UNITARY REPRESENTATIONS DEFINED BY BOUNDARY CONDITIONS—THE CASE OF $\mathfrak{sl}(2, R)$

BY

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§ 1. Introduction

Let $\mathfrak{g}$ be a Lie algebra over R, the field of real numbers, and σ, a $\mathfrak{g}$-module in a Hilbert space $\mathcal{H}$. If the domain of σ is dense, one can define an adjoint module $\sigma^\dagger$ in $\mathcal{H}$ such that

$$(\sigma(a)f, g) = (f, \sigma^\dagger(a^\dagger)g)$$

for all $f \in \mathcal{D}(\sigma)$, $g \in \mathcal{D}(\sigma^\dagger)$, $a \in \mathcal{U}[\mathfrak{g}]$, (see Appendix A for notation and details). The module σ is said to be symmetric or (infinitesimally) unitary if $\sigma \subset \sigma^\dagger$ and self-adjoint if $\sigma = \sigma^\dagger$. The importance of self-adjointness comes from the fact that dT is a self-adjoint module (see Appendix A). Here T is a unitary representation of the simply connected group corresponding to $\mathfrak{g}$, and dT is the usual $\mathfrak{g}$-module with the set of C^∞-vectors of T as its domain. Calling a $\mathfrak{g}$-module exact if it is equal to dT for some T, a natural problem would be to determine all exact extensions of a given symmetric $\mathfrak{g}$-module. The theory here is analogous to the theory of self-adjoint extensions of a single unbounded symmetric operator. In fact if dim $\mathfrak{g} = 1$, it is well known that $\mathfrak{g}$-module is exact if and only if it is self-adjoint. For the general case, self-adjointness is necessary but not sufficient for exact-

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ness. (See Appendix A.) However there are many interesting cases where self-adjointness alone is enough to assure integrability to the group. In this paper one such module for $\mathfrak{sl}(2, \boldsymbol{R})$ is studied in detail, and all its self-adjoint extensions are obtained. Since the extensions are determined by boundary conditions, it is natural to consider the corresponding group representations as being defined by the boundary conditions. An interesting feature is that all non-trivial unitary irreducible representations are obtained by determining all self-adjoint extensions of the module σ_λ. Although the representations of G have been known for a long time this way of deriving them appears to be new. It is an interesting problem to find modules similar to σ_λ for other groups as well. This question will be pursued in future papers.

A brief description of the contents follows. Generalities about $\mathfrak{g}$-modules and some basic results which are used repeatedly are collected together in Appendix A. In Section 3, the basic homomorphism ϱ_λ of $\mathfrak{sl}(2, \boldsymbol{R})$ into differential operators and the modules σ_λ, $\sigma_\lambda^\pm$ are defined. All self-adjoint extensions are determined in Theorems 1, 2 and Lemma 10. They are shown to be integrable (Theorem 3). Their unitary equivalence classes are identified in Theorem 4. Theorems 5 and 6 describe the set of C^∞ vectors, K-eigenbasis and the group operators for the representations $T_\lambda^\pm$, which correspond to self-adjoint extensions of $\sigma_\lambda^\pm$. Theorems 7, 8 and 10 do the same for the representations $T_{\lambda,\delta,\delta'}$ (here δ, δ' parametrize self-adjoint extensions of σ_λ). Theorem 9 is an auxiliary result which determines all representations of G in $L^2(R)$ with a given restriction to a parabolic subgroup. Theorem 11 gives the intertwining operator between the unitary principal series and the representations $T_{\lambda,\delta,\delta'}$, when λ is imaginary.

The methods and results of this paper will be used to obtain explicit decomposition of (1) the tensor product of two discrete series representations of $\mathfrak{sl}(2, \boldsymbol{R})$ ([12]), (2) the Weil representation associated to a quadratic form [13]. In fact the present paper grew out of an attempt to get such an explicit decomposition. In this connection we refer to [11] for another approach to the same problem.

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§ 2. Preliminaries on $\mathfrak{sl}_2$

Let G denote the simply connected Lie group with Lie algebra $\mathfrak{sl}(2, \boldsymbol{R})$. Let X, H, Y denote the standard basis of $\mathfrak{sl}(2, \boldsymbol{R})$, i.e.

$$X=\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix},\quad H=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix},\quad Y=\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}.$$

Then $[H, X]=2X$, $[H, Y]=-2Y$, and $[X, Y]=H$. Write $\varkappa(\theta)=\exp\theta(X-Y)$, $h(t)=\exp tH$, and $u(s)=\exp sX$. Here we have written exp, rather than $\exp_G$ for the exponential map of $\mathfrak{sl}(2, \mathbf{R})$ into G. Put $K=\{\varkappa(\theta)\,|\,\theta\in R\}$, $A=\{h(t)\,|\,t\in R\}$ and $N=\{u(s)\,|\,s\in R\}$. Then K, A, N are closed subgroups of G and $G=K\cdot A\cdot N$. Write

$$w=\exp\frac{\pi}{2}(X-Y), \quad \gamma=\exp\pi(X-Y)=w^2 \tag{1}$$

then the center $Z(G)$ is the cyclic group $=\{\gamma^n\,|\,n\in\mathbf{Z}\}$. Also

$$\operatorname{Ad} w\cdot H=-H, \quad \operatorname{Ad} w\cdot X=-Y, \quad \operatorname{Ad} w\cdot Y=-X. \tag{2}$$

Let $\mathcal{U}[\mathfrak{sl}_2]$ denote the universal enveloping algebra of $\mathfrak{sl}(2, \mathbf{C})$. The following basis of $\mathfrak{sl}(2, \mathbf{C})$ will be frequently used

$$X'=(iH+X+Y)/2, \quad H'=i(X-Y), \quad Y'=(-iH+X+Y)/2. \tag{3}$$

Then $\{X', H', Y'\}$ is another Lie triple. Let

$$\Omega=(H+1)^2+4YX=(H-1)^2+4XY \tag{4}$$

Then Ω generates the center of $\mathcal{U}[\mathfrak{sl}_2]$.

Let $\mathcal{E}(G)$ denote the set of equivalence classes of irreducible unitary representations of G. These are all known (Bargman [1], Kunze-Stein [6], Pukhansky [10] and Sally [15]). In any irreducible representation the center of the group and the center of the algebra $\mathcal{U}[\mathfrak{sl}_2]$ are mapped into scalars and they can be used to parametrize them. It is known that Spec H' and Spec Ω determine the unitary equivalence classes. Then the points of $\mathcal{E}(G)$ can be parametrized as follows:

1. $\omega(\eta, \lambda)$ where λ^2 and η are real and $|\operatorname{Re}\lambda|+|\eta|\leqslant 1$, with inequality holding if λ is real. These representations are characterized by Spec $H'=\eta+2\mathbf{Z}$, Spec $\Omega=\lambda^2$ and Spec $\gamma=e^{-i\eta\pi}$. Also $\omega(\eta, -\lambda)=\omega(\eta, \lambda)$, $\omega(\eta+2, \lambda)=\omega(\eta, \lambda)$.

2. $\omega^{\pm}(\lambda)$, where λ is real and $\lambda+1>0$. Here Spec $H'=\pm(\lambda+1+2\mathbf{N})$, Spec $\Omega=\lambda^2$ and Spec $\gamma=\exp(\mp i(\lambda+1)\pi)$.

3. $\omega(0, 1)$, the class of the trivial representation.

The class $\omega(\eta, \lambda)$ is known as the principal series when λ is purely imaginary, and as the complementary series when λ is real. The classes $\omega^{\pm}(\lambda)$ are the so-called discrete series. It is then known that representations of the class $\omega^{\pm}(\lambda)$ are integrable (or matrix entries belong to $L^1(G/Z(G))$) when $\lambda>1$ and are square integrable when $\lambda>0$. It may be of some interest to note that although $\omega^+(\lambda)$ and $\omega^-(\lambda)$ are not unitarily equivalent, they are physically (viewpoint of physicists) equivalent since there is an anti-unitary isomorphism between representations of the two classes.

§ 3. The homomorphism ϱ_λ and the modules σ_λ

For each $\lambda \in \mathbf{C}$, define the following differential operators on $R' = R\backslash\{0\}$:

$$\varrho_\lambda(H) = 2t\partial_t + 1, \quad \varrho_\lambda(X) = -it$$
$$\varrho_\lambda(X) = -i(t\partial_t^2 + \partial_t - \lambda^2/4t) \tag{5}$$

Then one checks easily that these operators satisfy the same commutation rules as the Lie triple $\{X, H, Y\}$. Thus ϱ_λ extends as an algebra homomorphism of $\mathcal{U}[\mathfrak{sl}_2]$ into differential operators on R'. A simple calculation shows that

$$\varrho_\lambda(\Omega) = \lambda^2$$

The formal transpose and adjoint with respect to Lebesgue measure on R is easily checked to be

$${}^t\varrho_\lambda(H) = -\varrho_\lambda(H), \quad {}^t\varrho_\lambda(X) = \varrho_\lambda(X), \quad {}^t\varrho_\lambda(Y) = \varrho_\lambda(Y)$$

In particular, if λ^2 is real the operators $\varrho_\lambda(Z)^* = -\varrho_\lambda(Z)$ for each $Z \in \mathfrak{sl}(2, \boldsymbol{R})$, i.e. they are formally skew adjoint. The natural domain of ϱ_λ is $C^\infty(R')$. In this part we shall determine all sub-modules of ϱ_λ in $L^2(R')$ which are integrable to a group representation. We also discuss simultaneously the $\mathfrak{g}$-modules on $R_\pm$, defined by $\sigma_\lambda^\pm = (C^\infty(R_\pm), \varrho_\lambda)$. Define

$$\sigma_\lambda = (C_c^\infty(R'), \varrho_\lambda), \quad \sigma_\lambda^\pm = (C_c^\infty(R_\pm), \varrho_\lambda). \tag{6}$$

Then σ_λ is a $\mathfrak{g}$-module in $L^2(R)$ and its adjoint module is described in the following lemma.

LEMMA 1. *The domain of the adjoint module* $\mathcal{D}(\sigma_\lambda^\dagger) = \{f \in C_c^\infty(R') \mid \varrho_\lambda(a) f \in L^2$, *for all* a *in* $\mathcal{U}[\mathfrak{sl}_2]\}$. *Moreover,* $\varrho_\lambda^\dagger(a) f = \varrho_\lambda(a) f$ *for all* $f \in \mathcal{D}(\sigma_\lambda^\dagger)$. *If* $f \in \mathcal{D}(\sigma_\lambda^\dagger)$, *then for each* $t_0 > 0$

$$\sup_{|t| > t_0} |t^m (t\partial_t)^n f| < \infty$$

for all $m, n \in \mathbf{N}$. *Similar results hold for the modules* $\sigma_\lambda^\dagger$ *in* $L^2(R_\pm)$.

Proof. The description of $\mathcal{D}(\sigma_\lambda^\dagger)$ is just Weyl's lemma. (See Appendix A.) Let $f \in \mathcal{D}(\sigma_\lambda^\dagger)$. Then

$$|f(s)| \leqslant \int_s^\infty |\partial_t f|\, dt \leqslant s^{-1/2} \|t\partial_t f\|_2$$
$$\leqslant \tfrac{1}{2} s^{-1/2} \|\varrho_\lambda(H-1) f\|_2.$$

The lemma follows by replacing f by $\varrho_\lambda(X^m (H-1)^n) f$.

§ 4. Eigenfunctions of $\varrho_\lambda(H')$

Suppose $\varrho_\lambda(H')f=\xi f$ for some $\xi\in\mathbf{C}$. Then f satisfies the Tricomi's equation

$$\left\{\partial_t^2+\frac{1}{t}\partial_t-1+\xi/t-\lambda^2/4t^2\right\}f=0.$$

If we put $f=2|t|^{-1/2}g(2t)$ (see [3], p. 251) then g satisfies the Whittaker's equation

$$[\partial_t^2-1/4+\varkappa/t+(1-\mu^2)/t^2]g=0$$

where $\varkappa=\xi/2$, $\mu=\lambda/2$. On any connected interval $W_{\varkappa,\mu}(t)$, $W_{-\varkappa,\mu}(-t)$ are a basis of solutions of the Whittaker's equation, where $W_{\varkappa,\mu}$ is the Whittaker's function. Also $W_{\varkappa,\mu}(t)=O(t^{\varkappa/2}e^{-t/2})$ as $t\to\infty$, so that a solution which is in L^2 near ∞ has to be a multiple of $W_{\varkappa,\mu}(t)$. Thus we introduce the function

$$L_{\varkappa,\mu}(t)=(2t)^{-1/2}W_{\varkappa,\mu}(2t),\quad t>0. \tag{7}$$

Then

$$f=\begin{cases}c_1L_{\xi/2,\lambda/2}(t), & t>0\\ c_2L_{-\xi/2,\lambda/2}(-t), & t<0\end{cases}$$

for suitable constants c_1, c_2. The function f will be in L^2 when ξ and λ are such that $L_{\xi/2,\lambda/2}(t)$ is in L^2 near 0. Now the point $t=0$ is a regular singular point of the Tricomi's equation with indices $\pm\lambda/2$. Now $L_{\xi/2,\lambda/2}(t)=e^{-t}(2t)^{\lambda/2}\Psi(a,c;2t)$ with $a=(1-\xi+\lambda)/2$ and $c=1+\lambda$, Ψ being Tricomi's function. ([3], p. 255, Vol. I.) From fractional power series expansion of Ψ near 0 ([3], p. 257, Vol. I) one obtains those of L and the results are summarized in the following

LEMMA 2. *There exist convergent power series $P_j(t)$ depending on λ such that for $t>0$*

$$L_{\xi/2,\lambda/2}(t)=\begin{cases}t^{\lambda/2}P_1(t)+t^{-\lambda/2}P_2(t), & \text{if}\quad \lambda\notin\mathbf{Z}\\ t^{\lambda/2}(P_1(t)\ln t+P_3(t)\}+t^{-\lambda/2}P_2(t) & \text{for}\quad \lambda\in\mathbf{Z},\quad \lambda\neq 0.\end{cases}$$

Moreover, define

$$c(\xi,\lambda)=\Gamma(-\lambda)2^{\lambda/2}\{\Gamma((1-\xi-\lambda)/2)\}^{-1}$$

for $\lambda\notin\mathbf{N}$, and $c(\xi,n)$ as the residue of $c(\xi,\lambda)$ at $\lambda=n$ (for example $c(\xi,0)=-\{\Gamma((1-\xi)/2)\}^{-1}$). Then $P_1(0)=c(\xi,\lambda)$ for all λ, and $P_2(0)=c(\xi,-\lambda)$ for all $\lambda\notin-\mathbf{N}$. When $\lambda=0$ we have

$$L_{\xi/2,0}(t)=\begin{cases}Q_1(t), & \xi-1\in 2\mathbf{N}\\ Q_2(t)\ln|t|+Q_3(t), & \text{if}\quad \xi-1\notin 2\mathbf{N}\end{cases}$$

where $Q_1(t)$ is a polynomial, $Q_1(0)=(-1)^{(\xi-1)/2}$ and Q_2, Q_3 are convergent power series with $Q_2(0)=c(\xi, 0)$ and $Q_3(0)=c(\xi, 0)d(\xi)$, where $d(\xi)=\psi((1-\xi)/2)-2\psi(0)+\ln 2$, ψ is the logarithmic derivative of the Γ-function.

From the above lemma, the following corollary is immediate.

COROLLARY 1. *If $|\mathrm{Re}\,\lambda|<1$, then $L_{\xi/2,\lambda/2}\in L^2(R_+)$ for all $\xi\in\mathbf{C}$. If $\mathrm{Re}\,\lambda\geqslant 1$, then $L_{\xi/2,\lambda/2}\in L^2(R_+)$ if and only if $c(\xi, -\lambda)=0$ or if and only if $\xi\in 1+\lambda+2\mathbf{N}$.*

COROLLARY 2. *Let $V_\lambda(\xi)\,(V_\lambda^{\pm}(\xi))$ denote the linear space of eigenfunctions of $\varrho_\lambda(H')$ in $L^2(R)\,(L^2(R_{\pm}))$ for the eigen-value ξ. Then*

(i) $V_\lambda(\xi)=V_\lambda^+(\xi)+V_\lambda^-(\xi)$,

(ii) $\dim V_\lambda^{\pm}(\xi)=1$ *for all* $\xi\in\mathbf{C}$ *if* $|\mathrm{Re}\,\lambda|<1$ *and*

(iii) *for* $\lambda\geqslant 1$, $\dim V_\lambda^{\pm}(\xi)=1$ *if* $\xi\in\pm(1+\lambda+2\mathbf{N})$, $\dim V_\lambda^{\pm}(\xi)=0$ *otherwise.*

From the above corollary $V_\lambda^{\pm}(\pm i)=0$ when $\lambda\geqslant 1$, and so we have

COROLLARY 3. *The symmetric operators $\sigma_\lambda(H')$, $\sigma_\lambda^{\pm}(H')$ are essentially self-adjoint in $L^2(R)$ and $L^2(R_{\pm})$ respectively when λ is real and $\geqslant 1$.*

The situation is very different when $|\mathrm{Re}\,\lambda|<1$. In the following paragraphs we shall find all $\mathfrak{sl}_2$-modules σ such that $\sigma_\lambda\subset\sigma\subset\sigma_\lambda^\dagger$ for which $\sigma(H')$ is essentially self-adjoint.

§ 5. Boundary forms

For each $a\in\mathcal{U}$ and $f, g\in\mathcal{D}(\sigma_\lambda^\dagger)$, let B_λ denote the boundary form of the module σ_λ, i.e.

$$B_\lambda(a\colon f\colon g)=(\varrho_\lambda(a)f, g)-(f, \varrho_\lambda(a^\dagger)g)$$

(see the appendix for the identities satisfied by B_λ). If $Z=t_1H+t_2X$, with t_1, t_2 real, then it is easy to check that none of the eigenfunctions of $\varrho_\lambda(Z)$ for real eigenvalues are in L^2 and so the operators $\sigma_\lambda(Z)$ are essentially skew adjoint. Thus $B_\lambda(Z\colon f\colon g)\equiv 0$ and so from the identities satisfied by B_λ (see Lemma A.2) it follows that

$$B_\lambda(a\colon f\colon g)=0$$

for all $f, g\in\mathcal{D}(\sigma_\lambda^\dagger)$ and a of the form H^mX^n. Thus it is sufficient to study $B_\lambda(H'^n\colon f\colon g)$. Now

$$\varrho_\lambda(H')=-\partial_t\circ t\circ\partial_t+t+\lambda^2/4t$$

so that

$$\begin{aligned}(d/dt)tW(f, g)&=(\varrho_\lambda(H')f)g-f\varrho_\lambda(H')g\\&=-i\{(\varrho_\lambda(Y)f)g-f\varrho_\lambda(Y)g\}\end{aligned}\qquad(8)$$

where $W(f, g) = f\partial_t g - g\partial_t f$ is the Wronskian. Thus

$$B_\lambda(H': f: \bar{g}) = -iB_\lambda(Y: f: \bar{g}) = -\{tW(f, g)|_-^+\}. \tag{9}$$

Here $\varphi|_-^+ = \lim_{t\to 0+} \varphi(t) - \lim_{t\to 0-} \varphi(t)$.

§ 6. Boundary values

We assume that λ^2 is real and $|\operatorname{Re}\lambda| < 1$. To consider all different cases simultaneously we use the following device. Define

$$a_\lambda(t) = \begin{cases} |t|^{-\lambda/2}, \lambda \neq 0 \\ \ln |t|, \lambda = 0 \end{cases} \tag{10}$$

and define

$$A_1^\pm(f) = \lim_{t\to 0\pm} \{tW(f, a_\lambda(t))\}$$

$$A_2^\pm(f) = \lim_{t\to 0\pm} \{tW(f, |t|^{\lambda/2})\}$$

Lemma 3. *For $f \in \mathcal{D}(\sigma_\lambda^\dagger)$, $A_j^\mp(f)$, $j = 1, 2$, exist and*

$$A_1^+(f) = (\pm i) \int_{R_\pm} a_\lambda(t) \varrho_\lambda(Y) f\, dt$$

$$A_2^\pm(f) = (\pm i) \int_{R_\pm} |t|^{\lambda/2} \varrho_\lambda(Y) f\, dt.$$

Moreover, for a suitable choice of $\varphi^\pm \in \mathcal{S}(R')$, we have

$$A_1^\pm(f) = iB(Y: f: a_\lambda(t)\varphi^\pm)$$

$$A_2^\pm(f) = iB(Y: f: |t|^{\lambda/2}\varphi^\pm)$$

Proof. Note that $\varrho_\lambda(Y)a_\lambda(t) = 0$, $\varrho_\lambda(Y)|t|^{\lambda/2} = 0$, the first two formulae follow from (8) and the fact that f is rapidly decreasing at ∞ (cf. Lemma 1). The last two follow from (9).

The following class of functions is somewhat more convenient to work with than $\mathcal{D}(\sigma_\lambda^\dagger)$. For any open subset U of R denote $\mathcal{S}(U)$ as the Schwartz space of U, i.e. $\mathcal{S}(U) = \{f \in C^\infty(U) \mid \sup_U |t^m \partial_t^n f| < \infty \text{ for all } m, n \in \mathbf{N}\}$. If $f \in \mathcal{S}(U)$, then f and its derivatives have limits as t approaches a boundary point of U. It is somewhat more convenient to work with classes of functions in $\mathcal{D}(\sigma_\lambda^\dagger)$ which have asymptotic expansions at the boundary. For this purpose we introduce the class $\mathfrak{X}_\lambda$.

Let $\mathfrak{X}_\lambda$ denote the class of functions $f \in C^\infty(R')$ such that

(i) for each $\delta>0$, $\sup\{|t^m\partial_t^n f|:|t|>\delta\}<\infty$

(ii) there exists $\delta>0$ and functions $f_1\ f_2\in\mathcal{S}(R')$ such that for $0<|t|<\delta$

$$f(t)=|t|^{\lambda/2}f_1(t)+a_\lambda(t)f_2(t) \tag{11}$$

Also write $\mathfrak{X}_\lambda^\pm=|t|^{\lambda/2}\mathcal{S}(R_\pm)$.

Lemma 4. *(i) $\mathfrak{X}_\lambda\subset L^2(R)$ and $\mathfrak{X}_\lambda^\pm\subset L^2(R_\pm)$. (ii) $\varrho_\lambda(Z)\mathfrak{X}_\lambda\subset\mathfrak{X}_\lambda$, and $\varrho_\lambda(Z)\mathfrak{X}_\lambda^\pm\subset\mathfrak{X}_\lambda^\pm$ for all $Z\in\mathfrak{sl}(2,\mathbf{R})$. In particular, $\mathfrak{X}_\lambda\subset\mathcal{D}(\sigma_\lambda^\dagger)$ and $\mathfrak{X}_\lambda^\pm\subset\mathcal{D}((\sigma_\lambda^\pm)^\dagger)$.*

Proof. Only (ii) needs checking. This follows from the following. Let $\alpha=\pm\lambda/2$. Put $\tilde\varrho_\lambda=|t|^{-\alpha/2}\circ\varrho_\lambda\circ|t|^{\alpha/2}$. Then $\tilde\varrho_\lambda(H)=2t\partial_t+1+\alpha$, $\tilde\varrho_\lambda(X)=-it$, and $\tilde\varrho_{\lambda_1}(Y)=-i(t\partial_t^2+(1+\alpha)\partial_t)$. This proves (ii) for $\lambda\neq0$. For $\lambda=0$, define $\tilde\varrho_0=(\ln|t|)^{-1}\circ\varrho_0\circ(\ln|t|)$, then $\tilde\varrho_0(H)=2t\partial_t+1-(\ln|t|)^{-1}$, $\tilde\varrho_0(X)=-it$ and $\tilde\varrho_0(Y)=-i(t\partial_t^2+\partial_t+2(\ln|t|)^{-1}\partial_t)$. Thus $\varrho_0(Z)\mathfrak{X}_0\subset\mathfrak{X}_0$ for all $Z\in\mathfrak{sl}(2,\mathbf{R})$. The rest is clear.

The following lemma is easily checked, by direct calculation.

Lemma 5. *Let $|\mathrm{Re}\,\lambda|<1$. Suppose that $f=|t|^{\lambda/2}f_1+a_\lambda(t)f_2$, $g=|t|^{\lambda/2}g_1+a_\lambda(t)g_2$ with $f_j,g_j\in\mathcal{S}(R')$, then*

$$\{tW(f,g)\}_-^+=c_\lambda(f_1g_2-g_1f_2)|_-^+$$

where c_λ is the constant $\equiv tW(|t|^{\lambda/2},a_\lambda(t))$, $=-\lambda$ if $\lambda\neq0$ and $=1$ if $\lambda=0$. In particular, $B_\lambda(H':f:\bar g)=-c_\lambda(f_1g_2-f_2g_1)_-^+$.

Lemma 6. *Assume $|\mathrm{Re}\,\lambda|<1$. Let $f\in\mathfrak{X}_\lambda$ have the local expansion $f=|t|^{\lambda/2}f_1+a_\lambda(t)f_2$ near 0. Then*

$$A_2^\pm(f)=-c_\lambda f_2(0\pm),\quad A_1^\pm(f)=c_\lambda f_1(0\pm).$$

Moreover, for all $f,g\in\mathcal{D}(\sigma_\lambda^\dagger)$

$$c_\lambda B_\lambda(H':f:g)=A_1^+(f)A_2^+(\bar g)-A_2^+(f)A_1^+(\bar g)-A_1^-(f)A_2^-(\bar g)+A_2^-(f)A_1^-(\bar g).$$

Proof. The first part of the lemma follows from the definition and the fact $c_\lambda\equiv tW(|t|^{\lambda/2},a_\lambda)$. Next the formula for $B_\lambda(H':f:g)$ follows from the previous lemma when $f,g\in\mathfrak{X}_\lambda$. To prove it in the general case note that the eigenspaces $V_\lambda(\xi)\subset\mathfrak{X}_\lambda$ and from known results about the adjoint of a symmetric operator, it follows that $\mathcal{D}(\sigma_\lambda(H')^*)=\mathcal{D}(\mathrm{Cl}\,\sigma_\lambda(H'))+V_\lambda(i)+V_\lambda(-i)$. Thus, given $\varphi,\psi\in\mathcal{D}(\sigma_\lambda^\dagger)$, it follows that there exist $\varphi_1,\psi_1\in\mathfrak{X}_\lambda$ such that $\varphi-\varphi_1$, $\psi-\psi_1\in\mathcal{D}(\mathrm{Cl}\,\sigma_\lambda(H'))$. But $B_\lambda(H':\varphi,\psi)=B_\lambda(H':\varphi_1:\psi_1)$. Finally observe that $A_j^\mp(f)=0$ for all $f\in C_c^\infty(R')$ and thus $A_j^\mp\equiv0$ on $\mathcal{D}(\mathrm{Cl}\,\sigma_\lambda(H'))\cap\mathcal{D}(\sigma_\lambda^\dagger)$ by Lemma 3. The formula for $B_\lambda(H':f:g)$ thus follows from the corresponsing formula for $f,g\in\mathfrak{X}_\lambda$.

LEMMA 7. *Define the sesquilinear form* $F_\lambda(x, y)$ *on* $\mathbf{C}^4$ *as follows:*

$$F_\lambda(x, y) = \begin{cases} x_1\bar{y}_1 - x_2\bar{y}_2 - x_3\bar{y}_3 + x_4\bar{y}_4, & \lambda^2 < 0 \\ x_1\bar{y}_2 - x_2\bar{y}_1 - x_3\bar{y}_4 + x_4\bar{y}_3, & \text{if } \lambda^2 \geqslant 0, \ |\lambda| < 1. \end{cases}$$

Define Λ_0: $\mathcal{D}(\sigma_\lambda^\dagger) \to \mathbf{C}^4$ *by setting*

$$\Lambda_0(f) = (A_1^+(f), A_2^+(f), A_1^-(f), A_2^-(f))$$

then Λ_0 *maps onto and*

$$c_\lambda B_\lambda(H'\colon f\colon g) = F_\lambda(\Lambda_0(f), \Lambda_0(g)) \tag{12}$$

for all $f, g \in \mathcal{D}(\sigma_\lambda^\dagger)$.

Proof. This follows from the previous lemma if you note the following. If $\lambda^2 < 0$, then $A_1^\pm(\bar{g}) = \overline{A_2^\pm(g)}$, and if $\lambda^2 \geqslant 0$, $|\mathrm{Re}\,\lambda| < 1$ then $A_1^\pm(\bar{g}) = \overline{A_1^\pm(g)}$, $A_2^\pm(\bar{g}) = \overline{A_2^\pm(g)}$. From the formulas for $A_j^\pm$ in Lemma 6, if follows that Λ_0 maps $\mathfrak{X}_\lambda$ onto $\mathbf{C}^4$.

LEMMA 8. (i) $\Lambda_0(\varrho_\lambda(X)f) = 0$ *for all* $f \in \mathcal{D}(\sigma_\lambda^\dagger)$ *and* (ii) *there exists a matrix* $M_\lambda \in GL(4, \mathbf{C})$ *such that* $\Lambda_0(\varrho_\lambda(H)f) = M_\lambda \cdot \Lambda_0(f)$. *Also* $M_\lambda = \mathrm{diag}\,(1+\lambda, 1-\lambda, 1+\lambda, 1-\lambda)$ *if* $\lambda \neq 0$ *and*

$$M_0 = \begin{pmatrix} 1 & -2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Proof. To show that $A_j^\pm(\varrho_\lambda(X)f) = 0$, it is sufficient to show that $\lim_{t\to 0} tW(tf, \psi) = 0$, where $\psi = |t|^{\lambda/2}$ or $a_\lambda(t)$. Now $tW(tf, \psi) = -tf\psi + t^2W(f, \psi)$. Now $t^2W(f, \psi) \to 0$ as $t \to 0$. Now $f \in \mathcal{D}(\sigma_\lambda^\dagger)$ implies that $f = O(|t|^{-1/2})$, and thus $tf\psi = O(|t|^{1-\lambda/2})$ if $\lambda \neq 0$, and $O(|t|^{1/2} \ln |t|)$ if $\lambda = 0$. Thus in all cases $\lim tf\psi = 0$. This proves that $\Lambda_0(\varrho_\lambda(X)f) = 0$.

To prove (ii) we use the identity (see Lemma A.2) satisfied by boundary forms. Thus $2B_\lambda(Y\colon f\colon g) = B_\lambda(YH - HY\colon f\colon g) = B_\lambda(Y\colon \varrho_\lambda(H)f\colon g) + B_\lambda(Y\colon f\colon \varrho_\lambda(H)g)$ since (see section 5), $B_\lambda(H\colon \cdot\colon \cdot) \equiv 0$. Thus $B_\lambda(Y\colon \varrho_\lambda(H)f\colon g) = B_\lambda(Y\colon f\colon \varrho_\lambda(2-H)g)$. Now let $\varphi \in C_c^\infty(R)$ such that $\varphi = 1$ around 0, then

$$\varrho_\lambda(2-H)|t|^{\lambda/2}\varphi = (-2t\partial_t)|t|^{\lambda/2}\varphi = (1-\lambda)|t|^{\lambda/2}$$

in a neighborhood of 0. $A_2^\pm(\varrho_\lambda(H)f) = iB_\lambda(Y\colon \varrho_\lambda(H)f\colon |t|^{\lambda/2}\varphi) = (1-\lambda)A_2^\pm(f)$. Again $\varrho_\lambda(2-H)(a_\lambda(t)\varphi) = (1+\lambda)|t|^{-\lambda/2}$ around 0 if $\lambda \neq 0$, and $= \ln|t| - 2$ near 0 if $\lambda = 0$. Thus $A_1^\pm(\varrho_\lambda(H)f) = (1+\lambda)A_1^\pm(f)$ if $\lambda \neq 0$, $= A_1^\pm(f) - 2A_2^\pm(f)$ if $\lambda = 0$. Thus $\Lambda_0(\varrho_\lambda(H)f) = M_\lambda \cdot \Lambda_0(f)$, where M_λ is given in the lemma. This completes the proof.

LEMMA 9. *Let the boundary values Λ_m be defined by $\Lambda_m(f) = \Lambda_0(\varrho_\lambda(Y)^m f)$. Then we have*

(i) $\Lambda_m \circ \varrho_\lambda(X) = -m(M_\lambda + m - 1)\cdot\Lambda_{m-1}$

(ii) $\Lambda_m \circ \varrho_\lambda(H) = (M_\lambda + 2)\cdot\Lambda_m$

(iii) $\Lambda_m \circ \varrho_\lambda(Y) = \Lambda_{m+1}$

(iv) $c_\lambda B_\lambda(H'^m\colon f\colon g) = \sum_{r=0}^{m-1} F_\lambda(\Lambda_r(f), \Lambda_{m-r-1}(g))$

(v) *Let $v_m \in \mathbf{C}^4$ be arbitrary. Then there exists an $f \in \mathfrak{X}_\lambda$ such that $\Lambda_m(f) = v_m$, $m \in \mathbf{N}$.*

Proof. The first statement follows from the identity $Y^m X = XY^m - m(H + m - 1)\,Y^{m-1}$ in $\mathcal{U}[\hat{\mathfrak{s}}\mathfrak{l}_2]$. The statements (ii) and (iii) are obvious. The part (iv) follows from the identity (iii) of Lemma A.2, satisfied by boundary forms and Lemma 7. Finally let $f \in \mathfrak{X}_\lambda$ and suppose that

$$f = |t|^{\lambda/2} f_1 + a_\lambda f_2 \quad \text{near} \quad 0, \quad \text{with} \quad f_1, f_2 \in \mathcal{S}(R').$$

Put $\varrho_\lambda(Y)^m f = |t|^{\lambda/2} f_{1,m} + a_\lambda f_{2,m}$. Then we have the formula

$$|t|^{-\alpha/2} \circ \varrho_\lambda(Y) \circ |t|^{\alpha/2} = -i(t\partial_t^2 + (1+\alpha)\partial_t)$$

One checks easily by induction that

$$\{(t\partial_t^2 + (1+a)\partial_t)\}^m = \left\{\prod_{j=1}^{m-1} (t\partial_t + j + a)\right\} \partial_t^m = D_{m,a}, \text{ say.}$$

Then $f_{1,m} = (-i)^m D_{m,\lambda/2} f_1$, so that $f_{1,m}(\pm 0) = (-i)^m (1+\alpha)_m (\partial_t^m f_1)(\pm 0)$. If $\lambda \neq 0$, we have similarly $f_{2,m}(\pm 0) = (-1)^m (1+\alpha)_m \partial_t^m f_2(\pm 0)$. If $v_m = \Lambda_m(f)$, then

$$v_m = c_\lambda(f_{1,m}(0+), f_{2,m}(0+), f_{1,m}(0-), f_{2,m}(0-)).$$

Put

$$w_m = (\partial_t^m f_1(0+), \partial_t^m f_2(0+), \partial_t^m f_1(0-), \partial_t^m f_2(0-)).$$

Then

$$w_m = i^m \{(1+\alpha)_m\}^{-1} c_\lambda^{-1} \cdot v_m.$$

By Borel's theorem ([6], p. 30), one can choose $f_1, f_2 \in \mathcal{S}(R')$ with values w_m for the derivatives at 0. Thus there exist $f \in \mathfrak{X}_\lambda$, such that $\Lambda_m(f) = v_m$. In the case $\lambda = 0$

$$\begin{pmatrix} f_{1,m} \\ f_{2,m} \end{pmatrix} = \begin{pmatrix} t\partial_t + 1 & 3 \\ 0 & t\partial_t + 1 \end{pmatrix} \begin{pmatrix} -i\partial_t & f_{1,m-1} \\ -i\partial_t & f_{2,m-1} \end{pmatrix}$$

One checks by induction on m

$$\begin{pmatrix} f_{1,m} \\ f_{2,m} \end{pmatrix} = \begin{pmatrix} t\partial_t + 1 & 2 \\ 0 & t\partial_t + 1 \end{pmatrix} \cdots \begin{pmatrix} t\partial_t + m & 2 \\ 0 & t\partial_t + m \end{pmatrix} \begin{pmatrix} (-i\partial_t)^m f_1 \\ (-i\partial_t)^m f_2 \end{pmatrix}$$

From which one gets easily

$$f_{1,m}(0\pm) = m!\{(-i\partial_t)^m f_1(0\pm) + 2(1+\tfrac{1}{2}+ \dots +1/m)(-i\partial_t)^m f_2(0\pm)\}$$
$$f_{2,m}(0\pm) = m!(-i)^m(\partial_t^m f_2)(0\pm)$$

An argument similar to the case $\lambda \neq 0$, now gives that there exists $f \in \mathfrak{X}_\lambda$ such that $\Lambda_m(f) = v_m$, for all m.

§ 7. Self-adjoint extensions

With these preparations we can now obtain all the self-adjoint extensions of σ_λ.

THEOREM 1. (i) *Let σ be an $\mathfrak{sl}_2$-module such that $\sigma_\lambda \subset \sigma \subset \sigma_\lambda^\dagger$. Then $\sigma_\lambda \subset \sigma^\dagger \subset \sigma_\lambda^\dagger$. Let $E(\sigma)$ denote the subspace of $\mathbf{C}^4$ defined by $E(\sigma) = \{\Lambda_0(f) \mid f \in \mathcal{D}(\sigma)\}$. Then $M_\lambda \cdot E(\sigma) = E(\sigma)$ and $\mathcal{D}(\sigma^\dagger) = \{f \in \mathcal{D}(\sigma_\lambda^\dagger) \mid \Lambda_m(f) \in E(\sigma)^\perp$ for all $m \in \mathbf{N}\}$.*

(ii) *Conversely let $E \subset \mathbf{C}^4$ be such that $M_\lambda \cdot E = E$. Let a $\mathfrak{sl}_2$-module $\sigma \subset \sigma_\lambda^\dagger$ be defined by $\mathcal{D}(\sigma) = \{f \in \mathcal{D}(\sigma_\lambda^\dagger) \mid \Lambda_m(f) \in E$, for all $m \in \mathbf{N}\}$. Then $E(\sigma) = E$ and $\sigma^{\dagger\dagger} = \sigma$. In particular, the map $\sigma \to E(\sigma)$ is a bijection of self-adjoint $\mathfrak{sl}_2$-modules σ such that $\sigma_\lambda \subset \sigma$ and subspaces E such that* (a) *$M_\lambda \cdot E = E$ and* (b) *$E = E^\perp$. Here the orthogonal complement is with respect to the form F_λ introduced in Lemma 7.*

Proof. Since $\Lambda_0(\varrho_\lambda(H)f) = M_\lambda \cdot \Lambda_0(f)$ and M_λ is invertible, it follows that $M_\lambda \cdot E(\sigma) = E(\sigma)$. From the relation $\Lambda_m \circ \varrho_\lambda(X) = -m(M_\lambda + m - 1)\Lambda_{m-1}$, it follows by induction on m, that $\Lambda_m(\mathcal{D}(\sigma)) = E(\sigma)$ for all m. Now $\sigma_\lambda \subset \sigma \subset \sigma_\lambda^\dagger$ implies that $\sigma_\lambda \subset \sigma^\dagger \subset \sigma_\lambda^\dagger$ and $\mathcal{D}(\sigma^\dagger) = \{f \in \mathcal{D}(\sigma_\lambda^\dagger) \mid B_\lambda(a\colon g\colon f) = 0$ for all $a \in \mathcal{U}$ and $g \in \mathcal{D}(\sigma)\}$. Since $B_\lambda(H^m X^n\colon g\colon f) = 0$ for all g and f it follows that $f \in \mathcal{D}(\sigma^\dagger)$ if and only if $B_\lambda(H'^m\colon g\colon f) = 0$ for all m. From part (iv) of previous lemma, this is equivalent to

$$\sum_{r=0}^{n-1} F_\lambda(\Lambda_r(g), \Lambda_{n-r-1}(f)) = 0$$

for all $n \in \mathbf{N}$, and $g \in \mathcal{D}(\sigma)$. Since the range $\Lambda_m(\mathcal{D}(\sigma)) = E(\sigma)$, it follows by induction on n that the above identity holds if and only if $\Lambda_m(f) \in E(\sigma)^\perp$, for all $m \in \mathbf{N}$. That $E(\sigma^\dagger) = E(\sigma)^\perp$ follows from the statement (v) of the previous lemma.

To prove (ii), note first that $\mathcal{D}(\sigma)$ is invariant under $\varrho_\lambda(Z)$, for all $Z \in \mathfrak{sl}(2, \boldsymbol{R})$. In fact this follows from the properties (i)–(iii) of the previous lemma. Since $\Lambda_m(f) = 0$ for $f \in \mathcal{D}(\sigma_\lambda)$, it follows that $\sigma_\lambda \subset \sigma \subset \sigma_\lambda^\dagger$. Thus σ is a well defined $\mathfrak{sl}_2$-module. From the first part it follows that $\mathcal{D}(\sigma^{\dagger\dagger}) = \{f \in \mathcal{D}(\sigma_\lambda^\dagger) \mid \Lambda_m(f) \in E(\sigma^\dagger)^\perp$, for all $m\}$. Since $E(\sigma^\dagger) = E^\perp$ and $E^{\perp\perp} = E$, it follows that $\sigma = \sigma^{\dagger\dagger}$. The rest is clear.

COROLLARY. $\mathcal{D}(\sigma_\lambda^{\dagger\dagger}) = \{f \in \mathcal{D}(\sigma_\lambda^\dagger) \mid \Lambda_m(f) = 0 \text{ for all } m \in \mathbf{N}\}$.

Proof. For $E(\sigma_\lambda^\dagger) = C^4$ and thus $E(\sigma_\lambda^{\dagger\dagger}) = \{0\}$, the corollary follows from this.
The next lemma describes all such subspaces.

LEMMA 10. *Let $\lambda^2 < 1$. For each λ, the following is a complete list of all subspaces E such that $M_\lambda \cdot E = E$ and $E = E^\perp$.*

Case 1. *$\lambda^2 < 0$. In this case E is of the form $E_{\delta,\delta'} = \mathbf{C}(e_1 + \delta e_3) + \mathbf{C}(e_2 + \delta' e_4)$, where $\delta, \delta' \in \mathbf{C}$ are such that $|\delta| = |\delta'| = 1$.*

Case 2. *$0 < \lambda^2 < 1$. There are two classes. Case 2a. E is of the form $E_{\delta,\delta'} = \mathbf{C}(e_1 = \delta e_3) + \mathbf{C}(e_2 + \delta' e_4)$, where $\delta' \bar{\delta} = 1$. Case 2b. E is one of the following $E_{13} = \mathbf{C}e_1 + \mathbf{C}e_3$, $E_{14} = \mathbf{C}e_1 + \mathbf{C}e_4$, $E_{23} = \mathbf{C}e_2 + \mathbf{C}e_3$ and $E_{24} = \mathbf{C}e_2 + \mathbf{C}e_4$.*

Case 3. *$\lambda = 0$. There are two classes. Case 3a. E is of the form*

$$E_{\delta,\delta'} = \{(x_1, x_2, x_3, x_4) \in \mathbf{C} \mid x_4 - \delta x_2 = 0 \quad and \quad \delta x_1 + \delta' x_2 - x_3 = 0\},$$

where $|\delta| = 1$, $\delta' \bar{\delta}$ is real. Case 3b. $E = E_{13}$.

Proof. In the statement of the Lemma, e_j is the standard basis of $\mathbf{C}^4$. Put $W_1 = \mathbf{C}e_1 + \mathbf{C}e_3$ and $W_2 = \mathbf{C}e_2 + \mathbf{C}e_4$. We consider each case separately.

Case 1. $\lambda^2 < 0$. Here the form F_λ (see Lemma 7) is symmetric, and M_λ is a diagonal matrix, with W_1, W_2 being the eigenspaces. Thus $E = E \cap W_1 = E \cap W_2$. In this case $W_1^\perp = W_2$ so that we must have $\dim E \cap W_j = 1$, $j = 1, 2$. Let $v_1 = a_1 e_1 + a_3 e_3$, $v_2 = a_2 e_2 + a_4 e_4$ be a basis of E. Then $F_\lambda(v_1, v_1) = |a_1|^2 - |a_3|^2 = 0$; $F_\lambda(v_2, v_2) = -|a_2|^2 + |a_4|^2 = 0$. Thus E is of the form $E_{\delta,\delta'}$. One checks that $(E_{\delta,\delta'})^\perp = E_{\delta,\delta'}$.

Case 2. $\lambda^2 > 0$. In this case the form F_λ is symplectic, but W_j are still the eigenspaces of M_λ. Consider first the case $\dim E \cap W_1 = 1$. Suppose $v_1 = a_1 e_1 + a_3 e_3$, $v_2 = a_2 e_2 + a_4 e_4$, $E = Cv_1 + Cv_2$. Then $B(v_1, v_2) = a_1 \bar{a}_2 - a_3 \bar{a}_4 = 0$. Now consider the case where none of the a_j's are zero. In this case E is of the form $E_{\delta,\delta'} = \mathbf{C}(e_1 + \delta e_3) + \mathbf{C}(e_2 + \delta' e_4)$ with δ, δ' satisfying $\delta \bar{\delta}' = 1$. One checks that $E^\perp = E$.

Case 2b. $\dim E \cap W_1 = 1$. But one of a_j's is zero. Suppose $a_1 = 0$. Then $a_3 \neq 0$, so that $a_4 = 0$. Thus in this case $E = \mathbf{C}e_2 + \mathbf{C}e_3$. Similarly you get the other possibilities listed.

Case 3. In this case M_0 (see Lemma 8) is unipotent. Write $M_0 = \exp(-2C)$, then C is nilpotent and $C^2 = 0$. Also $W_1 = \mathbf{C}e_1 + \mathbf{C}e_3 = \{v \in \mathbf{C}^4 \mid Cv = 0\}$.

Case 3*a*. $\dim E \cap W_1 = 1$. Then E has a basis v_1, v_2 such that $Cv_1 = 0$, $Cv_2 = v_1$. Thus $v_1 = a_1 e_1 + a_3 e_3$ and $v_2 = \Sigma b_j e_j$. Then $b_2 = a_1$, and $b_4 = a_3$. $F_0(v_1, v_2) = a_1 \bar{b}_2 - a_3 \bar{b}_4 = |a_1|^2 - |a_3|^2 = 0$. $F_0(v_2, v_2) = b_1 \bar{b}_2 - b_2 \bar{b}_1 - b_3 \bar{b}_4 + b_4 \bar{b}_3 = 0$. Thus we may suppose $E = \mathbf{C}(e_1 + \delta e_3) + \mathbf{C}(\delta' e_3 + e_2 + \delta e_4)$, where $\delta, \delta' \in \mathbf{C}$. Then $M_0 E = E$. The condition $E^\perp = E$ gives that $|\delta| = 1$ and $\bar{\delta}\delta'$ is real. Thus in this case $E = E_{\delta,\delta'}$.

Case 3*b*. If $\dim E \cap W_1 = 2$, then $E = \mathbf{C}e_1 + \mathbf{C}e_3$, is the only solution in this case.

Definition. Let $\sigma_{\lambda,\delta,\delta'}$ be the $\mathfrak{sl}_2$-module which is self-adjoint and is defined by the boundary conditions $\sigma_\lambda \subset \sigma_{\lambda,\delta,\delta'} \subset \sigma_\lambda^\dagger$ and

$$\mathcal{D}(\sigma_{\lambda,\delta,\delta'}) = \{f \in \mathcal{D}(\sigma_\lambda^\dagger) \mid \Lambda_m(f) \in E_{\delta,\delta'}, \text{ for all } m \in \mathbf{N}\}. \tag{13}$$

Here $E_{\delta,\delta'}$ is defined in the above lemma, and δ, δ' satisfy the appropriate conditions (depending on λ) stated there.

Remark. In this connection note that the module σ_λ depends only on λ^2, while $\sigma_{\lambda,\delta,\delta'}$ depends on λ. In fact we have ($\lambda \neq 0$),

$$\sigma_{-\lambda,\delta,\delta'} = \sigma_{\lambda,\delta',\delta} \tag{14}$$

This may be seen as follows. Writing $A_1^+(f\colon \lambda)$ for $A_1^+(f)$ to denote its dependence on λ, it is clear that $A_1^+(f\colon -\lambda) = A_2^+(f\colon \lambda)$, if $\lambda \neq 0$. From this it follows that $\Lambda_0(f\colon -\lambda) \in E_{\delta,\delta'}$ if and only if $\Lambda_0(f\colon \lambda) \in E_{\delta',\delta}$, proving (14).

Remark. Let V denote the unitary operator $Vf = c_1 f$ if $t > 0$, $= c_2 f$ if $t < 0$, where $|c_1| = |c_2| = 1$. Let $\sigma = V \circ \sigma_{\lambda,\delta,\delta'} \circ V^{-1}$. Then it is easily checked that

$$\sigma = \sigma_{\lambda,\delta c_2/c_1,\delta' c_2/c_1}. \tag{15}$$

THEOREM 2. (i) *For each λ real and $\lambda + 1 > 0$, there exists a unique self-adjoint $\mathfrak{sl}_2$-module $\mu_\lambda^\pm$ in $L^2(R_\pm)$ such $\mathcal{D}(\mu_\lambda^\pm) \supset \mathcal{X}_\lambda^\pm$.*

(ii) *For $\lambda = 0$, and for $\lambda \geq 1$, $\sigma_\lambda^\pm$ has a unique self-adjoint extension and for $-1 < \lambda < 1$, $\lambda \neq 0$, $\sigma_\lambda^\pm$ has exactly two self-adjoint extensions namely $\mu_\lambda^\pm$, $\mu_{-\lambda}^\pm$.*

Proof. Case 1. $\lambda \geq 1$. In this case $\sigma_\lambda^\pm(Z)$ is essentially self-adjoint, for all $Z \in \mathfrak{sl}(2, \boldsymbol{R})$ (Cor. 3 of Lemma 2). Hence if $\mu_\lambda^\pm$ is a self-adjoint extension of $\sigma_\lambda^\pm$, then by Lemma A.5 $(\sigma_\lambda^\pm)^\dagger = \mu_\lambda^\pm$, and $\mu_\lambda^\pm$ is the unique self-adjoint extension of $\sigma_\lambda^\pm$.

Case 2. $-1 < \lambda < 1$. Consider the self-adjoint module σ in $L^2(R)$ such that $\sigma_\lambda \subset \sigma$ and $E(\sigma) = E_{13}$. Then it is clear that $\mathcal{D}(\sigma) = \mathcal{D}(\sigma) \cap L^2(R_+) + \mathcal{D}(\sigma) \cap L^2(R_-)$ where we consider

$L^2(R_\pm)$ as subspaces of $L^2(R)$. Define $\sigma^\pm$ as $\mathfrak{sl}_2$-modules in $L^2(R_\pm)$, $\mathcal{D}(\sigma^\pm) = \mathcal{D}(\sigma) \cap L^2(R_\pm)$. Then σ self-adjoint means that $\sigma^\pm$ are self-adjoint in $L^2(R_\pm)$. From the definition of $E(\sigma)$, it follows that $\mathcal{D}(\sigma^\pm) \supset \mathfrak{X}_\lambda^\pm$. Define $\mu_\lambda^\pm = \sigma^\pm$. Then $\mu_\lambda^\pm$ are self-adjoint and $\mathcal{D}(\mu_\lambda^\pm) \supset \mathfrak{X}_\lambda^\pm$. To prove uniqueness suppose μ' is another self-adjoint module in $L^2(R_+)$ such that $\mathcal{D}(\mu') \supset \mathfrak{X}_\lambda^+$. Consider $\sigma' = \mu' + \mu_\lambda^-$. Then σ' is self-adjoint and $E(\sigma) \supset E_{13}$. Therefore $E(\sigma') = E_{13}$ and $\sigma = \sigma'$, implying $\mu' = \mu_\lambda^+$. The other cases are handled similarly.

LEMMA 11. *Let $\sigma_\lambda \subset \sigma \subset \sigma_\lambda^\dagger$ and $\sigma^\dagger = \sigma$. Then the operator $\sigma(H')$ is essentially self-adjoint.*

Proof. Now $\sigma = \sigma^\dagger$ implies that $\sigma(H')$ is a symmetric operator. Also $\sigma(H')^* \subset \sigma_\lambda(H')^*$. Suppose ξ is not real and $\sigma(H')^* f = \xi f$. Then $\sigma_\lambda(H')^* f = \xi f$ and so $\varrho_\lambda(H') f = \xi f$, i.e., $f \in \mathfrak{X}_\lambda$ (see Lemma 2). Also $f \in \mathcal{D}(\sigma(H')^*)$. Thus $(\varrho_\lambda(H')g, f) - (g, \varrho_\lambda(H')f) = 0$ for all $g \in \mathcal{D}(\sigma)$ or $B_\lambda(H': g: f) = 0$ for all $g \in \mathcal{D}(\sigma)$. Thus $F_\lambda(\Lambda_0(g), \Lambda_0(f)) = 0$, for all $g \in \mathcal{D}(\sigma)$. This implies that $\Lambda_0(f) \in E(\sigma)^\perp = E(\sigma)$, since σ is self-adjoint. Finally, by the identity satisfied by boundary forms, it follows that

$$B_\lambda(H'^n: g: f) = \sum_{r=0}^{n-1} B(H': \varrho(H')^r g: \xi^{n-r-1} f) = 0$$

for all $g \in \mathcal{D}(\sigma)$, and all n. Using formula (iv) of Lemma 9 it follows by induction on n, that $\Lambda_n(f) \in E(\sigma)$ for all $n \in \mathbf{N}$. Thus $f \in \mathcal{D}(\sigma^\dagger) = \mathcal{D}(\sigma)$. This contradicts symmetry of $\sigma(H')$, since ξ is not real. Thus $f = 0$. This completes the proof.

LEMMA 12. *Let $\sigma_\lambda \subset \sigma \subset \sigma_\lambda^\dagger$ and $\sigma^\dagger = \sigma$. Then the Spec $\sigma(H')$ is discrete. Moreover, all the eigenfunctions of $\mathrm{Cl}\,\sigma(H')$ are actually in $\mathcal{D}(\sigma)$.*

Proof. Let R_ζ denote the resolvent of the closure of the operator $\sigma(H')$, i.e., $R_\zeta = (\mathrm{Cl}\,\sigma(H') - \zeta)^{-1}$. We shall show that for each $f \in C_c^\infty(R')$, $R_\zeta f$ is a meromorphic function of ζ. This implies, by a well known formula for the spectral measure in terms of the resolvent, that the spectrum is discrete.

Let $\zeta \in \mathbf{C}$, ζ not real. Two linearly independent solutions of $\varrho_\lambda(H') f = \zeta f$ may be chosen as follows. The Whittaker's function has an analytic continuation to a domain containing the upper half-plane and R', and so we may define $y_1 = (2z)^{-1/2} W_{\varkappa.\mu}(2z)$, $y_2 = (2z)^{-1/2} W_{-\varkappa.\mu}(-z)$ where $\varkappa = \zeta/2$, $\mu = \lambda/2$. Then y_1, y_2 are solutions of the equation $\varrho_\lambda(H') f = \zeta f$. Also from the formula for the Wronskian of Whittaker's functions, it follows that $W(y_1, y_2) = z^{-1} e^{-i\zeta\pi/2}$. Thus, $tW(y_1, y_2) = e^{-i\zeta\pi/2}$ is a constant. Define $K_\xi(t, s) = e^{-i\varepsilon\pi/2} y_1(t) y_2(s)$ if $s < t$ and $e^{i\zeta\pi/2} y_2(t) y_1(s)$ if $s > t$. Then it follows from standard methods in differential equations that the function $g_1 = K_\xi f: t \to \int_{R'} K_\xi(t, s) f(s)\, ds$ is a solution of $(\varrho_\lambda(H') - \zeta) g_1 = f$, in R', whenever $f \in C_c(R')$. Actually $K_\xi f \in \mathfrak{X}_\lambda$ when $f \in C_c^\infty(R')$, since $y_1\,(y_2)$

is rapidly decreasing as $t \to \infty$ (as $t \to -\infty$). If $R_\zeta f = g$, then it is clear that $(\varrho_\lambda(H') - \zeta)(g - g_1) = 0$. Thus $g - g_1$ is an L^2-eigenfunction and so $g - g_1 \in \mathfrak{X}_\lambda$. Thus $g - g_1 = b_1(f) y_1$ if $t > 0$, $= b_2(f) y_2$ if $t < 0$, where b_1, b_2 are constants depending on f. Now $g \in \mathfrak{X}_\lambda$ and $g \in \mathcal{D}(\mathrm{Cl}\,\sigma(H'))$ means that $B_\lambda(H' : \varphi : g) = 0$ for all $\varphi \in \mathcal{D}(\sigma)$. Thus the boundary condition to be satisfied by g is that $\Lambda_0(g) \in E(\sigma)$. Or the constants are to be determined from the condition $\Lambda_0(K_\zeta f) + b_1(f)\Lambda_0(y_1^+) + b_2(f)\Lambda_0(y_2^-) \in E(\sigma)$ where $y_1^+ = y_1$ if $t > 0$, and $= 0$ if $t < 0$. Since $\Lambda_0(K_\zeta f)$, $\Lambda_0(y_1^+)$ and $\Lambda_0(y_2^-)$ are meromorphic in ζ (in fact, see Lemma 2, they involve only Γ-functions), it follows that $b_1(f)$, $b_2(f)$ depend meromorphically on ζ. Thus the function $\zeta \to (R_\zeta f, \varphi)$ is a meromorphic function of ζ. This proves then the spectrum is discrete. Since L^2-eigenfunctions of the operator $\varrho_\lambda(H')$ are in $\mathfrak{X}_\lambda$, it follows that, if $\sigma(H')^* f = \xi f$, then $f \in \mathfrak{X}_\lambda$ and $B_\lambda(H' : \varphi : f) = 0$ for all $\varphi \in \mathcal{D}(\sigma)$. But this implies that $B_\lambda(H'^m : \varphi : f) = 0$ for all m, and so $f \in \mathcal{D}(\sigma^\dagger) = \mathcal{D}(\sigma)$. This completes the proof.

In the above lemma we are dealing with the case λ^2 real and < 1. If λ is real and $\geqslant 1$, then we have already seen in Corollary 3 of Lemma 2 that $\sigma_\lambda^\pm(H')$ is essentially self-adjoint in $L^2(R_\pm)$. The proof of the above lemma actually gives the following for this case.

LEMMA 13. *Let $\lambda \geqslant 1$. Then spectrum of $\sigma_\lambda^\pm(H')$ is discrete. All the eigenfunctions of* $\mathrm{Cl}\,\sigma_\lambda^\pm(H')$ *are in* $\mathfrak{X}_\lambda^\pm$.

Proof. In this case $R_\zeta = \{\mathrm{Cl}\,\sigma_\lambda^\pm(H') - \zeta\}^{-1}$, and as in the previous lemma, we have $R_\zeta f = K_\zeta f$ in this case, for all $f \in C_c^\infty(R_\pm)$. Thus spec $\sigma_\lambda(H')$ is discrete.

Combining the previous discussion with Nelson's theorem, we have

THEOREM 3. (i) *Let σ be a self-adjoint $\mathfrak{sl}_2$-module in $L^2(R)$ (in $L^2(R_\pm)$) such that $\sigma_\lambda \subset \sigma(\sigma_\lambda^\pm \subset \sigma)$, then there exists a unique unitary representation T of the simply connected Lie group of $\mathfrak{sl}(2, \mathbf{R})$ in $L^2(R)$ (in $L^2(R_\pm)$) such that $dT = \sigma$.*

(ii) Let the representations $T_{\lambda,\delta,\delta'}$, $T_\lambda^\pm$ be defined by $dT_{\lambda,\delta,\delta'} = \sigma_{\lambda,\delta,\delta'}$ and $dT_\lambda^\pm = \mu_\lambda^\pm$. *Then $T_{\lambda,\delta,\delta'}$ and $T_\lambda^\pm$ are all irreducible.*

(iii) *If T is the unitary representation such that $E(dT) = E_{13}$, then $T = T_\lambda^+ \oplus T_\lambda^-$, if $\lambda \geqslant 0$. Similar results hold for other subspaces listed in Lemma* 10.

Proof. Let $\Delta = H^2 + (X + Y)^2 + (X - Y)^2$. Then $\Delta = \Omega - 1 - 2H'^2$. Then $\varrho_\lambda(\Delta) = \lambda^2 - 1 - 2\varrho_\lambda(H')^2$. From Lemmas 11–13, it follows that $\sigma(H')$ is essentially self-adjoint and has discrete spectrum. Moreover, all the eigenfunctions belong to $\mathcal{D}(\sigma)$. From this it is clear that $\sigma(H'^2)$ is also essentially self-adjoint. So Nelson's theorem now gives (i). The case $\sigma \supset \sigma_\lambda^\pm$ is discussed similarly.

To prove (ii) note that if T is any unitary representation such that $\sigma_\lambda \subset dT$, then

$T(\exp sH)f: t \to e^s f(e^{2s}t)$, and $T(\exp sX)f: t \to e^{-ist}f(t)$, Any bounded operator which commutes with these two one parameter groups must be scalars on each of the subspaces $L^2(R_\pm)$. If the bounded operator commutes with T, then $A\mathcal{D}(dT) \subset \mathcal{D}(dT)$ also. Thus if the boundary condition $E(\sigma)$ relates the boundary values on R_+ and R_-, the two scalars on $R_\pm$ must coincide. This proves that $T_{\lambda,\delta,\delta'}$ is irreducible. Similarly $T_\lambda^\pm$ is always irreducible. The rest is clear.

THEOREM 4. (i) *Spec* $dT_\lambda^\pm \cdot (H') = \pm(\lambda + 1 + 2\mathbf{N})$ *for all* λ *real* $\lambda + 1 > 0$. *The unitary equivalence class of* $T_\lambda^\pm$ *is* $\omega^\pm(\lambda)$.

(ii) Spec $dT_{\lambda,\delta,\delta'}(H') = \xi + 2\mathbf{Z}$ where ξ, δ, δ' *and* λ *are related as follows: Case* 1. *Suppose* $\lambda \neq 0$, *then*

$$\frac{\cos \pi(\xi-\lambda)/2}{\cot \pi(\xi+\lambda)/2} = \frac{\delta'}{\delta} \quad \text{or} \quad \frac{\sin \pi(\mu+\lambda)/2}{\sin \pi(\mu-\lambda)/2} = e^{i\pi\xi}$$

where $\delta/\delta' = e^{-i\mu\pi}$. *Case* 2. $\lambda = 0$. *In this case* $\pi \tan \pi\xi/2 = \delta'/\delta$. *In each of these cases* ξ *in the spectrum can be chosen uniquely so that* $|\xi| \leqslant 1$ *and if* λ *is real, then* $|\lambda| + |\xi| < 1$. *The unitary equivalence class of* $T_{\lambda,\delta,\delta'}$ *is* $\omega(\xi, \lambda)$.

Proof. (i) In this case $\xi \in \operatorname{Spec} dT_\lambda^+(H')$ if and only if $L_{\xi/2,\lambda/2} \in \mathfrak{X}_\lambda^+$. From Lemma 2 this happens if and only if $c(\xi, -\lambda) = 0$, i.e. iff $\xi \in \lambda + 1 + 2\mathbf{N}$. A similar argument works for the case T_λ^-.

Proof of (ii). In this case $\xi \in \operatorname{Spec} dT_{\lambda,\delta,\delta'}(H')$ if and only if there exists an $f \in \mathfrak{X}_\lambda$, $\varrho_\lambda(H')f = \xi f$ such that $\Lambda_0(f) \in E_{\delta,\delta'}$. Now $f = \alpha L_{\xi/2,\delta/2}(t)$ for $t > 0$, $= \beta L_{\xi/2,\lambda/2}(-t)$ if $t < 0$. Then we have two cases.

Case 1. Suppose $\lambda^2 < 1$, $\lambda \neq 0$. Then from Lemma 2, we have the following

$$\Lambda_0(f) = (\alpha c(\xi, \lambda), \alpha c(\xi, -\lambda), \beta c(-\xi, \lambda), \beta c(-\xi, -\lambda))$$

Thus ξ belongs to the spectrum if and only if $\beta c(-\xi, \lambda) = \delta\alpha c(\xi, \lambda)$ and $\beta c(-\xi, \lambda) = \delta'\alpha c(\xi, -\lambda)$ since $c(\xi, \lambda)$, $c(-\xi, -\lambda)$ cannot both be zero simultaneously it follows that both α and $\beta \neq 0$. (Note $|\delta\delta'| > 0$.) Thus ξ belongs to the spectrum if and only if

$$\frac{c(-\xi, \lambda)}{c(\xi, \lambda)} \frac{c(\xi, -\lambda)}{c(-\xi, -\lambda)} = \frac{\delta}{\delta'}.$$

Now

$$c(\xi, \lambda)c(-\xi, -\lambda) = \Gamma(-\lambda)\Gamma(\lambda)\{\Gamma(1-\xi-\lambda)/2)\Gamma((1+\xi+\lambda)/2)\}^{-1}$$
$$= (\pi^{-1})\Gamma(-\lambda)\Gamma(\lambda) \cos \pi(\xi+\lambda)/2.$$

Thus we have

$$\frac{\cos \pi(\xi-\lambda)/2}{\cos \pi(\xi+\lambda)/2}=\frac{\delta}{\delta'}.$$

If you put $\delta/\delta'=e^{-i\mu\pi}$ and simplify this expression above, we get the equivalent formulation given in the theorem. It is then easy to check that there exists a ξ in the spectrum such that $|\mathrm{Re}\,\lambda|+|\xi|<1$, when λ is real.

Case 2. Suppose $\lambda=0$. In this case we have from Lemma 2 that

$$\begin{aligned}\Lambda_0(f) &= (\alpha(-1)^{(\xi-1)/2},\ 0,\ \beta c(-\xi,0)d(-\xi),\ \beta c(-\xi,0)) \quad \text{or}\\ &= (\alpha c(\xi,0)d(\xi),\ \alpha c(\xi,0),\ (-1)^{(1+\xi)/2},\ 0) \quad \text{or}\\ &= (\alpha c(\xi,0)d(\xi),\ \alpha c(\xi,0),\ \beta c(-\xi,0)d(-\xi),\ \beta c(-\xi,0))\end{aligned}$$

where the first expression holds if $\xi-1\in 2\mathrm{N}$, the second one holds if $-(\xi+1)\in 2\mathrm{N}$, and the last one is valid if $\xi\notin\pm(1+2\mathrm{N})$. One checks easily from the definition of $E_{\delta,\delta'}$ (see Lemma 10) that in the first two cases $\Lambda_0(f)\notin E_{\delta,\delta'}$. Thus $\xi\notin\pm(1+2\mathrm{N})$ and $\Lambda_0(f)\in E_{\delta,\delta'}$ implies

$$\beta c(-\xi,0)=\delta\alpha c(\xi,0)$$

$$\delta\alpha c(\xi,0)d(\xi)+\delta'\alpha c(\xi,0)-\beta c(-\xi,0)d(-\xi)=0$$

Note that α, β cannot both be zero. If $\alpha=0$, then $c(-\xi,0)=0$, which is impossible since $\xi\notin\pm(1+2\mathrm{N})$. Thus both $\alpha\neq 0$, $\beta\neq 0$. Thus we have

$$d(\xi)-d(-\xi)=\delta'/\delta$$

Now $d(\xi)-d(-\xi)=\psi((1-\xi)/2)-\psi((1+\xi)/2)=\pi\tan\pi\xi/2$, since ψ is the logarithmic derivative of the Γ-function. Thus $\pi\tan(\pi\xi)/2=\delta'/\delta$. The rest is clear.

§ 8. Bases of eigenfunctions—the discrete series

We next obtain a basis of K-finite vectors for each of the representations $T_\lambda^\pm$. We begin with

LEMMA 14. *Let* $L_{\xi/2,\lambda/2}(t)=(2t)^{-1/2}W_{\xi/2,\lambda/2}(2t)$ $(t>0)$ *be the eigenfunctions of* $\varrho_\xi(H')$ *introduced earlier. Then*

(i) $\varrho_\lambda(H')L_{\xi/2,\lambda/2}=\xi L_{\xi/2,\lambda/2}$

(ii) $\varrho_\lambda(X')L_{\xi/2,\lambda/2}=-iL_{(\xi+2)/2,\lambda/2}$

(iii) $\varrho_\lambda(Y')L_{\xi/2,\lambda/2}=-a(a-c+1)L_{(\xi-2)/2,\lambda/2}$

where $a=(1-\xi+\lambda)/2$, $c=1+\lambda$, X' *and* Y' *are defined by* (3).

Proof. Let $\Psi(a, c; x)$ denote Tricomi's confluent hypergeometric function. Then the following identities for Ψ are known (see [3], p. 258)

$$(x\partial_x - x + c - a)\Psi = -\Psi(a-1, c; x)$$
$$(x\partial_x + a)\Psi = a(a-c+1)\Psi(a+1, c; x).$$

Since $L_{\xi/2,\lambda/2}(t) = (2t)^{(c-1)/2}e^{-t}\Psi(a, c; 2t)$, the lemma follows from the identities satisfied by Ψ.

THEOREM 5. (i) *Let λ be real and $\lambda+1>0$. Define $\psi_\xi^+(t) = \{c(\xi, \lambda)\}^{-1}L_{\xi/2,\lambda/2}(t)$, $t>0$. Then ψ_ξ^+, $\xi \in \lambda+1+2\mathrm{N}$ is a K-eigenbasis in $L^2(R_+)$ for the representation T_λ^+. Another expression for ψ_ξ^+ is the following*

$$\psi_\xi^+(t) = \left\{\binom{n+\lambda}{n}\right\}^{-1} e^{-t}t^{\lambda/2}L_n^{(\lambda)}(2t) \tag{13}$$

where $\xi = \lambda + 1 + 2n$. Also

$$(\psi_\xi^+, \psi_\xi^+) = \left\{\binom{n+\lambda}{\lambda}\right\}^{-1} \Gamma(\lambda+1)\, 2^{-(\lambda+1)}.$$

(ii) *If $\mathcal{S}(R_+)$ denotes the Schwartz space of R_+ then $\mathcal{D}(dT_\lambda^+) = \{t^{\lambda/2}f \mid f \in \mathcal{S}(R_+)\}$.*

(iii) *If J is the anti-unitary isomorphism $Jf: t \to \overline{f(-t)}$, $f \in L^2(R_-)$, then $T_\lambda^- = J^{-1} \circ T_\lambda^+ \circ J$.*

Proof. This theorem can be proved independently of the earlier development. The expression for ψ_ξ^+ in terms of the Laguerre polynomials follows from the identity $\Psi(-n, 1+\lambda; x) = n!(-1)^n L_n^{(\lambda)}(x)$ (see [3], p. 268). One could deduce this directly, since the differential equation $\varrho_\lambda(H')f = \xi f$ reduces to that of Laguerre polynomials, by putting $f = e^{-t}t^{\lambda/2}g$. To prove (ii) note that $t^{\nu/2}\mathcal{S}(R_+) \subset \mathcal{D}((\sigma_\lambda^+)^*) = \mathcal{D}(dT_\lambda^+)$. Let f be a C^∞-vector. Then we know (see Lemma 1) $f \in C^\infty(R_+)$ and $\varrho_\lambda(a)f \in L^2(R_+)$ for all $a \in \mathcal{U}$. From Lemma 1 it follows that f is rapidly decreasing at ∞. Now let $f = \Sigma a_\xi \psi_\xi^+$ be the eigenfunction expansion of f. It follows then that $a_\xi \xi^k = (\varrho_\lambda(H')^k f, \psi_\xi^+)$. Now $(\psi_\xi^+, \psi_\xi^+) = O(n^{-\lambda})$ if $\xi = \lambda+1+2n$. Thus $a_\xi = O(n^{-k})$ for every k. Now it is known that $|L_n^{(\lambda)}(x)| \leq Cn^\mu$ for all x, $0<x<1$ (see Szegö [16], p. 176), where $\mu = \mathrm{Max}\,(1/2-1/4, \lambda/2)$. Also $(d/dx)\, L_n^{(\lambda)} = -L_{n-1}^{(\lambda)}$ ([16], p. 101). Thus the series

$$\Sigma a_\xi (d/dt)^r L_n^{(\lambda)}(2t)$$

converges absolutely and uniformly in $(0, 1/2)$. Then the function $\varphi(t) = \Sigma a_\xi L_n^{(\lambda)}(2t) \in C^\infty[0, 1/2)$ and $f = t^{\lambda/2}e^{-t}\varphi$ in $0<t<1/2$, proving $f \in t^{\lambda/2}\mathcal{S}(R_+)$.

For the part (iii), it is easy to check that $\sigma_\lambda^- = J^{-1} \circ \sigma_\lambda^+ \circ J$, and since representations T, such that $\mathcal{D}(dT) \supset \mathfrak{X}_\lambda^\pm$ are unique, the result follows.

The following theorem is known. We state it and sketch a proof since it fits in naturally with the development here, and we will need it for another paper. (Foor $SL(2, \boldsymbol{R})$ see Kunze and Stein [6], Vilenkin [17], and for simply connected covering group of $SL(2, \boldsymbol{R})$ see Sally [15].)

THEOREM 6. *The unitary representations T_λ^+ of the simply connected group G of $SL(2, \boldsymbol{R})$ may be described by the formulae*

(i) $T_\lambda^+(h_s)f\colon t \to e^s f(e^{2s}t)$

(ii) $T_\lambda^+(u_s)f\colon t \to e^{-ist}f(t)$

(iii) $T_\lambda^+(w)f = e^{-i(\lambda+1)\pi/2}H_\lambda f$, *where H_λ is Hankel transform*

$$H_\lambda f = \operatorname*{l.i.m.}_{a\to\infty} \int_0^a f(s)\, J_\lambda(2(st)^{1/2})\, ds.$$

Proof. The first two statements are clear. The last one can be proved in several ways. It is known that the Hankel transform is a unitary operator and self-reciprocal or $H_\lambda^2 =$ identity. It is thus sufficient to check that $H_\lambda \psi_\xi^+ = \exp(-i\xi\pi)\psi_\xi^+$. But this follows from a known integral formula. (See [4], p. 42, No. (3).) One could also prove it by observing that the operator $\sigma_\lambda(Y)$ with domain $t^{\lambda/2}\mathcal{S}(R_+)$ is essentially skew-adjoint, and the operator H_λ is really the spectral map (or 'diagonalizing' operator) for $\sigma_\lambda(Y)$. In other words $H_\lambda \circ \sigma_\lambda(Y) \circ H_\lambda^{-1} = \sigma_\lambda(X)$ (see Dunford and Schwartz [2], p. 1535). Since $T_\lambda^+(w)$ is also a spectral map, it follows that the operator $T_\lambda^+(w)\circ H_\lambda^{-1}$ commutes with $\sigma_\lambda(X)$, $\sigma_\lambda(Y)$ and hence with $\sigma_\lambda(Z)$ for all $Z \in \mathfrak{sl}(2, \boldsymbol{R})$. Thus the operator $T_\lambda^+(w)\circ H_\lambda^{-1}$ is a scalar. The scalar can be evaluated by evaluating $T_\lambda^+(w)f$ and $H_\lambda \cdot f$, for $f = e^{-t}t^{\lambda/2} = \psi_{\lambda+1}^+$. We omit the details.

Remark 1. It is easy to calculate the matrix entry

$$(T_\lambda^+(h_t)\,\psi_{\lambda+1}^+, \psi_{\lambda+1}^+) = \int_0^\infty e^{-y\cosh t} y^\lambda\, dy = \Gamma(\lambda+1)/(\cosh t)^{\lambda+1}.$$

Since any element of G can be written in the form $\varkappa(\theta_1)h(t)\varkappa(\theta_2)$ with $t \geqslant 0$, and the Haar measure in this decomposition is $|e^{2t} - e^{-2t}|\, d\theta_1\, dt\, d\theta_2$, it follows that $T_\lambda^+ \in L^1(G/Z)$ if $\lambda > 1$ and $T_\lambda^+ \in L^2$ if $\lambda > 0$. These are well known.

Remark 2. Let $P = ZAN$ be the minimal parabolic subgroup of G. One has the Bruhat decomposition $G = P \cup NwP$. G is then generated by P and w and the relations satisfied by w are $w^2 = \gamma$, $wh_tw^{-1} = h_{-t}$ and for $t \neq 0$,

$$wu_tw = u_{-1/t}wu_{-t}h_{\ln|t|}\cdot\gamma^{m(t)} \tag{16}$$

where the integer $m(t) = 1$ if $t > 0$ and $= 0$ if $t < 0$. This can be checked directly for $SL(2, \boldsymbol{R})$. However, for the simply connected covering group, the third relation above is not so obvious. One could presumably use Bargmann's parametrization of G to verify this. Another method would be to use the representations T_λ^+ for this purpose. For example, let $t>0$, define

$$F(y,t)=\{T_\lambda^+(wu_tw)\psi_{\lambda+1}^+\}(y)\cdot e^{i\pi(\lambda+1)}.$$

Then the identity for wu_tw gives the following functional equation

$$e^{i\pi(\lambda+1)(m-1/2)}F(y,t)=t^{-1}e^{iy/t}F(y/t^2,-1/t).$$

The integer m can be determined from the above identity. In fact

$$F(y,t)=\int_0^\infty e^{-x(1+it)}x^{\lambda/2}J_\lambda(2(xy)^{1/2})\,dx$$

and from formula (10), ([4], p. 29), it follows that

$$F(y,t)=y^{\lambda/2}(1+it)^{-(\lambda+1)}\exp(-y/1+it).$$

From this the value of the integer m is easily calculated to be $=1$. Thus $m(t)=1$ for $t>0$. Also it is easy to check directly that $m(t)+m(-t)=1$. Thus $m(t)=0$ for $t<0$, proving the identity completely.

Remark 3. Another formula for $T_\lambda^+(w)$ is known. (See Sally [15], for details.) It may be described as follows. Let $M: L^2(R^+)\to L^2(R)$ be the unitary isomorphism given by the Mellin transform

$$Mf: x\to\frac{1}{\sqrt{2\pi}}\int_0^\infty t^{-(\frac{1}{2}+ix)}f(t)\,dt \tag{17}$$

for $f\in C_c(R_+)$. Then $T_\lambda^+(w)=W_\lambda\cdot V$, where V is the operator

$$Vf: t\to\frac{1}{t}f\left(\frac{1}{t}\right)$$

and $MW_\lambda M^{-1}$ is the operator of multiplication by the function

$$e^{-i\pi(\lambda+1)/2}\,\Gamma\left(\frac{1+\lambda-2ix}{2}\right)\Big/\Gamma\left(\frac{1+\lambda+2ix}{2}\right).$$

This can be checked easily from the following facts.

(i) $VT_\lambda^+(h_s)V^{-1}=T_\lambda^+(h_{-s})$ and thus W_λ commutes with h_s for all s.

(ii) $MW_\lambda M^{-1}$ is thus a multiplication operator. Finally the multiplier function can be evaluated by using the fact that $e^{-t}t^{\lambda/2}$ is an eigenfunction for $T_\lambda^+(w)$. We omit the details.

§ 9. The representations $T_{\lambda,\delta,\delta'}$

The representations $T_{\lambda,\delta,\delta}'$ were defined infinitesimally by the condition $dT_{\lambda,\delta,\delta'}=\sigma_{\lambda,\delta,\delta'}$, i.e.

$$\mathcal{D}(dT_{\lambda,\delta,\delta'})=\{f\in\mathcal{D}(\sigma_\lambda^\dagger)\,|\,\Lambda_m(f)\in E_{\delta,\delta'},\ \text{for all } m\in\mathbf{N}\}.$$

Here λ^2 is real and <1. $E_{\delta,\delta'}$ is defined in Lemma 10. A K-eigenbasis for the representation is given in the following.

LEMMA 15. *For each $\xi\in\operatorname{Spec} dT_{\lambda,\delta,\delta'}(H')$, define*

$$\psi_{\lambda,\xi}(t)=\begin{cases}\{c(\xi,\lambda)\}^{-1}L_{\xi/2,\lambda/2}(t), & t>0\\ \delta\{c(-\xi,\lambda)\}^{-1}L_{-\xi/2,\lambda/2}(-t), & t<0.\end{cases}$$

Then $\psi_{\lambda,\xi}$ is an eigenbasis of $dT_{\lambda,\delta,\delta'}(H')$. Moreover, the following formulae hold:

(i) $dT_{\lambda,\delta,\delta'}(H')\psi_{\lambda,\xi}=\xi\psi_{\lambda,\xi}$,

(ii) $dT_{\lambda,\delta,\delta'}(X')\psi_{\lambda,\xi}=\{i(1+\xi+\lambda)/2\}\psi_{\lambda,\xi+2}$,

(iii) $dT_{\lambda,\delta,\delta'}(Y')\psi_{\lambda,\xi}=\{-i(1-\xi+\lambda)/2\}\psi_{\lambda,\xi-2}$.

Proof. Since ξ is an eigenvalue, the eigenfunction f_ξ is of the form $c_1L_{\xi/2,\lambda/2}(t)$, $t>0$ and $c_2L_{-\xi/2,\lambda/2}(-t)$, for $t<0$. The constants c_1, c_2 are to be determined from the condition $\Lambda_0(f)\in E_{\delta,\delta'}$. From the local expansion of $L_{\xi/2,\lambda/2}$ it is easy to check that $\Lambda_0(\psi_{\lambda,\xi})\in E_{\delta,\delta'}$. Since the multiplicities are one, it follows that $\{\psi_{\lambda,\xi}\}$ is an eigenbasis. The formulae (i)–(iii) follow from Lemma 14.

Remark. The basis $\psi_{\lambda,\xi}$ is not orthonormal. If $\lambda=i\nu$, ν real $\neq 0$, then the identity $(X'\cdot\psi_{\lambda,\xi},\psi_{\lambda,\xi+2})=-(\psi_{\lambda,\xi},Y'\cdot\psi_{\lambda,\xi+2})$ gives $(\psi_{\lambda,\xi+2},\psi_{\lambda,\xi+2})=(\psi_{\lambda,\xi},\psi_{\lambda,\xi})$, for all ξ. Thus all the functions $\psi_{\lambda,\xi}$ have the same norm. Using the formula (40) on page 409 of [4], one can show that in this case

$$(\psi_{\lambda,\xi},\psi_{\lambda,\xi})=\frac{\nu\tan h(\nu\pi/2)\sec^2(\pi\xi/2)}{1+\tan^2(\pi\xi/2)\tan h^2(\nu\pi/2)}. \tag{19}$$

The norm of $\psi_{\lambda,\xi}$ can be evaluated for other values of λ, but it is no longer independent of ξ.

We next describe C^∞-vectors of the representation.

THEOREM 7. *Suppose $\lambda\neq 0$. Then $f\in\mathcal{D}(\sigma_\lambda^\dagger)$ if and only if*

(i) *$f\in C^\infty(R')$ and $\sup_{|t|>t_0}|t^m\partial_t^n f|<\infty$ for all $t_0>0$ and $m,n\in\mathbf{N}$.*

(ii) *Let $\alpha=\pm\lambda$. Then $\lim_{t\to 0\pm}|t|^{\alpha/2}\varrho_\lambda(a)f$ exists for all a belonging to the right ideal $(H-1-\alpha)\mathcal{U}+X\mathcal{U}$*

Moreover, $f \in \mathcal{D}(dT_{\lambda,\delta,\delta'})$ or is a C^∞-vector for the representation if and only if

$$\lim_{t\to 0-} |t|^{\alpha/2} \varrho_\lambda(a) f = \delta_\alpha \lim_{t\to 0+} |t|^{\alpha/2} \varrho_\lambda(a) f \tag{20}$$

for all $a \in (H-1-\alpha)\mathfrak{U} + X\mathfrak{U}$, $\alpha = \pm\lambda$, where $\delta_\alpha = \delta$ for $\alpha = \lambda$, $\delta_\alpha = \delta'$ for $\alpha = -\lambda$.

Proof. Suppose $f \in \mathcal{D}(\sigma_\lambda^\dagger)$. Then f satisfies (i) by Lemma 1. Moreover, $f(t) = O(|t|^{-1/2})$ as $t \to 0$ so that $|t|^{\alpha/2} \varrho_\lambda(a) f = 0$ for all $a \in X\mathfrak{U}$. Next

$$\begin{aligned} A_1^\pm(f) &= \lim_{t\to 0\pm} tW(f, |t|^{-\lambda/2}) \\ &= \lim_{t\to 0\pm} (-1/2)|t|^{-\lambda/2} \varrho_\lambda(H-1+\lambda) f. \end{aligned}$$

Similarly

$$A_2^\pm(f) = (-1/2) \lim_{t\to 0\pm} |t|^{\lambda/2} \varrho_\lambda(H-1-\lambda) f.$$

Thus if $f \in \mathcal{D}(\sigma_\lambda^\dagger)$, then (ii) holds. Conversely suppose f satisfies (i) and (ii). Then $\varrho_\lambda(H-1-\alpha) f \in L^2$ for $\alpha = \pm\lambda$ and thus $f \in L^2$. Thus $\varrho_\lambda(a) f \in L^2$ for all $a \in \mathfrak{U}$, or $f \in \mathcal{D}(\sigma_\lambda^\dagger)$. Next note that if $f \in \mathcal{D}(\sigma_\lambda^\dagger)$, then

$$\lim_{t\to 0\pm} |t|^{\alpha/2} \varrho_\lambda(a) f = 0$$

if $a \in X\mathfrak{U}$. On the other hand, the result $\Lambda_0 \circ \varrho_\lambda(H) = M_\lambda \cdot \Lambda_0$ of Lemma 8, implies that

$$\lim_{t\to 0\pm} |t|^{\alpha/2} \varrho_\lambda\{(H-1-\alpha)^2 a\} f = 0$$

for all $a \in \mathfrak{U}$. Thus when $f \in \mathcal{D}(\sigma_\lambda^\dagger)$, the condition (20) is satisfied when a is of the form $(H-1-\alpha)^2\mathfrak{U} + X\mathfrak{U}$. Now $E_{\delta,\delta'} = \{(x_1, x_2, x_3, x_4) \in \mathbb{C}^4 \mid x_3 = \delta x_1, x_4 = \delta' x_2\}$. Also $\Lambda_m = \Lambda_0 \circ \varrho_\lambda(Y)^m$, where $\Lambda_0 = (A_1^+, A_2^+, A_1^-, A_2^-)$. Thus if $f \in \mathcal{D}(\sigma_\lambda^\dagger)$, then f is a C^∞-vector if and only if it satisfies $\Lambda_m(f) \in E_{\delta,\delta'}$; this is equivalent to the condition (20).

THEOREM 8. *Suppose $\lambda = 0$. Then $f \in \mathcal{D}(\sigma_\lambda^\dagger)$ if and only if*

(i) *$f \in C^\infty(R')$ and $\sup_{|t|>t_0} |t^m \partial_t^n f| < \infty$ for all $t_0 > 0$ and $m, n \in \mathbb{N}$.*
(ii) *$\lim_{t\to 0\pm} \varrho_0(a) f$ exists for all $a \in (H-1)\mathfrak{U} + X\mathfrak{U}$.*
(iii) *$\lim_{t\to 0\pm} \{2 - \ln|t| \varrho_0(H-1)\} \varrho_0(a) f$ exists for all $a \in \mathfrak{U}$. Moreover, f is a C^∞-vector for $T_{0,\delta,\delta'}$ if and only if*
(iv) *$\lim_{t\to 0-} \varrho_0(a) f = \delta \lim_{t\to 0+} \varrho_0(a) f$ for all $a \in (H-1)\mathfrak{U} + X\mathfrak{U}$ and*
(v) *$\lim_{t\to 0-} \{2 - \ln|t| \varrho_0(H-1)\} \varrho_0(a) f$
$= \delta \lim_{t\to 0+} \{2 - \ln|t| \varrho_0(H-1)\} \varrho_0(a) f - \delta' \lim_{t\to 0+} \varrho_0\{(H-1)a\} f$ for all $a \in \mathfrak{U}$.*

This theorem is proved the same way as the previous one. We omit the details.

THEOREM 9. *Let π be a continuous (not necessarily unitary) representation of G in $L^2(R)$ such that*

(i) $\pi(h_s)f: t \to e^s f(e^{2s}t)$

(ii) $\pi(u_s)f: t \to e^{-i\cdot st}f(t)$ *for all* $s \in R$. *Then* $C_c^\infty(R') \subset \mathcal{D}(d\pi)$ *and there exist complex constants* λ, μ *such that*

$$d\pi | C_c^\infty(R_+) \supset \sigma_\lambda^+, \quad d\pi | C_c^\infty(R_-) \supset \sigma_\mu^-.$$

In particular, if π is unitary, then there are only two possibilities: (a) *π is irreducible and $\pi = T_{\lambda,\delta,\delta'}$ for some λ, δ, δ';* (b) *π is reducible and $\pi = T_\lambda^+ \oplus T_\mu^-$, for some λ, μ.*

Proof. We first show that $\mathcal{D}_0 = \mathcal{D}(d\pi)$—the set of C^∞-vectors of π is an $\mathcal{S}(R)$-module; i.e., if $f \in \mathcal{S}(R)$, $\varphi \in \mathcal{D}_0$, then $f\varphi \in \mathcal{D}_0$. Let $\pi_1 = \pi | AN$. Then one checks by Weyl's lemma (see Appendix A) that $\mathcal{D}(d\pi_1) = \{f \in C^\infty(R') \,|\, t^m(t\partial_t)^n f \in L^2(R)$, for all $m, n \in \mathbf{N}\}$, and $d\pi_1(H)f = (2t\partial_t + 1)f$, and $(d\pi_k)(X)f = -itf$. Clearly $\mathcal{D}_0 \subset \mathcal{D}(d\pi_1) \subset C^\infty(R')$. Next $\mathcal{D}_0$ is a complete locally convex vector space in the semi-norms, $f \to \|d\pi \cdot (a) f\|_2$, $a \in \mathcal{U}$. Moreover, $\pi(x)\mathcal{D}_0 \subseteq \mathcal{D}_0$ and $\pi | \mathcal{D}_0$ is a continuous representation of G. If $\varphi \in \mathcal{D}_0$, $g \in L^1(R)$, and if

$$\int |g(s)| \, \nu(\pi(u_s) \cdot \varphi) \, ds < \infty$$

for all continuous semi-norms ν on $\mathcal{D}_0$, then $\hat{g}\varphi = \int g(s)\pi(u_s)\varphi ds \in \mathcal{D}_0$. Now suppose $a \in \mathcal{U}$, $\deg a \leqslant r$, and $\nu(f) = \|d\pi(a) \cdot f\|_2$, $f \in \mathcal{D}_0$. From the properties of universal enveloping algebras, it follows that if $a_1, a_2, \ldots, a_m \in \mathcal{U}$ is a basis of the subspace of elements of degree $\leqslant r$, then there exist polynomials $p_1, \ldots, p_m$ in s such that

$$\mathrm{Ad}\, u^{-1} \cdot a = \Sigma p_j(s) a_j.$$

Thus

$$\nu(\pi(u_s)\varphi) \leqslant \Sigma |p_j(s)| \, \|d\pi(a_j)\varphi\|.$$

It thus follows that if $\int |g(s)| \, |s|^r ds < \infty$ for all $r \in \mathbf{N}$, then $\hat{g}\varphi \in \mathcal{D}_0$, where $\hat{g}$ denotes the Fourier transform of g. In particular, $\mathcal{S}(R)\varphi \subset \mathcal{D}_0$. In particular, $C_c^\infty(R')\mathcal{D}_0 \subset \mathcal{D}_0$.

Since $\pi(h_t)\mathcal{D}_0 \subset \mathcal{D}_0$, it follows that for each $t_0 \in R'$, there exists $\varphi \in \mathcal{D}_0$ such that $\varphi(t_0) \neq 0$, and hence a $f_{t_0} \in C_c^\infty(R') \cap \mathcal{D}_0$, such that $f_{t_0} = 1$ in a neighborhood of t_0. This implies easily that $C_c^\infty(R') \subset \mathcal{D}_0$. Let $D = d\pi(Y)$. It will be shown next that D is a second order differential operator. In fact $\pi(u_s^{-1}) D\pi(u_s)\varphi = d\pi(e^{-s\,\mathrm{ad}X}Y)\varphi = d\pi(Y \pm sH - s^2X)\varphi = \varphi_1 - s\varphi_2 - s^2\varphi_3$, where $\varphi_1 = D\varphi$, $\varphi_2 = d\pi(H)\varphi$, $\varphi_3 = d\pi(X)\varphi$. Thus

$$D\pi(u_s)\varphi = e^{-ist}\varphi_1 - se^{-ist}\varphi_2 - s^2 e^{-ist}\varphi_3.$$

Let

$$f(t) = \int g(s)\, e^{-ist}\, ds, \text{ with } g \in \mathcal{S}(R).$$

Then

$$D(f\varphi) = \int g(s)\, D\pi(u_s)\, \varphi\, ds = f\varphi_1 - i(\partial_t f)\, \varphi_2 + (\partial_t^2 f)\, \varphi_3.$$

Now suppose $f \in C_c^\infty(R')$ and $\varphi \in C_c^\infty(R')$ with $\varphi = 1$ on supp f. Then $f\varphi = f$, $\varphi_2 = 1$ on supp f, $\varphi_3 = -it$ on supp f. Thus

$$Df = f\varphi_1 - (i\partial_t f) - it(\partial_t^2 f)$$

where $f\varphi_1 = fD\varphi$, for any φ in $C_c^\infty(R')$ equal to 1 on supp f. From this it follows that $D \mid C_c^\infty(R')$ is a second order differential operator of the form $-i(t\partial_t^2 + \partial_t + \psi)$. Now the commutation rule $[H, Y] = -2Y$, gives, that there exist constants c_1, c_2 such that $\psi = c_1/t$ for $t > 0$, $= c_2/t$ for $t < 0$. Thus $d\pi \mid C_c^\infty(R_+) \supset \sigma_\lambda^+$, $d\pi \mid C_c^\infty(R_-) \supset \sigma_\mu^-$ for some λ, μ. Unitarity of π implies λ^2, μ^2 are real. If π is irreducible, then $d\pi(\Omega) =$ constant, so that $\lambda = \mu$ in this case. In this case $dT_{\lambda,\delta,\delta'}$ are the only irreducible self-adjoint extensions of σ_λ. Thus $\pi = T_{\lambda,\delta,\delta'}$ for some λ, δ, δ'. The second case is proved similarly.

For the representations $T_{\lambda,\delta,\delta'}$, an analogue of Theorem 6 can be given via the two-component Mellin transform. (See also Remark 3 following Theorem 6.) (In this connection see Sally [15], Vilenkin [17], for another approach.) Let $M: L^2(R_+) \to L^2(R)$ be the Mellin transform defined earlier (17). Define $M': L^2(R_-) \to L^2(R)$ by $M'f = Mf^0$, $f^0(t) = f(-t)$. Put $M_2 f = (Mf, M'f)$. Then M_2 gives a unitary isomorphism of $L^2(R')$ with $L^2(R) \otimes \mathbf{C}^2$ and is called the two-component Mellin transform. Let $A(x) = (a_{ij}(x))_{1 \leq i,j \leq 2}$ be a unitary matrix valued function on R. Define the operator $I \otimes A$ on $L^2(R) \otimes \mathbf{C}^2$ as follows:

$$I \otimes A \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} a_{11}(x)\, f_1(x) + a_{12}(x)\, f_2(x) \\ a_{21}(x)\, f_1(x) + a_{22}(x)\, f_2(x) \end{pmatrix}.$$

We call $I \otimes A$ a matrix multiplication operator.

THEOREM 10. *Let V be the unitary operator $Vf: t \to |t|^{-1} f(1/t)$, in $L^2(R)$.* Put $W_{\lambda,\delta,\delta'} = T_{\lambda,\delta,\delta'}(w)\, V$. *Then*

$$M_2 \circ W_{\lambda,\delta,\delta'} \circ M_2^{-1} = I \otimes A_{\lambda,\delta,\delta'}$$

for a suitable matrix-function $A_{\lambda,\delta,\delta'}$, and its matrix entries have explicit formulas in terms of the gamma and hypergeometric functions.

Proof. For the sake of brevity we use the module notation and write h_s instead of the operator $T_{\lambda,\delta,\delta'}(h_s)$. One checks easily that $Vh_s V^{-1}=h_{-s}$ and $wh_s w^{-1}=h_{-s}$ in G. So it follows that the operator $W_{\lambda,\delta,\delta'}$ commutes with the operator h_s. However, $M_2 h_s M_2^{-1}$ is just the operator of multiplication by $\exp(-2isx)$. Thus it follows from known results that $W_{\lambda,\delta,\delta'}$ is a matrix multiplication operator. To determine this matrix we use that $\psi_{\lambda,\xi}$ are eigenfunctions. Note $M_2 V M_2^{-1}: \varphi\to\varphi^0$ for $\varphi\in L^2(R)\otimes\mathbf{C}^2$, $\varphi^0(x)=\varphi(-x)$. Now put

$$g_{\lambda,\xi}(x)=\frac{1}{\sqrt{2\pi}}\int_0^\infty t^{-(1/2+ix)}\psi_{\lambda,\xi}(t)\,dt=M\psi_{\lambda,\xi}.$$

Then

$$M_2\psi_{\lambda,\xi}=\begin{pmatrix} g_{\lambda,\xi}(x)\\ \delta g_{\lambda,-\xi}(x)\end{pmatrix}$$

Since $\psi_{\lambda,\xi}$ is an eigenfunction for the eigenvalue $\exp(-\frac{1}{2}i\pi\xi)$, it follows that

$$A\begin{pmatrix} g_{\lambda,\xi}(-x)\\ \delta g_{\lambda,-\xi}(-x)\end{pmatrix}=e^{-i\pi\xi/2}\begin{pmatrix} g_{\lambda,\xi}(x)\\ \delta g_{\lambda,-\xi}(x)\end{pmatrix}$$

for all ξ in the spectrum. Now it is easy to check that if

$$A\begin{pmatrix}\alpha_1\\ \alpha_2\end{pmatrix}=\begin{pmatrix}\beta_1\\ \beta_2\end{pmatrix},\quad A\begin{pmatrix}\alpha_1'\\ \alpha_2'\end{pmatrix}=\begin{pmatrix}\beta_1'\\ \beta_2'\end{pmatrix}$$

then

$$A=\begin{pmatrix}\beta_1 & \beta_1'\\ \beta_2 & \beta_2'\end{pmatrix}\begin{pmatrix}\alpha_1 & \alpha_1'\\ \alpha_2 & \alpha_2'\end{pmatrix}^{-1}.$$

Let ξ, ξ' be in the spectrum of H' and $\xi\neq\xi'$. Then we have, using the above formulas,

$$\begin{pmatrix}1, & 0\\ 0, & 1/\delta\end{pmatrix}A\begin{pmatrix}1, & 0\\ 0, & \delta\end{pmatrix}=Q(x)\begin{pmatrix}e^{i\pi\xi/2} & 0\\ 0 & e^{-i\pi\xi'/2}\end{pmatrix}Q(-x)^{-1}$$

where

$$Q(x)=\begin{pmatrix}g_{\lambda,\xi}(x) & g_{\lambda,\xi'}(x)\\ g_{\lambda,-\xi}(x) & g_{\lambda,-\xi'}(x)\end{pmatrix}.$$

From the formula for Mellin transform of $W_{\varkappa,\mu}(x)$ ([4], p. 337), it follows that

$$g_{\lambda,\xi}(x)=\frac{2^b}{\sqrt{2\pi}\,\Gamma(-\lambda)}\frac{\Gamma(a)\Gamma(b)\Gamma(c-a)}{\Gamma(c)}F(a,b;c:\tfrac{1}{2})$$

where $a=(1+\lambda-2ix)/2$, $b=(1-\lambda-2ix)/2$, $c=1-ix-\xi/2$. This completes the proof.

Finally we mention here the intertwining operator connecting $T_{\lambda,\delta,\delta'}$ with the unitary principal series. Let $\eta\otimes e^\lambda$ denote the representation $\gamma^m h_t u_s\to e^{i\eta m\pi-\lambda t}$ of P and $U_{\eta,\lambda}=$

$\mathrm{Ind}_{P\uparrow G}\,\eta\otimes e^{\lambda}$. The following realization of $U_{\eta,\lambda}$ in $L^2(R)$ is well known. It comes from using $G=P\cup NwP$ and using

$$f\to\int_R f(u_t wP)\,dt$$

as the quasi-invariant measure in G/P. In fact, we have

$$U_{\eta,\lambda}(x)\,f\colon t\to\left|-ct+a\right|^{-(1-\lambda)}e^{i\eta m(x^{-1},t)\pi}f\left(\frac{dt-b}{-ct+a}\right) \tag{21}$$

where $\sigma(x)=\begin{bmatrix}a & b\\ c & d\end{bmatrix}$, and $x\to\sigma(x)$ is the covering homomorphism and the integer valued function $m(x,t)$ is defined as follows:

$$x\cdot u_t\cdot w\begin{cases}\in P, & \text{if}\quad ct+d=0\\ =u_{(at+b)/(ct+d)}wp(x,t), & \text{if}\quad ct+d\neq 0.\end{cases}$$

The element $p(x,t)\in P$ and $m(x,t)$ is defined uniquely by

$$p(x,t)=u_{-c(ct+d)}h_{\ln|ct+d|}\gamma^{m(x,t)}. \tag{23}$$

We have already seen that $m(w,t)=1$ if $t>0$, $=0$ if $t<0$ and $m(x,t)$ for arbitrary x can be found by using the identities

$$m(x\cdot y,t)=m(y,t)+m(x,\sigma(y)\cdot t)\quad\text{and}\quad m(\gamma^r y,t)=r\quad\text{if}\quad y\in AN.$$

Let $\mathfrak{F}$ denote the Fourier transform and let F_λ denote the operator $F_\lambda=|t|^{\lambda/2}\circ\mathfrak{F}$, $F_\lambda^{-1}=\mathfrak{F}^{-1}\circ|t|^{-\lambda/2}$. Then F_λ is unitary for $\lambda\in iR$ and for $f\in C_c(R)$

$$F_\lambda f\colon t\to|t|^{\lambda/2}\frac{1}{\sqrt{2\pi}}\int_R e^{-ityt}f(y)\,dy$$

and

$$F_\lambda^{-1}f\colon x\to\frac{1}{\sqrt{2\pi}}\int_R e^{ixt}f(t)\,|t|^{-\lambda/2}dt.$$

We then have

THEOREM 11. *Let $\lambda\in iR$, $-1<\eta\leqslant 1$ and suppose $(\lambda,\eta)\neq(0,1)$. Then*

$$T_{\lambda,\delta,\delta'}=F_\lambda\circ U_{\eta,\lambda}\circ F_\lambda^{-1}$$

where $\delta=-1$, $\delta'=-\{\cos\pi(\eta+\lambda)/2\}/\{\cos(\eta-\lambda)/2\}$ if $\lambda\neq0$, and $\delta=-1$, $\delta'=-\pi\tan(\pi\eta/2)$ if $\lambda=0$.

Proof. Let $S=F_\lambda\circ U_{\eta,\lambda}\circ F_\lambda^{-1}$. Then direct calculation shows that $S(h_s)f\colon t\to e^s f(e^{2s}t)$ and $S(u_s)f\colon t\to e^{-ist}f(t)$. Also $U_{\eta,\lambda}$ is known to be irreducible. Thus by Theorem 9, it follows that there exists δ, δ' such that $T_{\lambda,\delta,\delta'} = S = F_\lambda \circ U_{\eta,\lambda} \circ F_\lambda^{-1}$. We have to find δ and δ'. These are obtained by comparing K eigenfunctions in both. First of all Spec $dU_{\eta,\lambda}(H') = \eta + 2\mathbf{Z}$. Comparing it with spectrum of $dT_{\lambda,\delta,\delta'}(H')$ we have (see Theorem 4)

$$\delta'/\delta = \{\cos \pi(\eta+\lambda)/2\}/\{\cos \pi(\eta-\lambda)/2\} \quad \text{if} \quad \lambda \neq 0 \quad \text{and} \quad = \pi \tan \pi\eta/2, \quad \text{if} \quad \lambda = 0.$$

Next consider the representation $dU_{\eta,\lambda}$. From the formulas for the one-parameter groups $U_{\eta,\lambda}(\exp sZ)$, where $Z = H$ or X or Y, it is not difficult to show that $C_c^\infty(R) \subset \mathcal{D}(dU_{\eta,\lambda})$ and $dU_{\eta,\lambda} \supset \tau_\lambda$ where $\mathcal{D}(\tau_\lambda) = C_c^\infty(R)$ and

$$\tau_\lambda(H) = -(2t\partial_t + 1 - \lambda), \quad \tau_\lambda(X) = -\partial_t, \quad \text{and} \quad \tau_\lambda(Y) = (t^2\partial_t + (1-\lambda)t).$$

By Weyl's lemma $\mathcal{D}(\tau_\lambda^\dagger) = \{f \in C^\infty(R) \mid \tau_\lambda(a) f \in L^2(R), \text{ for all } a \in \mathcal{U}\}$ and for $f \in \mathcal{D}(\tau_\lambda^\dagger)$, $\tau_\lambda^\dagger(a) f = \tau_\lambda(a) f$. Since $U_{\eta,\lambda}$ is unitary, it follows that $dU_{\eta,\lambda} \subset \tau_\lambda^\dagger$. Thus the eigenfunctions $\{dU_{\eta,\lambda}(H') f = \xi f\} \subset \{f \in C^\infty(R) \mid \tau_\lambda(H') f = \xi f\}$. By direct computation, these eigenspaces are one-dimensional and if f_ξ is an eigenfunction then

$$f_\xi = (1+it)^{-(1-\lambda-\xi)/2}(1-it)^{-(1-\lambda+\xi)/2}$$

for $\xi \in \eta + 2Z$. Thus there exists a constant β_ξ such that $F_\lambda f_\xi = \beta_\xi \psi_{\lambda,\xi}$, where $\psi_{\lambda,\xi}$ is the eigenfunction introduced in Lemma 15. From formula (12) on page 119 of [4], it follows that

$$F_\lambda f_\xi = \begin{cases} -\sqrt{2\pi}\, 2^{\lambda/2}\{\Gamma((1-\lambda+\xi)/2)\}^{-1} L_{\xi/2,\lambda/2}(t), & t > 0 \\ \sqrt{2\pi}\, 2^{\lambda/2}\{\Gamma((1-\lambda-\xi)/2)\}^{-1} L_{-\xi/2,\lambda/2}(-t), & t < 0. \end{cases}$$

Comparing this with the formula for $\psi_{\lambda,\xi}$, one gets that $\delta = -1$ always. This completes the proof.

Appendix A. On extensions of symmetric $\mathfrak{g}$-modules

Let $\mathcal{U}$ be an associative algebra over $\mathbf{C}$ and τ an involutory, conjugate linear anti-automorphism of $\mathcal{U}$. We write a^τ instead of $\tau(a)$ for $a \in \mathcal{U}$. A $\mathcal{U}$-module σ consists of a complex vector space $\mathcal{D}(\sigma)$, called the domain of σ, and linear operators $\sigma(a)\colon \mathcal{D}(\sigma) \to \mathcal{D}(\sigma)$, such that $a \to \sigma(a)$ is a homomorphism of $\mathcal{U}$. If σ_1, σ_2 are two such modules, we say $\sigma_1 \subset \sigma_2$ if $\mathcal{D}(\sigma_1) \subset \mathcal{D}(\sigma_2)$ and $\sigma_1(a) = \sigma_2(a) \mid \mathcal{D}(\sigma_1)$, for all $a \in \mathcal{U}$. Let $\mathcal{H}$ be a complex separable Hilbert space. We shall be mainly concerned with $\mathcal{U}$-modules in $\mathcal{H}$, i.e., σ such that $\mathcal{D}(\sigma) \subset \mathcal{H}$. In the following we write $(\cdot,\cdot)$ for the inner product on $\mathcal{H}$ and use standard terminology for operators in a Hilbert space.

Definition of the adjoint module. Let σ be a densely defined $\mathcal{U}$-module, i.e., $\mathcal{D}(\sigma)$ is dense in $\mathcal{H}$. Then a vector g belongs to the domain of the adjoint of σ or $g \in \mathcal{D}(\sigma^\tau)$ if for each $a \in \mathcal{U}$, there is g_a such that $(\sigma(a)f, g) = (f, g_a)$ for all $f \in \mathcal{D}(\sigma)$. Then the vector g_a is unique and we define $\sigma^\tau(a)g = ga^\tau$. It is then clear that if $g \in \mathcal{D}(\sigma^\tau)$, $g_a \in \mathcal{D}(\sigma^\tau)$ for all a and $(g_a)_b = g_{ab}$. Thus σ^τ is a $\mathcal{U}$-module called the adjoint of σ. Also note that $\mathcal{D}(\sigma^\tau) = \cap \{\mathcal{D}(\sigma(a)^*) \mid a \in \mathcal{U}\}$ and $\sigma^\tau(a) \subset (\sigma(a^\tau))^*$.

A $\mathcal{U}$-module σ is said to be τ-symmetric (or simply symmetric when it is clear from the context what τ is meant) if $\sigma \subset \sigma^\tau$. This is equivalent to the statement

$$(\sigma(a)f, g) = (f, \sigma(a^\tau)g)$$

for all $f, g \in \mathcal{D}(\sigma)$.

The following lemma is proved the same way as in the case of a single operator.

LEMMA A.1. (i) *If σ_1 is densely defined and $\sigma_1 \subset \sigma_2$, then $\sigma_2^\tau \subset \sigma_1^\tau$,*

(ii) *If σ and σ^τ are both densely defined, then $\sigma^{\tau\tau} = (\sigma^\tau)^\tau$ exists, $\sigma \subset \sigma^{\tau\tau}$ and $\sigma^{\tau\tau\tau} = \sigma^\tau$.*

LEMMA A.2. *Let σ be a densely defined, symmetric $\mathcal{U}$-module. Let $B_\sigma(a\colon f\colon g) = (\sigma^\tau(a)f, g) - (f, \sigma^\tau(a^\tau)g)$ for $a \in \mathcal{U}$, $f, g \in \mathcal{D}(\sigma^\tau)$. Then the boundary forms B_σ satisfy the following identities:*

(i) *$B_\sigma(a\colon f\colon g) = 0$ if either f or $g \in \mathcal{D}(\sigma)$,*

(ii) *$B_\sigma(ab\colon f\colon g) = B_\sigma(a\colon \sigma^\tau(b)f\colon g) + B_\sigma(b\colon f\colon \sigma^\tau(a^\tau)g)$ and*

(iii) *$B_\sigma(a^n\colon f\colon g) = \sum_{r=0}^{n-1} B_\sigma(a\colon \sigma_\tau(a)^r f\colon \sigma^\tau(a^\tau)^{n-r-1}g)$. Moreover, $\sigma \subset \sigma^{\tau\tau} \subset \sigma^\tau$ and $\mathcal{D}(\sigma^{\tau\tau}) = \{f \in \mathcal{D}(\sigma^\tau) \mid B_\sigma(a\colon f\colon g) = 0$ for all $g \in \mathcal{D}(\sigma^\tau)\}$.*

(iv) *If $\sigma \subset \sigma_1 \subset \sigma^\tau$, then $\sigma \subset \sigma_1^\tau \subset \sigma^\tau$ and $\mathcal{D}(\sigma_1^\tau) = \{f \in \mathcal{D}(\sigma^\tau) \mid B_\sigma(a\colon f\colon g) = 0$ for all $a \in \mathcal{U}$ and $g \in \mathcal{D}(\sigma_1)\}$.*

The proofs are all straightforward and are omitted.

There are two prime examples of the pairs $(\mathcal{U}, \tau)$. In the first one, let $\mathcal{U} = \mathbf{C}[t]$, the polynomial algebra in one indeterminate t, and τ is defined by $\tau(t) = t$. In this case, a $\mathcal{U}$-module is defined by an operator A with domain $\mathcal{D}$ such that $A\mathcal{D} \subset \mathcal{D}$ and $\sigma(t^n) = A^n$. In this case, $\mathcal{D}(\sigma^\tau) = \cap\, \mathcal{D}(A^{*n}) = \mathcal{D}^\infty(A^*)$, and τ symmetry coincides with the usual notion of symmetry for a single operator.

The second example arises naturally in representation theory of Lie groups. Let $\mathfrak{g}$ be a Lie algebra over R, and $\mathfrak{g}_c$ its complexification. Let $\mathcal{U} = \mathcal{U}(\mathfrak{g}_c)$ denote the universal enveloping algebra of $\mathfrak{g}_c$. There exists a unique conjugate linear involutory anti-automorphism $a \to a^\dagger$ of $\mathcal{U}$ onto itself, such that $X^\dagger = -X$ for all $X \in \mathfrak{g}$. When dealing with Lie algebras, the pair $(\mathcal{U}(\mathfrak{g}_c), \dagger)$ is the one we shall be concerned with. Since a $\mathfrak{g}$-module extends uniquely to a $\mathcal{U}(\mathfrak{g}_c)$-module, we shall treat the concepts synonymously.

Let G be the simply connected Lie group with Lie algebra $\mathfrak{g}$. Let T be a continuous representation of G in $\mathcal{H}$. We define dT as the $\mathfrak{g}$-module whose domain is the collection of all C^∞-vectors of the representation T, and when $v \in \mathcal{D}(dT)$, $dT \cdot (X)v = (d/dt)|_{t=0} T(\exp tX)v$.

The following lemma is well known and is stated here only for ease of reference (see Warner [18], chapter 4).

LEMMA A.3. (i) *A vector $v \in \mathcal{D}(dT)$ if and only if the function $x \to (T(x)v, v')$ is a C^∞-function on G, for each $v' \in \mathcal{H}$.*

(ii) *For each $X \in \mathfrak{g}$, the closure of $dT(X)$ is the infinitesimal generator of the one parameter group $t \to T(\exp tX)$ and*

$$\mathcal{D}(dT) = \cap \{\mathcal{D}(A_{j1} \dots A_{jr}) \mid r = 1, 2, \dots; A_j = \mathrm{Cl}\, dT \cdot (X_j),$$
$$\text{where } X_1, \dots, X_n \text{ is a basis of } \mathfrak{g}\}.$$

LEMMA A.4. *Let σ be a $\mathfrak{g}$-module and T a unitary representation such that $\sigma \subset dT$. Assume that $\mathcal{D}(\sigma)$ is dense and $T(x)\mathcal{D}(\sigma) \subset \mathcal{D}(\sigma)$ for all $x \in G$. Then,* (i) *$\sigma^\dagger = dT$;* (ii) *If $z \in \mathrm{cent}\, \mathcal{U}[\mathfrak{g}_c]$ and $z^\dagger = z$, then $\sigma(z)$ is essentially self-adjoint;* (iii) *If $X \in \mathfrak{g}$; then $\sigma(X)$ is essentially skew adjoint.*

Proof. Now $\sigma \subset dT$ implies that $(dT)^\dagger \subset \sigma^\dagger$. Since T is unitary, we have $dT \subset (dT)^\dagger$. Thus it is sufficient to check that $\mathcal{D}(\sigma^\dagger) = \mathcal{D}(dT)$. Let $f \in \mathcal{D}(\sigma^\dagger)$ and let $g \in \mathcal{H}$ be arbitrary. It is sufficient to verify that the function $x \to \varphi_{g,f}(x) = (g, T(x)f)$ is C^∞. Let $g_n \in \mathcal{D}(\sigma)$ be such that $g_n \to g$ in $\mathcal{H}$. Then

$$\varphi_{g_n,f}(x; \eta) = (dT(\eta^0)\, T(x^{-1}) g_n, f) = (T(x^{-1}) g_n, \sigma(\eta^0)^* f), \quad \text{since} \quad \mathcal{D}(\sigma)$$

is G-invariant. Thus

$$\varphi_{g_n,f}(x; \eta) = \varphi_{g_n, \sigma^\dagger(\eta) f}(x)$$

for all $\eta \in \mathcal{U}[\mathfrak{g}]$. Thus $\varphi_{g_n,f}(x; \eta)$ converges uniformly on compact subsets to $\varphi_{g, \sigma^\dagger(\eta) f}(x)$. Since $\varphi_{g_n,f}$ is C^∞, it follows that $\varphi_{g,f}$ is also C^∞ or $f \in \mathcal{D}(dT)$. The rest of the statements of the lemma are known, cf. [8] or [9].

COROLLARY 1. $(dT)^\dagger = dT$.

COROLLARY 2. *With the same assumptions on σ as in Lemma A.4, let $\sigma_1 \subset \sigma$ be a $\mathfrak{g}$-submodule such that $\mathcal{D}(\sigma_1)$ is dense. Then $\sigma_1 \subset \sigma \subset dT \subset \sigma_1^\dagger$ and if $\mathcal{D} = \{f \in \mathcal{D}(\sigma_1^\dagger) \mid (\sigma_1^\dagger(\eta) g, f) = (g, \sigma_1^\dagger(\eta^\dagger) f)$ for all $g \in \mathcal{D}(\sigma)$ and all $\eta \in \mathcal{U}\}$, then $\mathcal{D} = \mathcal{D}(dT)$.*

Proof. The first corollary follows from Lemma A.4 by taking $\sigma = dT$. Since $\sigma_1 \subset \sigma \subset dT$, it follows that $(dT)^\dagger = dT \subset \sigma_1^\dagger$. Since $\sigma \subset dT \subset \sigma_1^\dagger$, it follows that $\mathcal{D}(dt) \subset \mathcal{D}$. On the other hand $\mathcal{D} \subset \mathcal{D}(\sigma^\dagger)$. But $\sigma^\dagger = dT$, so that $\mathcal{D}(dt) = \mathcal{D}$.

Definition A.1. A symmetric $\mathfrak{g}$-module σ is said to be integrable if there exists a continuous unitary representation T of the simply connected Lie group in $\mathcal{H}$ such that $\sigma \subset dT$. It is said to be exact if $\sigma = dT$.

We note that if $\sigma = dT_1 = dT_2$, then $T_1 = T_2$. In fact, $T_1(\exp tX)$ and $T_2(\exp tX)$ have the same infinitesimal generator, so that $T_1(\exp X) = T_2(\exp X)$ for all $X \in \mathfrak{g}$. However, if $\sigma \subset dT$, dT is not in general unique. In this connection we note the following

LEMMA A.5. *Let σ be a densely defined $\mathfrak{g}$-module such that $\sigma(X_j)$ is essentially skew-adjoint for a basis X_j of $\mathfrak{g}$. Then $\sigma^{\dagger\dagger} = \sigma^\dagger$. In particular, if $\sigma \subset dT$, then $dT = \sigma^\dagger$, and T is unique.*

Proof. If $\sigma(X_j)$ is essentially skew-adjoint, it follows that the boundary forms $B_\sigma(X_j\colon f\colon g) = 0$ for all f, $g \in \mathcal{D}(\sigma^\dagger)$ and, therefore, $B_\sigma(a\colon f\colon g) = 0$ for all $a \in \mathcal{U}$, from identities satisfied by the boundary forms B_σ (see Lemma A.2). Thus $\sigma^{\dagger\dagger} = \sigma^\dagger$, by the same lemma. Now if $\sigma \subset dT$, then $\sigma \subset dT \subset \sigma^\dagger$, and so $\sigma^{\dagger\dagger} \subset dT \subset \sigma^\dagger$ or $dT = \sigma^\dagger$.

The following is just a reformulation in our notation of a theorem of Nelson ([8]).

THEOREM. *Let $X_1, \ldots, X_n$ be a basis of $\mathfrak{g}$ and σ a $\mathfrak{g}$-module in $\mathcal{H}$. Then $\sigma = dT$ for some continuous unitary representation T if and only if* (i) *$\sigma = \sigma^\dagger$, and* (ii) *$\sigma(\Delta)$ is essentially self-adjoint, where $\Delta = X_1^2 + \ldots + X_n^2$.*

Remark. An example of Nelson (see [8], section 11) may be interpreted in our notation as follows: there exists $\mathfrak{g}$-module σ, of a two-dimensional abelian Lie algebra such that $\sigma = \sigma^\dagger$, but σ is not integrable. Whether such examples exist for semi-simple $\mathfrak{g}$ is not known. In this connection it might be of interest to note that all self-adjoint extensions of the module σ_λ considered in Section 3 are integrable to the group (cf. Theorem 3).

Let M be a C^∞-manifold and μ a C^∞-density on M, i.e., $\mu\colon (U, \varphi) \to \mu_{U,\varphi}$ a map from local charts (U, φ) to C^∞-functions on U, such that $\mu_{U,\varphi}$ transforms like the modulus of the Jacobian under change of coordinates. We also denote by μ, the corresponding measure induced on M. Let $M' \subset M$ be an open subset of M and D a smooth differential operator on M'. Then there exists a smooth differential operator ${}^tD$, called the transpose of D, such that

$$\int Df \cdot g\, d\mu = \int f \cdot {}^tD g\, d\mu$$

for all f, $g \in C_c^\infty(M')$. The operator $D^* g = ({}^t D \bar{g})^-$ is called the formal adjoint of D, and if $(f, g) = \int f \bar{g} d\mu$, then $(Df, g) = (f, D^* g)$ for all $f, g \in C_c^\infty(M')$.

Let $X \to \varrho(X)$ be a homomorphism of $\mathfrak{g}$ into differential operators on M', such that

(i) $\varrho(X)^* = -\varrho(X)$, for all $X \in \mathfrak{g}$,

(ii) For each $m_0 \in M'$, there exists an $a_0 \in \mathcal{U}(\mathfrak{g}_c)$, such that the operator $\varrho(a_0)$ is elliptic in a neighborhood of m_0.

Let $\sigma = (C_c^\infty(M'), \varrho)$. Then σ is a symmetric $\mathfrak{g}$-module in $L^2(\mu)$.

WEYL'S LEMMA. *With the above notation* $\mathcal{D}(\sigma^\dagger) = \{f \in C^\infty(M') \mid \varrho(a) f \in L^2(\mu)$, *for all* $a \in \mathcal{U}(\mathfrak{g}_c)\}$. *Moreover,* $\sigma^\dagger(a) f = \varrho(a) f$, *for all* $f \in \mathcal{D}(\sigma^\dagger)$.

Proof. This lemma is quite classical and we include a sketch of proof, for lack of adequate reference. It follows easily from the regularity theorem for elliptic operators. In fact, suppose $f \in \mathcal{D}(\sigma^\dagger)$. Let u_f be the distribution $\varphi \to (\varphi, f)$, $\varphi \in C_c^\infty(M')$; ${}^t\varrho(a_0)\, u_f(\varphi) = u_f(\varrho(a_0)\varphi) = (\varphi, f_{a_0})$. Thus the distribution ${}^t\varrho(a_0^n) u_f$ is an L^2-function for all n; since $\varrho(a_0)$ is elliptic in a neighborhood of $m_0 \in M'$, it follows from regularity theorem that u_f is locally a C^∞-function near m_0. Thus $f \in C^\infty(M')$ and $(\varrho(a)\varphi, f) = (\varphi, \varrho(a^\dagger) f)$ for all $a \in \mathcal{U}$. This proves the lemma.

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