x,\, y)\, d\, F^*(e_x)\, d\, u^*(e_y), \quad \alpha'' = \int\limits_{e_y} \int\limits_{(D_x)} L(x,\, y)\, \Phi(x)\, d\, u^*(e_x)\, d\, u^*(e_y) \tag{7. 8}$$

is immaterial; one has

$$|\alpha'| \leqq \int\limits_{(D_x)} \left|\int\limits_{e_y} L(x,\, y)\, d\, u^*(e_y)\right| d\, v^*(e_x) \leqq [u^*(e_y)]^{\frac{1}{2}} \int\limits_{(D_x)} L_{(e_y)}(x)\, d\, v^*(e_x) \tag{7. 8 a}$$

$$\leqq [u^*(e_y)]^{\frac{1}{2}} \int\limits_{(D_x)} L(x)\, d\, v^*(e_x), \quad L^2_{(e_y)}(x) = \int\limits_{e_y} L^2(x,\, y)\, d\, u^*(e_y)$$

and

$$(7.\,8\,\mathrm{b})\qquad |\alpha''| \leqq \int\limits_{e_y} \Big| \int\limits_{(D_x)} L(x, y)\, \Phi(x)\, d\,u^*(e_x) \Big|\, d\,u^*(e_y)$$

$$\leqq \int\limits_{e_y} \Big[\int\limits_{(D_x)} L^2(x, y)\, d\,u^*(e_x) \Big]^{\frac{1}{2}} \Big[\int\limits_{(D_x)} |\Phi(x)|^2\, d\,u^*(e_x) \Big]^{\frac{1}{2}} du^*(e_y)$$

$$\leqq \Big[\int\limits_{(D_x)} |\Phi(x)|^2\, d\,u^*(e_x) \Big]^{\frac{1}{2}} \int\limits_{e_y} L(y)\, d\,u^*(e_y).$$

In consequence of (7. 7) it is inferred that

$$(7.\,8\,\mathrm{c})\qquad \int\limits_{(D_x)} \Big[\int\limits_{e_y} L(x, y)\, d\,u^*(e_y) \Big] [\Phi(x)\, d\,u^*(e_x) + d\,F^*(e_x)]$$

$$= \int\limits_{(D_x)} \Big[\int\limits_{e_y} L(x, y)\, d\,u^*(e_y) \Big]\, d\,\psi^*(e_x).$$

Multiplying both members of (7. 6) by $d\,u^*(e_y)$ and integrating over e_y we obtain

$$(7.\,8\,\mathrm{d})\qquad \int\limits_{e_y} \Phi(y)\, d\,u^*(e_y) = \lambda \int\limits_{e_y} \Big[\int\limits_{(D_x)} L(x, y)\, \Phi(x)\, d\,u^*(e_x)\, d\,u^*(e_y) + \int\limits_{e_y} q(y)\, d\,u^*(e_y).$$

Now by (7. 6 a)

$$\int\limits_{e_y} q(y)\, d\,u^*(e_y) = \lambda \int\limits_{e_y} \int\limits_{(D_x)} L(x, y)\, d\,F^*(e_x)\, d\,u^*(e_y)$$

$$= \lambda\, \alpha' = \lambda \int\limits_{(D_x)} \Big[\int\limits_{e_y} L(x, y)\, d\,u^*(e_y) \Big]\, d\,F^*(e_x).$$

We add $F^*(e_y)$ to both members of (7. 8 d) and changing the order of integration obtain

$$\psi^*(e_y) = \lambda \int\limits_{(D_x)} \Big[\int\limits_{e_y} L(x, y)\, d\,u^*(e_y) \Big]\, \Phi(x)\, d\,u^*(e_x)$$

$$+ \lambda \int\limits_{(D_x)} \Big[\int\limits_{e_y} L(x, y)\, d\,u^*(e_y) \Big]\, d\,F^*(e_x) + F^*(e_y),$$

which in consequence of (7. 8 c) yields

$$\psi^*(e_y) = \lambda \int\limits_{(D_x)} \Big[\int\limits_{e_y} L(x, y)\, d\,u^*(e_y) \Big]\, d\,\psi^*(e_x) + F^*(e_y).$$

Lemma 7. 1. *In the case* (T) (*Definition* 5. 1) *and under Hypothesis* 7. 4 *every solution* $\Phi(y)$, *such that*

$$\int_{(D_y)} \Phi^2(y)\, d\, u^*(e_y)$$

exists, of the equation (7. 6), (7. 6 a) *gives rise to a solution*

$$\psi^*(e_y) = F^*(e_y) + \int_{e_y} \Phi(y)\, d\, u^*(e_y)$$

of the integral equation (7. 3 a).

Examining the converse situation, let $\psi^*(e_y)$ be a solution of (7. 3 a); offhand there is no assurance that $\psi^*(e_y)$ will be of the form (7. 7) where $\Phi^2(y)$ is integrable $\{u^*\}$ and $\Phi(y)$ satisfies (7. 6), (7. 6 a). In other words *there is no assurance that Lemma* 7. 1 *supplies all the solutions of* (7. 3 a) *from those* of (7. 6), (7. 6 a) (the latter problem being of form (7. 3)).

Hence it appears necessary to study (7. 3 a) directly.

In the meanwhile, a type of a converse to the Lemma 7. 1 is embodied in the following result.

Lemma 7. 2. *In the case* (T) *and under Hypothesis* 7. 4 *let* $\psi^*(e_y)$ *be a solution of* (7. 3 a) *such that* $\psi^*(e_y) - F^*(e_y)$ *is absolutely continuous* $\{u^*\}$ *and such that the function* $\Gamma(y)$ *from the resulting relation*

$$\psi^*(e_y) = F^*(e_y) + \int_{e_y} \Gamma(y)\, d\, u^*(e_y) \tag{7. 9}$$

($\Gamma(y)$ *integrable* $\{u^*\}$) *is such that*

$$\int_{(D_x)} L(x)\, |\, \Gamma(x)\, |\, d\, u^*(e_y) \tag{7. 9 a}$$

exists. Then $\Gamma(y)$ *will be a solution of* (7. 6), (7. 6 a) *almost everywhere* $\{u^*\}$.

In fact, substituting (7. 9) into (7. 3 a) we obtain

$$\int_{e_y} \Gamma(y)\, d\, u^*(e_y) = \lambda \int_{(D_x)} \left[\int_{e_y} L(x, y)\, d\, u^*(e_y) \right] d\, F^*(e_x) + \lambda \beta_1,$$

where

$$\beta_1 = \int_{(D_x)} \left[\int_{e_y} L(x, y)\, d\, u^*(e_y) \right] \Gamma(x)\, d\, u^*(e_x). \tag{7. 9 b}$$

In consequence of the statement with respect to α' of (7. 8) and by (7. 6 a)

$$(7.\ 9\ \text{c})\qquad \int_{e_y} \Gamma(y)\, d\, u^*(e_y) = \int_{e_y} \left[\lambda \int_{(D_x)} L(x, y)\, d\, F^*(e_x)\right] d\, u^*(e_y) + \lambda\, \beta_1$$
$$= \int_{e_y} q(y)\, d\, u^*(e_y) + \lambda\, \beta_1.$$

In view of the existence of (7. 9 a) the order of integration in β_1 may be changed; we have

$$|\beta_1| \leqq \int_{(D_x)} |\Gamma(x)|\, [u^*(e_y)]^{\frac{1}{2}} \left[\int_{e_y} L^2(x, y)\, d\, u^*(e_y)\right]^{\frac{1}{2}} d\, u^*(e_x)$$
$$\leqq [u^*(e_y)]^{\frac{1}{2}} \int_{(D_x)} L(x)\, |\Gamma(x)|\, d\, u^*(e_x).$$

Thus by virtue of (7. 9 c) it is inferred that the Lemma holds as stated. We note that *this result takes place even with the condition regarding* (7. 9 a) *dropped, provided that* $\Gamma(x)$ *is such that in* (7. 9 b) *the order of integration can be changed.*

8. **Direct Treatment of Problem (7. 3 a).**

Let O *denote the set of points in the complex* λ*-plane* consisting of all the points not on the axis of reals as well as of the points on the axis of reals not belonging to the closure of the set of points represented by the characteristic values

$$(8.\ 1)\qquad \lambda_{n, j}\quad (n = n_1, n_2, \ldots; j = 1, 2, \ldots; \lim_i n_i = \infty).$$

When λ is in O we have

$$(8.\ 1\ \text{a})\qquad |\lambda - \lambda_{n, j}| \geqq \delta(\lambda) > 0\quad (n = n_1, n_2, \ldots; j = 1, 2, \ldots),$$

where $\delta(\lambda)$ is independent of n_i and j.

Let $\psi_n^*(e_y)$ be a solution, for λ in O, of the problem (7. 1 a). By (G) formula (5. 17) is applicable yielding

$$(8.\ 2)\qquad \psi_n^*(e_y) = F^*(e_y) + \lambda \int_{(D_x)} \left(\int_{e_y} L_n(x, y)\, d\, u^*(e_y)\right) d\, F^*(e_x)$$
$$- \sum_{k=1}^{\infty} \frac{1}{(\lambda - \lambda_{n, k})\, \lambda_{n, k}}\, \lambda^2\, \sigma_{n, k}\, \psi_{n, k}^*(e_y),$$

where

$$\sigma_{n,k} = \int\limits_{(D_x)} \varphi_{n,k}(x)\, d\, F^*(e_x).$$

In fact, if $\psi_n^*(e_y)$ were not expressible by the second member of (8. 2), the difference between $\psi_n^*(e_y)$ and this member would be a function $\omega_n^*(e_y)$ such that

$$\omega_n^*(e_y) = \lambda \int\limits_{(D_x)} \left[\int\limits_{e_y} L_n(x,\, y)\, d\, u^*(e_y) \right] d\, \omega_n^*(e_x),$$

which contrary to (8. 1 a) would imply that λ is a characteristic value.

By virtue of a process involved in (7. 8 a) and by (8. 1 a) one has

$$(8.\ 3) \qquad |\psi_n^*(e_y)| \leqq v^*(e_y) + |\lambda| [u^*(e_y)]^{\frac{1}{2}} \int\limits_{(D_x)} L_{(e_y)}(x)\, d\, v^*(e_x) + \frac{1}{\delta(\lambda)} |\lambda|^2 s_n$$

where

$$s_n = \sum_k \left| \frac{\sigma_{n,k}}{\lambda_{n,k}} \psi_{n,k}^*(e_y) \right|.$$

Substituting

$$\varphi_{n,k}(x) = \lambda_{n,k} \int\limits_{(D_z)} L_n(x,\, z)\, \varphi_{n,k}(z)\, d\, u^*(e_x)$$

into $\sigma_{n,k}$ we obtain

$$\begin{aligned}(8.\ 3\ \mathrm{a}) \quad s_n &= \sum_k |\psi_{n,k}^*(e_y)| \left| \int\limits_{(D_x)} \left[\int\limits_{(D_z)} L_n(x,\, z)\, \varphi_{n,k}(z)\, d\, u^*(e_z) \right] d\, F^*(e_x) \right| \\ &= \sum_k |\psi_{n,k}^*(e_y)| \left| \int\limits_{(D_z)} \left[\int\limits_{(D_x)} L_n(x,\, z)\, d\, F^*(e_x) \right] \varphi_{n,k}(z)\, d\, u^*(e_x) \right| \\ &\leqq \left[\sum_k |\psi_{n,k}^*(e_y)^2| \right]^{\frac{1}{2}} \left[\sum_k \left| \int\limits_{(D_z)} \left[\int\limits_{(D_x)} L_n(x,\, z)\, d\, F^*(e_x) \right] \varphi_{n,k}(z)\, d\, u^*(e_z) \right|^2 \right]^{\frac{1}{2}}.\end{aligned}$$

Now

$$\psi_{n,k}^*(e_y) = \int\limits_{e_y} \varphi_{n,k}(y)\, d\, u^*(e_y) = \int\limits_{(D_y)} q(y)\, \varphi_{n,k}(y)\, d\, u^*(e_y),$$

where $q(y)$ is unity in e_y and zero in $(D_y) - e_y$; hence by Bessel's inequality

(8. 3 b) $$\sum_k |\psi^*_{n,k}(e_y)|^2 \leqq \int_{(D_y)} q^2(y)\, d\, u^*(e_y) = \int_{e_y} d\, u^*(e_y) = u^*(e_y).$$

On the other hand, the square of the second factor of the last member in (8. 3 a) is equal or is less than

$$\int_{(D_z)} \left[\int_{(D_x)} L_n(x, y)\, d\, F^*(e_x)\right]^2 d\, u^*(e_z) \leqq \int_{(D_z)} \left[\int_{(D_x)} |L_n(x, z)|\, d\, v^*(e_x)\right]^2 d\, u^*(e_z)$$

$$= \int_{(D_z)} \int_{(D_x)} \int_{(D_s)} |L_n(x, z)\, L_n(s, z)|\, d\, v^*(e_x)\, d\, v^*(e_s)\, d\, u^*(e_z)$$

$$= \int_{(D_x)} \int_{(D_s)} \left[\int_{(D_z)} |L_n(x, z)\, L_n(s, z)|\, d\, u^*(e_z)\right] d\, v^*(e_x)\, d\, v^*(e_s)$$

$$\leqq \int_{(D_x)} \int_{(D_s)} L_n(x)\, L_n(s)\, d\, v^*(e_x)\, d\, v^*(e_s) = \left[\int_{(D_x)} L(x)\, d\, v^*(e_x)\right]^2.$$

Whence in consequence of (8. 3 a) and (8. 3 b)

$$s_n \leqq [u^*(e_y)]^{\frac{1}{2}} \int_{(D_x)} L(x)\, d\, v^*(e_x)$$

and, finally, (8. 3) yields

(8. 4) $$|\psi^*_n(e_y)| \leqq v^*(e_y) + \beta(e_y) = v^*(e_y) + |\lambda|\, [u^*(e_y)]^{\frac{1}{2}} \cdot$$
$$\cdot \int_{(D_x)} L_{(e_y)}(x)\, d\, v^*(e_x) + \frac{|\lambda|^2}{\delta(\lambda)} [u^*(e_y)]^{\frac{1}{2}} \int_{(D_x)} L(x)\, d\, v^*(e_x).$$

If in (8. 2) the term $F^*(e_y)$ is transposed to the first member and, with this modification, the subsequent steps are repeated we obtain the following result.

Lemma 8. 1. *The approximating solutions $\psi^*_n(e_y)$ satisfy, for λ in the set O, the 'compactness' inequalities*

(8. 4 a) $$|\psi^*_n(e_y) - F^*(e_y)| \leqq \beta(e_y) \qquad (cf.\ (8.\ 4))$$

(for $n = n_1, n_2 \ldots$ and for all $\{B\}$-sets $e_y < (D_y)$), implying that the $\psi^*_n(e_y) - F^*(e_y)$ are absolutely continuous $\{u^*\}$, uniformly with respect to n.

The term 'compactness', here, is justified in view of the *de la V. Poussin-Frostman's* theorem, by virtue of which (8. 4 a) implies existence of an infinite subsequence $\{\psi^*_{m_j}(e_y)\}$ and of an additive set-function $\psi^*(e_y)$ such that

$$\psi^*(e_y) = \lim \psi^*_{m_j}(e_y) = F^*(e_y) + A^*(e_y), \tag{8. 5}$$

where

$$|A^*(e_y)| \leqq \beta(e_y) \tag{8. 5 a}$$

and, consequently, $A^*(e_y)$ is absolutely continuous $\{u^*\}$. Convergence to the limit in (8. 5) takes place on all $\{B\}$-sets on whose frontiers $A^*(e_y)$ vanishes — thus, on all $\{B\}$-sets on whose frontiers $u^*(e_y)$ vanishes. We also have

$$\psi^*_n(e_y) = F^*(e_y) + A^*_n(e_y), \quad |A^*_n(e_y)| \leqq \beta(e_y). \tag{8. 5 b}$$

Turning to the equation satisfied by $\psi^*_n(e_y)$,

$$\psi^*_n(e_y) = \lambda \int_{(D_x)} \left[\int_{e_y} L_n(x, y)\, d\,u^*(e_y)\right] d\,\psi^*_n(e_x) + F^*(e_y), \tag{8. 6}$$

in the limit (as $n = m_j \to \infty$) one obtains

$$\psi^*(e_y) = \lambda \lim_n \int_{(D_x)} \left[\int_{e_y} L_n(x, y)\, d\,u^*(e_y)\right] d\,\psi^*_n(e_x) + F^*(e_y), \tag{8. 6 a}$$

where $\psi^*(e_y)$ is the function (8. 5). It is of importance to find conditions under which

$$\lim_n \int_{(D_x)} \left[\int_{e_y} L_n(x, y)\, d\,u^*(e_y)\right] d\,\psi^*_n(e_x) = \int_{(D_x)} \left[\int_{e_y} L(x, y)\, d\,u^*(e_y)\right] d\,\psi^*(e_x), \tag{8. 6 b}$$

since the latter relation would imply that $\psi^*(e_x)$ of (8. 5) is a solution of the problem (7. 3 a). It is known (cf. De la V. Poussin's book (VP; p. 11)) that

$$\lim_n \int_{(D)} q_n(x)\, d\,\mu_n(e_x) = \int_{(D)} q(x)\, d\,\mu(e_x) \tag{8. 7}$$

when the $\mu_n(e)$ are additive functions of sets $\{B\}$, $\mu_n \leqq A$, $\mu_n \to \mu$, the μ_n and μ vanish on the frontier of (D), while continuous functions $q_n(x)$ converge uniformly to $q(x)$ (necessarily continuous). As remarked in (VP), continuity of $q(x)$ in (D) is essential. On writing

$$q_n(x) = \int_{e_y} L_n(x, y)\, d\, u^*(e_y), \quad q(x) = \int_{e_y} L(x, y)\, d\, u^*(e_y)$$

it is seen that the conditions of the above theorem do not hold, inasmuch as $q(x)$ is not necessarily continuous in the closed set (D_x).

We let

$$r_n(e_y, e_z) = \int_{e_y} \int_{e_z} L_n(z, y)\, d\, u^*(e_z)\, d\, \psi_n^*(e_y). \tag{8. 8}$$

Substituting the expression obtained from (8. 2)

$$\psi_n^*(e_y) = F^*(e_y) + \lambda\, \alpha_{n,1} - \lambda^2\, \alpha_{n,2},$$

where

$$\alpha_{n,1} = \int_{e_y} \left[\int_{(D_x)} L_n(x, y)\, d\, F^*(e_x) \right] d\, u^*(e_y),$$

$$\alpha_{n,2} = \int_{e_y} \left[\sum_k \frac{\sigma_{n,k}}{(\lambda - \lambda_{n,k})\, \lambda_{n,k}}\, \varphi_{n,k}(y) \right] d\, u^*(e_y),$$

it is inferred that

$$r_n(e_y, e_z) = r_{n,1} + \lambda\, r_{n,2} - \lambda^2\, r_{n,3} \tag{8. 8 a}$$

with

$$r_{n,1} = \int_{e_y} \int_{e_z} L_n(z, y)\, d\, u^*(e_z)\, d\, F^*(e_y),$$

$$r_{n,2} = \int_{e_y} \int_{e_z} \int_{(D_x)} L_n(z, y)\, L_n(x, y)\, d\, F^*(e_x)\, d\, u^*(e_z)\, d\, u^*(e_y)$$

and

$$r_{n,3} = \int_{e_y} \int_{e_z} L_n(z, y)\, d\, u^*(e_z) \left[\sum_k \frac{\sigma_{n,k}}{\lambda - \lambda_{n,k}}\, \frac{\varphi_{n,k}(y)}{\lambda_{n,k}} \right] d\, u^*(e_y).$$

Here

$$|r_{n,1}| \leqq \int_{e_y} \left| \int_{e_z} L_n(z, y)\, d\, u^*(e_z) \right| d\, v^*(e_y) \leqq [u^*(e_z)]^{\frac{1}{2}} \int_{e_y} L_{(e_z)}(y)\, d\, v^*(e_y) \tag{8. 8 b}$$

(notation of (7. 8 a)) and

$$|r_{n,2}| = \left| \int\limits_{(D_x)} \int\limits_{e_z} \left(\int\limits_{e_y} L_n(z, y) L_n(x, y)\, d\, u^*(e_y) \right) d\, u^*(e_z)\, d\, F^*(e_x) \right|$$

$$\leqq \int\limits_{(D_x)} |\cdots|\, d\, v^*(e_x) \leqq \int\limits_{(D_x)} \int\limits_{e_z} \left| \int\limits_{e_y} L_n(z, y) L_n(x, y)\, d\, u^*(e_y) \right| d\, u^*(e_z)\, d\, v^*(e_x)$$

Now

$$\left| \int\limits_{e_y} L_n(z, y) L_n(x, y)\, d\, u^*(e_y) \right| \leqq \left[\int\limits_{e_y} L_n^2(z, y)\, d\, u^*(e_y) \right]^{\frac{1}{2}} \left[\int\limits_{e_y} L_n^2(x, y)\, d\, u^*(e_y) \right]^{\frac{1}{2}}$$

$$\leqq \left[\int\limits_{e_y} L^2(z, y)\, d\, u^*(e_y) \right]^{\frac{1}{2}} \left[\int\limits_{e_y} L^2(x, y)\, d\, u^*(e_y) \right]^{\frac{1}{2}} = L_{(e_y)}(z)\, L_{(e_y)}(x).$$

Hence

$$(8.\ 8\,\mathrm{c}) \qquad |r_{n,2}| \leqq \int\limits_{(D_x)} \int\limits_{e_z} L_{(e_y)}(z)\, L_{(e_y)}(x)\, d\, u^*(e_z)\, d\, v^*(e_x)$$

$$= \left[\int\limits_{(D_x)} L_{(e_y)}(x)\, d\, v^*(e_y) \right] \left[\int\limits_{e_z} L_{(e_y)}(z)\, d\, u^*(e_z) \right].$$

Turning to $r_{n,3}$ one obtains

$$|r_{n,3}| = \left| \int\limits_{e_y} \int\limits_{e_z} L_n(z, y)\, d\, u^*(e_z) \left[\sum_k \frac{1}{\lambda - \lambda_{n,k}} \int\limits_{(D_x)} \frac{\varphi_{n,k}(x)}{\lambda_{n,k}}\, d\, F^*(e_x)\, \varphi_{n,k}(y) \right] d\, u^*(e_y) \right|$$

$$= \left| \int\limits_{(D_x)} \int\limits_{e_z} d\, F^*(e_x)\, d u^*(e_z) \left[\sum_k \frac{1}{\lambda - \lambda_{n,k}} \left(\frac{\varphi_{n,k}(x)}{\lambda_{n,k}} \right) \int\limits_{e_y} L_n(z, y)\, \varphi_{n,k}(y)\, d\, u^*(e_y) \right] \right|$$

$$\leqq \int\limits_{(D_x)} \int\limits_{e_z} d\, v^*(e_x)\, d\, u^*(e_z)\, |[\cdots]|.$$

For λ in O we have (8. 1 a) and, accordingly,

$$|r_{n,3}| \leqq \frac{1}{\delta(\lambda)} \int\limits_{(D_x)} \int\limits_{e_z} d\, v^*(e_x)\, d\, u^*(e_z) \left\{ \sum_k \left| \frac{\varphi_{n,k}(x)}{\lambda_{n,k}} \right| \left| \int\limits_{e_y} L_n(z, y)\, \varphi_{n,k}(y)\, d\, u^*(e_y) \right| \right\}$$

$$\leqq \frac{1}{\delta(\lambda)} \int\limits_{(D_k)} \int\limits_{e_z} d\, v^*(e_x)\, d\, u^*(e_z) \left[\sum_k \left| \frac{\varphi_{n,k}(x)}{\lambda_{n,k}} \right|^2 \right]^{\frac{1}{2}} \left[\sum_k \left| \int\limits_{e_y} L_n(z, y)\, \varphi_{n,k}(y)\, d\, u^*(e_y) \right|^2 \right]^{\frac{1}{2}}.$$

Now

$$\frac{\varphi_{n,k}(x)}{\lambda_{n,k}} = \int_{(D_v)} L_n(v, x)\,\varphi_{n,k}(v)\,d\,u^*(e_v)$$

and consequently by virtue of Bessel's inequality

$$\sum_k \left|\frac{\varphi_{n,k}(x)}{\lambda_{n,k}}\right|^2 \leqq \int_{(D_v)} L_n^2(v, x)\,d\,u^*(e_v) \leqq L^2(x).$$

Thus

$$|r_{n,3}| \leqq \frac{1}{\delta(\lambda)} \int_{(D_x)} \int_{e_z} d\,v^*(e_x)\,d\,u^*(e_z)\,L(x)\,L_{(e_y)}(z)$$

and, finally

$$|r_{n,3}| \leqq \frac{1}{\delta(\lambda)} \left[\int_{(D_x)} L(x)\,d\,v^*(e_x)\right]\left[\int_{e_z} L_{(e_y)}(z)\,d\,u^*(e_z)\right]. \tag{8. 8 d}$$

In view of (8. 8 a)—(8. 8 d) it is observed that the function of (8. 8) satisfies the inequality

$$\begin{aligned} |r_n(e_y, e_z)| \leqq \beta'(e_y, e_z) = [u^*(e_z)]^{\frac{1}{2}} \int_{e_y} L_{(e_z)}(y)\,d\,v^*(e_y) \\ + |\lambda| \left[\int_{(D_x)} L_{(e_y)}(x)\,d\,v^*(e_x)\right]\left[\int_{e_z} L_{(e_y)}(z)\,d\,u^*(e_z)\right] + \\ + \frac{|\lambda|^2}{\delta(\lambda)} \left[\int_{(D_x)} L(x)\,d\,v^*(e_x)\right]\left[\int_{e_z} L_{(e_y)}(z)\,d\,u^*(e_z)\right]. \end{aligned} \tag{8. 9}$$

Now

$$L_{(e_z)}^2(y) = \int_{e_z} L^2(z, y)\,d\,u^*(e_z) \leqq L^2(y).$$

Hence from (8. 9) we obtain the simpler inequality

$$\begin{aligned} |r_n(e_y, e_z)| \leqq \beta(e_y, e_z) = [u^*(e_z)]^{\frac{1}{2}} \int_{e_y} L(y)\,d\,v^*(e_y) \\ + \left[|\lambda| + \frac{|\lambda|^2}{\delta(\lambda)}\right] \alpha' \int_{e_z} L_{(e_y)}(z)\,d\,u^*(e_z), \quad \alpha' = \int_{(D_x)} L(x)\,d\,v^*(e_x). \end{aligned} \tag{8. 9 a}$$

By virtue of (8. 5 b) and of the relation subsequent to (8. 8)

$$A_n^*(e_y) = \lambda\,\alpha_{n,1} - \lambda^2\,\alpha_{n,2}.$$

Hence on writing

$$\varrho_n(e_y, e_z) = \int_{e_y}\int_{e_z} L_n(z, y)\,d\,u^*(e_z)\,d\,A_n^*(e_y) \tag{8. 10}$$

and repeating the steps subsequent to (8. 8), we now obtain the same result as before, but with $r_{n,1}$ replaced by zero; thus

$$|\varrho_n(e_y, e_z)| \leq \left[|\lambda| + \frac{|\lambda|^2}{\delta(\lambda)}\right]\alpha' \int_{e_z} L_{(e_y)}(z)\,d\,u^*(e_z). \tag{8. 10 a}$$

By (8. 5 b), (8. 6) and (8. 10)

$$\begin{aligned} A_n^*(e_y) &= \lambda \int_{(D_x)}\left[\int_{e_y} L_n(x,\ y)\,d\,u^*(e_y)\right] d\,F^*(e_x) \\ &+ \lambda \int_{(D_x)}\left[\int_{e_y} L_n(x,\ y)\,d\,u^*(e_y)\right] d\,A_n^*(e_x) = \lambda\,r'_{n,1}(e_y) + \lambda\,\varrho_n(D_x, e_y). \end{aligned} \tag{8. 11}$$

Now

$$\int_{e_y} L_n(x, y)\,d\,u^*(e_y) \to \int_{e_y} L(x, y)\,d\,u^*(e_y)$$

and

$$\left|\int_{e_y} L_n(x,\ y)\,d\,u^*(e_y)\right| \leqq [u^*(e_y)]^{\frac{1}{2}}\left[\int_{e_y} L^2(x,\ y)\,d\,u^*(e_y)\right]^{\frac{1}{2}} \leqq [u^*(e_y)]^{\frac{1}{2}} L(x),$$

the last member here being integrable $\{v^*\}$; consequently

$$\lim_n r'_{n,1}(e_y) = \int_{(D_x)}\int_{e_y} L(x, y)\,d\,u^*(e_y)\,d\,F^*(e_x).$$

On taking account of (8. 5) and letting $n = m_j$ in (8. 11) $\to \infty$, we accordingly derive

$$A^*(e_y) - \lambda \int_{(D_x)}\int_{e_y} L(x, y)\,d\,u^*(e_y)\,d\,F^*(e_x) = \lim_n \lambda\,\varrho_n(D_x, e_y) \tag{8. 11 a}$$

on every $\{B\}$-set on whose frontier u^* vanishes. We thus know that the limit in the second member above exists — it remains to find its form.

Let e_y be a fixed $\{B\}$-set whose closure lies in $(D_y)^0$ and on whose frontier u^* vanishes. Let ω_x be a closed domain in $(D_x)^0$ on whose frontier u^* also vanishes. By (8. 11)

$$(8.\ 12)\qquad A_n^*(e_y) - \lambda \int\limits_{(D_x)} \Big[\int\limits_{e_y} L_n(x, y)\, d\, u^*(e_y)\Big]\, d\, F^*(e_x)$$

$$- \lambda \int\limits_{\omega_x} \Big[\int\limits_{e_y} L_n(x, y)\, d\, u^*(e_y)\Big]\, d\, A_n^*(e_x) = \lambda \int\limits_{(D_x)-\omega_x} \Big[\int\limits_{e_y} L_n(x, y)\, d\, u^*(e_y)\Big]\, d\, A_n^*(e_x)$$

and, in the limit (as $n = m_j \to \infty$)

$$(8.\ 12\,\mathrm{a})\qquad A^*(e_y) - \lambda \int\limits_{(D_x)} \int\limits_{e_y} L(x, y)\, d\, u^*(e_y)\, d\, F^*(e_x)$$

$$- \lambda \int\limits_{\omega_x} \Big[\int\limits_{e_y} L(x, y)\, d\, u^*(e_y)\Big]\, d\, A^*(e_x) = \lim_n \lambda\, \varrho_n((D_x) - \omega_x, e_y)$$

(cf. (8. 10)), provided

$$(8.\ 13)\qquad \lim_n \int\limits_{\omega_x} \Big[\int\limits_{e_y} L_n(x, y)\, d\, u^*(e_y)\Big]\, d\, A_n^*(e_x) = \int\limits_{\omega_x} \Big[\int\limits_{e_y} L(x, y)\, d\, u^*(e_y)\Big]\, d\, A^*(e_x).$$

To establish (8. 13) we note that, depending on ω_x and e_y, there exists a number n' so that

$$(8.\ 13\,\mathrm{a})\qquad L_n(x, y) = L_{n'}(x, y) = L(x, y)$$

for all $n \geqq n'$ when x is in ω_x and y is in e_y. Hence (8. 13) will hold if

$$(8.\ 13\,\mathrm{b})\qquad \lim_n \int\limits_{\omega_x} \Big[\int\limits_{e_y} L(x, y)\, d\, u^*(e_y)\Big]\, d\, A_n^*(e_x) = \int\limits_{\omega_x} \Big[\int\limits_{e_y} L(x, y)\, d\, u^*(e_y)\Big]\, d\, A^*(e_x).$$

By (8. 13 a) $|L(x, y)| \leqq n'$ for x in ω_x and y in e_y; moreover,

$$\int\limits_{e_y} L(x, y)\, d\, u^*(e_y)$$

is continuous in x for x in the closed domain ω_x. On the other hand,

$$\lim_n A_n^* = A^*, \quad |A_n^*(e)| \leqq \beta(e) \quad (n = m_1, m_2, \ldots; \text{ cf. } (8.4));$$

the set-functions A_n^*, A^* being absolutely continuous $\{u^*\}$, these functions vanish on the frontier of ω_x, together with u^*. Accordingly, (8, 13 b) is seen to hold in view of the satisfied conditions of Theorem (8. 7). This establishes (8. 12 a).

By virtue of (8. 10 a)

$$|\varrho_n((D_x) - \omega_x, e_y)| \leqq \left[|\lambda| + \frac{|\lambda|^2}{\delta(\lambda)}\right] \alpha' \int_{e_y} L_{((D_x)-\omega_x)}(y)\, d\, u^*(e_y).$$

Letting $n = m_j \to \infty$ and taking account of (8. 12 a) one obtains

$$(8.\ 14) \qquad \left| A^*(e_y) - \lambda \int_{(D_x)} \int_{e_y} L(x, y)\, d\, u^*(e_y)\, d\, F^*(e_x) \right.$$

$$\left. - \lambda \int_{\omega_y} \left[\int_{e_y} L(x, y)\, d\, u^*(e_y) \right] d\, A^*(e_x) \right| = |\lambda|\, |\lim \varrho_n((D_x) - \omega_x, e_y)|$$

$$\leqq \left[|\lambda|^2 + \frac{|\lambda|^2}{\delta(\lambda)}\right] \alpha' \int_{e_y} L_{((D_x)-\omega_x)}(y)\, d\, u^*(e_y)$$

$$\leqq \left[|\lambda|^2 + \frac{|\lambda|^3}{\delta(\lambda)}\right] \alpha' \int_{(D_y)} \left[\int_{(D_x-\omega_x)} L^2(x, y)\, d\, u^*(e_x) \right]^{\frac{1}{2}} d\, u^*(e_y).$$

A sequence of closed domains $\omega_{x,j}$ $(j = 1, 2, \ldots)$ can be always found so that $\omega_{x,j} < (D_x)^0$,

$$(8.\ 15) \qquad \omega_{x,1} < \omega_{x,2} < \cdots$$

and

$$(8.\ 15\text{ a}) \qquad \lim_j \omega_{x,j} = (D_x)^0.$$

Definition 8. 1. *The set-function $u^*(e_x)$ will be said to be regular with respect to the frontier of (D_x) if for some sequence of domains $\omega_{x,j} < (D_x)^0$, satisfying* (8. 15), (8. 15 a), *u^* vanishes on the frontier of each $\omega_{x,j}$ and if u^* vanishes on the frontier of (D_x) as well.*

Assuming u^* regular, in accordance with the above, on letting in (8. 14 a) $\omega_x = \omega_{x,j} \to (D_x)^0$, we obtain

$$A^*(e_y) - \lambda \int\limits_{(D_x)} \int\limits_{e_y} L(x, y)\, d\, u^*(e_y)\, d\, F^*(e_x) - \lambda \int\limits_{(D_x)^0} \left[\int\limits_{e_y} L(x, y)\, d\, u^*(e_y) \right] d\, A^*(e_x) = 0 .$$

Here $(D_x)^0$ may be replaced by (D_x) since A^*, as established before, is zero on the frontier of (D_x), if u^* is. Whence the function ψ^* of (8. 5) satisfies (7. 3 a).

We have the following Existence Theorem.

Theorem 8. 1. *We consider kernels of form* (T) (*Definition* 5. 1) *and assume Hypothesis* 7. 4. *Let, moreover,* u^* *be regular with respect to the frontier of* (D_x) (*Definition* 8. 1). *Let* λ *be in the set* O, *introduced at the beginning of this section. The additive function*

$$\psi^*(e_y) = F^*(e_y) + A^*(e_y),$$

obtained by the limiting process of (8. 5), *is a solution of the equation*

$$\psi^*(e_y) = \lambda \int\limits_{(D_x)} \left[\int\limits_{e_y} L(x, y)\, d\, u^*(e_y) \right] d\, \psi^*(e_y) + F^*(e_y) \tag{7. 3 a}$$

for every $\{B\}$*-set* e_y, *whose closure is in* $(D_x)^0$ *and on whose frontier* u^* *vanishes. Furthermore,* $|A^*(e_y)| \leqq \beta(e_y)$ (*see* (8. 4)).

Inasmuch as

$$\lim_n \int\limits_{e_y} L_n(x, y)\, d\, u^*(e_y) = \int\limits_{e_y} L(x, y)\, d\, u^*(e_y), \qquad \left| \int\limits_{e_y} L_n(x, y)\, d\, u^*(e_y) \right| \leqq [u^*(e_y)]^{\frac{1}{2}} L(x),$$

while the second integral (7. 4 a) exists, one has

$$\lim_n \int\limits_{(D_x)} \left[\int\limits_{e_y} L_n(x, y)\, d\, u^*(e_y) \right] d\, F^*(e_x) = \int\limits_{(D_x)} \left[\int\limits_{e_y} L(x, y)\, d\, u^*(e_y) \right] d\, F^*(e_x). \tag{8. 15¹}$$

Thus by (8. 2)

$$\lim_n \psi^*_{n_j}(e_y) = F^*(e_y) + \lambda \int\limits_{(D_x)} \left[\int\limits_{e_y} L(x, y)\, d\, u^*(e_y) \right] d\, F^*(e_x) - \lambda^2 B^*(e_y) \tag{8. 15¹ a}$$

where

$$B^*(e_y) = \lim_n B^*_n(e_y), \qquad B^*_n(e_y) = \sum_{k=1}^{\infty} \frac{\sigma_{n,k}}{(\lambda - \lambda_{n,k})\,\lambda_{n,k}}\, \psi^*_{n,k}(e_y). \tag{8. 15¹ b}$$

The limit in (8. 1 b) (and hence in (8. 15^1 a)) will certainly exist when $n = m_j \to \infty$, where the m_j are from (8. 5) and when the sets considered are those on whose frontiers u^* vanishes. However, quite apart from these considerations, (8. 15^1 a) will hold (for $n = n_j \to \infty$) whenever (8. 15^1 b) holds, as $n = n_j \to \infty$.

By the relation subsequent to (8. 2) and with (6. 2), (6. 2 a) in view, we may rewrite $B_n^*(e_y)$ as follows

$$B_n^*(e_y) = \sum_k \frac{1}{\lambda - \lambda_{n,k}} \int\limits_{(D_x)} \frac{\varphi_{n,k}(x)}{\lambda_{n,k}} \psi_{n,k}^*(e_y)\, d\, F^*(e_x).$$

Now

$$\frac{\varphi_{n,k}(x)}{\lambda_{n,k}} = \int\limits_{(D_z)} L_n(x, z)\, \varphi_{n,k}(z)\, d\, u^*(e_z)$$

and hence

$$B_n^*(e_y) = \sum_k \frac{1}{\lambda - \lambda_{n,k}} \int\limits_{(D_x)} \int\limits_{(D_z)} L_n(x, z)\, \varphi_{n,k}(z)\, \psi_{n,k}^*(e_y)\, d\, u^*(e_z)\, d\, F^*(e_x).$$

Designating by $\theta_n^*(x, e_y/\lambda)$ the additive function of sets $\{B\}$ which on figures (τ) coincides with $\tau\theta_n(x, \tau'\lambda)$ of (6. 3), one deduces that

(8. 16)
$$\theta_n^*(x, e_y/\lambda) = \sum_{0<\lambda_{n,k}<\lambda} \varphi_{n,k}(x)\, \psi_{n,k}^*(e_y) \qquad (\text{for } \lambda > 0),$$
$$\theta_n^*(x, e_y/\lambda) = -\sum_{\lambda\leq\lambda_{n,k}<0} \varphi_{n,k}(x)\, \psi_{n,k}^*(e_y) \qquad (\text{for } \lambda < 0)$$

and $\theta_n^*(x, e_y/0) = 0$. Accordingly, rewriting $B_n^*(e_y)$ in terms of a Stieltjes integral we obtain

(8. 17)
$$B_n^*(e_y) = \int\limits_{-\infty}^{\infty} \frac{1}{\lambda - \mu}\, d_\mu\, \sigma_n(e_y/\mu),$$

$$\sigma_n(e_y/\mu) = \int\limits_{(D_x)} \int\limits_{(D_z)} L_n(x, z)\, \theta_n^*(z, e_y/\mu)\, d\, u^*(e_z)\, d\, F^*(e_x)$$

when λ is distinct from the $\lambda_{n,j}$. Finally we put (8. 17) in the form

(8. 17 a)
$$B_n^*(e_y) = B_{n,1}(e_y) + B_{n,2}(e_y),$$

$$B_{n,1}(e_y) = \int_{-\infty}^{\infty} \frac{1}{\lambda - \mu} d_\mu \sigma'_n(e_y/\mu), \quad B_{n,2}(e_y) = \int_{-\infty}^{\infty} \frac{1}{\lambda - \mu} d_\mu \sigma''_n(e_y/\mu),$$

(8. 17 b) $$\sigma'_n(e_y/\mu) = \int_{(D_x)} \int_{(D_z)} L(x, z)\, \theta^*_n(z, e_y/\mu)\, d\, u^*(e_z)\, d\, F^*(e_x),$$

(8. 17 c) $$\sigma''_n(e_y/\mu) = \int_{(D_x)} \int_{(D_z)} [L_n(x, z) - L(x, z)]\, \theta^*_n(z, e_y/\mu)\, d\, u^*(e_z)\, d\, F^*(e_x).$$

The following Lemma will be helpful.

Lemma 8. 2. *Let* v^* *be from* (7. 2 a). *Suppose that* $H(x, z)$ *is such that the integrals*

(8. 18) $$H^2(x) = \int_{(D_z)} H^2(x, z)\, d\, u^*(e_z), \quad \int_{(D_x)} H(x)\, d\, v^*(e_x)$$

exist. On writing

(8. 18 a) $$i_n(e_y/\mu) = \int_{(D_x)} \int_{(D_z)} \theta^*_n(z, e_y/\mu)\, H(x, z)\, d\, u^*(e_z)\, d\, F^*(e_x)$$

we shall have

(8. 18 b) $$|\, i_n(e_y/\mu)\,|, \ V_{\lambda_1}^{\lambda_2} i_n(e_y/\mu) \leqq [u^*(e_y)]^{\frac{1}{2}} \left[\int_{(D_x)} H(x)\, d\, v^*(e_x) \right],$$

where $V_{\lambda_1}^{\lambda_2}$ *refers to variation, in* μ, *on any finite interval* (λ_1, λ_2).

Let the l_j be such that

$$\lambda_1 \leqq l_0 < l_1 < \cdots < l_m \leqq \lambda_2.$$

We form the sum

$$V_{n,m} = \sum_{\nu=1}^{m} \left| \int_{(D_x)} \int_{(D_z)} [\theta^*_n(z, e_y/l_\nu) - \theta^*_n(z, e_y/l_{\nu-1})]\, H(x, z)\, d\, u^*(e_z)\, d\, F^*(e_x) \right|$$

$$= \sum_{\nu=1}^{m} \left| \int_{(D_x)} \int_{(D_z)} \left(\sum_k^{(\nu)} \varphi_{n,k}(z)\, \psi^*_{n,k}(e_y) \right) H(x, z)\, d\, u^*(e_z)\, d\, F^*(e_x) \right|;$$

here the summation symbol with a superscript is over values k such that $l_{\nu-1} \leqq \lambda_{n,k} < l_\nu$. One has

$$V_{n,m} = \sum_{\nu=1}^{m} \left| \sum_k^{(\nu)} \psi_{n,k}^*(e_y) \int_{(D_x)} \left[\int_{(D_z)} H(x, z)\, \varphi_{n,k}(z)\, d\, u^*(e_z) \right] d\, F^*(e_x) \right.$$

$$\leqq \sum_k |\psi_{n,k}^*(e_y)| \left| \int_{(D_z)} \left[\int_{(D_x)} H(x, z)\, d\, F^*(e_x) \right] \varphi_{n,k}(z)\, d\, u^*(e_z) \right|$$

$$\leqq \left[\sum_k |\psi_{n,k}^*(e_y)|^2 \right]^{\frac{1}{2}} \left[\sum_k \left| \int_{(D_z)} \left[\int_{(D_x)} H(x, z)\, d\, F^*(e_x) \right] \varphi_{n,k}(z)\, d\, u^*(e_z) \right|^2 \right]^{\frac{1}{2}}.$$

By (8. 3 b) and in view of Bessel's inequality

$$V_{n,m} \leqq [u^*(e_y)]^{\frac{1}{2}} \left[\int_{(D_z)} \left[\int_{(D_x)} |H(x, z)|\, d\, v^*(e_x) \right]^2 d\, u^*(e_z) \right]^{\frac{1}{2}}.$$

For the double integral last displayed we have

$$\int_{(D_z)} \int_{(D_x)} \cdots = \int_{(D_z)} \int_{(D_x)} \int_{(D_s)} |H(x, z)\, H(s, z)|\, d\, v^*(e_x)\, d\, v^*(e_s)\, d\, u^*(e_z)$$

$$= \int_{(D_x)} \int_{(D_s)} \left[\int_{(D_z)} |H(x, z)\, H(s, z)|\, du^*(e_z) \right] d\, v^*(e_x)\, d\, v^*(e_z)$$

$$\leqq \int_{(D_x)} \int_{(D_s)} \left[\int_{(D_z)} H^2(x, z)\, d\, u^*(e_z) \right]^{\frac{1}{2}} \left[\int_{(D_z)} H^2(s, z)\, d\, u^*(e_z) \right]^{\frac{1}{2}} d\, v^*(e_x)\, d\, v^*(e_s)$$

$$= \int_{(D_x)} \int_{(D_s)} H(x)\, H(s)\, d\, v^*(e_x)\, d\, v^*(e_s).$$

Thus

$$V_{n,m} \leqq [u^*(e_y)]^{\frac{1}{2}} \left[\int_{(D_x)} H(x)\, d\, v^*(e_x) \right].$$

This inequality, together with the fact that $\theta^*(z, e_y/0) = 0$, implies (8. 18 b), which establishes the Lemma.

In view of this Lemma from (8. 17 c) we obtain

(8. 19) $$V_{\lambda_1}^{\lambda_2} \sigma_n''(e_y/\mu) \leqq [u^*(e_y)]^{\frac{1}{2}}\, \sigma_n'',$$

$$\sigma_n'' = \int_{(D_x)} \left[\int_{(D_z)} (L_n(x, z) - L(x, z))^2\, d\, u^*(e_z) \right]^{\frac{1}{2}} d\, v^*(e_x).$$

Now

$$\lim_n \int_{(D_z)} [L_n(x, z) - L(x, z)]^2 \, d\, u^*(e_z) = 0$$

because $|L_n(x, z) - L(x, z)|^2 \leqq 4 |L^2(x, z)|$, where $4|L^2(x, z)|$ is integrable, in z, $\{B, u^*\}$; moreover,

$$\left[\int_{(D_z)} [L_n(x, z) - L(x, z)]^2 \, d\, u^*(e_z)\right]^{\frac{1}{2}} \leqq 2\, L(x)$$

where the last member is integrable $\{B, v^*\}$. Hence

(8. 19 a) $$\lim_n \sigma_n'' = 0.$$

Let λ be real in O; then (8. 1 a) holds and, in consequence of (8. 16)

$$d_\mu \, \sigma_n''(e_y/\mu) = 0 \qquad \left(\lambda - \frac{\delta}{2} \leqq \mu \leqq \lambda + \frac{\delta}{2}\right).$$

Take l sufficiently great so that $-l < \lambda - \frac{\delta}{2} < \lambda + \frac{\delta}{2} < l$. Then

$$\int_{-l}^{l} \frac{1}{\lambda - \mu} d_\mu \, \sigma_n''(e_y/\mu) = \left(\int_{-l}^{\lambda - \frac{\delta}{2}} + \int_{\lambda + \frac{\delta}{2}}^{l}\right) \cdots$$

In the path of integration displayed in the last member $(\lambda - \mu)^{-1}$ is continuous as a function of μ and

$$\left|\frac{1}{\lambda - \mu}\right| \leqq \frac{2}{\delta}.$$

Thus by (8. 19)

$$\left|\int_{-l}^{l} \frac{1}{\lambda - \mu} d_\mu \, \sigma_n''(e_y/\mu)\right| \leqq \frac{2}{\delta(\lambda)} [u^*(e_y)]^{\frac{1}{2}} \sigma_n''$$

and, accordingly, on letting $l \to +\infty$ we obtain

$$|B_{n,2}(e_y)| \leqq \frac{2}{\delta(\lambda)} [u^*(e_y)]^{\frac{1}{2}} \sigma_n''.$$

In view of (8. 19 a) this implies that

(8. 20) $$\lim_n B_{n,2}(e_y) = 0;$$

the same result is obtained for λ non real.

Lemma 8. 3. *If $q(x, z)$ is such that the integrals*

$$q^2(x) = \int_{(D_z)} q^2(x, z)\, d\,u^*(e_z), \quad \int_{(D_x)} q(x)\, d\,v^*(e_x)$$

exist, then

$$\lim_n \int_{(D_x)} \left[\int_{(D_z)} \theta_n^*(z, e_y/\mu)\, q(x, z)\, d\,u^*(e_z)\right] d\,F^*(e_x)$$
$$= \int_{(D_x)} \left[\int_{(D_z)} \theta^*(z, e_y/\mu)\, q(x, z)\, d\,u^*(e_z)\right] d\,F^*(e_x)$$

as $n = n_j \to \infty$, the n_j being from (6. 14).

On writing

$$\theta_n^{**}(e_x, e_y/\mu) = \int_{e_x} \theta_n^*(x, e_y/\mu)\, d\,u^*(e_x),$$

by (8. 3 b) it is inferred that

$$|\theta_n^{**}(e_x, e_y/\mu)|^2 = \left|\sum_k{}' \psi_{n,k}^*(e_x)\, \psi_{n,k}^*(e_y)\right|^2 \leqq \sum_k{}' |\psi_{n,k}^*(e_x)|^2 \sum_k{}' |\psi_{n,k}^*(e_y)|^2 \leqq u^*(e_x)\, u^*(e_y).$$

On the other hand, (6. 4) gives us

$$\int_{(D_z)} \theta_n^*(z, e_x/\mu)\, \theta_n^*(z, e_y/\mu)\, d\,u^*(e_z) = \pm \int_{e_x}\int_{e_y} \theta_n(x, y/\mu)\, d\,u^*(e_x)\, d\,u^*(e_y) = \pm\, \theta_n^{**}(e_x, e_y/\mu)$$

and, in particular,

(8. 21) $$\int_{(D_z)} \theta_n^{*2}(z, e/\mu)\, d\,u^*(e_z) = |\theta_n^{**}(e, e/\mu)| \leqq u^*(e)$$

for $\{B\}$-sets e.

We have

(8. 21 a) $$\lim \theta_{n_j}^*(z, e_s/\mu) = \theta^*(z, e_s/\mu) = \int_{e_s} \theta(z, s/\mu)\, d\,u^*(e_s)$$

inasmuch as

$$\theta_n^*(z, e_s/\mu) = \int_{e_s} \theta_n(z, s/\mu)\, d\, u^*(e_s),$$

while (6. 14) holds and $|\theta_n| \leqq |\mu|^2 L(z) L(s)$ (see (6. 11 a)), where the last member is integrable, in s, $\{B, u^*\}$.

By Theorem 4. 3 the relations (8. 21). (8. 21 a) imply that

$$g_n(x) = \int_{(D_z)} \theta_n^*(z, e_y/\mu)\, q(x, z)\, d\, u^*(e_z) \to \int_{(D_z)} \theta^*(z, e_y/\mu)\, q(x, z)\, d\, u^*(e_z) = g(x)$$

as $n = n_j \to \infty$; moreover, by virtue of (8. 21)

$$|g_n(x)| \leqq q(x) \left[\int_{(D_z)} \theta_n^{*2}(z, e/\mu)\, d\, u^*(e_z)\right]^{\frac{1}{2}} \leqq q(x)[u^*(e)]^{\frac{1}{2}},$$

where the last member is integrable (in x) $\{B, v^*\}$, by hypothesis. Whence the conclusion of the Lemma follows at once.

Under Hypothesis 7. 4 one may take $q(x, z) = L(x, z)$, obtaining from this Lemma the following result for the functions (8. 17 b):

$$(8.\ 22) \qquad \lim \sigma'_{n_j}(e_y/\mu) = \sigma'(e_y/\mu) = \int_{(D_x)} \int_{(D_z)} L(x, z)\, \theta^*(z, e_y/\mu)\, d\, u^*(e_z)\, d\, F^*(e_x)$$

for all $\{B\}$-sets $e_y < (D_y)$. Furthermore, by Lemma 8. 2 (with $H(x, z) = L(x, z)$)

$$(8.\ 22\ \text{a}) \qquad V_{-l}^{l}\, \sigma'_n(e_y/\mu) \leqq [u^*(e_y)]^{\frac{1}{2}} \int_{(D_x)} L(x)\, d\, v^*(e_x) = a_0.$$

Let λ be real in O; then, for l sufficiently great,

$$\int_{-l}^{l} \frac{1}{\lambda - \mu}\, d_\mu\, \sigma'_n(e_y/\mu) = \left(\int_{-l}^{\lambda - \frac{\delta}{2}} + \int_{\lambda + \frac{\delta}{2}}^{l}\right) \ldots,$$

inasmuch as $d_\mu\, \sigma'_n(e_y/\mu) = 0$ in $\left(\lambda - \dfrac{\delta}{2},\ \lambda + \dfrac{\delta}{2}\right)$. By (8. 22), (8. 22 a) Helly's theorem is applicable yielding

$$(8.\ 23) \qquad \lim_{n_j} \int_{-l}^{l} \frac{1}{\lambda - \mu} d_\mu \sigma_n'(e_y/\mu) = \int_{-l}^{l} \frac{1}{\lambda - \mu} d_\mu \sigma'(e_y/\mu);$$

the same will hold, for similar reasons, for λ non real.

Consider now, for λ in O, the functions

$$r_{n,\,l} = \int_{l}^{+\infty} \frac{1}{\lambda - \mu} d_\mu \sigma_n'(e_y/\mu), \quad r_{n,\,-l} = \int_{-\infty}^{-l} \frac{1}{\lambda - \mu} d_\mu \sigma_n'(e_y/\mu),$$

where we take l so that

$$-\,l < \Re\lambda < l.$$

Then $l \pm \Re\lambda > 0$ and, by (8. 22 a),

$$|r_{n,\,l}| \leqq \frac{a_0}{l - \Re\lambda}, \quad |r_{n,\,-l}| \leqq \frac{a_0}{l + \Re\lambda}.$$

Thus, given ε (> 0) we can choose $l = l_\varepsilon$ so that

$$\left| \left(\int_{-\infty}^{-l} + \int_{l}^{+\infty} \right) \frac{1}{\lambda - \mu} d_\mu \sigma_n'(e_y/\mu) \right| < \varepsilon$$

for all n. Together with (8. 23) this implies that

$$\lim_{n_j} \int_{-\infty}^{\infty} \frac{1}{\lambda - \mu} d_\mu \sigma_n'(e_y/\mu) = \int_{-\infty}^{\infty} \frac{1}{\lambda - \mu} d_\mu \sigma'(e_y/\mu).$$

Accordingly, on taking account of (8. 17 a), (8. 20) we obtain the relation

$$\lim B_{n_j}^*(e_y) = \int_{-\infty}^{\infty} \frac{1}{\lambda - \mu} d_\mu \sigma'(e_y/\mu)$$

for all $\{B\}$-sets $e_y < (D_y)$; in consequence of (8. 15 a) and (8. 15 b) the following result is established.

Theorem 8. 2. *Let λ be in the set O (see beginning of this section). Every solution, referred to in Theorem 8. 1, has a spectral representation*

$$(8.\ 24)\quad \psi^*(e_y) = F^*(e_y) + \lambda \int\limits_{(D_x)} \left[\int\limits_{e_y} L(x, y)\, d\, u^*(e_y) \right] d\, F^*(e_x)$$
$$- \lambda^2 \int\limits_{-\infty}^{\infty} \frac{1}{\lambda - \mu}\, d_\mu \int\limits_{(D_x)} \int\limits_{(D_z)} L(x, z)\, \theta^*(z, e_y/\mu)\, d\, u^*(e_z)\, d\, F^*(e_x).$$

The function $\psi^*(e_y)$ *will satisfy the integral equation* (7. 3 a) *for every* $\{B\}$*-set* e_y, *whose closure is in* $(D_y)^0$, *even if* u^* *does not vanish on the frontier of* e_y.

The truth of the statement subsequent to (8. 24) follows from the fact that for the solution ψ^*, referred to in the Theorem, validity of the limiting relation (8. 5) (with $m_j = n_j$) can be asserted not only for $\{B\}$-sets e_y on whose frontiers u^* vanishes — as has been done previously — but, more generally, for sets e_y on whose frontiers u^* is not required to vanish. With this in mind we repeat all the developments from (8. 5) up to Theorem 8. 1, arriving at the result as stated in Theorem 8. 2.

9. The non Homogeneous Problem (7. 3).

Let $f(x)$ be any function as described in connection with (5. 3), (5. 3 a). The $c_{n,k}$ of (5. 16) are expressible in the form

$$(9.\ 1)\qquad c_{n,k} = \int\limits_{(D_y)} f_n(y)\, \varphi_{n,k}(y)\, d\, u^*(e_y).$$

Definition 9. 1. *Let* T *be the set of points in the complex* λ*-plane such that*

$$(9.\ 1\ \text{a})\qquad \sum_k \left| \frac{c_{n,k}}{\lambda - \lambda_{n,k}} \right|^2 \leqq A^2(\lambda) < +\infty \qquad (n = n_1, n_2, \ldots)$$

where $\lim n_j = \infty$ *and* $A(\lambda)$ *is independent of* n.

In the set O by virtue of (8. 1 a) one has

$$(9.\ 2)\quad \sum_k \left| \frac{c_{n,k}}{\lambda - \lambda_{n,k}} \right|^2 \leqq \frac{1}{\delta^2(\lambda)} \sum_k |c_{n,k}|^2 \leqq \frac{1}{\delta^2(\lambda)} \int\limits_{(D_y)} |f_n(y)|^2\, d\, u^*(e_y)$$
$$\leqq \frac{1}{\delta^2(\lambda)} \int\limits_{(D_y)} |f(y)|^2\, d\, u^*(e_y).$$

Hence *the set* T *contains the set* O; T may possibly depend on f, while O is independent of f. Both sets contain all non real λ.

With λ in T let $\varphi_n(x)$ be a solution, such that $\varphi_n^2(x)$ is integrable $\{B, u^*\}$, of the approximating non homogeneous problem (7. 1); thus

$$\varphi_n(x) = \lambda \int\limits_{(D_y)} L_n(x, y)\, \varphi_n(y)\, d\, u^*(e_y) + f_n(x). \tag{9. 3}$$

Since λ is taken distinct from the $\lambda_{n,k}$ it follows that $\varphi_n(x)$ is essentially unique and is expressible in the form (5. 16). Multiplying the two members of the letter relation by $\bar{\varphi}_n(x)\, d\, u^*(e_x)$ and integrating we obtain

$$\int\limits_{(D_x)} |\varphi_n(x)|^2\, d\, u^*(e_x) = \int\limits_{(D_x)} f_n(x)\, \bar{\varphi}_n(x)\, d\, u^*(e_x) - \lambda \sum_{k=1}^{\infty} \frac{c_{n,k}}{\lambda - \lambda_{n,k}} \int\limits_{(D_x)} \bar{\varphi}_n(x)\, \varphi_{n,k}(x)\, d\, u(e_x)$$

in consequence of a permissible interchange of order of summation and integration. Hence

$$\int\limits_{(D_x)} |\varphi_n(x)|^2\, d\, u^*(e_x) \leqq \left[\int\limits_{(D_x)} |f_n(x)|^2\, d\, u^*(e_x)\right]^{\frac{1}{2}} \left[\int\limits_{(D_x)} |\varphi_n(x)|^2\, d\, u^*(e_x)\right]^{1}$$
$$+ |\lambda| \left[\sum_k \left|\frac{c_{n,k}}{\lambda - \lambda_{n,k}}\right|^2\right]^{\frac{1}{2}} \left[\sum_k \left|\int\limits_{(D_x)} \bar{\varphi}_n(x)\, \varphi_{n,k}(x)\, d\, u^*(e_x)\right|^2\right]^{\frac{1}{2}}$$

and, in consequence of (9. 1 a) and of Bessel's inequality,

$$\int\limits_{(D_x)} |\varphi_n(x)|^2\, d\, u^*(e_x) \leqq \left[\int\limits_{(D_x)} |f(x)|^2\, d\, u^*(e_x)\right]^{\frac{1}{2}} \left[\int\limits_{(D_x)} |\varphi_n(x)|^2\, d\, u^*(e_x)\right]^{\frac{1}{2}}$$
$$+ |\lambda|\, A(x) \left[\int\limits_{(D_x)} |\varphi_n(x)|^2\, d\, u^*(e_x)\right]^{\frac{1}{2}}.$$

Accordingly we have the following result.

Lemma 9. 1. *Let λ be in the set T (Definition 9. 1). For the solutions $\varphi_n(x)$ of the approximating problems (9. 3) one has*

$$\int\limits_{(D_x)} |\varphi_n(x)|^2\, d\, u^*(e_x) \leqq B^2(\lambda), \tag{9. 4}$$

$$B(\lambda) = \left[\int\limits_{(D_x)} |f^2(x)|\, d\, u^*(e_x)\right]^{\frac{1}{2}} + |\lambda|\, A(\lambda) \qquad (cf.\ (9.\ 1\ a))$$

for $n = n_1, n_2, \ldots$

By (9. 4) Theorem 4. 2 is applicable enabling us to assert that there exists a function $\varphi(x)$ to which a subsequence $\{\varphi_{m_j}(x)\}$ of $\{\varphi_{n_j}(x)\}$ converges weakly, in the sense of Definition 4. 1, while

$$\int\limits_{(D_x)} |\varphi(x)|^2 \, d\, u^*(e_x) \leqq B^2(\lambda). \tag{9. 5}$$

We have

$$\lim \int\limits_{e_x} \varphi_{m_j}(x) \, d\, u^*(e_x) = \int\limits_{e_x} \varphi(x) \, d\, u^*(e_x) = \Phi(e_x) \tag{9. 5 a}$$

for all $\{B\}$-sets e_x on whose frontiers u^* is zero; here $\Phi(e_x)$ is absolutely continuous $\{u^*\}$.

In consequence of the second part of Theorem 4. 1, applicable by virtue of the fact that

$$|L_{m_j}(x, y)| \leqq |L(x, y)|,$$

where $L^2(x, y)$ is integrable $\{B, u^*\}$ (in y), we have

$$\lim \int\limits_{(D_y)} L_{m_j}(x, y) \, \varphi_{m_j}(y) \, d\, u^*(e_y) = \int\limits_{(D_y)} L(x, y) \, \varphi(y) \, d\, u^*(e_y).$$

Hence the limit,

$$\lim \varphi_{m_j}(x) = \varphi'(x), \tag{9. 5 b}$$

exists in the ordinary sense and

$$\varphi'(x) = \lambda \int\limits_{(D_y)} L(x, y) \, \varphi(y) \, d\, u^*(e_y) + f(x). \tag{9. 6}$$

Now

$$\int\limits_{e_x} |\varphi_{m_j}(x)|^2 \, d\, u^*(e_x) \leqq B^2(\lambda);$$

whence by (9. 5 b) and in view of Theorem 4. 3

$$\lim \int\limits_{e_x} \varphi_{m_j}(x) \, d\, u^*(e_x) = \int\limits_{e_x} \varphi'(x) \, d\, x = \Phi'(e_x); \tag{9. 6 a}$$

therefore, in consequence of (9. 5 a), $\Phi = \Phi'$ and

$$\varphi(x) = \varphi'(x)$$

except on a set e_0 such that $u^*(e_0) = 0$.

Thus $\varphi(x)$ is a solution of (7. 3) almost everywhere $\{u^*\}$. We shall now show that at least one of these functions $\varphi(x)$ admits of a 'spectral' representation. For this purpose we envisage the representation of $\varphi_n(x)$ by the third member in (5. 16). One has

$$\varphi(x) = f(x) + \lambda \int\limits_{(D_y)} L(x, y) f(y)\, d\, u^*(e_y) - \lambda^2 \varphi_0(x) \tag{9. 7}$$

where

$$\varphi_0(x) = \lim_n \lambda_n(x), \quad \lambda_n(x) = \sum_{k=1}^{\infty} \frac{1}{\lambda - \lambda_{n,k}}\, c_{n,k} \frac{\varphi_{n,k}(x)}{\lambda_{n,k}}, \tag{9. 7 a}$$

whenever $n \to \infty$ through a sequence of values such that the limit in (9. 17 a) exists; the values $n = m_j$ of (9. 5 a), for instance, will suffice. By (9. 1) and since

$$\frac{\varphi_{n,k}(x)}{\lambda_{n,k}} = \int\limits_{(D_z)} L_n(z, x)\, \varphi_{n,k}(z)\, d\, u^*(e_z)$$

it follows that

$$\alpha_n(x) = \sum_k \frac{1}{\lambda - \lambda_{n,k}} \int\limits_{(D_y)} \int\limits_{(D_z)} f_n(y)\, L_n(z, x)\, \varphi_{n,k}(y)\, \varphi_{n,k}(z)\, d\, u^*(e_z)\, d\, u^*(e_y).$$

Thus, in view of (6. 1), (6. 1 a).

$$\lambda_n(x) = \int\limits_{-\infty}^{\infty} \frac{1}{\lambda - \mu}\, d_\mu\, \sigma_n(x/\mu), \tag{9. 7 b}$$

$$\sigma_n(x/\mu) = \int\limits_{(D_y)} \int\limits_{(D_z)} \theta_n(y, z/\mu) f_n(y)\, L_n(z, x)\, d\, u^*(e_z)\, d\, u^*(e_y).$$

We shall write

$$\sigma_n(x/\mu) = \sigma_n'(x/\mu) + \sigma_n''(x/\mu) + \sigma_n'''(x/\mu), \tag{9. 7 c}$$

$$\sigma_n'(x/\mu) = \int\limits_{(D_y)} \int\limits_{(D_z)} \theta_n(y, z/\mu) f(y)\, L(z, x)\, d\, u^*(e_z)\, d\, u^*(e_y),$$

$$\sigma_n''(x/\mu) = \int\limits_{(D_y)} \int\limits_{(D_z)} \theta_n(y, z/\mu) f_n(y) (L_n(z, x) - L(z, x))\, d\, u^*(e_y)\, d\, u^*(e_z),$$

$$\sigma_n'''(x/\mu) = \int\limits_{(D_y)} \int\limits_{(D_z)} \theta_n(y, z/\mu) (f_n(y) - f(y))\, L(z, x)\, d\, u^*(e_y)\, d\, u^*(e_z).$$

By (6. 18 c)

$$\text{(9. 8)} \qquad \lim \sigma_n'(x/\mu) = \int\limits_{(D_y)} \int\limits_{(D_z)} \theta(y, z/\mu) f(y)\, L(z, x)\, d\, u^*(e_z)\, d\, u^*(e_y) = \sigma'(x/\mu),$$

as $n = n_j$ (n_j from (6. 14)) $\to \infty$. In view of (6. 19 b) (with $\psi_n(\lambda)$ from (6. 19 a)) we have

$$\text{(9. 8 a)} \qquad V_{\lambda_1}^{\lambda_2} \sigma_n'(x/\mu) \leqq \left[\int\limits_{(D_y)} f^2(y)\, d\, u^*(e_y)\right]^{\frac{1}{2}} \left[\int\limits_{(D_z)} L^2(z, x)\, d\, u^*(e_z)\right]^{\frac{1}{2}},$$

$$\text{(9. 8 b)} \qquad V_{\lambda_1}^{\lambda_2} \sigma_n''(x/\mu) \leqq \left[\int\limits_{(D_y)} f_n^2(y)\, d\, u^*(e_y)\right]^{\frac{1}{2}} \left[\int\limits_{(D_z)} [L_n(z, x) - L(z, x)]^2\, d\, u^*(e_z)\right]^{\frac{1}{2}},$$

$$\text{(9. 8 c)} \qquad V_{\lambda_1}^{\lambda_2} \sigma_n'''(x/\mu) \leqq \left[\int\limits_{(D_y)} (f_n(y) - f(y))^2\, d\, u^*(e_y)\right]^{\frac{1}{2}} \left[\int\limits_{(D_z)} L^2(z, x)\, d\, u^*(e_z)\right]^{\frac{1}{2}}.$$

On writing

$$\text{(9. 9)} \qquad \lambda_n(x) = \lambda_n'(x) + \lambda_n''(x) + \lambda_n'''(x),$$

$$\lambda_n'(x) = \int_{-\infty}^{\infty} \frac{1}{\lambda - \mu}\, d_\mu\, \sigma_n'(x/\mu), \qquad \lambda_n''(x) = \int_{-\infty}^{\infty} \frac{1}{\lambda - \mu}\, d_\mu\, \sigma_n''(x/\mu),$$

$$\lambda_n'''(x) = \int_{-\infty}^{\infty} \frac{1}{\lambda - \mu}\, d_\mu\, \sigma_n'''(x/\mu)$$

it is observed that, for λ in O,

$$\text{(9. 9 a)} \qquad \lim \lambda_n''(x) = \lim \lambda_n'''(x) = 0;$$

this is established by following a procedure analogous to that from (8. 19) to (8. 20) and by making use of the fact that the second members in (9. 8 b), (9. 8 c) are independent of λ_1, λ_2 and tend to zero as $n \to \infty$.

It is to be noted that the second member in (9. 8 a) is independent of n, λ_1, λ_2. With the aid of (9. 8) and (9. 8 a) and following steps similar to those in the text from (8. 22 a) to Theorem 8. 2, it is inferred that

$$\lim \lambda'_{n_j}(x) = \int_{-\infty}^{\infty} \frac{1}{\lambda - \mu} d_\mu \sigma'(x/\mu)$$

for λ in O. On taking note of (9. 9), (9. 9 a), (9. 7 c), (9. 7 a) and (9. 7) we sum the above developments as follows.

Theorem 9. 1. *Let λ be in the set T (Definition 9. 1). Let the $\varphi_n(x)$ $(n_1, n_2, \ldots)$ be approximating solutions satisfying (9. 3) (Lemma 9. 4 will hold). For some subsequences $\{\varphi_{m_j}(x)\}$ we have $\lim \varphi_{m_j}(x) = \varphi(x)$ (in the ordinary sense); every such function $\varphi(x)$ will be a solution of the equation*

$$\varphi(x) = \lambda \int_{(D_y)} L(x, y)\, \varphi(y)\, d\, u^*(e_y) + f(x)$$

almost everywhere $\{u^*\}$; $\varphi(x)$ satisfies the inequality (9. 5).

When λ is in the subset O of T any such solution admits a spectral representation

$$\varphi(x) = f(x) + \lambda \int_{(D_y)} L(x, y) f(y)\, d\, u^*(e_y) \tag{9. 10}$$
$$- \lambda^2 \int_{-\infty}^{\infty} \frac{1}{\lambda - \mu}\, d_\mu \int_{(D_y)} \int_{(D_z)} \theta(y, z/\mu) f(y)\, L(z, x)\, d\, u^*(e_z)\, d\, u^*(e_y),$$

where θ is a suitable spectrum; in fact, θ is the limit of an appropriate subsequence of the sequence $\{\theta_{m_j}\}$, where the m_j are from (9. 5 b). On the other hand, (9. 10) will still represent a solution when θ is any spectrum $[\theta = \lim \theta_{k_j}]$, provided that the set O is defined for the sequence $k_1, k_2, \ldots$.

In view of (5. 16)

$$\varphi_n(x) = f_n(x) - \lambda \int_{-\infty}^{\infty} \frac{1}{\lambda - \mu}\, d_\mu \int_{(D_y)} f_n(y)\, \theta_n(x, y/\mu)\, d\, u^*(e_y).$$

Hence it is natural to inquire whether it is possible to replace (9. 10) by the simpler representation

$$f(x) - \lambda \int_{-\infty}^{\infty} \frac{1}{\lambda - \mu}\, d_\mu \int_{(D_y)} f(y)\, \theta(x, y/\mu)\, d\, u^*(e_y). \tag{9. 11}$$

When the kernel is (T^*) (Definition 6. 1) an analogue of the theorem in (C; p. 43) will hold, enabling transformation of the second member in (9. 10) into the expression (9. 11). Offhand as much cannot be said for the more general kernels (T).

We shall prove that *the solutions, referred to in Theorem* 9. 1, *admit of a representation*

$$\varphi(x) \sim f(x) - \lambda \int_{-\infty}^{\infty} \frac{1}{\lambda - \mu}\, d_\mu \int_{(D_y)} f(y)\, \theta(x, y/\mu)\, d\, u^*(e_y) \tag{9. 12}$$

for λ *in* O; *here and in the sequel* $\sim$ *is to denote convergence in the mean square.*

To prove this we introduce the notation

$$\tau_n(x/\mu) = \int_{(D_y)} f_n(y)\, \theta_n(x, y/\mu)\, d\, u^*(e_y), \qquad \tau(x/\mu) = \int_{(D_y)} f(y)\, \theta(x, y/\mu)\, d\, u^*(e_y)$$

and proceed to establish that, for $l (> 0)$ finite,

$$\lim_{n_j} \int_{-l}^{l} \frac{1}{\lambda - \mu}\, d_\mu\, \tau_n(x/\mu) = \int_{-l}^{l} \frac{1}{\lambda - \mu}\, d_\mu\, \tau(x/\mu), \tag{9. 13}$$

when λ is in O. Let us take λ real, first. With $\delta(\lambda)$ from (8. 1 a) one has $d_\mu\, \tau_n(x/\mu) = 0$ for

$$\lambda - \frac{\delta}{2} \leqq \mu \leqq \lambda + \frac{\delta}{2}$$

and (with l suitably great)

$$\int_{-l}^{l} \frac{1}{\lambda - \mu}\, d_\mu\, \tau_n(x/\mu) = \left(\int_{-l}^{\lambda - \frac{\delta}{2}} + \int_{\lambda + \frac{\delta}{2}}^{l} \right) \cdots \tag{9. 14}$$

Now

$$\tau_n(x/\mu) = \tau_n'(x/\mu) + \tau_n''(x/\mu), \tag{9. 14 a}$$

$$\tau'_n(x/\mu) = \int_{(D_y)} f(y)\,\theta_n(x,\,y/\mu)\,d\,u^*(e_y),$$

$$\tau''_n(x/\mu) = \int_{(D_y)} (f_n(y) - f(y))\,\theta_n(x,\,y/\mu)\,d\,u^*(e_y).$$

By (6. 19) see (6. 17))

$$V_{-l}^{l}\,\tau''_n(x/\mu) \leqq l\,L(x)\left[\int_{(D_y)} |f_n(y) - f(y)|^2\,d\,u^*(e_y)\right] = l\,\alpha_n(x). \tag{9. 14 b}$$

Hence

$$\left|\int_{-l}^{l} \frac{1}{\lambda - \mu}\,d_\mu\,\tau''_n(x/\mu)\right| = \left|\int_{-l}^{\lambda - \frac{\delta}{2}} + \int_{\lambda + \frac{\delta}{2}}^{l} \cdots\right| \leqq \frac{2}{\delta}\,V_{-l}^{l}\,\tau''_n(x/\mu) \leqq \frac{2}{\delta}\,l\,\alpha_n(x). \tag{9. 15}$$

Since $\lim\limits_n \alpha_n(x) = 0$ it follows that

$$\lim_n \int_{-l}^{l} \frac{1}{\lambda - \mu}\,d_\mu\,\tau''_n(x/\mu) = 0. \tag{9. 15 a}$$

The same is established by analogous methods and using (9. 14 b), when λ is non real. By (6. 21 a)

$$\lim_{n_j} \int_{-l}^{l} \frac{1}{\lambda - \mu}\,d_\mu\,\tau'_n(x/\mu) = \lim_{n_j}\left[\int_{-l}^{\lambda - \frac{\delta}{2}} + \int_{\lambda + \frac{\delta}{2}}^{l}\right]\cdots \tag{9. 15 b}$$

$$= \left(\int_{-l}^{\lambda - \frac{\delta}{2}} + \int_{\lambda + \frac{\delta}{2}}^{l}\right) \frac{1}{\lambda - \mu}\,d_\mu\,\tau(x/\mu) = \int_{-l}^{l} \frac{1}{\lambda - \mu}\,d_\mu\,\tau(x/\mu)$$

for λ real, in O; this will hold for λ non real, as well. In the derivation of (9. 15 b) use was made of the fact that $d_\mu\,\tau(x/\mu) = 0$ in $\left(\lambda - \frac{\delta}{2},\ \lambda + \frac{\delta}{2}\right)$, inasmuch as θ has this property. Now

$$\int_{-l}^{l} \frac{1}{\lambda - \mu} d_\mu \tau_n(x/\mu) = \int_{-l}^{l} \frac{1}{\lambda - \mu} d_\mu \tau_n'(x/\mu) + \int_{-l}^{l} \frac{1}{\lambda - \mu} d_\mu \tau_n''(x/\mu);$$

accordingly, the relations (9. 15 a), (9. 15 b) are seen to imply (9. 13).

On writing

$$(9.\ 16) \qquad \varphi_n(x) = \varphi_n(x, l) + r_n(x, l), \quad \varphi_n(x, l) = f_n(x) - \lambda \int_{-l}^{l} \frac{1}{\lambda - \mu} d_\mu \tau_n(x/\mu),$$

$$r_n(x, l) = -\lambda \left(\int_{-\infty}^{-l} + \int_{l}^{\infty} \right) \frac{1}{\lambda - \mu} d_\mu \tau_n(x/\mu),$$

it is observed that the limits

$$(9.\ 16\text{ a}) \quad \lim_{n_j} \varphi_{n_j}(x) = \varphi(x), \quad \lim_{n_j} \varphi_{n_j}(x, l) = \varphi(x, l) = f(x) - \lambda \int_{-l}^{l} \frac{1}{\lambda - \mu} d_\mu \tau(x/\mu)$$

exist — the first in consequence of earlier developments, the second in view of (9. 13); hence the limit

$$(9.\ 16\text{ b}) \qquad \lim_{n_j} r_{n_j}(x, l) = r(x, l) \qquad (\lambda \text{ in } O)$$

exists, as well. In accordance with this one may write

$$(9.\ 16\text{ c}) \qquad \varphi(x) = \varphi(x, l) + r(x, l)$$

and

$$(9.\ 17) \qquad \int_{(D_x)} |\varphi(x) - \varphi(x, l)|^2 \, d u^*(e_x) = \int_{(D_x)} |r(x, l)|^2 \, d u^*(e_x).$$

Now, taking $l > |\Re \lambda|$ and noting that

$$r_n(x, l) = \sum_k^{(l)} \frac{-\lambda}{\lambda - \lambda_{n,k}} \int_{(D_y)} f_n(y) \varphi_{n,k}(x) \varphi_{n,k}(y) \, d u^*(e_y),$$

where summation is with respect to k, corresponding to the intervals $(-\infty, -l)$, $(l, +\infty)$, we have

$$|r_n(x, l)|^2 = |\lambda|^2 \sum_k^{(l)} \sum_j^{(l)} \frac{1}{\lambda - \lambda_{n,k}} \frac{1}{\bar\lambda - \lambda_{n,j}} \int_{(D_y)} \int_{(D_s)} f_n(y) \bar f_n(s) \cdot$$

$$\cdot \varphi_{n,k}(x) \varphi_{n,j}(x) \varphi_{n,k}(y) \varphi_{n,j}(s) \, d u^*(e_y) \, d u^*(e_s).$$

Multiplying by $d u^*(e_x)$, recalling the orthonormal character of the $\varphi_{n,k}(x)$ one obtains

$$\int\limits_{(D_x)} | r_n(x,\ l)|^2 \, d u^*(e_x)$$
$$= \sum_k^{(l)} \frac{|\lambda|^2}{|\lambda - \lambda_{n,k}|^2} \int\limits_{(D_y)} \int\limits_{(D_s)} f_n(y) \bar{f}_n(s) \varphi_{n,k}(y) \varphi_{n,k}(s) \, d u^*(e_y) \, d u^*(e_s).$$

Thus

$$\int\limits_{(D_x)} | r_n(x,\ l)|^2 \, d u^*(e_x) = \left(\int\limits_{-\infty}^{-l} + \int\limits_{l}^{\infty}\right) \frac{|\lambda|^2}{|\lambda - \mu|^2} d_\mu w_n(\mu),$$

$$w_n(\mu) = \int\limits_{(D_y)} \int\limits_{(D_s)} \theta_n(y,\ s/\mu) f_n(y) \bar{f}_n(s) \, d u^*(e_y) \, d u^*(e_s).$$

By (6. 19 b)

$$V_{\lambda_1}^{\lambda_2} w_n(\mu) \leqq \int\limits_{(D_y)} |f_n(y)|^2 \, d u^*(e_y) \leqq \int\limits_{(D_y)} |f(y)|^2 \, d u^*(e_y).$$

Now, with $-l < \Re\lambda < l$, it is observed that

$$(9.\ 18) \qquad \left|\frac{1}{\lambda - \mu}\right| \leqq \frac{1}{l - \Re\lambda} \quad (\mu \geqq l), \qquad \left|\frac{1}{\lambda - \mu}\right| \leqq \frac{1}{l + \Re\lambda} \quad (\mu \leqq -l).$$

Whence

$$\int\limits_{(D_x)} | r_n(x,\ l)|^2 \, d u^*(e_x) \leqq [(l - \Re\lambda)^{-2} + (l + \Re\lambda)^{-2}] \, |\lambda|^2 \, V w_n(\mu)$$
$$\leqq [(l - \Re\lambda)^{-2} + (l + \Re\lambda)^{-2}] \, |\lambda|^2 \int\limits_{(D_y)} |f(y)|^2 \, d u^*(e_y) = r(l),$$

where $r(l) \to 0$, *as* $l \to +\infty$, *and* $r(l)$ *is independent of* n. For the limit (9. 16 b) we accordingly obtain the inequality

$$\int\limits_{(D_x)} | r(x,\ l)|^2 \, d u^*(e_x) \leqq r(l).$$

Therefore, on letting l in (9. 17) approach infinity, it is deduced that

$$\varphi(x,\ l) \sim \varphi(x) \qquad \text{(in } (D_x));$$

$\varphi(x,\ l)$ being defined in (9. 16 a), the truth of (9. 12) is now made evident.

A still different representation holds. In fact, we shall establish that, *for λ in O, the solutions, referred to in Theorem 9. 1, are representable in the form*

$$(9.\ 19)\qquad \varphi(x) = f(x) - \lambda D_x \left[\int_{-\infty}^{\infty} \frac{1}{\lambda - \mu} d_\mu \int_{(D_y)} f(y)\, \theta^*(e_x, y/\mu)\, d\,u^*(e_y) \right]$$

almost everywhere $\{u^*\}$; *here* D_x *is the symbol of a 'regular' (in the sense of* (S; 152—156) *for instance) set-derivative with* u^* *used as the measure function.*

We have $f_n(y) \to f(y)$ and, by (8. 21 a) $\theta^*_{n_j} \to \theta^*$. On the other hand, $|f_n(y)| \leqq |f(y)|$, where $|f(y)|^2$ is integrable $\{B, u^*\}$ and

$$\int_{(D_y)} \theta_n^{*2}(e_x, y/\mu)\, d\,u^*(e_y) \leqq u^*(e_x)$$

in view of (8. 21). Hence by virtue of Theorem 4. 3, on writing

$$\zeta_n(e_x/\mu) = \int_{(D_y)} f_n(y)\, \theta_n^*(e_x, y/\mu)\, d\,u^*(e_y)$$

one obtains

$$(9.\ 20)\qquad \lim \zeta_{n_j}(e_x/\mu) = \zeta(e_x/\mu) = \int_{(D_y)} f(y)\, \theta^*(e_x, y/\mu)\, d\,u^*(e_y).$$

Form the sum

$$V_{n,m} = \sum_{\nu=1}^{m} \left| \int_{(D_y)} f_n(y) \left(\theta_n^*(e_x, y/l_\nu) - \theta_n^*(e_x, y/l_{\nu-1})\right) d\,u^*(e_y) \right|$$

$(\lambda_1 = l_0 < l_1 < \cdots < l_m = \lambda_2)$. Then

$$V_{n,m} = \sum_{\nu=1}^{m} \left| \sum_k^{(\nu)} \int_{(D_y)} f_n(y)\, \psi^*_{n,k}(e_x)\, \varphi_{n,k}(y)\, d\,u^*(e_y) \right|$$

$$\leqq \sum_k |\psi^*_{n,k}(e_x)| \left| \int_{(D_y)} f_n(y)\, \varphi_{n,k}(y)\, d\,u^*(e_y) \right|$$

$$\leqq \left[\sum_k |\psi^*_{n,k}(e_x)|^2 \right]^{\frac{1}{2}} \left[\sum_k \left| \int_{(D_y)} f_n(y)\, \varphi_{n,k}(y)\, d\,u^*(e_y) \right|^2 \right]^{\frac{1}{2}}.$$

Hence by Bessel's inequality and in view of (8. 3 b)

$$V_{\lambda_1}^{\lambda_2} \zeta_n(e_x/\mu) \leqq [u^*(e_x)]^{\frac{1}{2}} \left[\int\limits_{(D_y)}\!\!\int |f(y)|^2 \, d\, u^*(e_y) \right]^{\frac{1}{2}} = \zeta, \tag{9. 20 a}$$

where $\zeta (< +\infty)$ is independent of n and λ_1, λ_2.

By (9. 20), (9. 20 a) and Helly's theorem for λ real, in O, we have

$$\lim_{n_j} \left(\int\limits_{-l}^{\lambda - \frac{\delta}{2}} + \int\limits_{\lambda + \frac{\delta}{2}}^{l} \right) \frac{1}{\lambda - \mu} \, d_\mu \zeta_n(e_x/\mu) = \left(\int\limits_{-l}^{\lambda - \frac{\delta}{2}} + \int\limits_{\lambda + \frac{\delta}{2}}^{l} \right) \frac{1}{\lambda - \mu} \, d_\mu \zeta(e_x/\mu)$$

and, for λ non real,

$$\lim_{n_j} \int\limits_{-l}^{l} \frac{1}{\lambda - \mu} \, d_\mu \zeta_n(e_x/\mu) = \int\limits_{-l}^{l} \frac{1}{\lambda - \mu} \, d_\mu \zeta(e_x/\mu). \tag{9. 20 b}$$

Inasmuch as $\zeta_n(e_x/\mu)$ is constant, as a function of μ, interior the interval $(\lambda - \delta, \lambda + \delta)$ the same will be true for $\zeta(e_x/\mu)$ cf. (9. 20)); consequently (9. 20 b) will hold for *all* λ in O. Now, with $l > |\Re \lambda|$ one has (9. 18) and, by virtue of (9. 20 a),

$$\left| \left(\int\limits_{-\infty}^{-l} + \int\limits_{l}^{+\infty} \right) \frac{1}{\lambda - \mu} \, d_\mu \zeta_n(e_x/\mu) \right| \leqq [(l - \Re \lambda)^{-1} + (l + \Re \lambda)^{-1}] \; V \zeta_n(e_x/\mu)$$

$$\leqq [(l - \Re \lambda)^{-1} + (l + \Re \lambda)^{-1}] \, \zeta \quad (n = 1, 2, \ldots).$$

The last member here is independent of n and approaches zero, as $l \to \infty$; together with (9. 20 b), this fact implies that

$$\lim_{n_j} \int\limits_{-\infty}^{\infty} \frac{1}{\lambda - \mu} \, d_\mu \zeta_n(e_x/\mu) = \int\limits_{-\infty}^{\infty} \frac{1}{\lambda - \mu} \, d_\mu \zeta(e_x/\mu) \tag{9. 21}$$

for λ in O.

Taking the indefinite integral of the two members in the formula preceding (9. 11) and observing that certain changes in the order of integration are permissible, we obtain

$$\int\limits_{e_x} \varphi_n(x) \, d\, u^*(e_x) = \int\limits_{e_x} f_n(x) \, d\, u^*(e_x) - \lambda \int\limits_{-\infty}^{\infty} \frac{1}{\lambda - \mu} \, d_\mu \zeta_n(e_x/\mu).$$

It had been previously established that (9. 5 b) implies (9. 6 a); thus

$$\lim \int_{e_x} \varphi_{n_j}(x)\, d\, u^*(e_x) = \int_{e_x} \varphi(x)\, d\, u^*(e_x).$$

On the other hand, $|f_n(x)| \leqq |f(x)|$, where $|f(x)|$ is integrable $\{B, u^*\}$. Hence in view of (9. 21)

$$\int_{e_x} \varphi(x)\, d\, u^*(e_x) = \int_{e_x} f(x)\, d\, u^*(e_x) - \lambda \int_{-\infty}^{\infty} \frac{1}{\lambda - \mu}\, d_\mu\, \zeta(e_x/\mu)$$

(see (9. 20)), from which the conclusion (9. 19) follows as stated.

Using (9. 3) and Lemma 9. 1 we obtain

$$(9.\ 22) \qquad |\varphi_n(x)| \leqq |\lambda|\, L(x)\, B(\lambda) + |f(x)|, \quad |\varphi_n(x) - f_n(x)| \leqq |\lambda|\, L(x)\, B(\lambda).$$

One also has

$$|(\varphi_n(x_1) - f(x_1)) - (\varphi_n(x) - f(x))| \leqq |\lambda| \left[\int_{(D_y)} (L_n(x_1, y) - L_n(x, y))^2\, d\, u^*(e_y)\right]^{\frac{1}{2}} B(\lambda).$$

When the kernel is (T^*) from the above it is inferred that

$$(9.\ 22\ \mathrm{a}) \qquad |(\varphi_n(x_1) - f(x_1)) - (\varphi_n(x) - f(x))| \\ \leqq |\lambda| \left[\int_{(D_y)} (L(x_1, y) - L(x, y))^2\, d\, u^*(e_y)\right]^{\frac{1}{2}} B(\lambda)$$

for x_1, x in any closed set ω_0, $< (D_x)^0$, and for all $n \geqq n'$, where n' depends on ω_0.

In view of (9. 22 a) *the solutions* $\varphi(x)$ *referred to in Theorem* 9. 1 *are continuous in* x *for* x *in* $(D_x)^0$, *provided* $L(x, y)$ *is a kernel* (T^*).

In the case of (T^*) the approximating sequences can be so selected (using Vitali's theorem) that limits $\varphi(x)$, satisfying equations (7. 3), are continuous in x (in $(D_x)^0$) for every λ in the set T and are analytic in λ at all the interior points of T (for every x in $(D_x)^0$); such results are analogous to those obtained for certain kernels in (C).

10. Operators O and their Applications.

We shall first consider equations (7. 3 a) for kernels (T), when the integrals of Hypothesis 7. 4 exist and when u^* is 'regular' (Definition 8. 1) with respect to the frontier of (D).

On writing

$$\psi^*(e_y) - F^*(e_y) = A(e_y), \quad \psi_n^*(e_y) - F^*(e_y) = A_n(e_y) \tag{10. 1}$$

in place of (7. 1 a) and (7. 3 a) we have

$$A_n(e_y) = \lambda \int_{(D_x)} \left[\int_{e_y} L_n(x, y)\, d\, u^*(e_y) \right] d\, A_n(e_x) + \lambda\, g_n(F^*/e_y), \tag{10. 2}$$

$$A(e_y) = \lambda \int_{(D_x)} \left[\int_{e_y} L(x, y)\, d\, u^*(e_y) \right] d\, A(e_x) + \lambda\, g(F^*/e_y) \tag{10. 2 a}$$

where

$$g_n(F^*/e_y) = \int_{(D_x)} \left[\int_{e_y} L_n(x, y)\, d\, u^*(e_y) \right] d\, F^*(e_x), \tag{10. 2 b}$$

$$g(F^*/e_y) = \int_{(D_x)} \left[\int_{e_y} L(x, y)\, d\, u^*(e_y) \right] d\, F^*(e_x). \tag{10. 2 c}$$

Consider now two equations (10. 2):

$$\begin{aligned} {}_1A_n(e_y) &= \lambda \int_{(D_x)} \left[\int_{e_y} L_n(x, y)\, d\, u^*(e_y) \right] d\, {}_1A_n(e_x) + \lambda\, g_n({}_1F^*/e_y), \\ {}_2A_n(e_y) &= \lambda \int_{(D_x)} \left[\int_{e_y} L_n(x, y)\, d\, u^*(e_y) \right] d\, {}_2A_n(e_x) + \lambda\, g_n({}_2F^*/e_y); \end{aligned} \tag{10. 3}$$

the decompositions of the additive functions of $\{B\}$-sets ${}_iF^*$ being

$${}_iF^*(e_x) = {}_iF_1^*(e_x) - {}_iF_2^*(e_x), \quad {}_iF_1^*(e_x) > 0, \quad {}_iF_2^*(e_x) \geqq 0, \tag{10. 3 a}$$

we write

$$v_i^*(e_x) = {}_iF_1^*(e_x) + {}_iF_2^*(e_x) \qquad (i = 1, 2) \tag{10. 3 b}$$

and we consider only such functions F^*, ${}_iF^*$ (as the case may be) for which the integrals

(10. 3 c)
$$\int\limits_{(D_x)} L(x)\,d\,v^*(e_x), \quad \int\limits_{(D_x)} L(x)\,d\,v_i^*(e_x)$$

exist (in agreement with Hypothesis 7. 4); *such functions will be termed admissible with respect to* $L(x, y)$.

Set-functions ${}_iA_n(e_y)$ $(i = 1, 2)$ satisfying (10. 3) exist, for λ in O; moreover, by (8. 4 a)

$$|{}_iA_n(e_y)| \leqq \beta(e_y) \qquad (n = n_1, n_2, \ldots)$$

for $i = 1, 2$ and for λ in O, the function $\beta(e_y)$ being defined in (8. 4). The following identity is satisfied:

(10. 4)
$$\int\limits_{(D_x)}\int\limits_{(D_z)} L_n(x, z)\,d\,{}_1F^*(e_z)\,d\,{}_2A_n(e_x) = \int\limits_{(D_x)}\int\limits_{(D_z)} L_n(x, z)\,d\,{}_2F^*(e_z)\,d\,{}_1A_n(e_x).$$

To establish this we note (see (7. 9)) that

$${}_iA_n(e_y) = \int\limits_{e_y} {}_i\Gamma_n(y)\,d\,u^*(e_y) \qquad (i = 1, 2);$$

here ${}_i\Gamma_n(y)$ is integrable $\{u^*\}$. The integrals

$$\int\limits_{(D_x)} L_n(x)\,|{}_i\Gamma_n(x)|\,d\,u^*(e_x)$$

$(i = 1, 2;\ n = 1, 2, \ldots)$, where

$$L_n^2(x) = \int\limits_{(D_y)} L_n^2(x, y)\,d\,u^*(e_y),$$

exist. Accordingly, by virtue of Lemma 7. 2 (applied to ${}_i\Gamma_n(y)$), one has

(10. 5)
$${}_i\Gamma_n(y) = \lambda \int\limits_{(D_x)} L_n(x, y)\,{}_i\Gamma_n(x)\,d\,u^*(e_x) + {}_iq_n(y),$$

$${}_iq_n(y) = \lambda \int\limits_{(D_z)} L_n(z, y)\,d\,{}_iF^*(e_z) \qquad \text{(cf. (7. 6), (7. 6 a))}.$$

From these integral equations it is deduced that

$$\int_{(D_x)} {}_1q_n(x)\,{}_2\Gamma_n(x)\,d\,u^*(e_x) = \int_{(D_x)} {}_2q_n(x)\,{}_1\Gamma_n(x)\,d\,u^*(e_x), \tag{10. 5 a}$$

which is an identity precisely analogous to a result in the classical theory. In view of the relation connecting ${}_iA_n$ and ${}_i\Gamma_n$ from (10. 5 a) it is inferred that

$$\int_{(D_x)} {}_1q_n(x)\,d\,{}_2A_n^*(e_x) = \int_{(D_x)} {}_2q_n(x)\,d\,{}_1A_n^*(e_x).$$

Finally, replacement of the ${}_iq_n(x)$ by their expressions (10. 5) will yield the identity (10. 4) (for $\lambda \neq \lambda_{n,j}$; $j = 1, 2, \ldots$).

Let λ be in O. In this set we have (8. 1 a) for $n = n_1, n_2, \ldots$ ($n_s \to +\infty$, as $s \to \infty$). In the sequence $\{n_s\}$ there exist subsequences $\{m_s\}$ such that

$$\lim \theta_{m_s}(x, y/\lambda) = \theta(x, y/\lambda). \tag{10. 6}$$

Of course there may be more than one spectral function θ. For a fixed set O the sequences $\{m_s\}$, for which a relation (10. 6) holds, depend on the kernel $L(x, y)$ only. We observe now that the result of Theorem 8. 2 amounts to the following.

Let $A_{m_j}(e_x)$ be the solution (λ in O) of the equation (10. 2) (with (10. 2 b) and $n = m_j$). *The limit*

$$\lim A_{m_j}(e_x) = A(e_x) = \psi^*(e_x) - F^*(e_x) \tag{10. 7}$$

exists for $\{B\}$*-sets* e_x*, whose closure is in* $(D_x)^0$*, provided* u^* *is regular with respect to the frontier of* (D_x) *and provided that* $F^*(e_x)$ *is admissible with respect to* $L(x, y)$*; moreover, under these conditions* $A(e_x)$ *satisfies* (10. 2 a) *and*, in view of the representation (8. 24),

$$\begin{aligned} A(e_y) = O(\lambda, e_y/F^*) = {} & \lambda \int_{(D_x)} \left[\int_{e_y} L(x, y)\,d\,u^*(e_y) \right] d\,F^*(e_x) \\ & - \lambda^2 \int_{-\infty}^{\infty} \frac{1}{\lambda - \mu}\,d_\mu \int_{(D_x)} \int_{(D_z)} L(x, z)\,\theta^*(z, e_y/\mu)\,d\,u^*(e_z)\,d\,F^*(e_x) \end{aligned} \tag{10. 7 a}$$

for λ *in the set* O (*Hypothesis* (7. 4) *assumed*).

The operator $O(\lambda, e_y/\ldots)$ depends on L, θ^*. Now θ^* (rather, θ) has been defined by (10. 6); the sequence $\{m_s\}$ therein involved being independent of any possible choice of F^*, it follows that the choice of θ^* in $O(\lambda, e_y/\ldots)$ is independent of F^*. Hence, *the operator* $O(\lambda, e_y/\ldots)$, *defined by the last member in* (10. 7 a), *is independent of* F^*; *this is a linear operator which is defined for all additive functions* F^*, *admissible with respect to* $L(x, y)$ (u^* being regular with respect to the frontier of (D_x)).

On letting in (10. 3) $n = m_j$ and on taking account of (10. 7), (10. 7 a), in the limit we obtain

$$\lim_{m_j} {}_iA_{m_j}(e_x) = {}_iA(e_x) = O(\lambda, e_x/{}_iF^*) \qquad (i = 1, 2),$$

where ${}_iA(e_x)$ is a solution of (10. 2 a) for $F^* = {}_iF^*$. Moreover, *one will have, whenever* ${}_1F^*$, ${}_2F^*$ *are admissible with respect to* $L(x, y)$,

$$\int_{(D_x)}\int_{(D_z)} L(x, z)\, d\,{}_1F^*(e_z)\, d\,O(\lambda, e_x/{}_2F^*) = \int_{(D_x)}\int_{(D_z)} L(x, z)\, d\,{}_2F^*(e_z)\, d\,O(\lambda, e_x/{}_1F^*). \tag{10. 8}$$

To establish this identity, we put in (10. 4) $n = m_j$ and pass to the limit. We shall now proceed to justify the latter step.

With the first member of (10. 4) in view, let us form

$$\varrho_n(e_x, e_z) = \int_{e_x}\int_{e_z} L_n(x, z)\, d\,{}_1F^*(e_z)\, d\,{}_2A_n(e_x).$$

Repeating the reasoning subsequent to (8. 8), replacing $d\,u^*(e_z)$ by $d\,{}_1F^*(e_z)$ (eventually, by $d\,v_1^*(e_z)$) and ψ_n^* by ${}_2A_n = {}_2\psi_n^* - {}_2F^*$, in place of (8. 9 a) one obtains (for λ in O)

$$\begin{gathered} |\varrho_n(e_x, e_z)| \leqq \left[|\lambda| + \frac{|\lambda|^2}{\delta(\lambda)}\right] \alpha_1 \int_{e_z} L_{(e_x)}(z)\, d\,v_1^*(e_z) = \beta(e_x, e_z) \\ \alpha_1 = \int_{(D_s)} L(s)\, d\,v_2^*(e_s), \quad L^2_{(e_x)}(z) = \int_{e_x} L^2(x, z)\, d\,u^*(e_x). \end{gathered} \tag{10. 9}$$

The first member in (10. 4) is of the form

$$\varrho_n(D_x, D_z). \tag{10. 9 a}$$

We assume that the set-function $v_1^*(e_z)$, corresponding to ${}_1F^*(e_z)$ vanishes on the frontier of (D_z). There exists a sequence of closed domains

$$\zeta_{z,1} < \zeta_{z,2} < \ldots,$$

each lying in the interior of (D_z), such that

$$\lim_{\nu} \zeta_{z,\nu} = (D_z)^0.$$

With $u^*(e_x)$ regular with respect to the frontier of (D_x), we designate by $\omega_{x,r}$ $(r = 1, 2, \ldots)$ a sequence of closed domains, each in $(D_x)^0$, such that $\lim \omega_{x,r} = (D_x)^0$, while $u^* = 0$ on their frontiers (also, $u^* = 0$ on the frontier of (D_x)). Now, by (10. 9)

$$|\varrho_{m_j}(D_x, D_z)| \leqq \beta(D_x, D_z) < +\infty \qquad (j = 1, 2, \ldots),$$

the m_j being the subscripts involved in the definition of $O(\lambda, e_y/\ldots)$ (cf. (10. 7), (10. 7 a)). Hence the sequence $\{m_j\}$ contains an infinite subsequence $\{k_j\}$ for which the limit

(10. 10) $$\lim_{j} \varrho_{k_j}(D_x, D_z) = \sigma$$

exists. Clearly, if in (10. 7) m_j is replaced by k_j the resulting operator $O(\lambda, e_y/\ldots)$ will be unchanged.

One has

(10. 10 a) $$\varrho_k(D_x, D_z) = \varrho_k(D_x, \zeta_{z,\nu}) + \varrho_k(D_x, D_z - \zeta_{z,\nu}) = \varrho_k(\omega_{x,r}, \zeta_{z,\nu}) + R_k^{r,\nu},$$

where

$$R_k^{r,\nu} = \varrho_k(D_x - \omega_{x,r}, \zeta_{z,\nu}) + \varrho_k(\omega_{x,r}, D_z - \zeta_{z,\nu}) + \varrho_k(D_x - \omega_{x,r}, D_z - \zeta_{z,\nu}).$$

In consequence of (10. 9)

$$|R_k^{r,\nu}| \leqq \beta(D_x - \omega_{x,r}, \zeta_{z,\nu}) + \beta(\omega_{x,r}, D_z - \zeta_{z,\nu}) + \beta(D_x - \omega_{x,r}, D_z - \zeta_{z,\nu}).$$

Accordingly, introducing the expression for β, after some simplifications we obtain

(10. 10 b) $$|R_k^{r,\nu}| \leqq \left[|\lambda| + \frac{|\lambda|^2}{\delta(\lambda)}\right] \alpha_1 \left[2 \int\limits_{(D_z)} L_{(D_x - \omega_{x,r})}(z)\, d\, v_1^*(e_z) + \int\limits_{(D_z - \zeta_{z,\nu})} L(z)\, d\, v_1^*(e_z)\right],$$

where

$$L^2_{(D_x - \omega_{x,r})}(z) = \int\limits_{(D_x - \omega_{x,r})} L^2(x, z)\, d\, u^*(e_x).$$

Since $\omega_{x,r} \to (D_x)^0$, while u^* vanishes on the frontier of (D_x) one has

$$\lim_{r} L^2_{(D_x - \omega_{x,r})}(z) = 0.$$

Now the integral

$$\int_{(D_z)} L(z)\, d\, v_1^*(e_z)$$

exists, while

$$L_{(D_x - \omega_{x,\, r})}(z) \leqq L(z);$$

hence the first integral displayed in (10. 10 b) approaches zero as $r \to \infty$. On the other hand, $\zeta_{z,\, \nu} \to (D_z)^0$ (as $\nu \to \infty$), while v_1^* is zero on the frontier of (D_z); whence

$$\lim_{\nu} \int_{(D_z - \zeta_{z,\, \nu})} L(z)\, d\, v_1^*(e_z) = 0.$$

Accordingly, an implication of (10. 10 b), essential for our purposes, is that

(10. 10 c) $$|R_k^{r,\, \nu}| \leqq \varrho^{r,\, \nu},$$

where $\varrho^{r,\, \nu}$ *is independent of* k and

(10. 10 d) $$\lim_{r,\, \nu} \varrho^{r,\, \nu} = 0;$$

the order of the limiting processes in (10. 10 d) is immaterial.

We have $L_n(x, z) = L(x, z)$, for x in $\omega_{x,\, r}$ and z in $\zeta_{z,\, \nu}$, when $n \geqq n'$, where n' may depend on r and ν. Thus

$$\varrho_n(\omega_{x,\, r}, \zeta_{z,\, \nu}) = \int_{\omega_{x,\, r}} \Big[\int_{\zeta_{z,\, \nu}} L(x, z)\, d\, F_1(e_z)\Big]\, d_2\, A_n(e_x) \qquad (n \geqq n').$$

The function of x

$$\int_{\zeta_{z,\, \nu}} L(x, z)\, d\, F_1(e_z)$$

is continuous in the closed domain $\omega_{x,\, r}$. It is also observed that the sequence of functions $\{{}_2A_n(e_x)\}$ is bounded, uniformly with respect to n, and that the ${}_2A_{k_j}(e_x)$, as well as the limit

$$\lim_{k_j} {}_2A_{k_j}(e_x) = {}_2A\,(e_x),$$

vanishes on the frontier of $\omega_{x,\, r}$, inasmuch as u^* does so (we note that, in accordance with section 8, the ${}_2A_{k_j}(e_x)$ and ${}_2A(e_x)$ are absolutely continuous $\{u^*\}$. Hence by the theorem referred to in connection with (8. 7)

$$\lim_{k_j} \varrho_{k_j}(\omega_{x,r}, \zeta_{z,\nu}) = \varrho(\omega_{x,r}, \zeta_{z,\nu}) = \int_{\omega_{x,r}} \int_{\zeta_{z,\nu}} L(x, z)\, d\,{}_1F^*(e_z)\, d\,{}_2A(e_x).$$

Hence by (10. 10 a) and (10. 10)

$$\lim_{k_j} R_{k_j}^{r,\nu} = \sigma - \varrho(\omega_{x,r}, \zeta_{z,\nu})$$

exists. In view of (10. 10 c)

$$|\sigma - \varrho(\omega_{x,r}, \zeta_{z,\nu})| \leqq \varrho^{r,\nu}$$

and, by virtue of (10. 10 d),

$$\sigma = \lim_{r,\nu} \varrho(\omega_{x,r}, \zeta_{z,\nu}) = \int_{(D_x)^0} \int_{(D_z)^0} L(x, z)\, d\,{}_1F^*(e_z)\, d\,{}_2A(e_x).$$

In the last member, here, $(D_x)^0$ and $(D_z)^0$ may be replaced by (D_x) and (D_z), respectively, since v_1^* and u^* (hence, ${}_2A$) vanish on the frontiers of these domains. Thus, on making use of (10. 10), we obtain

$$(10.\ 11) \qquad \lim \int_{(D_x)} \int_{(D_z)} L_{k_j}(x, z)\, d\,{}_1F^*(e_z)\, d\,{}_2A_{k_j}(e_x) = \int_{(D_x)} \int_{(D_z)} L(x, z)\, d\,{}_1F^*(e_z)\, d\,{}_2A(e_x);$$

in view of the remark subsequent to (10. 10) one may replace ${}_2A(e_x)$ by $O(\lambda, e_x/{}_2F^*)$. By a similar procedure it can be shown that for some subsequence $\{s_j\}$ of $\{k_j\}$

$$\lim \int_{(D_x)} \int_{(D_z)} L_{s_j}(x, z)\, d\,{}_2F^*(e_z)\, d\,{}_1A_{s_j}(e_x) = \int_{(D_x)} \int_{(D_z)} L(x, z)\, d\,{}_2F^*(e_z)\, d\,{}_1A(e_x),$$

where

$${}_1A(e_x) = O(\lambda, e_x/{}_1F^*).$$

The s_j are introduced at the step corresponding to (10. 10). It is observed that (10. 11) will hold when the k_j are replaced by the s_j. We are ready to formulate the following result.

Theorem 10. 1. *Let λ be in a set O and let $u^*(e_x)$ be regular (Definition 8. 1) with respect to the frontier of (D_x). Corresponding to every spectrum (10. 6) there* *exists an operator* $O(\lambda, e_y/\ldots)$ *of the form* (10. 7 a). *Whenever* F^* *is admissible with respect to* $L(x, y)$ (*in accordance with the text in connection with* (10. 3 c)), *the set-function*

$$A(e_x) = O(\lambda, e_x/F^*)$$

will constitute a solution, for λ *in* O, *of the non homogeneous problem* (10. 2 a) (*cf.* (10. 2 c)). *This operator satisfies the identity*

$$\text{(10. 12)} \quad \int_{(D_x)} \int_{(D_z)} L(x, z)\, d\, {}_1F^*(e_z)\, d\, O(\lambda, e_x/{}_2F)$$

$$= \int_{(D_x)} \int_{(D_z)} L(x, z)\, d\, {}_2F^*(e_z)\, d\, O(\lambda, e_x/{}_1F^*),$$

for λ *in* O, *whenever set-functions* $v_1^*(e_x)$, $v_2^*(e_x)$, *corresponding to* ${}_1F^*$, ${}_2F^*$ (*cf.* (10. 3 b), (10. 3 a)), *are admissible with respect to* $L(x, y)$ *and vanish on the frontier of* (D_x).

When F^* *is admissible with respect to* $L(x, y)$,

$$\psi^*(e_x) = O^*(\lambda, e_x/F^*) = F^*(e_x) + O(\lambda, e_x/F^*)$$

is a solution of (7. 1 a) (*for* λ *in* O). *Whenever, in addition to the previously stated conditions,* ${}_1F^*$, ${}_2F^*$ *are such that the order of integration in*

$$\int_{(D_x)} \int_{(D_z)} L(x, z)\, d\, {}_1F^*(e_z)\, d\, {}_2F^*(e_x)$$

is immaterial, one has (*for* λ *in* O)

$$\text{(10. 12 a)} \quad \int_{(D_x)} \int_{(D_z)} L(x, z)\, d\, {}_1F^*(e_z)\, d\, O^*(\lambda, e_x/{}_2F^*)$$

$$= \int_{(D_x)} \int_{(D_z)} L(x, z)\, d\, {}_2F^*(e_z)\, d\, O^*(\lambda, e_x/{}_1F^*).$$

With the aid of the above identities we shall establish the following uniqueness theorem.

Theorem 10. 2. *Let* λ *be in a set* O *and* u^* *be regular with respect to the frontier of* (D_x). *Assume, moreover, that* $L(x, y)$ *is a kernel of the form* (T), *definite in the sense that the integral*

(10. 13) $$Q(\gamma) = \int_{(D_x)} \int_{(D_z)} L(x, z)\, d\gamma(e_z)\, d\bar{\gamma}(e_x)$$

is distinct from zero for all additive fonctions γ (possibly complex valued) of $\{\beta\}$-sets, not identically zero, and such that this integral maintains the same sign for all such functions. This assertion is made for functions γ for which conditions of the form (10. 16), (10. 16 a) *hold and for which, of course,* $Q(\gamma)$ *exists.*

(1°). *If for a non real λ^* the equation*

(10. 14) $$\psi^*(e_y) = \lambda \int_{(D_x)} \int_{e_y} L(x, y)\, du^*(e_y)\, d\psi^*(e_x)$$

has $\psi^*(e_y) = 0$ as the only solution the same is true for all non real λ.

(2°). *The number of distinct solutions of the equation* (10. 14) *is the same for all non real λ.*

(3°). *If for $\lambda = \lambda_1$, fixed in O, the problem* (10. 14) *has no solutions (distinct from zero) the same is true for all non real λ.*

(4_0). *The number m of solutions of* (10. 14) *for any real λ in O is equal to or is greater than the number n of distinct solutions for non real values of λ.*

If (1°) does not hold there exists a function $\psi^*(e_y) \neq 0$, satisfying (10. 14) for some *non real* λ, distinct from λ^*. One has

$$\psi^*(e_y) - \lambda^* \int_{(D_x)} \int_{e_y} L(x, y)\, du^*(e_y)\, d\psi^*(e_x) = {}_1F^*(e_y) \equiv \left(1 - \frac{\lambda^*}{\lambda}\right) \psi^*(e_y).$$

Since the solution of the homogeneous problem, for λ^*, is uniquely zero by the hypothesis of (1°), one may rewrite the above in the form

(10. 15) $$O^*(\lambda^*, e_y/{}_1F^*) = \psi^*(e_y).$$

Now

$$\bar{\psi}^*(e_y) - \bar{\lambda} \int_{(D_x)} \int_{e_y} L(x, y)\, du^*(e_y)\, d\bar{\psi}^*(e_x) = 0,$$

$$\bar{\psi}^*(e_y) - \lambda^* \int_{(D_x)} \int_{e_y} L(x, y)\, du^*(e_y)\, d\bar{\psi}^*(e_x) = {}_2F^*(e_y) \equiv \left(1 - \frac{\lambda^*}{\bar{\lambda}}\right) \bar{\psi}^*(e_y).$$

Thus, in view of the remark leading to (10. 15), we have

(10. 15 a) $$O^*(\lambda^*, e_y/_2F^*) = \bar{\psi}^*(e_y).$$

In consequence of the identity (10. 12 a)

$$\left(1 - \frac{\lambda^*}{\lambda}\right) \int\limits_{(D_x)} \int\limits_{(D_z)} L(x, z)\, d\psi^*(e_z)\, d\bar{\psi}^*(e_x)$$
$$= \left(1 - \frac{\lambda^*}{\bar{\lambda}}\right) \int\limits_{(D_x)} \int\limits_{(D_z)} L(x, z)\, d\bar{\psi}^*(e_z)\, d\psi^*(e_x)$$

and, with the aid of the notation (10. 13), it is deduced that

(10. 15 b) $$\left(1 - \frac{\lambda^*}{\lambda}\right) Q(\psi^*) = \left(1 - \frac{\lambda^*}{\bar{\lambda}}\right) Q(\bar{\psi}^*).$$

Now, inasmuch as $\psi^* \neq 0$ and $L(x, y)$ has been assumed definite, we have $Q(\psi^*) \neq 0$. On the other hand, by virtue of the symmetry of $L(x, y)$ and of the possible interchange of the order of integration in the repeated integral defining $Q(\psi^*)$, we have

(10. 15 c) $$Q(\psi^*) = Q(\bar{\psi}^*).$$

Thus, from (10. 15 b) it is inferred that $\lambda = \bar{\lambda}$, which implies a contradiction. Accordingly (1°) has been established.

Here and in the sequel it will be understood that only those solutions ψ^* *of the homogeneous problem* (10. 14) *are considered for which the order of integration in* $Q(\psi^*)$ (*cf.* (10. 13) *is immaterial, the integral*

(10. 16) $$\int\limits_{(D_x)} L(x)\, d\psi^*(e_x)$$

exists, while for the functions $\psi_1^*(\geqq 0)$, $\psi_2^*(\geqq 0)$, *from the decomposition* $\psi^* = \psi_1^* - \psi_2^*$, *we have*

(10. 16 a) $$\psi_1^* = 0, \quad \psi_2^* = 0 \qquad (\text{on frontier of } (D_x)).$$

Incidentally, the latter condition is certainly satisfied when ψ^* is absolutely continuous $\{u^*\}$, inasmuch as $u^*(e_x)$ being regular with respect to the frontier of (D_x), ψ^* will vanish on the frontier of (D_x) together with u^*.

The above conditions enable rigorous justification of the steps involved in the proof of (1°). Thus, it is noted that ${}_1F^*$, ${}_2F^*$ are zero on the frontier of (D_x) in so far as (10. 16 a) holds — this enables application of the identity (10. 12 a). On the other hand, the inversions (10. 15), (10. 15 a) are possible, in accordance with Theorem 10. 1, in view of (10. 16).

To prove (2°), suppose that for non real λ^*, λ, on the same side of the axis of reals, the homogeneous problem has n and $m (> n)$ distinct solutions, respectively:

$$\psi_1^*, \ldots \psi_n^* \text{ (for } \lambda^*); \; \psi_1, \ldots \psi_m \qquad \text{(for } \lambda).$$

'Distinct', here and throughout, means of course 'linearly independent'. We form solutions for λ by writing

$$\psi = \sum_1^m c_j \psi_j,$$

where the c_j are constants (not all zero). One has

$$\psi(e_y) - \lambda^* \int_{(D_x)} \int_{e_y} L(x, y)\, d\, u^*(e_y)\, d\, \psi(e_x) = \left(1 - \frac{\lambda^*}{\lambda}\right) \psi(e_y) \equiv {}_1F^*(e_y),$$

$$\overline{\psi}(e_y) - \lambda^* \int_{(D_x)} \int_{e_y} L(x, y)\, d u^*(e_y)\, d\, \overline{\psi}(e_y) = \left(1 - \frac{\lambda^*}{\overline{\lambda}}\right) \overline{\psi}(e_y) \equiv {}_2F^*(e_y).$$

Together with the ψ_j the function ψ satisfies (10. 16), (10. 16 a). Hence, on one hand,

$$O^*(\lambda^*, e_x/{}_1F^*) = \psi(e_x) + \Phi_1(e_x),$$

$$O^*(\lambda^*, e_x/{}_2F^*) = \overline{\psi}(e_x) + \Phi_2(e_x),$$

where Φ_1, Φ_2 are certain solutions for λ^*; on the other hand, in consequence of identity (10. 12 a), applicable since the real and imaginary parts of ${}_1F^*$, ${}_2F^*$ satisfy (10. 16), (10. 16 a),

$$\left(1 - \frac{\lambda^*}{\lambda}\right) \int_{(D_x)} \int_{(D_z)} L(x, z)\, d\, \psi(e_z)\, [d\, \overline{\psi}(e_x) + d\, \Phi_2(e_x)]$$

$$= \left(1 - \frac{\lambda^*}{\overline{\lambda}}\right) \int_{(D_x)} \int_{(D_z)} L(x, z)\, d\, \overline{\psi}(e_z)\, [d\, \psi(e_x) + d\, \Phi_1(e_x)].$$

Since $m > n$ one may choose the c_j so that

$$\int_{(D_x)}\int_{(D_z)} L(x, z)\, d\,\overline{\psi}(e_z)\, d\,\Phi(e_x) = 0 \tag{10.17}$$

for all solutions Φ (for λ^*) of the homogeneous problem. We then obtain

$$\left(1 - \frac{\lambda^*}{\lambda}\right)\int_{(D_x)}\int_{(D_z)} L(x, z)\, d\,\psi(e_z)\, d\,\Phi_2(e_x) + \left(1 - \frac{\lambda^*}{\lambda}\right) Q(\psi) = \left(1 - \frac{\lambda^*}{\overline{\lambda}}\right) Q(\psi),$$

inasmuch as $Q(\overline{\psi}) = Q(\psi)$ (cf. (10. 15 c)). Accordingly

$$\left(\frac{1}{\lambda^*} - \frac{1}{\lambda}\right)\int_{(D_x)}\int_{(D_z)} L(x, z)\, d\,\psi(e_z)\, d\,\Phi_2(e_x) = \left(\frac{1}{\lambda} - \frac{1}{\overline{\lambda}}\right) Q(\psi). \tag{10.18}$$

Now, Φ_2 being a solution of the homogeneous problem for λ^*, one obtains

$$\overline{\Phi}_2(e_y) - \overline{\lambda}^* \int_{(D_x)}\int_{e_y} L(x, y)\, d\,u^*(e_y)\, d\,\overline{\Phi}_2(e_x) = 0$$

and

$$\overline{\Phi}_2(e_y) - \lambda^* \int_{(D_x)}\int_{e_y} L(x, y)\, d\,u^*(e_y)\, d\,\overline{\Phi}_2(e_x) = \left(1 - \frac{\lambda^*}{\overline{\lambda}^*}\right)\overline{\Phi}_2(e_y) \equiv {}_3F^*(e_y).$$

Since $\psi(e_x)$ and $O^*(\lambda^*, e_x/{}_2F^*)$ satisfy conditions of the form (10. 16), (10. 16 a), the functions $\overline{\Phi}_2(e_y)$, ${}_3F^*(e_y)$ also satisfy these conditions. Hence Theorem 10.1 is applicable to the above relation, yielding

$$O^*(\lambda^*, e_y/{}_3F) = \overline{\Phi}_2(e_y) + \Phi_3(e_y),$$

where Φ_3 is a solution of the homogeneous problem for λ^*,

$$O^*(\lambda^*, e_y/{}_2F) = \overline{\psi}(e_y) + \Phi_2(e_y)$$

(from the preceding) and

$$\left(1 - \frac{\lambda^*}{\overline{\lambda}}\right)\int_{(D_x)}\int_{(D_z)} L(x, z)\, d\,\overline{\psi}(e_z)\,[d\,\overline{\Phi}_2(e_x) + d\,\Phi_3(e_x)]$$

$$= \left(1 - \frac{\lambda^*}{\overline{\lambda}^*}\right)\int_{(D_x)}\int_{(D_z)} L(x, z)\, d\,\overline{\Phi}_2(e_z)\,[d\,\overline{\psi}(e_x) + d\,\Phi_2(e_x)].$$

On making use of (10. 17) it is inferred that

$$\left(1-\frac{\lambda^*}{\lambda}\right)\int_{(D_x)}\int_{(D_z)} L(x, z)\, d\,\overline{\psi}(e_z)\, d\,\overline{\Phi}_2(e_x)$$

$$= \left(1-\frac{\lambda^*}{\bar{\lambda}^*}\right)\int_{(D_x)}\int_{(D_z)} L(x, z)\, d\,\overline{\Phi}_2(e_z)\, d\,\overline{\psi}(e_x) + \left(1-\frac{\lambda^*}{\bar{\lambda}^*}\right) Q(\Phi_2).$$

The two double integrals here displayed are identical in view of the symmetry of $L(x, z)$ and of the possible change of order of integration. Hence

$$\left(\frac{1}{\bar{\lambda}^*}-\frac{1}{\lambda}\right)\int_{(D_x)}\int_{(D_z)} L(x, z)\, d\,\overline{\psi}(e_z)\, d\,\overline{\Phi}_2(e_x) = \left(\frac{1}{\lambda^*}-\frac{1}{\bar{\lambda}^*}\right) Q(\Phi_2)$$

and, by virtue of (10. 18)

$$\left(\frac{1}{\bar{\lambda}}-\frac{1}{\lambda}\right) Q(\psi) = \left(\frac{1}{\lambda^*}-\frac{1}{\bar{\lambda}^*}\right) Q(\Phi_2). \tag{10. 19}$$

$L(x, y)$ is definite and $Q(\psi) \neq 0$. If $\Phi_2 \neq 0$, then $Q(\Phi_2)$ will have the sign of $Q(\psi)$. Accordingly, (10. 19) presents a contradiction, inasmuch as the numbers

$$\frac{1}{\bar{\lambda}}-\frac{1}{\lambda},\quad \frac{1}{\lambda^*}-\frac{1}{\bar{\lambda}^*}$$

are of opposite sign. The case when λ, λ^* are on opposite sides of the axis of reals is covered by a remark analogous to that used for a similar occasion in (C). Thus, part (2°) of the theorem has been established.

The above proofs of (1°), (2°) are partly analogous to certain developments in (C; Chapter II).

As we turn to the demonstration of (3°), (4°), lines of procedure are suggested to us by (T; pp. 604—607). Thus, to prove (3°) we first make the observation that if λ_1 is non real the conclusion of (3°) is a consequence of (1°). Hence, if (3°) were not true there would be on hand a real λ_1, in O, for which (10. 14) has no solutions ($\neq 0$), while for some non real λ there exists a function ψ^*, distinct from zero, satisfying (10. 14) (for this value of λ). Accordingly

$$\psi^*(e_y) - \lambda_1 \int_{(D_x)}\int_{e_y} L(x, y)\, d\,u^*(e_y)\, d\,\psi^*(e_x) = \left(1-\frac{\lambda_1}{\lambda}\right)\psi^*(e_y) \equiv {}_1F^*(e_y).$$

The equation obtained by replacing, here, ${}_1F^*$ by zero has no solutions distinct from zero. Hence in view of Theorem 10. 1

$$\psi^*(e_y) = O^*(\lambda_1, e_y/{}_1F^*).$$

We also have

$$\overline{\psi}^*(e_y) - \lambda_1 \int\limits_{(D_x)} \int\limits_{e_y} L(x, y)\, d\, u^*(e_y)\, d\, \overline{\psi}^*(e_x) = \left(1 - \frac{\lambda_1}{\overline{\lambda}}\right) \overline{\psi}^*(e_y) = {}_2F^*(e_y)$$

and

$$\overline{\psi}^*(e_y) = O^*(\lambda_1, e_y/{}_2F^*).$$

By virtue of identity (10. 12 a)

$$\left(1 - \frac{\lambda_1}{\lambda}\right) \int\limits_{(D_x)} \int\limits_{(D_z)} L(x, z)\, d\, \psi^*(e_z)\, d\, \overline{\psi}^*(e_x) = \left(1 - \frac{\lambda_1}{\overline{\lambda}}\right) \int\limits_{(D_x)} \int\limits_{(D_z)} L(x, z)\, d\, \overline{\psi}^*(e_z)\, d\, \psi^*(e_x);$$

that is,

$$\left(1 - \frac{\lambda_1}{\lambda}\right) Q(\psi^*) = \left(1 - \frac{\lambda_1}{\overline{\lambda}}\right) Q(\overline{\psi}^*) = \left(1 - \frac{\lambda_1}{\overline{\lambda}}\right) Q(\psi^*).$$

Here $Q(\psi^*) \neq 0$. We arrive at a contradiction, which establishes (3°).

To demonstrate (4°) (compare with (T; pp. 606—607)) suppose that (4°) does not hold. Then there exists a real value λ, in O, for which there are m distinct solutions; for these one has

$$\psi_j(e_y) = \lambda \int\limits_{(D_x)} \int\limits_{e_y} L(x, y)\, d\, u^*(e_y)\, d\, \psi_j^*(e_x) \qquad (j = 1, \ldots m),$$

while for a non real λ^* there are n, $n > m$, distinct solutions for which

$$\psi_\nu^*(e_y) = \lambda^* \int\limits_{(D_x)} \int\limits_{e_y} L(x, y)\, d\, u^*(e_y)\, d\, \psi_\nu^*(e_x) \qquad (\nu = 1, \ldots n).$$

The ψ_j may be considered to be real-valued. We construct a solution (for λ^*)

$$\psi^*(e_y) = \sum_1^n c_\nu\, \psi_\nu^*(e_y),$$

where the c_ν are constants, not all zero, chosen so that

$$\int_{(D_x)} \int_{(D_z)} L(x, z)\, d\, \psi^*(e_z)\, d\, \overline{\psi}^*(e_x) = 0 \tag{10. 20}$$

for *all* solutions ψ of the homogeneous problem for the value λ. In fact,

$$\psi(e_x) = \sum_1^m d_j\, \psi_j(e_x)$$

so that (10. 20) is seen to hold if

$$\sum_{j=1}^m \sum_{\nu=1}^n c_\nu\, \overline{d}_j \int_{(D_x)} \int_{(D_z)} L(x, z)\, d\, \psi_\nu^*(e_z)\, d\, \overline{\psi}_j(e_x) = 0$$

for all constants d_j, that is provided

$$\sum_{\nu=1}^n c_\nu \int_{(D_x)} \int_{(D_z)} L(x, z)\, d\, \psi_\nu^*(e_z)\, d\, \overline{\psi}_j(e_x) = 0 \qquad (j = 1, \ldots m);$$

the latter relations can be secured in so far as it has been assumed that $n > m$.

We have

$$\psi^*(e_y) - \lambda^* \int_{(D_x)} \int_{e_y} L(x, y)\, d\, u^*(e_y)\, d\, \psi^*(e_x) = 0,$$

$$\overline{\psi}^*(e_y) - \overline{\lambda}^* \int_{(D_x)} \int_{e_y} L(x, y)\, d\, u^*(e_y)\, d\, \overline{\psi}^*(e_x) = 0;$$

thus

$$\psi^*(e_y) - \lambda \int_{(D_x)} \int_{e_y} L(x, y)\, d\, u^*(e_y)\, d\, \psi^*(e_x) = \left(1 - \frac{\lambda}{\lambda^*}\right) \psi^*(e_y) = F(e_y),$$

$$\overline{\psi}^*(e_y) - \lambda \int_{(D_x)} \int_{e_y} L(x, y)\, d\, u^*(e_y)\, d\, \overline{\psi}^*(e_x) = \left(1 - \frac{\lambda}{\overline{\lambda}^*}\right) \overline{\psi}^*(e_y) = \bar{F}(e_y)$$

and, by virtue of Theorem 10. 1,

$$\psi^*(e_y) = O^*(\lambda,\ e_y/F) + \gamma_1(e_y),$$

$$\overline{\psi}^*(e_y) = O^*(\lambda,\ e_y/\bar{F}) + \gamma_2(e_y),$$

where γ_1, γ_2 are certain solutions of (10. 14) for λ, satisfying conditions of the form (10. 16), (10. 16 a) (since ψ^*, $O^*(\lambda, e_y/F)$ do). Inasmuch as

$$O^*(\lambda, e_y/\bar{F}) = \bar{O}^*(\lambda, e_y/F),$$

one has $\gamma_2(e_y) = \bar{\gamma}_1(e_y)$. In consequence of the identity (10. 12 a)

$$\int\limits_{(D_x)}\int\limits_{(D_z)} L(x, z)\, d\, F(e_z)\, d\, O^*(\lambda, e_x/\bar{F}) = \int\limits_{(D_x)}\int\limits_{(D_z)} L(x, z)\, d\, \bar{F}(e_z)\, d\, O^*(\lambda, e_x/F)$$

and

$$\left(1 - \frac{\lambda}{\lambda^*}\right)\int\limits_{(D_x)}\int\limits_{(D_z)} L(x, z)\, d\, \psi^*(e_z)\, d\, [\bar{\psi}^*(e_x) - \bar{\gamma}_1(e_x)]$$

$$= \left(1 - \frac{\lambda}{\bar{\lambda}^*}\right)\int\limits_{(D_x)}\int\limits_{(D_z)} L(x, z)\, d\, \bar{\psi}^*(e_z)\, d\, [\psi^*(e_x) - \gamma_1(e_x)].$$

Whence, in view of the remark in connection with (10. 20),

$$\left(1 - \frac{\lambda}{\lambda^*}\right) Q(\psi^*) = \left(1 - \frac{\lambda}{\bar{\lambda}^*}\right) Q(\psi^*);$$

here $\psi^* \neq 0$ and hence $Q(\psi^*) \neq 0$; consequently λ^* must be real contrary to the previously made supposition. This completes the proof of (4°) and of the theorem.

11. Operators O (continued).

It will be said that $L(x, y)$ is of class I in a subset O' of a set O if for every λ in O' the homogeneous problem (10. 14) has no solutions distinct from zero.

A class of additive functions (possibly complex-valued) of $\{B\}$-sets will be said to form a *class* E if (1°) every $F(e_x)$ of the class vanishes on the frontier of (D_x), (2°) every $F(e_x)$ of the class is 'admissible' with respect to $L(x, y)$ (cf. (10. 3 c)) and (3°) for every pair of functions, F_1, F_2, of the class the order of integration in the repeated integral

$$\int\limits_{(D_x)}\int\limits_{(D_z)} L(x, z)\, d\, F_1(e_z)\, d\, F_2(e_x) \tag{11. 1}$$

is immaterial. The above is implied to refer to the non negative components in the decompositions (into differences) of the real and imaginary parts of the functions involved.

Theorem 11. 1. *Let $u^*(e_x)$ be regular (Definition 8. 1) with respect to the frontier of (D_x). In order that a kernel $L(x, y)$ should be of class I in a subset O' of a set O it is necessary that*

$$\int_{(D_x)}\int_{(D_z)} L(x, z)\, d\left[\int_{(D_s)}\int_{e_z} L(s, z)\, d\, u^*(e_z)\, d\, F_1(e_s)\right] d\, F_2(e_x) \tag{11. 2}$$
$$= \int_{(D_x)}\int_{(D_z)} L(x, z)\, d\left[\int_{(D_s)}\int_{e_z} L(s, z)\, d\, u^*(e_z)\, d\, F_2(e_s)\right] d\, F_1(e_x)$$

for all F_1, F_2 of class E such that the functions

$$\int_{(D_x)}\int_{e_y} L(x, y)\, d\, u^*(e_y)\, d\, F_1(e_x),\quad \int_{(D_x)}\int_{e_y} L(x, y)\, d\, u^*(e_y)\, d\, F_2(e_x) \tag{11. 2 a}$$

are also of class E. Conversely, if (11. 2) holds for all $F_1, F_2 < E$ such that the functions (11. 2 a) $< E$ and if $L(x, y)$ is definite (Theorem 10. 2), the kernel $L(x, y)$ will be of class I for all non real values λ.

To prove the first part of the theorem let F_1, F_2 be a pair of functions as described subsequent to (11. 2). Then the functions G_1, G_2, defined by the relations

$$\begin{aligned} F_1(e_y) - \lambda \int_{(D_x)}\int_{e_y} L(x, y)\, d\, u^*(e_y)\, d\, F_1(e_x) &= G_1(e_y), \\ F_2(e_y) - \lambda \int_{(D_x)}\int_{e_y} L(x, y)\, d\, u^*(e_y)\, d\, F_2(e_x) &= G_2(e_y) \end{aligned} \tag{11. 3}$$

(λ in O'), will belong to a class E, inasmuch as the G_i are differences of functions of class E. Under the supposition that $L(x, y)$ is of class I in O', inversion of (11. 3) with the aid of Theorem 10. 1 is possible, yielding

$$F_1(e_x) = O^*(\lambda, e_x/G_1),\quad F_2(e_x) = O^*(\lambda, e_x/G_2).$$

Hence, the requisite conditions of this theorem being satisfied, the identity (10. 12 a) takes the form (when applied to G_1, G_2)

$$\int_{(D_x)}\int_{(D_z)} L(x, z)\, d\, G_1(e_z)\, d\, F_2(e_x) = \int_{(D_x)}\int_{(D_z)} L(x, z)\, d\, G_2(e_z)\, d\, F_1(e_x).$$

Substituting, here, the expressions for the G_i from (11. 3), we obtain

$$(11.\ 3\,\mathrm{a})\qquad \int\limits_{(D_x)}\Bigg\{\int\limits_{(D_z)} L(x, z)\Big[d\,F_1(e_z) - \lambda\, d \int\limits_{(D_s)}\int\limits_{e_z} L(s, z)\, d\,u^*(e_z)\, d\,F_1(e_s)\Big]\Bigg\}\, d\,F_2(e_x)$$

$$= \int\limits_{(D_x)}\Bigg\{\int\limits_{(D_z)} L(x, z)\Big[d\,F_2(e_z) - \lambda\, d \int\limits_{(D_s)}\int\limits_{e_z} L(s, z)\, d\,u^*(e_z)\, d\,F_2(e_s)\Big]\Bigg\}\, d\,F_1(e_x).$$

Now

$$\int\limits_{(D_x)}\int\limits_{(D_z)} L(x, z)\, d\,F_1(e_z)\, d\,F_2(e_x) = \int\limits_{(D_x)}\int\limits_{(D_z)} L(x, z)\, d\,F_2(e_z)\, d\,F_1(e_x);$$

this is established by interchanging x, z in the first member (for instance), by noting that $L(z, x) = L(x, z)$ and by changing the order of integration, the latter operation being permissible since F_1, F_2 are of class E. From (11. 3 a) one then derives the relation (11. 2).

Suppose now that (11. 2) *holds for all* F_1, F_2 *of* E *for which the functions* (11. 2 a) *belong to* E. If the conclusion of the latter part of the theorem is not true then there exists a non real value λ and a solution $\Phi(e_y)$, $\neq 0$, of the homogeneous problem (10. 14) for this λ. In this connection, it is to be recalled that a solution of (10. 14) is always implied to satisfy conditions of the form stated in conjunction with (10. 16), (10. 16 a). One has

$$(11.\ 4)\qquad \Phi(e_y) - \lambda \int\limits_{(D_x)}\int\limits_{e_y} L(x, y)\, d\,u^*(e_y)\, d\,\Phi(e_x) = 0,$$

$$(11.\ 4\,\mathrm{a})\qquad \bar{\Phi}(e_y) - \bar{\lambda} \int\limits_{(D_x)}\int\limits_{e_y} L(x, y)\, d\,u^*(e_y)\, d\,\bar{\Phi}(e_x) = 0.$$

Using notation (10. 13) we have

$$(11.\ 5)\qquad Q(\Phi) = \int\limits_{(D_s)}\int\limits_{(D_y)} L(s, y)\, d\,\Phi(e_y)\, d\,\bar{\Phi}(e_s) = \int\limits_{(D_s)}\int\limits_{(D_y)} L(s, y)\, d\,\bar{\Phi}(e_y)\, d\,\Phi(e_s).$$

Substitution of (11. 4) in the second member and of (11. 4 a) in the third member of (11. 5) will yield

$$(11.\ 5\,\mathrm{a})\qquad \frac{1}{\lambda}\, Q(\Phi) = \int\limits_{(D_s)}\Bigg\{\int\limits_{(D_y)} L(s, y)\, d\Big[\int\limits_{(D_x)}\int\limits_{e_y} L(x, y)\, d\,u^*(e_y)\, d\,\Phi(e_x)\Big]\Bigg\}\, d\,\bar{\Phi}(e_s),$$

$$\frac{1}{\bar{\lambda}} Q(\Phi) = \int\limits_{(D_s)}\left\{\int\limits_{(D_y)} L(s, y)\, d\left[\int\limits_{(D_x)}\int\limits_{e_y} L(x, y)\, d\, u^*(e_y)\, d\, \bar{\Phi}(e_x)\right]\right\} d\, \Phi(e_s).$$

Now Φ, $\bar{\Phi}$ are members of E; in view of (11. 4), (11. 4 a) the functions (11. 2 a), formed with

$$F_1 = \Phi, \quad F_2 = \bar{\Phi}$$

will also be members of E. In consequence of the previous hypothesis (11. 2) this would imply that the second members in (11. 5 a) are equal; hence

$$\left(\frac{1}{\lambda} - \frac{1}{\bar{\lambda}}\right) Q(\Phi) = 0$$

and, necessarily, $Q(\Phi) = 0$. With $L(x, y)$ definite, from the latter equality it is inferred that $\Phi = 0$, contrary to a provious supposition. The contradiction establishes the theorem.

Theorem 11. 1 is of the type of a result given in (C; pp. 75—77); however, essential features of difference are to be noted.

We observe that (11. 2) is a condition allowing interchange of certain limiting processes.

We turn now to equations (7. 3), *for kernels* (T). An equation (7. 3) is approximated by (7. 1). With λ in a set O we have (8. 1 a) satisfied for an infinite sequence $\{n_s\}$; for a subsequence $\{m_s\}$ of $\{n_s\}$ a spectrum θ is defined in (10. 6). Correspondingly, let $\varphi_{m_s}(x)$ be the solution (λ in O) of (7. 1); thus

$$\varphi_{m_s}(x) = \lambda \int\limits_{(D_y)} L_{m_s}(x, y)\, \varphi_{m_s}(y)\, d\, u^*(e_y) + f_{m_s}(x),$$

where $f_{m_s}(x)$ is a function approximating (for $s \to \infty$) to $f(x)$ in the sense specified before. The representation (9. 10) of a solution $\varphi(x)$ of (7. 3), formed with the spectrum θ mentioned above, yields an operator $O(\lambda, x/\ldots)$,

$$(11.\ 6) \quad \lim_s \varphi_{m_s}(x) = \varphi(x) = O(\lambda, x/f) = f(x) + \lambda \int\limits_{(D_y)} L(x, y) f(y)\, d\, u^*(e_y)$$
$$- \lambda^2 \int\limits_{-\infty}^{\infty} \frac{1}{\lambda - \mu}\, d_\mu \int\limits_{(D_y)}\int\limits_{(D_z)} \theta(y, z/\mu) f(y)\, L(z, x)\, d\, u^*(e_z)\, d\, u^*(e_y).$$

The sequeuce $\{m_s\}$ and, hence, θ is independent of f. Thus *the operator* $O(\lambda, x/\ldots)$ *is independent of* f.

For λ in the set O the operator $O(\lambda, x/\ldots)$ yields particular solutions of non homogeneous problems; in fact, *for every function f, such that $|f|^2$ is integrable $\{B, u^*\}$, a solution almost everywhere $\{u^*\{$ of*

$$\varphi(x) = \lambda \int_{(D_y)} L(x, y)\, \varphi(y)\, d\, u^*(e_y) + f(x) \qquad (\lambda \text{ in } O)$$

is furnished by

$$\text{(11. 6 a)} \qquad \varphi(x) = O(\lambda, x/f).$$

Here and throughout functions $f(x)$, or corresponding functions differently designated, in non homogeneous problems (7. 3) will be understood to be such that the functions $f_n(x)$ (cf. (5. 3 a)), approximating to $f(x)$, are continuous in (D_x).

Consider now a pair of equations (7. 3)

$$\text{(11. 7)} \qquad \varphi(x) = \lambda \int_{(D_y)} L(x, y)\, \varphi(y)\, d\, u^*(e_y) + {}_i f(x) \qquad (i = 1, 2),$$

where the $|{}_i f|^2$ are integrable $\{B, u^*\}$. For the solutions ${}_i\varphi_n$ of the corresponding approximating equations,

$$\text{(11. 7 a)} \qquad {}_i\varphi_n(x) = \lambda \int_{(D_y)} L_n(x, y)\, {}_i\varphi_n(y)\, d\, u^*(e_y) + {}_i f_n(x) \qquad (i = 1, 2;\ \lambda \text{ in } O),$$

we obtain without difficulty the relation

$$\text{(11. 7 b)} \qquad \int_{(D_y)} {}_1\varphi_n(y)\, {}_2 f_n(y)\, d\, u^*(e_y) = \int_{(D_y)} {}_2\varphi_n(y)\, {}_1 f_n(y)\, d\, u^*(e_y).$$

In (11. 7 b) we let $n = m_s \to \infty$; on taking account of (11. 6) *one obtains the following identity, satisfied by operators $O(\lambda, x/\ldots)$,*

$$\text{(11. 8')} \qquad \int_{(D_y)} {}_2 f(y)\, O(\lambda, y/{}_1 f)\, d\, u^*(e_y) = \int_{(D_y)} {}_1 f(y)\, O(\lambda, y/{}_2 f)\, d\, u^*(e_y)$$

(λ in O); this is valid whenever ${}_1 f^2$, ${}_2 f^2$ are integrable $\{B, u^*\}$.

Passage to the limit under the integral signs in (11. 7 b) is justified in view of the inequality

$$\int_{(D_x)} |\varphi_{m_s}|^2\, d\, u^*(e_x) \leqq B^2(\lambda) \qquad (\lambda \text{ in } O),$$

taking place in accordance with Lemma 9. 1, and in consequence of the relations

$$\lim_s {}_i\varphi_{m_s}(y) = O(\lambda, y/{}_if), \quad |{}_if_{m_s}(y)| \leqq |{}_if(y)|, \quad \lim_s {}_if_{m_s}(y) = {}_if(y),$$

where the $|{}_if(y)|^2$ are integrable $\{B, u^*\}$; in fact, under these conditions Theorem 4. 3 is applicable.

Theorem 11. 2. *With $L(x, y)$ of type (T) consider the problem*

$$\varphi(x) = \lambda \int_{(D_x)} L(x, y)\, \varphi(y)\, d u^*(e_y). \tag{11. 8}$$

(1°). *If for a non real λ^* (11. 8) has $\varphi(x) = 0$ (almost everywhere $\{u^*\}$) as the only solution the same is true for all non real λ.*

[2°]. *The number of distinct solutions of (11. 8) is the same for all non real λ.*

(3°). *If for $\lambda = \lambda_1$, fixed in a set O, the problem (11. 8) has no solutions (distinct from zero almost everywhere $\{u^*\}$) the same will hold for all non real λ.*

(4°). *The number of solutions of (11. 8) for any real λ in O is equal or is greater than the number of distinct solutions for non real values of λ.*

Note. Only those solutions $\varphi(x)$ are envisaged for which $|\varphi(x)|^2$ is integrable $\{B, u^*\}$.

The proofs of (1°), (2°) may be effected with the aid of the identity (11. 8′), following closely the lines of the corresponding proofs in (C; Chapter II).

Parts (3°), (4°) of the above Theorem may be demonstrated on the basis of (11. 8′), following the lines of reasoning given in (T; pp. 604—607). We shall omit any further details of proof.

The analogue to Theorem 11. 1 for the problem (11. 8) is as follows.

In order that $L(x, y)$ be of class I in a subset O' of a set O (i. e. that (11. 8) should for every λ in O' have no solutions distinct from zero almost everywhere $\{u^*\}$), *it is necessary that the order of integration in*

$$\int_{(D_x)} \int_{(D_z)} L(x, z) f_1(x) f_2(z)\, d u^*(e_x)\, d u^*(e_z) \tag{11. 9}$$

should be immaterial for all f_1, f_2 such that the functions

$$|f_1|^2,\ |f_2|^2, \left|\int_{(D_y)} L(x, y) f_1(y)\, d u^*(e_y)\right|^2, \left|\int_{(D_y)} L(x, y) f_2(y)\, d u^*(e_y)\right|^2 \tag{11. 9a}$$

are integrable $\{B, u^*\}$.

If the condition stated in connection with (11. 9), (11. 9 a) *holds, the kernel* $L(x, y)$ *will be of class I for all non real* λ.

The above result is analogous to and constitutes an extension of the theorem in (C; pp. 75—77). To prove the first part of the theorem let $f_i(x)$ $(i = 1, 2)$ be two functions satisfying the statement in connection with (11. 9 a). Then the $q_i(x)$,

$$q_i(x) = f_i(x) - \lambda \int_{(D_y)} L(x, y) f_i(y)\, d u^*(e_y) \qquad (\lambda \text{ in } O';\ i = 1, 2), \tag{11. 10}$$

are such that the $|q_i(x)|^2$ are integrable $\{B, u^*\}$. With $L(x, y) < I$ (in O'), (11. 6 a) will yield

$$f_i(x) = O(\lambda, x/q_i)$$

(in O') and, accordingly, by virtue of (11. 8′)

$$\int_{(D_y)} q_2(y)\, O(\lambda, y/q_1)\, d u^*(e_y) = \int_{(D_y)} q_1(y)\, O(\lambda, y/q_2)\, d u^*(e_y).$$

Substituting, here, (11. 10) one obtains the necessary condition.

The sufficient condition is established precisely as in (C; pp. 75—76).

12. Some more General Kernels.

If we restrict ourselves to equations (7. 3), where $L(x, y)$ is such that the integral $L^2(y)$ (cf. (5. 4 a)) exists, while the continuity conditions imposed subsequent to (5. 4 a) on kernels of type (T) are deleted, we can still obtain the greater part of the results established above for the problem (7. 3) (homogeneous or non homogeneous). The approximating kernels $L_n(x, y)$, as defined subsequent to (5. 6), will not necessarily be continuous; however, $L_n(x, y)$ will be measurable $\{B, u^*\}$ and will be uniformly bounded with respect to (x, y) (x in (D_x), y in (D_y)). At the same time one may drop the continuity condition on $f(x)$, in (7. 3), and require only that $|f|^2$ be integrable $\{B, u^*\}$. If one then considers the approximating equation

$$\varphi_n(x) = \lambda \int_{(D_y)} L_n(x, y)\, \varphi_n(y)\, d u^*(e_y) + f(x), \tag{12. 1}$$

it is observed that the essential features of the Gunther theory will continue to hold for (12. 1); that is, a theory of Fredholm type will be applicable to (12. 1). To demonstrate this fact one needs only to justify the limiting processes (involving, in this case, determinants whose elements are repeated Lebesgue-Stieltjes integrals) involved in the usual development of the Fredholm theory. We shall not go into the details of this. On the basis of the theory, relating to (12. 1), the main part of the previously established developments for (7. 3) is extended to the present somewhat more general problem.

Suppose now that the integral $L^2(y)$ (5. 4 a) *does not exist, while the symmetric kernel* $L(x, y)$ *is merely measurable* $\{B, u^*\}$ *for* (x, y) *in* $[(D_x), (D_y)]$. Then the problem

$$\varphi(x) = \lambda \int_{(D_y)} L(x, y)\, \varphi(y)\, d\, u^*(e_y) + f(x) \tag{12. 2}$$

(with $f(x)$ zero or not; $|f|^2$ integrable $\{B, u^*\}$) could not be said to be solluble, unless some indirect additional conditions were introduced. Accordingly, let us assume that corresponding to $L(x, y)$ there is a linear operator $T_x(\xi/h(x))$ (ξ a parameter), analogous to an operator introduced by *Carleman* in (C; p. 138), for which the following five conditions hold:

$$\int_{(D_y)} |\, T_x(\xi/L(x, y))\,|^2\, d\, u^*(e_y) \tag{12. 3}$$

exists;

$$|\, T_x(\xi/L_n(x, y))\,| < \gamma(\xi, y), \tag{12. 3 a}$$

where $\gamma(\xi, y)$ is independent of n and $|\gamma(\xi, y)|^2$ is integrable (in y) $\{B, u^*\}$;

$$\lim_n T_x(\xi/L_n(x, y)) = T_x(\xi/L(x, y)); \tag{12. 3 b}$$

$$\lim_n T_x(\xi/f_n(x)) = T_x(\xi/f(x)), \tag{12. 3 c}$$

whenever $f_n \to f$ weakly (Definition 4. 1);

$$\int_{(D_y)} T_x(\xi/L_n(x, y))\, \varphi(y)\, d\, u^*(e_y) = T_x\left(\xi \Big/ \int_{(D_y)} L_n(x, y)\, \varphi(y)\, d\, u^*(e_y)\right) \tag{12. 3 d}$$

for all $\varphi(y)$ with $|\varphi|^2$ integrable $\{B, u^*\}$.

The conditions (12. 3) — (12. 3 d) are of the form of those in (T; p. 586) given for kernels therein designated as of Type 1. C.

Corresponding to (12. 1) and (12. 2) we shall have the equations

$$(12.\ 4) \qquad T_x(\xi/\varphi_n(x)) = \lambda \int_{(D_y)} T_x(\xi/L_n(x, y))\, \varphi_n(y)\, d\, u^*(e_y) + T_x(\xi/f(x)),$$

$$(12.\ 5) \qquad T_x(\xi/\varphi(x)) = \lambda \int_{(D_y)} T_x(\xi/L(x, y))\, \varphi(y)\, d\, u^*(e_y) + T_x(\xi/f(x)).$$

In view of the condition (12. 3) one may envisage solutions $\varphi(y)$, with $|\varphi|^2$ integrable $\{\beta, u^*\}$, of the problem (12. 5), as well as of the homogeneous problem

$$(12.\ 5\ \mathrm{a}) \qquad T_x(\xi/\varphi(x)) = \lambda \int_{(D_y)} T_x(\xi/L(x, y))\, \varphi(y)\, d\, u^*(e_y).$$

Consider the function

$$(12.\ 6) \qquad \Omega_n(e_x, e_y/\lambda) = \int_{e_x} \int_{e_y} \theta_n(x, y/\lambda)\, d\, u^*(e_x)\, d\, u^*(e_y),$$

where $\theta_n(x, y/\lambda)$ is the function so denoted in section 6. The function (12. 6) is identical with θ_n^{**}, introduced subsequent to Lemma 8. 3 (in this connection see (8. 16)). The symbol in the first member of (12. 6) has been brought in, in in place of θ_n^{**}, because the function in question plays a role analogous to that of a function designated as Ω in (C; Chapter IV). A number of essential features of difference is to be noted. For instance, our $\Omega_n(e_x, e_y/\lambda)$ (and, eventually, limits $\Omega(e_x, e_y/\lambda)$) is a function of $\{\beta\}$-sets e_x, e_y, while in case of (C) the corresponding functions depend on points x, y, instead.

In consequence of some of the developments subsequent to Lemma 8.3 one has

$$(12.\ 7) \qquad |\Omega_n(e_x, e_y/\lambda)| \leqq [u^*(e_x)]^{\frac{1}{2}}\, [u^*(e_y)]^{\frac{1}{2}}$$

(all real λ).

In consequence of (12. 6), (6. 1), (6. 1 a) and since

$$\psi^*_{n,k}(e_y) = \int_{e_y} \varphi_{n,k}(y)\, d\, u^*(e_y)$$

we have

$$(12.8)\qquad \begin{aligned} \Omega_n(e_x, e_y/\lambda) &= \sum_{0<\lambda_{n,k}<\lambda} \psi^*_{n,k}(e_x)\,\psi^*_{n,k}(e_y) \qquad &(\text{for } \lambda > 0),\\ \Omega_n(e_x, e_y/\lambda) &= -\sum_{\lambda\leqq\lambda_{n,k}<0} \psi^*_{n,k}(e_x)\,\psi^*_{n,k}(e_y) \qquad &(\text{for } \lambda < 0) \end{aligned}$$

and $\Omega_n = 0$ for $\lambda = 0$.

Subdividing the interval $(-l, l)(l > 0)$, as described subsequent to (6. 4), we obtain

$$V_n = \sum_{j=1}^{q} | \Omega_n(e_x, e_y/l_j) - \Omega_n(e_x, e_y/l_{j-1}) | \leqq \sum_k^{(1)} | \psi^*_{n,k}(e_x)\,\psi^*_{n,k}(e_y) |,$$

where the summation in the last member is over values k for which $-l \leqq \lambda_{n,k} < l$. Hence, in consequence of (8. 3 b),

$$V_n^2 \leqq \sum_k | \psi^*_{n,k}(e_x) |^2 \sum_k | \psi^*_{n,k}(e_y) |^2 \leqq u^*(e_x)\, u^*(e_y)$$

and

$$(12.9)\qquad V_{-l}^{l}\, \Omega_n(e_x, e_y/\lambda) \leqq [u^*(e_x)]^{\frac{1}{2}} [u^*(e_y)]^{\frac{1}{2}}.$$

Let $V_n(e_x, e_y/\lambda)$ be the variation, on $(-l, \lambda)$, for λ in $(-l, l)$ of the function $\Omega_n(e_x, e_y/\lambda)$. On writing

$$(12.9\,\text{a})\qquad \begin{aligned} \Omega_n(e_x, e_y/\lambda) &= \Omega_{n,1}(e_x, e_y/\lambda) - \Omega_{n,2}(e_x, e_y/\lambda),\\ 2\,\Omega_{n,1}(e_x, e_y/\lambda) &= V_n(e_x, e_y/\lambda) + \Omega_n(e_x, e_y/\lambda),\\ 2\,\Omega_{n,2}(e_x, e_y/\lambda) &= V_n(e_x, e_y/\lambda) - \Omega_n(e_x, e_y/\lambda), \end{aligned}$$

it is observed that the functions $\Omega_{n,i}(e_x, e_y/\lambda)$ $(i = 1, 2)$ are monotone non decreasing in λ (on $(-l, l)$); moreover, in view of (12. 7), (12. 9),

$$(12.9\,\text{b})\qquad | \Omega_{n,i}(e_x, e_y/\lambda) | \leqq [u^*(e_x)]^{\frac{1}{2}} [u^*(e_y)]^{\frac{1}{2}}$$

$(i = 1, 2;\ \lambda$ on $(-l, l))$. Using a suitable adaptation of the De la V. Poussin-Frostman theorem and of (12; 9 b) we infer existence of a subsequence (n_j) and of functions $\Omega_{:1}$, $\Omega_{:2}$ so that

$$\lim_{n_j} \Omega_{n_j:i}(e_x, e_y/\lambda) = \Omega_{:i}(e_x, e_y/\lambda) \qquad (\lambda \text{ on } (-l, l))$$

for all $\{B\}$-sets e_x, e_y (in (D_x), (D_y)) on whose frontiers u^* vanishes. *We shall have*

$$\lim_{n_j} \Omega_{n_j}(e_x, e_y/\lambda) = \Omega(e_x, e_y/\lambda) = \Omega_{:1} - \Omega_{:2} \tag{12. 10}$$

for sets e_x, e_y *on whose frontiers* u^* *vanishes; furthermore,*

$$V_\alpha^\beta \Omega(e_x, e_y/\lambda) \leqq [u^*(e_x)]^{\frac{1}{2}} [u^*(e_y)]^{\frac{1}{2}} \tag{12. 10 a}$$

(*for any real interval* $(-l, l)$) *and*

$$|\Omega(e_x, e_y/\lambda)| \leqq [u^*(e_x)]^{\frac{1}{2}} [u^*(e_y)]^{\frac{1}{2}}. \tag{12. 10 b}$$

Convergence in (12. 10) is first established for any interval $(-l, l)$ and then extended to $(-\infty, +\infty)$; (12. 10 b) implies that Ω, together with Ω_n, is absolutely continuous $\{u^*\}$ in e_x and in e_y,

Using the developments of section 4, with the aid of the above function Ω the following results (up to (12. 13 b)) will be stated without details of proof.

One has

$$\lim_{n_j} \int_{(D_y)} h(y) \left[\int_{e_x} \theta_{n_j}(s, y/\lambda)\, d\, u^*(e_s)\right] d\, u^*(e_y) \tag{12. 11}$$

$$= \lim_{n_j} \int_{e_y} \left[\int_{(D_y)} \theta_{n_j}(s, y/\lambda)\, d\, u^*(e_y)\right] d\, u^*(e_s) = \int_{(D_y)} h(y)\, D_y\, \Omega(e_x, e_y/\lambda)\, d\, u^*(e_y),$$

whenever $|h|^2$ is integrable $\{B, u^*\}$; *here* D_y *is symbol of set-derivation* (regular in the sense of Lebesgue (S; 152—156), u^* being the measure function) *and yields a function of* y. When, in addition, g^2 is integrable $\{B, u^*\}$, one has

$$\lim_{n_j} \int_{(D_x)} \int_{(D_y)} \theta_{n_j}(x, y/\lambda)\, g(x)\, h(y)\, d\, u^*(e_x)\, d\, u^*(e_y) \tag{12. 11 a}$$

$$= \int_{(D_x)} g(x) \left[D_x \int_{(D_y)} D_y\, \Omega(e_x, e_y/\lambda)\, h(y)\, d\, u^*(e_y)\right] d\, u^*(e_x),$$

$$V_\alpha^\beta \int_{(D_x)} g(x) \left[D_x \int_{(D_y)} D_y\, \Omega(e_x, e_y/\lambda)\, h(y)\, d\, u^*(e_y)\right] d\, u^*(e_x) \tag{12. 11 b}$$

$$\leqq \left[\int_{(D_x)} h^2(x)\, d\, u^*(e_x)\right]^{\frac{1}{2}} \left[\int_{(D_x)} g^2(x)\, d\, u^*(e_x)\right]^{\frac{1}{2}},$$

$$\lim_{n_j} \int_{\alpha}^{\beta} \alpha(\lambda)\, d_\lambda \int_{(D_x)} \int_{(D_y)} \theta_{n_j}(x, y/\lambda)\, g(x)\, h(y)\, d\, u^*(e_x)\, d\, u^*(e_y)$$
$$= \int_{\alpha}^{\beta} \alpha(\lambda)\, d_\lambda \int_{(D_x)} g(x)\, D_x \left[\int_{(D_y)} D_y\, \Omega(e_x, e_y/\lambda)\, h(y)\, d\, u^*(e_y) \right] d\, u^*(e_x), \tag{12. 11 c}$$

whenever $\alpha(\lambda)$ is continuous on the closed finite interval (α, β). Moreover,

$$\int_{\alpha}^{\beta} \alpha(\lambda)\, d_\lambda \int_{(D_x)} g(x)\, D_x \left[\int_{(D_y)} D_y\, \Omega(e_x, e_y/\lambda)\, h(y)\, d\, u^*(e_y) \right] d\, u^*(e_x)$$
$$= \int_{\alpha}^{\beta} g(x)\, D_x \left[\int_{\alpha}^{\beta} \alpha(\lambda)\, d_\lambda \int_{(D_y)} D_y\, \Omega(e_x, e_y/\lambda)\, h(y)\, d\, u^*(e_y) \right] d\, u^*(e_x)$$
$$= \int_{(D_y)} g(x)\, D_x \left[\int_{(D_y)} D_y \left\{ \int_{\alpha}^{\beta} \alpha(\lambda)\, d_\lambda\, \Omega(e_x, e_y/\lambda) \right\} h(y)\, d\, u^*(e_y) \right] d\, u^*(e_x). \tag{12. 12}$$

The generalized *Bessel's inequality* is of the form

$$\int_{-\infty}^{\infty} d_\lambda \int_{(D_x)} h(x)\, D_x \left[\int_{(D_y)} D_y\, \Omega(e_x, e_y/\lambda)\, h(y)\, d\, u^*(e_y) \right] d\, u^*(e_x) \leqq \int_{(D_x)} h^2(x)\, d\, u^*(e_x). \tag{12. 13}$$

In agreement with *Carleman's* terminology (C; p. 136), if (12. 13) holds with the equality sign for all $h(x)$, for which $h^2(x)$ is integrable $\{B, u^*\}$, then Ω is termed *closed*; the inequality (12. 13) then could appropriately be called generalized *Perceval's identity*, in which case

$$\int_{-\infty}^{\infty} d_\lambda \int_{(D_x)} g(x)\, D_x \left[\int_{(D_y)} D_y\, \Omega(e_x, e_y/\lambda)\, h(y)\, d\, u^*(e_y) \right] d\, u^*(e_x) = \int_{(D_x)} g(x)\, h(x)\, d\, u^*(e_x), \tag{12. 13 a}$$

whenever g^2, h^2 are integrable $\{B, u^*\}$. This yields the generalized *Fourier* expansion

(12. 13 b)
$$f(x) = D_x \int_{-\infty}^{\infty} d_\lambda \int_{(D_y)} D_y\, \Omega(e_x, e_y/\lambda) f(y)\, d\, u^*(e_y)$$

almost everywhere $\{u^*\}$, for all f for which f^2 is integrable $\{B, u^*\}$.

Let $\varphi_n(x)$ be the solution, for λ in a set O (cf. (8. 1 a)), of the equation (12. 1). The inequality (9. 4) of Lemma 9. 1 will hold; thus

(12. 14)
$$\int_{(D_x)} |\varphi_n(x)|^2\, d\, u^*(e_x) \leqq B^2(\lambda) \qquad (\lambda \text{ in } O;\ n = n_1, n_2, \ldots),$$

where (cf. (9. 4), (9. 1 a) and statement subsequent to (9. 1 a))

(12. 14 a)
$$B(\lambda) = \left[1 + \frac{|\lambda|}{\delta(\lambda)}\right] \left[\int_{(D_x)} |f(x)|^2\, d\, u^*(e_x)\right]^{\frac{1}{2}}.$$

By Theorem 4.2 the sequence $\{n_j\}$ contains a subsequence $\{m_j\}$ so that the sequence $\{\varphi_{m_j}(x)\}$ converges in the weak sense (Definition 4. 1) to a function $\varphi(x)$ for which

(12. 14 b)
$$\int_{(D_x)} |\varphi(x)|^2\, d\, u^*(e_x) \leqq B^2(\lambda),$$

λ having a value fixed in O. If in (12. 4) we let $n = m_j \to \infty$, letting φ_n denote the solution of (12. 1) in consequence of the conditions (12. 3)—(12. 3 d) and of some of the theorem of section 4, it is inferred that $\varphi(x)$ *satisfies the equation* (12. 5).

The above solution $\varphi(x)$ can be represented in the form

(12. 15)
$$\varphi(x) = f(x) + \lambda D_x \int_{-\infty}^{\infty} \frac{1}{\mu - \lambda}\, d_\mu \int_{(D_y)} f(y)\, D_y\, \Omega(e_x, e_y/\mu)\, d\, u^*(e\ ) = O(\lambda, x/f)$$

almost everywhere $\{u^*\}$ in (D_x) (λ in O). This result is an extension of a formula in (C; p. 139), for λ non real, and of a result in (T; p. 600), for λ possibly real in O. The operator $O(\lambda, x/\ldots)$ can be chosen independent of f.

With $O(\lambda, x/\ldots)$ from (12. 15) in place of the operator so designated in (11. 6), *one obtains an analogue of the identity* (11. 8′), *as well as uniqueness properties, relating to the homogeneous problem* (12. 5 a) (λ *in* O), *closely similar to* *those given in Theorem* 11. 2 — the latter under the supposition that $T_x(\xi/h(x))$ is real whenever $h(x)$ (with h^2 integrable $\{B, u^*\}$) is real, while

$$\overline{T}_x(\xi/h(x)) = T_x(\xi/\bar{h}(x))$$

(cf. (T; pp. 607, 608)). There is also an analogue of the result [(11. 9), (11. 9 a)].

For the problem now under consideration one may obtain extensions of all those results in (C; Chapter IV) which are not based on continuity properties of the kernels involved.

13. Integral Equations of the first Kind.

In this section use is made of the notions and results of section 12. The problem now under consideration is similar to the one considered by Trjitzinsky in (T; pp. 611—619).

With $L(x, y)$ a kernel of the type specified in section 12 and $T_x(\xi/\ldots)$ denoting a corresponding operator, subject to (12. 3)—(12. 3 d), we consider the equation

$$\int_{(D_y)} L(x, y)\,\varphi(y)\,d\,u^*(e_y) = f(x), \tag{13. 1}$$

where f^2 *is integrable* $\{B, u^*\}$, and the corresponding problem

$$\int_{(D_y)} T_x(\xi/L(x, y))\,\varphi(y)\,d\,u^*(e_y) = T_x(\xi/f(x)). \tag{13. 2}$$

All solutions in question (of (13, 2) or (13. 3)) are implied to have squares integrable $\{B, u^*\}$.

Theorem 13. 1. (1°). *Suppose the equation*

$$\int_{(D_y)} T_x(\xi/L(x, y))\,\varphi(y)\,d\,u^*(e_y) = 0 \tag{13. 3}$$

is closed in the sense that it has no solution $\varphi(y)$ *for which*

$$\int_{(D_y)} \varphi^2(y)\,d\,u^*(e_y) \neq 0.$$

(2°). *Suppose, moreover, that for an infinite subsequence* (n_j) *of* (n) *one has*

$$\Gamma_n = \sum_k \lambda_{n,k}^2 f_{n,k}^2 \leqq A < \infty \qquad (n = n_1, n_2, \ldots), \tag{13. 4}$$

where A *is independent of* n_j $(j = 1, 2, \ldots)$ *and*

$$f_{n,k} = \int_{(D_s)} f(s)\, \varphi_{n,k}(s)\, d\,u^*(e_s). \tag{13. 4 a}$$

The equation (13. 2) *will have a unique solution* $\varphi(x)$ *with* φ^2 *integrable* $\{B, u^*\}$. *If*

$$\sum_k \lambda_{n,k}^{2+2\eta} f_{n,k}^2 \leqq B < \infty \quad (n = n_1, n_2, \ldots;\ \textit{some}\ \eta > 0), \tag{13. 4 b}$$

this solution will be representable in the form

$$\varphi(x) = D_x \int_{-\infty}^{\infty} \lambda\, d_\lambda \left[\int_{(D_s)} f(s)\, D_s\, \Omega(e_x,\ e_s/\lambda)\, d\,u^*(e_s) \right] \tag{13. 5}$$

almost everywhere $\{u^*\}$.

Uniqueness of solution of (13. 2) follows from the fact that a difference of two distinct solutions, that is of solutions differing on a set of points of positive $\{u^*\}$ measure, would be a solution of (13. 3), contrary to hypothesis.

Adapting the developments of (C), leading to a result in (C; p. 142), to our problem we infer without difficulty that condition (1°) of the Theorem implies that $\Omega(e_x, e_y/\lambda)$ is 'closed'; accordingly, (12. 13) will hold with the equality sign, as well as (12. 13 b) — for all $h(x)$, $f(x)$, whose squares are integrable $\{B, u^*\}$. This fact does not necessarily imply that $\Omega_n(e_x, e_y/\lambda)$, that is the sequence $\varphi_{n,1}(x)$, $\varphi_{n,2}(x), \ldots$, is closed.

We define $\varphi_n(x)$ by the relation

$$\varphi_n(e_x) = \int_{e_x} \varphi_n(x)\, d\,u^*(e_x) = \int_{-\infty}^{\infty} \lambda\, d_\lambda \left[\int_{(D_s)} f(s)\, D_s\, \Omega_n(e_x,\ e_s/\lambda)\, d\,u^*(e_s) \right]. \tag{13.6}$$

In view of (12. 6), (6. 1) and (6. 1 a)

$$\varphi_n(e_x) = \sum_k \lambda_{n,k} f_{n,k} \int_{e_x} \varphi_{n,k}(x)\, d\,u^*(e_x) \qquad (n = n_j). \tag{13. 6 a}$$

The series, here involved, and the integral of the last member in (13. 6) converge because $\varphi_{n,1}(x)$, $\varphi_{n,2}(x)$, ... is an orthonormal sequence, while the series in (13. 4) is implied to be convergent by hypothesis. It will be essential for our purposes to consider the situation in greater detail.

As remarked before, the integral

$$\int_{e_x} \varphi_{n,k}(x)\, d\, u^*(e_x)$$

is the 'Fourier' coefficient (with respect to $\varphi_{n,k}(x)$) of $q_{e_x}(x)$, the latter function being defined as unity in e_x and as zero in $(D_x) - e_x$. Hence by Bessel's inequality

$$\sum_k \left[\int_{e_x} \varphi_{n,k}(x)\, d\, u^*(e_x)\right]^2 \leqq \int_{(D_x)} q_{e_x}^2(x)\, d\, u^*(e_x) = u^*(e_x).$$

Thus on taking account of (13. 4) from (13. 6a) it is inferred that

$$(13.\ 6\,\mathrm{b}) \qquad |\varphi_n(e_x)|^2 \leqq \sum_k \lambda_{n,k}^2 f_{n,k}^2 \sum_k \left[\int_{e_x} \varphi_{n,k}(x)\, d\, u^*(e_x)\right]^2 \leqq A\, u^*(e_x)$$

$(n = n_1, n_2, \ldots)$. Consequently, in view of the *De la V. Poussin-Frostman* theorem, for an infinite subsequence (m_j) of (n_j) one has

$$(13.\ 7) \qquad \lim \varphi_{m_j}(e_x) = \varphi(e_x), \quad |\varphi(e_x)|^2 \leqq A\, u^*(e_x)$$

for all $\{B\}$-sets (in D_x)) on whose frontiers u^* vanishes, the limiting additive function of $\{B\}$-sets $\varphi(e_x)$ being defined for all $\{B\}$-sets e_x in (D_x); by the second relation (13. 7) $\varphi(e_x)$ *is absolutely continuous* $\{u^*\}$. Hence

$$(13.\ 7\,\mathrm{a}) \qquad \varphi(x) = D_x\, \varphi(e_x)$$

is a function defined almost everywhere $\{u^*\}$, measurable $\{B, u^*\}$ over (D_x); furthermore,

$$(13.\ 7\,\mathrm{b}) \qquad \varphi(e_x) = \int_{e_x} \varphi(x)\, d\, u^*(e_x).$$

We shall prove the following Lemma.

Lemma 13. 1. *Let* $\psi_1(x)$, $\psi_2(x)$, ... *be a sequence ortho normal in* (D_x) (*the measure function being* u^*). *Let* $c_1^2 + c_2^2 + \cdots = c^2 < \infty$. *The function* $\psi(x) = D_x \psi(e_x)$, *where*

$$\psi(e_x) = \sum_1^\infty c_\nu \int_{e_x} \psi_\nu(x)\, d\, u^*(e_x),$$

has the properties

$$(1^\circ) \qquad \int_{(D_x)} \psi(x)\, \psi_\nu(x)\, d\, u^*(e_x) = c_\nu \qquad (\nu = 1, 2, \ldots),$$

$$(2^\circ) \qquad \psi(x) \sim \Sigma\, c_\nu\, \psi_\nu(x) \qquad (\textit{in } (D_x);\ \textit{also see } (\beta),\ \textit{below}).$$

Note. It is to be recalled that $\sim$ denotes convergence in the mean square.

The series representing $\psi(e_x)$ converges in consequence of previous remarks; since

$$|\psi(e_x)| \leqq c\,[u^*(e_x)]^{\frac{1}{2}},$$

existence of $\psi(x)$ is likewise assured. For the function

$$\psi(j, x) = \sum_1^j c_\nu\, \psi_\nu(x)$$

we have

$$\int_{(D_x)} \psi^2(j, x)\, d\, u^*(e_x) = c_1^2 + \cdots + c_j^2 \leqq c^2;$$

moreover, in consequence of the definition of $\psi(e_x)$

$$\int_{e_x} \psi(j, x)\, d\, u^*(e_x) = \sum_1^j c_\nu \int_{e_x} \psi_\nu(x)\, d\, u^*(e_x) \to \psi(e_x) = \int_{e_x} \psi(x)\, d\, u^*(e_x)$$

(as $j \to \infty$). Hence

$$\psi(j, x) \to \psi(x) \qquad (\text{as } j \to \infty)$$

in the *weak* sense; in view of (4. 6)

$$(\alpha) \qquad \int_{(D_x)} \psi^2(x)\, d\, u^*(e_x) \leqq c^2.$$

Accordingly, on noting that

$$\int_{(D_x)} \psi(j, x) \psi_m(x) \, d u^*(e_x) = c_m \qquad (j \geqq m)$$

and on letting $j \to \infty$, we observe that passage to the limit under the integral sign is permissible, yielding

$$\int_{(D_x)} \psi(x) \psi_m(x) \, d u^*(e_x) = c_m,$$

that is, the relation (1°) of the Lemma. We form

$$R(j) = \int_{(D_x)} [\psi(x) - \psi(j, x)]^2 \, d u^*(e_x).$$

In consequence of (1°)

$$R(j) = \int_{(D_x)} \psi^2(x) \, d u^*(e_x) - \sum_1^j c_\nu^2 \geqq 0.$$

Hence, in the limit (as $j \to \infty$), one obtains

$$\int_{(D_x)} \psi^2(x) \, d u^*(e_x) \geqq c^2;$$

together with (α), this implies

$$(\beta) \qquad \int_{(D_x)} \psi^2(x) \, d u^*(e_x) = c_1^2 + c_2^2 + \cdots$$

and, accordingly, $\lim R(j) = 0$, which yields (2°); the Lemma is thus established.

The above result picks out a particular function (of interest for our purposes) amongst the functions whose existence is asserted in the *Riesz-Fisher* theorem.

In accordance with Lemma 13. 1 for the function $\varphi_n(x)$ of (13. 6) we have

$$\varphi_n(x) \sim \sum_k \lambda_{n,k} f_{n,k} \varphi_{n,k}(x) \qquad (n = n_j);$$

(13. 7 c)

$$\int_{(D_x)} \varphi_n^2(x) \, d u^*(e_x) = \Gamma_n \leqq A \qquad \text{(cf. (13. 4))}.$$

The latter inequalities, together with (13. 7) imply that

$$\lim \varphi_{m_j}(x) = \varphi(x) \tag{13. 8}$$

in the *weak* sense. The m_j here are the subscripts from (13. 7) and may be chosen so that

$$\lim \theta_{m_j}(x, y/\lambda) = \theta(x, y/\lambda)$$

exists — this is merely a matter of chosing a suitable subsequence, if necessary, of the original sequence.

We define $f(n, x)$ by the relation

$$f(n, x) = \int_{(D_s)} L_n(x, s)\, \varphi_n(s)\, d\, u^*(e_s). \tag{13. 9}$$

In view of (13. 7 c)

$$\varphi_n(s) \sim \varphi_n(l/s) \qquad (\text{as } l \to \infty); \tag{13. 9 a}$$

and

$$\varphi_n(l/s) = \sum_k^{(l)} \lambda_{n,k} f_{n,k}\, \varphi_{n,k}(s),$$

where the summation is with respect to k over values of k for which

$$-l \leqq \lambda_{n,k} < l.$$

On writing

$$f(l, n, x) = \int_{(D_s)} L_n(x, s)\, \varphi_n(l/s)\, d\, u^*(e_s) \tag{13. 9 b}$$

and on taking note of (13, 9 a), passage to the limit under the integral sign being justifiable, we obtain the relation

$$\lim_l f(l, n, x) = \int_{(D_s)} L_n(x, s)\, \varphi_n(s)\, d\, u^*(e_s) = f(n, x) \tag{13. 9 c}$$

(as $l \to +\infty$). On the other hand,

$$f(l, n, x) = \sum_k^{(l)} \lambda_{n,k} f_{n,k} \int_{(D_s)} L_n(x, s)\, \varphi_{n,k}(s)\, d\, u^*(e_s) = \sum_k^{(l)} f_{n,k}\, \varphi_{n,k}(x)$$

so that in consequence of (13. 9 c) one has

$$\sum_k f_{n,k}\,\varphi_{n,k}(x) = f(n, x), \tag{13. 9 d}$$

convergence of the series to $f(n, x)$ being in the ordinary sense; in particular,

$$\int\limits_{(D_x)} f^2(n, x)\,d\,u^*(e_x) = \sum_k f_{n,k}^2 \leqq \int\limits_{(D_x)} f^2(x)\,d\,u^*(e_x) \tag{13. 10}$$

for $n = n_1, n_2, \ldots$.

We have

$$\int\limits_{e_x} f(n, x)\,d\,u^*(e_x) = \sum_k f_{n,k} \int\limits_{e_x} \varphi_{n,k}(x)\,d\,u^*(e_x)$$

$$= \sum_k \int\limits_{(D_s)} f(s)\left[\int\limits_{e_x} \varphi_{n,k}(s)\,\varphi_{n,k}(x)\,d\,u^*(e_x)\right] d\,u^*(e_s)$$

$$= \int\limits_{-\infty}^{\infty} d_\lambda\,\sigma_n(e_x/\lambda), \quad \sigma_n(e_x/\lambda) = \int\limits_{(D_s)} f(s)\,D_s\,\Omega_n(e_s, e_x/\lambda)\,d\,u^*(e_s).$$

Thus, for $l > 0$,

$$\int\limits_{e_x} f(n, x)\,d\,u^*(e_x) = \int\limits_{-l}^{l} d_\lambda\,\sigma_n(e_x/\lambda) + \left(\int\limits_{-\infty}^{-l} + \int\limits_{l}^{\infty}\right) d_\lambda\,\sigma_n(e_x/\lambda). \tag{13. 11}$$

Now, in (13. 11),

$$\left(\int\limits_{-\infty}^{-l} + \int\limits_{l}^{\infty}\right)\cdots = \sum{}' f_{n,k} \int\limits_{e_x} \varphi_{n,k}(x)\,d\,u^*(e_x),$$

where the summation is over values of k corresponding to the intervals $(-\infty, -l)$, $(l, +\infty)$; for these values of k

$$|\lambda_{n,k}| \geqq l;$$

accordingly

$$\left|\left(\int\limits_{-\infty}^{-l} + \int\limits_{l}^{\infty}\right)\cdots\right| = \left|\sum{}' (\lambda_{n,k} f_{n,k}) \frac{1}{\lambda_{n,k}} \int\limits_{e_x} \varphi_{n,k}(x)\,d\,u^*(e_x)\right|$$

$$\leqq \frac{1}{l} \sum{}' |\lambda_{n,k} f_{n,k}| \left|\int\limits_{e_x} \varphi_{n,k}(x)\,d\,u^*(e_x)\right| \leqq \frac{1}{l} \sum_k \cdots$$

$$\leqq \frac{1}{l}\left[\sum_k |\lambda_{n,k} f_{n,k}|^2\right]^{\frac{1}{2}} \left[\sum_k \left|\int\limits_{e_x} \varphi_{n,k}(x)\,d\,u^*(e_x)\right|^2\right]^{\frac{1}{2}}$$

and, by virtue of (13. 4) and of the inequality preceding (13. 6 b),

(13. 11 a) $$\left|\left(\int_{-\infty}^{-l} + \int_{l}^{\infty}\right) d_\lambda\, \sigma_n\,(e_x/\lambda)\right| \leqq \frac{1}{l} A^{\frac{1}{2}} [u^*(e_x)]^{\frac{1}{2}}.$$

We now turn to the first term in the second member of (13. 11). In consequence of (12. 11 c)

(13. 11 b) $$\lim_{m_j} \int_{-l}^{l} d_\lambda\, \sigma_{m_j}(e_x/\lambda) = \int_{-l}^{l} d_\lambda\, \sigma(e_x/\lambda),$$

$$\sigma(e_x/\lambda) = \int_{(D_s)} f(s)\, D_s\, \Omega(e_s, e_x/\lambda)\, d\, u^*(e_s).$$

In fact, to demonstrate this one needs only to replace α, β, $\alpha(\lambda)$, $g(x)$ in (12. 11 c) by $-l$, l, 1, $f(x)$, respectively; and to let $h(y) = 1$ in e_y and $h(y) = 0$ in $(D_y) - e_y$ (e_y being a fixed $\{B\}$-set).

By (3. 11)—(3. 11 b)

$$\lim_{m_j} \int_{e_x} f(m_j, x)\, d\, u^*(e_x) = \int_{-\infty}^{\infty} d_\lambda \int_{(D_s)} f(s)\, D_s\, \Omega(e_s, e_x/\lambda)\, d\, u^*(e_s).$$

As noted before, Ω is closed; thus (12. 13 b) holds and one has

(13. 12) $$\lim_{m_j} \int_{e_x} f(m_j, x)\, d\, u^*(e_x) = \int_{e_x} f(x)\, d\, u^*(e_x).$$

In consequence of (13. 10) and (13. 12)

(13. 13) $$f(m_j, x) \to f(x) \qquad (\text{as } m_j \to \infty)$$

weakly in the sense of Definition 4. 1.

By (12. 3 d) from (13. 9) one obtains

$$\int_{(D_s)} T_x(\xi/L_n(x, s))\, \varphi_n(s)\, d\, u^*(e_s) = T_x(\xi/f(n, x)).$$

In view of (13. 13) the property (12. 3 c) will yield

$$\lim_{m_j} T_x(\xi/f(m_j, x)) = T_x(\xi/f(x)).$$

On the other hand, by virtue of (12. 3 b), (12. 3 a), (13. 8) Theorem 4. 1 (the second part) is applicable so as to give the limiting relation

$$\lim_{m_j} \int_{(D_s)} T_x(\xi/L_{m_j}(x, s))\, \varphi_{m_j}(s)\, d\, u^*(e_s) = \int_{(D_s)} T_x(\xi/L(x, s))\, \varphi(s)\, d\, u^*(e_s).$$

Accordingly, *the function* $\varphi(x)$ *of* (13. 8) *is a solution* (*the solution, in fact*) *of our problem* (13. 2); $\varphi^2(x)$ *is integrable* $\{B, u^*\}$.

By (13. 6)

$$\varphi_n(e_x) = \int_{-l}^{l} \lambda\, d_\lambda\, \sigma_n(e_x/\lambda) + \left(\int_{-\infty}^{-l} + \int_{l}^{\infty}\right) \lambda\, d_\lambda\, \sigma_n(e_x/\lambda), \tag{13. 14}$$

where $\sigma_n(e_x/\lambda)$ is from the relation preceding (13. 11). By virtue of (12. 11 c)

$$\lim_{m_j} \int_{-l}^{l} \lambda\, d_\lambda\, \sigma_{m_j}(e_x/\lambda) = \int_{-l}^{l} \lambda\, d_\lambda\, \sigma(e_x/\lambda) \qquad \text{(cf. (13. 11 b))}. \tag{13. 14 a}$$

On the other hand, by (13. 6 a)

$$\left(\int_{-\infty}^{-l} + \int_{l}^{\infty}\right) \cdots = \sum_k{}' \lambda_{n,k}\, f_{n,k} \int_{e_x} \varphi_{n,k}(x)\, d\, u^*(e_x)$$

(summation corresponding to $(-\infty, -l)$, $(l, +\infty)$) and, under (13. 4 b), one has

$$\begin{aligned}
\left|\left(\int_{-\infty}^{-l} + \int_{l}^{\infty}\right) \cdots \right| &= \left| \sum_k{}' (\lambda_{n,k}^{1+\eta} f_{n,k}) \frac{1}{\lambda_{n,k}^{\eta}} \int_{e_x} \varphi_{n,k}(x)\, d\, u^*(e_x) \right| \\
&\leqq \frac{1}{l^\eta} \sum_k{}' |\lambda_{n,k}^{1+\eta} f_{n,k}| \left| \int_{e_x} \varphi_{n,k}(x)\, d\, u^*(e_x) \right| \\
&\leqq \frac{1}{l^\eta} \left[\sum_k |\lambda_{n,k}^{2+2\eta} f_{n,k}^2|\right]^{\frac{1}{2}} \left[\sum_k \left|\int_{e_x} \varphi_{n,k}(x)\, d\, u^*(e_x)\right|^2\right]^{\frac{1}{2}}
\end{aligned}$$

so that

$$\left|\left(\int_{-\infty}^{-l} + \int_{l}^{\infty}\right) \lambda\, d_\lambda\, \sigma_n(e_x/\lambda)\right| \leqq l^{-\eta} B^{\frac{1}{2}} \cdot [u^*(e_x)]^{\frac{1}{2}}. \tag{13. 14 b}$$

We note (13. 7); the relations (13. 14)—(13. 14 b) therefore imply

$$\varphi(e_x) = \int_{-\infty}^{\infty} \lambda \, d_\lambda \, \sigma(e_x/\lambda).$$

On taking account of (13. 7 a) we finally obtain the representation (13. 5), thus completing the proof of the theorem.

The developments of this section supersede those of section 6 in (T; 611—619). It is to be noted that *the passage to the limit under the integral sign in formula* (6. 31 c) *of* (T; p. 618) *is not necessarily justifiable;* accordingly, *the italics preceding Theorem 6. 2 in* (T; p. 619) *cannot be considered as established* (*even though closure of Ω will take place as stated*). *However, the concluding Theorem 6. 2 of* (T) *is correct as formulated;* this may be inferred in view of our present Theorem 13. 1.

We shall conclude this work with a few general remarks. The integral problem (7. 3 a), where the unknown is a function of sets $\{B\}$, is singular in the sense that the kernel behaves in a manner irregular, according to various hypotheses involved, in the neighborhood of the frontier of the domain (D_x); for this problem the frontier of (D_x) is, so to say, a 'singular set'. It is possible to formulate the problem so that the 'singular set' is any measurable subset of (D_x) (with some points possibly in the interior of (D_x)), while the essential features of the theory of (7. 3 a) (and of the corresponding homogeneous problem), as developed in these pages, continue to hold (with appropriate modifications). For instance, those of the results which make use of the hypothesis of 'regularity' of u^* with respect to the frontier of (D_x) (Definition 8. 1) would have to be restated under the supposition of 'regularity' of u^* with respect to the 'singular set'. We shall not go into the details of formulating such an extension of the notion of 'regularity'. In order to make use of the results of Gunther we did assume in these pages that u^* (also F^*) is continuous as a function of sets $\{B\}$ (i. e. $u^*(e) \to 0$ with the diameter of e) — this enabled application of Lemma 3. 2, leading to the desired connection. This condition on u^* (and F^*) may be weakened by taking account of the text from (3. 11) to (3. 15). As previously observed, for a part of our developments continuity conditions of kernels $L(x, y)$ may be deleted — we need merely to secure discreteness of the characteristic values and orthogonality of the characteristic functions of the approximating kernels.