SLICING AND INTERSECTION THEORY FOR CHAINS ASSOCIATED WITH REAL ANALYTIC VARIETIES

BY

ROBERT M. HARDT

Brown University, Providence, R.I., U.S.A. (1)

Contents

1. Introduction

In [F2] H. Federer exhibited the classical complex algebraic varieties as integral currents and applied techniques of geometric measure theory to give new formulations of the algebraic geometer's concepts of dimension, tangent cone and intersection. Wishing to extend such notions to larger classes of geometric objects, he gave geometric-measure-theoretic characterizations of the dimension of a real analytic variety and of the tangent cone of a real analytic chain ([F, 3.4.8, 4.3.18]); he also conjectured in [F, 4.3.20] that the theory of slicing, which has enjoyed several applications in geometric measure theory ([FF, 3.9], [F1], [A], [F2, 3], [B1], [B2], [B3], [F]), could be used to construct a viable intersection theory for real analytic chains. This is the aim of the present paper.

Let $t \geq n$ be integers and M be a separable oriented real analytic manifold. A t dimensional locally integral flat current ([F, 4.1.24]) T in M is called a *t dimensional analytic chain in M* if M can be covered by open sets U for which there exist t and $t-1$ dimensional real analytic subvarieties V and W of U with $U \cap \operatorname{spt} T \subset V$ and $U \cap \operatorname{spt} \partial T \subset W$. It then

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follows from [F, 4.2.28] that T is a locally finite sum of chains corresponding to integration over certain t dimensional oriented analytic submanifolds of M. If f is an analytic map from M into $\mathbf{R}^n$, then for almost all y in $\mathbf{R}^n$ the *slice of* T *in* $f^{-1}\{y\}$, denoted $\langle T, f, y\rangle$ is a $t-n$ chain in M defined by the relative differentiation of measures (3.5, [F, 4.3], [F2, 3.5]; in case T corresponds to integration over an oriented analytic submanifold N of M, the slice $\langle T, f, y\rangle$ for almost all y, is the $t-n$ chain given by integration along the oriented fiber $f^{-1}\{y\} \cap N$.). Let Y be the set of those y in $\mathbf{R}^n$ for which the dimensions of $f^{-1}\{y\} \cap \operatorname{spt} T$ and $f^{-1}\{y\} \cap \operatorname{spt} \partial T$ do not exceed $t-n$ and $t-n-1$ respectively. We prove in 4.3 our basic result:

SLICING THEOREM. *The function which associates* $\langle T, f, y\rangle$ *with* y *maps* Y *into the* $t-n$ *dimensional analytic chains in* M *and is continuous with respect to the topology of the locally integral flat chains in* M.

It follows in § 5 that if S and T are analytic chains in M and the dimensions of $\operatorname{spt} S \cap \operatorname{spt} T$, $\operatorname{spt} \partial S \cap \operatorname{spt} T$, and $\operatorname{spt} S \cap \operatorname{spt} \partial T$ are not unusually large, then the *intersection of* S *and* T, denoted $S \cap T$, is well-defined by slicing the Cartesian product $S \times T$, in any coordinate neighborhood, by the subtraction map. The resulting *real analytic intersection theory* is then characterized in 5.8–5.11 by certain classical algebraic formulae.

To prove the Slicing theorem we employ the proposition:

If A *is a real analytic subvariety of* M *and* K *is a compact subset of* M, *then there exists an integer* I *such that*

$$\operatorname{card}(K \cap A \cap f^{-1}\{y\}) \leq I$$

whenever $y \in \mathbf{R}^n$ *and* $\dim(A \cap f^{-1}\{y\}) \leq 0$.

The existence of such a bound (which apparently was previously unknown even in the analogous complex case) is established in 2.9(1) following a description in 2.4 of analytic mappings of bounded semianalytic sets. The lemma in 3.1 whose statement and proof are essentially due to H. Federer, is intended to supplement the discussion of [F, 4.3.16]. The proofs of 4.7 and 5.8(11) are also due to Federer. An application of the Slicing theorem to the chains associated with the zero sets of real polynomial mappings is given in 4.8. For the case of positive holomorphic chains, the theorem in 6.5 on the continuity of slicing is more general than 4.3. The counterexample in 6.6 to the corresponding proposition for real analytic chains is a modified version of an example of H. Federer.

The origins of intersection theory go back to the paper [KR] of Kronecker in which he associated an integer-valued index to certain systems of functions of several variables. In [LE] Lefschetz gave an algebraic topological definition for certain intersections of simplicial chains and discussed briefly intersections of real and complex analytic objects. The case of complex algebraic chains has been studied by many algebraic geometers (for example [C], [W], [SA], [SE]). Complex holomorphic intersections have been treated in [BH], [D], [K1], and [K2]. [BH] also contains an intersection theory for the cycles modulo two defined by the real parts of holomorphic sets. The real analytic chains which we consider include each of the above cases. Their supports correspond to arbitrary real analytic sets which may fail to be either coherent or C-analytic ([N, pp. 93–109], [WB, pp. 152–156]). The methods employed in [F2], [F], [K1], [K2], and the present paper are all based on geometric measure theory, notably H. Federer's theory of slicing.

Most of the references will be from [F]. We refer to [F2] mainly for theorem 3.17 and to [N] for some elementary properties of holomorphic sets used in § 6. Most of the notation is also from [F] (see his glossaries on pp. 669–671). In addition for any two maps $f\colon A\to B$, $g\colon A\to C$ we use the symbol

$$f \boxdot g$$

to denote the map which sends $a\in A$ onto $(f(a), g(a))\in B\times C$.

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2. Analytic blocks and analytic fibers

Let M be a separable m dimensional real analytic Riemannian manifold. For $\varrho>0$ let $\mathcal{H}^\varrho$ denote the ϱ dimensional Hausdorff measure induced by the Riemannian metric ([KN, p. 157], [F, 2.10.2]). Whenever t is a nonnegative integer with $m>t$ (respectively, $m=t$) and G is a subset of M, we call G a *t dimensional analytic block in M* if there exist an open set U in M, with Clos $G\subset U$, and real-valued functions $g_0, g_1, \ldots, g_{m-t}$ (resp., g_0) analytic in U so that G is one of the connected components of the set

$$U\cap\{x\colon g_1(x)=\ldots=g_{m-t}(x)=0\}\sim U\cap\{x\colon g_0(x)=0\}$$

(resp., $U\sim U\cap\{x\colon g_0(x)=0\}$) and for each $x\in G$, the sequence $Dg_1(x), \ldots, Dg_{m-t}(x)$ is linearly independent (compare [F, 3.4.5]). We shall be interested in the class $\mathcal{S}(M)$ of those subsets of M which are locally finite unions of analytic blocks in M of various dimensions; thus $A\in\mathcal{S}(M)$ if and only if there exist analytic blocks $G_1, G_2, \ldots$ in M so that $A=\bigcup_{j=1}^\infty G_j$ and $\{j\colon G_j\cap K\neq\emptyset\}$ is finite for every compact $K\subset M$. It follows from the results of S. Łojaseiwicz in [LO3, pp. 40–70] that $\mathcal{S}(M)$ coincides with the class of *semianalytic sets* in M as defined and studied in [LO1], [LO2], and [LO3]. Moreover [LO3] contains the complete proofs of many interesting properties of such sets. However we shall refer only to [F], notably [F, 3.4.5–3.4.12, 4.2.28] for our discussion of $\mathcal{S}(M)$ first because [F] contains all those facts relevant for our purposes and second because the I.H.E.S. course notes [LO3] are not as readily available to the reader.

2.1. LEMMA. *If* A, $B \in \mathcal{S}(M)$, *then:*

(1) $A \cup B \in \mathcal{S}(M)$.

(2) $A \cap B \in \mathcal{S}(M)$.

(3) $A \sim B \in \mathcal{S}(M)$.

(4) $A \times B \in \mathcal{S}(M \times M)$.

(5) *For any connected component* C *of* A, $C \in \mathcal{S}(M)$.

(6) *For any real-valued function* g *analytic in a neighborhood of* Clos A, $A \cap \{x: g(x) = 0\} \in \mathcal{S}(M)$.

Proof. (1), (4) and (5) are clear. For any of the sets D which occur in (2), (3), or (6) and any point $x \in \operatorname{Clos} D$ we may, by [F, 3.4.9], find an open neighborhood U_x of x so that

first, there exists an analytic isomorphism h of U_x into $\mathbf{R}^m$ with $h(x) = 0$, and

second, there exist real analytic subvarieties V_x and W_x of U_x so that $U_x \cap D$ is the union of some finite family of connected components of $V_x \cap W_x$.

Then we apply the local theory of [F, 3.4.8(11), 3.4.9] to $h(V_x \sim W_x) = h(V_x) \sim h(W_x)$ and select a possibly smaller open neighborhood U_x^* of x so that $U_x^* \cap D \in \mathcal{S}(M)$. By the paracompactness of M and [F, 3.4.9] we may choose a locally finite refinement $\{U_1, U_2, \ldots\}$ of the cover $\{U_x^*: x \in \operatorname{Clos} D\}$ of Clos D such that $U_1, U_2, \ldots \in \mathcal{S}(M)$, hence

$$D = \bigcup_{j=1}^{\infty} (U_j \cap D) \in \mathcal{S}(M).$$

2.2. *Dimension.* Recalling [F, 3.4.8(3)], we define, for $\varnothing \neq E \subset M$, the *real analytic dimension of* E, denoted dim E, as

$\sup_{x \in M} \inf \{\dim \alpha: \alpha$ is the germ of an analytic variety at x and α contains the germ of E at $x\}$;

in addition, we define $\dim \varnothing = -1$. Then for any two subsets E and F of M we have, by [F, 3.4.8(14)], the equation

$$\dim (E \cup F) = \sup \{\dim E, \dim F\}.$$

We will say that a point $x \in M$ is a *regular point for a set* $A \in \mathcal{S}(M)$ if there exists a neighborhood U of x so that $U \cap A$ is a connected real analytic submanifold of M. From [F, 3.4.8(11) (13) (14) (16), 3.4.9] we infer that if $\varnothing \neq A \in \mathcal{S}(M)$, then the following four expressions are *equivalent characterizations of* dim A:

(1) $\sup\{\dim[\operatorname{Tan}(A, x)]: x$ is a regular point of $A\}$,

(2) $\sup\{k$: there exists a k dimensional analytic block G in M with $G \subset A\}$,

(3) $\inf\{k: A \cap K$ is k rectifiable for every compact $K \subset M\}$,

(4) $\sup\{\varrho: \mathcal{H}^\varrho(A) > 0\}$.

We will also use the following two important facts for $\varnothing \neq A \in \mathcal{S}(M)$:

(5) $\dim(\operatorname{Clos} A \sim A) < \dim A$;

(6) $\dim(A \sim A \cap \{x: x$ is a regular point of $A\}) < \dim A$.

(5) follows from [F, 3.4.8(16)] by reasoning as in [F, 4.2.28] and (6) follows from [F, 3.4.10].

From (4) and [F, 2.10.25] we infer:

(7) If f is an analytic map of a neighborhood of Clos E into $\mathbf{R}^n$, then for $\mathcal{L}^n$ almost all y in $\mathbf{R}^n$

$$\dim(E \cap f^{-1}\{y\}) \leqslant \sup\{-1, \dim E - n\}.$$

2.3. Lemma. *If E is a subset of M and U is a neighborhood of* Clos E, *then there exists a closed set $A \in \mathcal{S}(M)$ such that*

$$E \subset A \subset U \text{ and } \dim A = \dim E.$$

Proof. This is obvious in case $\dim E = -1$. We assume inductively that 2.3 with E replaced by F is true for all subsets F of M with $\dim F < \dim E$. By applying [F, 3.4.8(11), 3.4.9] and the paracompactness of M as in the proof of 2.1, we choose a locally finite open cover $\{U_1, U_2, \ldots\}$ of Clos E and $B_1, B_2, \ldots \in \mathcal{S}(M)$ so that

$$U_j \cap E \subset B_j \subset U_j \cap U, \quad \dim B_j \leqslant \dim E$$

for $j \in \{1, 2, \ldots\}$, hence

$$B = \bigcup_{j=1}^{\infty} B_j \in \mathcal{S}(M), \quad E \subset B \subset U, \quad \text{and } \dim B = \dim E.$$

Since, by 2.2(5), $\dim(\operatorname{Clos} B \sim B) < \dim E$, we may use induction to choose a closed set $C \in \mathcal{S}(M)$ so that

$$\operatorname{Clos} B \sim B \subset C \subset U, \quad \dim C < \dim E,$$

and we take $A = B \cup C$ to finish the proof.

2.4. We will prove by induction on t that the following two propositions hold for every nonnegative integer t.

Proposition ($\mathbf{A}_t$). *If A is the union of finitely many analytic blocks in* $\mathbf{R}^m$ *such that* Clos A *is compact and* $\dim A \leqslant t$, *f is an analytic map of a neighborhood of* Clos A *into* $\mathbf{R}^n$, *and*

$$R = A \cap \{x\colon x \text{ is a regular point of } A \text{ and } \dim Df(x)[\operatorname{Tan}(A, x)] = t\},$$

then there exist a compact set Q in some Euclidean space, with $\dim Q \leqslant t-1$, and an analytic map q of a neighborhood of Q into $\mathbf{R}^n$ such that $R \sim f^{-1}[q(Q)]$ and $f(R) \sim q(Q)$ are t dimensional analytic submanifolds of $\mathbf{R}^m$ and $\mathbf{R}^n$ having only finitely many connected components, and f maps each connected component of $R \sim f^{-1}[q(Q)]$ isomorphically onto a connected component of $f(R) \sim q(Q)$.

PROPOSITION $(\mathbf{B}_t)$. *If m, n, A, and f are as in Proposition $(\mathbf{A}_t)$ and if $n = t+1$, then the set*

$$\mathbf{R}^{t+1} \sim f(A)$$

has only a finite number of connected components.

In case $t = 0$, the set A is finite, and the truth of Proposition $(\mathbf{A}_0)$ is evident. We will show in 2.8 that

$$(\mathbf{A}_t) \text{ implies } (\mathbf{B}_t) \text{ for every integer } t \geqslant 0$$

and in 2.9 that

$$(\mathbf{B}_{t-1}) \text{ implies } (\mathbf{A}_t) \text{ for every integer } t \geqslant 1.$$

First we prove two lemmas.

2.5. LEMMA. *If m, n, t, f, A, and R are as in Proposition $(\mathbf{A}_t)$ and if $\mathcal{G}$ is a finite family of real-valued functions analytic in a neighborhood of $\operatorname{dmn} f$, then there exists a compact set $B \in \mathcal{S}(\mathbf{R}^m)$ such that*

$$B \subset \operatorname{dmn} f,\ \dim B \leqslant t-1,\ \operatorname{Clos} R \sim R \subset B,\ B \cup (A \sim R) \text{ is compact},\ B \cup (A \sim R) \in \mathcal{S}(M),$$

and for every component C of

$$R \sim B = A \sim [B \cup (A \sim R)]$$

and every $g \in \mathcal{G}$ the function $g | C$ is either strictly negative, or identically zero, or strictly positive.

Proof. Letting Θ denote the collection of all maps from $\mathcal{G}$ to the set $\{-1, 0, 1\}$, we verify with the aid 2.1(1) (2) (3) (5) (6) that for each $\theta \in \Theta$

$$A_\theta = A \cap \{x\colon \operatorname{sign} g(x) = \theta(g) \text{ for } g \in \mathcal{G}\} \in \mathcal{S}(\mathbf{R}^m)$$

and that $A = \bigcup_{\theta \in \Theta} A_\theta$ is a partition of A. By [F, 3.4.8(11), 3.4.9] and the compactness of $\operatorname{Clos} A$, there exist a positive integer J, and open cover $\{U_1, U_2, \ldots, U_J\}$ of $\operatorname{Clos} A$, and, for each $j \in \{1, 2, \ldots, J\}$ and $\theta \in \Theta$, a finite family $\Gamma_{j,\theta}$ of disjoint analytic blocks in $\mathbf{R}^m$ such that

$$U_j \subset \operatorname{dmn} f \text{ and } U_j \cap A_\theta = \bigcup \Gamma_{j,\theta};$$

whence the decomposition

$$U_j \cap A = \bigcup \Gamma_j \text{ where } \Gamma_j = \bigcup_{\theta \in \Theta} \Gamma_{j,\theta}$$

is a partition of $U_j \cap A$ into analytic blocks in $\mathbf{R}^m$.

Fixing $j \in \{1, 2, ..., J\}$ we will now prove that *if* $G \in \Gamma_j$, *then*

$$\textit{either } G \cap R = \varnothing \textit{ or } \dim(G \sim R) \leqslant t-1.$$

For this purpose we assume $G \in \Gamma_j$, $G \cap R \neq \varnothing$, and $\dim G = t$ and infer from [F, 3.1.18] that $G \cap R$ is open relative to G, hence $\dim(G \cap R) = t$. Then choosing, according to 2.2(5) and 2.3, a compact set $D_j \in \mathcal{S}(\mathbf{R}^m)$ so that $\dim D_j \leqslant t-1$ and

$$\bigcup \{\operatorname{Clos} H \sim H\colon H \in \Gamma_j\} \subset D_j \subset \operatorname{dmn} f,$$

we note that $(G \cap R) \sim D_j$ is nonempty because

$$t = \dim(G \cap R) \leqslant \sup \{\dim[(G \cap R) \sim D_j], \dim D_j\}.$$

To estimate the dimension of $(G \sim R) \sim D_j = (G \sim D_j) \sim R$ we observe that every point in $G \sim D_j$ is a regular point of A because

$$(G \sim D_j) \cap \operatorname{Clos} H = \varnothing \text{ for any } H \in \Gamma_j \sim \{G\}.$$

Choosing a neighborhood U of $\operatorname{Clos} G$ and real-valued functions $g_0, g_1, ..., g_{m-t}$ which describe G as in the definition in § 2 and letting $f_1, ..., f_n$ be the real-valued functions such that

$$f(x) = (f_1(x), ..., f_n(x)) \in \mathbf{R}^n \quad \text{for } x \in \operatorname{dmn} f,$$

we associate with each $\lambda \in \Lambda(n, t)$ the real-valued analytic function

$$\phi_\lambda = |Dg_1 \wedge ... \wedge Dg_{m-t} \wedge Df_{\lambda(1)} \wedge ... \wedge Df_{\lambda(t)}|^2.$$

Then the function $\phi = \sum_{\lambda \in \Lambda(n,t)} \phi_\lambda$ is analytic on $U \cap \operatorname{dmn} f$ and satisfies the condition

$$\begin{aligned} \varnothing \neq (G \sim D_j) \cap R &= (G \sim D_j) \cap \{x\colon \dim Df(x)[\operatorname{Tan}(G, x)] = t\} \\ &= (G \sim D_j) \cap \{x\colon \phi(x) \neq 0\} \in \mathcal{S}(\mathbf{R}^m), \end{aligned}$$

hence the real analytic dimension of

$$(G \sim R) \sim D_j = (G \sim D_j) \cap \{x\colon \phi(x) = 0\} \subset G \cap \{x\colon \phi(x) = 0\}$$

does not exceed $t-1$ by virtue of [F, 3.4.8(15), 3.1.24]. Consequently

$$\dim(G\sim R)=\sup\{\dim(G\sim R\sim D_j),\ \dim D_j\}\leqslant t-1.$$

Next, we let $\Gamma=\bigcup_{j=1}^{J}\Gamma_j$, recall 2.2(5), and apply 2.3 with

$$E=(\bigcup\{G\sim R\colon G\in\Gamma,\ G\cap R\neq\emptyset\})\cup(\bigcup\{\text{Clos}\,G\sim G\colon G\in\Gamma\})$$

and U equal to some compact neighborhood of Clos E in dmn f to choose a compact set $B\in\mathcal{S}(\mathbf{R}^m)$ with $E\subset B\subset \text{dmn} f$ and $\dim B\leqslant t-1$.

We infer that $R\subset\bigcup\{G\colon G\in\Gamma,\ G\cap R\neq\emptyset\}$, hence

$$\text{Clos}\,R\sim R\subset\bigcup\{\text{Clos}\,G\sim R\colon G\in\Gamma,\ G\cap R\neq\emptyset\}$$

$$\subset\bigcup\{(\text{Clos}\,G\sim G)\cup(G\sim R)\colon G\in\Gamma,\ G\cap R\neq\emptyset\}\subset B.$$

Since B is the union of finitely many analytic blocks in $\mathbf{R}^m$, so is the set

$$B\cup(A\sim R)=B\cup[\bigcup\{G\colon G\in\Gamma,\ G\cap R=\emptyset\}].$$

Moreover $B\cup(A\sim R)$ is compact, because $A\sim R$ is closed relative to A, and hence

$$\text{Clos}\,(A\sim R)\sim(A\sim R)\subset\text{Clos}\,A\sim A\subset\bigcup\{\text{Clos}\,G\sim G\colon G\in\Gamma\}\subset B.$$

Finally we assume that $g\in\mathcal{G}$ and that C is a connected component of $R\sim B$. Since, by 2.2(1), $\dim C=t$, there exist $j\in\{1,2,\ldots,J\}$, $\theta\in\Theta$, and $G\in\Gamma_{j,\theta}$ so that

$$C\cap G\neq\emptyset\ \text{ and }\ \dim G=t.$$

Observing first that $C\cap G$ is open relative to C because

$$C\subset A\supset G,\quad \dim C=\dim A=\dim G,$$

and any point in $C\cap G$ is a regular point for C, A, and G, and second that $C\cap G$ is closed relative to C because

$$C\cap(\text{Clos}\,G\sim G)\subset C\cap B=\emptyset,$$

we conclude that $C\cap G=C$, $C\subset G$, hence $\text{sign}\,g(x)=\theta(g)$ for all $x\in C$, and the proof of 2.5 is complete.

2.6. Lemma. *Let s, n, m_1, m_2 be nonnegative integers. If, for each $i\in\{1,2\}$, Q is a compact element of $\mathcal{S}(\mathbf{R}^{m_i})$, $\dim Q_i\leqslant s$, and q_i is an analytic map of a neighborhood of Q_i into $\mathbf{R}^n$, then there exists a compact set $Q\in\mathcal{S}(\mathbf{R}^{m_1}\times\mathbf{R}^{m_2})$, with $\dim Q\leqslant s$, and an analytic map q of a neighborhood of Q into $\mathbf{R}^n$ such that*

$$q(Q)=q_1(Q_1)\cup q_2(Q_2).$$

Proof. Choosing $a \in \mathbf{R}^{m_1} \sim Q_1$ and $b \in \mathbf{R}^{m_2} \sim Q_2$ and letting

$$Q = (Q_1 \times \{b\}) \cup (\{a\} \times Q_2) \subset \mathbf{R}^{m_1} \times \mathbf{R}^{m_2},$$

we see that $Q \in \mathcal{S}(\mathbf{R}^{m_1} \times \mathbf{R}^{m_2})$, that $\dim Q \leqslant s$, and that there exists an analytic map q of a neighborhood of Q so that

$$q(x, b) = q_1(x) \text{ for } x \in Q_1,$$

$$q(a, y) = q_2(y) \text{ for } y \in Q_2.$$

2.7. *Proof that proposition* $(\mathbf{A}_t)$ *implies proposition* $(\mathbf{B}^t)$ *for* $t \geqslant 0$.

We infer from 2.2(6) that

$$\dim[A \cap \{x: x \text{ is not a regular point of } A\}] \leqslant t-1$$

and from [F, 3.1.18] that

$$\mathcal{H}^t[f(A \cap \{x: x \text{ is a regular point of } A \text{ and } \dim Df(x)[\operatorname{Tan}(A, x)] < t\})] = 0,$$

hence

$$X = [f(A) \sim f(R)] \cup q(Q) \subset f(A \sim R) \cup q(Q)$$

has $\mathcal{H}^t$ measure zero. Moreover Proposition $(\mathbf{A}_t)$ implies that $f(A) \sim X = f(R) \sim q(Q)$ is a t dimensional analytic submanifold of $\mathbf{R}^{t+1}$ having only a finite number of connected components.

In *case* $f(A) \subset X$, the set $\mathbf{R}^{t+1} \sim f(A)$ is connected. In fact, we define, for each $a \in \mathbf{R}^{t+1} \sim X$, the open analytic map

$$\psi_a: \mathbf{R}^{t+1} \sim \{a\} \to \mathbf{S}^t$$

by $\psi_a(y) = (y-a)/|y-a|$ whenever $y \in \mathbf{R}^{t+1} \sim \{a\}$. Fixing $a \in \mathbf{R}^{t+1} \sim X$, we observe that on the one hand, by 2.2(7),

$$X \cap \psi_a^{-1}\{\xi\} = \varnothing \quad \text{for } \mathcal{H}^t \text{ almost all } \xi \in \mathbf{S}^t$$

while on the other hand

$$\operatorname{Int} \psi_a(C) \neq \varnothing \quad \text{whenever } C \text{ is a component of } \mathbf{R}^{t+1} \sim f(A).$$

Hence there exists $\xi \in \psi_a(C)$ with $X \cap \psi_a^{-1}\{\xi\} = \varnothing$, and so the closed half-line

$$\mathbf{R}^{t+1} \cap \{y: (y-a) \bullet \xi = |y-a|\}$$

lies in $\mathbf{R}^{t+1} \sim X$ and connects a with C.

From now on we assume that $f(A) \sim X$ *is nonempty*, and we observe that the proof of 2.8 reduces to demonstrating the following two assertions:

(1) *For every component D of $f(A) \sim X$ there are at most two components C of $\mathbf{R}^{t+1} \sim f(A)$ with $D \cap \operatorname{Clos} C \neq \varnothing$.*

(2) *For every component C of $\mathbf{R}^{t+1} \sim f(A)$ there exists at least one component D of $f(A) \sim X$ with $D \cap \operatorname{Clos} C \neq \varnothing$.*

In fact, these two assertions imply that

the number of components of $\mathbf{R}^{t+1} \sim f(A) \leq$ twice the number of components of $f(A) \sim X$.

To prove (1) we assume that D is a component of $f(A) \sim X$ and verify that if C is a component of $\mathbf{R}^{t+1} \sim f(A)$ for which $D \cap \operatorname{Clos} C \neq \varnothing$, then $D \subset \operatorname{Clos} C$. Clearly $D \cap \operatorname{Clos} C$ is closed relative to D. To see that $D \cap \operatorname{Clos} C$ is also open relative to D, we let $d \in D \cap \operatorname{Clos} C$ and choose by [F, 3.1.19(1), 3.1.24] a neighborhood U of d in $\mathbf{R}^{t+1}$ such that $U \cap f(A) = U \cap D$ along with an analytic isomorphism h of U onto the open ball $\mathbf{U}(0, 1)$ in $\mathbf{R}^{t+1}$ such that $h(d) = 0$ and

$$h(U \cap D) = \mathbf{U}(0, 1) \cap \{z: \mathbf{e}_1 \bullet z = 0\}.$$

Then
$$U \cap C \cap \{y: \mathbf{e}_1 \bullet h(y) \neq 0\} \neq \varnothing,$$

hence
$$\text{either } U \cap \{y: \mathbf{e}_1 \bullet h(y) > 0\} \subset C \text{ or } U \cap \{y: \mathbf{e}_1 \bullet h(y) < 0\} \subset C.$$

In either case we conclude that $U \cap D \cap \operatorname{Clos} C = U \cap D$ is a neighborhood of d relative to D. Whence $D \subset \operatorname{Clos} C$.

Now suppose d_1 and d_2 are two points in D. From the previous paragraph we see that for each $i \in \{1, 2\}$ there are at most two components C_i, C_i^* of $\mathbf{R}^{t+1} \sim f(A)$ whose closures contain d_i and that therefore

$$D \subset \operatorname{Clos} C_1 \cap \operatorname{Clos} C_1^* \cap \operatorname{Clos} C_2 \cap \operatorname{Clos} C_2^*,$$

hence either $C_1 = C_2$, $C_1^* = C_2^*$ or $C_1 = C_2^*$, $C_1^* = C_2$, and (1) now follows.

To prove (2) we assume that C is a component of $\mathbf{R}^{t+1} \sim f(A)$ and choose a point $b \in f(A) \sim X$. Since, by 2.2(4) (5),

$$\mathcal{H}^t(\operatorname{Clos} A \sim A) = 0,$$

we may, according to [F, 2.10.11], select a point $\xi \in \psi_b(C)$ so that

$$[X \cup f(\operatorname{Clos} A \sim A)] \cap \psi_b^{-1}\{\xi\} = \varnothing$$

to conclude that the closed half-line

$$L = \mathbf{R}^{t+1} \cap \{y: (y-b) \bullet \xi = |y-b|\}$$

connects b with C and that the set $L \cap [f(A) \sim X] = L \cap f(\operatorname{Clos} A)$ is nonempty and closed. If $c \in C \cap L$, then there exists a point $d \in L \cap [f(A) \sim X]$ for which

$$|d-c| = \inf\{|e-c| : e \in L \cap [f(A) \sim X]\}.$$

Therefore $d \in \operatorname{Clos} C$, and (2) follows by choosing that component D of $f(A) \sim X$ which contains d.

2.8. *Proof that proposition* $(\mathbf{B}_{t-1})$ *implies proposition* $(\mathbf{A}_t)$ *for* $t \geq 1$. The proof will consist of two applications of Proposition $(\mathbf{B}_{t-1})$ and a construction using various Cartesian products of $\mathbf{R}^m$. Throughout 2.8 we assume that the set B is chosen as in 2.5 with $\mathcal{G} = \emptyset$. The first use of $(\mathbf{B}_{t-1})$ will be made in proving:

(1) *There exists an integer* I *such that*

$$\operatorname{card}(R \cap f^{-1}\{y\}) \leq I \text{ for all } y \in \mathbf{R}^n.$$

For this we consider three cases:

Case 1, $n < t$. Here $R = \emptyset$ and we take $I = 0$.

Case 2, $n = t$. Here we recall [F, 3.1.18], note that for each $w \in \mathbf{R}^t \sim f(B)$ the fiber

$$R \cap f^{-1}\{w\} = \operatorname{Clos} R \cap f^{-1}\{w\}$$

is compact and discrete, hence finite, and observe that $f|(R \sim f^{-1}[f(B)])$ is a covering map because $\operatorname{Clos} R \sim R \subset B$. By Proposition $(\mathbf{B}_{t-1})$ the set $\mathbf{R}^t \sim f(B)$ has only a finite number of components, and therefore there exists an integer I for which

$$\operatorname{card}(R \cap f^{-1}\{w\}) \leq I \quad \text{whenever } w \in \mathbf{R}^t \sim f(B).$$

Suppose now that there exist a point $y \in \mathbf{R}^t$ and a subset F of $R \cap f^{-1}\{y\}$ whose cardinality is $I+1$. For each $x \in F$ we choose, according to [F, 3.1.18], a neighborhood U_x of x in $\mathbf{R}^m$ so that $U_x \cap A$ is a connected t dimensional analytic submanifold of $\mathbf{R}^m$ and $f|(U_x \cap A)$ is an analytic isomorphism. Since $\mathcal{H}^t[f(B)] = 0$ and since $\bigcap_{x \in F} f(U_x \cap A)$ is a neighborhood of y, we may choose a point

$$w \in [\bigcap_{x \in F} f(U_x \cap A)] \sim f(B)$$

to obtain the contradiction

$$I + 1 = \operatorname{card} F \leq \operatorname{card}(R \cap f^{-1}\{w\}) \leq I,$$

and Case 2 follows.

Case 3, $n > t$. If (1) is false, then there exists a countable set $E \subset \mathbf{R}^n$ for which

$$\sup_{e \in E} \operatorname{card}(R \cap f^{-1}\{e\}) = \infty.$$

Since $R\cap f^{-1}(E)$ is also countable we may find an $n-t$ dimensional vector subspace P of $\mathbf{R}^n$ such that for all $x\in R\cap f^{-1}(E)$

$$P\cap Df(x)[\operatorname{Tan}(A,x)]=\{0\}.$$

Choosing $p\in \mathbf{O}^*(n,t)$ so that $\ker p=P$, we see that $R\cap f^{-1}(E)$ is contained in the set

$$R'=A\cap\{x\colon x \text{ is a regular point of } A \text{ and } \dim D(p\circ f)(x)[\operatorname{Tan}(A,x)]=t\},$$

hence

$$\sup_{e\in E}\operatorname{card}\,[R'\cap(p\circ f)^{-1}\{p(e)\}]\geqslant\sup_{e\in E}\operatorname{card}\,(R\cap f^{-1}\{e\})=\infty.$$

This contradicts Case 2 with f and R replaced by $p\circ f$ and R', and finishes the proof of (1).

For the construction of Q and q we will use for each $i\in\{1, 2, ..., I\}$ the set

$$A_i=(R\sim B)^i\cap\{(x_1,\dots,x_i):f(x_1)=\dots=f(x_i)\ \text{ and } \prod_{\lambda\in\Lambda(i,2)}|x_{\lambda(1)}-x_{\lambda(2)}|^2\neq 0\},$$

which is an element of $\mathcal{S}([\mathbf{R}^m])^i$ by virtue of 2.1(2) (3) (4) (6), and the analytic map f_i: $(\operatorname{dmn} f)^i\to\mathbf{R}^n$ given by $f_i(x_1, ..., x_i)=f(x_1)$ for $(x_1, ..., x_i)\in(\operatorname{dmn} f)^i$.

We first make the observation that

$$\dim A_i\leqslant t.$$

In fact, by 2.2(2) there exists an analytic block $G\subset A_i$ with $\dim G=\dim A_i$. To compute $\dim G$ we recall from [F, 3.1.18] that for any $b\in G$ for which

$$\dim Df_i(b)\,[\operatorname{Tan}\,(G,\mathrm{b})]=\sup_{a\in G}\dim Df_i(a)\,[\operatorname{Tan}\,(G,a)]$$

one has the equation

$$\dim\operatorname{Tan}(G,b)=\dim\operatorname{Tan}[G\cap f_i^{-1}\{f_i(b)\},b]+\dim Df_i(b)[\operatorname{Tan}(G,b)].$$

Since

$$\operatorname{card}\,[G\cap f_i^{-1}\{f_i(b)\}]\leqslant\operatorname{card}\,[A_i\cap f_i^{-1}\{f_i(b)\}]$$

is finite, $\dim\operatorname{Tan}[G\cap f_i^{-1}\{f_i(b)\},b]\leqslant 0$. On the other hand, $\dim Df_i(b)[\operatorname{Tan}(G,b)]\leqslant t$ because f factors as $f\circ p_{(1)}$ where $p_{(1)}(x_1, ..., x_i)=x_1$ for $(x_1, ..., x_i)\in(\operatorname{dmn} f)^i$. Consequently

$$\dim A_i\leqslant\dim\operatorname{Tan}(G,b)\leqslant 0+t.$$

Next, we define for every $i\in\{1, 2, ..., I\}$

$$R_i=A_i\cap\{a\colon a \text{ is a regular point of } A_i \text{ and } \dim Df_i(a)[\operatorname{Tan}(A_i,a)]=t\},$$

$$\mathcal{G}_i=\{g_{j,\lambda}\colon j\in\{1,2,...,m\},\ \lambda\in\Lambda(i,2)\}$$

where for each $j\in\{1, 2, ..., m\}$ and $\lambda\in\Lambda(i, 2)$

$$g_{j,\lambda}\colon (\mathbf{R}^m)^i \to R,\ g_{j,\lambda}(x_1, \ldots, x_i) = \mathbf{e}_j \bullet (x_{\lambda(1)} - x_{\lambda(2)}) \text{ for } (x_1, \ldots, x_i) \in (\mathbf{R}^m)^i,$$

and we apply 2.5 with $\mathbf{R}^m$, f, A, R, and $\mathcal{G}$ replaced by $(\mathbf{R}^m)^i$, f_i, A_i, R_i, and $\mathcal{G}_i$ to choose a compact set $B_i \in \mathcal{S}([\mathbf{R}^m]^i)$ such that

$$B_i \subset (\operatorname{dmn} f)^i,\ \dim B_i \leq t-1,\ \operatorname{Clos} R_i \sim R_i \subset B_i,$$

$$B_i \cup (A_i \sim R_i) \in \mathcal{S}([\mathbf{R}^m]^i),\ B_i \cup (A_i \sim R_i) \text{ is compact,}$$

and for every component C of $R_i \sim B_i = A_i \sim [B_i \cup (A_i \sim R_i)]$ and every $g \in \mathcal{G}_i$, the function $g | C$ is either strictly negative, or identically zero, or strictly positive.

Setting $Q_i = B_i \cup (A_i \sim R_i)$, we note that

$$\dim Q_i \leq t-1.$$

In fact, since
$$Q_i \sim B_i = [B_i \cup (A_i \sim R_i)] \sim B_i \in \mathcal{S}([\mathbf{R}^m]^i),$$

there exists by 2.2(2) an analytic block $H \subset Q_i \sim B_i$ with $\dim H = \dim(Q_i \sim B_i)$. If $\dim H = t$, then we may, according to [F, 3.1.18] and 2.2(6), select a point $b \in H$ such that b is a regular point of A_i and

$$\dim Df_i(b)[\operatorname{Tan}(H, b)] = \sup_{a \in H} \dim Df_i(a)[\operatorname{Tan}(H, a)]$$

to obtain the contradiction

$$\begin{aligned}\dim H &= \dim \operatorname{Tan}(H, b) \\ &= \dim \operatorname{Tan}(H \cap f_i^{-1}\{f_i(b)\}, b) + \dim Df_i(b)[\operatorname{Tan}(H, b)] \leq 0 + t - 1.\end{aligned}$$

because $A_i \cap f_i^{-1}\{f_i(b)\}$ is finite and $b \in A_i \sim R_i$. Thus

$$\dim Q_i = \sup\{\dim(Q_i \sim B_i),\ \dim B_i\} \leq t-1.$$

Recalling 2.6 we choose a compact set

$$Q \in \mathcal{S}(\mathbf{R}^m \times \mathbf{R}^m \times [\mathbf{R}^m]^2 \times \ldots \times [\mathbf{R}^m]^i),$$

with $\dim Q \leq t-1$, and an analytic map q of a neighborhood of Q into $\mathbf{R}^n$ such that

$$q(Q) = f(B) \cup f_1(Q_1) \cup \ldots \cup f_i(Q_i).$$

To verify that all the conclusions of Proposition $(\mathbf{A}_t)$ hold, it will be sufficient to prove the following three statements.

(2) *For every component C of $R \sim f^{-1}[q(Q)]$*

$$\text{Clos } f(C) \sim f(C) \subset q(Q)$$

and $f|C$ *is an analytic isomorphism.*

(3) *For every two components* C *and* D *of* $R \sim f^{-1}[q(Q)]$ *either* $f(C) \cap f(D) = \varnothing$ *or* $f(C) = f(D)$.

(4) *The set* $R \sim f^{-1}[q(Q)]$ *has only finitely many connected components.*

To prove (2) we assume that C is a component of $R \sim f^{-1}[q(Q)]$, note that Clos C is compact, and conclude that

$$\begin{aligned}\text{Clos } f(C) \sim f(C) &\subset f(\text{Clos } C \sim C)\\ &\subset f([\text{Clos } R \sim R] \cup f^{-1}[q(Q)]) \subset f(B \cup f^{-1}[q(Q)]) \subset q(Q).\end{aligned}$$

Moreover to show that $f|C$ is an analytic isomorphism it suffices to note that $C \subset R$, recall [F, 3.1.18], and prove that $f|C$ is one-to-one.

For this purpose we define, for each pair of integers h, i with $I \geq h \geq i \geq 1$ and each $\mu \in \Lambda(h, i)$, the map

$$p_\mu: (\text{dmn } f)^h \to (\text{dmn } f)^i$$

so that $p_\mu(x_1, \ldots, x_h) = (x_{\mu(1)}, \ldots, x_{\mu(i)})$ for $(x_1, \ldots, x_h) \in (\text{dmn } f)^h$, we let Ω_i, for each $i \in \{1, 2, \ldots, I\}$, denote the family of connected components of

$$R_i \sim \bigcup \{p_\nu(Q_k): k \in \{i, i+1, \ldots, I\}, \nu \in \Lambda(k, i)\},$$

and we make the observation:

(5) *If* $I \geq h \geq i \geq 1$ *are integers,* $\mu \in \Lambda(h, i)$, $E \in \Omega_h$, $F \in \Omega_i$, *and* $F \cap p_\mu(E) \neq \varnothing$, *then*

$$F \subset p_\mu(E).$$

In fact, $F \cap p_\mu(E)$ is closed relative to F because

$$\begin{aligned}&F \cap [\text{Clos } p_\mu(E) \sim p_\mu(E)] \subset F \cap p_\mu(\text{Clos } E \sim E)\\ &\subset F \cap p_\mu[(\text{Clos } R_h \sim R_h) \cup (\bigcup \{p_\nu(Q_k): k \in \{h, \ldots, I\}, \nu \in \Lambda(k, h)\})]\\ &\subset F \cap [\bigcup \{p_\nu(Q_k): k \in \{i, \ldots, I\}, \nu \in \Lambda(k, i)\}] = \varnothing.\end{aligned}$$

To see that $F \cap p_\mu(E)$ is also open relative to F we assume $e \in F \cap p_\mu(E)$ and choose distinct points $x_1, \ldots, x_h$ in the fiber $R \cap f^{-1}\{f_i(e)\}$ such that $(x_1, \ldots, x_h) \in E$ and $e = (x_{\mu(1)}, \ldots, x_{\mu(i)})$. Since dim $F = t = \dim A_i$, there exists a neighborhood U of e in $(\mathbf{R}^m)^i$ with $U \cap F = U \cap A_i$. Furthermore $p_\mu|E$ has constant rank t because $f_h|E = f_i \circ (p_\mu|E)$ does, and so

$$p_\mu[E \cap p_\mu^{-1}(U)] \subset U \cap A_i = U \cap F$$

is a neighborhood of e relative to F in $F \cap p_\mu(E)$. Thus (5) follows by the connectedness of F.

Returning to the proof of (2) we show that there exists an integer $i \in \{1, 2, \ldots, I\}$ so that

(6) $\operatorname{card}(R \cap f^{-1}\{y\}) = i$ *for every* $y \in f(C)$. In fact fix a point $w \in f(C)$ and select $i \in \{1, 2, \ldots, I\}$ and $(v_1, v_2, \ldots, v_i) \in (\mathbf{R}^m)^i$ so that

$$\operatorname{card}(R \cap f^{-1}\{w\}) = i, \; v_1 \in C, \text{ and } R \cap f^{-1}\{w\} = \{v_1, v_2, \ldots, v_i\}.$$

Choosing $F \in \Omega_i$ and $\mu \in \Lambda(i, 1)$ so that $(v_1, \ldots, v_i) \in F$ and $\mu(1) = 1$, we infer from (5) that $C \subset p_\mu(F)$ because C is contained in some element of Ω_1. Hence if $y \in f(C)$, then

$$h = \operatorname{card}(R \cap f^{-1}\{y\}) \geq i.$$

If $h > i$, then we may choose $(x_1, \ldots, x_h) \in (\mathbf{R}^m)^h$, $E \in \Omega_h$, and $\nu \in \Lambda(h, 1)$ so that

$$x_1 \in C, \; R \cap f^{-1}\{y\} = \{x_1, \ldots, x_h\}, \; (x_1, \ldots, x_h) \in E, \text{ and } \nu(1) = 1,$$

deduce from (5) that $C \subset p_\nu(E)$ and obtain the contradiction

$$\operatorname{card}(R \cap f^{-1}\{w\}) \geq h,$$

and (6) follows.

Next choosing $i \in \{1, 2, \ldots, I\}$, $F \in \Omega_i$, and $\mu \in \Lambda(i, 1)$ as in the previous paragraph, we observe that

(7) $f_i | [F \cap p_\mu^{-1}(C)]$ *is one-to-one.*

In fact otherwise by (6) there exist a $y \in f(C)$ and points $x_1, x_2, \ldots, x_i \in R \cap f^{-1}\{y\}$, and a permutation $\sigma \neq \mathbf{1}_{\{1,2,\ldots,i\}}$ of $\{1, 2, \ldots, i\}$ so that

$$(x_1, \ldots, x_i) \in F \text{ and } (x_{\sigma(1)}, \ldots, x_{\sigma(i)}) \in F.$$

Accordingly
$$\mathbf{e}_j \bullet (x_{\sigma(h)} - x_h) \neq 0$$

for some $j \in \{1, 2, \ldots, m\}$ and $h \in \{1, 2, \ldots, i\}$. Defining

$$H = \{1, 2, \ldots, i\} \cap \{k\colon \operatorname{sign}[\mathbf{e}_j \bullet (x_k - x_h)] = \operatorname{sign}[\mathbf{e}_j \bullet (x_{\sigma(h)} - x_h)]\}$$

we observe that $h \notin H$ and $\sigma(h) \in H$ and that for each $k \in \{1, 2, \ldots, i\}$

$$\operatorname{sign}[\mathbf{e}_j \bullet (x_{\sigma(k)} - x_{\sigma(h)})] = \operatorname{sign}[\mathbf{e}_j \bullet (x_k - x_h)]$$

because F is a connected subset of $R_i \sim B_i$ and hence

$$\operatorname{sign} g_{j,\lambda}(x_{\sigma(1)}, \ldots, x_{\sigma(i)}) = \operatorname{sign} g_{j,\lambda}(x_1, \ldots, x_i)$$

where $\lambda \in \Lambda(i, 2)$ and $\operatorname{im} \lambda = \{h, k\}$. Using the equation

$$\mathbf{e}_j \bullet (x_{\sigma(k)} - x_h) = \mathbf{e}_j \bullet (x_{\sigma(k)} - x_{\sigma(h)}) + \mathbf{e}_j \bullet (x_{\sigma(h)} - x_h)$$

for every $k \in H$, we infer that $\sigma(H) \subset H$, hence $\sigma(H) = H$, which contradicts $\sigma(h) \in H \sim \sigma(H)$, and we conclude that the map $f_i | [F \cap p_\mu^{-1}(C)]$ is, indeed, one-to-one.

Then since $C \subset p_\mu(F)$ and since

$$f_i | [F \cap p_\mu^{-1}(C)] = (f | C) \circ (p_\mu | [F \cap p_\mu^{-1}(C)]),$$

the map $f | C$ is also one-to-one, and the proof of (2) is complete.

To prove (3) we assume that C and D are connected components of $R \sim f^{-1}[q(Q)]$ with $f(C) \cap f(D) \neq \varnothing$. To see that $f(C) = f(D)$ we note that $f(C) \cup f(D)$ is connected and that $f(C) \cap f(D)$ is closed relative to $f(C) \cup f(D)$ because

$$\begin{aligned}
&[f(C) \cup f(D)] \cap (\operatorname{Clos}[f(C) \cap f(D)] \sim [f(C) \cap f(D)]) \\
&\quad \subset [\mathbf{R}^n \sim q(Q)] \cap ([\operatorname{Clos} f(C) \sim f(C)] \cup [\operatorname{Clos} f(D) \sim f(D)]) \\
&\quad \subset [\mathbf{R}^n \sim q(Q)] \cap f[(\operatorname{Clos} C \sim C) \cup (\operatorname{Clos} D \sim D)] \\
&\quad \subset [\mathbf{R}^n \sim q(Q)] \cap f([\operatorname{Clos} R \sim R] \cup f^{-1}[q(Q)]) \\
&\quad \subset [\mathbf{R}^n \sim q(Q)] \cap f(B \cup f^{-1}[q(Q)]) \subset [\mathbf{R}^n \sim q(Q)] \cap q(Q) = \varnothing.
\end{aligned}$$

On the other hand if $y \in f(C) \cap f(D)$ and $R \cap f^{-1}\{y\} = \{x_1, \ldots, x_i\}$, then there exist $F \in \Omega_i$, $\mu \in \Lambda(i, 1)$, and $\nu \in \Lambda(i, 1)$ so that

$$(x_1, \ldots, x_i) \in F, \quad x_{\mu(1)} \in C, \quad \text{and } x_{\nu(1)} \in D.$$

We infer from (5) that

$$C \subset p_\mu(F) \text{ and } D \subset p_\nu(F)$$

and from (7) that f_i maps $F \cap p_\mu^{-1}(C)$ isomorphically onto $f(C)$ and $F \cap p_\nu^{-1}(D)$ isomorphically onto $f(D)$, and we conclude that

$$f(C) \cap f(D) = f_i[F \cap p_\mu^{-1}(C) \cap p_\nu^{-1}(D)]$$

is open relative to $f(C)$, to $f(D)$, and hence to $f(C) \cup f(D)$. Therefore $f(C) = f(D)$ and (3) follows.

For the proof of (4) we will make a second application of Proposition $(\mathbf{B}_{t-1})$ by considering three cases.

Case 1, $n < t$. Here $R = \varnothing$.

Case 2, $n = t$. Here we need only observe that $f | (E \sim f^{-1}[q(Q)])$ is a covering map with finite fibers because

$$\text{Clos } R \sim R \subset B \subset f^{-1}[q(Q)]$$

and that the set $\mathbf{R}^t \sim q(Q)$ has only a finite number of connected components by Proposition $(\mathbf{B}_{t-1})$.

Case 3, $n > t$. Here we first choose $p \in \mathbf{O}^*(n, t)$ so that the set

$$R' = A \cap \{x\colon x \text{ is a regular point of } A \text{ and } \dim D(p \circ f)(x)[\operatorname{Tan}(A, x)] = t\}$$

satisfies $\dim (R \sim R') \leq t-1$. For this purpose we choose a countable dense subset V of R, let P be an $n-t$ dimensional vector subspace of $\mathbf{R}^n$ with

$$P \cap Df(v)[\operatorname{Tan}(A, v)] = \{0\} \quad \text{for all } v \in V,$$

and choose $p \in \mathbf{O}^*(n, t)$ so that $\ker p = P$. It follows that $\dim(R \sim R') \leq t-1$, because otherwise we may first apply 2.5 with f, R, and $\mathcal{G}$ replaced by $p \circ f$, R', and $\varnothing$ to choose a compact set $B' \in \mathcal{S}(\mathbf{R}^m)$ so that

$$\dim B' \leq t-1,\ R' \sim B' \in \mathcal{S}(\mathbf{R}^m),\ \text{Clos } R' \sim R' \subset B',$$

$$(R \sim R') \sim (B \cup B') = (R \sim B) \sim (R' \sim B') \sim B' \in \mathcal{S}(\mathbf{R}^m),$$

$$\dim [(R \sim R') \sim (B \cup B')] = t,$$

and then apply 2.2(2) with A replaced by $(R \sim R') \sim (B \cup B')$ to choose a t dimensional analytic block G satisfying the contradictory conditions

$$G \subset (R \sim R') \sim (B \cup B'), \quad \varnothing \neq G \cap V \subset G \cap R'.$$

Next we choose by 2.3 a compact set $Y \in \mathcal{S}(\mathbf{R}^m)$ so that

$$(R \sim R') \cup B' \subset Y \quad \text{and} \quad \dim Y \leq t-1,$$

and observe that

$$X = R' \cap (p \circ f)^{-1}(p[f(Y) \cup q(Q)])$$

has $\mathcal{H}^t$ measure zero because $\mathcal{H}^t(p[f(Y) \cup q(Q)]) = 0$ and because R' may be covered by countably many sets U open in $\mathbf{R}^m$ such that $(p \circ f) | (U \cap R')$ is an analytic isomorphism whose inverse is Lipschitzian. Moreover $(p \circ f) | (R' \sim X)$ is a covering map with finite fibers because

$$\text{Clos } R' \sim R' \subset B' \subset Y.$$

Since, according to 2.6 and Proposition $(\mathbf{B}_{t-1})$, $\mathbf{R}^t \sim p[f(Y) \cup q(Q)]$ has only a finite number of components, so does $R' \sim X$. Finally every component of $R \sim f^{-1}[q(Q)]$ contains at least one component of $R' \sim X$ because

$$R \sim f^{-1}[q(Q)] \sim (R' \sim X) \subset (R \sim R') \cup X$$

has $\mathcal{H}^t$ measure zero. This completes the proof of (4) and hence of 2.8.

2.9. COROLLARY. *If A is the union of finitely many analytic blocks in M,* Clos A *is compact,* $\dim A = t$, *and f is an analytic map of a neighborhood of* Clos A *into* $\mathbf{R}^n$, *then:*

(1) *There exists an integer J such that*

$$\operatorname{card}(A \cap f^{-1}\{y\}) \leqslant J \text{ whenever } \dim(A \cap f^{-1}\{y\}) \leqslant 0.$$

(2) *In case $t > n$, there exists an integer J^* such that*

$$\mathcal{H}^{t-n}(A \cap f^{-1}\{y\}) \leqslant J^* \text{ whenever } \dim(A \cap f^{-1}\{y\}) \leqslant t-n.$$

Proof. It is sufficient to prove the corollary *in case* $M = \mathbf{R}^m$.

In this case we prove (1) by induction on t. For $t = -1$ we take $J = 0$, and for $t = 0$ we take $J = \operatorname{card} A$. We now assume that $t > 0$ and that (1) with A replaced by B is true whenever $B \in \mathcal{S}(M)$, Clos B is compact and $\dim B < t$. We may also assume, without loss of generality, that A itself is a t dimensional analytic block in M and that A is described by $U, g_0, g_1, \dots, g_{m-t}$ as in the definition in § 2.

Letting $f_1, \dots, f_n$ be the real-valued functions such that

$$f(x) = (f_1(x), \dots, f_n(x)) \quad \text{for } x \in \operatorname{dmn} f,$$

we define for each $s \in \{1, \dots, t\}$ and $\lambda \in \Lambda(n, s)$ the real-valued analytic function

$$\phi_\lambda = |Dg_1 \wedge \dots \wedge Dg_{m-t} \wedge Df_{\lambda(1)} \wedge \dots \wedge Df_{\lambda(s)}|^2.$$

Then the analytic functions

$$\phi_0 = 1, \quad \phi_s = \sum_{\lambda \in \Lambda(n,s)} \phi_\lambda \quad \text{for } s \in \{1, \dots, t\}$$

satisfy the condition

$$r_x = \sup\{s \colon \phi_s(x) \neq 0\} = \dim Df(x)[\operatorname{Tan}(A, x)] \quad \text{for } x \in A.$$

Letting $r = \sup_{x \in A} r_x$ we infer from 2.1(6) that

$$B = A \cap \{x \colon r_x < r\} = A \cap \{x \colon (\phi_r + \dots + \phi_t)(x) = 0\} \in \mathcal{S}(M)$$

and from [F, 3.4.8(15), 3.1.24] that $\dim B < t$. Choosing an integer j such that

$$\operatorname{card}(B \cap f^{-1}\{y\}) \leqslant j \text{ whenever } \dim(B \cap f^{-1}\{y\}) \leqslant 0,$$

we consider the two cases:

Case 1, $r < t$. Here it suffices to take $J = j$ because any point $x \in A \sim B$ is a generic point of rank r for $f|A$, hence

$$\dim[A \cap f^{-1}\{f(x)\}] \geqslant t - r > 0$$

by [F, 3.1.18].

Case 2, $r=t$. Here we let

$$R = A \sim B = A \cap \{x: \dim Df(x)[\operatorname{Tan}(A, x)] = t\},$$

choose I as in 2.8(1), and take $J = I + j$.

To prove (2) we recall, for each $\lambda \in \Lambda(m, t-n)$, the projection $\mathbf{p}_\lambda: \mathbf{R}^m \to \mathbf{R}^{t-n}$ defined in [F, 1.7.4], apply (1) with f replaced by $f \boxdot \mathbf{p}_\lambda$ to choose an integer J_λ such that

$$\operatorname{card}(A \cap f^{-1}\{y\} \cap \mathbf{p}_\lambda^{-1}\{z\}) \leqslant J_\lambda \text{ whenever } \dim(A \cap f^{-1}\{y\} \cap \mathbf{p}_\lambda^{-1}\{z\}) \leqslant 0,$$

and set

$$J^* = \sum_{\lambda \in \Lambda(m, t-n)} J_\lambda \mathcal{L}^{t-n}[\mathbf{p}_\lambda(A)].$$

If $y \in \mathbf{R}^n$ and $\dim(A \cap f^{-1}\{y\}) \leqslant t-n$, then by 2.2(7)

$$\dim(A \cap f^{-1}\{y\} \cap \mathbf{p}_\lambda^{-1}\{z\}) \leqslant 0$$

for all $\lambda \in \Lambda(m, t-n)$ and $\mathcal{L}^{t-n}$ almost all $z \in \mathbf{R}^{t-n}$; using 2.2(3) and [*F*, 3.2.27], we conclude

$$\mathcal{H}^{t-n}(A \cap f^{-1}\{y\}) \leqslant \sum_{\lambda \in \Lambda(m, t-n)} \int \operatorname{card}(A \cap f^{-1}\{y\} \cap \mathbf{p}_\lambda^{-1}\{z\})\, d\mathcal{L}^{t-n} z \leqslant J^*.$$

2.10. COROLLARY. *Suppose that* $E \subset M$, Clos E *is compact*, $\dim E \leqslant t$, *and* f *is an analytic map of a neighborhood of* Clos E *into* $\mathbf{R}^n$.

(1) *If* $t \leqslant n$, *then there exists an integer* J *such that*

$$\operatorname{card}(E \cap f^{-1}\{y\}) \leqslant J \text{ for } \mathcal{H}^t \text{ almost all } y \in \mathbf{R}^n.$$

(2) *If* $t \geqslant n$, *then there exists an integer* J^* *such that*

$$\mathcal{H}^{t-n}(E \cap f^{-1}\{y\}) \leqslant J^* \text{ for } \mathcal{L}^n \text{ almost all } y \in \mathbf{R}^n.$$

Proof. We note by [F, 2.10.35], that on $\mathbf{R}^t$ the two measures $\mathcal{H}^t$, $\mathcal{L}^t$ coincide, and on $\mathbf{R}^m$ $\mathcal{H}^0$ equals counting measure. Then we choose, according to 2.3, a compact set $A \in \mathcal{S}(M)$ with $E \subset A \subset \operatorname{dmn} f$ and $\dim A = \dim E$, and we apply [F, 2.10.11] and 2.9.

2.11. *Remark. Propositions* ($\mathbf{A}_t$) *and* ($\mathbf{B}_t$) *remain true if* $\mathbf{R}^m$ *is replaced by an* m *dimensional analytic manifold* M. In fact, there exist a positive integer J and for each $j \in \{1, 2, ..., J\}$ an open subset U_j of M along with an analytic isomorphism h_j of a neighborhood of Clos U_j into $\mathbf{R}^m$ so that $h_j(U_j) = \mathbf{U}(0, 1)$ and

$$\operatorname{Clos} A \subset U_1 \cup U_2 \cup \dots \cup U_J.$$

For each $j \in \{1, 2, ..., J\}$ we select a point $a_j \in \mathbf{R}^m \sim h_j(\operatorname{Clos} U_j)$, define

$$A^* = \bigcup_{j=1}^{J}[\{a_1\} \times \dots \times \{a_{j-1}\} \times h_j(U_j \cap A) \times \{a_{j+1}\} \times \dots \times \{a_J\}] \subset \mathbf{R}^m)^J,$$

note that $A^* \in \mathcal{S}([\mathbf{R}^m]^J)$ by 2.1(1) (2) (4) (5) (6), choose an analytic map ϕ of a neighborhood of Clos A^* so that

$$\phi(a_1, \dots, a_{j-1}, z, a_{j+1}, \dots, a_J) = h_j^{-1}(z)$$

whenever $j \in \{1, 2, \dots, J\}$ and $z \in h_j(U_j \cap A)$, and define $f^* = f \circ \phi$ and

$$R^* = A^* \cap \{w: w \text{ is a regular point of } A^* \text{ and } \dim Df^*(w)[\operatorname{Tan}(A^*, w)] = t\}.$$

To prove Proposition $(\mathbf{A}_t)$ and $(\mathbf{B}_t)$ with $\mathbf{R}^m$ replaced by M, we apply Propositions $(\mathbf{A}_t)$ and $(\mathbf{B}_t)$ with $\mathbf{R}^m$, A, f replaced by $(\mathbf{R}^m)^J$, A^*, f^* and choose Q and q accordingly. We observe that

$$f(A) = f^*(A^*), \quad f(R) = f^*(R^*),$$

hence $f(R) \sim q(Q) = f^*(R^*) \sim q(Q)$ is a t dimensional analytic submanifold of $\mathbf{R}^t$. Also

$$\text{the number of components of } R \sim f^{-1}[q(Q)] \\ \leq \text{the number of components of } R^* \sim (f^*)^{-1}[q(Q)].$$

In fact, if C is a component of $R \sim f^{-1}[q(Q)]$ and C^* is a component of $R^* \sim (f^*)^{-1}[q(Q)]$ with $\phi(C^*) \cap C \neq \emptyset$, then $\phi(C^*)$, being a connected subset of $R \sim f^{-1}[q(Q)]$, is contained in C. Moreover in this case

$$\phi(C^*) = C \quad \text{and } f|C \text{ is an analytic isomorphism}$$

because C is connected, $f^*|C^* = f \circ (\phi|C^*)$ is an analytic isomorphism, $\phi|C^*$ is an analytic isomorphism, $\phi(C^*)$ is open relative to C, and

$$\begin{aligned} C \cap [\operatorname{Clos} \phi(C^*) \sim \phi(C^*)] &\subset C \cap \phi[\operatorname{Clos} C^* \sim C^*] \\ &\subset C \cap f^{-1}[f^*(\operatorname{Clos} C^* \sim C^*)] \subset C \cap f^{-1}[\operatorname{Clos} f^*(C^*) \sim f^*(C^*)] \\ &\subset C \cap f^{-1}([\operatorname{Clos} f^*(R^*) \sim f^*(R^*)] \cup q[Q]) \\ &= C \cap f^{-1}([\operatorname{Clos} f(R) \sim f(R)] \cup q[Q]) = \emptyset. \end{aligned}$$

3. Some properties of the groups $\mathcal{F}_t^{\mathrm{loc}}(M)$ and $\mathbf{I}_t^{\mathrm{loc}}(M)$

In this section let M be a separable Riemannian manifold of class ∞, and let t be a nonnegative integer. We will consider the vectorspaces $\mathcal{D}^t(M)$ and $\mathcal{D}_t(M)$ of *t dimensional differential forms with compact support in M and t dimensional currents in M, the mass norm* $\mathbf{M}$ *on* $\mathcal{D}_t(M)$, the abelian subgroups $\mathcal{R}_t(M)$ and $\mathbf{I}_t(M)$ of $\mathcal{D}_t(M)$ consisting of *t dimensional rectifiable* and *integral currents in M*, and the group $\mathcal{F}_t(M)$ of *t dimensional integral flat chains in M* defined by

$$\mathcal{F}_t(M) = \mathcal{D}_t(M) \cap \{R + \partial S : R \in \mathcal{R}_t(M),\ S \in \mathcal{R}_{t+1}(M)\}.$$

These concepts are discussed thoroughly in Chapter Four of [F] in case M is an open subset of a Euclidean space. For an arbitrary separable Riemannian manifold of class ∞ $\mathcal{D}^t(M)$, $\mathcal{D}_t(M)$, $\mathbf{M}$, $\mathcal{R}_t(M)$, and $\mathbf{I}_t(M)$ have been used in [F2] and are easily defined by reformulating [F, 4.1]. Moreover most of the results of [F, 4.1–4.3] have been written so as to be readily adaptable to Riemannian manifolds. As in [F, 4.1.24] we may consider the localized versions of each of these groups by defining the group

$$\mathcal{R}_t^{\text{loc}}(M) \ \ [\text{resp. } \mathbf{I}_t^{\text{loc}}(M), \text{ resp. } \mathcal{F}_t^{\text{loc}}(M)]$$

of *t dimensional locally rectifiable currents* [resp. *locally integral currents*, resp. *locally integral flat chains*] *in M* as the collection of all currents $T \in \mathcal{D}_t(M)$ such that for every $x \in M$ there exists a current $Q \in \mathcal{R}_t(M)$ [resp. $\mathbf{I}_t(M)$, resp. $\mathcal{F}_t(M)$] with $x \notin \operatorname{spt}(T-Q)$. Consequently

$$\begin{array}{ccccc} \mathbf{I}_t^{\text{loc}}(M) & \subset & \mathcal{R}_t^{\text{loc}}(M) & \subset & \mathcal{F}_t^{\text{loc}}(M) \\ \cup & & \cup & & \cup \\ \mathbf{I}_t(M) & \subset & \mathcal{R}_t(M) & \subset & \mathcal{F}_t(M). \end{array}$$

As in [F, 4.3.16] we topologize the group $\mathcal{F}_t^{\text{loc}}(M)$ by associating with each pair (U, δ) such that

$$U \text{ is open, Clos } U \text{ is compact, } \delta > 0,$$

a *basic neighborhood of* 0 $\mathbf{N}(U, \delta)$ consisting of those currents $T \in \mathcal{F}_t^{\text{loc}}(M)$ for which there exist $R \in \mathcal{R}_t(M)$ and $S \in \mathcal{R}_{t+1}(M)$ with

$$\operatorname{spt}(T - R - \partial S) \subset M \sim U,\ \mathbf{M}(R) + \mathbf{M}(S) < \delta.$$

This definition has the following three consequences.

(1) *If* $U_1 \subset U_2 \subset \ldots$ *are open sets having compact closures in M and* $\bigcup_{j=1}^{\infty} U_j = M$, *then the collection*

$$\{\mathbf{N}(U_j, j^{-1}) : j = 1, 2, \ldots\}$$

forms a countable neighborhood basis at 0.

(2) *If f is a locally Lipschitzian map of M into a Riemannian manifold N of class* ∞, $K \subset M$, *and the map* $f|K$ *is proper, then the induced homomorphism* $f_\#$ *maps*

$$\mathcal{F}_t^{\text{loc}}(M) \cap \{T : \operatorname{spt} T \subset K\}$$

continuously into $\mathcal{F}_t^{\text{loc}}(N)$.

(3) *In case* $t \geq 1$ *the boundary operator* ∂ *maps* $\mathcal{F}_t^{\text{loc}}(M)$ *continuously into* $\mathcal{F}_{t-1}^{\text{loc}}(M)$.

3.1. Lemma (Extending representations). *If, for each* $i \in \{1, 2\}$, U_i *and* V_i *are open subsets of* M *so that* Clos V_i *is a compact subset of* U_i, *then there exists a positive number* ϱ *such that whenever*

$$T \in \mathfrak{F}_t^{\text{loc}}(M),\ R_i \in \mathcal{R}_t(M),\ S_i \in \mathcal{R}_{t+1}(M),$$

$$\textit{and } \operatorname{spt}(T - R\ - \partial S\ ) \subset M \sim U_i \textit{ for } i \in \{1, 2\}$$

one may find $R \in \mathcal{R}_t(M)$, $S \in \mathcal{R}_{t+1}(M)$ *with*

$$\operatorname{spt}(R - R_1) \cup \operatorname{spt}(S - S_1) \subset M \sim V_1,$$

$$\operatorname{spt}(T - R - \partial S) \subset M \sim (V_1 \cup V_2),$$

$$\mathbf{M}(R) + \mathbf{M}(S) \leqslant \varrho[\mathbf{M}(R_1) + \mathbf{M}(S_1) + \mathbf{M}(R_2) + \mathbf{M}(S_2)].$$

Proof. We choose $\alpha \in \mathcal{D}^0(M)$ with im $\alpha \subset \{y: 0 \leqslant y \leqslant 1\}$, $\alpha(x) = 0$ for $x \in V_1$, $a(x) = 1$ for $x \in V_2 \sim U_1$, and set $\varrho = 2 + 2\operatorname{Lip}(\alpha)$. Noting that

$$K = (\operatorname{Clos} V_2) \cap \{x: 1/4 \leqslant \alpha(x) \leqslant 3/4\} \subset (U_1 \cap U_2) \sim V_1,$$

we also choose $\beta \in \mathcal{D}^0(M)$ with im $\beta \subset \{y: 0 \leqslant y \leqslant 1\}$,

$$\operatorname{spt} \beta \subset (U_1 \cap U_2) \sim V_1, \quad K \cap \operatorname{spt}(1 \sim \beta) = \varnothing.$$

We will now show that there exists a number r so that $0 < r < 1$ and

$$(S_2 - S_1) \llcorner \{x: \beta(x) > r\} \in \mathbf{I}_{t+1}(M).$$

For this purpose we choose $\gamma \in \mathcal{D}^0(M)$ with

$$\operatorname{im} \gamma \subset \{y: 0 \leqslant y \leqslant 1\},\ \operatorname{spt} \gamma \subset (U_1 \cap U_2) \sim V_1,\ \operatorname{spt}(\beta) \cap \operatorname{spt}(1 - \gamma) = \varnothing,$$

and we observe that $(S_2 - S_1) \llcorner \gamma$ is a normal current in M ([F, p. 358]) because, for each $\phi \in \mathcal{D}^t(U)$,

$$\partial[(S_2 - S_1) \llcorner \gamma](\phi) = (S_2 - S_1)(\gamma d\phi) = (\partial S_2 - \partial S_1)(\gamma\phi) - (S_2 - S_1)(\phi \wedge d\gamma),$$

$$\operatorname{spt}(\gamma\phi) \subset (U_1 \cap U_2) \sim V_1,$$

$$(R_1 + \partial S_1)(\gamma\phi) = T(\gamma\phi) = (R_2 + \partial S_2)(\gamma\phi),$$

hence
$$\mathbf{M}(\partial[(S_2 - S_1) \llcorner \gamma]) \leqslant \mathbf{M}(R_2 - R_1) + \operatorname{Lip}(\gamma)\mathbf{M}(S_2 - S_1) < \infty.$$

Noting that the discussions of [F, 4.2.1, 4.3.4, 4.3.6] apply to Riemannian manifolds, we choose r such that $0 < r < 1$ and

$$\langle (S_2 - S_1) \llcorner \gamma, \beta, r+ \rangle = \langle (S_2 - S_1) \llcorner \gamma, \beta, r \rangle \in \mathbf{I}_t(M),$$

hence $Q = [(S_2 - S_1) \llcorner \gamma] \llcorner \{x: \beta(x) > r\} \in \mathbf{I}_{t+1}(M)$; inasmuch as $\{x: \beta(x) > r\} \subset \operatorname{spt} \beta \subset \{x: \gamma(x) = 1\}$ we find that

$$Q = (S_2 - S_1) \llcorner \{x: \beta(x) > r\} \in \mathbf{I}_{t+1}(M).$$

Next we remark that $K \subset \{x: \beta(x) > r\}$,

$$\operatorname{spt}(S_2 - S_1 - Q) \subset M \sim K, \text{ and } \mathbf{M}(Q) \leqslant \mathbf{M}(S_1) + \mathbf{M}(S_2),$$

and choose s so that

$$1/4 < s < 3/4, \ \langle Q, \alpha, s+\rangle \in \mathbf{I}_t(M), \ \mathbf{M}\langle Q, \alpha, s+\rangle \leqslant 2 \operatorname{Lip}(\alpha) \mathbf{M}(Q).$$

Defining

$$R = R_1 + (R_2 - R_1) \llcorner \{x: \alpha(x) > s\} + \langle Q, \alpha, s+\rangle,$$

$$S = S_1 + (S_2 - S_1) \llcorner \{x: \alpha(x) > s\},$$

we readily obtain the mass estimate

$$\begin{aligned} \mathbf{M}(R) + \mathbf{M}(S) &\leqslant 2\mathbf{M}(R_1) + \mathbf{M}(R_2) + 2\mathbf{M}(S_1) + \mathbf{M}(S_2) + 2 \operatorname{Lip}(\alpha)[\mathbf{M}(S_1) + \mathbf{M}(S_2)] \\ &\leqslant \varrho[\mathbf{M}(R_1) + \mathbf{M}(S_1) + \mathbf{M}(R_2) + \mathbf{M}(S_2)] \end{aligned}$$

and the inclusion

$$\operatorname{spt}(R - R_1) \cup \operatorname{spt}(S - S_1) \subset M \sim V_1$$

because $V_1 \cap [(\operatorname{spt} \alpha) \cup \operatorname{spt} Q] = \emptyset$. In order to verify that $\operatorname{spt}(T - R - \partial S) \subset M \sim (V_1 \cup V_2)$ we suppose $\phi \in \mathcal{D}^t(M)$ and consider three special cases.

Case 1, $\operatorname{spt} \phi \subset (V_1 \cup V_2) \cap \{x: \alpha(x) < s\}$. Here $R(\phi) = R_1(\phi)$, $S(d\phi) = S_1(d\phi)$, and $\operatorname{spt} \phi \subset U_1$,

$$\text{hence } T(\phi) = (R_1 + \partial S_1)(\phi) = (R + \partial S)(\phi).$$

Case 2, $\operatorname{spt} \phi \subset (V_1 \cup V_2) \cap \{x: \alpha(x) > s\}$. Here $R(\phi) = R_2(\phi)$, $S(d\phi) = S_2(d\phi)$, and $\operatorname{spt} \phi \subset U_2$,

$$\text{hence } T(\phi) = (R_2 + \partial S_2)(\phi) = (R + \partial S)(\phi).$$

Case 3, $\operatorname{spt} \phi \subset (V_1 \cup V_2) \cap \{x: 1/4 < \alpha(x) < 3/4\}$. Here $\operatorname{spt} \phi \subset K \subset U_1 \cap U_2$, $T(\phi) = (R_1 + \partial S_1)(\phi)$. Letting σ denote the characteristic function of $\{x: \alpha(x) > s\}$ we infer that

$$\langle Q, \alpha, s+\rangle = (\partial Q) \llcorner \sigma - \partial(Q \llcorner \sigma),$$

hence

$$\begin{aligned} (R + \partial S - T)(\phi) &= (R_2 - R_1)(\sigma\phi) + \langle Q, \alpha, s+\rangle(\phi) + \partial[(S_2 - S_1) \llcorner \sigma](\phi) \\ &= (R_2 - R_1)(\sigma\phi) + (\partial Q)(\sigma\phi) - Q(\sigma d\phi) + (S_2 - S_1)(\sigma d\phi) \\ &= (R_2 - R_1 + \partial Q)(\sigma\phi) + (S_2 - S_1 - Q)(\sigma d\phi) = 0 \end{aligned}$$

because $\operatorname{spt}(\sigma\phi) \cup \operatorname{spt}(\sigma d\phi) \subset K$ while

$$\operatorname{spt}(S_2 - S_1 - Q) \subset M \sim K, \quad \operatorname{spt}[R_2 - R_1 - \partial(S_2 - S_1)] \subset M \sim (U_1 \cap U_2) \subset M \sim K.$$

3.2. Corollary.

(1) *If* $\mathcal{U}$ *is a cover for* M *consisting of open sets having compact closures, then the collection of all* $\mathbf{N}(U, \delta)$ *corresponding to* $U \in \mathcal{U}$ *and* $\delta > 0$ *forms a neighborhood subbasis of* $\mathcal{F}_t^{\text{loc}}(M)$ *at* 0.

(2) *Whenever* $U_1, U_2, \ldots$ *are open sets having compact closures in* M *and* $\bigcup_{j=1}^{\infty} U_j = M$ *we may exhibit* $\mathcal{F}_t^{\text{loc}}(M)$ *with its topology as a complete metric space by defining the distance between two points* T_1, T_2 *of* $\mathcal{F}_t^{\text{loc}}(M)$ *as*

$$\mathbf{dist}\,(T_1, T_2) = \sum_{j=1}^{\infty} [\delta_j / 2^j (1 + \delta_j)] \text{ where } \delta_j = \inf\{\delta : T_1 - T_2 \in \mathbf{N}(U_j, \delta)\}.$$

(3) $\mathcal{F}_t(M) = \mathcal{F}_t^{\text{loc}}(M) \cap \{T : \text{spt } T \text{ is compact}\}$.

(4) $\mathcal{F}_t(M)$ *is dense in* $\mathcal{F}_t^{\text{loc}}(M)$.

(5) *For each* $T \in \mathcal{F}_t^{\text{loc}}(M)$ *there exist* $R \in \mathcal{R}_t^{\text{loc}}(M)$ and $S \in \mathcal{R}_{t+1}^{\text{loc}}(M)$ *such that* $T = R + \partial S$.

Proof. To prove (1) we assume that U is an open subset of M, Clos U is compact and $\delta > 0$. We choose

first, open sets $U_1, U_2, \ldots, U_J$ in $\mathcal{U}$ so that

$$\text{Clos } U \subset U_1 \cup U_2 \cup \ldots \cup U_J,$$

second, open sets $V_1, V_2, \ldots, V_J$ so that

$$\text{Clos } U \subset V_1 \cup V_2 \cup \ldots \cup V_J, \text{ Clos } V_j \subset U_j \text{ for } j \in \{1, 2, \ldots, J\},$$

and third, open sets $W_0, W_1, \ldots, W_J$ so that $W_0 = \emptyset$

and
$$\text{Clos } (\bigcup_{i=1}^{j} V_i) \subset W_j \subset \text{Clos } W_j \subset W_{j-1} \cup U_j$$

for every $j \in \{1, 2, \ldots, J\}$. Then for each $j \in \{2, 3, \ldots, J\}$ we apply 3.1 with U_1, V_1, U_2, V_2 replaced by $W_{j-2} \cup U_{j-1}$, W_{j-1}, U_j, V_j to choose an appropriate positive number ϱ_j, and we set

$$\varepsilon_j = \delta / [2^{(J-j+1)} \varrho_j \varrho_{j+1} \cdots \varrho_J] \quad \text{and} \quad \varepsilon_1 = \varepsilon_2.$$

To prove the inclusion

$$\mathbf{N}(U_1, \varepsilon_1) \cap \mathbf{N}(U_2, \varepsilon_2) \cap \ldots \cap \mathbf{N}(U_J, \varepsilon_J) \subset \mathbf{N}(U, \delta)$$

we assume $T \in \mathbf{N}(U_1, \varepsilon_1) \cap \ldots \cap \mathbf{N}(U_J, \varepsilon_J)$, choose for each $j \in \{1, 2, \ldots, J\}$ currents $R_j \in \mathcal{R}_t(M)$ $S_j \in \mathcal{R}_{t+1}(M)$ so that

$$\text{spt}(T - R_j - \partial S_j) \subset M \sim U_j, \; \mathbf{M}(R_j) + \mathbf{M}(S_j) < \varepsilon_j$$

and inductively select currents $P_1, P_2, \ldots, P_J$ in $\mathcal{R}_t(M)$ and $Q_1, Q_2, \ldots, Q_J$ in $\mathcal{R}_{t+1}(M)$ so that $P_1 = R_1$, $Q_1 = S_1$, and

$$\operatorname{spt}(T-P_j-\partial Q_j)\subset M\sim(W_{j-1}\cup V_j),$$

$$\mathbf{M}(P_j)+\mathbf{M}(Q_j)<\varrho_j(\varepsilon_j+\varrho_{j-1}(\varepsilon_{j-1}+\dots+\varrho_2(\varepsilon_2+\varepsilon_1)\dots))$$

for every $j\in\{2, 3, \dots, J\}$ by using 3.1 with

$$U_1,\ V_1,\ U_2,\ V_2,\ \varrho,\ R_1,\ S_1,\ R_2,\ S_2$$

replaced by

$$W_{j-2}\cup U_{j-1}, W_{j-1},\ U_j,\ V_j,\ \varrho_j,\ P_{j-1},\ Q_{j-1},\ R_j,\ S_j.$$

In particular

$$\operatorname{spt}(T-P_J-\partial Q_J)\subset M\sim(W_{J-1}\cup V_J)\subset M\sim U,$$

$$\mathbf{M}(P_J)+\mathbf{M}(Q_J)<\varrho_J(\varepsilon_J+\varrho_{J-1}(\varepsilon_{J-1}+\dots+\varrho_2(\varepsilon_2+\varepsilon_1)\dots))=\delta,$$

hence $T\in\mathbf{N}(U,\delta)$.

To prove (2) we note that the metric **dist** defines the topology of $\mathcal{F}_t^{\mathrm{loc}}(M)$ by virtue of (4). To show completeness we assume that $T_1, T_2, \dots$ is a **dist** Cauchy sequence in $\mathcal{F}_t^{\mathrm{loc}}(M)$, observe that $\mathbf{R}$ is complete and that the topology of $\mathcal{F}_t^{\mathrm{loc}}(M)$ is stronger than the relative topology induced by the inclusion $\mathcal{F}_t^{\mathrm{loc}}(M)\subset\mathcal{D}_t(M)$, and then let T be the functional defined on $\mathcal{D}^t(M)$ by the condition

$$T(\phi)=\lim_{i\to\infty}T_i(\phi)\quad\text{for }\phi\in\mathcal{D}^t(M).$$

Clearly T is linear; to see that $T\in\mathcal{F}_t^{\mathrm{loc}}(M)$ we pass to a subsequence and relabel so that

$$\mathbf{dist}\,(T_{i+1},T_i)<2^{-i}\quad\text{for } i\in\{1,2,\dots\}.$$

Choosing for each $i\in\{1,2,\dots\}$ and $j\in\{1,2,\dots\}$ currents $R_{i,j}\in\mathcal{R}_t(M)$ and $S_{i,j}\in\mathcal{R}_{t+1}(M)$ so that

$$\operatorname{spt}(T_{i+1}-T_i-R_{i,j}-\partial S_{i,j})\subset M\sim U_j,$$

$$\sum_{i=j+1}^{\infty}[\mathbf{M}(R_{i,j})+\mathbf{M}(S_{i,j})]\leq\sum_{i=j+1}^{\infty}2^{j-i}/(1-2^{j-i})<\infty,$$

we conclude that

$$R_j=\sum_{i=1}^{\infty}R_{i,j}\in\mathcal{R}_t^{\mathrm{loc}}(M),\ \ S_j=\sum_{i=1}^{\infty}S_{i,j}\in\mathcal{R}_{t+1}^{\mathrm{loc}}(M),\ \ T_1+R_j+\partial S_j\in\mathcal{F}_t^{\mathrm{loc}}(M),$$

$T(\phi)=(T_1+R_j+\partial S_j)(\phi)$ whenever $\phi\in\mathcal{D}^t(M)$ with $\operatorname{spt}\phi\subset U_j$,

$$\mathbf{dist}\,(T_i,T)\to 0\quad\text{as } i\to\infty.$$

The proofs of (3), (4), and (5) are similar to that of (1).

3.3. Whenever $U\subset W$ are open subsets of M there is a continuous monomorphism $\mathcal{D}^t(U)\to\mathcal{D}^t(W)$ which sends $\phi\in\mathcal{D}^t(U)$ to $\phi\cup[(W\sim U)\times\{0\}]\in\mathcal{D}^t(W)$. For every $T\in\mathcal{D}_t(W)$ the image of T under the dual linear map $\mathcal{D}_t(W)\to\mathcal{D}_t(U)$ will be called the *restriction of*

T to $\mathcal{D}^t(U)$ and denoted $T|U$. Moreover whenever $S \in \mathcal{D}^t(U)$ and spt S is a compact subset of U we define the *extension of S to $\mathcal{D}^t(W)$* to be the unique current $T \in \mathcal{D}_t(W)$ for which spt $T \subset$ spt S and $T|U = S$.

3.4. COMPACTNESS THEOREM. *If $\mathcal{C}$ is a subset of $\mathbf{I}_t^{\mathrm{loc}}(M)$ and if for every compact $K \subset M$ there exists an integer I such that*

$$(\|T\| + \|\partial T\|)(K) \leqslant I \quad \textit{whenever } T \in \mathcal{C},$$

then $\mathcal{C}$ is relatively compact in $\mathbf{I}_t^{\mathrm{loc}}(M)$ with respect to the topology of $\mathcal{F}_t^{\mathrm{loc}}(M)$.

Proof. We choose $U_1, V_1, h_1, U_2, V_2, h_2, \ldots$ so that $\bigcup_{j=1}^{\infty} V_j = M$ and for each $j \in \{1, 2, \ldots\}$ U_j is an open subset of M, h_j is a Lipschitzian analytic isomorphism from U_j onto the open ball $\mathbf{U}(0, 2) \subset \mathbf{R}^m$ with $h_j(V_j) = \mathbf{U}(0, 1)$, and we define the function

$$u\colon \mathbf{U}(0, 2) \to \mathbf{R}, \quad u(y) = |y| \ \text{ for } y \in \mathbf{U}(0, 2).$$

Suppose that $T_1, T_2, \ldots$ is a sequence of elements of $\mathcal{C}$. To show that this sequence contains a subsequence convergent in $\mathbf{I}_t^{\mathrm{loc}}(M)$, we recall [F, 4.2.1, 4.3.4, 4.3.6] and for each $i \in \{1, 2, \ldots\}$ and $j \in \{1, 2, \ldots\}$ select $r_{i,j}$ so that $1 < r_{i,j} < 2$,

$$\langle h_{j\#}(T_i|U_j), u, r_{i,j}-\rangle = \langle h_{j\#}(T_i|U_j), u, r_{i,j}\rangle \in \mathbf{I}_{t-1}[\mathbf{U}(0, 2)],$$
$$\mathbf{M}\,\langle h_{j\#}(T_i|U_j), u, r_{i,j}-\rangle \leqslant [\mathrm{Lip}\,(h_j)]^t \|T_i\| (U_j \cap \{x\colon 1 < |h_j(x)| < 2\}),$$

and deduce that

$$R_{i,j} = [h_{j\#}(T_i|U_j)] \llcorner \mathbf{U}(0, r_{i,j}) \in \mathbf{I}_t[\mathbf{U}(0, 2)],$$
$$\mathbf{M}(R_{i,j}) \leqslant [\mathrm{Lip}\,(h_j)]^t \|T_i\| (U_j),$$
$$\mathbf{M}(\partial R_{i,j}) \leqslant [\mathrm{Lip}\,(h_j)]^{t-1} [\mathrm{Lip}\,(h_j)\, \|T_i\| + \|\partial T_i\|]\, (U_j).$$

We may now inductively select currents $R_1, R_2, \ldots \in \mathbf{I}_t[\mathbf{U}(0, 2)]$ and strictly increasing maps $\alpha_1, \alpha_2, \ldots$ of $\{1, 2, \ldots\}$ into $\{1, 2, \ldots\}$ so that for every $j \in \{1, 2, \ldots\}$ im $\alpha_{j+1} \subset$ im α_j and

$$R_{\alpha_j(i),j} \to R_j \quad \text{in } \mathcal{F}_t^{\mathrm{loc}}[\mathbf{U}(0, 2)] \text{ as } i \to \infty$$

by repeatedly applying the Compactness theorem of [F, 4.2.17(2)] to extract convergent subsequences first from the sequence

$$R_{1,1}, R_{2,1}, R_{3,1}, \ldots,$$

then consecutively from the sequences

$$R_{\alpha_j(1),j+1}, R_{\alpha_j(2),j+1}, R_{\alpha_j(3),j+1}, \ldots \quad \text{for } j = 1, 2, 3, \ldots.$$

Letting, for each $i \in \{1, 2, \ldots\}$ and $j \in \{1, 2, \ldots\}$, $T_{i,j}$ and S_j denote the extensions to $\mathcal{D}^t(M)$ of $(h_j^{-1})_\# R_{i,j}$ and $(h_j^{-1})_\# R_j$ respectively, we infer that

$$T_{\alpha_j(i),j} \to S_j \quad \text{in } \mathcal{F}_t^{\text{loc}}(M) \text{ as } i \to \infty.$$

It also follows that whenever $j<k$ are positive integers

$$\operatorname{spt}(S_j - S_k) \subset M \sim (V_j \cap V_k)$$

because $\operatorname{im} \alpha_k \subset \operatorname{im} \alpha_j$: thus there exists $S \in \mathbf{I}_t^{\text{loc}}(M)$ characterized by the condition

$$\operatorname{spt}(S - S_j) \subset M \sim V_j \quad \text{whenever } j \in \{1, 2, \ldots\}.$$

Moreover the subsequence

$$T_{\alpha_i(i)} \to S \quad \text{in } \mathcal{F}_t^{\text{loc}}(M) \text{ as } i \to \infty$$

because for each $j \in \{1, 2, \ldots\}$

$$T_{\alpha_i(i)} - S = (T_{\alpha_i(i)} - T_{\alpha_j(i)}) + (T_{\alpha_j(i)} - T_{\alpha_j(i),j}) + (T_{\alpha_j(i),j} - S_j) + (S_j - S),$$

$$\operatorname{spt}(T_{\alpha_j(i)} - T_{\alpha_j(i),j}) \cup \operatorname{spt}(S_j - S) \subset M \sim V_j,$$

$$T_{\alpha_i(i)} - T_{\alpha_j(i)} \to 0 \text{ and } T_{\alpha_j(i),j} - S_j \to 0 \text{ as } i \to \infty,$$

and 3.2(1) is applicable with the cover $\mathcal{U} = \{V_1, V_2, \ldots\}$.

3.5. *Slicing.* Letting $Y_1, \ldots, Y_n$ be the standard coordinate functions on $\mathbf{R}^n$, we abbreviate the *standard n form* $\Omega = DY_1 \wedge \ldots \wedge DY_n$ *on* $\mathbf{R}^n$, and we recall from [F, 4.3.1], [F2, 3.5] that

if $T \in \mathcal{F}_t^{\text{loc}}(M)$, *then for* $\mathcal{L}^n$ *almost all* y *in* $\mathbf{R}^n$ *there exists a current* $\langle T, f, y\rangle \in \mathcal{D}_{t-n}(M)$, *called the slice of* T *in* $f^{-1}\{y\}$ *and defined by the formula*

$$\langle T, f, y\rangle(\psi) = \lim_{\varrho \to 0+} (T \llcorner f^\#[\mathbf{B}(y, \varrho) \wedge \Omega]/[\boldsymbol{\alpha}(n)\varrho^n])(\psi)$$

whenever $\psi \in \mathcal{D}^{t-n}(M)$.

For such y we readily verify the following four statements:

(1) $\operatorname{spt}\langle T, f, y\rangle \subset f^{-1}\{y\} \cap \operatorname{spt} T$.

(2) $\partial\langle T, f, y\rangle = (-1)^n \langle \partial T, f, y\rangle$ in case $t > n$.

(3) $\langle T, f, y\rangle | U = \langle T|U, f|U, y\rangle$ whenever U is an open subset of M.

(4) $h_\#\langle T, f, y\rangle = \langle h_\# T, f \circ h^{-1}, y\rangle \in \mathcal{D}_{t-n}(N)$ whenever h is a diffeomorphism of class ∞ from M onto a manifold N of class ∞.

Section 4.3 of [F], to which we shall often refer, contains a comprehensive discussion of slicing including several applications. In addition, Section 3.17 of [F2] provides a basic theorem concerning the existence and continuity of zero dimensional slices of integral currents. We shall present here a complete restatement of this theorem with some slight improvement to include those cases where the domain of the slicing function $\langle T, f, \cdot\rangle$ is not necessarily open.

3.6. THEOREM. *Let $f: M \to \mathbf{R}^n$ be a locally Lipschitzian map and $T \in \mathbf{I}_t^{\mathrm{loc}}(M)$ satisfy $f|\operatorname{spt} T$ is proper and $\operatorname{spt} T \sim \operatorname{spt} \partial T$ is locally-connected. Suppose μ and ν are positive integers with*

$$\Theta^t(\|T\|, x) \leq \mu \quad \textit{whenever } x \in M,$$

$$\mathcal{L}^n[\mathbf{R}^n \cap \{y: \operatorname{card}(f^{-1}\{y\} \cap \operatorname{spt} T) > \nu\}] = 0.$$

Let G be the class of all nonempty connected open subsets of $\mathbf{R}^n \sim f(\operatorname{spt} \partial T)$ and

$$Y = [\mathbf{R}^n \sim f(\operatorname{spt} \partial T)] \cap \{y: \operatorname{card}(f^{-1}\{y\} \cap \operatorname{spt} T) < \infty\};$$

for each $W \in G$ let $\Gamma(W)$ denote the set of all components of $f^{-1}(W) \cap \operatorname{spt} T$ and

$$\Gamma^*(W) = \{V \cap f^{-1}(Y): V \in \Gamma(W)\};$$

also let

$$H = \bigcup\{\Gamma(W): W \in G\}, \quad H^* = \bigcup\{\Gamma^*(W): W \in G\}.$$

Then the following nine conclusions hold:

(1) *For each $V \in \Gamma(W)$ there exists an integer $\Delta(V)$ such that*

$$f_{\#}(T \llcorner V) = \Delta(V)\mathbf{E}^n \llcorner W.$$

(2) *If $V \in \Gamma(W)$ and $\Delta(V) \neq 0$, then $f(V) = W$.*

(3) $\operatorname{card}[\Gamma(W) \cap \{V: \Delta(V) \neq 0\}] \leq \nu$.

(4) *If $W \subset W'$ belong to G, and $V' \in \Gamma(W')$, then*

$$\Delta(V') = \sum_{V' \supset V \in \Gamma(W)} \Delta(V).$$

(5) *H^* is a base for the relative topology of $f^{-1}(Y) \cap \operatorname{spt} T$.*

(6) *If $x \in f^{-1}(Y) \cap \operatorname{spt} T$, then $\Delta(V)$ has the same value, hereafter denoted $\Delta(x)$, for all sufficiently small neighborhoods V of x belonging to H.*

(7) *For every Borel set $E \subset \mathbf{R}^n \sim f(\operatorname{spt} \partial T)$*

$$\mathbf{M}[T \llcorner f^{\#}(\Omega \llcorner E)] \leq \mu\nu\mathcal{L}^n(E).$$

(8) *For every $y \in Y$*

$$\langle T, f, y\rangle = \sum_{x \in f^{-1}\{y\} \cap \operatorname{spt} T} \Delta(x)\,\boldsymbol{\delta}_x, \quad \mathbf{M}\langle T, f, y\rangle \leq \mu\nu.$$

(9) *The function mapping*

$$y \in Y \text{ onto } \langle T, f, y\rangle \in \mathcal{F}_0^{\mathrm{loc}}(M)$$

is continuous.

Proof. Only a few modifications in the proof of [F2, 3.17] are required. The proofs of (1), (2), (4), and (7) are essentially the same. For (3) we note that $\mathcal{L}^n$ almost all points in W have at most ν counterimages in spt T. To prove (5) let $x \in f^{-1}(Y) \cap \operatorname{spt} T$, choose V_1, V_2, ... as in the proof of [F2, 3.17(5)], and verify that

$$\bigcap_{j=1}^{\infty} \operatorname{Clos}[V_j \cap f^{-1}(Y)] = \{x\}.$$

(6) and (8) now follow as in the proof of [F2, 3.17(6)(8)].

For the proof of (9) we fix $y \in Y$ and $\varepsilon > 0$ and abbreviate $f^{-1}\{y\} \cap \operatorname{spt} T = F$. Corresponding to every $x \in F$ is a set $U_x \in H$ which satisfies the conditions: $x \in U_x$,

$$\operatorname{diam} U_x < \inf\{\varepsilon/\mu\nu, \tfrac{1}{2} \operatorname{distance}(\{x\}, F \sim \{x\})\}, \quad \Delta(U_x) = \Delta(x).$$

Since

$$\varrho = \operatorname{distance}[\{y\}, f(\operatorname{spt} T \sim \bigcup_{x \in F} U_x)] > 0,$$

we may choose $W \in G$ with $y \in W \subset \mathbf{U}(y, \varrho)$ and set

$$V_x = U_x \cap f^{-1}(W) \in \Gamma(W) \quad \text{for } x \in F$$

so that $f^{-1}(W) \cap \operatorname{spt} T = \bigcup_{x \in F} V_x$.

Let $w \in W$. For each $v \in f^{-1}\{w\} \cap \operatorname{spt} T$ we select that $x \in F$ for which $v \in V_x$, choose a Lipschitzian curve β_v: $[0, 1] \to M$ of length ([F, 3.2.46], [KN, p. 157]) less than $\varepsilon/\mu\nu$ so that $\beta_v(0) = x$ and $\beta_v(1) = v$, define the current

$$S = \sum_{v \in f^{-1}\{W\} \cap \operatorname{spt} T} \Delta(v)\ \beta_{v\#}[0, 1] \in \mathbf{I}_1(M),$$

and verify by (4), (6), and (8) that

$$\begin{aligned} \partial S &= \sum_{x \in F} \sum_{v \in f^{-1}\{w\} \cap V_x} \Delta(v)(\boldsymbol{\delta}_v - \boldsymbol{\delta}_x) \\ &= \sum_{v \in f^{-1}\{w\} \cap \operatorname{spt} T} \Delta(v)\, \boldsymbol{\delta}_v - \sum_{x \in F} \Delta(x)\, \boldsymbol{\delta}_x = \langle T, f, w\rangle - \langle T, f, y\rangle \end{aligned}$$

and by (6), (1), and (7) that

$$\mathbf{M}(S) \leqslant \sum_{v \in f^{-1}\{w\} \cap \operatorname{spt} T} |\Delta(v)| \operatorname{length} \beta_v \leqslant \mu\nu(\varepsilon/\mu\nu) = \varepsilon.$$

4. Slicing analytic chains

Suppose M is a separable m dimensional analytic Riemannian manifold. We call T a *t dimensional analytic chain in M* if and only if

$$T \in \mathcal{F}_t^{\mathrm{loc}}(M), \dim(\operatorname{spt} T) \leq t, \dim(\operatorname{spt} \partial T) \leq t-1.$$

In case $t>0$, ∂T is consequently a $t-1$ dimensional analytic chain in M. It follows from [F, 4.2.28] that every analytic chain T is representable as a locally finite sum of chains which correspond to integration over t dimensional oriented analytic blocks in M. This decomposition plus [F, 3.4.8(11)] implies that T is an element of $\mathbf{I}_t^{\mathrm{loc}}(M)$, that $\dim(\operatorname{spt} T) = t$ whenever $T \neq 0$, and that $K \cap \operatorname{spt} T$ is t rectifiable for every compact set $K \subset M$.

4.1. Lemma. *If U is an open subset of $\mathbf{R}^m$, s is a positive integer, $S \in \mathcal{F}_s^{\mathrm{loc}}(U)$, and $\psi \in \mathcal{D}^s(U)$, then*

$$S(\psi) = \sum_{\lambda \in \Lambda(m,s)} \int \langle S, \mathbf{p}_\lambda | U, z\rangle \langle \mathbf{e}_\lambda, \psi\rangle \, d\mathcal{L}^s z.$$

Proof. Letting Ω denote the standard s form on $\mathbf{R}^s$, we infer from [F, 4.1.6, 4.3.2 (1)] that

$$S(\psi) = S[\sum_{\lambda \in \Lambda(m,s)} \langle \mathbf{e}_\lambda, \psi\rangle \wedge (\mathbf{p}_\lambda|U)^\# \Omega] = \sum_{\lambda \in \Lambda(m,s)} [S \llcorner (\mathbf{p}_\lambda|U)^\# \Omega] \langle \mathbf{e}_\lambda, \psi\rangle$$

$$= \sum_{\lambda \in \Lambda(m,s)} \int \langle S, \mathbf{p}_\lambda | U, z\rangle \langle \mathbf{e}_\lambda, \psi\rangle \, d\mathcal{L}^s z.$$

4.2. Lemma. *If T is a t dimensional analytic chain in M, f is an analytic map from M into $\mathbf{R}^n$, and $t \geq n$, then for every compact set $K \subset M$ there exists a positive integer I such that*

$$(\|\langle T, f, y\rangle\| + \|\partial \langle T, f, y\rangle\|)(K) \leq I$$

whenever $y \in \mathbf{R}^n$ and $\langle T, f, y\rangle \in \mathcal{D}_{t-n}(M)$.

Proof. Choosing by [F, 4.2.28, 3.4.8(13)] a positive integer μ so that

$$\Theta^t(\|T\|, x) \leq \mu \quad \text{whenever } x \in K$$

and choosing J^* as in 2.11(2) with $E = K \cap \operatorname{spt} T$, we infer from [F, 4.3.6, 4.3.8, 2.9.2, 2.9.7] that for $\mathcal{L}^n$ almost all $w \in \mathbf{R}^n$ the following statements hold true:

$$\mathcal{H}^{t-n}[f^{-1}\{w\} \cap K \cap \operatorname{spt} T] \leq J^*, \quad \langle T, f, w\rangle \in \mathbf{I}_{t-n}^{\mathrm{loc}}(M),$$

$$\Theta^{t-n}(\|\langle T, f, w\rangle\|, x) \leq \mu \Theta^{t-n}(\mathcal{H}^{t-n} \llcorner f^{-1}\{w\} \cap \operatorname{spt} T, x) \text{ for } \mathcal{H}^{t-n} \text{ almost all } x \in K,$$

hence

$$\|\langle T, f, w\rangle\|(K) \leq \int_K \Theta^{t-n}(\|\langle T, f, w\rangle\|, x)\, d\mathcal{H}^{t-n} \leq \mu J^*.$$

For an arbitrary point $y \in \mathbf{R}^n$ for which $\langle T, f, y\rangle \in \mathcal{D}_{t-n}(M)$ we refer to [F, 4.1.5, 4.3.1, 4.3.2(2)] to conclude that

$$\|\langle T, f, y\rangle\| (K) \leq \liminf_{\varrho\to 0+} \left(\int_{\mathbf{B}(y,\varrho)} \|\langle T, f, w\rangle\| (K)\, d\mathcal{L}^n w/[\boldsymbol{\alpha}(n)\,\varrho^n]\right) \leq \mu J^*.$$

A similar argument for $\partial\langle T, f, y\rangle = (-1)^n\langle \partial T, f, y\rangle$ finishes the proof.

4.3. Slicing Theorem. *If T, f, n are as in 4.2 and if*

$$Y = \mathbf{R}^n \cap \{y\colon \dim(f^{-1}\{y\} \cap \operatorname{spt} T) \leq t-n \text{ and } \dim(f^{-1}\{y\} \cap \operatorname{spt} \partial T) \leq t-n-1\},$$

then the function which associates $\langle T, f, y\rangle$ with y maps Y into the $t-n$ dimensional analytic chains in M and is continuous in the topology of $\mathcal{F}^{\mathrm{loc}}_{t-n}(M)$.

Proof. We will first prove 4.3 assuming that *M is an open subset of $\mathbf{R}^m$ and* spt *T is compact*, by considering two cases.

Case 1, $t=n$. Here we remark that $\operatorname{spt} T \sim \operatorname{spt} \partial T$ is locally connected by virtue of [F, 4.2.28, 3.4.8(11)], choose, according to [F, 4.2.28, 3.4.8(13)] and 2.10(1) positive integers μ and ν so that

$$\Theta^n(\|T\|, x) \leq \mu \quad \text{whenever } x \in M,$$
$$\mathcal{L}^n[\mathbf{R}^n \cap \{y\colon \operatorname{card}(f^{-1}\{y\} \cap \operatorname{spt} T) > \nu\}] = 0,$$

and then apply 3.6(9).

Case 2, $t>n$. From 2.2(7) and Case 1 we infer that for each $y \in Y$ the statements

$$\dim(f^{-1}\{y\} \cap \mathbf{p}_\lambda^{-1}\{z\} \cap \operatorname{spt} T) \leq 0,$$
$$f^{-1}\{y\} \cap \mathbf{p}_\lambda^{-1}\{z\} \cap \operatorname{spt} \partial T = \varnothing,$$
$$\langle T, f \boxdot (\mathbf{p}_\lambda | M), (y, z)\rangle \in \mathbf{I}_0(M)$$

hold true for all $\lambda \in \Lambda(m, t-n)$ and $\mathcal{L}^{t-n}$ almost all $z \in \mathbf{R}^{t-n}$, and we observe that if $\phi \in \mathcal{D}^0(M)$, then the function mapping z onto $\langle T, f \boxdot (\mathbf{p}_\lambda | M), (y, z)\rangle(\phi)$ is defined, continuous, and bounded except for an $\mathcal{L}^{t-n}$ null set, and is hence $\mathcal{L}^{t-n}$ summable. We deduce that for each $y \in Y$ the linear functional on $\mathcal{D}^{t-n}(M)$ defined by

$$L_y(\psi) = \sum_{\lambda \in \Lambda(m,t-n)} \int \langle T, f, \boxdot (\mathbf{p}_\lambda | M), (y, z)\rangle \langle \mathrm{e}_\lambda, \psi\rangle \, d\mathcal{L}^{t-n} z \quad \text{for } \psi \in \mathcal{D}^{t-n}(M)$$

is an element of $\mathcal{D}_{t-n}(M)$ because we may apply 4.2 with f and K replaced by $f \boxdot (\mathbf{p}_\lambda | M)$ and $\operatorname{spt} \psi$ to obtain the estimates

$$\|\langle T, f \boxdot (\mathbf{p}_\lambda | M), (y, z)\rangle\| (\operatorname{spt} \psi) \leq I \quad \text{whenever } \langle T, f \boxdot (\mathbf{p}_\lambda | M), (y, z)\rangle \in \mathcal{D}_0(M),$$

$$L_y(\psi) \leqslant I\mathbf{M}(\psi) \sum_{\lambda\in\Lambda(m,t-n)} \mathcal{L}^{t-n}[\mathbf{p}_\lambda(\operatorname{spt}\psi)].$$

Moreover for each $\psi \in \mathcal{D}^{t-n}(M)$ the function mapping $y \in Y$ onto $L_y(\psi) \in \mathbf{R}$ is continuous. In fact let $e \in Y$ and E be a countable subset of Y containing e. Recalling 2.2(7), we note that for $\mathcal{L}^{t-n}$ almost all $z \in \mathbf{R}^{t-n}$ the two conditions

$$\dim(f^{-1}\{y\} \cap \mathbf{p}_{\lambda|\mathbf{E}}^{-1}\{z\} \cap \operatorname{spt} T) \leqslant 0, \quad f^{-1}\{y\} \cap \mathbf{p}_{\lambda|\mathbf{E}}^{-1}\{z\} \cap \operatorname{spt} \partial T = \varnothing$$

hold whenever $y \in E$ and $\lambda \in \Lambda(m, t-n)$, and we may apply the above estimate, Lebesgue's bounded convergence theorem ([F, 2.4.9]), and Case 1 to conclude that

$$\begin{aligned}
\lim_{E\ni y\to e} L_y(\psi) &= \lim_{E\ni y\to e} \sum_{\lambda\in\Lambda(m,t-n)} \int \langle T, f \boxdot (\mathbf{p}_\lambda | M), (y,z)\rangle \langle \mathbf{e}_\lambda, \psi\rangle \, d\mathcal{L}^{t-n} z \\
&= \sum_{\lambda\in\Lambda(m,t-n)} \int \lim_{E\ni y\to e} \langle T, f \boxdot (\mathbf{p}_\lambda | M), (y,z)\rangle \langle \mathbf{e}_\lambda, \psi\rangle \, d\mathcal{L}^{t-n} z \\
&= \sum_{\lambda\in\Lambda(m,t-n)} \int \langle T, f \boxdot (\mathbf{p}_\lambda | M), (e,z)\rangle \langle \mathbf{e}_\lambda, \psi\rangle \, d\mathcal{L}^{t-n} z = L_e(\psi).
\end{aligned}$$

We next observe by 2.2(7), [F, 4.3.6, 4.3.5] and Fubini's theorem that for $\mathcal{L}^n$ almost all $a \in \mathbf{R}^n$

$$a \in Y, \quad \langle T, f, a\rangle \in \mathcal{J}_{t-n}(M),$$

$$\langle\langle T, f, a\rangle, \mathbf{p}_\lambda | M, z\rangle = \langle T, f \boxdot (\mathbf{p}_\lambda | M), (a,z)\rangle \quad \text{for } \mathcal{L}^{t-n} \text{ almost all } z \in \mathbf{R}^{t-n}$$

hence we deduce from 4.1, for all $\psi \in \mathcal{D}^{t-n}(M)$, the equation

$$\begin{aligned}
\langle T, f, a\rangle(\psi) &= \sum_{\lambda\in\Lambda(m,t-n)} \int \langle\langle T, f, a\rangle, (\mathbf{p}_\lambda | M), z\rangle \langle \mathbf{e}_\lambda, \psi\rangle \, d\mathcal{L}^{t-n} z \\
&= \sum_{\lambda\in\Lambda(m,t-n)} \int \langle T, f \wedge (\mathbf{p}_\lambda | M), (a,z)\rangle \langle \mathbf{e}_\lambda, \psi\rangle \, d\mathcal{L}^{t-n} z = L_a(\psi).
\end{aligned}$$

For an arbitrary point $y \in Y$ and all $\psi \in \mathcal{D}^{t-n}(M)$, recall [$F$, 4.3.2(1)] to compute

$$\begin{aligned}
\lim_{\varrho\to 0+} (T \llcorner f^{\#}[\mathbf{B}(y,\varrho) \wedge \Omega] / [\boldsymbol{\alpha}(n)\varrho^n])(\psi) &= \lim_{\varrho\to 0+} \left[\int_{\mathbf{B}(y,\varrho)} \langle T, f, a\rangle(\psi) \, d\mathcal{L}^n a\right] \Big/ [\boldsymbol{\alpha}(n)\varrho^n] \\
&= \lim_{\varrho\to 0+} \left[\int_{\mathbf{B}(y,\varrho)} L_a(\psi) \, d\mathcal{L}^n a\right] \Big/ [\boldsymbol{\alpha}(n)\varrho^n] = L_y(\psi),
\end{aligned}$$

and conclude that $$\langle T, f, y\rangle = L_y \in \mathcal{D}_{t-n}(M).$$

To infer that $\langle T, f, y\rangle \in \mathbf{I}_{t-n}^{\text{loc}}(M)$ we let

$$W = Y \cap \{w: \langle T, f, w\rangle \in \mathbf{I}_{t-n}(M)\}$$

and note that $y \in \operatorname{Clos} W$ because by 2.2(7), [F, 4.3.6] $\mathcal{L}^n(\mathbf{R}^n \sim W) = 0$; then, observing that the set of currents $\{\langle T, f, w\rangle: w \in W\}$ in $\mathcal{F}^{\mathrm{loc}}_{t-n}(M)$ relatively compact in $\mathbf{I}^{\mathrm{loc}}_{t-n}(M)$ by reason of 4.2 and 3.4, we see that the convergence

$$\langle T, f, w\rangle \to \langle T, f, y\rangle \quad \text{as } w \to y \quad \text{in } W,$$

which occurs in the weak topology of $\mathcal{D}_{t-n}(M)$, occurs also in the topology of $\mathcal{F}^{\mathrm{loc}}_{t-n}(M)$ and that the limiting current $\langle T, f, y\rangle$ is therefore a locally integral current.

As a consequence $W = Y$, and we also conclude from this compactness argument that on Y the function $\langle T, f, \cdot\rangle$ is continuous in the topology of $\mathcal{F}^{\mathrm{loc}}_{t-n}(M)$. Thus the proof of Case 2 is complete.

The transition to the general case of an *arbitrary separable analytic Riemannian manifold M and analytic chain T in M* is only technical. Let E be a countable subset of Y. Choosing $u, U_1, V_1, h_1, U_2, V_2, h_2, \ldots$ as in 3.4, we recall 2.2(7) to select for each $j \in \{1, 2, \ldots\}$ a number r_j so that $1 < r_j < 2$,

$$\dim [U_j \cap (u \circ h_j)^{-1}\{r_j\} \cap \operatorname{spt} T] \leq t - 1,$$

$$\dim [U_j \cap (u \circ h_j)^{-1}\{r_j\} \cap f^{-1}\{e\} \cap \operatorname{spt} T] \leq t - n - 1 \quad \text{for all } e \in E$$

and we infer that the current

$$R_j = [h_{j\#}(T \mid U_j)] \llcorner \mathbf{U}(0, r_j)$$

is a t dimensional analytic chain in $\mathbf{U}(0, 2)$, that $\operatorname{spt} R_j$ is compact, and that the inequalities

$$\dim [(f \circ h_j^{-1})^{-1}\{e\} \cap \operatorname{spt} R_j] \leq t - n,$$

$$\dim [f \circ h_j^{-1})^{-1}\{e\} \cap \operatorname{spt} \partial R_j] \leq t - n - 1$$

hold whenever $e \in E$; applying the previous discussion with M, T, and f replaced by $\mathbf{U}(0, 2)$, R_j, and $f \circ h_j^{-1}$ and recalling 3.5(4), 3.(2), we conclude that

$$\langle (h_j^{-1})_\# R_j, f, e\rangle = (h_j^{-1})_\# \langle R_j, f \circ h_j^{-1}, e\rangle \in \mathbf{I}_{t-n}(U_j)$$

whenever $e \in E$ and that the function $\langle (h_j^{-1})_\# R_j, f, \cdot\rangle$ is $\mathcal{F}^{\mathrm{loc}}_{t-n}(U_j)$ continuous on E. Consequently if S_j denotes the extension of $(h_j^{-1})_\# R_j$ to $\mathcal{D}^t(M)$, then

$$\operatorname{spt}(T - S_j) \subset M \sim V_j, \quad \langle S_j, f, e\rangle \in \mathbf{I}_{t-n}(M) \quad \text{whenever } e \in E,$$

and the function $\langle S_j, f, \cdot\rangle$ is $\mathcal{F}^{\mathrm{loc}}_{t-n}(M)$ continuous on E.

To show that $\langle T, f, \cdot\rangle$ is a $\mathcal{F}^{\mathrm{loc}}_{t-n}(M)$ continuous function from E into $\mathbf{I}^{\mathrm{loc}}_{t-n}(M)$, it will be sufficient to prove that, for each $e \in E$, $\langle T, f, e\rangle \in \mathcal{D}_{t-n}(M)$, then observe that

$$\operatorname{spt}(\langle T, f, e\rangle - \langle S_j, f, e\rangle) \subset \operatorname{spt}(T - S_j) \subset M \sim V_j \quad \text{whenever } j \in \{1, 2, \ldots\},$$

and apply 3.2(1) with $t = t - n$ and $\mathcal{U} = \{V_1, V_2, \ldots\}$. For this purpose we use a partition of unity $\phi_1, \phi_2, \ldots$ so that

$$\phi_j \in \mathcal{D}^0(M) \text{ and } \operatorname{spt} \phi_j \subset V_j \text{ for } j \in \{1, 2, \ldots\},$$

$$\{j : K \cap \operatorname{spt} \phi_j \neq \emptyset\} < \infty \text{ for every compact } K \subset M, \text{ and } \sum_{j=1}^{\infty} \phi_j = 1,$$

compute for each $e \in E$ and $\psi \in \mathcal{D}^{t-n}(M)$, the limit

$$\begin{aligned}
&\lim_{\varrho \to 0+} (T \llcorner f^{\#}[\mathbf{B}(e, \varrho) \wedge \Omega] / [\boldsymbol{\alpha}(n)\, \varrho^n])(\psi) \\
&\quad = \lim_{\varrho \to 0+} (T \llcorner f^{\#}[\mathbf{B}(e, \varrho) \wedge \Omega] / [\boldsymbol{\alpha}(n)\, \varrho^n]) \left(\sum_{j=1}^{\infty} \phi_j \psi\right) \\
&\quad = \lim_{\varrho \to 0+} \sum_{j=1}^{\infty} (S_j \llcorner f^{\#}[\mathbf{B}(e, \varrho) \wedge \Omega] / [\boldsymbol{\alpha}(n)\, \varrho^n])(\phi_j \psi) = \sum_{j=1}^{\infty} \langle S_j, f, e\rangle (\phi_j \psi),
\end{aligned}$$

and apply the characterization of $\mathcal{D}_{t-n}(M)$ given on p. 345 of [F].

The proof of 4.3 is completed by noting the arbitrariness of E and making the observation that if $y \in Y$, then

$$\langle T, f, y\rangle \in \mathbf{I}^{\mathrm{loc}}_{t-n}(M) \subset \mathcal{F}^{\mathrm{loc}}_{t-n}(M),$$

$$\operatorname{spt}\langle T, f, y\rangle \subset f^{-1}\{y\} \cap \operatorname{spt} T, \quad \operatorname{spt} \partial\langle T, f, y\rangle \subset f^{-1}\{y\} \cap \operatorname{spt} \partial T,$$

and thus $\langle T, f, y\rangle$ is a $t-n$ dimensional analytic chain in M.

4.4. Corollary. *If $t \geq n \geq l \geq 0$ are integers, N is a separable n dimensional analytic Riemannian manifold, T is a t dimensional analytic chain in M,*

$$M \xrightarrow{\ f\ } N \xrightarrow{\ g\ } \mathbf{R}^l$$

are analytic maps, and $f \mid \operatorname{spt} T$ is proper, then

$$\langle f_{\#} T, g, z\rangle = f_{\#}\langle T, g \circ f, z\rangle$$

whenever $z \in \mathbf{R}^l$ satisfies the two conditions

$$\dim [(g \circ f)^{-1}\{z\} \cap \operatorname{spt} T] \leq t - l, \quad \dim [(g \circ f)^{-1}\{z\} \cap \operatorname{spt} \partial T] \leq t - l - 1.$$

Proof. In case $\operatorname{spt} T$ is *compact*, the corollary follows from 4.3 and [F, 4.3.1, 4.3.2(7) (1)].

To prove the *general case* we choose a cover $\mathcal{V}$ of N consisting of sets V for which there exist an open set U containing V along with an analytic isomorphism h of U onto $\mathbf{U}(0, 2) \subset \mathbf{R}^n$ such that $h(V) = \mathbf{U}(0, 1)$, and we define the function

$$v\colon \mathbf{U}(0,2) \to \mathbf{R},\ v(a) = |a| \quad \text{for } a \in \mathbf{U}(0,2).$$

For each $V \in \mathfrak{V}$ we use 2.2(7) to choose r so that $1 < r < 2$ and

$$\dim [f^{-1}(U) \cap (v \circ h \circ f)^{-1}\{r\} \cap \operatorname{spt} T) \leq -1,$$

$$\dim [f^{-1}(U) \cap (v \circ h \circ f)^{-1}\{r\} \cap (g \circ f)^{-1}\{z\} \cap \operatorname{spt} T] \leq t - l - 1,$$

infer that $S = T \llcorner (h \circ f)^{-1}[\mathbf{U}(0,r)]$ is a t dimensional analytic chain in M with compact support, that

$$\dim [(g \circ f)^{-1}\{z\} \cap \operatorname{spt} S] \leq t - l, \quad \dim [(g \circ f)^{-1}\{z\} \cap \operatorname{spt} \partial S] \leq t - l - 1,$$

and hence that

$$(f_\# T) | V = (f_\# S) | V,\ \langle f_\# S, g, z\rangle = f_\# \langle S, g \circ f, z\rangle,\ f_\# \langle S, g \circ f, z\rangle | V = f_\# \langle T, g \circ f, z\rangle | V.$$

We conclude first, by use of a partition of unity, that $\langle f_\# T, g, z\rangle \in \mathbf{I}_{t-l}^{\text{loc}}(N)$, and second that $\langle f_\# T, g, z\rangle | V = f_\# \langle T, g \circ f, z\rangle | V$ for every $V \in \mathfrak{V}$, and 4.4 follows.

4.5. COROLLARY. *If s, n, and l are nonnegative integers with $s \geq n + l \geq 0$, S is an s dimensional analytic chain in M,*

$$f\colon M \to \mathbf{R}^n, \quad g\colon M \to \mathbf{R}^l$$

are analytic maps, and

$$A = \mathbf{R}^n \cap \{y\colon \dim (f^{-1}\{y\} \cap \operatorname{spt} S) \leq s - n \text{ and } \dim (f^{-1}\{y\} \cap \operatorname{spt} \partial S) \leq s - n - 1\},$$

$$B = \mathbf{R}^l \cap \{z\colon \dim (g^{-1}\{z\} \cap \operatorname{spt} S) \leq s - l \text{ and } \dim (g^{-1}\{z\} \cap \operatorname{spt} \partial S) \leq s - l - 1\},$$

$$C = (\mathbf{R}^n \times \mathbf{R}^l) \cap \{(y,z)\colon \dim (f^{-1}\{y\} \cap g^{-1}\{z\} \cap \operatorname{spt} S) \leq s - n - l \text{ and}$$
$$\dim (f^{-1}\{y\} \cap g^{-1}\{z\} \cap \operatorname{spt} \partial S) \leq s - n - l - 1\},$$

then

$$\langle\langle S, f, a\rangle, g, b\rangle = \langle S, f \boxdot g, (a,b)\rangle \quad \text{whenever } (a,b) \in (A \times \mathbf{R}^l) \cap C,$$

$$\langle\langle S, f, a\rangle, g, b\rangle = (-1)^{nl} \langle\langle S, g, b\rangle, f, a\rangle \quad \text{whenever } (a,b) \in (A \times B) \cap C.$$

Proof. To prove the *first conclusion* we consider the set

$$D = C \cap \{(y,z)\colon \langle\langle S, f, y\rangle, g, z\rangle = \langle S, f \boxdot g, (y,z)\rangle\},$$

and make the observation:

If $a \in A$, then $(a,z) \in D$ for $\mathcal{L}^l$ almost all $z \in \mathbf{R}^l$. In fact, we note by 2.2(7), [F, 4.3.5] and Fubini's theorem that the set

$$E = \mathbf{R}^n \cap \{y\colon (y,z) \in D \text{ for } \mathcal{L}^l \text{ almost all } z \in \mathbf{R}^l\}$$

satisfies $\mathcal{L}^n(\mathbf{R}^n \sim E) = 0$, hence $a \in \operatorname{Clos} E$. Letting Ω denote the standard l form on $\mathbf{R}^l$, we use [F, 4.3.2(1)], 4.3, 4.2, and Lebesgue's bounded convergence theorem to see that for each $\psi \in \mathbf{D}^{s-n-l}(M)$

$$\int \langle\langle S, f, a\rangle, g, z\rangle (\psi)\, d\mathfrak{L}^l z = \langle S, f, a\rangle [(g^\# \Omega) \wedge \psi] = \lim_{E \ni y \to a} \langle S, f, y\rangle [(g^\# \Omega) \wedge \psi]$$
$$= \lim_{E\ni y \to a} \int \langle\langle S, f, y\rangle, g, z\rangle (\psi)\, d\mathfrak{L}^l z = \lim_{E\ni y\to a} \int \langle S, f \boxdot g\,(y, z)\rangle (\psi)\, d\mathfrak{L}^l z$$
$$= \int \lim_{E\ni y\to a} \langle S, f\boxdot g\,(y,z)\rangle (\psi)\, d\mathfrak{L}^l z = \int \langle S, f\boxdot g\,(a, z)\rangle (\psi)\, d\mathfrak{L}^l z.$$

The observation now follows, thanks to the arbitrary nature of ψ and 2.2(7).

Consequently if $(a, b) \in (A \times \mathbf{R}^l) \cap C$, then

$$b \in \mathrm{Clos}[\mathbf{R}^l \cap \{z\colon (a, z) \in D\}],$$

and we again use 4.3 to conclude that

$$\langle\langle S, f, a\rangle, g, b\rangle = \lim_{(a,z)\in D,\, z\to b} \langle\langle S, f, a\rangle, g, z\rangle$$
$$= \lim_{(a,z)\in D,\, z\to b} \langle S, f\boxdot g\,(a,z)\rangle = \langle S, f\boxdot g\,(a, b)\rangle.$$

To prove the *second conclusion* we assume that $(a, b) \in (A \times B) \cap C$, note that the map

$$h\colon \mathbf{R}^n \times \mathbf{R}^l \to \mathbf{R}^l \times \mathbf{R}^n,\ h(y, z) = (z, y) \quad \text{for } (y, z) \in \mathbf{R}^n \times \mathbf{R}^l$$

has determinant $(-1)^{nl}$, recall [F, 4.3.2(6)], and make two applications of the first conclusion to deduce that

$$\langle\langle S, f, a\rangle, g, b\rangle = \langle S, f\boxdot' g, (a, b)\rangle = (-1)^{nl} \langle S, g\boxdot f, (b, a)\rangle = (-1)^{nl} \langle\langle S, g, b\rangle, f, a\rangle.$$

4.6. *Example.* Consider the real-valued analytic function h on $\mathbf{R}^3$ given by

$$h(x, y, z) = x^2 + y^2 z^2 - y^2 \quad \text{for } (x, y, z) \in \mathbf{R}^3$$

and the two-dimensional analytic submanifold of $\mathbf{R}^3$

$$H = \mathbf{R}^3 \cap \{(x, y, z)\colon h(x, y, z) = 0,\ y \neq 0\}.$$

The current $S = \partial(\mathbf{E}^3 \llcorner \{(x, y, z)\colon h(x, y, z) < 0\}$ is a two-dimensional analytic chain in $\mathbf{R}^3$ with $\partial S = 0$ and

$$\mathrm{spt}\, S = \mathrm{Clos}\, H = H \cup \{(0, 0, z)\colon -1 \leqslant z \leqslant 1\}.$$

Defining the two maps

$$f\colon \mathbf{R}^3 \to \mathbf{R},\ g\colon \mathbf{R}^3 \to \mathbf{R},\quad f(x, y, z) = x,\ g(x, y, z) = y$$

for $(x, y, z) \in \mathbf{R}^3$, we recall 3.5(2) and compute

$$\langle S, f, 0\rangle = -\partial\langle \mathbf{E}^3 \llcorner \{(x, y, z:)\ h(x, y, z) < 0\}, f, 0\rangle = -\partial[\langle \mathbf{E}^3, f, 0\rangle \llcorner \{(x, y, z): h(x, y, z) < 0\}]$$
$$= -\partial[(\boldsymbol{\delta}_0 \times \mathbf{E}^2) \llcorner \{(0, y, z):\ -1 \leqslant z \leqslant 1\}]$$
$$= \boldsymbol{\delta}_0 \times \mathbf{E}^1 \times \boldsymbol{\delta}_1 - \boldsymbol{\delta}_0 \times \mathbf{E}^1 \times \boldsymbol{\delta}_{-1},$$

$$\langle S, g, 0\rangle = -\partial\langle \mathbf{E}^3 \llcorner \{(x, y, z): h(x, y, z) < 0\}, g, 0\rangle = -\partial[\langle \mathbf{E}^3, g, 0\rangle \llcorner \{(x, y, z): h(x, y, z) < 0\}]$$
$$= -\partial[(\mathbf{E}^1 \times \boldsymbol{\delta}_0 \times \mathbf{E}^1) \llcorner \{(0, 0, z):\ -1 \leqslant z \leqslant 1\}] = -\partial[0] = 0.$$

Hence

$$\langle\langle S, f, 0\rangle, g, 0\rangle = \boldsymbol{\delta}_{(0,0,1)} - \boldsymbol{\delta}_{(0,0,-1)} \neq 0 = \langle\langle S, g, 0\rangle, f, 0\rangle$$

even though

$$\dim (f^{-1}\{0\} \cap \operatorname{spt} S) = \dim (g^{-1}\{0\} \cap \operatorname{spt} S) = 1,$$
$$\dim (g^{-1}\{0\} \cap \operatorname{spt} \langle S, f, 0\rangle) = 0, \quad \dim (f^{-1}\{0\} \cap \operatorname{spt} \langle S, g, 0\rangle) = -1.$$

4.7. THEOREM. *If L and M are l and m dimensional separable analytic manifolds, $h: L \times M \to \mathbf{R}^n$ is an analytic map,*

$$h_w: M \to \mathbf{R}^n,\ h_w(x) = h(w, x) \ \textit{for}\ w \in L \text{ and } x \in M,$$

T is a t dimensional analytic chain in M with $t \geqslant n$,

$$\dim [(L \times \operatorname{spt} T) \cap h^{-1}\{0\}] \leqslant l + t - n,$$
$$\dim [(L \times \operatorname{spt} \partial T) \cap h^{-1}\{0\}] \leqslant l + t - n - 1,$$
$$W = L \cap \{w: \dim (h_w^{-1}\{0\} \cap \operatorname{spt} T) \leqslant t - n \text{ and } \dim (h_w^{-1}\{0\} \cap \operatorname{spt} \partial T) \leqslant t - n - 1\},$$

then the function mapping

$$w \in W \ \textit{onto}\ \langle T, h_w, 0\rangle \in \mathcal{F}_{t-n}^{\text{loc}}(M)$$

is continuous.

Proof. We assume L is an open subset of $\mathbf{R}^l$, let $S = \mathbf{E}^l \lfloor L$, and let

$$\lambda: L \times M \to L, \quad \mu: L \times M \to M$$

be the projections. Also let $\sigma_w: M \to L \times M$ be given by $\sigma_w(x) = (w, x)$ so that $h \circ \sigma_w = h_w$ and $\mu \circ \sigma_w = \mathbf{1}_M$. For each $w \in W$ we note that σ_w is proper and use 4.5 and 4.4 to compute

$$\langle S \times T, \lambda \boxdot h, (w, 0)\rangle = \langle\langle S \times T, \lambda, w\rangle, h, 0\rangle = \langle \boldsymbol{\delta}_w \times T, h, 0\rangle = \langle \sigma_{w\#} T, h, 0\rangle = \sigma_{w\#}\langle T, h_w, 0\rangle,$$

hence

$$\mu_\# \langle S \times T, \lambda \boxdot h, (w, 0)\rangle = \langle T, h_w, 0\rangle.$$

From 4.3 we see that the function mapping $w \in W$ to $\langle S \times T, \lambda \boxdot h, (w, 0)\rangle$ is continuous. For any open set V having compact closure in L we observe that the map $\mu \mid (\operatorname{Clos} V) \times M$ is proper; hence, by 3.(2), $\mu_\# \langle S \times T, \lambda \boxdot h, (\cdot, 0)\rangle$ is continuous on $V \cap W$.

4.8. Letting k be a nonnegative integer, we apply 4.7 to give precise form to the idea that *the variety of common zeros of a system of real-valued polynomials in several variables of degrees not exceeding k depends continuously on the coefficients of the polynomials* (Compare [F, 4.3.12]).

Let $m \geqslant n$ be positive integers and let L be the collection of all polynomial maps w from $\mathbf{R}^m$ to $\mathbf{R}^n$ for which degree $w \leqslant k$ ([F, 1.10.4]). L is a real vector space of dimension

$$l = n \sum_{i=0}^{k} \binom{i+m-1}{m-1}.$$

Also let
$$W = L \cap \{w: \dim w^{-1}\{0\} \leqslant m-n\}.$$

THEOREM. *The function mapping*

$$w \in W \quad \textit{onto} \quad \langle \mathbf{E}^m, w, 0\rangle \in \mathcal{F}^{\mathrm{loc}}_{m-n}(\mathbf{R}^m)$$

is continuous.

Proof. Defining the analytic map

$$h: L \times \mathbf{R}^m \to \mathbf{R}^n,\ h(w, x) = w(x) \quad \text{for } w \in L \text{ and } x \in \mathbf{R}^m$$

we observe that im $Dh(w, x) = \mathbf{R}^n$ for all $w \in L$ and $x \in \mathbf{R}^m$

because
$$h(w + c(y), x) = h(w, x) + y \quad \text{for } y \in \mathbf{R}^n$$

where $c(y)$ is the constant function mapping $\mathbf{R}^m$ onto $\{y\}$, hence

$$\langle (c(y), 0), Dh(w, x)\rangle = y \quad \text{for } y \in \mathbf{R}^n.$$

Thus by [F, 3.1.18] the set $h^{-1}\{0\}$ is a $l+m-n$ dimensional analytic submanifold of $\mathbf{R}^m$, and we may apply 4.7 with $M = \mathbf{R}^m$, $t = m$, and $T = \mathbf{E}^m$.

4.9. *Remark.* The notions of analytic block, $\mathcal{S}(M)$, real analytic dimension, slicing, and analytic chain do not depend on the Riemannian metric. Thus the statements of Propositions $(\mathbf{A}_t)$ $(\mathbf{B}_t)$, Corollary 2.9(1), the Slicing theorem with its corollaries, and Theorem 4.7 do not depend on the existence of a particular Riemannian metric. On the other hand, different Riemannian metrics are likely to give rise to different bounds, J^* in 2.9(2) and I in 4.2.

5. Intersections of analytic chains

In this section we assume that M and N are separable m and n dimensional orientable analytic Riemannian manifolds with orienting m and n vectorfields ξ_M and ξ_N and let

$$\mathcal{M} = \mathcal{H}^m \wedge \xi_M \quad \text{and} \quad \mathcal{N} = \mathcal{H}^n \wedge \xi_N$$

be the corresponding *orienting m and n cycles for M and N*. We shall repeatedly use the functions

$$f: \mathbf{R}^m \times \mathbf{R}^m \to \mathbf{R}^m, \quad \tilde{f}: \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}^n,$$

$$g: \mathbf{R}^m \to \mathbf{R}^m \times \mathbf{R}^m, \quad \gamma: M \to M \times M, \quad \tilde{\gamma}: N \to N \times N,$$

$$\mu: M \times N \to M, \quad \nu: M \times N \to N, \quad \mu_1: M \times M \to M, \quad \mu_2: M \times M \to M$$

given by $f(u_1, u_2) = u_1 - u_2$, $\tilde{f}(v_1, v_2) = v_1 - v_2$, $g(u_1) = (u_1, u_1)$, $\gamma(x) = (x, x)$, $\tilde{\gamma}(y) = (y, y)$, $\mu(x, y) = x$, $\nu(x, y) = y$, $\mu_1(w, x) = w$, $\mu_2(w, x) = x$ for $(u_1, u_2) \in \mathbf{R}^m \times \mathbf{R}^m$, $(v_1, v_2) \in \mathbf{R}^n \times \mathbf{R}^n$, $u_1 \in \mathbf{R}^m$, $x \in M$, $y \in N$, $(x, y) \in M \times N$, and $(w, x) \in M \times M$.

Whenever $Q \in \mathcal{F}_q^{\text{loc}}(M)$, $R \in \mathcal{F}_r^{\text{loc}}(M)$, $Q \times R \in \mathcal{F}_{q+r}^{\text{loc}}(M \times M)$, and $q + r \geqslant m$ we shall say that *the intersection of Q and R exists* provided there exists a current $Q \cap R \in \mathcal{D}_{q+r-m}(M)$ characterized by the condition:

(1) *If U is an open subset of M and h is an orientation-preserving analytic isomorphism from U onto some open subset of $\mathbf{R}^m$, then*

$$(\gamma | U)_\#[(Q \cap R) | U] = (-1)^{(m-q)r} \langle (Q \times R) | (U \times U), f \circ (h \times h), 0 \rangle.$$

(Compare [F, 4.3.20]). For an s dimensional analytic chain S in M and a t dimensional analytic chain T in M we shall say that

$$\{S, T\} \textit{ intersect suitably}$$

if and only if

$$s + t \geqslant m, \quad \dim(\operatorname{spt} S \cap \operatorname{spt} T) \leqslant s + t - m,$$

$$\dim [(\operatorname{spt} \partial S \cap \operatorname{spt} T) \cup (\operatorname{spt} S \cap \operatorname{spt} \partial T)] \leqslant s + t - m - 1.$$

In 5.1–5.4 we will prove that

(2) *if $\{S, T\}$ intersect suitably, then the intersection of S and T exists and $S \cap T$ is an $s + t - m$ dimensional analytic chain in M.*

Moreover in 5.8–5.11 we prove various intersection formulae and discuss how these properties characterize the resulting real analytic intersection theory.

5.1. Lemma. *If $b: M \to \mathbf{R}^k$ and $c: N \to \mathbf{R}^l$ are locally Lipschitzian maps, $Q \in \mathcal{F}_q^{\text{loc}}(M)$, $R \in \mathcal{F}_r^{\text{loc}}(N)$, $y \in \mathbf{R}^k$, $z \in \mathbf{R}^l$, $\langle Q, b, y \rangle \in \mathcal{D}_{q-k}(M)$, and $\langle R, c, z \rangle \in \mathcal{D}_{r-l}(N)$, then*

$$\langle Q, b, y \rangle \times R = \langle Q \times R, b \circ \mu, y \rangle,$$

$$Q \times \langle R, c, z \rangle = (-1)^{ql} \langle Q \times R, c \circ \nu, z \rangle.$$

Proof. Whenever i is an integer with $q - k + r \geqslant i \geqslant 0$, $\alpha \in \mathcal{D}^{q-k+r-i}(M)$, and $\beta \in \mathcal{D}^i(N)$ we deduce that

$$(b\circ\mu)_{\#}[(Q\times R)\llcorner(\mu^{\#}\alpha\wedge\nu^{\#}\beta)]=0 \text{ in case } i\neq r,$$
$$=(-1)^{rk}b_{\#}(Q\llcorner\alpha)\cdot R(\beta) \text{ in case } i=r.$$

Noting that these currents are representable by integration according to [F, 4.1.18] and recalling [F, 4.3.1], we see that if ω is a bounded Baire form of degree k on $\mathbf{R}^k$, then

$$[(Q\times R)\llcorner(b\circ\mu)^{\#}\omega](\mu^{\#}\alpha\wedge\nu^{\#}\beta)=0 \text{ in case } i\neq r,$$
$$=(Q\llcorner b^{\#}\omega)(\alpha)R(\beta) \text{ in case } i=r.$$

Therefore by [F, 4.1.8]

$$(Q\times R)\llcorner(b\circ\mu)^{\#}\omega=(Q\llcorner b^{\#}\omega)\times R,$$

and the first conclusion follows. The proof of the second is similar.

5.2. Lemma. *If b_1 and b_2 are analytic maps of M into $\mathbf{R}^n$ satisfying the conditions*

$$F=b_1^{-1}\{0\}=b_2^{-1}\{0\},\ \dim F\leqslant m-n,\ \langle\mathcal{M},b_1,0\rangle=\langle\mathcal{M},b_2,0\rangle$$

and if R is an r dimensional analytic chain in M with $r\geqslant n$,

$$\dim(F\cap\operatorname{spt}R)\leqslant r-n,\ \dim(F\cap\operatorname{spt}\partial R)\leqslant r-n-1,$$

then

$$\langle R,b_1,0\rangle=\langle R,b_2,0\rangle.$$

Proof. By 3.5(3) (4) it suffices to consider the special case when *M is an open subset of $\mathbf{R}^m$ and $\mathcal{M}=\mathbf{E}^m|M$.* In this case we infer from [F, 4.3.20] that

$$R=R\cap\mathcal{M}=(-1)^{(m-r)m}\mu_{2\#}\langle R\times\mathcal{M},f|(M\times M),0\rangle,$$

note that the restriction of μ_2 to the set

$$\operatorname{spt}\langle R\times\mathcal{M},f|(M\times M),0\rangle\subset\gamma(M)$$

is a proper map, and then refer to 4.4, 4.5, and 5.1 to see that for $i\in\{1,2\}$

$$\begin{aligned}(-1)^{(m-r)m}\langle R,b_i,0\rangle&=\langle\mu_{2\#}\langle R\times\mathcal{M},f|(M\times M),0\rangle,b_i,0\rangle\\&=\mu_{2\#}\langle\langle R\times\mathcal{M},f|(M\times M),0\rangle,b_i\circ\mu_2,0\rangle\\&=(-1)^{mn}\mu_{2\#}\langle\langle R\times\mathcal{M},b_i\circ\mu_2,0\rangle,f|(M\times M),0\rangle\\&=(-1)^{mn+rn}\mu_{2\#}\langle R\times\langle\mathcal{M},b_i,0\rangle,f|(M\times M),0\rangle.\end{aligned}$$

5.3. Lemma. *If $h\colon U\to\mathbf{R}^m$ is an analytic coordinate system (as in 5.(1)), then*

$$\langle(\mathcal{M}\times\mathcal{M})|(U\times U),f\circ(h\times h),0\rangle=(\gamma|U)_{\#}(\mathcal{M}|U).$$

Proof. We let $V = \operatorname{im} h$, $Q = \mathcal{M} \mid U$, $R = \mathbf{E}^m \mid V$, note that $h_\# Q = R$ because h preserves orientation, and consider the commutative diagram:

$$\begin{array}{ccccc} U \times U & \xrightarrow{\;h \times h\;} & V \times V & \xrightarrow{\;f \mid (V \times V)\;} & \mathbf{R}^m \\ \uparrow{\scriptstyle \gamma \mid U} & & \uparrow{\scriptstyle g \mid V} & & \\ U & \xrightarrow{\;\;h\;\;} & V & & \end{array}$$

Observing that $h \times h$ is a proper map, we use 4.4 and [F, 4.3.20] (which implies $\mathbf{E}^m \cap \mathbf{E}^m = \mathbf{E}^m$, $\langle \mathbf{E}^m \times \mathbf{E}^m, f, 0\rangle = g_\# \mathbf{E}^m$) to compute

$$\begin{aligned} (h \times h)_\# \langle Q \times Q, f \circ (h \times h), 0\rangle &= \langle (h \times h)_\# (Q \times Q), f \mid (V \times V), 0\rangle \\ &= \langle R \times R, f \mid (V \times V), 0\rangle = (g \mid V)_\# R = (g \mid V)_\# h_\# Q = (h \times h)_\# (\gamma \mid U)_\# Q. \end{aligned}$$

Since $(h \times h)_\#$ is univalent, $\langle Q \times Q, f \circ (h \times h), 0\rangle = (\gamma \mid U)_\# Q$.

5.4. Returning to the proof of 5.(2) we assume that for each $i \in \{1, 2\}$

U_i is an open subset of M and h_i is an orientation-preserving analytic isomorphism from U_i onto some open subset of $\mathbf{R}^m$,

and make the abbreviations

$$U^* = U_1 \cap U_2,\; h_i^* = h_i \mid U^* \text{ for } i \in \{1, 2\}.$$

We infer from 5.3 that

$$\begin{aligned} \langle (\mathcal{M} \times \mathcal{M}) \mid (U^* \times U^*), f \circ (h_1^* \times h_1^*), 0\rangle &= (\gamma \mid U^*)_\# (\mathcal{M} \mid U^*) \\ &= \langle (\mathcal{M} \times \mathcal{M}) \mid (U^* \times U^*), f \circ (h_2^* \times h_2^*), 0\rangle. \end{aligned}$$

Then observing that γ maps $\operatorname{spt} S \cap \operatorname{spt} T$ and $(\operatorname{spt} \partial S \cap \operatorname{spt} T) \cup (\operatorname{spt} S \cap \operatorname{spt} \partial T)$ isomorphically onto $\operatorname{spt}(S \times T)$ and $\operatorname{spt} \partial(S \times T)$ respectively and that

$$[f \circ (h_1^* \times h_1^*)]^{-1}\{0\} = \gamma(U^*) = [f \circ (h_2^* \times h_2^*)]^{-1}\{0\}$$

has real analytic dimension m, we apply 5.2 with M, $\mathcal{M}$, m, b_i, n, F, and R replaced by $U^* \times U^*$, $(\mathcal{M} \times \mathcal{M}) \mid (U^* \times U^*)$, $2m$, $f \circ (h_i^* \times h_i^*)$, m, $\gamma(U^*)$, and $S \times T$ to conclude that

$$\begin{aligned} \langle (S \times T) \mid (U_1 \times U_1), f \circ (h_1 \times h_1), 0\rangle \mid (U^* \times U^*) &= \langle (S \times T) \mid (U^* \times U^*), f \circ (h_1^* \times h_1^*), 0\rangle \\ &= \langle (S \times T) \mid (U^* \times U^*), f \circ (h_2^* \times h_2^*), 0\rangle \\ &= \langle (S \times T) \mid (U_2 \times U_2), f \circ (h_2 \times h_2), 0\rangle \mid (U^* \times U^*). \end{aligned}$$

Thus $S \cap T$ is, indeed, well-defined by conditions 5.(1). Moreover $S \cap T$ is an $s+t-m$ dimensional analytic chain in M because the real analytic dimensions of

$$\operatorname{spt}(S \cap T) \subset \operatorname{spt} S \cap \operatorname{spt} T \quad \text{and} \quad \operatorname{spt} \partial(S \cap T) \subset (\operatorname{spt} \partial S \cap T) \cup (\operatorname{spt} S \cap \operatorname{spt} \partial T)$$

do not exceed $s+t-m$ and $s+t-m-1$ respectively and because, by condition 5.(1), the current $(S \cap T) | U$ is the $[\mu_1 | (U \times U)]_\#$ image of a locally integral flat slice, hence an element of $\mathcal{J}^{\mathrm{loc}}_{s+t-m}(U)$.

5.5. Lemma. *If Q, R are q, r dimensional analytic chains in M, $\{Q, R\}$ intersect suitably, $b: M \to \mathbf{R}^n$ is an analytic map, $q+r \geqslant m+n$, $y \in \mathbf{R}^n$, and*

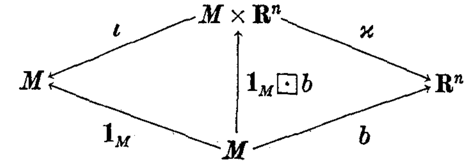
$$\dim(b^{-1}\{y\} \cap \operatorname{spt} R) \leqslant r-n, \quad \dim(b^{-1}\{y\} \cap \operatorname{spt} \partial R) \leqslant r-n-1,$$
$$\dim(b^{-1}\{y\} \cap \operatorname{spt} Q \cap \operatorname{spt} R) \leqslant q+r-m-n,$$
$$\dim(b^{-1}\{y\} \cap \operatorname{spt} \partial Q \cap \operatorname{spt} R) \leqslant q+r-m-n-1,$$
$$\dim(b^{-1}\{y\} \cap \operatorname{spt} Q \cap \operatorname{spt} \partial R) \leqslant q+r-m-n-1,$$

then
$$Q \cap \langle R, b, y \rangle = \langle Q \cap R, b, y \rangle.$$

Proof. We may assume M is an open subset of $\mathbf{R}^m$ and $\mathcal{M} = \mathbf{E}^m | M$. Then we use 5.(1), 5.1, 4.5, and 4.4 to compute

$$\begin{aligned} Q \cap \langle R, b, y \rangle &= (-1)^{(m-q)(r-n)} \mu_{2\#} \langle Q \times \langle R, b, y \rangle, f | (M \times M), 0 \rangle \\ &= (-1)^{(m-q)r-mn} \mu_{2\#} \langle \langle Q \times R, b \circ \mu_2, y \rangle, f | (M \times M), 0 \rangle \\ &= (-1)^{(m-q)r} \mu_{2\#} \langle \langle Q \times R, f | (M \times M), 0 \rangle, b \circ \mu_2, y \rangle \\ &= (-1)^{(m-q)r} \langle \mu_{2\#} \langle Q \times R, f | (M \times M), 0 \rangle, b, y \rangle = \langle Q \cap R, b, y \rangle. \end{aligned}$$

5.6. Noting that the definition of the intersection chain depends on slicing, we observe how, conversely, the slice is expressible in terms of intersection by using, for any analytic map b of M into $\mathbf{R}^n$, the commutative diagram



where ι and $\varkappa$ are the projections, and proving the lemma:

Lemma. *If R is an r dimensional analytic chain in M, $r \geqslant n$, $y \in \mathbf{R}^n$, and*

$$\dim(b^{-1}\{y\} \cap \operatorname{spt} R) \leqslant r-n, \quad \dim(b^{-1}\{y\} \cap \operatorname{spt} \partial R) \leqslant r-n-1,$$

then
$$\langle R, b, y \rangle = (-1)^{rn} \iota_\#([\mathbf{1}_M \boxdot b)_\# R] \cap [\mathcal{M} \times \boldsymbol{\delta}_y]).$$

Proof. Observing that $\boldsymbol{\delta}_y = \langle \mathbf{E}^n, \mathbf{1}_{\mathbf{R}^n}, y\rangle$, we deduce from 5.1, 5.5, [F, 4.3.20], and 4.4 that

$$\begin{aligned}(-1)^{mn}\iota_\#([(\mathbf{1}_M\square b)_\# R] \cap [\mathcal{M} \times \boldsymbol{\delta}_y]) &= \iota_\#([(\mathbf{1}_M\square b)_\# R \cap \langle \mathcal{M} \times \mathbf{E}^n, \varkappa, y\rangle) \\ &= \iota_\# \langle [(\mathbf{1}_M\square b)_\# R] \cap [\mathcal{M} \times \mathbf{E}^n], \varkappa, y\rangle = \iota\ \ \langle (\mathbf{1}_M\square b\ _\# R, \varkappa, y\rangle = \langle R, b, y\rangle.\end{aligned}$$

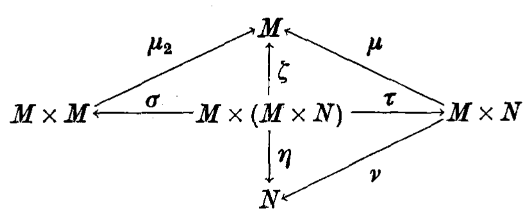
5.7. Lemma. *Suppose that M and N are open subsets of $\mathbf{R}^m$ and $\mathbf{R}^n$, $\mathcal{M} = \mathbf{E}^m | M$, $\mathcal{N} = \mathbf{E}^n | N$. If L is an analytic chain in $M \times N$ and*

$$\begin{aligned}&\sigma: M \times (M \times N) \to M \times M, \quad \tau: M \times (M \times N) \to M \times N,\\ &\tilde{\sigma}: (M \times N) \times N \to M \times N, \quad \tilde{\tau}: (M \times N) \times N \to N \times N\end{aligned}$$

are given by $\sigma(w, (x, y)) = (w, x)$, $\tau(w, (x, y)) = (x, y)$, $\tilde{\sigma}((x, y), z) = (x, y)$, $\tilde{\tau}((x, y), z) = (y, z)$ *whenever* $w \in M$, $(x, y) \in M \times N$, *and* $z \in N$, *then*

$$\tau_\# \langle \mathcal{M} \times L, f \circ \sigma, 0\rangle = L = (-1)^{(l+n)n}\tilde{\sigma}_\# \langle L \times \mathcal{N}, f \circ \tilde{\tau}, 0\rangle.$$

Proof. We consider the commutative diagram



let Ω denote the standard m form on $\mathbf{R}^m$, and define, for each $\varrho > 0$, the form

$$\Omega_\varrho = [\Omega \llcorner \mathbf{B}(0, \varrho)] / [\boldsymbol{\alpha}(m)\varrho^m].$$

If k is an integer with $0 \leqslant k \leqslant l$, $\alpha \in \mathcal{D}^{l-k}(M)$, and $\beta \in \mathcal{D}^k(N)$ we may use [F, 4.3.1, 4.3.2(7), 4.3.20] to compute

$$\begin{aligned}&\tau_\# \langle \mathcal{M} \times L, f \circ \sigma, 0\rangle\, (\mu^\# \alpha \wedge \nu^\# \beta)\\ &\qquad = \langle \mathcal{M} \times L, f \circ \sigma, 0\rangle\, (\zeta^\# \alpha \wedge \eta^\# \beta)\\ &\qquad = \lim_{\varrho \to 0} (\mathcal{M} \times L)\, [(f \circ \sigma)^\# \Omega_\varrho \wedge \zeta^\# \alpha \wedge \eta^\# \beta]\\ &\qquad = (-1)^{k(l-k+m)} \lim_{\varrho \to 0} [(\mathcal{M} \times L) \llcorner \eta^\# \beta]\, [(f \circ \sigma)^\# \Omega_\varrho \wedge \zeta^\# \alpha]\\ &\qquad = (-1)^{k(l-k)} \lim_{\varrho \to 0} ([\mathcal{M} \times (L \llcorner \nu^\# \beta)] \llcorner [f \circ \sigma]^\# \Omega_\varrho)\, (\zeta^\# \alpha)\\ &\qquad = (-1)^{k(l-k)} \lim_{\varrho \to 0} (\mu_2 \circ \sigma)_\# ([\mathcal{M} \times (L \llcorner \nu^\# \beta)] \llcorner [f \circ \sigma]^\# \Omega_\varrho)\, (\alpha)\\ &\qquad = (-1)^{k(l-k)} \lim_{\varrho \to 0} \mu_{2\#} ([\mathcal{M} \times \mu_\# (L \llcorner \nu^\# \beta)] \llcorner [f | (M \times M)]^\# \Omega_\varrho)\, (\alpha)\\ &\qquad = (-1)^{k(l-k)} \mu_{2\#} \langle \mathcal{M} \times \mu_\# (L \llcorner \nu^\# \beta), f | (M \times M), 0\rangle\, (\alpha)\\ &\qquad = (-1)^{k(l-k)} \mu_\# (L \llcorner \nu^\# \beta)\, (\alpha) = L(\mu^\# \alpha \wedge \nu^\# \beta).\end{aligned}$$

Recalling from [F, 4.1.3] that the differential forms $\mu^{\#}\alpha \wedge \nu^{\#}\beta$ corresponding to $k \in \{0, 1, ..., l\}$, $\alpha \in \mathcal{D}^{l-k}(M)$, and $\beta \in \mathcal{D}^{k}(N)$ generate a dense vectorsubspace of $\mathcal{D}^{l}(M \times N)$, we conclude that

$$\tau_{\#}\langle \mathfrak{M} \times L, f \circ \sigma, 0 \rangle = L.$$

The proof of the second equation is similar.

5.8. INTERSECTION FORMULAE. *If R, S, T are r, s, t dimensional analytic chains in M and if P, Q are p, q dimensional analytic chains in N, then the following twelve statements hold:*

If $\{S, T\}$ intersect suitably, then

(0) *$S \cap T$ is an $s+t-m$ dimensional analytic chain in M,*

(1) *$S \cap (jT) = j(S \cap T)$ for any integer j,*

(2) (*anticommutativity*) *$S \cap T = (-1)^{(m-s)(m-t)} T \cap S$,*

(3) (*restriction*) *$(S \cap T) | U = (S | U) \cap (T | U)$ for every open subset U of M,*

(4) (*isomorphic invariance*) *$\phi_{\#}(S \cap T) = (\phi_{\#}S) \cap (\phi_{\#}T)$, for every orientation-preserving analytic isomorphism ϕ of M onto an oriented analytic manifold, and*

(5) (*reduction to the diagonal*)

$$\gamma_{\#}(S \cap T) = (-1)^{(m-s)t}(S \times T) \cap \gamma_{\#}\mathfrak{M}.$$

(6) (*projection formulae*) *Suppose L is an analytic chain in $M \times N$. If $\mu | \operatorname{spt} L$ is proper and $\{L, R \times \mathfrak{N}\}$ intersect suitably, then the intersection of $\mu_{\#}L$ and R exists and*

$$(\mu_{\#}L) \cap R = \mu_{\#}[L \cap (R \times \mathfrak{N})] \in \mathbf{I}^{\text{loc}}_{l+r-m}(M).$$

If $\nu | \operatorname{spt} L$ is proper and $\{\mathfrak{M} \times Q, L\}$ intersect suitably, then the intersection of Q and $\nu_{\#}L$ exists and

$$Q \cap (\nu_{\#}L) = \nu_{\#}[(\mathfrak{M} \times Q) \cap L] \in \mathbf{I}^{\text{loc}}_{l+q-n}(N).$$

(7) (*associativity*) *If $\{R, S\}$ intersect suitably, $\{S, T\}$ intersect suitably, and*

$\dim(\operatorname{spt} R \cap \operatorname{spt} S \cap \operatorname{spt} T) \leq r+s+t-2m$,

$$\dim[(\operatorname{spt} \partial R \cap \operatorname{spt} S \cap \operatorname{spt} T) \cup (\operatorname{spt} R \cap \operatorname{spt} \partial S \cap \operatorname{spt} T) \cup (\operatorname{spt} R \cap \operatorname{spt} S \cap \operatorname{spt} \partial T)] \leq r+s+t-2m-1,$$

then $(R \cap S) \cap T = R \cap (S \cap T)$

(8) $\mathfrak{M} \cap T = T = T \cap \mathfrak{M}$.

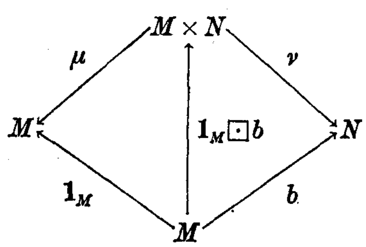
(9) (*boundary formula*) *If* $\{S, T\}$ *intersect suitably and* $s+t>m$, *then*

$$\begin{aligned}\partial(S\cap T) &= (-1)^{m-t}(\partial S)\cap T+S\cap(\partial T) && \text{in case } s>0<t,\\ &= (-1)^{m}(\partial S)\cap T && \text{in case } s>0=t,\\ &= S\cap(\partial T) && \text{in case } s=0<t.\end{aligned}$$

(10) (*Cartesian product formula*) *If* $\{R, S\}$ *intersect suitably and* $\{P, Q\}$ *intersect suitably, then*

$$(R\cap S)\times(P\cap Q)=(-1)^{(m-r)(n-q)}(R\times P)\cap(S\times Q).$$

(11) (*inverse mapping formula*) *Let* $b: M\to N$ *be an analytic mapping and consider the commutative diagram*



If Q *satisfies the two conditions*

$$\dim b^{-1}(\operatorname{spt} Q)\leqslant q+m-n,\quad \dim b^{-1}(\operatorname{spt}\partial Q)\leqslant q+m-n-1,$$

then $b^{\#}Q=\mu_{\#}[(\mathcal{M}\times Q)\cap(\mathbf{1}_M\boxdot b)_{\#}\mathcal{M}]$ *is a* $q+m-n$ *dimensional analytic chain in* M; *moreover if* $b\,|\operatorname{spt} R$ *is proper and*

$$\dim[b^{-1}(\operatorname{spt} Q)\cap\operatorname{spt} R]\leqslant q+r-n,$$

$$\dim([b^{-1}(\operatorname{spt} Q)\cap\operatorname{spt}\partial R]\cup[b^{-1}(\operatorname{spt}\partial Q)\cap\operatorname{spt} R])\leqslant q+r-n-1,$$

then the intersection of Q *and* $b_{\#}R$ *exists and*

$$Q\cap b_{\#}R=b_{\#}[(b^{\#}Q)\cap R]\in\mathbf{I}^{\mathrm{loc}}_{q+r-n}(N).$$

Proof of (0) (1) (3) (4). (0) is proven in 5.4, (1) (3) follow from the definition 5.(1), and (4) follows from 5.(1), 3.5(4).

Proof of (9). (9) follows from 5.(1), 3.5(2), and the remark ([F, 4.1.8]) that if $s+t>0$, then

$$\begin{aligned}\partial(S\times T) &= (\partial S)\times T+(-1)^{s}S\times(\partial T) && \text{in case } s>0<t,\\ &= (\partial S)\times T && \text{in case } s>0=t,\\ &= S\times(\partial T) && \text{in case } s=0<t.\end{aligned}$$

Proof of (6). First we consider the *special case*

$$M \text{ is an open subset of } \mathbf{R}^m,\ \mathcal{M} = \mathbf{E}^m | M,$$
$$N \text{ is an open subset of } \mathbf{R}^n,\ \mathcal{N} = \mathbf{E}^n | N.$$

Letting $\tilde{\sigma}$, $\tilde{\tau}$ be as in 5.7, we define the maps

$$F\colon (M \times N) \times (M \times N) \to \mathbf{R}^m \times \mathbf{R}^n, \quad \alpha\colon (M \times N) \times (M \times N) \to M,$$
$$\beta\colon [(M \times N) \times N] \times M \to (M \times N) \times (M \times N),$$
$$\beta_M\colon [(M \times N) \times N] \times M \to M \times M, \quad \beta_N\colon [(M \times N) \times N] \times M \to N \times N$$

by
$$F((x, y), (w, z)) = (x - y, w - z), \quad \alpha((x, y), (w, z)) = x,$$
$$\beta([(x, y), z], w) = ((x, y), (w, z)), \quad \beta_M([(x, y), z], w) = (x, w),$$
$$\beta_N([(x, y), z], w) = (y, z) \quad \text{for } (x, y) \in M \times N,\ w \in M, \text{ and } z \in N.$$

Observing that β and

$$\beta_M | ([(\operatorname{spt} L) \times N] \times M) \cap (f \circ \beta_N)^{-1}\{0\}$$

are proper maps and that

$$F \circ \beta = (f \circ \beta_M) \boxdot (\tilde{f} \circ \beta_N), \quad \alpha \circ \beta = \mu_1 \circ \beta_M, \quad \beta_M = (\mu \circ \tilde{\sigma}) \times \mathbf{1}_M,$$

we infer from 5.(1), [F, 4.1.8], 4.4, 4.5, 5.1, and 5.7 that

$$\begin{aligned}
&(-1)^{(m+n-l)(r+n)} \mu_\#[L \cap (R \times \mathcal{N})] \\
&\quad = \alpha_\# \langle L \times (R \times \mathcal{N}), F, (0, 0)\rangle \\
&\quad = (-1)^{rn} \alpha_\# \langle \beta_\#[(L \times \mathcal{N}) \times R], F, (0, 0)\rangle \\
&\quad = (-1)^{rn} (\mu_1 \circ \beta_M)_\# \langle (L \times \mathcal{N}) \times R, (f \circ \beta_M) \boxdot (\tilde{f} \circ \beta_N), (0, 0)\rangle \\
&\quad = (-1)^{(r+m)n} (\mu_1 \circ \beta_M)_\# \langle\langle (L \times \mathcal{N}) \times R, \tilde{f} \circ \beta_N, 0\rangle, f \circ \beta_M, 0\rangle \\
&\quad = (-1)^{(r+m)n} \mu_{1\#} \langle \beta_{M\#} \langle (L \times \mathcal{N}) \times R, \tilde{f} \circ \beta_N, 0\rangle, f | (M \times M), 0\rangle \\
&\quad = (-1)^{(r+m)n} \mu_{1\#} \langle [(\mu \circ \tilde{\sigma})_\# \langle L \times \mathcal{N}, \tilde{f} \circ \tilde{\tau}, 0\rangle] \times R, f | (M \times M), 0\rangle \\
&\quad = (-1)^{(r+m+l+n)n} \mu_{1\#} \langle (\mu_\# L) \times R, f | (M \times M), 0\rangle = (-1)^{(r+m+l+n)n+(m-l)r} (\mu_\# L) \cap R.
\end{aligned}$$

To prove the *general case* we assume $h\colon U \to \mathbf{R}^m$ is an analytic coordinate system for M (as in 5.(1)) and verify the formula

$$(-1)^{(m-l)r} \langle [(\mu_\# L) \times R] | (U \times U), f \circ (h \times h), 0\rangle = (\gamma | U)_\# (\mu_\# [L \cap (R \times \mathcal{N})] | U) \in \mathbf{I}^{\text{loc}}_{l+r-m}(U \times U).$$

To do this it suffices to assume that $\operatorname{Clos} U$ is compact, hence $\nu[\mu^{-1}(\operatorname{Clos} U) \cap \operatorname{spt} L]$ is compact. Letting v be as in 4.4 we choose h_1, U_1, $V_1, \ldots, h_J$, U_J, V_J so that

$$\nu[\mu^{-1}(\text{Clos } U) \cap \text{spt } L] \subset V_1 \cup \dots \cup V_J \subset N,$$

and for each $j \in \{1, \dots, J\}$, $V_j \subset U_j \subset N$ and h_j is an orientation-preserving analytic isomorphism from U_j onto $\mathbf{U}(0, 2) \subset \mathbf{R}^n$ with $h_j(V_j) = \mathbf{U}(0, 1)$. For each $j \in \{1, \dots, J\}$ we choose r_j so that $1 < r_j < 2$,

$$\dim [(U \times U_j) \cap (v \circ h_j \circ \nu)^{-1}\{r_j\} \cap \text{spt } L] \leqslant l - 1,$$
$$\dim [(U \times U_j) \cap (v \circ h_j \circ \nu)^{-1}\{r_j\} \cap \mu^{-1}(\text{spt } R) \cap \text{spt } L] \leqslant l + r - m - 1,$$

and define the current

$$L_j = [L \,|\, (U \times N)] \llcorner (U \times [U_j \cap \{y : |h_j(y)| < r_j\} \sim \bigcup_{i=1}^{j-1} U_i \cap \{y : |h_i(y)| \leqslant r_i\}]),$$

hence L_j is an l dimensional analytic chain in $U \times N$, $\text{spt } L_j \subset U \times U_j$, $\{L_j, (R \,|\, U) \times \mathcal{N}\}$ intersect suitably, and $L \,|\, (U \times N) = L_1 + \dots + L_J$ by [F, 4.1.20].

Using for each $j \in \{1, \dots, J\}$ the commutative diagram

$$\begin{array}{ccc} U \times U_j & \xrightarrow{\quad h \times h_j \quad} & h(U) \times \mathbf{U}(0,2) \\ \Big\downarrow {\scriptstyle \mu | (U \times U_j)} & & \Big\downarrow {\scriptstyle \text{projection}} \\ U & \xrightarrow{\quad h \quad} & h(U) \end{array}$$

(3), 4.4, and the special case considered before, we find that

$$\langle ([\mu \,|\, (U \times N)]_\# L_j) \times (R \,|\, U), f \circ (h \times h), 0 \rangle$$
$$= (-1)^{(m-l)r} (\gamma \,|\, U)_\# [\mu \,|\, (U \times N)]_\# (L_j \cap [(R \,|\, U) \times \mathcal{N}]) \in \mathbf{I}^{\text{loc}}_{l+r-m}(U \times U),$$

and the desired formula follows by linearity.

From this formula we conclude first that the intersection of $\mu_\# L$ and R exists and second that

$$(\mu_\# L) \cap R = \mu_\# [L \cap (R \times \mathcal{N})] \in \mathbf{I}^{\text{loc}}_{l+r-m}(M).$$

The proof of the second formula is similar.

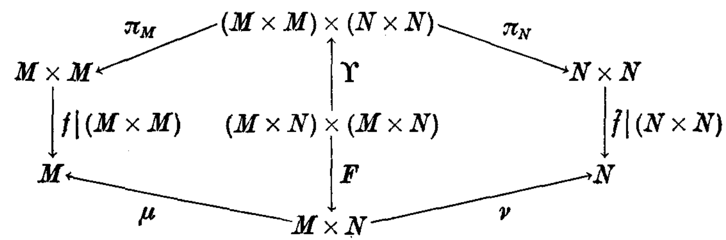
Proof of (2) (5) (7) (8) (10). By (3) (4) we may assume without loss of generality that M and N are open subsets of $\mathbf{R}^m$ and $\mathbf{R}^n$, $\mathcal{M} = \mathbf{E}^m \,|\, M$, and $\mathcal{N} = \mathbf{E}^n \,|\, N$.

For statements (2) and (8) it suffices to argue as in [F, 4.3.20] with f, g replaced by $f \,|\, (M \times M)$, $g \,|\, M$.

To prove (5) we use 5.(1), 5.3, 5.5, and (8) to compute

$$(S \times T) \cap \gamma_\# \mathcal{M} = (S \times T) \cap \langle \mathcal{M} \times \mathcal{M}, f \,|\, (M \times M), 0 \rangle = \langle (S \times T) \cap (\mathcal{M} \times \mathcal{M}), f \,|\, (M \times M), 0 \rangle$$
$$= \langle S \times T, f \,|\, (M \times M), 0 \rangle = (-1)^{(m-s)t} (S \cap T).$$

To prove (10) we consider the commutative diagram



where $\Upsilon((w, y), (x, z)) = ((w, x), (y, z))$ and π_M, π_N are the projections, and then conclude from 5.(1), 5.1, 4.5, and 4.4 that

$$\begin{aligned}
&(-1)^{(m-r)s+(n-p)q+(r+s-m)n}(\gamma\times\tilde{\gamma})_\#[(R\cap S)\times(P\cap Q)]\\
&\quad=(-1)^{(r+s-m)n}\langle R\times S, f|(M\times M), 0\rangle\times\langle P\times Q, \tilde{f}|(N\times N), 0\rangle\\
&\quad=\langle\langle R\times S, f|(M), 0\rangle\times(P\times Q), \tilde{f}\circ\pi_N, 0\rangle\\
&\quad=\langle\langle(R\times S)\times(P\times Q), f\circ\pi_M, 0\rangle, \tilde{f}\circ\pi_N, 0\rangle\\
&\quad=(-1)^{sp}\langle\Upsilon_\#[(R\times P)\times(S\times Q)], (f\circ\pi_M)\boxdot(\tilde{f}\circ\pi_N), (0,0)\rangle\\
&\quad=(-1)^{sp}\Upsilon_\#\langle(R\times P)\times(S\times Q), F, (0,0)\rangle\\
&\quad=(-1)^{sp+(m+n-r-p)(s+q)}(\gamma\times\tilde{\gamma})_\#[(R\times P)\cap(S\times Q)].
\end{aligned}$$

To prove (7) we use 5.(1), (2), (6), 5.5, and (10) to deduce that

$$\begin{aligned}
&(-1)^{(m-s)t}R\cap(S\cap T)\\
&\quad=R\cap\mu_{1\#}\langle S\times T, f|(M\times M), 0\rangle\\
&\quad=(-1)^{(m-r)m}\mu_{1\#}(R\times\mathcal{M})\cap\langle S\times T, f|(M\times M), 0\rangle]\\
&\quad=(-1)^{(m-r)m}\mu_{1\#}\langle(R\times\mathcal{M})\cap(S\times T), f|(M\times M), 0\rangle\\
&\quad=(-1)^{(m-r)t}\mu_{1\#}\langle(R\cap S)\times(\mathcal{M}\cap T), f|(M\times M), 0\rangle\\
&\quad=(-1)^{(m-r)t}\mu_{1\#}\langle(R\cap S)\times T, f|(M\times M), 0\rangle=(-1)^{(m-r)t+(m-r-s+m)t}(R\cap S)\cap T.
\end{aligned}$$

Proof of (11). The chain $b^\# Q$ is an analytic chain because $b^\# Q\in\mathbf{I}^{\text{loc}}_{q+m-n}(M)$ and

$$\operatorname{spt}(b^\# Q)\subset b^{-1}(\operatorname{spt} Q),\quad \operatorname{spt}(\partial b^\# Q)\subset b^{-1}(\operatorname{spt}\partial Q).$$

Using (6) and (8), we obtain

$$\begin{aligned}
&\nu_\#[(\mathcal{M}\times Q)\cap(\mathbf{1}_M\boxdot b)_\# R]=Q\cap\nu_\#[(\mathbf{1}_M\boxdot b)_\# R]=Q\cap b_\# R,\\
&\mu_\#([(\mathbf{1}_M\boxdot b)_\#\mathcal{M}]\cap[R\times\mathcal{N}])=(\mu_\#[(\mathbf{1}_M\boxdot b)_\#\mathcal{M}])\cap R=\mathcal{M}\cap R=R.
\end{aligned}$$

Since $\operatorname{spt}([(\mathbf{1}_M\boxdot b)_\#\mathcal{M}]\cap[R\times\mathcal{N}])$ is contained in

$$b = (M \times N) \cap \{(x, b(x)) : x \in M\}$$

and $[(\mathbf{1}_M \boxdot b) \circ \mu] | b = \mathbf{1}_b$, $(b \circ \mu) | b = \nu | b$ we infer from (7) and (6) that

$$\begin{aligned}
(\mathbf{1}_M \boxdot b)_\# R &= (\mathbf{1}_M \boxdot b)_\# \mu_\# ([(\mathbf{1}_M \boxdot b)_\# \mathcal{M}] \cap [R \times \mathcal{N}]) = [(\mathbf{1}_M \boxdot b)_\# \mathcal{M}] \cap [R \times \mathcal{N}], \\
Q \cap b_\# R &= \nu_\# [(\mathcal{M} \times Q) \cap ([(\mathbf{1}_M \boxdot b)_\# \mathcal{M}] \cap [R \times \mathcal{N}])] \\
&= b_\# \mu_\# ([(\mathcal{M} \times Q) \cap (\mathbf{1}_M \boxdot b)_\# \mathcal{M}] \cap [R \times \mathcal{N}]) \\
&= b_\# (\mu_\# [(\mathcal{M} \times Q) \cap (\mathbf{1}_M \boxdot b)_\# \mathcal{M}] \cap R) = b_\# [(b^\# Q) \cap R].
\end{aligned}$$

5.9. *Example.* Choosing the oriented planes

$$R = \boldsymbol{\delta}_0 \times \mathbf{E}^1 \times \mathbf{E}^1 \in \mathbf{I}_2^{\text{loc}}(\mathbf{R}^3), \quad T = \mathbf{E}^1 \times \boldsymbol{\delta}_0 \times \mathbf{E}^1 \in \mathbf{I}_2^{\text{loc}}(\mathbf{R}^3)$$

and the analytic chain S from 4.6, we infer from 5.8(8) (9) (10) that

$$\begin{aligned}
(R \cap S) \cap T &= [R \cap \partial(\mathbf{E}^3 \llcorner \{(x, y, z) : h(x, y, z) < 0\}] \cap T \\
&= (\partial[(\boldsymbol{\delta}_0 \times \mathbf{E}^1 \times \mathbf{E}^1) \llcorner \{(0, y, z) : z^2 < 1\}]) \cap T \\
&= -\partial([(\boldsymbol{\delta}_0 \times \mathbf{E}^1 \times \mathbf{E}^1) \cap (\mathbf{E}^1 \times \boldsymbol{\delta}_0 \times \mathbf{E}^1)] \llcorner \{(0,0,z) : z^2 < 1\}) \\
&= -\partial[([\boldsymbol{\delta}_0 \times \mathbf{E}^1) \cap (\mathbf{E}^1 \times \boldsymbol{\delta}_0)] \times \mathbf{E}^1) \llcorner \{(0, 0, z) : z^2 < 1\}] \\
&= \partial[(\boldsymbol{\delta}_0 \times \boldsymbol{\delta}_0 \times \mathbf{E}^1) \llcorner \{(0, 0, z) : z^2 < 1\}] \\
&= \boldsymbol{\delta}_{(0,0,1)} - \boldsymbol{\delta}_{(0,0,-1)} \neq 0 = -R \cap \partial(0) \\
&= -R \cap \partial[(\mathbf{E}^1 \times \boldsymbol{\delta}_0 \times \mathbf{E}^1) \llcorner \{(0, 0, z) : z^2 < 1\}] = R \cap (S \cap T)
\end{aligned}$$

even though each of the four pairs

$$\{R, S\}, \{S, T\}, \{R \cap S, T\}, \{R, S \cap T\}$$

intersect suitably.

5.10. Assuming that $I \in \{2, 3, \ldots\}$ and that for each $i \in \{1, \ldots, I\}$ T_i is a t_i dimensional analytic chain in M, we are motivated by 4.5, 5.8(7), 5.9 to say that $\{T_1, \ldots, T_I\}$ *intersect suitably* if and only if

$$\sum_{i=1}^{I} t_i \geq (I-1)m, \quad \dim \left(\bigcap_{i=1}^{I} \operatorname{spt} T_i \right) \leq \left(\sum_{i=1}^{I} t_i \right) - (I-1)m,$$

$$\dim \bigcup_{i=1}^{I} (\operatorname{spt} T_1 \cap \ldots \cap \operatorname{spt} T_{i-1} \cap \operatorname{spt} \partial T_i \cap \operatorname{spt} T_{i+1} \cap \ldots \cap \operatorname{spt} T_I) \leq \left(\sum_{i=1}^{I} t_i \right) - (I-1)m - 1,$$

and in this case to define the *I-fold intersection of* $T_1, \ldots, T_I$, denoted

$$T_1 \cap \ldots \cap T_I,$$

by the condition:

If U, h *are as in* 5.(1) *and*

$$F : (\mathbf{R}^m)^I \to (\mathbf{R}^m)^{I-1}, \quad \Gamma : M \to M^I,$$

$F(u_1, \ldots, u_I) = (u_1 - u_2, \ldots, u_{I-1} - u_I)$, $\Gamma(x) = (x, \ldots, x)$ *for* $(u_1, \ldots, u_I) \in (\mathbf{R}^m)^I$ *and* $x \in M$, *then*

$$(\Gamma | U)_\# [(T_1 \cap \ldots \cap T_I) | U]$$
$$= (-1)^\theta \langle (T_1 \times \ldots \times T_I) | (U \times \ldots \times U), F \circ (h \times \ldots \times h), (0, \ldots, 0) \rangle$$

where
$$\theta = \sum_{i=2}^{I} t_i \left[(i-1)m - \sum_{j=1}^{i-1} t_j \right].$$

From 4.5, 4.4 it then follows, for instance, that

$$T_1 \cap T_2 \cap T_3 = (T_1 \cap T_2) \cap T_3$$

whenever $\{T_1, T_2, T_3\}$ and $\{T_1, T_2\}$ intersect suitably.

5.11. INTERSECTION AXIOMS. A *real analytic intersection theory* $\mathfrak{I}$ *is a rule which associates with every triple* $(\mathfrak{M}, S, T)$ *such that*

there exists an m dimensional separable, orientable real analytic manifold M,

$\mathfrak{M}$ *is an orienting m cycle for* M,

S *is an s dimensional analytic chain in* M,

T *is a t dimensional analytic chain in* M, *and*

$\{S, T\}$ *intersect suitably in* M

an $s+t-m$ *dimensional analytic chain* $\mathfrak{I}_\mathfrak{M}(S, T)$ *in* M *so that the following eight conditions hold:*

If $M, \mathfrak{M}, S, T$ *are as above, then*

(1) $\mathfrak{I}_\mathfrak{M}(S, jT) = j\mathfrak{I}_\mathfrak{M}(S, T)$ *for every integer* j,

(2) $\mathfrak{I}_\mathfrak{M}(S, T) = (-1)^{(m-s)(m-t)} \mathfrak{I}_\mathfrak{M}(T, S)$,

(3) $\mathfrak{I}_\mathfrak{M}(S, T) | U = \mathfrak{I}_{\mathfrak{M}|U}(S|U, T|U)$ *for every open subset* U *of* M,

(4) $\phi_\# \mathfrak{I}_\mathfrak{M}(S, T) = \mathfrak{I}_{\phi_\# \mathfrak{M}}(\phi_\# S, \phi_\# T)$ *for every analytic isomorphism* ϕ *of* M *onto an analytic manifold, and*

(5) $\gamma_\# \mathfrak{I}_\mathfrak{M}(S, T) = (-1)^{(m-s)t} \mathfrak{I}_{\mathfrak{M} \times \mathfrak{M}}(S \times T, \gamma_\# \mathfrak{M})$ *where* $\gamma: M \to M \times M$ *is given by* $\gamma(x) = (x, x)$ *for* $x \in M$.

(6) *If* R, S, T *are* r, s, t *dimensional analytic chains in* M *such that* $\{R, S\}$, $\{S, T\}$, *and* $\{R, S, T\}$ *intersect suitably, then*

$$\mathfrak{I}_\mathfrak{M}[\mathfrak{I}_\mathfrak{M}(R, S), T] = \mathfrak{I}_\mathfrak{M}[R, \mathfrak{I}_\mathfrak{M}(S, T)].$$

(7) *If* N *is a separable, orientable real analytic manifold with orienting cycle* $\mathfrak{N}$, L *is an analytic chain in* $M \times N$, $\mu: M \times N \to M$ *is the projection,* $\mu | \operatorname{spt} T$ *is proper,* $\mu_\# L$ *and* R *are analytic chains in* M, *and* $\{L, R \times \mathfrak{N}\}$ *intersect suitably, then*

$$\mathfrak{I}_m(\mu_\# L, R) = \mu_\# \mathfrak{I}_{m\times n}(L, R\times \mathcal{N}).$$

(8) $\mathfrak{I}_{\mathbf{E}^0}(\mathbf{E}^0, \mathbf{E}^0) = \mathbf{E}^0$ *where* $\mathbf{E}^0$ *is the orienting* 0 *cycle for* $\mathbf{R}^0 = \{0\}$ *defined by* $\mathbf{E}^0(\psi) = \psi(0)$ *for every function* $\psi: \mathbf{R}^0 \to \mathbf{R}$.

THEOREM. *There exists a unique real analytic intersection theory.*

Proof. Existence has been proven in 5.8(0) (1) (2) (3) (4) (5) (6) (7) (8).

To show uniqueness we assume $\mathfrak{I}$ is a real analytic intersection theory and M, $\mathcal{M}$, S, T are as above, we observe by (3) (1) (2)

$$\operatorname{spt} \mathfrak{I}_m(S, T) \subset (\operatorname{spt} S) \cap \operatorname{spt} T,$$

and then we prove the equation

$$\mathfrak{I}_m(S, T) = S \cap T$$

by considering seven cases.

Case 1, $S = \boldsymbol{\delta}_x$ *for some* $x \in M$, $T = \mathcal{M}$. *Here* $S \cap T = \boldsymbol{\delta}_x$ by 5.8(8), and

$$\mathfrak{I}_m(S, T) = i\boldsymbol{\delta}_x \text{ for some integer } i$$

by the above observation and [F, 4.1.26]. To show that i equals one, we define the maps

$$\phi: M \to \mathbf{R}^0 \times M,\ \phi(w) = (0, w) \text{ for } w \in M,$$
$$\pi_0: \mathbf{R}^0 \times M \to \mathbf{R}^0,\ \pi_0(0, w) = 0 \text{ for } (0, w) \in \mathbf{R}^0 \times M,$$

and use (4) (7) (8) to compute

$$i\mathbf{E}^0 = \pi_{0\#}\phi_\#(i\boldsymbol{\delta}_x) = \pi_{0\#}\phi_\#\mathfrak{I}_m(S, T) = \pi_{0\#}\mathfrak{I}_{\mathbf{E}^0\times m}(\mathbf{E}^0 \times \boldsymbol{\delta}_x, \mathbf{E}^0 \times \mathcal{M}) = \mathfrak{I}_{\mathbf{E}^0}(\mathbf{E}^0, \mathbf{E}^0) = \mathbf{E}^0.$$

Case 2, $S = \mathcal{M}$, $T = \mathcal{M}$. Here $S \cap T = \mathcal{M}$. If U is a connected open subset of $M \sim \operatorname{spt} \partial \mathfrak{I}_m(S, T)$, then

$$\mathfrak{I}_m(S, T) \,|\, U = j\mathcal{M} \,|\, U \text{ for some integer } j$$

by [F, 4.1.31]. Letting $x \in U$ we infer from (1) (3) (6) and Case 1 that j equals one by computing

$$j\boldsymbol{\delta}_x = \mathfrak{I}_{m|U}[\boldsymbol{\delta}_x, \mathfrak{I}_m(S, T)|U] = \mathfrak{I}_{m|U}[\boldsymbol{\delta}_x, \mathfrak{I}_{m|U}(\mathcal{M}|U, \mathcal{M}|U)]$$
$$= \mathfrak{I}_{m|U}[\mathfrak{I}_{m|U}(\boldsymbol{\delta}_x, \mathcal{M}|U), \mathcal{M}|U] = \mathfrak{I}_{m|U}(\boldsymbol{\delta}_x, \mathcal{M}|U) = \boldsymbol{\delta}_x.$$

Consequently

$$\dim \operatorname{spt} [\mathfrak{I}_m(S, T) - \mathcal{M}] \leq \dim \operatorname{spt} \partial\mathfrak{I}_m(\mathcal{M}, \mathcal{M}) \leq m - 1,\ \mathfrak{I}_m(S, T) - \mathcal{M} \in \mathcal{F}_m^{\mathrm{loc}}(M),$$

hence $\mathfrak{I}_m(S, T) - \mathcal{M} = 0$ by [F, 4.1.20].

Case 3, $M=\mathbf{R}^m$, $\mathcal{M}=\mathbf{E}^m=T$, $S=\langle \mathbf{E}^m, \alpha, y\rangle$ *for some* $\alpha\in\mathbf{O}^*(m, m-s)$, $y\in\mathbf{R}^{m-s}$. Here $S\cap T=S$. By use of (1) (2) (4) we may replace $M, \mathcal{M}, S, T$ by $\mathbf{R}^s\times\mathbf{R}^{m-s}$, $\mathbf{E}^s\times\mathbf{E}^{m-s}$, $\mathbf{E}^s\times\boldsymbol{\delta}_0$, $\mathbf{E}^s\times\mathbf{E}^{m-s}$. For any connected open subsets V and W of $\mathbf{R}^s$ and $\mathbf{R}^{m-s}$ with

$$(V\times W)\cap \operatorname{spt}\partial\mathfrak{I}_{\mathcal{M}}(S, T)=0$$

we infer from [F, 4.1.31] that

$$\mathfrak{I}_{\mathcal{M}}(S, T)\,|\,(V\times W)=kS\,|\,(V\times W) \text{ for some integer } k,$$

we let $\pi_V\colon V\times W\to V$ be the projection, and we compute from (3) (7) and Case 2 that

$$\begin{aligned} k\mathbf{E}^s\,|\,V=\pi_{V\#}[\mathfrak{I}_{\mathcal{M}}(S, T)\,|\,(V\times W)] &= \pi_{V\#}\mathfrak{I}_{(\mathbf{E}^s|V)\times(\mathbf{E}^{m-s}|W)}[(\mathbf{E}^s\,|\,V)\times\boldsymbol{\delta}_0, (\mathbf{E}^s\,|\,V)\times(\mathbf{E}^{m-s}\,|\,W)] \\ &= \mathfrak{I}_{\mathbf{E}^s|V}(\mathbf{E}^s\,|\,V, \mathbf{E}^s\,|\,V)=\mathbf{E}^s\,|\,V. \end{aligned}$$

Hence $\operatorname{spt}[\mathfrak{I}_{\mathcal{M}}(S, T)-S]\subset\operatorname{spt}\partial\mathfrak{I}_{\mathcal{M}}(S, T)$, and $\mathfrak{I}_{\mathcal{M}}(S, T)=S$ by [F, 4.1.20].

Case 4, $M=\mathbf{R}^m$, $\mathcal{M}=\mathbf{E}^m$, $S=\langle\mathbf{E}^m, \alpha, y\rangle$, $T=\langle\mathbf{E}^m, \beta, z\rangle$ *for some* $\alpha\in\mathbf{O}^*(m, m-s)$, $y\in\mathbf{R}^{m-s}$, $\beta\in\mathbf{O}^*(m, m-t)$, $z\in\mathbf{R}^{m-t}$. Here either $\alpha^{-1}\{y\}\cap\beta^{-1}\{z\}$ is empty, in which case

$$S\cap T=0=\mathfrak{I}_{\mathcal{M}}(S, T)$$

by (3), or $\dim(\alpha^{-1}\{y\}\cap\beta^{-1}\{z\})=s+t-m$, in which case we may, by (1) (2) (4), replace $M, \mathcal{M}, S, T$ by $\mathbf{R}^t\times\mathbf{R}^{m-t}$, $\mathbf{E}^t\times\mathbf{E}^{m-t}$, $\langle\mathbf{E}^t, \varepsilon, 0\rangle\times\mathbf{E}^{m-t}$, $\mathbf{E}^t\times\boldsymbol{\delta}_0$ for some $\varepsilon\in\mathbf{O}^*(t, m-s)$. Then

$$S\cap T=(-1)^{(m-s)(m-t)}\langle\mathbf{E}^t, \varepsilon, 0\rangle\times\boldsymbol{\delta}_0$$

by 5.8(10) (8). For connected open sets V and W of $\mathbf{R}^t$ and $\mathbf{R}^{m-t}$ with $(V\times W)\cap\operatorname{spt}\partial\mathfrak{I}_{\mathcal{M}}(S, T)$ empty,

$$\mathfrak{I}_{\mathcal{M}}(S, T)\,|\,(V\times W)=l[(\langle\mathbf{E}^t, \varepsilon, 0\rangle\,|\,V)\times\boldsymbol{\delta}_0] \text{ for some integer } l.$$

To see that l equals $(-1)^{(m-s)(m-t)}$ we compute, with the aid of (2) (3) (7) and Case 3, that

$$\begin{aligned} (-1)^{(m-s)(m-t)}l\langle\mathbf{E}^t, \varepsilon, 0\rangle\,|\,V &= \pi_{V\#}[\mathfrak{I}_{\mathcal{M}}(T, S)\,|\,(V\times W)] \\ &= \mathfrak{I}_{\mathbf{E}^t|V}(\mathbf{E}^t\,|\,V, \langle\mathbf{E}^t, \varepsilon, 0\rangle\,|\,V)=\mathfrak{I}_{\mathbf{E}^t}(\mathbf{E}^t, \langle\mathbf{E}^t, \varepsilon, 0\rangle)\,|\,V=\langle\mathbf{E}^t, \varepsilon, 0\rangle\,|\,V. \end{aligned}$$

Case 5, M *is an open subset of* $\mathbf{R}^m$, $\mathcal{M}=\mathbf{E}^m\,|\,M$, $s+t=m$. Here we abbreviate $X=(\operatorname{spt} S)\cap\operatorname{spt} T$ and note by [F, 4.1.24] that there exist integers i_x, j_x for each $x\in X$ so that

$$S\cap T=\sum_{x\in X} i_x\boldsymbol{\delta}_x, \quad \mathfrak{I}_{\mathcal{M}}(S, T)=\sum_{x\in X} j_x\boldsymbol{\delta}_x.$$

We fix $x\in X$ and define the map

$$e\colon \mathbf{R}^m \times \mathbf{R}^m \to \mathbf{R}^m,\ e(u_1, u_2) = u_1 + u_2 \text{ for } (u_1, u_2) \in \mathbf{R}^m \times \mathbf{R}^m.$$

Choosing ϱ, σ so that

$$0 < \varrho < \tfrac{1}{2} \operatorname{distance} [\{x\}, (\mathbf{R}^m \sim M) \cup (X \sim \{x\})],$$
$$0 < \sigma < \inf \{\operatorname{distance} (\gamma[\mathbf{B}(x, \varrho)], \operatorname{spt} \partial[S \times T]),$$
$$\operatorname{distance} (\gamma[\mathbf{B}(x, \varrho) \sim \mathbf{U}(x, \varrho)], \operatorname{spt} [S \times T])\}$$

and abbreviating $U = \mathbf{U}(0, \sigma) \times \mathbf{U}(2x, 2\varrho) \subset \mathbf{R}^m \times \mathbf{R}^m$, we find that the set $V = (f \boxdot e)^{-1}(U)$ is a nonempty open subset of $(M \times M) \sim \operatorname{spt} \partial(S \times T)$ and that the map

$$f \mid V \cap \operatorname{spt}(S \times T)$$

is proper. Moreover by [F, 4.3.2.(1)]

$$(f \mid V)_\# [(S \times T) \mid V] = (-1)^{(m-s)t} i_x \mathbf{E}^m \mid \mathbf{U}(0, \sigma)$$

because $\langle (S \times T) \mid V, f \mid V, \cdot \rangle (1)$,being a continuous, integer-valued function on $\mathbf{U}(0, \sigma)$ has constant value

$$\langle (S \times T) \mid V, f \mid V, 0 \rangle (1) = (-1)^{(m-s)t} i_x.$$

Factoring $f \mid V$ as $\pi_1 \circ [(f \boxdot e) \mid V]$ where $\pi_1\colon U \to \mathbf{U}(0, \sigma)$ is the projection, we use Case 1 and (1) (2) (3) (4) (5) to conclude that

$$\begin{aligned}
(-1)^{(m-s)t} i_x \boldsymbol{\delta}_0 &= \mathfrak{I}_{m \mid \mathbf{U}(0,\sigma)} [\boldsymbol{\delta}_0, (-1)^{(m-s)t} i_x \mathbf{E}^m \mid \mathbf{U}(0, \sigma)] \\
&= \mathfrak{I}_{m \mid \mathbf{U}(0,\sigma)} [(-1)^{(m-s)t} i_x \mathbf{E}^m \mid \mathbf{U}(0, \sigma), \boldsymbol{\delta}_0] \\
&= \pi_{1\#} \mathfrak{I}_{(m \times m) \mid \mathbf{U}} [(f \boxdot e)_\# (S \times T) \mid U, (\boldsymbol{\delta}_0 \times \mathcal{M}) \mid U] \\
&= \pi_{1\#} (\mathfrak{I}_{m \times m} [(f \boxdot e)_\# (S \times T), \boldsymbol{\delta}_0 \times \mathcal{M}] \mid U) = (f \mid V)_\# [\mathfrak{I}_{m \times m} (S \times T, \gamma_\# \mathcal{M}) \mid V] \\
&= (-1)^{(m-s)t} (f \mid V)_\# [\mathfrak{I}_m (S, T) \mid V] = (-1)^{(m-s)t} j_x \boldsymbol{\delta}_0
\end{aligned}$$

Case 6, M is an open subset of $\mathbf{R}^m$, $\mathcal{M} = \mathbf{E}^m \mid M$, $s + t > m$. Here we first observe that if $\alpha \in \mathbf{O}^*(m, s+t-m)$, $y \in \mathbf{R}^{s+t-m}$, $P = \langle \mathbf{E}^m, \alpha, y \rangle \mid M$, $\{T, P\}$ intersect suitably, and $\{S, T, P\}$ intersect suitably, then by (5) (7) (6) (3), Case 5, and Case 4,

$$\begin{aligned}
(-1)^{(m-s)t} [\mathfrak{I}_m (S, T) - (S \cap T)] \cap P &= (-1)^{(m-s)t} (\mathfrak{I}_m [\mathfrak{I}_m (S, T), P] - [S \cap T] \cap P) \\
&= \mathfrak{I}_m [\mu_{1\#} \mathfrak{I}_{m \times m} (S \times T, \gamma_\# \mathcal{M}), P] - (\mu_{1\#} [(S \times T) \cap \gamma_\# \mathcal{M}]) \cap P \\
&= \mu_{1\#} (\mathfrak{I}_{m \times m} [\mathfrak{I}_{m \times m} (S \times T, \gamma_\# \mathcal{M}), P \times \mathcal{M}] - [(S \times T) \cap \gamma_\# \mathcal{M}] \cap [P \times \mathcal{M}]) \\
&= \mu_{1\#} (\mathfrak{I}_{m \times m} [S \times T, \mathfrak{I}_{m \times m} (\gamma_\# \mathcal{M}, P \times \mathcal{M})] - [S \times T] \cap [(\gamma_\# \mathcal{M}) \cap (P \times \mathcal{M})]) \\
&= \mu_{1\#} (\mathfrak{I}_{m \times m} [S \times T, (\gamma_\# \mathcal{M}) \cap (P \times \mathcal{M})] - [S \times T] \cap [(\gamma_\# \mathcal{M}) \cap (P \times \mathcal{M})]) \\
&= \mu_{1\#} (0) = 0.
\end{aligned}$$

For each $\lambda \in \Lambda(m, s+t-m)$ both

$$\{T, \langle \mathbf{E}^m, \mathbf{p}_\lambda, z\rangle | M\} \text{ and } \{S, T, \langle \mathbf{E}^m, \mathbf{p}_\lambda, z\rangle | M\}$$

intersect suitably for $\mathcal{L}^{s+t-m}$ almost all $z \in \mathbf{R}^{s+t-m}$, hence by 5.5, 5.8(8)

$$\langle \mathfrak{I}_m(S, T) - S \cap T, \mathbf{p}_\lambda | M, z\rangle = [\mathfrak{I}_m(S, T) - S \cap T] \cap [\langle \mathbf{E}^m, \mathbf{p}_\lambda, z\rangle | M] = 0,$$

and we conclude from 4.1 that $\mathfrak{I}_m(S, T) = S \cap T$.

Case 7, general case. Here we apply Case 5, Case 6, and (3) (4).

6. Slicing positive holomorphic chains

We have studied the continuity of the real analytic slice $\langle T, f, y\rangle$ with respect to y in 4.3 and with respect to f in 4.7. Continuity with respect to T, on the other hand, even when the dimensions of spt $T \cap f^{-1}\{y\}$ and spt $\partial T \cap f^{-1}\{y\}$ do not become unusually large, is in general false, as is shown by the example in 6.6. Affirmative results, however, may be obtained in the analogous *complex holomorphic case.*

In this section we assume that M is a separable complex m dimensional complex manifold. A current $T \in \mathcal{F}_{2t}^{\text{loc}}(M)$ is called a *complex t dimensional holomorphic chain in M* if $\partial T = 0$ and if M can be covered by open sets U for which there exists a complex t dimensional holomorphic subvariety H of U with $U \cap \operatorname{spt} T \subset H$. It follows that T is a $2t$ dimensional analytic chain in M. We will say that T is *positive* if and only if for $\|T\|$ almost all $x \in M$ the simple $2t$ vector $\mathbf{T}(x)$ is complex and positive ([F, 4.1.28, 1.6.6]). By [F, 4.2.29] the support of a holomorphic chain in M is a holomorphic subset of M, because the closure of any connected component of the set of regular points of a holomorphic set is also holomorphic ([N, p. 67]).

J. King has characterized in [K2] complex t dimensional positive holomorphic chains as those currents $T \in \mathcal{R}_{2t}^{\text{loc}}(M)$ for which $\partial T = 0$ and $\mathbf{T}(x)$ is complex and positive for $\|T\|$ almost all $x \in M$; he has also described complex holomorphic intersection theory and has proven the complex analogue of the Slicing theorem of 4.3. Here we propose to prove a more general statement (6.5) by exploiting the fact that in $\mathbf{C}^m$ such chains are *area minimizing currents* ([F, 5.4.1, 5.4.19]).

6.1. LEMMA. *Suppose $U \subset \mathbf{C}^m$, $V \subset \mathbf{C}^n$, $W \subset \mathbf{C}^m \times \mathbf{C}^n$ are open sets, V is connected,* Clos$(U \times V)$ *is a compact subset of W, and $q\colon U \times V \to V$ is the projection. If R is a positive complex n dimensional holomorphic chain in W,*

$$[(\text{Bdry } U) \times \text{Clos } V] \cap \operatorname{spt} R = \varnothing,$$

and $S = R \,|\, (U \times V)$, then there exists an integer k such that for all $v \in V$

$$\operatorname{card}(q^{-1}\{v\} \cap \operatorname{spt} S) \leq k, \ \mathbf{M}\langle S, q, v\rangle = k.$$

Moreover if R_j, for each $j \in \{1, 2, \ldots\}$, is a positive complex n dimensional holomorphic chain in W,

$$S_j = R_j | (U \times V), \text{ and } R_j \to R \text{ in } \mathcal{F}_{2n}^{\text{loc}}(W) \text{ as } j \to \infty,$$

then there exists an integer J such that for all $j \geq J$

$$[(\operatorname{Bdry} U) \times \operatorname{Clos} V] \cap \operatorname{spt} R_j = \varnothing,$$

$$\operatorname{card}(q^{-1}\{v\} \cap \operatorname{spt} S_j) \leq k, \ \mathbf{M}\langle S_j, q, v\rangle = k \text{ for } v \in V.$$

Furthermore for each $v \in V$

$$\langle S_j, q, v\rangle \to \langle S, q, v\rangle \text{ as } j \to \infty \text{ in } \{J, J+1, \ldots\}.$$

Proof. For every $v \in V$, $q^{-1}\{v\} \cap \operatorname{spt} S$ is a compact holomorphic subset of $U \times V$ and is hence finite ([N, p. 52]). Therefore 4.3 implies that the function $\langle S, q, \cdot\rangle$ is $\mathcal{F}_0^{\text{loc}}(U \times V)$ continuous on V.

For $(u, v) \in \operatorname{spt} S$ we define the integer

$$\Delta(u, v) = [\langle S, q, v\rangle \llcorner \{(u, v)\}](1),$$

and recall 3.6 to see that the inequality $\Delta(u, v) > 0$ may be verified

first, in case (u, v) is a regular point of spt S

because by [F, 1.6.6],

$$\det [Dq(u, v) | \operatorname{Tan}(\operatorname{spt} S, (u, v))] > 0,$$

then, in general by 3.6(4) (6).

It follows that $\mathbf{M}\langle S, q, v\rangle = \langle S, q, v\rangle(1)$ is a continuous, positive integer-valued function on V, hence has constant value k for some positive integer k; moreover by 3.6(2) (4)

$$\operatorname{card}(q^{-1}\{v\} \cap \operatorname{spt} S) \leq k \text{ for } v \in V, \quad q_{\#} S = k\mathbf{E}^{2n} | V$$

where we have identified $\mathbf{C}^n$ with $\mathbf{R}^{2n}$.

Next we refer to [F, 5.4.19] to see that $R, R_1, R_2, \ldots$ are area minimizing currents and apply [F, 5.4.2] with $H = (\operatorname{Bdry} U) \times \operatorname{Clos} V$ to conclude that the set $A = \{j\colon H \cap \operatorname{spt} R_j \neq \varnothing\}$ is finite. For integers $j > \sup A$ there exist positive integers k_j such that

$$\mathbf{M}\langle S_j, q, v\rangle = j_j \quad \text{for } v \in V, \ q_{\#} S_j = k_j \mathbf{E}^{2n} | V;$$

moreover since $q | \operatorname{spt} S_j$ is proper for $j > \sup A$ and

$$q_{\#}S_j \to q_{\#}S \text{ as } j \to \infty \text{ in } \{\sup A+1, \sup A+2, \ldots\},$$

we may choose an integer $J > \sup A$ so that $k_j = k$ for all integers $j \geqslant J$.

To complete the proof we fix $v \in V$, $\varepsilon > 0$, abbreviate

$$F = U \cap \{u: (u, v) \in \operatorname{spt} S\}, \quad F_j = U \cap \{u: (u, v) \in \operatorname{spt} S_j\},$$

choose for each $u \in F$ an open convex neighborhood U_u of u such that $\operatorname{Clos} U_u \subset U$,

$$\operatorname{diam} U_u < \inf\{\varepsilon/k, \tfrac{1}{2} \operatorname{distance}(\{u\}, F \sim \{u\})\},$$

and then select a connected open neighborhood Y of v so that $\operatorname{Clos} Y \subset V$ and

$$K = (\operatorname{Clos} U \sim \bigcup_{u \in F} U_u) \times \operatorname{Clos} Y$$

does not intersect $\operatorname{spt} S$. Applying [F, 5.4.2] again, this time with $H = K$, we choose an integer $J^* \geqslant J$ such that for $j \geqslant J^*$ and $u \in F$

$$K \cap \operatorname{spt} S_j = \varnothing, \quad q_{\#}[S_j \llcorner (U_u \times Y)] = q_{\#}[S \llcorner (U_u \times Y)] = \Delta(u, v)(\mathbf{E}^{2n} \llcorner Y),$$

hence $[\langle S_j, q, v\rangle \llcorner (U_u \times Y)](1) = \Delta(u, v)$. For each $j \geqslant J^*$ and $w \in F_j$ we choose that $u \in F$ for which $w \in U_u$ and define the current

$$Q_{j,w} = (\langle S_j, q, v\rangle \llcorner \{(w, v)\})(1)[(u, v), (w, v)] \in \mathbf{I}_1(U \times V)$$

to conclude that

$$\mathbf{M}(\sum_{w \in F_j} Q_{j,w}) \leqslant \sum_{u \in F} \Delta(u, v) \operatorname{diam} U_u \leqslant [\sum_{u \in F} \Delta(u, w)]\, \varepsilon/k = \varepsilon,$$

$$\partial(\sum_{w \in F_j} Q_{j,w}) = \sum_{u \in F} \sum_{w \in F_j \cap U_u} (\langle S_j, q, v\rangle \llcorner \{(w, v)\})(1)[\boldsymbol{\delta}_{(w,v)} - \boldsymbol{\delta}_{(u,v)}]$$

$$= \langle S_j, q, v\rangle - \sum_{u \in F} \Delta(u, v)\, \boldsymbol{\delta}_{(u,v)} = \langle S_j, q, v\rangle - \langle S, q, v\rangle.$$

6.2. *Notations.* Let $\mathbf{U}(m)$ denote the unitary group of all $\mathbf{C}$ linear isometries of $\mathbf{C}^m$ and $\mathbf{u}(m)$ denote the associated Haar measure. We shall use the usual $\mathbf{C}$ base

$$\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_m$$

of $\mathbf{C}^m$ given by $\varepsilon_1 = (1, 0, \ldots, 0)$, $\varepsilon_2 = (0, 1, 0, \ldots, 0)$, $\ldots$, $\varepsilon_m = (0, \ldots, 0, 1)$ and the dual $\mathbf{C}$ base

$$\alpha_1, \alpha_2, \ldots, \alpha_m$$

of $\Lambda^1_{\mathbf{C}}(\mathbf{C}^m, \mathbf{C})$. Whenever $2m \geqslant l \geqslant 0$ are integers, the products

$$\alpha_{\mu,\nu} = \alpha_{\mu(1)} \wedge \ldots \wedge \alpha_{\mu(k)} \wedge \bar{\alpha}_{\nu(1)} \wedge \ldots \wedge \bar{\alpha}_{\nu(l-k)}$$

corresponding to all $k \in \{0, 1, \ldots, l\}$, $\mu \in \Lambda(m, k)$, and $\nu \in \Lambda(m, l-k)$ form a $\mathbf{R}$ base for $\Lambda^l(\mathbf{C}^m, \mathbf{C})$. In case $s \in \{1, 2, \ldots, m\}$ and $\lambda \in \Lambda(m, s)$ we also define

$$\varepsilon_\lambda = \varepsilon_{\lambda(1)} \wedge \mathbf{i}\varepsilon_{\lambda(1)} \wedge \ldots \wedge \varepsilon_{\lambda(s)} \wedge \mathbf{i}\varepsilon_{(\lambda)s} \in \wedge_{2s} \mathbf{C}^m,$$

$$\boldsymbol{\pi}_\lambda : \mathbf{C}^m \to \mathbf{C}^s, \boldsymbol{\pi}_\lambda(w_1, \ldots, w_m) = (w_{\lambda(1)}, \ldots, w_{\lambda(s)}) \text{ for } (w_1, \ldots, w_m) \in \mathbf{C}^m.$$

6.3. Lemma. *If D is a complex s dimensional holomorphic subset of some open subset of $\mathbf{C}^m$ and $0 \in D$, the for $\mathbf{u}(m)$ almost all $g \in \mathbf{U}(m)$ there exists an open ball B about 0 in $\mathbf{C}^m$ such that*

$$B \cap (\boldsymbol{\pi}_\lambda \circ g)^{-1}\{0\} \cap D = \{0\}$$

whenever $\lambda \in \Lambda(m, s)$.

Proof. (Compare [F, 3.2.48]). Let

$$\mathbf{S} = \mathbf{C}^m \cap \{(w_1, \ldots, w_m) : w_1 \bar{w}_1 + \ldots + w_m \bar{w}_m = 1\},$$

fix a point $c \in \mathbf{S}$, and consider the map

$$\Phi : \mathbf{U}(m) \to \mathbf{S}, \ \Phi(g) = g(c) \quad \text{for } g \in \mathbf{U}(m).$$

Recalling [F, 3.2.47] one readily finds a neighborhood W of c in $\mathbf{S}$ along with real analytic isomorphisms

$$\Phi^{-1}[g(W)] \simeq g(W) \times \Phi^{-1}\{c\} \quad \text{for all } g \in \mathbf{U}(m).$$

Consequently, setting $\mu = \dim \mathbf{U}(m) - 2m + 1$,

$$X = \Phi^{-1}[\mathbf{S} \cap \operatorname{Tan}(D, 0)], \quad Y = \Phi^{-1}[\mathbf{S} \cap \bigcup_{\lambda \in \Lambda(m,s)} \boldsymbol{\pi}_\lambda^{-1}\{0\}],$$

and noting that $\dim[\operatorname{Tan}(D, 0) \cap \mathbf{S}] \leqslant 2s - 1$ by [F, 3.4.11], we infer

$$\dim X \leqslant 2s - 1 + \mu, \quad \dim Y \leqslant 2m - 2s - 1 + \mu,$$

hence

$$\dim (X \times Y) \leqslant 2m - 2 + 2\mu.$$

Using the map

$$\Psi : X \times Y \to \mathbf{U}(m), \ \Psi(x, y) = y \circ x^{-1} \text{ for } (x, y) \in X \times Y,$$

we see that, whenever $g \in \mathbf{U}(m)$,

$$\Psi^{-1}\{g\} = \{(x, g \circ x) : x \in \Phi^{-1}[\mathbf{S} \cap \operatorname{Tan}(D, 0) \cap g^{-1}(\bigcup_{\lambda \in \Lambda(m,s)} \boldsymbol{\pi}_\lambda^{-1}\{0\})]\}$$

and apply [F, 2.10.11, 2.7.7] to conclude that for $\mathbf{u}(m)$ almost all $g \in \mathbf{U}(m)$

$$\dim \Psi^{-1}\{g\} \leqslant (2m-2+2\mu)-(\mu+2m-1) = \mu-1,$$

hence
$$\mathbf{S} \cap \operatorname{Tan}(D,0) \cap g^{-1}\Big(\bigcup_{\lambda\in\Lambda(m,s)} \boldsymbol{\pi}_\lambda^{-1}\{0\}\Big) = \varnothing$$

because $\dim \Phi^{-1}\{a\} = \mu$ whenever $a \in \mathbf{S}$. Reference to [F, 3.1.21] completes the proof.

6.4. LEMMA. *If W is an open subset of $\mathbf{C}^m$, s is a positive integer, S is a complex s dimensional holomorphic chain in W, and $\psi \in \mathcal{D}^{2s}(W)$, then*

$$S(\psi) = \sum_{\lambda\in\Lambda(m,s)} \int \langle S, \boldsymbol{\pi}_\lambda | W, z\rangle \langle \varepsilon_\lambda, \psi\rangle \, d\mathcal{L}^{2s} z.$$

Proof. Recalling [F, 1.6.6], we observe that if $\sigma \in \Lambda_{2s}\mathbf{C}^m$ is complex, $\mu \in \Lambda(m,k)$, and $\nu \in \Lambda(m, 2s-k)$, then

$$\langle \sigma, a_{\mu,\nu}\rangle = 0 \quad \text{unless } k = s \text{ and } \mu = \nu.$$

Noting that for $\|S\|$ almost all $x \in W$ the simple $2s$ vector $\mathbf{S}(x)$ is complex and letting Ω be the standard $2s$ form on $\mathbf{C}^s = \mathbf{R}^{2s}$, we infer from [F, 4.1.6, 4.3.2(1)] that

$$S(\psi) = S\Big[\sum_{\varkappa\in\Lambda(2m,2s)} \langle \mathbf{e}_\varkappa, \psi\rangle \wedge (\mathbf{p}_\varkappa | W)^{\#}\Omega\Big] = S\Big[\sum_{\lambda\in\Lambda(m,s)} \langle \varepsilon_\lambda, \psi\rangle \wedge (\boldsymbol{\pi}_\lambda|W)^{\#}\Omega\Big]$$

$$= \sum_{\lambda\in\Lambda(m,s)} [S \llcorner (\boldsymbol{\pi}_\lambda | W)^{\#}\Omega] \langle \varepsilon_\lambda, \psi\rangle = \sum_{\lambda\in\Lambda(m,s)} \int \langle S, \boldsymbol{\pi}_\lambda | W, z\rangle \langle \varepsilon_\lambda, \psi\rangle \, d\mathcal{L}^{2s} z.$$

6.5. THEOREM. *If $f\colon M \to \mathbf{C}^n$* is holomorphic, *$t \geqslant n$, and $\mathcal{J}$ is the set of all positive complex t dimensional holomorphic chains T in M for which*

$$\dim (f^{-1}\{0\} \cap \operatorname{spt} T) \leqslant 2t - 2n,$$

then the function on $\mathcal{J}$ which sends

$$T \text{ to } \langle T, f, 0\rangle$$

is continuous with respect to the topologies of $\mathcal{F}_{2t}^{\mathrm{loc}}(M)$ and $\mathcal{F}_{2t-2n}^{\mathrm{loc}}(M)$.

Proof. By 3.5(3) (4) and 3.2(1) we may assume that *M is an open subset of $\mathbf{C}^m$.*

Suppose that $T_0, T_1, T_2, \ldots$ are elements of $\mathcal{J}$ and that

$$T_j \to T_0 \ \text{ in } \ \mathcal{F}_{2t}^{\mathrm{loc}}(M) \ \text{ as } \ j \to \infty.$$

To show that $\langle T_j, f, 0\rangle$ approaches $\langle T_0, f, 0\rangle$ as j approaches ∞, it suffices by 3.4 and 3.2(1) to prove the following local result:

For every point $x \in M$ there exist an open neighborhood
U of x in M and a positive number I
satisfying the two conditions:

(1) $\mathbf{M}\langle T_j|U, f|U, 0\rangle \leq I$ *for* $j \in \{0, 1, \ldots\}$.
(2) *For each* $\psi \in \mathcal{D}^{2t-2n}(U)$

$$\langle T_j|U, f|U, 0\rangle(\psi) \to \langle T_0|U, f|U, 0\rangle(\psi) \text{ as } j \to \infty.$$

This we prove by considering four cases.

Case 1, $f(x) \neq 0$. Here we take $U = M \sim f^{-1}\{0\}$, hence $\langle T_j|U, f|U, f|U, 0\rangle = 0$ for every $j \in \{0, 1, \ldots\}$.

Case 2, $x \notin \operatorname{spt} T_0$. Here we take any neighborhood U of x such that Clos U is a compact subset of $M \sim \operatorname{spt} T_0$ and apply [F, 5.4.2] with $H = \operatorname{Clos} U$ to infer that

$$A = \{j : (\operatorname{Clos} U) \cap \operatorname{spt} T_j \neq \varnothing\}$$

is finite, hence

$$\langle T_j|U, f|U, 0\rangle = \langle T_0|U, f|U, 0\rangle = 0$$

for $j > \sup A$.

Case 3, $x \in f^{-1}\{0\} \cap \operatorname{spt} T_0$ *and* $t = n$. Here we choose first, an open neighborhood U of x with compact closure in M and

$$(\operatorname{Bdry} U) \cap f^{-1}\{0\} \cap \operatorname{spt} T_0 = \varnothing,$$

then, an open ball V about 0 in $\mathbf{C}^n$ of radius less than

$$\operatorname{distance}[(\operatorname{Bdry} U) \times \{0\}, (\mathbf{1}_M \boxdot f)(\operatorname{spt} T_0)],$$

hence

$$[(\operatorname{Bdry} U) \times (\operatorname{Clos} V)] \cap (\mathbf{1}_M \boxdot f)(\operatorname{spt} T_0) = \varnothing.$$

Letting $p: U \times V \to U$, $q: U \times V \to V$ be the projections and defining the holomorphic chains

$$S_j = [(\mathbf{1}_M \boxdot f)_\# T_j] | (U \times V) \quad \text{for} \quad j \in \{0, 1, \ldots\},$$

we infer statements (1) and (2) from 6.1, 3.(2), and the equation

$$\langle T_j | U, f | U, 0\rangle = p_\# \langle S_j, q, 0\rangle \quad \text{for} \quad j \in \{0, 1, \ldots\}$$

which follows from 4.4.

Case 4, $x \in f^{-1}\{0\} \cap \operatorname{spt} T_0$ *and* $t > n$. Here we assume without loss of generality that

$x=0\in\mathbf{C}^m$, and we apply 6.3 with $D=f^{-1}\{0\}\cap\operatorname{spt} T_0$, $s=t-n$ to choose $g\in\mathbf{U}(m)$ and an open ball B about 0 in M so that

$$B\cap[f\boxdot(\boldsymbol{\pi}_\lambda\circ g)]^{-1}\{(0,0)\}\cap\operatorname{spt} T_0=\{0\}$$

whenever $\lambda\in\Lambda(m,t-n)$. In order to apply 6.4 and 6.1 we choose for each $\lambda\in\Lambda(m,t-n)$ the map $\lambda^*\in\Lambda(m,m-t+n)$ for which $\operatorname{im}\lambda^*=\{1,\ldots,m\}\sim\operatorname{im}\lambda$ and define the two maps.

$$\mathbf{C}^m\times\mathbf{C}^n\xrightarrow{\phi_\lambda}\mathbf{C}^{m-t+n}\times(\mathbf{C}^n\times\mathbf{C}^{t-n})\xrightarrow{\mu_\lambda}\mathbf{C}^m$$

so that $\phi_\lambda(x,y)=(\boldsymbol{\pi}_{\lambda^*}(x),(y,\boldsymbol{\pi}_\lambda(x)))$, $\mu_\lambda\circ\phi_\lambda(x,y)=x$, for $(x,y)\in\mathbf{C}^m\times\mathbf{C}^n$ and consider the holomorphic chains

$$R_{j,\lambda}=(\phi_\lambda\circ[(g|M)\boxdot f])_\# T_j\quad\text{for } j\in\{0,1,\ldots\}.$$

Noting that

$$[\boldsymbol{\pi}_{\lambda^*}(B)\times\{(0,0)\}]\cap\operatorname{spt} R_{0,\lambda}=\{(0,(0,0))\},$$

we choose open neighborhoods U_λ of 0 in $\mathbf{C}^{m-t+n}$, V_λ of $(0,0)$ in $\mathbf{C}^n\times\mathbf{C}^{t-n}$ so that

$$\operatorname{Clos}(U_\lambda\times V_\lambda)\subset\phi_\lambda(B\times\mathbf{C}^n),\quad[(\operatorname{Bdry} U_\lambda)\times(\operatorname{Clos} V_\lambda)]\cap\operatorname{spt} R_{0,\lambda}=\varnothing,$$

we let $p_\lambda\colon U_\lambda\times V_\lambda\to U_\lambda$, $q_\lambda\colon U_\lambda\times V_\lambda\to V_\lambda$ be the projections, and we apply 6.1 with R, R_j, U, V, q replaced by $R_{0,\lambda}$, $R_{j,\lambda}$, U_λ, V_λ, q_λ to find integers I_λ, J_λ such that for every $v\in V_\lambda$

$$\mathbf{M}\langle R_{j,\lambda}|(U_\lambda\times V_\lambda),q_\lambda,v\rangle\leqslant I_\lambda$$

whenever $j\in\{J_\lambda,J_\lambda+1,\ldots\}$ and for every $\psi\in\mathcal{D}^0(U_\lambda\times V_\lambda)$

$$\langle R_{j,\lambda}|(U_\lambda\times V_\lambda),q_\lambda,v\rangle(\psi)\to\langle R_{0,\lambda}|(U_\lambda\times V_\lambda),q_\lambda,v\rangle(\psi)$$

as $j\to\infty$ in $\{J_\lambda,J_\lambda+1,\ldots\}$.

Letting

$$I=\sum_{\lambda\in\Lambda(m,t-n)}I_\lambda\mathcal{L}^{2t-2n}[\boldsymbol{\pi}_\lambda(B)],$$

U be an open neighborhood of 0 in $\mathbf{C}^m$, and V be an open neighborhood of 0 in $\mathbf{C}^n$ such that $Y_\lambda=\phi_\lambda(U\times V)\subset U_\lambda\times V_\lambda$ for every $\lambda\in\Lambda(m,t-n)$, we readily obtain statements (1) and (2) from Lebesgue's bounded convergence theorem and the equation

$$\begin{aligned}\langle T_j|U,f|U,0\rangle(\psi)&=(g|U)_\#\langle T_jU,f|U,0\rangle[(g|U)^{\#-1}\psi]\\&=\langle(g|U)_\#(T_j|U),f\circ g^{-1}|g(U),0\rangle[(g|U)^{\#-1}\psi]\\&=\sum_{\lambda\in\Lambda(m,t-n)}\int\langle(g|U)_\#(T_j|U),[(f\circ g^{-1})\boxdot\boldsymbol{\pi}_\lambda]|g(U),(0,z)\rangle\langle\varepsilon_\lambda,(g|U)^{\#-1}\psi\rangle\,d\mathcal{L}^{2t-2n}z\end{aligned}$$

$$= \sum_{\lambda\in\Lambda(m,t-n)} \int \langle(\mu_\lambda|Y_\lambda)_\#(R_{j,\lambda}|Y_\lambda), [(f\circ g^{-1})\boxdot\pi_\lambda]|g(U), (0,z)\rangle\langle\varepsilon_\lambda, (g|U)^{\#-1}\psi\rangle\, d\mathcal{L}^{2t-2n}z$$

$$= \sum_{\lambda\in\Lambda(m,t-n)} \int (\mu_\lambda|Y_\lambda)_\#\langle R_{j,\lambda}|Y_\lambda, q_\lambda|Y_\lambda, (0,z)\rangle\langle\varepsilon_\lambda, (g|U)^{\#-1}\psi\rangle\, d\mathcal{L}^{2t-2n}z$$

for $j\in\{0, 1, \ldots\}$ and $\psi\in\mathcal{D}^{2t-2n}(U)$ which follows from 4.4, 4.5, and 6.4 applied with $W=g(U)$, $s=t-n$, $S=\langle(g|U)_\#(T_j|U), f\circ g^{-1}|g(U), 0\rangle$. This completes the proof.

6.6. *Example. The real analytic analogue of* 6.5 *is false.* In fact, let S, f, g be as in 4.6 and for each $0\neq\varepsilon\in\mathbf{R}$ let

$$Q_\varepsilon = (\mathbf{E}^1\times\boldsymbol{\delta}_\varepsilon\times\mathbf{E}^1)\llcorner\{(x,\varepsilon,z): x^2\varepsilon^{-2}+z^2<1\}.$$

Then by 3.5(2)

$$\langle S, g, \varepsilon\rangle = \partial Q_\varepsilon, \quad \mathbf{M}(Q_\varepsilon) = \pi|\varepsilon|,$$

$$\langle\langle S, g, \varepsilon\rangle, f, 0\rangle = \boldsymbol{\delta}_{(0,\varepsilon,1)} - \boldsymbol{\delta}_{(0,\varepsilon,-1)},$$

hence

$$\lim_{\varepsilon\to 0}\langle\langle S, g, \varepsilon\rangle, f, 0\rangle = \boldsymbol{\delta}_{(0,0,1)} - \boldsymbol{\delta}_{(0,0,-1)} \neq 0 = \langle 0, f, 0\rangle = \langle\lim_{\varepsilon\to 0}\langle S, g, \varepsilon\rangle, f, 0\rangle,$$

even though $\partial\langle S, g, y\rangle = 0$ and

$$\dim(f^{-1}\{0\}\cap\operatorname{spt}\langle S, g, y\rangle)\leq 0$$

for all $y\in\mathbf{R}$.

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