CONCORDANCE CLASSES OF REGULAR $O(n)$-ACTIONS ON HOMOTOPY SPHERES

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0. Introduction

A basic approach in the study of transformation groups is to compare smooth actions of compact Lie groups on homotopy spheres with linear actions on standard spheres. This paper examines actions of the orthogonal group, $O(n)$, on homotopy spheres. We consider only those actions which resemble certain fixed linear actions insofar as their isotropy groups and normal representations are concerned. We are then able to classify such actions, up to concordance, by comparing them directly, via an equivariant map, with their linear

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counterpart. The linear actions that we use as models are $k\varrho_n + \underline{m}$, where $k\varrho_n$ denotes the standard action of $O(n)$ on k-tuples of vectors in $\mathbf{R}^n$ and $\underline{m}$ denotes the trivial m-dimensional representation. A smooth action of $O(n)$ on a manifold M is *regular* if its orbit types and normal representations occur among those of $k\varrho_n$. In this situation, we shall also say that the $O(n)$-action on M is *k-axial*. Any isotropy group of a k-axial action is conjugate to a standardly embedded $O(n-i)$, for some i, $0 \leqslant i \leqslant \min(n, k)$. (We shall usually be assuming that $n \geqslant k$.) Thus, a k-axial action has at most $k+1$ different orbit types, and they are linearly ordered. We shall denote by ${}_iM$ the submanifold which is fixed by the subgroup $O(n-i) \subset O(n)$.

Given an $O(n)$-action on a homotopy sphere, there are simple conditions which imply that it is regular. For example, if the principal orbit type is $O(n)/O(n-k)$, with $n>k$, then the action is k-axial, [16]. A similar result holds for an $O(n)$-action on an h-cobordism between two homotopy spheres.

A k-axial $O(n)$-action on a homotopy sphere Σ must resemble a linear model more closely than is obvious, *a priori*. For example, it follows from the theory of P. A. Smith that ${}_i\Sigma$ is a homology sphere, where the coefficients are taken to be $\mathbf{Z}$ if $(n-i)$ is even or $\mathbf{Z}/2$ if $(n-i)$ is odd. Also, if $\dim({}_0\Sigma) = m-1$ (the empty set has dimension -1), then it follows from a formula of A. Borel, that $\dim({}_i\Sigma) = (ki+m-1)$, for all i with $0 \leqslant i \leqslant n$. Thus, Σ $(={}_n\Sigma)$ has dimension $(kn+m-1)$, and the fixed point sets of the various isotropy groups are homology spheres of the same dimension as the corresponding fixed sub-spheres in the linear action $k\varrho_n + \underline{m}$ restricted to S^{kn+m-1}.

Two regular $O(n)$-manifolds M and M' are *concordant* if there is a regular $O(n)$-action on a h-cobordism W, such that its restriction to ∂W is (oriented) equivalent to $M \coprod (-M')$. Let $\Theta^1(k, n, m)$ denote the set of concordance classes of k-axial $O(n)$-actions on homotopy spheres(1) of dimension $(kn+m-1)$. For $m>0$, it is an abelian group under equivariant connected sum. Our goal is to compute this group (for $n \geqslant k$). The first two authors carried out a similar program for regular $U(n)$- or $Sp(n)$-actions in [11].

Suppose that $O(n)$ acts k-axially on a homotopy sphere Σ^{kn+m-1} and that $n \geqslant k$. In Theorem 5.2, we construct a certain parallelizable manifold V^{kn+m}, with k-axial $O(n)$-action and with $\partial V = \Sigma$. We consider the submanifolds ${}_iV$, $0 \leqslant i \leqslant n$. The boundary of ${}_iV$ is ${}_i\Sigma$, and as we remarked above, ${}_i\Sigma$ is an R_ε-homology sphere, where $\varepsilon = (-1)^{n-i}$, $R_+ = \mathbf{Z}$, and $R_- = \mathbf{Z}_{(2)}$. Roughly speaking, our main result is that the concordance class of Σ is completely determined by the intersection and self-intersection forms of ${}_0V$ and ${}_1V$. More precisely, if $p = ki + m = \dim({}_iV)$, then we define an invariant $\sigma_i (= \sigma_i(\Sigma))$ in the surgery

(1) All our results remain valid for homology h-cobordism classes of regular $O(n)$-actions on integral homology spheres.

group $L_p(R_\varepsilon)$, as follows. If p is odd, then $\sigma_i=0$. If $p\equiv 2(4)$, then σ_i is the Arf-Kervaire invariant associated to a quadratic form on the middle dimensional homology of ${}_iV$ with $\mathbf{Z}/2$ coefficients. If $p\equiv 0(4)$, then σ_i is the Witt class of the intersection form on the torsion-free part of the middle dimensional homology of ${}_iV$. In this case, if $\varepsilon=+1$, then σ_i can be identified with one-eighth of the index of the bilinear form (this is an integer); while, if $\varepsilon=-1$, then σ_i takes values in W, the Witt group of symmetric, bilinear forms which are even and non-singular over $\mathbf{Z}_{(2)}$. Eventually the following facts will be established:

(1) σ_i depends only on the concordance class of Σ and hence, defines a map $\sigma_i\colon \Theta^1(k, n, m)\to L_p(R_\varepsilon)$, which is a homomorphism for $m>0$.

(2) If k is odd, then $\sigma_i=0$.

(3) If k is even, then $\sigma_i=\sigma_{i+2}$.(1)

(4) If k is even, then $c(\sigma_i)=c(\sigma_{i+1})$, where $c\colon L_*(R_\varepsilon)\to\mathbf{Z}/2$ is the Arf-Kervaire homomorphism.

(5) For $m>0$, the σ_i's can assume any possible value subject to the relations (2), (3) and (4).

(6) If $\sigma_0(\Sigma)=0=\sigma_1(\Sigma)$, then (provided $k\neq 2$ and neither ${}_0V$ nor ${}_1V$ has dimension 4), Σ is concordant to a sphere with linear action.

Thus, for k odd, $m\neq 4$ and $(k, m)\neq(3, 1)$, every Σ^{kn+m-1} is concordant to a sphere with linear action(2); while, for k even, $k\neq 2$ and $m\neq 0, 4$, the following sequence is exact:

$$0\longrightarrow\Theta^1(k,n,m)\xrightarrow{(\sigma_0,\,\sigma_1)} L_m(R_\varepsilon)\oplus L_{m+k}(R_{-\varepsilon})\xrightarrow{c+c}\mathbf{Z}/2.$$

The result for $k=2$ is slightly different. In this case, we cannot use merely σ_0 and σ_1. It is necessary to take algebraic refinements of them (linking forms on Siefert surfaces). Thus, in this case the result is that the enhanced σ_0 and σ_1 determine the concordance class of Σ^{2n+m-1} and tie the groups $\Theta^1(2, n, m)$ to knot cobordism groups.

When $m=0$, a result similar to the generic one holds. It is necessary, however, to reinterpret σ_0 as the number of fixed points of the action on V (counted with sign). If k is odd, every action is concordant to a linear one. If k is even and $n\equiv 0(2)$, then

$$\Theta^1(k,n,0)\xrightarrow{(\sigma_0,\,\sigma_1)}(\{\pm 1\}, L_k(\mathbf{Z}_{(2)}))$$

is injective (if $k\neq 4$). Its image is all pairs $(\pm 1, \sigma_1)$ such that the Kervaire invariant of σ_1, $c(\sigma_1)$, is zero. If k is even and n is odd, then

(1) For k even, σ_i and σ_{i+2} take values in the same surgery group.

(2) The case $(k, m)=(1, 3)$ follows from other considerations.

$$\Theta^1(k, n, 0) \xrightarrow{(\sigma_0, \sigma_1)} (\text{odd integers}, L_k(\mathbf{Z}))$$

is injective (if $k \neq 4$). Its image is all pairs (d, σ_1) such that $c(d) = c(\sigma_1)$. (Here $c(d)$ means the Kervaire invariant of the normal map of d points to 1 point, i.e., $c(d) \neq 0$ if and only if $d \equiv \pm 3(8)$.)

These results lead to a calculation of the groups $\Theta^1(k, n, m)$ in all but a few exceptional cases. We tabulate the groups in the next three theorems.

THEOREM. *Suppose that $n \geqslant k$ and that $m \neq 0, 4$.*

(a) *If $k \equiv 0(4)$, then*

$$\Theta^1(k, n, m) = \begin{cases} \mathbf{Z} + \overline{W}; & m \equiv 0(4) \\ \mathbf{Z}/2; & m \equiv 2(4) \\ 0; & m \equiv 1(2). \end{cases}$$

($\overline{W} = \text{kernel } (c\colon W \to \mathbf{Z}/2)$.)

(b) *If k is odd and $(k, m) \neq (3, 1)$, then $\Theta^1(k, n, m) = 0$.*

(c) *If $k \equiv 2(4)$ and $k \neq 2$, then*

$$\Theta^1(k, n, m) = \begin{cases} \mathbf{Z}; & m + 2n \equiv 0(4) \\ W; & m + 2n \equiv 2(4) \\ 0; & m + 2n \equiv 1(2). \end{cases}$$

THEOREM. *Suppose that $m \neq 0, 2, 4$ and that $n \geqslant 2$.*

$$\Theta^1(2, n, m) = \begin{cases} G_+; & m + 2n \equiv 0(4) \\ G_-; & m + 2n \equiv 2(4) \\ 0; & m + 2n \equiv 1(2). \end{cases}$$

(*The groups $G_\pm$ are the "algebraic knot cobordism groups".*)

THEOREM. *Suppose that $k \neq 4$ and that $n \geqslant k$.*

$$\Theta^1(k, n, 0) = \begin{cases} \{\pm 1\}; & k \text{ odd} \\ \{\pm 1\}; & n \text{ even}, k \equiv 2(4) \\ \{\pm 1\} \times \overline{W}; & n \text{ even}, k \equiv 0(4) \\ \Omega; & n \text{ odd}, k \equiv 2(4) \\ \ker\,(c + c) \subset \Omega \times W; & n \text{ odd}, k \equiv 0(4). \end{cases}$$

(*Here Ω is the odd integers.*)

The case $k \neq 2$ and $m \neq 0$ is proved in Section 12. The case $k=2$ is dealt with in Section 13, and the case $m=0$ in Section 14.

Actually, in the case of mono-axial actions ($k=1$), these results have been known for at least fifteen years, see [17] and [27]. The case of bi-axial actions ($k=2$), perhaps the most interesting, has been studied extensively, see [4], [5], [6], [14], [15], and [18]. In this case, for n even, the above results are due to Bredon [6].

Let $S\colon \Theta^1(k, n, m) \to \Theta_{kn+m-1}$ be the natural map. As we have seen, the image of S is contained in bP_{kn+m}, the subgroup consisting of h-cobordism classes of homotopy spheres which bound parallelizable manifolds. From (2), (3), and (4) above we immediately deduce the following:

(A) If k is odd or if m is odd, then Σ^{kn+m-1} is h-cobordant to the standard sphere.

(B) If k is even and m is even, then the following diagram commutes whenever $i \equiv n(2)$:

$$\begin{tikzcd} \Theta^1(k, n, m) \arrow[rr, "S"] \arrow[dr, "\sigma_i"'] & & bP_{kn+m} \\ & L_{ki+m}(\mathbf{Z}) = L_{kn+m}(\mathbf{Z}) \arrow[ur, "b"'] & \end{tikzcd}$$

Thus, if n is even, the h-cobordism class of Σ is determined either by the index or Arf-Kervaire invariant of ${}_0V$; while if n is odd, it is determined either by the index or Arf-Kervaire invariant of ${}_1V$. It follows, from (5), that S is onto bP_{kn+m} provided $m>0$.(1) (If $m=0$ and n is odd, then S is again onto; while if n is even, S is the zero map.)

An interesting corollary of the above calculations is that the homomorphism $\omega_*\colon \Theta^1(k, n+1, m) \to \Theta^1(k, n, m+k)$, defined by restricting the $O(n+1)$-action to $O(n)$, is an isomorphism (under mild hypotheses on n, m, and k).

The first nine sections contain preliminary material about regular actions. The main point of introducing this material is to reduce the concordance question on homotopy spheres to questions in surgery theory. In the remaining six sections these questions are answered and consequences are derived.

Any smooth G-manifold is stratified by the submanifolds consisting of those orbits of a given type (or "normal type"). This stratification projects to one for the orbit space. If M is a k-axial $O(n)$-manifold, with $n \geqslant k$, then the strata can be indexed by $\{i \in \mathbf{Z} \mid 0 \leqslant i \leqslant k\}$; M_i denotes the stratum of orbits of type $O(n)/O(n-i)$.

The reduction of the concordance question to surgery is accomplished as follows.

(1) This can be seen directly by considering actions on Brieskorn varieties.

First, it is shown that there is an equivariant "stratified" map $F: (V^{kn+m}, \Sigma^{kn+m-1}) \to (D^{kn+m}, S^{kn+m-1})$, where $O(n)$ acts linearly on (D^{kn+m}, S^{kn+m-1}) via $k\varrho_n + \underline{m}$. If $m>0$, then we may also assume that F is a degree one normal map. The proof of this result is explained in Section 8. Let A, B, K and L denote the orbit spaces of V, Σ, D and S, respectively, and let $f: (A, B) \to (K, L)$ be the induced map of orbit spaces. Necessary and sufficient conditions are given for $F: (V, \Sigma) \to (D, S)$ to induce an isomorphism on integral homology. These conditions are stated in terms of the induced maps $f|A_i: (A_i, B_i) \to (K_i, L_i)$ on each stratum. One condition is that, for each i, $f|A_i$ must induce an isomorphism on homology with coefficients in $\mathbf{Z}/2$. The other condition involves the homology with coefficients in $\mathbf{Z}$ of the "double branched cover of $A_i \cup A_{i-1}$ along A_{i-1}". The precise result is stated as Theorem 7.1.

Our program, then, is to successively successfully do surgery on the $f|A_i$, relative to $f|B_i$, to achieve these homology conditions. If this is done (and if the top stratum of A is made simply connected), then we will have replaced V by a contractible $O(n)$-manifold. Hence, Σ^{kn+m-1} will be concordant to S^{kn+m-1} with the linear action. *A priori*, there may be an obstruction to surgery on each stratum. It will be proved, however, that most of these obstructions either vanish or are indeterminant (i.e. can be made to vanish by appropriate choice of surgery on the lower strata). This is the case for all the obstructions when k is odd, and is the case for all but the obstructions at levels 0 and 1 when k is even. The obstructions at levels 0 and 1 are identified with σ_0 and σ_1.

As stated above, this program is very close to what was done in [11] for regular $U(n)$- and $Sp(n)$-actions. For such actions, the strata of the orbit space of the linear model are simply connected; and at each stage we are required to do surgery to an integral homology isomorphism. The fact that the surgery obstruction on each stratum (except for the bottom one) either vanishes or is indeterminant essentially follows from well-known product formulae in the surgery theory of simply connected manifolds. Thus, for regular $U(n)$- and $Sp(n)$-actions the necessary results in surgery are completely straightforward.

For regular $O(n)$-actions the situation is more complicated because:

(1) the strata of the linear orbit space usually have fundamental group $\mathbf{Z}/2$(1),

(2) the strata alternate between being orientable and non-orientable,

(3) in the fiber bundle relating one stratum to the boundary of the next the fundamental group of the base can act non-trivially on the homology of the fiber, and

(4) we are required to do surgery to achieve a mixture of $\mathbf{Z}$- and $\mathbf{Z}/2$-conditions on homology.

(1) The case $k=2$ is distinguished by the fact that the 1-stratum has fundamental group $\mathbf{Z}$.

The process by which almost all of the surgery obstructions "cancel" must, therefore, be more sophisticated than in the $U(n)$ and $Sp(n)$ cases. This cancellation process is based on three different product formulae, which are proved in section 15. The first, 15.1, concerns $\mathbf{CP}^{2l}$-bundles where the fundamental group of the base acts non-trivially on $H_*(\mathbf{CP}^{2l})$. The second, 15.3, concerns $\mathbf{RP}^{2l}$-bundles. The third, 15.5, concerns $\mathbf{RP}^{2l+1}$-bundles. In all cases we have a normal map between the total spaces of such bundles which covers a normal map between the bases. The product formula relates the surgery obstructions on base and total space.

Our calculation of the concordance groups is not quite complete. The case $m=4$ leads to four dimensional surgery, and the group of concordance classes must be enlarged by a group associated with $\Theta_3^{\mathbf{Z}}$ or $\Theta_3^{\mathbf{Z}/2}$. (Here Θ_3^R means the group of R-homology 3-spheres with those that bound R-homology disks set equal to zero.) The case $(k, m)=(2, 2)$ is intimately related to classical knot cobordism. Thus, the classification of concordance classes of regular actions in these cases depends on the solution of these outstanding low dimensional surgery problems.

Finally, it should be emphasized that the two "ends" of a concordance need not be equivariantly diffeomorphic. However, our classification of $O(n)$-actions up to concordance does clarify what problems occur in understanding the equivariant diffeomorphism question. One might hope, naively, for the orbit space of a concordance to be equivalent (as a stratified space) to the product of one end with the unit interval. If this happens, then, of course, the two ends are equivariantly diffeomorphic. However, all one can say in general, is that each stratum of the orbit space of a concordance is a $\mathbf{Z}/2$-homology h-cobordism between its two ends. Thus, for example the integral homology of its ends may be different. Also, the fundamental group of such a cobordism may be different from that of either end. In general, such discrepancies in fundamental group and integral homology occur. Thus, the classification of regular $O(n)$-actions on homotopy spheres up to equivariant diffeomorphism would seem to involve difficult questions concerning $\mathbf{Z}/2$- homology h-cobordisms.

1. Stratification by normal orbit type

In this section, we review some general definitions from [10] and [33].

Let G be a compact Lie group. Consider pairs (H, V), where H is a closed subgroup and V is an H-module with no invariant non-zero vectors. Two such pairs (H, V) and (H', V') are *equivalent* if the corresponding G-vector bundles $G\times_H V$ and $G\times_{H'} V'$ are isomorphic. (This just means that H and H' are conjugate and that there is a compatible linear isomorphism from V to V'.) A resulting equivalence class is called a *normal orbit type*.

Now, suppose that G acts smoothly on a manifold M. Let B be the orbit space and $\pi: M \to B$ the natural projection. For $x \in M$, G_x is the isotropy group and S_x is the slice representation. The *normal representation* N_x is the G_x-module S_x/F_x, where $F_x \subset S_x$ is the subspace which is fixed by G_x. The *normal orbit type of* x is the equivalence class of (G_x, N_x). A *stratum of* M is the set of points of a given normal orbit type and a *stratum of* B is the image of a stratum of M. If α is a normal orbit type, then $\dot{M}_\alpha$ and $\dot{B}_\alpha$ denote the corresponding strata. It follows from the Differentiable Slice Theorem that $\dot{M}_\alpha$ and $\dot{B}_\alpha$ are both smooth manifolds and that $\pi|\dot{M}_\alpha: \dot{M}_\alpha \to \dot{B}_\alpha$ is the projection map of a smooth fiber bundle (the fibers are orbits). N_x is the fiber at x of the normal bundle of $\dot{M}_\alpha$ in M.

If α' and α are normal orbit types, then $\alpha' \leqslant \alpha$ if α occurs as a normal orbit type in $G \times_H V$, where (H, V) is a representative for α'. This defines a partial ordering on the set of normal orbit types. Clearly,

$$\text{closure}\,(\dot{M}_\alpha) = \bigcup_{\beta \leqslant \alpha} \dot{M}_\beta.$$

In [10] and [19] it is shown how to attach, in a canonical fashion, a boundary to each stratum of M (or of B) obtaining a manifold with corners called a "closed stratum". The method is based on the following construction.

Suppose that M is a differentiable manifold with corners and that $A \subset M$ is a proper submanifold with corners. (Here proper means that A has a smooth tubular neighborhood in M which is smoothly isomorphic to the total space of a vector bundle over A, the normal bundle of A in M, $\nu_{A\subset M}$.) Define $\hat{M}_A$, M "blown up" along A, to be $(M-A) \cup S\nu_{A\subset M}$, where $S\nu_{A\subset M}$ is the sphere bundle associated to the normal bundle. $\hat{M}_A$ naturally inherits the structure of a smooth manifold with corners. If W has a smooth G-action and A is invariant, then $\hat{M}_A$ has a natural smooth G-action.

If A is a minimal stratum of M, then it is a proper invariant submanifold, so $\hat{M}_A$ is a G-manifold with one less stratum. One can continue by blowing up a minimal stratum of $\hat{M}_A$, etc. To construct the *closed α-stratum of* M, one blows up all the strata of index less than α and then takes the α-stratum of the resulting manifold with corners. The result is denoted by M_α. It is a manifold with corners with interior equal to the original stratum. A *closed stratum of* B is the orbit space of a closed stratum of M.

Let N_α denote the normal bundle of M_α in the appropriate blow-up of M. If $\beta > \alpha$, then let $\partial_\alpha M_\beta$ be the closure (in M_β) of the β-stratum of the sphere bundle associated to N_α. We define $\partial_\alpha B_\beta$ similarly. If $X = \partial M$, then

$$\partial(M_\beta) = X_\beta \cup \bigcup_{\alpha<\beta} \partial_\alpha M_\beta$$

and

$$\partial(B_\beta) = \pi(X_\beta) \cup \bigcup_{\alpha<\beta} \partial_\alpha B_\beta.$$

Suppose that (H, V) is a representative for α and that $Y = G \times_H V$. The projection $\partial_\alpha M_\beta \to B_\alpha$ is a smooth fiber bundle with fiber $\partial_\alpha Y_\beta$, and $\partial_\alpha B_\beta \to B_\alpha$ is a fiber bundle with fiber the orbit space of $\partial_\alpha Y_\beta$.

A smooth equivariant map $h: M \to M'$ of G-manifolds is *stratified at* x if $G_x = G_{h(x)}$ and if the differential of h at x induces an isomorphism $N_x \cong N_{h(x)}$ An equivariant map is *stratified* if it is stratified at each point. If an equivariant map h is stratified, then $h(\dot{M}_\alpha) \subset \dot{M}'_\alpha$ and the differential of h induces an equivariant linear bundle map from the normal bundle of $\dot{M}_\alpha$ in M to the normal bundle of $\dot{M}'_\alpha$ in M'. A key observation is that *the restriction of an equivariant stratified map to any given stratum extends to a map between the corresponding closed strata. Moreover, this extension is a bundle map on each face.* (This is proved in [10].) There is a similar notion of a "stratified map" between two orbit spaces. In order to define this notion, it is first necessary to discuss the local structure of orbit spaces.

The orbit space B has an induced "smooth structure" obtained by defining a function $g: U \to \mathbf{R}$ (U an open subset of B) to be *smooth* if $g \circ \pi$ is smooth. A continuous map $\varphi: B \to B'$ is *smooth* if it pulls back smooth functions on open sets in B' to smooth functions on open sets in B (see [4], [9], [32]). From the ring of germs of smooth functions which vanish at $b \in B$, one can define (d'après Zariski) the cotangent space at b and its dual, the tangent space $T_b B$. Let TB denote the union of all the tangent spaces. By the Slice Theorem, $b = \pi(x)$ has a neighborhood in B which is smoothly isomorphic to S_x/G_x. It follows from a result of G. Schwarz [32], that the linear orbit space S_x/G_x can be identified with a certain semialgebraic subset of some Euclidean space $\mathbf{R}^s$. This defines an embedding of $T(S_x/G_x)$ into $T\mathbf{R}^s$ which is linear on each tangent plane. It induces a topology on $T(S_x/G_x)$ and thus one for TB. In general, $TB \to B$ is *not* a vector bundle since the dimension of $T_b B$ need not be locally constant. However, the restriction of TB to any stratum is a vector bundle, and the ordinary tangent bundle of the stratum is a sub-bundle. The quotient of $TB \mid \dot{B}_\alpha$ by TB_α is called the *normal bundle of* B_α *in* B and is denoted by $\dot{\nu}_\alpha(B)$.

A smooth map $f: B \to B'$ of orbit spaces is *stratified*(1) if it preserves the stratification and if for each index α, the induced map $f_*: \dot{\nu}_\alpha(B) \to \dot{\nu}_\alpha(B')$ is a bundle map (that is, a fiberwise linear isomorphism).

(1) Such maps are called "weakly stratified" in [10] and "normally transverse" in [33].

2. Regular actions

Suppose that M and X are smooth G-manifolds. We say that M *is modeled on* X if the normal orbit types of G on M occur among those of G on X. Equivalently, M is modeled on X if given non-negative integers m and m' such that $m-m'=\dim X-\dim M$, then every orbit of $M\times\mathbf{R}^m$ has an open invariant neighborhood isomorphic to an open invariant neighborhood in $X\times\mathbf{R}^{m'}$. (Here, G acts trivially on the second factors.)

If one is interested in smooth actions on spheres or on disks, then it is natural to study actions which are modeled on various linear actions. The linear action

$$k\varrho_n: O(n)\times M(n, k)\to M(n, k)$$

is defined as the action of $O(n)$ on the vector space of n by k matrices by matrix multiplication on the left. Alternatively, it is the natural action of $O(n)$ on k-tuples of vectors in $\mathbf{R}^n$.

Definition 2.1. A smooth $O(n)$-manifold M is *k-axial* if it is modeled on $M(n, k)$. We shall also say that M is a *regular* $O(n)$-manifold.

Remark. Taking as linear models either the natural action of $U(n)$ or k-tuples of vectors in $\mathbf{C}^n$ or of $Sp(n)$ on k-tuples of vectors in quaternionic n-space, one defines k-axial $U(n)$-manifolds and k-axial $Sp(n)$-manifolds in a similar fashion.

Given a matrix $x\in M(n, k)$, the column vectors span a subspace $P\subset\mathbf{R}^n$. The isotropy group G_x is the orthogonal group $O(P^\perp)$. The row vectors of x span a subspace $Q\subset\mathbf{R}^k$. One can show that the normal representation at x is the natural action of $O(P^\perp)$ on $\operatorname{Hom}(Q^\perp, P^\perp)$. It follows that the normal representation at x is equivalent to $O(n-i)$ acting on $M(n-i, k-i)$ for some i. Let i denote the equivalence class of $(O(n-i), M(n-i, k-i))$. As we have just seen the strata of a k-axial $O(n)$-manifold are indexed by integers i, such that $0\leqslant i\leqslant \min(n, k)$. The i-stratum of $M(n, k)$ is the set of matrices of rank i.

Next we consider the orbit spaces of the linear models. Let $S(k)$ be the vector space of k by k symmetric matrices and let $B(k)\subset S(k)$ be the subset of positive semidefinite matrices. Consider the polynomial mapping π: $M(n, k)\to S(k)$ defined by $\pi(x)={}^tx\cdot x$, where ${}^tx$ is the transpose of x. If $g\in O(n)$, then $\pi(gx)={}^tx\cdot g^{-1}\cdot g\cdot x=\pi(x)$. Consequently, π is constant on orbits and therefore, induces a map $\bar\pi$: $M(n, k)/O(n)\to S(k)$. It is straightforward to check the following:

(a) The image of π is contained in $B(k)$.
(b) π maps the i-stratum of $M(n, k)$ onto the set of matrices in $B(k)$ of rank i.
(c) In particular, if $n\geqslant k$, then π maps $M(n, k)$ onto $B(k)$.

LEMMA 2.2. *The map $\bar{\pi}$: $M(n, k)/O(n) \to B(k)$ is a smooth isomorphism onto its image.*

Proof. The entries of $\pi(x)$ are homogeneous quadratic polynomials in the entries of x. According to [35] these polynomials generate the ring of $O(n)$-invariant polynomials on $M(n, k)$. Under this hypothesis, the lemma becomes a special case of the main result in [32].

Henceforth, we identify the orbit space of $M(n, k)$ with its image in $B(k)$ and the orbit map with π. In particular, if $n \geqslant k$, then $M(n, k)/O(n)$ is identified with $B(k)$. Let $\dot{B}_i(k)$ denote the i-stratum of $B(k)$. In view of (b), $\dot{B}_i(k)$ is the space of positive semi-definite matrices of rank i.

Facts about $B(k)$ immediately translate into local information about orbit spaces of regular actions. We make a few observations.

(1) $B(k)$ is a convex cone with non-empty interior in $S(k)$.
(2) $B(k)$ is homeomorphic to Euclidean half-space of dimension $\frac{1}{2}k(k+1)$.
(3) $T(B(k))$ is identified with $B(k) \times S(k)$.

We leave the verification of this to the reader. As a consequence we have the following.

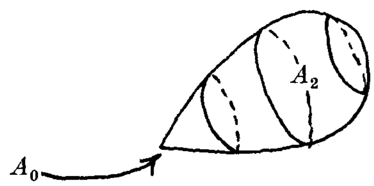
LEMMA 2.3. *Suppose that B is the orbit space of a k-axial $O(n)$-action and that $n \geqslant k$. Then B is homeomorphic to a manifold with boundary (the boundary being the union of the singular strata). Moreover, $TB = \bigcup T_b B$ is a (locally trivial) vector bundle over B.*

It should be emphasized that B is *not* smoothly isomorphic to a smooth manifold with boundary; rather the singular strata have neighborhoods which are differentiably modeled on neighborhoods of the singular strata in $B(k)$.

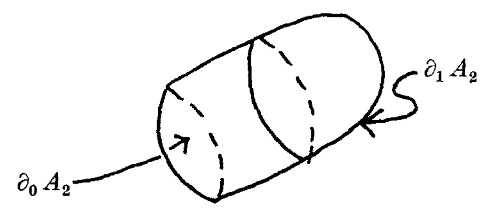
Example 2.4. Suppose that

$$\begin{pmatrix} x & z \\ z & y \end{pmatrix}$$

represents a matrix in $S(2) \cong \mathbf{R}^3$. Then $B(2) = \{(x, y, z) \mid x \geqslant 0, y \geqslant 0, xy - z^2 \geqslant 0\}$ is a solid three dimensional cone. Consider the orbit space A of a *bi*-axial $O(n)$-action on a closed $(2n+m)$-manifold, where $n \geqslant 2$. Then A is locally isomorphic to $B(2) \times \mathbf{R}^n$. As a space, A is an $(m+3)$-manifold with boundary. It has three strata. The image of the principal orbits, $\dot{A}_2$, is the interior of A. The fixed point set $\dot{A}_0 = A_0$ is a closed m-manifold embedded in ∂A, and $\dot{A}_1 = \partial A - A_0$. Away from A_0, A is a smooth manifold with boundary; however, there is a differentiable singularity along A_0.



Picture of A



Picture of A_2

LEMMA 2.5. *$\dot{B}_i(k)$ is a fiber bundle over the Grassmannian of i-planes in k-space with the fiber over a plane P being the space of positive definite forms on P. Thus,*

$$\dot{B}_i(k) = \dot{B}_i(i) \times_{O(i)} [O(k)/O(k-i)]$$

where $O(i)$ acts naturally on $O(k)/O(k-i)$ on the left and on $\dot{B}_i(i)$ by conjugation. A similar formula holds for the closed stratum.

Proof. The lemma states that a positive semi-definite form $z \in \dot{B}_i(k)$ is determined by the following data:

(1) an i-plane in $\mathbf{R}^k$, and

(2) a positive definite form on the i-plane.

Let R_z denote the radical of z. The i-plane is $(R_z)^\perp$. The form $z\,|\,(R_z)^\perp$ is positive definite.

COROLLARY 2.6. *If $i=0$ or $i=k$, then $\pi_1(B_i(k))=0$. If $i=1$ and $k=2$, then $\pi_1(B_i(k)) \cong \mathbf{Z}$. In all other cases $\pi_1(B_i(k)) = \mathbf{Z}/2$.*

COROLLARY 2.7. *As a special case of 2.5 we have $B_1(k) = [0, \infty) \times \mathbf{RP}^{k-1}$. Thus $\partial_0 B_1(k) = \mathbf{RP}^{k-1}$. This is important because for any k-axial $O(n)$ manifold with quotient A, $\partial_i A_{i+1} \to A_i$ is a fiber bundle with fiber $\partial_0 B_1(k-i) \cong \mathbf{RP}^{k-i-1}$.*

3. Double branched covers

As before, suppose that $O(n)$ acts k-axially on a manifold M with orbit space B. In this section we describe a way of functorially associating to M, for each integer i with $0 < i \leqslant \min\{n, k+1\}$, a smooth involution γ on a manifold with corners $E_i = E_i(M)$. In fact, $E_i(M)$ is the double branched cover of $B_i \cup B_{i-1}$ along B_{i-1}, and γ is natural involution on the cover. More explicitly, we have the following:

(i) The fixed point set of γ on E_i is B_{i-1}.

(ii) If $\tilde{B}_i$ is E_i blown up along B_{i-1}, then $\tilde{B}_i/\gamma \cong B_i$.

(iii) $E_i/\gamma \cong B_i \cup_p B_{i-1}$, where $p\colon \partial_{i-1} B_i \to B_{i-1}$ is the canonical projection.

These manifolds will play an important role in our study of regular $O(n)$-actions. Here is the construction. Let $O(i)\times O(n-i)\subset O(n)$ be the standard embedding. Denote the fixed point set of $O(n-i)$ by[1]

$$_iM = M^{O(n-i)}.$$

Then $O(i)$ acts smoothly and k-axially on $_iM$. The strata of $_iM$ have index less than or equal to min (i, k). Blow up the strata of $_iM$ of index less than $i-1$ to obtain a manifold with corners $_i\hat{M}$. If $i\leqslant k$, then $_i\hat{M}$ has only two non-empty strata (i and $i-1$). If $i=k+1$, then only the k-stratum is non-empty; while if $i>k+1$, $_i\hat{M}$ is empty. Define

$$E_i(M) = {_i\hat{M}}/SO(i).$$

Notice that $O(i)$ acts freely on the i-stratum of $_i\hat{M}$. On the $(i-1)$-stratum the isotropy group is conjugate to $O(1)$. Since $SO(i)\cap O(1)=\{1\}$, the action of $SO(i)$ on $_i\hat{M}$ is free. It follows that the orbit space E_i is a smooth manifold with corners. It has "faces" $\partial_0 E_i$, $\partial_1 E_i$, ..., $\partial_{i-2}E_i$, where $\partial_j E_i$ denotes the fiber bundle over B_j, which arises from applying this construction to the normal sphere bundle of M_j.

The group $O(i)/SO(i)\cong\mathbf{Z}/2$ acts smoothly on E_i and the correspondence $M\to E_i(M)$ is clearly a functor from the category of k-axial $O(n)$-manifolds and equivariant, stratifed maps to the category of involutions on manifolds with corners, and equivariant stratified maps. Moreover, if $F\colon M\to M'$ is stratified, then the restriction of $E_i(F)$ to any face is a bundle map. One verifies routinely that $E_i/\gamma=B_i\cup_p B_{i-1}$, and hence, that E_i is the double branched cover of $B_i\cup_p B_{i-1}$ along B_i.

Let $E_i(k)$ denote the result of this construction applied to the linear model, that is, $E_i(k)=E_i(M(n, k))$.

Example 3.1. $E_0(k)=B_0(k)$ is a point. $E_1(k)=\mathbf{R}^k$, and the involution is $x\mapsto -x$. To obtain $E_2(k)$, one first blows up the origin of $M(2, k)$ obtaining $[0, \infty)\times S^{2k-1}$ and then divides out by the action of $SO(2)$. Thus,

$$E_2(k)\cong[0, \infty)\times\mathbf{CP}^{k-1}$$

and

$$\partial_0 E_2(k)\cong\mathbf{CP}^{k-1}.$$

The involution on $E_2(k)$ is given by complex conjugation on $\mathbf{CP}^{k-1}$. Thus, the fixed point set of the involution is $[0, \infty)\times\mathbf{RP}^{k-1}\cong B_1(k)$. We do not know of a similar convenient description of $E_i(k)$ for $i>2$.

[1] N.B. $_iM$ should not be confused with M_i which is the closed i-stratum.

LEMMA 3.2. *$E_i(k)$ is simply connected. If $i<k$ and $(i, k)\neq(1, 2)$, then $\tilde{B}_i(k)$ is simply connected.*

Proof. This is immediate from 2.5 and the above description of $E_i(k)$.

In general, the fiber of $\partial_i E_{i+j}\to B_i$ is $\partial_0 E_j(k-i)$. Thus, the fiber of $\partial_i E_{i+2}\to B_i$ is $\mathbf{CP}^{k-i-1}$. We consider the action of the fundamental group of B_i on the fiber of $\partial_i E_{i+2}\to B_i$.

LEMMA 3.3. *The double covering $\tilde{B}_i\to B_i$ defines a homomorphism $\varphi: \pi_1(B_i)\to \mathbf{Z}/2$. Let τ be the non-trivial action of $\mathbf{Z}/2$ on $H_*(\mathbf{CP}^{k-i-1})$ (as a co-ring). Then $\pi_1(B)$ acts on $H_*(\mathbf{CP}^{k-i-1})$ by $\tau\circ\varphi$.*

Proof. Let SN_i be the normal sphere bundle to M_i in M. Then $SN_i\to B_i$ is a bundle with fiber $O(n)\times_{O(n')}S^{n'k'-1}$ (where $k'=k-i$ and $n'=n-i$ and $S^{n'k'-1}$ is the unit sphere in $M(n', k')$). The structure group reduces to $O(i)\times O(k')$ with $O(i)$ acting on $O(n)$ via $O(i)\subset O(i)\times O(n')\subset O(n)$ and with $O(k')$ acting on $M(n', k')$ by right multiplication. By definition $\partial_i E_{i+2}=E_{i+2}(SN_i)$. $E_{i+2}(SN_i)$ is the result of applying the construction E_{i+2} fiberwise in the bundle $SN_i\to B_i$, and

$$\begin{aligned} E_{i+2}(O(n)\times_{O(n')}S^{n'k'-1}) &= \left\{\frac{O(i+2)}{SO(i+2)}\right\}\times_{O(2)}S^{2k'-1}\\ &= \mathbf{Z}/2\times_{O(2)}S^{2k'-1}\\ &\cong \mathbf{CP}^{k'-1}. \end{aligned}$$

From this it is clear that $O(i)\times O(k')$ acts on $\mathbf{CP}^{k'-1}$ as follows:

(1) The subgroup $SO(i)$ acts trivially on $\mathbf{CP}^{k'-1}$, and the induced action of $O(i)/SO(i)$ is by complex conjugation.

(2) The action of $O(k')$ is induced by the linear bi-axial action on $\mathbf{C}^{k'}$.

Since the $O(k')$ action is the restriction of the natural $U(k')$-action and since $U(k')$ is connected, it follows that $O(k')$ acts trivially on $H_*(\mathbf{CP}^{k'-1})$. Thus, $SO(i)\times O(k')$ is the subgroup which acts trivially on homology. Finally, it is easy to see that the two sheeted cover of B_i associated with the action of $(O(i)\times(O(k'))/(SO(i)\times O(k'))=\mathbf{Z}/2$ on $H_*(\mathbf{CP}^{k'-1})$ is $\tilde{B}_i$.

For technical reasons, we shall sometimes make the following assumption about a connected regular $O(n)$-manifold M.

Hypothesis 3.4. $M^{O(1)}$ is connected and $M^{O(2)}$ is non-empty.

This hypothesis is automatic if $n\geqslant k+2$. If $n=k+1$, it is equivalent to the condition that B_{k-1} is non-empty; while if $n=k$, it is equivalent to having B_{k-1} connected and B_{k-2} non-empty.

LEMMA 3.5. *Suppose that M is a k-axial $O(n)$-manifold with orbit space B, with $n \geqslant k$, and that 3.4 holds. Then M is simply connected if and only if B_k is simply connected.*

Proof. In general, if a G-space X has a connected orbit, then $\pi_1(X) \to \pi_1(X/G)$ is onto. (See page 91 in [4].) If $M^{O(1)} \neq \emptyset$, then M has a connected orbit. Hence, if $\pi_1(M)=0$, then $\pi_1(B)=0$. But, by 2.3, B is homeomorphic to B_k; hence, B_k is also simply connected.

We consider the converse. If $n \geqslant k+1$, then the union of the lower strata is codimension $(n-k+1)$. Thus by general position $\pi_1(M_k) \to \pi_1(M)$ is onto. M_k is a fiber bundle over B_k with fiber $O(n)/O(n-k)$. Since B_k is simply connected, π_1 (principal orbit)$\to \pi_1(M)$ is onto. If $n > k+1$, then the principal orbit is simply connected. If $n=k+1$, then the principal orbit can be deformed into an orbit of type $O(n)/O(2)$ (since $B_{k-1} \neq \emptyset$) and hence, its fundamental group is mapped trivially into M. If $n=k$, consider $\hat{M} = M_k \cup_p M_{k-1}$. The union of the lower strata $M - \operatorname{int}(\hat{M})$, is codimension 4 in M. Hence $\pi_1(\hat{M}) = \pi_1(M)$. On the other hand, $\hat{M}$ is a principal $SO(k)$-bundle over E_k, which, being the union of two copies of B_k along B_{k-1}, is simply connected. Hence, $\pi_1(SO(k)) \to \pi_1(M)$ is onto. Since $B_{k-2} \neq \emptyset$, an $SO(k)$-orbit can be deformed into an $SO(k)/SO(2)$-orbit; consequently, M is simply connected.

4. Orientations

An orientation for a regular $U(n)$- or $Sp(n)$-manifold induces an orientation for the fixed point set of each isotropy group and an orientation for each stratum of the orbit space. The situation is slightly more complicated for regular $O(n)$-actions. In this case there are essentially two independent orientations. One of these can be taken as an orientation for M and the other as an orientation for $M^{O(1)}$. Since the action is regular, $M^{O(2r)} = M^{T^r}$ where T^r is a maximal torus for $O(2r)$. Thus, an orientation for M determines one for each fixed point set of the form $M^{O(2r)}$ and an orientation for $M^{O(1)}$ determines one for each fixed point set of the form $M^{O(2r+1)} = \{M^{O(1)}\}^{O(2r)}$. Consequently, orientations for M and $M^{O(1)}$ determine orientations for each ${}_iM$ and for each E_i (where ${}_iM = M^{O(n-i)}$ and $E_i = {}_i\hat{M}/SO(i)$).

In light of this, we define an *orientation* for a regular $O(n)$-manifold M to be an orientation for M together with an orientation for $M^{O(1)}$. An equivariant stratified $F\colon M \to M'$ is *orientation-preserving* if it preserves both orientations. Similarly, an equivariant diffeomorphism is called an *oriented equivalence* if it is orientation preserving.

If both M and $M^{O(1)}$ are connected, then the regular $O(n)$-manifold has four possible orientations. Of course, it may happen that some of these are oriented equivalent.

Example 4.1. Suppose that $X = M(n, k) \times \mathbf{R}^m$. If $g \in GL(k)$, then define an equivariant linear isomorphism R_g: $M(n, k) \to M(n, k)$ by $R_g(x) = x \cdot g$. If g is a reflection, then $R_g \times \mathrm{id}$ has degree $(-1)^n$ on X and degree $(-1)^{n-1}$ on $X^{O(1)}$. If h: $\mathbf{R}^m \to \mathbf{R}^m$ is an orientation reversing diffeomorphism, then $\mathrm{id} \times h$ reverses both orientations. Consequently, for $m > 0$, all four orientations on X are equivalent. For $m = 0$ there are two distinct equivalence classes.

LEMMA 4.2. *Suppose that M is an oriented k-axial $O(n)$-manifold and that 3.4 holds. Then the involution on $E_i(M)$ is orientation preserving if and only if $(k-i+1)$ is even.*

Proof. Let m and m' be positive integers such that $m + \dim M(n, k) = m' + \dim M$. Proving the lemma for M is equivalent to proving it for $M \times \mathbf{R}^{m'}$. Using 3.4, we see that any point $x \in M \times \mathbf{R}^{m'}$ is contained in an invariant tubular neighborhood about the orbit of some point $y \in M^{O(2)} \times \mathbf{R}^{m'}$. But $G(y)$ and $G(y)^{O(1)}$ are both *connected*. Any such tubular neighborhood is equivalent to an open invariant neighborhood in $M(n, k) \times \mathbf{R}^m$. By the previous example, we can choose this equivalence to be orientation preserving. Thus, it suffices to prove the lemma for $M(n, k) \times \mathbf{R}^m$ or equivalently for $M(n, k)$. There is no involution for $i = 0$. For $i > 0$, $E_i(k)$ is connected and the involution has fixed point set of codimension $(k-i+1)$. The lemma follows.

COROLLARY 4.3. *Suppose that M is an oriented k-axial $O(n)$-manifold for which 3.4 holds. If $i = 0$, $i = k$ or if $(k-i+1)$ is even, then B_i has a canonical orientation. In all other cases, B_i is non-orientable (provided that B_{i-1} is nonempty).*

5. Pullbacks and the construction of V

We begin by stating a theorem of the first author concerning the existence of an equivariant stratified map from M to $M(n, k)$. We then derive corollaries in this section and the next by constructions similar to ones of Bredon, [6], for the special case $k = 2$.

As usual, suppose that $O(n)$ acts k-axially on M with orbit space B and $n \geq k$. The bundle of principal orbits $M_k \to B_k$ has fiber $O(n)/O(n-k)$ and structure group $O(k) = N_{O(n-k)}/O(n-k)$. Let $P_k \to B_k$ denote the associated principal $O(k)$-bundle. The next result is central to the theory of regular $O(n)$-actions.

THEOREM 5.1. *If $P_k \to B_k$ is a trivial bundle, then there exists an equivariant stratified map F: $M \to M(n, k)$. Moreover, equivariant stratified homotopy classes of such maps are in one-to-one correspondence with homotopy classes of trivializations of P_k.*

An outline of the proof can be found in [9].

Next, we recall the pullback construction of [6], [10] and [33]. Suppose that X is a smooth G-manifold and that C is a "local G-orbit space", (that is, C is locally isomorphic to the orbit space of a smooth G-manifold). Let $h: C \to X/G$ be a stratified map. Then the formal pullback,

$$h^*(X) = \{(c, x) \in C \times X \mid h(c) = \pi(x)\},$$

is a smooth G manifold over C. If Y is another smooth G-manifold over C and if $H: Y \to X$ is an equivariant stratified map covering $h: C \to X/G$, then there is a natural equivariant stratified map $Y \to h^*(X)$ covering the identity on C. It follows easily that $Y \to h^*(X)$ is an equivariant diffeomorphism. *Therefore, the statement that there exists an equivariant stratified map $H: Y \to X$ is equivalent to the statement that Y is the pullback or X via h.* In particular, Theorem 5.1 can be rephrased as stating that if P_k is a trivial bundle, then M is equivalent to some pullback of the linear model $M(n, k)$ via a stratified map $f: B \to B(k)$.

We assume for the remainder of this section that

$$M = f^*(M(n, k)).$$

Consider the chain

$$_0M \subset {_1M} \subset {_2M} \dots \subset {_nM} = M$$

where $_iM = M^{O(n-i)}$. Then $_iM = f^*(M(i, k))$. This last equation suggests how to extend the chain to the right. Thus, for $s > n$, $_sM$ is defined as $f^*(M(s, k))$. An orientation for M induces one for $_sM$.

Next we establish that M is an equivariant boundary.

THEOREM 5.2. *Let M, B, and $f: B \to B(k)$ be as above. Then M is the boundary of a k-axial $O(n)$-manifold V with orbit space A.*

(1) *A is homeomorphic to $B \times I$.*

(2) *$V = \tilde{f}^*(M(n, k))$, where $\tilde{f}: A \to B(k)$ is a stratified map extending f.*

Proof. Let $_iF: {_iM} \to M(i, k)$ be the natural $O(i)$-equivariant stratified map covering f. Then, for $j < i$, $_iF$ is transverse to $M(j, k)$ with inverse image $_jM$ and $_iF \mid (_jM) = {_jF}$. Consider the map $F = {_{n+1}F}: {_{n+1}M} \to M(n+1, k)$. Regarded as an $O(n)$-module, $M(n+1, k) = M(1, k) \times M(n, k)$ with trivial action on the first factor. Let $p: (M(1, k) - \{0\}) \times M(n, k) \to S^{k-1}$ be projection on the first factor followed by radial projection onto the unit sphere. Let $y \in S^{k-1}$ be a regular value of the following composition:

$$_{n+1}M - {_nM} \xrightarrow{F} M(n+1, k) - M(n, k) \xrightarrow{p} S^{k-1},$$

and let $\mathbf{R}_+ y \subset M(1, k)$ be the ray through y. Then F is transverse to $\mathbf{R}_+ y \times M(n, k)$. Set

$$V = F^{-1}(\mathbf{R}_+ y \times M(n, k)).$$

Since F is $O(n)$-equivariant, V is a smooth $O(n)$-manifold with boundary. We see that $\partial V = {}_nM = M$. Let $\tilde{F}$ denote the composition of $F \mid V$ with projection onto $M(n, k)$. Then $\tilde{F}$ is clearly equivariant and stratified. Let $\tilde{f}: A \to B(k)$ be the map of orbit spaces induced by $\tilde{F}$. Since $\tilde{F} \mid \partial V = {}_nF$, it follows that $\tilde{f} \mid B = f$. Since ${}_{n+1}M = \{(b, z) \in B \times M(n+1, k) \mid f(b) = {}^tz \cdot z\}$, we have

$$V = \{(b, s, x) \in B \times \mathbf{R}_+ \times M(n, k) \mid f(b) = s^2 e + {}^tx \cdot x\}$$

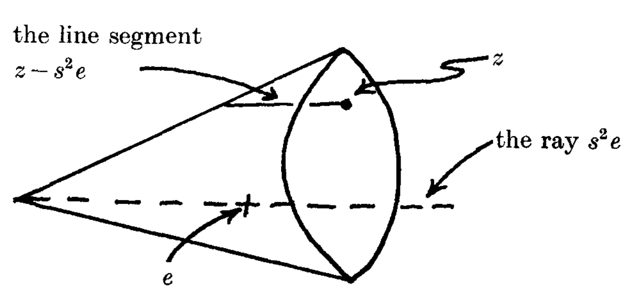
where $e = {}^ty \cdot y$. In other words, V is defined by the pullback square

$$\begin{array}{ccc} V & \xrightarrow{F \mid V} & \mathbf{R}_+ \times M(n, k) \\ \downarrow & & \downarrow \Lambda \\ B & \xrightarrow{f} & B(k) \end{array}$$

where $\Lambda(s, x) = s^2 e + {}^tx \cdot x$. Thus the orbit space A is defined by the following pullback square:

$$\begin{array}{ccc} A & \longrightarrow & \mathbf{R}_+ \times B(k) \\ \downarrow & & \downarrow \lambda \\ B & \xrightarrow{f} & B(k) \end{array}$$

where $\lambda(s, z) = s^2 e + z$. The fiber of λ at $z \in B(k)$ is $[0, \varepsilon]$, where $\varepsilon = \inf \{s \in \mathbf{R}_+ \mid z - s^2 e \notin B(k)\}$. It is clear (from the picture) that $\varepsilon = 0$, if and only if $z \in \partial B(k)$.



If $b \in B$, the fiber of $A \to B$ at b is identified with the fiber of λ at $f(b)$. Therefore, if b belongs to the interior of B, this fiber is an interval; while, if $b \in \partial B$, the fiber is a point. Consequently, A is homeomorphic to $B \times I$.

6. Equivariant framings

In this section, we relate the equivariant normal bundle of a regular $O(n)$ manifold to the normal bundle of its orbit space. The following non-standard terminology is adopted. An m-dimensional G-vector bundle over a G-space X is said to be *trivial*, if it is equivalent to $X \times \mathbf{R}^m$, where $\mathbf{R}^m$ has trivial G-action. Similarly, two G-vector bundles E and E' are *stably equivalent* if $E+F \cong E'+F'$, where F and F' are trivial.

As in the previous section, we assume that $M=f^*(M(n,k))$, where $f\colon B\to B(k)$ is stratified. Since B is locally modeled on $B(k)$ and since $B(k)$ is a subset of Euclidean space, it follows that B can be embedded in some Euclidean space (i.e., $i\colon B \hookrightarrow \mathbf{R}^p$ and the smooth structure on B is induced from the smooth structure on $\mathbf{R}^p$). This embedding induces a linear map $T_b(B) \hookrightarrow T_{i(b)}(\mathbf{R}^n)$. By 2.3, TB is a locally trivial vector bundle. Hence, the embedding $B \hookrightarrow \mathbf{R}^p$ induces an embedding of vector bundles $TB \hookrightarrow T\mathbf{R}^p|B$. The normal bundle of B in $\mathbf{R}^p$ is defined to be $(T\mathbf{R}^p|B)/TB$.

THEOREM 6.1. *Suppose that B embeds in $\mathbf{R}^p$ with normal bundle $\nu(B)$. Then M can be equivariantly embedded in the representation $\mathbf{R}^p \times M(n,k)$ with normal bundle stably equivalent to $\pi^*\nu(B)$ (as $O(n)$-vector bundles).*

Proof. We have $M \subset B \times M(n,k) \subset \mathbf{R}^p \times M(n,k)$. Consider the map $\varphi\colon B\times M(n,k) \to B(k)$ defined by $\varphi(b,x)=f(b)-\pi(x)$. One sees that 0 is a regular value of φ, and that $M=\varphi^{-1}(0)$. Hence, the normal bundle of M in $B\times M(n,k)$, being the pullback of the normal bundle of 0 in $B(k)$, is trivial.

If p is sufficiently large compared to the dimension of B and $\nu(B)$ is the normal bundle of B in $\mathbf{R}^p$, then $\nu(B)$ is called the *stable normal bundle of* B and $\pi^*\nu(B)$ is the *stable normal bundle of* M.

COROLLARY 6.2. *The stable normal bundle of M is (equivariantly) trivial if and only if the stable normal bundle of B is trivial. Moreover, there is a natural one-to-one correspondence between equivariant framings of the stable normal bundle of M and framings of the stable normal bundle of B.*

7. Implications of Smith Theory

Suppose that $F\colon M\to M'$ is an equivariant stratified map of regular $U(n)$- or $Sp(n)$-manifolds. Then F induces an isomorphism on homology if and only if each of the induced maps between corresponding strata of the orbit spaces induces an isomorphism on homology. The proof is an application of Smith Theory, Mayer-Vietoris sequences, and the

Comparison Theorem for spectral sequences. Smith Theory comes into play because the fixed point set of each isotropy group is equal to the fixed point set of its maximal torus.

If M and M' are regular $O(n)$-manifolds, then, by using $\mathbf{Z}/2$-tori, the same arguments show that F induces an isomorphism on homology with $\mathbf{Z}/2$ coefficients if and only if it induces a $\mathbf{Z}/2$-homology equivalence between corresponding strata of the orbit spaces. The corresponding result with integer coefficients is the following.

THEOREM 7.1. *Suppose that $F: M \to M'$ is a stratified map of regular $O(n)$-manifolds. Then F induces an isomorphism on integral homology if and only if for each integer i, $i \equiv n(2)$, the map $E_i(F): E_i(M) \to E_i(M')$ induces an isomorphism on integral homology.*

We shall also need the following related result.

THEOREM 7.2. *Suppose that $F: (M, \partial M) \to (M', \partial M')$ is a stratified map of regular $O(n)$-manifolds and that $F|\partial M: \partial M \to \partial M'$ induces an isomorphism on integral homology. Fix an integer i, $i \equiv n(2)$. Further suppose that for each j such that $j < i$ and $j \equiv n(2)$, the map $E_j(M) \to E_j(M')$ induces an isomorphism on homology. Then $\partial E_i(M) \to \partial E_i(M')$ induces an isomorphism on homology.*

These two results are proved in Appendix 2 of [9].(1)

Suppose that $g: X \to Y$ is an equivariant, stratified map of smooth $\mathbf{Z}/2$-manifolds (i.e. manifolds with involutions). Let $F \subset X$ and $F' \subset Y$ be the fixed point sets of the involutions; also let $\bar{X} = \hat{X}_F/(\mathbf{Z}/2)$ and $\bar{Y} = \hat{Y}_{F'}/(\mathbf{Z}/2)$. If g induces an isomorphism on homology, then it follows from Smith Theory, that $F \xrightarrow{g|F} F'$, $\hat{X}_F \xrightarrow{\hat{g}} \hat{Y}_F'$, and $\bar{X} \xrightarrow{\bar{g}} \bar{Y}$ all induce isomorphisms on $\mathbf{Z}/2$-homology. Thus, if $E_i(F): E_i(M) \to E_i(M')$ is a homology isomorphism, then both induced maps $f_i: B_i \to B_i'$ and $f_{i-1}: B_{i-1} \to B_{i-1}'$ are $\mathbf{Z}/2$-homology isomorphisms.

There is one case in which we can say more. The involution on X is a *reflection*, if F is of codimension one and disconnects X (i.e., if $\hat{X}_F \to \bar{X}$ is the trivial double cover). If $\mathbf{Z}/2$ acts by reflections on X and Y and $g: X \to Y$ induces an isomorphism on integral homology, then it follows easily that the maps $g|F: F \to F'$ and $\bar{g}: \bar{X} \to \bar{Y}$ also both induce isomorphisms on integral homology. As a corollary to this observation we have the following:

PROPOSITION 7.3. *Suppose that $F: M \to M'$ is an equivariant stratified map of oriented k-axial $O(n)$-manifolds with $n \geqslant k$ and that F induces an isomorphism on integral homology. Then the induced map between top strata $f_k: B_k \to B_k'$ is an isomorphism on integral homology.*

(1) In [9] E_i is called D_i.

If $k \equiv n(2)$, then the map between the next to top strata f_{k-1}: $B_{k-1} \to B'_{k-1}$ is also an isomorphism on integral homology.

Proof. If $k \equiv n(2)$, this follows from the above observation and the fact that $E_k(F)$ is an integral homology isomorphism. If $k \not\equiv n(2)$, it follows directly from 7.1 and the fact that $E_{k+1} = B_k$.

8. Actions on homology spheres

As usual, M denotes a k-axial $O(n)$-manifold and ${}_iM = M^{O(n-i)}$. The first basic observation from Smith Theory is the following:

PROPOSITION 8.1. *If M is a $\mathbf{Z}/2$-homology sphere, then so is ${}_iM$. If M is an integral homology sphere and if $(n-i)$ is even, then ${}_iM$ is also an integral homology sphere.*

LEMMA 8.2. *Suppose that M is a $\mathbf{Z}/2$-homology sphere and that $n \neq 1$. If the dimension of ${}_0M$ is $(m-1)$, then the dimension of ${}_iM$ is $(in+m-1)$, and in particular, M has dimension $(kn+m-1)$. (By convention the dimension of the empty set is -1.) Conversely, if the dimension of M is $(kn+m-1)$, then $m \geq 0$ and ${}_0M$ has dimension $(m-1)$.*

Proof. This follows from the well-known formula of Borel, [3], which relates the dimension of the fixed point set of $(\mathbf{Z}/2)^n$ to the dimensions of the fixed sets of subgroups of index two. The details of the argument can be found in [9] or [15].

For the remainder of this section we suppose that $O(n)$ acts k-axially on an integral homology sphere Σ^{kn+m-1} with orbit space B and that $n \geq k$.

PROPOSITION 8.3. *With the above hypotheses B_k is acyclic. If, in addition, Σ is simply connected, then B_k is contractible.*

Proof. Let us first consider $H_*(B_k) = H_*(\dot{B}_k)$. We shall use a theorem of *R.* Oliver [29], which asserts that the orbit space of a compact Lie group action on an acyclic manifold is acyclic. There are three cases.

Case 1. $m > 0$: Let x be a fixed point (by 8.2, the fixed point set of $O(n)$ on Σ is nonempty if $m > 0$). $\Sigma - \{x\}$ is acyclic the orbit space $(\Sigma - \{x\})/O(n)$ is an acyclic manifold with boundary. Hence, its top stratum is also acyclic; but this top stratum is $\dot{B}_k$.

Case 2. $m = 0$ *and* $n > k$: Consider the restriction of the $O(n)$-action to $O(n-1)$ and let $C = \Sigma/O(n-1)$. Then C is a manifold with boundary. By Case 1, it is acyclic. There is a natural projection p: $C \to B$ which sends an $O(n-1)$-orbit to its image as an $O(n)$-orbit.

If $y \in \dot{B}_i$, then $p^{-1}(y) \cong V_{n,i}/O(n-1)$, where $V_{n,i}$ is the Stiefel manifold of i-frames in n-space. The orbit space $V_{n,i}/O(n-1)$ is an i-disk if $i<n$ and an $(n-1)$-sphere when $i=n$ (see [9] page 78). We see that $p^{-1}(y)$ is always a disk, and therefore, that p is a homotopy equivalence. Thus, B (and hence, B_k) is acyclic.

Case 3. $m=0$ and $n=k$: In this case C is a compact manifold without boundary of dimension $l=\frac{1}{2}k(k+1)+k-2$. Let x be a fixed point of $O(k-1)$ on Σ. Since $C-\{\pi(x)\}$ is acyclic, it follows that C is a homology l-sphere. If $y \in \partial B$, then $p^{-1}(y)$ is a disk. Since the inclusion $i: \partial B \hookrightarrow B$ therefore factors through C, it follows that i_* is the zero map on homology. Poincaré duality implies that ∂B must be a homology sphere. Its dimension is $\frac{1}{2}k(k+1)-2$. Therefore, by Alexander duality, $C-p^{-1}(\partial B)$ has the homology of a sphere of dimension $l-(\frac{1}{2}k(k+1)-2)-1=k-1$. If $y \in \dot{B}_k$, then $p^{-1}(y) \cong O(k)/O(k-1)=S^{k-1}$. Therefore, $C-p^{-1}(\partial B) \to \dot{B}_k$ is an S^{k-1}-bundle, and the total space has non-vanishing homology only in dimension 0 and $k-1$. Consequently, $\dot{B}_k$ is acyclic.

Finally, note that if $\pi_1(\Sigma)=0$, then by 3.5, $\pi_1(B_k)=0$. As a result B_k is contractible.

COROLLARY 8.4. *With Σ and B as above, the principal orbit bundle $P_k \to B_k$ is a trivial fiber bundle. Up to homotopy there are exactly two trivializations of P_k (since $O(k)$ has two components).*

Consequently, the results of Section 5 apply to Σ. In particular, Σ is equivalent to a pullback of the linear model, and Σ is the boundary of the k-axial $O(n)$-manifold V of 5.2. Let A be the orbit space of V. Since A is homeomorphic to $B \times I$, A is also acyclic. Therefore, the tangent bundle and the stable normal bundle of A are both trivial. The results of Section 6 now apply. Thus, the stable normal bundle of V is trivial. This implies that V and $V^{O(1)}$ are orientable, so we assume, as of now, that we have picked an orientation for V and that Σ is oriented compatibly. We note that it also follows (from obstruction theory) that an oriented framing of the stable normal bundle of V is unique up to an equivariant homotopy.

Suppose that $g: M \to M'$ is an equivariant stratified map of oriented, regular $O(n)$-manifolds. Since g is transverse to ${}_iM'$, it follows that g and ${}_ig: {}_iM \to {}_iM'$ have the same degree up to sign. Clearly, by the way the orientations were defined, $\deg({}_ig)=\deg({}_jg)$ when $i \equiv j(2)$, (see Section 4). Hence, there are two independent degrees associated to g: $\deg(g)$ and $\deg({}_{n-1}g)$.

Let D^{kn+m} denote the unit disk in the linear action $k\varrho_n+\underline{m}$; and let S^{kn+m-1} be the unit sphere.

THEOREM 8.7. *Let Σ^{kn+m-1} be oriented in the sense of Section 4.*

(1) *If $m>0$, then there is an equivariant, stratified $F: \Sigma^{kn+m-1} \to S^{kn+m-1}$ such that* $\deg(F) = \deg({}_{n-1}F) = 1$. *Such a map is unique up to equivariant, stratified homotopy.*

(2) *Choose, once and for all, an orientation*[1] *for the linear model S^{kn-1}. If $m=0$, then there is an equivariant, stratified $F: \Sigma^{kn-1} \to S^{kn-1}$ which has positive degree on ${}_i\Sigma$ for every odd i. Such an F is unique up to equivariant stratified homotopy.*

Proof. The argument which was used to prove 5.1 shows that for $m>0$, F is determined up to homotopy by (1) the homotopy class of ${}_0F: {}_0\Sigma \to {}_0S$, and (2) by the choice of trivialization for the bundle of principal orbits. (See page 79 in [9].) We choose ${}_0F$ to be of degree $+1$. Since ${}_0S = S^{m-1}$, this determines ${}_0F$ up to homotopy. For the resulting map $F: \Sigma \to S$, $\deg({}_iF) = +1$ if $i \equiv 0(2)$ and $\deg({}_iF) = \pm 1$ if $i \equiv 1(2)$. If we change the trivialization of the bundle of principal orbits, this changes the sign of $\deg({}_iF)$ for $i \equiv 1(2)$. Hence, there is one choice of trivialization that makes $\deg({}_iF) = +1$ for all i.

If $m=0$, then the fixed point set is empty and the only choice required in defining F is a choice of trivialization of the bundle of principal orbits. Thus, there are exactly two such $F: \Sigma^{kn-1} \to S^{kn-1}$ up to equivariant, stratified homotopy. They differ by the automorphism of k-tuples of vectors in $\mathbf{R}^n$ obtained by sending $(x_1, x_2, \ldots, x_k)$ to $(-x_1, x_2, \ldots, x_k)$. This map has degree $(-1)^i$ on ${}_i\Sigma$. Therefore, the two possible maps have degrees of opposite sign on ${}_i\Sigma$ for all i odd.

We need to know necessary and sufficient conditions for Σ to be an homotopy sphere in terms of the map $F: \Sigma \to S$.

THEOREM 8.8. *Let $F: M^{kn+m-1} \to S^{kn+m-1}$ be an equivariant, stratified map of degree ± 1 and let $f: B \to L$ be the induced map of quotient spaces. Then M^{kn+m-1} is a homotopy sphere if and only if*

(1) $\pi_1(B_k) = 0$, *and*

(2) $E_i(F)_*: H_*(E_i(M); \mathbf{Z}) \to H_*(E_i(S); \mathbf{Z})$ *is an isomorphism for all i, with $i \equiv n(2)$.*

Proof. This is immediate from 3.5 and 7.1.

THEOREM 8.9. *Let $F: M^{kn-1} \to S^{kn-1}$ be an equivariant stratified map covering $f: B \to L$. Then M is a homotopy sphere if and only if*

(1) $\pi_1(B_k) = 0$,

(2) *when n is odd, $E_1(M)$ is an integral homology sphere, and*

(3) $E_i(F)_*: H_*(E_i(M); \mathbf{Z}) \to H_*(E_i(S); \mathbf{Z})$ *is an isomorphism for $i \geqslant 2$ and $i \equiv n(2)$.*

[1] Recall that if $m=0$, there are two inequivalent orientations for the linear action on D^{kn}.

Proof. Again, 3.5 implies that $\pi_1(M)=0$ if and only if $\pi_1(B_k)=0$. We show that conditions (2) and (3) are equivalent to M being an integral homology sphere.

If n is even and M is a homology sphere, then by (8.1), ${}_2M$ is a homology sphere and ${}_2M\to E_2(M)$ is a circle bundle. It follows that $E_2(M)$ is a homology $\mathbf{CP}^{k-1}$ and that the characteristic class of the circle bundle is a generator for $H^2(E_2(M))$. Since this circle bundle is induced by pulling back via $E_2(F)$: $E_2(M)\to E_2(S)$, if follows that $E_2(F)$ must be of degree ± 1. Conversely, if condition (3) holds (and n is still assumed to be even), then $E_2(F)$ is of degree ± 1. Consequently, when n is even, F itself is degree ± 1. Thus, this case follows from the previous theorem.

If n is odd, and M is an integral homology sphere, then so is $E_1(M)$. The double cover $E_1(M)\to B_1$ is induced by the map f_1: $B_1\to L_1=\mathbf{RP}^{k-1}$. Hence, f_1 must be of degree ± 1 modulo 2. Hence, F is of odd degree. We claim that if k is odd, then F must actually be of degree ± 1. Once we show this, the case k odd follows from the previous theorem. Consider ${}_2M$. It is a $\mathbf{Z}_{(2)}$-homology sphere and a circle bundle over $E_2(M)$. Hence, $E_2(M)$ is a $\mathbf{Z}_{(2)}$-homology $\mathbf{CP}^{k-1}$. The normal bundle of B_1 in $E_2(M)$ has a twisted Euler class in $H^{k-1}(B_1;\mathbf{Z}^-)$. (Recall, that since $k-1\equiv 0(2)$, B_1 is non-orientable.) There is an involution on $E_2(M)$ with B_1 as fixed point set. The G-signature theorem tells us that this twisted Euler class must be a generator in $H^{k-1}(B_1;\mathbf{Z}^-)$. Since this class is induced via the map f_1: $B_1\to\mathbf{RP}^{k-1}$, it follows that f_1 is degree ± 1 (with twisted coefficients). Hence $E_1(F)$, and consequently F, have degree ± 1.

This leaves the case when k is even and n is odd. Here, the map does not necessarily have to be of degree ± 1. We have shown, however, that it is of odd degree. Let $\hat{M}$ be M blown up along M_1, and let $\hat{S}$ be S blown up along S_1. We claim that condition (3) is equivalent to the condition that $E_i(\hat{F})$: $E_i(\hat{M})\to E_i(\hat{S})$ be an isomorphism on integral homology for $i\equiv n(2)$. This follows from 7.1, since $E_i(\hat{M})=E_i(M)$ for $i\geqslant 3$, and $E_1(\hat{M})=\varnothing$.

We claim that condition 2 is equivalent to the condition that $(F,\hat{F})_*$: $H_*(M,\hat{M})\to H_*(S,\hat{S})$ is an isomorphism in dimensions less than $kn-1$. First, note that we have a commutative diagram:

$$\begin{array}{ccc} H_*(M_1;\mathbf{Z}^-) & \xrightarrow{\text{Thom}\,\cong} & H_{*+(k-1)(n-1)}(M,\hat{M};\mathbf{Z}) \\ \big\downarrow F_* & & \big\downarrow (F,\hat{F})_* \\ H_*(S_1;\mathbf{Z}^-) & \xrightarrow{\text{Thom}\,\cong} & H_{*+(k-1)(n-1)}(S,\hat{S};\mathbf{Z}). \end{array}$$

Secondly, F: $M_1\to S_1$ is a bundle map covering f_1: $B_1\to L_1$. The fiber is S^{n-1}, and the action of $\pi_1(L_1)$ on S^{n-1} is via the antipodal map. (Actually, $S_1=S^{k-1}\times_{\mathbf{Z}/2}S^{n-1}$.) Using these facts, a simple calculation shows that F_*: $H_*(M_1;\mathbf{Z}^-)\to H_*(S_1;\mathbf{Z}^-)$ is an isomorphism for

$* < k+n-2$ if and only if $f_1: B_1 \to L_1$ induces an isomorphism on homology in degrees $< k-1$. This, in turn, is equivalent to B_1 being an integral homology $\mathbf{RP}^{k-1}$ and f_1 being of odd degree. This, of course, is equivalent to $E_1(M)$ being an integral homology sphere. Comparing these results via the long exact sequences for $(M, \hat{M})$ and $(S, \hat{S})$, we see that conditions (2) and (3) are equivalent to the fact that $F_*: H_*(M) \to H_*(S)$ is an isomorphism for $* < kn-1$. This is equivalent to M being an integral homology sphere.

COROLLARY 8.10. *$F: \Sigma^{kn-1} \to S^{kn-1}$ is always of odd degree on each stratum. If k is odd or if n is even, then the degree is ± 1.*

9. The concordance groups

Definition 9.1. The oriented k-axial $O(n)$-manifolds M and M' (of the same dimension) are *concordant* if there is an oriented k-axial $O(n)$-manifold W simple homotopy equivalent to $M \times I$(1) with the restriction of the action to ∂W being oriented equivalent to $M' \coprod -M$. (Here $-M$ means that both the orientation of M and $M^{O(1)}$ have been reversed.)

If M and M' have nonempty fixed point sets and if dim $M =$ dim M', then one defines the *equivariant connected sum, $M \# M'$*, by taking connected sum at two fixed points. The result is well-defined up to oriented equivalence.

We now come to our main object of study. Let $\Theta^1(k, n, m)$ denote the set of concordance classes of oriented k-axial $O(n)$-actions on homotopy spheres of dimension $(kn+m-1)$. More generally, if $G^d(n)$ stands for $O(n)$, $U(n)$ or $Sp(n)$ as $d=1, 2$ or 4, then define $\Theta^d(k, n, m)$ to be the set of concordance classes of oriented k-axial $G^d(n)$-actions on homotopy spheres or dimension $(dkn+m-1)$.

THEOREM 9.2. *For $m \geq 1$, the set $\Theta^d(k, n, m)$ is an abelian group under connected sum. An action on a homotopy sphere $\Sigma^{dkn+m-1}$ represents the zero element of this group if and only if it extends to a k-axial action on a contractible manifold. The inverse of $[\Sigma]$ is $[-\Sigma]$.*

The standard arguments can be used to prove this. (See for example, [22] and [4] page 339.)

For $n \geq k$ and $d=2$ or 4, the groups $\Theta^d(k, n, m)$ were calculated in [11]. The remainder of this paper is devoted to calculating $\Theta^1(k, n, m)$ for $n \geq k$, by a similar program, which we outline below.

The calculations for $m \neq 0$, 4 and $k \neq 2$ are made in Sections 11 and 12. The calculations for $k=2$ are in Section 13; while the results for $m=0$ and $m=4$ appear in Section 14.

(1) Of course, if dim $W \geq 6$, then W is diffeomorphic to $M \times I$.

We suppose for the remainder of this section that $m>0$. Let Σ^{kn+m-1} represent an element in $\Theta^1(k, n, m)$ and let $F\colon (V^{kn+m}, \Sigma^{kn+m-1}) \to (D^{kn+m}, S^{kn+m-1})$ be the equivariant stratified map with $\deg(F) = \deg({}_{n-1}F) = 1$ constructed in Theorem 8.7. We seek to alter V by an equivariant normal bordism relative to $F|\Sigma$ to make it contractible.

Let A, B, K, and L denote the orbit spaces of V, Σ, D, and S, respectively, and let $f\colon (A, B) \to (K, L)$ be the map induced by F. For each i, $0 \leqslant i \leqslant k$, let $f_i\colon (A_i, B_i) \to (K_i, L_i)$ be the induced map of closed strata. We note that f_i is naturally covered by a map of stable normal bundles, since:

(1) the map of equivariant stable normal bundles $\nu(V) \to \nu(D)$ induces a bundle map $\nu(A) \to \nu(K)$, and

(2) the normal bundle of A_i in A is mapped by a bundle map to the normal bundle of K_i in K.

Hence, since f_i is also of degree one, it is a normal map. Similarly, we see that for each i, $0 \leqslant i \leqslant k+1$, the induced map of double branched covers $E_i(f)\colon (E_i(A), E_i(B)) \to (E_i(K), E_i(L))$ is a $\mathbf{Z}/2$-equivariant stratified normal map. (Recall that a normal map $f\colon M \to N$ is a map of pairs $(M, \partial M) \to (N, \partial N)$ which is covered by a linear bundle map $\bar{f}\colon \nu_M \to \xi_N$ where ν_M is the stable normal bundle of M. For us normal maps will always be of degree 1 unless otherwise specified.)

The next result allows us to translate our problem of doing $O(n)$-equivariant surgery to a sequence of ordinary surgery problems.

THEOREM 9.3. *Suppose $F\colon (V^{kn+m}, \Sigma^{kn+m-1}) \to (D^{kn+m}, S^{kn+m-1})$ is chosen so that $f\colon B \to L$ is of degree one on each stratum. (This can always be done if $m>0$.) Then*

(1) *$(f_j|B_j)_*\colon H_*(B_j; \mathbf{Z}/2) \to H_*(L_j; \mathbf{Z}/2)$ is an isomorphism for all j, $0 \leqslant j \leqslant k$,*

(2) *$(E_j(f)|E_j(B))_*\colon H_*(E_j(B); \mathbf{Z}) \to H_*(E_j(L); \mathbf{Z})$ is an isomorphism for all j, $0 \leqslant j \leqslant k+1$, such that $j \equiv n(2)$, and*

(3) *$\pi_1(B_k) = 0$.*

Moreover, analogous conditions hold for the extension $f\colon A \to K$ if and only if the k-axial $O(n)$-manifold V^{kn+m} is contractible.

Proof. The map $F|\Sigma\colon \Sigma^{kn+m-1} \to S^{kn+m-1}$ is a homotopy equivalence. Thus, (2) follows from Theorem 7.1. Also, (1) is implied by (2). Condition (3) follows from Lemma 3.5. The statement about the extension $f\colon A \to K$ follows in a similar manner.

Our program is to successively try to do surgery on each $f_i\colon A_i \to K_i$ relative to $f_i|\partial A_i$ to achieve the conditions of 9.3. If we succeed, then we will have constructed a new orbit space $\bar{A}$ together with a stratified map $\bar{f}: (\bar{A}, B) \to (K, L)$ so that $\bar{f}|B = f|B$ and so that $\bar{V} = \bar{f}^*(D)$ is contractible. Initially, it will appear that surgery obstructions are encountered at each stratum; however, we will then show that all these obstructions either vanish or are indeterminant (tied to choices made in dealing with the lower strata) except for those encountered on the bottom two strata.

First we analyze the fundamental groups and orientations involved.

PROPOSITION 9.4. *$\pi_1(K_0) = \pi_1(K_k) = 0$. $\pi_1(K_i) = \mathbf{Z}/2$, provided $0 < i < k$ and $(k, i) \neq (2, 1)$;*

$$\pi_1(\partial K_i) = \pi_1(K_i) \quad \textit{if } i \geq 2 \textit{ or if } m \geq 3.$$

This results immediately from 2.5 and 2.6. From 4.3 we have the following:

PROPOSITION 9.5. *For $0 < i < k$, A_i and K_i are $(-1)^{k-i+1}$-orientable. For $i = 0$ or $i = k$, A_i and K_i are orientable.*

10. The relevant surgery groups

In this section we freely use the notation and results of [34]. Let R be a subring of $\mathbf{Q}$, π a group, and $w: \pi \to \mathbf{Z}/2$ a homomorphism. Recall that the surgery group $L_{2s}(R[\pi], w)$ is a Grothendieck group of triples (G, λ, μ), called hermitian forms. Here G is a free $R[\pi]$-module, λ is a non-singular $(-1)^s$-hermitian-symmetric pairing and μ is a quadratic refinement of λ. In this group hyperbolic forms are set equal to zero, [34] page 45. The surgery group $L_{2s+1}(R[\pi], w)$ is generally defined in terms of automorphisms of forms, but for finite groups π there is a description in terms of triples (T, l, q), called linking forms. T is a finite $R[\pi]$-module with short free resolution, l is a non-singular $(-1)^s$-hermitian-symmetric linking form, and q is a quadratic refinement, see [28] pages 32 and 33 and [30]. A linking form is *resolvable*, if it is induced by reducing a hermitian form (F, λ, μ) over $R[\pi]$ which is non-singular over $\mathbf{Q}[\pi]$, see [28] page 42. In the Grothendieck group $L_{2s+1}(R[\pi], w)$ resolvable linking forms are set equal to zero.

If we have a normal map $f: (M^m, \partial M) \to (N^m, \partial N)$, then we denote its kernel groups by $K_i(f; R[\pi_1(N)])$ or by $K_i(M; R[\pi_1(N)])$. If $f|\partial M$ is an $R[\pi_1(N)]$-homology isomorphism, then the kernel groups satisfy Poincaré duality.

If $m = 2s$, then after we do surgery below the middle dimension, we can assume that $K_i(M; R[\pi_1(N)]) = 0$ for $i < s$ and that $K_s(M; R[\pi_1(N)])$ is free. Geometric intersection produces a non-singular pairing λ; the bundle map covering f produces immersed cycles in the middle dimension with self-intersections which give the quadratic refinement μ, see [34] page 45–46. This triple determines the surgery obstruction $\sigma(f) \in L_{2s}(R[\pi_1(N)], w_1(N))$. If $m=2s+1$ and π is finite, then we can assume that $K_i(M; R[\pi_1(N)])=0$ for $i<s$ and $K_s(M; \mathbf{Q}[\pi_1(N)])=0$. Thus $K_s(M; R[\pi_1(N)])$ is finite and has a short free resolution. There is a geometrically defined linking pairing and self-linking pairing (see [28] pages 32–40) which determines $\sigma(f) \in L_{2s+1}(R[\pi_1(N)], w_1(N))$.

If $\sigma(f)=0$ and $m \geqslant 5$, then we can perform further surgery to construct a normal bordism from f to a normal map which induces an isomorphism on $R[\pi_1(N)]$-homology.(1) More specifically, if $m=2s$, and $f\colon M^m \to N^m$ is highly connected with H a subkernel ([34], page 45) for the intersection form of f, then there is a normal bordism $G\colon W \to N \times I$ to an $R[\pi_1(N)]$-homology equivalence with G highly connected and $K_{s+1}(G, f; R[\pi_1(N)])=H$. If $m=2s+1$, $f\colon M^m \to N^m$ is highly connected, and (F, λ, μ) is a resolution of the linking form of f, then there is a normal bordism $G\colon W \to N \times I$ from f to an $R[\pi_1(N)]$-homology equivalence which realizes the resolution. This means that G is highly connected and that the geometric intersection and self-intersection forms on $K_{s+1}(G; R[\pi_1(N)])$ are identified with λ and μ, respectively.

This completes our general discussion of the Wall groups. We turn now to the explicit computations which we need in order to do stratified surgery on regular $O(n)$-manifolds. It is easily seen that symmetric forms over $R \subset \mathbf{Q}$ have at most one quadratic refinement. Such a refinement will exist if and only if $\lambda(x, x) \in R$ is always divisible by 2 in R. In this case the unique refinement, μ, is given by $\mu(x)=\frac{1}{2}\lambda(x, x)$. Thus, $L_0(R)$ is a Grothendieck group of symmetric, non-singular, even matrices over R. If $R=\mathbf{Z}$, then such matrices, modulo hyperbolic ones, are classified by one eighth their index. Hence,

$$(\alpha_{ij}) \mapsto \tfrac{1}{8} I(\alpha_{ij})$$

induces an isomorphism $L_0(\mathbf{Z}) \cong \mathbf{Z}$. Denote by W the group of such matrices over $\mathbf{Z}_{(2)}$ modulo hyperbolic ones. There is a short exact sequence $0 \to \mathbf{Z} \to W \to T \to 0$ where

$$T = \sum_{\substack{p \text{ prime} \\ p \equiv 3(4)}} \mathbf{Z}/4 \oplus \sum_{\substack{p \text{ prime} \\ p \equiv 1(4)}} (\mathbf{Z}/2 \oplus \mathbf{Z}/2).$$

See [26].

As we have seen in Corollary 2.6, the fundamental group of a stratum in the standard linear model is either 0 or $\mathbf{Z}/2$. We need to do surgery with either $\mathbf{Z}_{(2)}$ or $\mathbf{Z}$ coefficients. The next three theorems give the calculations of the Wall groups in these cases.

(1) This is also true for $m \leqslant 3$ provided that the range is simply connected.

THEOREM 10.1.

(a)
$$L_i(\mathbf{Z}) = \begin{cases} \mathbf{Z} & i \equiv 0(4) \\ 0 & i \equiv 1(4) \\ \mathbf{Z}/2 & i \equiv 2(4) \\ 0 & i \equiv 3(4). \end{cases}$$

(b)
$$L_i(\mathbf{Z}_{(2)}) = \begin{cases} W & i \equiv 0(4) \\ 0 & i \equiv 1(4) \\ \mathbf{Z}/2 & i \equiv 2(4) \\ 0 & i \equiv 3(4). \end{cases}$$

The calculations of $L_i(\mathbf{Z})$ were made in [22]. The ones for $L_i(\mathbf{Z}_{(2)})$ are completely analogous. In both cases the isomorphism from L_2 to $\mathbf{Z}/2$ is the Arf invariant of the associated quadratic form over $\mathbf{Z}/2$, [1]. This obstruction is called the Arf-Kervaire invariant.

In the case $\pi = \mathbf{Z}/2$ we denote the two possible maps $w: \mathbf{Z}/2 \to \mathbf{Z}/2$ by $+$ and $-$ ($+$ denotes the trivial map). The non-trivial element in $\mathbf{Z}/2$ is denoted γ.

THEOREM 10.2.

(a)
$$L_i(\mathbf{Z}[\mathbf{Z}/2], +) = \begin{cases} \mathbf{Z} \oplus \mathbf{Z} & i \equiv 0(4) \\ 0 & i \equiv 1(4) \\ \mathbf{Z}/2 & i \equiv 2(4) \\ \mathbf{Z}/2 & i \equiv 3(4). \end{cases}$$

(b)
$$L_i(\mathbf{Z}[\mathbf{Z}/2], -) = \begin{cases} \mathbf{Z}/2 & i \equiv 0(2) \\ 0 & i \equiv 1(2). \end{cases}$$

These groups are calculated in [34] page 162. Both $L_{2l}(\mathbf{Z}[\mathbf{Z}/2], -)$ and $L_{4l+2}(\mathbf{Z}[\mathbf{Z}/2], +)$ are detected by the Arf-Kervaire invariant. The isomorphism $L_{4l+3}(\mathbf{Z}[\mathbf{Z}/2], +) \cong \mathbf{Z}/2$ is a codimension 1 Kervaire invariant. If $f: M^{4l+3} \to N^{4l+3}$ is a normal map with N orientable and $\pi_1(N) = \mathbf{Z}/2$, then dual to the generator of $H^1(N; \mathbf{Z}/2)$ is a submanifold $X^{4l+2} \subset N$. If f is transverse to X and if $f^{-1}(\partial X) \to \partial X$ induces an isomorphism on $\mathbf{Z}/2$-homology, then the surgery obstruction of f is equal to the Kervaire invariant of $f \mid (f^{-1}X)$.

The isomorphism $L_0(\mathbf{Z}[\mathbf{Z}/2], +) \to \mathbf{Z} \oplus \mathbf{Z}$ sends a form represented by the matrix $(\alpha_{ij} + \beta_{ij}\gamma)$ to
$$(\tfrac{1}{8} I(\alpha_{ij} + \beta_{ij}), \tfrac{1}{8} I(\alpha_{ij} - \beta_{ij})).$$

If (λ_{ij}) is an even, non-singular symmetric matrix over $\mathbf{Z}_{(2)}$, then its *reduction modulo 2* is the quadratic form over $\mathbf{Z}/2$ whose bilinear form is given by $(\lambda_{ij}) \bmod 2$ and whose quadratic refinement is defined by
$$\mu([x]) = \{\tfrac{1}{2}({}^t x \cdot (\lambda_{ij}) \cdot x)\} \bmod 2.$$

(Since (λ_{ij}) is even, it follows that ${}^t x\cdot\lambda_{ij}\cdot x$ is even.) Let $c\colon W\to \mathbf{Z}/2$ be the map which assigns to (λ_{ij}) the Arf invariant of its reduction modulo 2. An elegantly simple argument of Levine, [24], shows that $c(\lambda_{ij})$ is 0 if $\det(\lambda_{ij})\equiv\pm 1 \bmod 8\mathbf{Z}_{(2)}$ and is 1 if $\det(\lambda_{ij})\equiv \pm 3 \bmod 8\mathbf{Z}_{(2)}$.

THEOREM 10.3

(a) $$L_i(\mathbf{Z}_{(2)}[\mathbf{Z}/2],\,+)=\begin{cases}\overline{W\oplus W} & i\equiv 0(4)\\ 0 & i\equiv 1(4)\\ \mathbf{Z}/2 & i\equiv 2(4)\\ 0 & i\equiv 3(4).\end{cases}$$

(b) $$L_i(\mathbf{Z}_{(2)}[\mathbf{Z}/2],\,-)=\begin{cases}\mathbf{Z}/2 & i\equiv 0(2)\\ 0 & i\equiv 1(2).\end{cases}$$

(*Here* $\overline{W\oplus W}=\text{kernel}\,[(c+c)\colon W\oplus W\to\mathbf{Z}/2]$.)

Proof. The calculations in [34] for $L_i(\mathbf{Z}[\mathbf{Z}/2],\,-)$ are valid as well for $L_i(\mathbf{Z}_{(2)}[\mathbf{Z}/2],\,-)$. Possibly the simplest proof of (a) is to take advantage of the fact that the projective class group and the Whitehead group of $\mathbf{Z}_{(2)}[\mathbf{Z}/2]$ are 0. Under these hypotheses the fibered square

$$\begin{tikzcd}
\mathbf{Z}_{(2)}[\mathbf{Z}/2] \arrow[r, "r^+"] \arrow[d, "r^-"'] \arrow[dr, phantom, "\begin{array}{c} a+b\gamma\mapsto a+b\\ \downarrow\\ a-b\end{array}"] & \mathbf{Z}_{(2)} \arrow[d] \\
\mathbf{Z}_{(2)} \arrow[r] & \mathbf{Z}/2
\end{tikzcd}$$

leads to a long exact sequence of Wall groups (see [2], page 27):

(10.4) $$\ldots\to L_{i+1}(\mathbf{Z}/2)\to L_i(\mathbf{Z}_{(2)}[\mathbf{Z}/2])\xrightarrow{r^+ + r^-} L_i(\mathbf{Z}_{(2)})\oplus L_i(\mathbf{Z}_{(2)})\to\ldots$$

From (10.4) the computations follow easily. Note that the map

$$0\to L_0(\mathbf{Z}_{(2)}[\mathbf{Z}/2],\,+)\to W\oplus W$$

sends $(\alpha_{ij}+\beta_{ij}\gamma)$ to $((\alpha_{ij}+\beta_{ij}),\,(\alpha_{ij}-\beta_{ij}))$.

Sometimes the forms that occur in doing surgery on the next to the top stratum of a regular $O(n)$-manifold are intermediate between $\mathbf{Z}[\mathbf{Z}/2]$-forms and $\mathbf{Z}_{(2)}[\mathbf{Z}/2]$-forms. The reason is that we have a normal map which, when restricted to the boundary, is an isomorphism on $\mathbf{Z}$ and $\mathbf{Z}_{(2)}[\mathbf{Z}/2]$ homology and we wish to do surgery to make the map on the interior an isomorphism on $\mathbf{Z}$ and $\mathbf{Z}_{(2)}[\mathbf{Z}/2]$ homology. The fundamental group is $\mathbf{Z}/2$

and the manifolds are oriented. Before calculating the relevant surgery obstruction groups we need a lemma.

LEMMA 10.5. *Let T be a $\mathbf{Z}[\mathbf{Z}/2]$-module of odd order. (It has a short free resolution.) If (T, l, q) is a non-singular symmetric linking form over $\mathbf{Z}[\mathbf{Z}/2]$, then its class in $L_3(\mathbf{Z}[\mathbf{Z}/2], +)$ is trivial if and only if the order of T is of the form $8s \pm 1$.*[1]

Proof. Take a degree one normal map $f: X^{4k+3} \to Y^{4k+3}$ with $K_i(f; \mathbf{Z}[\mathbf{Z}/2]) = 0$ for $i < 2k+1$, and with $K_{2k+1}(f; \mathbf{Z}[\mathbf{Z}/2])$ equal to T. If the surgery obstruction of f is trivial, then there is a normal map $F: W^{4k+4} \to Y \times I$ from f to a homotopy equivalence. Consider $\tilde{F}: \tilde{W} \to \tilde{Y} \times I$. It is a normal map between spaces with free involutions, and is a $\mathbf{Z}_{(2)}$-equivalence on the boundary. Consider the non-equivariant surgery obstruction $\tau(\tilde{F})$ in $L_0(\mathbf{Z}_{(2)})$. This element has zero Arf-Kervaire invariant (since it is twice the Arf-Kervaire invariant of the normal map on the quotient spaces). Hence, Levines's result says that, after making $\tilde{F}$ highly connected, the determinant of the matrix for the intersection paiting on $K_{2k+2}(\tilde{F})$ is ± 1 modulo 8. This means that the order of $K_{2k+1}(\partial \tilde{F})$ is ± 1 modulo 8, i.e., the order of $K_{2k+1}(\tilde{f})$ is ± 1 modulo 8. This proves that if (T, l, q) represents 0 in $L_3(\mathbf{Z}[\mathbf{Z}/2], +)$, then the order of T is ± 1 modulo 8. From this, the result follows easily.

Let us return now to the calculation of the surgery group associated with the next to the top stratum.

THEOREM 10.6. *Let $f: (M^m, \partial M) \to (N^m, \partial N^m)$ be a normal map with $\pi_1(N) = \mathbf{Z}/2$ and N orientable. Suppose that $f | \partial M$ is a $\mathbf{Z}$ and $\mathbf{Z}_{(2)}[\mathbf{Z}/2]$ homology isomorphism. Then the obstruction to doing surgery to make f a $\mathbf{Z}$ and $\mathbf{Z}_{(2)}[\mathbf{Z}/2]$ homology isomorphism lies in a group $L_m^*(\mathbf{Z}_{(2)}[\mathbf{Z}/2], +)$. These groups are given by the following table:*

$$L_i^*(\mathbf{Z}_{(2)}[\mathbf{Z}/2], +) = \begin{cases} \mathbf{Z} \oplus \overline{W} \subset L_0(\mathbf{Z}_{(2)}[\mathbf{Z}/2], +) & i \equiv 0(4) \\ 0 & i \equiv 1(4) \\ \mathbf{Z}/2 \text{ (The Arf-Kervaire invariant)} & i \equiv 2(4) \\ 0 & i \equiv 3(4). \end{cases}$$

Proof. $K_i(\partial M; \mathbf{Z}[\mathbf{Z}/2])$ is, as an abelian group, odd torsion. Consequently, it decomposes as $K_i^+(\partial M) \oplus K_i^-(\partial M)$ where $\gamma \in \mathbf{Z}/2$ acts by multiplication by ± 1 on $K_i^{\pm}(\partial M)$. Since $K_i(\partial M; \mathbf{Z}) = 0$, $K_i^+(\partial M) = 0$.

Consider now the case $m = 2l+1$. Surgery below the middle dimension allows us to assume that $K_i(M; \mathbf{Z}[\mathbf{Z}/2])$ vanishes for $i < l$. Since $L_{2l+1}(\mathbf{Z}_{(2)}[\mathbf{Z}/2], +) = 0$, we can, in addition, assume that $K_l(M; \mathbf{Z}[\mathbf{Z}/2])$ is odd torsion, and hence, equal to $K_l^+ \oplus K_l^-$. All higher

[1] This lemma gives a different proof of the fact that there is no codimension one Kervaire invariant in $L_3(\mathbf{Z}_{(2)}[\mathbf{Z}/2], +)$.

kernel modules for f are of the form K_l^-. We wish to do surgery to kill K_l^+. The linking form on $K_l^+ \oplus K_l^-$ breaks up into the orthogonal sum of a pairing on K_l^+ and one on K_l^-. Since $K_i^+(\partial M)=0$ for all $i \neq l$ it follows that the linking form on K_l^+ is non-singular. Hence, $(K_l^+, l|K_l^+, g|K_l^+)$ determines an element in $L_{2l+1}(\mathbf{Z}[\mathbf{Z}/2], +)$. Assume, for the moment, that this element is trivial. Let (F, λ, μ) be a resolution over $\mathbf{Z}[\mathbf{Z}/2]$. We use this resolution to construct a normal bordism $G: W \to N \times I$ with one end being $f: M \to N$ and with

$$(1) \qquad K_i(W, M; \mathbf{Z}[\mathbf{Z}/2]) = \begin{cases} 0; & i \neq l+1 \\ F^*; & i = l+1 \end{cases}$$

$$(2) \qquad K_{l+1}(W)/(\operatorname{Im} K_{l+1}(M)) \to K_{l+1}(W, M) \quad \text{identified with ad } (\lambda).$$

Let $f': M' \to N$ be the other end of this normal bordism. From the kernel sequences for (W, M) and (W, M') we see that

$$K_i(f'; \mathbf{Z}[\mathbf{Z}/2]) = \begin{cases} K_i(f; \mathbf{Z}[\mathbf{Z}/2]) & i \neq l \\ K_l^-(f) & i = l. \end{cases}$$

Thus, f' induces an isomorphism on $\mathbf{Z}$ and $\mathbf{Z}_{(2)}[\mathbf{Z}/2]$ homology.

The above argument was predicated on the fact that $(K_l^+, l|K_l^+, q|K_l^+)$ determined the trivial element in $L_{2l+1}(\mathbf{Z}[\mathbf{Z}/2], +)$. If it does not, then $2l+1 \equiv 3(4)$. Form the normal bordism $M \times I \# K \to N \times I$ where K is the plumbing of two copies of the tangent bundle of S^{l+1}. Since $H_l(\partial K) \cong \mathbf{Z}/3$, this has the effect of adding a copy of a non-trivial linking form on $\mathbf{Z}/3$ to K_l^+. By the previous lemma this changes the surgery obstruction of $(K_l^+, l|K_l^+, q|K_l^+)$ to zero. This completes the proof that $L_{2l+1}^*(\mathbf{Z}_{(2)}[\mathbf{Z}/2], +)$ is zero.

We turn now to $L_{2l}^*(\mathbf{Z}_{(2)}[\mathbf{Z}/2], +)$. Our first aim is to show that if $\sigma(f) \in L_{2l}(\mathbf{Z}_{(2)}[\mathbf{Z}/2], +)$ is zero, then we can do surgery to kill simultaneously $K_l(f; \mathbf{Z}_{(2)}[\mathbf{Z}/2])$ and $K_l(f; Z)$. Showing this will show that $L_{2l}^*(\mathbf{Z}_{(2)}[\mathbf{Z}/2], +) \to L_{2l}(\mathbf{Z}_{(2)}[\mathbf{Z}/2], +)$ is an injection.

We can assume that $K_i(f; \mathbf{Z}[\mathbf{Z}/2]) = 0$ for $i < l$. The module $K_l(f; \mathbf{Z}[\mathbf{Z}/2])$ may have an odd torsion submodule, T. If so, the action of γ on T is multiplication by -1. The module $K_l(f; \mathbf{Z}[\mathbf{Z}/2])/T$ is a free $\mathbf{Z}[\mathbf{Z}/2]$-module and has a $(-1)^l$-hermitian-symmetric form (λ, μ) on it. It is embedded as a lattice in $K_l(f; \mathbf{Z}_{(2)}[\mathbf{Z}/2])$ with (λ, μ) being non-singular over $\mathbf{Z}_{(2)}[\mathbf{Z}/2]$. Suppose that over $\mathbf{Z}_{(2)}[\mathbf{Z}/2]$ the form is hyperbolic. Let

$$(H \oplus H^*, e, \mu) \overset{\varphi}{\cong} (K_l(f; \mathbf{Z}_{(2)}[\mathbf{Z}/2]), \lambda, \mu)$$

be an isomorphism. (Here H and H^* are dual under e and $\mu|H = 0$.) The map φ induces a splitting of $K_l(f; \mathbf{Z}[\mathbf{Z}/2])/T = K_l(f)/T$ into

$$\{\varphi(H) \cap (K_l(f)/T)\} \oplus \{\varphi(H^*) \cap (K_l(f)/T)\}.$$

The matrix for λ in this splitting is of the form

$$\left(\begin{array}{c|c} 0 & (\alpha_{ij}) \\ \hline (-1)^l(\alpha_{ji}) & 0 \end{array}\right)$$

where (α_{ij}) is a $\mathbf{Z}[\mathbf{Z}/2]$-matrix which is non-singular over $\mathbf{Z}_{(2)}[\mathbf{Z}/2]$. In addition, since $K_l(f|\partial M; \mathbf{Z})=0$, the integral matrix $(r^+(\alpha_{ij}))$ is non-singular. As a result, doing surgery on a $\mathbf{Z}[\mathbf{Z}/2]$-basis for $\varphi(H)\cap(K_l(f)/T)$ produces the required normal map.

This proves that $L^*_{2l}(\mathbf{Z}_{(2)}[\mathbf{Z}/2], +)$ injects into $L_{2l}(\mathbf{Z}_{(2)}[\mathbf{Z}/2], +)$. One sees easily that this inclusion is onto for $l\equiv 1(2)$, and has image $\mathbf{Z}\oplus\overline{W}$ for $l\equiv 0(2)$.

Let $R^\pm$ be the $R[\mathbf{Z}/2]$-module structure on R where γ acts by ± 1. For any $R[\mathbf{Z}/2]$-module C, define $C^\pm$ to be $C\otimes_{R[\mathbf{Z}/2]}R^\pm$ and define $r^\pm\colon C\to C^\pm$ to be the obvious reduction. If $\Lambda\colon M\times M\to C$ is a bilinear form, then let $\Lambda^\pm\colon M^\pm\times M^\pm\to C^\pm$ be defined by $\Lambda^\pm(r^\pm(m_1), r^\pm(m_2))=r^\pm(\Lambda(m_1, m_2))$. Also, let $r^\pm\colon L_0(R[\mathbf{Z}/2])\to L_0(R)$ be the induced map.

If M is a $\mathbf{Q}[\mathbf{Z}/2]$-module and $\lambda\colon M\times M\to\mathbf{Q}[\mathbf{Z}/2]$ is a bilinear form, then there is naturally associated a form over $\mathbf{Q}$, $\lambda_e\colon M\times M\to\mathbf{Q}$ defined by taking $\lambda_e(x, y)$ to be the coefficient of the identity of $\lambda(x, y)$. Let $E^\pm$ be the (± 1)-eigenspace for the action of γ on M. Then $M=E^+\oplus E^-$ is an orthogonal decomposition for λ_e. Furthermore, $E^\pm$ is canonically isomorphic to $M^\pm$, and under this identification $\lambda_e|E^\pm$ becomes $\frac{1}{2}\lambda^\pm$.

PROPOSITION 10.7. *Let $f\colon(M^{4m}, \partial M)\to(N^{4m}, \partial N)$ be a normal map. Suppose that $\pi_1(N)=\mathbf{Z}/2$, that N is orientable, and that $f|\partial M$ is an $R[\mathbf{Z}/2]$-homology isomorphism. Let $\tilde f\colon(\tilde M, \partial\tilde M)\to(\tilde N, \partial\tilde N)$ be the double cover of f, and let $\tau(\tilde f)$ denote its (non-equivariant) surgery obstruction. Then $\tau(\tilde f)=\sigma_+(f)+\sigma_-(f)$ in $L_0(R)$, where $\sigma_\pm(f)=r^\pm(\sigma(f))$.*

Proof. Suppose that f is highly connected and that (K, λ) is its middle dimensional intersection form over $R[\mathbf{Z}/2]$. The middle dimensional form for $\tilde f$ (over R) is then (K, λ_e). Hence, over $\mathbf{Q}$, the form for $\tilde f$ is isomorphic to $(K^+\otimes\mathbf{Q}, \frac{1}{2}\lambda^+)\oplus(K^-\otimes\mathbf{Q}, \frac{1}{2}\lambda^-)$. Since the map $L_0(R)\to L_0(\mathbf{Q})$ defined by sending (G, ψ) to $(G\otimes_R\mathbf{Q}, \frac{1}{2}\psi)$ is an injection, it follows that in $L_0(R)$ the form (K, λ) is equivalent to $(K^+, \lambda^+)\oplus(K^-, \lambda^-)$.

11. Statement of the main theorem

The main theorem concerns the calculation of $\Theta^1(k, n, m)$. Since the case $m=0$ is different in several ways, we postpone discussion of it until Section 14. Here, we make the following assumption:

Assumption 11.1. $m>0$.

For Sections 11 and 12, $F\colon (V^{kn+m}, \Sigma^{kn+m-1}) \to (D^{kn+m}, S^{kn+m-1})$ is as in 9.3. The surgery obstruction $\sigma({}_iF)$ lies in $K_{ki+m}(\mathbf{Z})$ for $i \equiv n(2)$ and in $L_{ki+m}(\mathbf{Z}_{(2)})$ for $i \not\equiv n(2)$. Set $\sigma_i(\Sigma)=\sigma({}_iF)$.

PROPOSITION 11.2. *The invariant $\sigma_i(\Sigma)$ depends only on the concordance class of Σ^{kn+m-1}. Moreover, $\sigma_i(\Sigma)$ is additive with respect to equivariant connected sum. Thus, for each integer i such that $i\equiv n(2)$ there is a homomorphism*

$$\sigma_i\colon \Theta^1(k, n, m) \to L_{ki+m}(\mathbf{Z}).$$

For each integer i such that $i \not\equiv n(2)$ there is a homomorphism

$$\sigma_i\colon \Theta^1(k, n, m) \to L_{ki+m}(\mathbf{Z}_{2)}).$$

Proof. Let W be a concordance from Σ to Σ'; let V' be the framed manifold bounded by Σ'; let $F'\colon (-V', -\Sigma') \to (D, S)$ be the equivariant stratified normal map of Theorem 8.7. By Theorems 5.1 and 6.2, we can find an equivariant stratified normal map $F''\colon W \to S\times I$. Since $F|\Sigma$ and $F'|(-\Sigma')$ are unique up to an equivariant stratified homotopy, we may assume that $F''|\partial W = F|\Sigma \coprod F'|(-\Sigma')$. Notice that $S^{kn+m} = D^{kn+m} \cup (S^{kn+m-1}\times I) \cup D^{kn+m}$. Let $Y^{kn+m} = V \cup W \cup (-V')$ and let $G = F \cup F'' \cup F'\colon Y^{kn+m} \to S^{kn+m}$. By Smith Theory, $F''|_iW$ is a homology equivalence with coefficients $\mathbf{Z}$ (for $i\equiv n(2)$) or $\mathbf{Z}_{(2)}$ (for $i\not\equiv n(2)$); hence, $\sigma({}_iF'')=0$. Using Theorem 5.2 we see that T^{kn+m} bounds a framed manifold X^{kn+m-1} and that G extends to an equivariant stratified normal map from X^{kn+m-1} to D^{kn+m-1}. Consequently,

$$\begin{aligned} 0 &= \sigma({}_iG) \\ &= \sigma({}_iF) + \sigma({}_iF'') - \sigma({}_iF') \\ &= \sigma({}_iF) - \sigma({}_iF'). \end{aligned}$$

This proves that σ_i depends only on the concordance class of Σ. It is clearly additive with respect to connected sum.

The following questions arises: What values can be assumed by the $\{\sigma_i\}$? This is answered by the following result.

PROPOSITION 11.3. *If k is odd, then $\sigma_i=0$. If k is even, then $\sigma_i=\sigma_{i+2}$ and $c(\sigma_{i+1}(\Sigma)) = c(\sigma_i(\Sigma))$, where c stands for the Arf-Kervaire invariant.*

(Note that if $k\equiv 0(2)$, then $\dim({}_iV) \equiv \dim({}_{i+2}V)$ (4).)

To deduce Proposition 11.3 one needs the fact that ${}_{i+2}V$ has a semi-free S^1-action with fixed point set ${}_iV$, and the fact that ${}_{i+1}V$ has a $\mathbf{Z}/2$-action with fixed point set ${}_iV$.

The former is obtained by restricting the action of $O(i)\times SO(2)\subset O(i+2)$ to the second factor, and the latter is obtained by restricting the action of $O(i)\times O(1)\subset O(i+1)$ to the second factor. The proposition now follows immediately from 15.4 and 15.5.

Let $R_+=\mathbf{Z}$ and $R_-=\mathbf{Z}_{(2)}$. Consider $(\sigma_0,\sigma_1)\colon \Theta^1(k,n,m)\to L_m(R_\varepsilon)\oplus L_{k+m}(R_{-\varepsilon})$, where $\varepsilon=(-1)^n$. We see, by 11.3, that the image of (σ_0,σ_1) is contained in the kernel of $c+c\colon L_m(R_\varepsilon)\oplus L_{k+m}(R_{-\varepsilon})\to\mathbf{Z}/2$. (Recall that $c\colon L_{2s}(R_\pm)\to\mathbf{Z}/2$ is the Arf invariant of the mod 2 reduction.)

The main result of this paper is the following.

THEOREM 11.4.

(I) *If k is odd, $m\neq 4$ and $(k,m)\neq(3,1)$, then $\Theta^1(k,n,m)=0$.*

(II) *If k is even, then*

$$\Theta^1(k,n,m)\xrightarrow{(\sigma_0,\,\sigma_1)} L_m(R_\varepsilon)\oplus L_{m+k}(R_{-\varepsilon})\xrightarrow{c_0+c_1}\mathbf{Z}/2$$

is exact. ($\varepsilon=(-1)^n$.) If $k\neq 2$ and $m\neq 4$, then (σ_0,σ_1) is injective.

The case (II) above leads to a calculation of most of the groups $\Theta^1(k,n,m)$ for k even.

THEOREM 11.5. *Suppose that $m\neq 4$, that $k\equiv 0(2)$, and that $k\neq 2$.*

(1) *If $k\equiv 0(4)$, then*

$$\Theta^1(k,n,m)=\begin{cases}\mathbf{Z}+\overline{W}; & m\equiv 0(4)\\ \mathbf{Z}/2; & m\equiv 2(4)\\ 0; & m\equiv 1(2).\end{cases}$$

(2) *If $k\equiv 2(4)$, then*

$$\Theta^1(k,n,m)=\begin{cases}\mathbf{Z}; & m+2n\equiv 0(4)\\ W; & m+2n\equiv 2(4)\\ 0; & m+2n\equiv 1(2).\end{cases}$$

(*Recall that $W=L_0(\mathbf{Z}_{(2)})$ and $\overline{W}$ is the kernel of $c\colon W\to\mathbf{Z}/2\to 0$.*)

12. Stratified surgery

Suppose, as before, that $F\colon(V^{kn+m},\Sigma^{kn+m-1})\to(D^{kn+m},S^{kn+m-1})$ is an equivariant stratified normal map, and that $f\colon(A,B)\to(K,L)$ is the map of orbit spaces induced by F. For each i, $0\leqslant i\leqslant k$, let $f_i\colon(A_i,B_i)\to(K_i,L_i)$ be the induced map of closed strata, and let $\bar f_i\colon(E_i(A),E_i(B))\to(E_i(K),E_i(L))$ be the induced map of double branched covers.

In this section we shall work with the $f_i\colon A_i\to K_i$ relative to $B\to L$. If we construct a normal bordism $\psi\colon Z_i\to K_i\times I$ from f_i to $g_i\colon C_i\to K_i$, relative to ∂A_i, then we can extend this to a stratified normal bordism $h\colon(G,B\times I)\to(K\times I,L\times I)$, where $G=A\times I\cup\psi^*\xi$

with ξ the cone bundle neighborhood of $K_i \times I$ in K. If we pull back the standard linear model over $K \times I$ by h, then we have an equivariant stratified normal bordism $(W, \Sigma \times I) \to (D^{kn+m} \times I, S^{kn+m} \times I)$. Hence, we can use a normal bordism of f_i: $A_i \to K_i$ to construct one for F: $V \to D$. Thus, as we do surgery on the f_i, we are actually doing equivariant surgery on V. Our goal is to do this in such a way that V becomes contractible, or more precisely, to understand the obstructions to making V contractible. All these obstructions are, in the end, described in terms of ordinary surgery obstructions for the f_i, as one would be led to expect by 9.3.

Here, we shall prove that almost all the obstructions to completing surgery on the f_i to achieve the conditions of 9.3 either vanish automatically or are indeterminant (i.e., can be made to vanish by an appropriate choice of surgery on the lower strata). The only ones which are meaningful and non-zero occur in the case when k is even and $i=0$ or 1. These obstructions will be identified with σ_0 and σ_1. A similar argument will show that we can construct a normal map F: $(V, \Sigma) \to (D, S)$ realizing any possible value of (σ_0, σ_1). This will prove Theorems 11.4 and 11.5.

Throughout this section we shall assume that $k>2$, that $m>0$ and that $m \neq 4$. In the case of mono-axial actions ($k=1$), the manifold V constructed in 5.2 is already contractible; hence, $\Theta^1(1, n, m)=0$. The investigation of the other cases is postponed until the next two sections.

Under the above hypothesis no stratum A_i can have dimension 4, and the only possible stratum with dimension <4 is A_0. Hence, if, when we are considering f_i: $A_i \to K_i$, the high dimensional obstruction vanishes, we will be able to do surgery on f_i to make it an equivalence.

Assume by induction that we have completed surgery through level $i-1$, i.e., assume that:

(I) $(f_j)_*$: $H_*(A_j; \mathbf{Z}_{(2)}[\pi]) \to H_*(K_j; \mathbf{Z}_{(2)}[\pi])$ is an isomorphism for all j such that $0 \leq j < i$, where $\pi = \pi_1(K_j)$.

(II) $(\tilde{f}_j)_*$: $H_*(E_j(A); \mathbf{Z}) \to H_*(E_j(K); \mathbf{Z})$ is an isomorphism for all j, $0 \leq j < i$, such that $j \equiv n(2)$.

We consider the problem of doing surgery on the normal map f_i: $(A_i, \partial A_i) \to (K_i, \partial K_i)$. There are two cases depending on whether $(n-i)$ is even or odd.

Case 1. $i \not\equiv n(2)$.

LEMMA 12.1. *Suppose that the inductive hypotheses* (I) *and* (II) *hold through level* $i-1$ *and that* $i \not\equiv n(2)$. *Let* $\pi = \pi_1(K_i)$. *Then*

(1) *$f_i|\partial A_i$: $\partial A_i \to \partial K_i$ induces an isomorphism on $\mathbf{Z}_{(2)}[\pi]$-homology.*

(2) *If $i=k$ or $i=k-1$, then $f_i|\partial A_i$ also induces an isomorphism on integral homology.*

Proof. By Theorem 7.2, the map $\partial E_{i+1}(A) \to \partial E_{i+1}(K)$ induces an isomorphism on integral homology. Since the map $f_i|\partial A_i$ is the induced map of fixed point sets of the involutions on $\partial E_i(A)$ and $\partial E_i(K)$, it follows that $f_i|\partial A_i$ induces an isomorphism on $\mathbf{Z}/2$-homology or, equivalently, on $\mathbf{Z}_{(2)}$-homology. By considering the Gysin sequence of the double cover $\partial \tilde{A}_i \to \partial A_i$, we see that $\tilde{f}_i|\partial \tilde{A}_i$ is also a $\mathbf{Z}/2$-homology isomorphism. This means that $f_i|\partial A_i$ induces an isomorphism on $\mathbf{Z}_{(2)}[\pi]$-homology. This proves (1). Statement (2) is a direct consequence of the proof of 7.3.

When $i \not\equiv n(2)$, we try to do surgery on f_i (relative to $f_i|\partial A_i$) to a homology equivalence with coefficients dictated by the above lemma. If $i \leqslant k-2$, the obstruction to completing this surgery lies in a Wall group of the form $L_s(\mathbf{Z}_{(2)}[\pi], \pm)$, where π is either $\mathbf{Z}/2$ or 0, $s=\dim A_i$, and the $\pm$ refers to the orientability of K_i. If $i=k$, the obstruction lies in the ordinary simply connected surgery group, $L_s(\mathbf{Z})$. If $i=k-1$, then the obstruction to completing surgery to a map which is simultaneously a homology equivalence with coefficients $\mathbf{Z}$ and $\mathbf{Z}_{(2)}[\pi]$, lies in the group $L_s^*(\mathbf{Z}_{(2)}[\pi], \pm)$ which was calculated in 10.6.

Case 2. $i \equiv n(2)$. We assume that surgery has been done so that conditions (I) and (II) hold. This implies that $\tilde{f}|\partial E_i(A)$: $\partial E_i(A) \to \partial E_i(K)$ induces an isomorphism on integral homology, and that f_{i-1}: $A_{i-1} \to K_{i-1}$ is a $\mathbf{Z}_{(2)}[\pi]$-equivalence. We want to arrange, by doing surgery on f_i: $A_i \to K_i$ relative to $f_i|\partial A_i$, that $\tilde{f}_i$: $E_i(A) \to E_i(K)$ is an integral equivalence. Put another way, we want to identify the obstruction to doing $\mathbf{Z}/2$-equivariant surgery on $\tilde{f}_i$: $E_i(A) \to E_i(K)$ relative to $\partial E_i(A) \cup A_{i-1}$. (If $i=0$, then there is no involution and $E_0(K)=K_0$ is simply connected. Thus, in this case, the surgery obstruction lies in $L_m(\mathbf{Z})$.) If the involutions on $E_i(A)$ and $E_i(K)$ were fixed point free or if f_{i-1}: $A_{i-1} \to K_{i-1}$ were a homotopy equivalence, then this would be an ordinary surgery problem with obstruction in $L_*(\mathbf{Z}[\mathbf{Z}/2], \pm)$. We claim that if f_{i-1}: $A_{i-1} \to K_{i-1}$ is highly connected (with one extra condition if $i=k-1$), then the obstruction for doing $\mathbf{Z}/2$-equivariant surgery on $\tilde{f}_i$ still lies in $L_*(\mathbf{Z}[\mathbf{Z}/2], \pm)$. To prove this we need a lemma.

LEMMA 12.2. *Suppose that F: $(V, \Sigma) \to (D, S)$ satisfies conditions* (I) *and* (II) *for $j<i$, that $i \equiv n(2)$, and that $i<k$. Let r be the dimension of A_{i-1}.*

(a) *We can do surgery on f_{i-1} relative to ∂A_{i-1} so that f_{i-1} satisfies condition* (I) *and also so that $K_*(f_{i-1}; \mathbf{Z}[\mathbf{Z}/2])$ is zero for $* \leqslant [(r-3)/2]$.*

(b) *If* $i=k-1$, *then* A_{i-1} *is non-orientable. If* $\dim A_{i-1}=2s$, *then we can assume that part* (a) *is satisfied, and that* $K_{s-1}(A_{i-1};\mathbf{Z}^-)=0$.

Proof. The normal method of doing surgery on f_{i-1} is to first make $K_*(f_{i-1};\mathbf{Z}[\mathbf{Z}/2])=0$ for $*\leqslant[(r-2)/2]$. Then, if the $\mathbf{Z}_{(2)}[\mathbf{Z}/2]$-surgery obstruction vanishes, further surgery in dimension $[r/2]$ will make it satisfy condition (I). Such surgery leaves $K_*(f_{i-1};\mathbf{Z}[\mathbf{Z}/2])$ unchanged for $*\leqslant[r-3/2]$. This proves part (a). Part (b) is an immediate consequence of Lemma 15.6.

The relevance of this is the following corollary.

COROLLARY 12.3. *Suppose that* $F:(V,\Sigma)\to(D,S)$ *satisfies the conclusion of Lemma* 12.2. *Further suppose that* $i\equiv n(2)$. *Let* s *be the dimension of* $E_i(K)$. *Then,* $K_*((E_i(A)-A_{i-1});\mathbf{Z})\to K_*(E_i(A);\mathbf{Z})$ *is an isomorphism for* $*<[s/2]$ *and onto for* $*=[s/2]$.

Proof. Note that if we blow up $E_i(A)$ along A_{i-1} the result is exactly the double cover of A_i, $\tilde{A}_i$. We may identify this blow-up with the complement of a tubular neighborhood of A_{i-1} in $E_i(A)$. Showing that $K_*(\tilde{A}_i;\mathbf{Z})\to K_*(E_i(A);\mathbf{Z})$ satisfies the above statement is equivalent to showing that $K_*(E_i(A),\tilde{A}_i;\mathbf{Z})=0$ for $*\leqslant[s/2]$. By the Thom isomorphism, $K_*(E_i(A),\tilde{A}_i;\mathbf{Z})\simeq K_{*-k+i-1}(A_{i-1};\mathbf{Z})$ where $\pi_1(A_{i-1})$ acts on the coefficients of the second kernel group by multiplication by $(-1)^{k-i}$. (Recall that the codimension of A_{i-1} in $E_i(A)$ is $(k-i+1)$.) Applying 12.2 gives the result.

Now perform equivariant surgery on $\tilde{A}_i\to\tilde{K}_i$ to make $K_*(\tilde{A}_i;\mathbf{Z})=0$ for $*<[s/2]$. By 12.3 this makes $K_*(E_i(A);\mathbf{Z})=0$ for $*<[s/2]$.

At this point we break the discussion into 2 cases $-\, s\equiv 0(2)$ and $s\equiv 1(2)$. In the first case, $s=2r$, the only remaining kernel module is $K_r(E_i(A);\mathbf{Z})$, It is a free abelian group and has a non-singular (integral) intersection form. In addition, there is an action of $\mathbf{Z}/2$ on $K_r(E_i(A);\mathbf{Z})$ coming from the involutions, γ, on $E_i(A)$ and $E_i(K)$. The intersection form is invariant under this action. This allows us to give $K_r(E_i(A);\mathbf{Z})$ the structure of a $\mathbf{Z}[\mathbf{Z}/2]$-module and to enhance the intersection form to a $\mathbf{Z}[\mathbf{Z}/2]$-valued form. We claim that $K_r(E_i(A);\mathbf{Z})$ is a free $\mathbf{Z}[\mathbf{Z}/2]$-module, and that the enhanced form is non-singular. If this is true, then it defines an element in $L_s(\mathbf{Z}[\mathbf{Z}/2],\pm)$. Since $K_r(E_i(A);\mathbf{Z})$ is a free abelian group, to show that it is a free $\mathbf{Z}[\mathbf{Z}/2]$-module it suffices to show that $K_r(E_i(A);\mathbf{Z}_{(2)})$ is a free $\mathbf{Z}_{(2)}[\mathbf{Z}/2]$-module. But $K_r(E_i(A);\mathbf{Z}_{(2)})=K_r(\tilde{A}_i;\mathbf{Z}_{(2)})$. The normal map $\tilde{f}_i|\tilde{A}_i$ is (1) a $Z_{(2)}$-homology equivalence on the boundary, (2) highly connected, and (3) an equivariant normal map between free $\mathbf{Z}/2$-actions. Thus, the usual results of surgery theory imply that $K_r(\tilde{A}_i;\mathbf{Z}_{(2)})$ is a free $\mathbf{Z}_{(2)}[\mathbf{Z}/2]$-module (see [34]). The pairing over $\mathbf{Z}[\mathbf{Z}/2]$ is automatically non-singular on $K_r(E_i(A);\mathbf{Z})$ since its integral counterpart is.

If the class determined by $K_r(E_i(A); \mathbf{Z})$ is trivial, then there is a subkernel $H \subset K_r(E_i(A); \mathbf{Z})$. Since $K_r(\tilde{A}_i; \mathbf{Z}) \to K_r(E_i(A); \mathbf{Z})$ is onto, we represent a $\mathbf{Z}[\mathbf{Z}/2]$-basis for H by disjointly embedded spheres in $\tilde{A}_i$. Equivariant surgery on these spheres and their images under the involution produces a new normal map $\tilde{f}'_i: E_i(A') \to E_i(K)$ which is an integral equivalence. Thus, if the obstruction associated to $\tilde{f}_i$ in $L_s(\mathbf{Z}[\mathbf{Z}/2], \pm)$ vanishes, then we can do equivariant surgery on $\tilde{f}_i: E_i(A) \to E_i(K)$ to make it an integral equivalence.

Now, let us consider the case $s = 2r+1$. Do surgery equivariantly on $\tilde{f}_i | \tilde{A}_i: \tilde{A}_i \to \tilde{K}_i$ until $K_*(E_i(A); \mathbf{Z}) = 0$ for $* < r$. Since $L_{2r+1}(\mathbf{Z}_{(2)}[\mathbf{Z}/2], \pm) = 0$, we can do further equivariant surgery in dimension r to make $K_r(\tilde{A}_i; \mathbf{Z}_{(2)}) = 0$. This makes $K_r(E_i(A); \mathbf{Z})$ an odd torsion group. It has a $\mathbf{Z}/2$-action and an equivariant, non-singular linking form. Hence, $K_r(E_i(A); \mathbf{Z})$ is a $\mathbf{Z}[\mathbf{Z}/2]$-module with a non-singular linking form. Since any $\mathbf{Z}[\mathbf{Z}/2]$-module of odd order has a short free resolution, it follows that $K_r(E_i(A))$ and its linking form define an element in $L_{2r+1}(\mathbf{Z}[\mathbf{Z}/2], \pm)$. If the class of this form vanishes, then we can do equivariant surgery on $\tilde{A}_i$ to realize a resolution of the form. (This uses the fact that $K_r(\tilde{A}_i) \to K_r(E_i(A))$ is onto.) The result of the surgery is to make $K_r(E_i(A); \mathbf{Z}) = 0$.

If we take the image of the obstruction defined above in $L_s(\mathbf{Z}_{(2)}[\mathbf{Z}/2], \pm)$, we clearly get the obstruction to making $f_i: A_i \to K_i$ a $\mathbf{Z}_{(2)}[\mathbf{Z}/2]$-homology equivalence. Since $L_{2r}(\mathbf{Z}[\mathbf{Z}/2], \pm) \to L_{2r}(\mathbf{Z}_{(2)}[\mathbf{Z}/2], \pm)$ is an injection, it follows that the obstruction associated to $\tilde{f}_i: E_i(A) \to E_i(K)$ by the above procedure is independent of the equivariant surgery which we performed on $E_i(A)$ when $s = 2r$ (provided, of course, that we work relative to $\partial E_i(A) \cup A_{i-1}$). The only case in which the surgery group $L_{2r+1}(\mathbf{Z}[\mathbf{Z}/2], \pm)$ is non-zero is when $2r+1 \equiv 3(4)$ and the involution is orientation preserving (i.e., $k-i+1 \equiv 0(2)$). The obstruction is also well-defined in this case as we point out in Remark 12.6, below.

We still must consider the case $i = k$, when $k \equiv n(2)$. Since $E_k(K)$ is the double of K_k along K_{k-1}, doing surgery to make $\tilde{f}_k$ an integral homology isomorphism is the same as making f_k and f_{k-1} integral equivalences. Completing surgery at level $(k-1)$ makes f_{k-1} and $f_k | \partial A_k$ integral equivalences. Hence, the surgery group for $\tilde{f}_k$ is $L_s(\mathbf{Z})$.

This completes the identification of the surgery groups for the $\tilde{f}_i$ ($i \equiv n(2)$), and shows how to associate to a normal map $F: (V, \Sigma) \to (D, S)$ a well-defined obstruction in $L_s(\mathbf{Z}[\mathbf{Z}/2], \pm)$ once surgery has been done so that conditions (I) and (II) hold through level $(i-1)$ and so that 12.2 holds for f_{i-1}.

We now show that if k is odd, or if $i > 1$, then any possible surgery obstruction either vanishes automatically or is indeterminant. The proof of this will be based on the following observations from Sections 1, 2 and 3:

(1) $\partial_i A_{i+1} \to A_i$ is a fiber bundle with fiber $\mathbf{RP}^{k-i-1}$.

(2) $\partial_i E_{i+2}(A) \to A_i$ is a fiber bundle with fiber $\mathbf{CP}^{k-i-1}$.

(3) The stratified map $f: A \to K$ induces bundle maps $f_{i+1} | \partial_i A_{i+1}: \partial_i A_{i+1} \to \partial_i K_{i+1}$ and $\partial_i E_{i+2}(A) \to \partial_i E_{i+2}(K)$ which cover f_i.

The relevance of these observations is due to the fact that there are "product formulae" which relate the surgery obstruction of a normal map to the obstruction of a bundle map which covers it. The precise statements and proofs of these results are postponed until Section 15 where they appear as Theorems 15.1, 15.2, 15.3, and 15.4.

The first result along this line shows that with one exception, all possible Arf-Kervaire invariants vanish or are indeterminant. The exception occurs when $i=0$ and k is even. Since the Arf-Kervaire invariant that arises as an obstruction to doing $\mathbf{Z}/2$-equivariant surgery on $\bar{f}_i$: $E_i(A) \to E_i(K)$ is equal to that of f_i: $A_i \to K_i$, it suffices to consider the Arf-Kervaire of f_i.

PROPOSITION 12.4. *Suppose that surgery has been completed through level* $(i-1)$ *and that* $\dim A_i = 2r$. *Consider the Arf-Kervaire invariant of* f_i: $A_i \to K_i$. *It vanishes*

(a) *if* $(k-i-1)$ *is even, or*

(b) *if* $i+1 \equiv n(2)$ *and* $(2r+k-i-1) \equiv 3(4)$.

(c) *In all other cases, provided* $i>0$, *it is indeterminant and can be made to vanish by changing the way surgery was done on* f_{i-1}.

Proof. (a) The map f_i: $A_i \to K_i$ is covered by the map of $\mathbf{RP}^{k-i-1}$-bundles $f_{i+1} | \partial_i A_{i+1}$: $\partial_i A_{i+1} \to \partial_i K_{i+1}$. Since the fiber is a projective space of even dimension, it follows from Theorem 15.3 that the Arf-Kervaire invariant of f_i equals that of $f_{i+1} | \partial_i A_{i+1}$. Next consider the map $f_{i+1} | \partial A_{i+1}$: $\partial A_{i+1} \to \partial K_{i+1}$. Its Arf-Kervaire invariant vanishes, since it is a boundary. On the other hand, it decomposes as $\bar{\partial} A_{i+1} \cup \partial_i A_{i+1} \to \bar{\partial} K_{i+1} \cup \partial_i K_{i+1}$, where $\bar{\partial} A_{i+1} = \overline{\partial A_{i+1} - \partial_i A_{i+1}}$. Since we have completed surgery through level $i-1$, $f_{i+1} | \bar{\partial} A_{i+1}$ is a $\mathbf{Z}/2$-homology equivalence. Consequently,

$$c(f_i) = c(f_{i+1} | \partial_i A_{i+1}) = c(f_{i+1} | \partial A_{i+1}) = 0.$$

This proves (a).

(b) Since surgery has been completed through level $i-1$ and $i+1 \equiv n(2)$, the map $\bar{f}_{i+1}$: $E_{i+1}(A) \to E_{i+1}(K)$ is an integral homology isomorphism on the boundary. There are involutions on $E_{i+1}(A)$ and $E_{i+1}(K)$ with fixed point sets A_i and K_i. Theorem 15.5 identifies the Arf-Kervaire invariant of f_i with that of $\tau(\bar{f}_{i+1})$. ($\tau(\bar{f}_{i+1})$ is the non-equivariant surgery obstruction of $\bar{f}_{i+1}$: $E_{i+1}(A) \to E_{i+1}(K)$, see 10.7.) Since $\dim E_{i+1}(A) = (2r+k-i) \equiv 0(4)$, and since $\bar{f}_{i+1}|\partial E_{i+1}(A)$ is an integral homology equivalence, it follows that the Arf-Kervaire invariant of $\tau(\bar{f}_{i+1})$ vanishes. This proves (b).

(c) Let h_{i-1}: $X_{i-1}\to K_{i-1}\times I$ be a normal bordism, relative to the boundary, from f_{i-1}: $A_{i-1}\to K_{i-1}$ to another map f'_{i-1}: $A'_{i-1}\to K_{i-1}$ which still satisfies conditions (I) and (II) at level $i-1$. Extend this to a stratified normal bordism h: $(X, B\times I)\to(K\times I, L\times I)$ and let f': $A'\to K$ be the "other end." Consider h_i: $X_i\to K_i\times I$. We have

$$\partial X_i = A_i \cup (\bar{\partial} A_i\times I)\cup h^*_{i-1}(\xi)\cup A'_i$$

where ξ is the $\mathbf{RP}^{k-i}$-bundle $\partial_{i-1}K_i\times I$. In part (c), $k-i$ is even; hence, by 15.3, $c(h_i|h^*_{i-1}(\xi))=c(h_{i-1})$. Since $\bar{\partial}A_i\times I$ is mapped by a $\mathbf{Z}/2$ equivalence we have that

$$0=c(h_i|\partial X_i)=c(f_i)+c(f'_i)+c(h_{i-1}).$$

Therefore, $c(f_i)$ is indeterminant provided we can choose h_{i-1}: $X_{i-1}\to K_{i-1}\times I$ to have non-zero Arf-Kervaire invariant. The dimension of $K_{i-1}\times I$ is $(2r-k+i)$. Since $(k-i)$ is even, $K_{i-1}\times I$ is orientable and of even dimension. If $(i-1)\not\equiv n(2)$, then since c: $L_{2*}(Z_{(2)})\to\mathbf{Z}/2$ is onto, we may choose h_{i-1} to have non-zero Arf-Kervaire invariant. If $(i-1)\equiv n(2)$ then the hypothesis of (c) implies that $(2r-k+i)\equiv 2(4)$. Hence, in this case, we can also arrange the Arf-Kervaire invariant h_{i-1} to be non-zero.

PROPOSITION 12.5. *Suppose that surgery has been done through level* $(i-1)$, *that* $\dim A_i\equiv 3(4)$, *and that* $i\equiv n(2)$. *If the surgery group associated to* $\bar{f}_i$: $E_i(A)\to E_i(K)$ *has a codimension one Kervaire invariant, then we can make the invariant vanish by changing the way we do surgery to make* f_{i-1}: $A_{i-1}\to K_{i-1}$ *a* $\mathbf{Z}_{(2)}[\mathbf{Z}/2]$-*equivalence.*

Proof. For $\bar{f}_i$ to have a codimension one Kervaire invariant we need $0<i<k$ and $k-i\equiv 1(2)$. (The second condition makes the involution on $E_i(K)$ orientation preserving.) We may first of all arrange that $\bar{f}_i$ is highly connected and that $K_r(E_i(A))$ is odd torsion (where $2r+1=\dim E_i(A)$). The dimension of A_{i-1} is odd. Thus, there is a normal bordism h_{i-1}: $X_{i+1}\to K_{i-1}$, relative to ∂A_{i-1}, between f_{i-1}: $A_{i-1}\to K_{i-1}$ and another highly connected $\mathbf{Z}_{(2)}[\mathbf{Z}/2]$-equivalence f'_{i-1}: $A'_{i-1}\to K_{i-1}$, so that the Arf-Kervaire invariant of h_{i-1} is non-zero. As before, extend this to a stratified normal bordism h: $X\to K\times I$ and call the other end f': $A'\to K$. By a further surgery on $\bar{f}'_i$ we may arrange that $\bar{f}'_i$: $E_i(A')\to E_i(K)$ is highly connected and that $K_r(E_i(A'))$ is odd torsion. We have $\bar{h}_i$: $E_i(X)\to E_i(K)\times I$. Since $E_i(X)$ and $E_i(K)\times I$ have involutions with fixed points X_{i-1} and $K_{i-1}\times I$, Theorem 15.5 implies that the Arf-Kervaire invariant of $\tau(\bar{h}_i)$ equals that of h_{i-1}. Since $\dim E_i(X)\equiv 0(4)$, if we

do (non-equivariant!) surgery on $\tilde{h}_i$ relative to $\partial E_i(X)$ to make it highly connected and look at the intersection pairing

$$K_{r+1}(E_i(X)) \otimes K_{r+1}(E_i(X)) \to \mathbf{Z}.$$

It is represented by a matrix of determinant congruent to ± 3 modulo 8. Hence, the order of $K_r(\partial E_i(X))$ is congruent to ± 3 modulo 8. But $K_r(\partial E_i(X)) = K_r(E_i(A)) \oplus K_r(E_i(A'))$. This implies that the orders of $K_r(E_i(A))$ and $K_r(E_i(A'))$ are, up to sign and modulo 8, different. Hence, by 10.5, the surgery obstructions for $\tilde{f}_i$ and $\tilde{f}_i'$ in $L_3(\mathbf{Z}[\mathbf{Z}/2], +)$ are different, i.e., the codimension 1 Kervaire invariant of $\tilde{f}_i$ is indeterminant.

Remark 12.6. This argument actually shows that the change in the codimension 1 Kervaire invariant of $\tilde{f}_i$ equals the Arf-Kervaire invariant of the bordism at level $(i-1)$. Hence, if we are working relative to $f_{i-1}: A_{i-1} \to K_{i-1}$, then we cannot change the codimension 1 Kervaire invariant of $\tilde{f}_i$, i.e., working relative to f_{i-1} the codimension 1 Kervaire invariant of $\tilde{f}_i$ is well-defined.

Now, we are left with obstructions only when A_i is orientable and $\dim A_i \equiv 0(4)$. Suppose, for the moment, that $0 < i < k$. In case 1, where $n \not\equiv i(2)$, we want to do surgery on $f_i: A_i \to K_i$ to make it a $\mathbf{Z}_{(2)}[\mathbf{Z}/2]$-equivalence. The obstruction lies in $L_0(\mathbf{Z}_{(2)}[\mathbf{Z}/2], +) = \overline{W \oplus W}$. By 12.4 the Arf-Kervaire invariant vanishes, so the obstruction actually lies in the subgroup $\overline{W} \oplus \overline{W}$. (If $i = k-1$, then the obstruction lies in $L_0^*(\mathbf{Z}_{(2)}[\mathbf{Z}/2], +) = \mathbf{Z} \oplus \overline{W}$.) In case 2, where $n \equiv i(2)$, the obstruction to doing equivariant surgery on $E_i(A)$ lies in $L_0(\mathbf{Z}[\mathbf{Z}/2], +) = \mathbf{Z} \oplus \mathbf{Z}$. Since $L_0(\mathbf{Z}[\mathbf{Z}/2], +) \to L_0(\mathbf{Z}_{(2)}[\mathbf{Z}/2], +)$ is an injection, it suffices to consider only the obstruction for making $f_i: A_i \to K_i$, a $\mathbf{Z}_{(2)}[\mathbf{Z}/2]$-equivalence, taking care to remember that when $i \equiv n(2)$ the obstruction lies in the subgroup $\mathbf{Z} \oplus \mathbf{Z} \subset \overline{W \oplus W}$.

For the top and bottom strata, the obstructions lie in somewhat different groups: $\sigma(f_k) \in L_0(\mathbf{Z})$; while $\sigma(f_0) \in L_0(\mathbf{Z})$ if $n \equiv 0(2)$, and $\sigma(f_0) \in L_0(\mathbf{Z}_{(2)}) = W$ if $n \equiv 1(2)$ (actually in the second case, provided $k \not\equiv 2(4)$, $\sigma(f_0) \in \overline{W}$ by 12.4).

We now show how almost all of these obstructions annihilate each other.

PROPOSITION 12.7. *Suppose that surgery has been done on* $F: (V, \Sigma) \to (D, S)$ *through level* $(i-1)$, *and suppose that* A_i *is orientable* (i.e., $(k-i-1) \equiv 0(2)$, $i = 0$, *or* $i = k$) *and that* $\dim A_i = 4r$. *Consider the obstruction* $\sigma(f_i)$ *to completing surgery at level i. If* $0 < i < k$, *then* $\sigma(f_i) = (\sigma_+(f_i), \sigma_-(f_i)) \in \overline{W} \oplus \overline{W}$.

(a) *If* $1 < i < k$ *and* $(k-i-1) \equiv 0(4)$, *then* $\sigma_+(f_i) = 0$ *and* $\sigma_-(f_i)$ *is indeterminant (tied to the stratum 2 levels down). If* $1 < i < k$ *and* $(k-i=1) \equiv 2(4)$, *then* $\sigma_-(f_i) = 0$ *and* $\sigma_+(f_i)$ *is indeterminant.*

(b) *If* $i=1$, *then* $\sigma_+(f_i)$ *vanishes when* $k-2\equiv 0(4)$, *and* $\sigma_-(f_i)$ *vanishes when* $k-2\equiv 2(4)$.

(c) *If* k *is odd, then* $\sigma(f_0)$ *vanishes.*

(d) $\sigma(f_k)$ *is indeterminant.*

Proof. Suppose that $0<i<k$ and that $(k-i-1)=2l$. Consider $\sigma(f_i)=(\sigma_+(f_i),\sigma_-(f_i))\in \overline{W}\oplus\overline{W}$ (remembering that when $i\equiv n(2)$ that (σ_+,σ_-) actually lies in $\mathbf{Z}\oplus\mathbf{Z}$). We have $\mathbf{CP}^{2l}$-bundles, $\partial_i E_{i+2}(A)\to A_i$ and $\partial_i E_{i+2}(K)\to K_i$. The map $\tilde{f}_i|\partial_i E_{i+2}(A)$ is a bundle map covering $f_i\colon A_i\to K_i$. By 3.3, the action of $\pi_1(K_i)$ on $H_*(\mathbf{CP}^{2l})$ is non-trivial. Hence, Theorem 15.1 tells us that $\sigma_\omega(f_i)=\tau(\tilde{f}_{i+2}|\partial_i E_{i+2}(A))$ in $L_0(\mathbf{Z}_{(2)})$. (Here, $\omega=(-1)^l$.) Since we have done surgery through level $(i-1)$, $K_*(\partial E_{i+2}(A);\mathbf{Z}_{(2)})\cong K_*(\partial_i E_{i+2}(A);\mathbf{Z}_{(2)})$. Hence, $\tau(\tilde{f}_{i+2}|\partial_i E_{i+2}(A))=\tau(\tilde{f}_{i+2}|\partial E_{i+2}(A))=0$. This proves (b) and the vanishing statements in (a).

Similarly, if $i\geqslant 2$, then $\partial_{i-2}E_i(A)\to A_{i-2}$ and $\partial_{i-2}E_i(K)\to K_{i-2}$ are $\mathbf{CP}^{k-i+1}$-bundles (where $k-i+1=2l+2$) and $\tilde{f}_i|\partial_{i-2}E_i(A)$ is a bundle map covering $f_{i-2}\colon A_{i-2}\to K_{i-2}$. Let $h_{i-2}\colon X_{i-2}\to K_{i-2}\times I$ be a normal bordism of f_{i-2}. Note that $K_{i-2}\times I$ is orientable and of dimension congruent to 0 modulo 4. If $i\not\equiv n(2)$, we can choose h_{i-2} to represent an arbitrary element in $\overline{W}\oplus\overline{W}\subset L_0(\mathbf{Z}_{(2)}[\mathbf{Z}/2])$; while, if $i\equiv n(2)$, it can represent an arbitrary element in $\mathbf{Z}+\mathbf{Z}$. Choose h_{i-2} so that $\sigma_{-\omega}(h_{i-2})=\tau(\tilde{f}_i)$ (if $i-2=0$ then choose $\sigma(h_0)=\tau(\tilde{f}_2)$). By Theorem 15.1 (or 15.2 if $i-2=0$), this will change $\tilde{f}_i\colon E_i(A)\to E_i(K)$ so that $\tau(\tilde{f}_i)=0$. However, in changing the map at level $(i-2)$ we have destroyed the fact that it is correct at level $(i-1)$. The dimension of A_{i-1} is even and A_{i-1} is non-orientable. Thus, the surgery obstruction for f_{i-1} is an Arf-Kervaire invariant. By Theorem 15.3, the Arf-Kervaire invariant that we introduce at level $(i-1)$ is equal to that of the normal bordism at level $(i-2)$. Since this bordism has obstruction in $\overline{W}\oplus\overline{W}$, its Arf-Kervaire invariant vanishes. Hence, we can complete surgery at level $(i-1)$. Since surgery at level $(i-1)$ only changes $\tilde{f}_i\colon E_i(A)\to E_i(K)$ by a normal bordism relative to $\partial E_i(A)$, it leaves $\tau(\tilde{f}_i)$ unchanged. Hence, we can assume that conditions (I) and (II) hold through level $(i-1)$, that f_{i-1} satisfies 12.2, and that $\tau(\tilde{f}_i)=0$. If we consider the image of $\tau(\tilde{f}_i)$ in $L_0(\mathbf{Z}_{(2)})=W$, then it is equal to $\tau(\tilde{f}_i|\tilde{A}_i)$. By 10.7, $\tau(\tilde{f}_i|\tilde{A}_i)=\sigma_+(f_i)+\sigma_-(f_i)$. Since, as we have already seen, $\sigma_\omega(f_i)$ vanishes automatically, it follows that in changing $\tau(\tilde{f}_i)$ to be zero, we have made $\sigma_{-\omega}(f_i)$ zero. This completes the proof of (a).

We turn now to the proof of (c). The map $f_0\colon A_0\to K_0$ is covered by a map of trivial $\mathbf{CP}^{k-1}$-bundles, $\partial_0 E_2(A)\to\partial_0 E_2(K)$. If k is odd, then Theorem 15.2 shows that $\sigma(f_0)=\tau(\tilde{f}_2|\partial_0 E_2(A))$ in $L_0(\mathbf{Z}_{(2)})$. But $\tau(\tilde{f}_2|\partial_0 E_2(A))=\tau(\tilde{f}_2|\partial E_2(A))=0$.

Part (d) is obvious since $\partial_{k-1}A_k\cong A_{k-1}$.

Now we can prove the main result, 11.4.

Proof of Theorem 11.4. (I) Suppose that k is odd and greater than 2. Then by 12.4, 12.5, and 12.7 all surgery obstructions either vanish or are indeterminant. We can therefore complete surgery provided no stratum has dimension 4. Thus, $\Theta^1(k, n, m)=0$, provided $m\neq 4$ and $(k, m)\neq(3, 1)$.

(II) Suppose that k is even and $k>2$. We want to show that

$$0 \longrightarrow \Theta^1(k, n, m) \xrightarrow{(\sigma_0, \sigma_1)} L_m(R_\varepsilon)\oplus L_{m+k}(R_{-\varepsilon}) \xrightarrow{c+c} \mathbf{Z}/2$$

is exact (provided $m\neq 4$). Here $\varepsilon=(-1)^n$, $R_+=\mathbf{Z}$, and $R_-=\mathbf{Z}_{(2)}$. First we show that (σ_0, σ_1) is injective. Let $F: (V, \Sigma)\to(D, S)$ be the equivariant stratified normal map covering $f: (A, B)\to(K, L)$. Suppose $\sigma_0(\Sigma)=0=\sigma_1(\Sigma)$. We must show that we can complete surgery on f. Note that ${}_0F=f_0: A_0\to B_0$. Hence, since $0=\sigma_0(\Sigma)=\sigma(f_0)$, we can complete surgery at level 0. Thus, we may assume that f_0 is an R_ε-homology equivalence. If dim $(A_1)=m+k\not\equiv 0(4)$, then by 12.4, 12.5 and 12.7, there are no further surgery obstructions. If $m+k\equiv 0(4)$ the only remaining obstruction is $\sigma_\omega(f_1)$, where $\omega=(-1)^{k/2}$. Since ${}_1F=E_1(f)$, we have that $0=\sigma_1(\Sigma)=\sigma(E_1(f))=\sigma(\tilde{f}_1)=\sigma_\omega(f_1)$. Hence, the map (σ_0, σ_1): $\Theta^1(k, n, m)\to L_m(R_\varepsilon)\oplus L_{m+k}(R_{-\varepsilon})$ is injective.

Next we must show that (σ_0, σ_1) is onto ker $[c+c: L_m(R_m(R_\varepsilon))\oplus L_{m+k}(R_{-\varepsilon})\to \mathbf{Z}/2]$. Let (α_0, α_1) be an arbitrary element of the kernel. We will construct k-axial actions on homotopy spheres Σ' and Σ'' so that:

(1) $\sigma_0(\Sigma')=\alpha_0$, and

(2) $\sigma_0(\Sigma'')=0$ and $\sigma_1(\Sigma'')$ is an arbitrary element of ker $[c_1: L_{m+k}(R_{-\varepsilon})\to \mathbf{Z}/2]$.

From these two special cases and additivity of the invariants, the general result follows. The idea in constructing both Σ' and Σ'' is to build a suitable stratified normal bordism from the identity map on L to $f: B\to L$ so that $f^*(S^{kn+m-1})$ is a homotopy sphere.

The construction of Σ'. Recall that $L_0=S^{m-1}$. We can find a normal bordism $h_0: G_0\to L_0\times I$ from id: $L_0\to L_0$ to a R_ε-homology equivalence $f_0: B_0\to L_0$ so that $\sigma(h_0)=\alpha_0$. Extend this to a stratified bordism $h: G\to L\times I$ and call the other end $f': B'\to L$. We claim that *without changing the bordism at level* 0, we can do stratified surgery on f' to a map $f: B\to L$ satisfying the conditions of 9.3 (so that $\Sigma'=f^*(S^{kn+m-1})$ is a homotopy sphere). Suppose m is even (otherwise the problem is trivial). By 12.4, 12.5, and 12.7 all surgery obstructions for f' either vanish or are indeterminant. If an obstruction is indeterminant, then it can be made zero by changing the stratified map at most two levels down. Hence, it suffices to check our claim on the 1-stratum and 2-stratum. Since dim $B_1=m-1+k\equiv 1(2)$, the only possible obstruction is the codimension one Kervaire invariant and this occurs only when n is odd and $m+k\equiv 0(4)$. By Remark 12.6, this codimension one Kervaire invariant is equal to $c(h_0)=c(\alpha_0)=c(\alpha_1)$. But $\alpha_1\in L_0(\mathbf{Z})$ and therefore, $c(\alpha_1)=0$. Thus, the obstruction at level 1 vanishes. The 2-stratum is non-orientable and hence, causes no problems since any non-vanishing Arf-Kervaire invariant can be made to vanish by changing the map one level down. This proves the claim and therefore, shows that we can construct Σ'.

Construction of Σ''. The argument is similar. Let $\beta\in\ker[c\colon L_{m+k}(R_{-\varepsilon})\to\mathbf{Z}/2]$. If $m+k\not\equiv 0(4)$, then $\beta=0$ and there is nothing to prove. So, suppose $m+k\equiv 0(4)$ and $k=2l$. Define $\hat\beta\in L_0(R_{-\varepsilon}[\mathbf{Z}/2],+)$ by

$$\hat\beta_+=\begin{cases}\beta;\ l \text{ even}\\ 0;\ l \text{ odd}\end{cases}\qquad \hat\beta_-=\begin{cases}0;\ l \text{ even}\\ \beta;\ l \text{ odd.}\end{cases}$$

We can construct a bordism (relative to the boundary)

$$h_1\colon G_1\to L_1\times I$$

from id: $(L_1,\partial_0L_1)\to(L_1,\partial_0L_1)$ to $f_1\colon(B_1,\partial_0B_1)\to(L_1,\partial_0L_1)$ with $\sigma(h_1)=\hat\beta$. As before, we assert that we can complete surgery without disturbing this bordism, and, as before, it suffices to check the 2-stratum and 3-stratum. By 15.3, the Arf-Kervaire invariant of the 2-stratum is equal to $c(h_1)=c(\beta)=0$. By Theorem 15.1 the part of the obstruction on the 3-stratum which does not automatically vanish is equal to $\hat\beta_-$ if l is even and to $\hat\beta_+$ if l is odd. In either case, it is zero. Hence, we can construct Σ'' with $\sigma_0(\Sigma'')=0$ and $\sigma_1(\Sigma'')=\beta$.

13. Bi-axial actions

In this section we calculate the groups $\Theta^1(2,n,m)$.

Suppose that F^k is a $\mathbf{Z}/2$ homology sphere, that Σ^{k+2} is the boundary of some contractible manifold (hence, Σ is an integral homology sphere), and that $F^k\subset\Sigma^{k+2}$. The pair (Σ^{k+2},F^k) is called an ε-knot if it satisfies condition $(\varepsilon)=(\pm)$ below:

$(+)$ F^k is an integral homology sphere.

$(-)$ The double branched cover of Σ along F, $\tilde\Sigma$, is an integral homology sphere.

Two ε-knots (Σ_0^{m+2},F_0^m) and (Σ_1^{m+2},F_1^m) are *ε-knot cobordant* if there is a pair (W^{m+3},L^{m+1}) such that W^{m+3} is an integral homology h-cobordism from Σ_0^{m+2} to Σ_1^{m+2}, L^{m+1} is a $\mathbf{Z}/2$-homology h-cobordism from F_0^m to F_1^m, and such that (W^{m+3},L^{m+1}) satisfies the condition analogous to (ε). Let HC_m^ε denote the abelian group (under connected sum) of such ε-knot cobordism classes.

If $O(n)$ acts bi-axially on a homotopy sphere Σ^{2n+m-1} with orbit space B, then the diffeomorphism class of B (as a stratified space) determines the equivariant diffeomorphism class of Σ^{2n+m-1}, see [15] and [18]. (From the point of view of this paper, the reason for this is that B admits a unique degree 1 map up to homotopy, into the orbit space of the linear model.) B is a manifold with boundary which has 3-strata—int B, $\partial B - B_0$, and B_0, (see 2.4). The pair $(\partial B, B_0)$, called the *orbit knot*, is an ε-knot, where $\varepsilon = (-1)^n$. The orbit space of a concordance yields an ε-knot cobordism of the orbit knots. Conversely, any ε-knot gives a bi-axial $O(n)$-manifold, well-defined up to concordance. Thus

$$\Theta^1(2, n, m) = HC^{\varepsilon}_{m-1}.$$

The group C_m of knot cobordism classes of homotopy m-spheres embedded in S^{m+2} has been defined and calculated by Kervaire [21] and Levine [25]. Their methods work equally well for HC^+_m (and in fact, $HC^+_m \cong C_m$ for $m > 3$). Our goal in this section is to provide a similar calculation of HC^-_m.

First, we recall the method of [21] and [25]. Any $\mathbf{Z}/2$-homology m-sphere, $F^m \subset \Sigma^{m+2}$ is the boundary of a framed submanifold $V^{m+1} \subset \Sigma^{m+2}$, called a "Siefert surface" for F^m. Let $F_*(V) = H_*(V)/\mathrm{Tor}\, H_*(V)$. In the case $m = 2l-1$, there is a bilinear form (a *Siefert form*) $\theta\colon F_l(V^{2l}) \otimes F_l(V^{2l}) \to \mathbf{Z}$ defined as: $\theta(\alpha, \beta)$ is the linking number of α with $i_*(\beta)$, where $i\colon V \to \Sigma - V$ is a small displacement of V in the positive normal direction.

If $A(x, y)$ is a bilinear form on a free abelian group, then for $\delta = \pm 1$ define a new form $A + \delta({}^tA)$ by $(A + \delta({}^tA))(x, y) = A(x, y) + \delta A(y, x)$. This new form is δ-symmetric. If we begin with $\theta\colon F_l(V^{2l}) \otimes F_l(V^{2l}) \to \mathbf{Z}$, then $\theta + (-1)^l({}^t\theta)$ is the usual intersection form on $F_l(V^{2l})$. Thus, $\theta + (-1)^l({}^t\theta)$ is non-singular over $\mathbf{Z}_{(2)}$. If F^{2l-1} is an integral homology sphere, then $\theta + (-1)^l({}^t\theta)$ is non-singular over $\mathbf{Z}$.

A bilinear form on a free abelian group is *null-cobordant*, if it vanishes on a direct summand of one half the rank. Two bilinear forms θ and θ' are cobordant, if the orthogonal direct sum of θ with $-\theta'$ is null cobordant. A form θ is said to be a *δ-form*, where $\delta = \pm 1$, if $\theta + \delta({}^t\theta)$ is non-singular over $\mathbf{Z}$.

Let G_δ be the group (under direct sum) of cobordism classes of δ-forms.

Kervaire proved that $HC^+_{2l} = 0$, and Levine showed that the map $\varphi_+\colon HC^+_{2l-1} \to G_\delta$, $\delta = (-1)^l$, defined by sending the cobordism class of a knot to the cobordism class of its Siefert form, is a well-defined homomorphism which is an isomorphism for $l > 2$. If $(\Sigma^{2l+1}, F^{2l-1})$ is a minus-knot, then $\theta + \delta({}^t\theta)$ may not be unimodular. However, we shall show below, in Corollary 13.5, that $\theta - \delta({}^t\theta)$ is unimodular, and that the map

$$\varphi_-\colon HC^-_{2l-1} \to G_{-\delta}$$

which associated to a minus-knot the cobordism class of its Seifert form is well defined.

The main result of this section is the following:

THEOREM 13.1.

(1) $HC^{\varepsilon}_{2l}=0$.

(2) *For* $l>2$, $\varphi_\varepsilon\colon HC^{\varepsilon}_{2l-1}\to G_{\varepsilon\delta}$, *is an isomorphism, where* $\delta=(-1)^l$.

(3) *The maps* $\varphi_+\colon HC_1^+\to G_+$, $\varphi_-\colon HC_1^-\to G_-\cap G_+$ *and* $\varphi_\varepsilon\colon HC_3^\varepsilon\to G_\varepsilon$ *are all epimorphisms.*

As we have remarked, the case $\varepsilon=+$ follows from the arguments of [21] and [25] almost without change. (There is an obvious modification for HC_3^+.) Hence, we shall not discuss this case further.

Of course, our interest in this result is the following corollary.

COROLLARY 13.2. *If* $n>2$ *and* $m\neq 2$ *or* 4 *then*

$$\Theta^1(2,n,m)=\begin{cases} G_+; & \text{if } m+2n\equiv 0(4)\\ G_-; & \text{if } m+2n\equiv 2(4)\\ 0; & \text{if } m+2n\equiv 1(2).\end{cases}$$

First we consider the even dimensional case.

PROPOSITION 13.3. $HC^-_{2l}=0$.

Proof. Let (Σ^{2l+2}, F^{2l}) be a minus-knot, and let W^{2l+3} be a parallelizable manifold that Σ bounds. Take a Seifert surface for F in Σ, L^{2l+1}, and deform it relative to F so that it becomes properly embedded in W. There is a normal map $f\colon (W,\Sigma)\to(D^{2l+3},S^{2l+2})$ which is transverse to $D^{2l+1}\times\{0\}$ with preimage L. Since $f|\partial L$ is a $\mathbf{Z}_{(2)}$-homology equivalence, we can do surgery to make $K_*(L;\mathbf{Z})=0$ for $*\leqslant l$ and $K_{l+1}(L;\mathbf{Z}_{(2)})=0$. We wish to do equivariant surgery on the double branced cover of W, $\hat{W}$ until it becomes contractible. As in 12.3, we see that $K_*(\hat{W}-L)\to K_*(\hat{W})$ is an isomorphism for $*\leqslant l$ and onto for $*=l+1$. Hence, if we do surgery on $W-L$ to make $K_*(W-L;\mathbf{Z}[\mathbf{Z}])=0$ for $*\leqslant l$ and $K_{l+1}(W-L;\mathbf{Z}[\mathbf{Z}/2])$ odd torsion, then $K_*(\hat{W};\mathbf{Z})=0$ for $*\leqslant l$ and $K_{l+1}(\hat{W};\mathbf{Z})$ is odd torsion. Since $\tilde{\Sigma}$ is an integral homology sphere, the linking form on $K_{l+1}(\hat{W};\mathbf{Z})$ is non-singular. As in Section 12, it defines an element in $L_{2l+3}(\mathbf{Z}[\mathbf{Z}/2],+)$. If the element it defines is trivial, then we can complete the surgery to make $K_*(\hat{W};\mathbf{Z})=0$. The only time that this element is not automatically zero is when $2l+3=3(4)$. In this case, as in 12.5, the obstruction (a codimension one Kervaire invariant) is indeterminant. Thus by changing the way in which we did surgery on L, we can make the obstruction vanish and hence, complete the surgery on W. Once we have made $\hat{W}$ contractible, the pair (W,L) becomes a minus-knot cobordism from (Σ, F) to zero.

Next we recall Bredon's description of a geometric construction which realizes the algebraic periodicity of knot cobordism, [5]. Let $B(2)$ denote the 3-dimensional cone of 2 by 2 symmetric matrices. Suppose that (Σ^{m+2}, F^m) is an ε-knot and that B^{m+3} is a contractible manifold with $\partial B^{m+3}=\Sigma^{m+2}$. Construct a local orbit space, also denoted by B^{m+3}, by replacing a tubular neighborhood of F^m with $F^m \times B(2)$. We can find a stratified map $f: B \to B(2)$, which is unique up to a stratified homotopy.(1) For each integer n, form the bi-axial $O(n)$-manifold

$$_nM = f^*(M(n, 2)).$$

Its associated orbit knot is (Σ^{m+2}, F^m). Consider the restriction of the $O(n+1)$-action on $_{n+1}M$ to $O(n)$, and let $(\Sigma^{m+4}, \tilde{\Sigma}^{m+2})$ be the associated orbit knot, i.e., let $\Sigma^{m+4}={}_2M/O(1)$ and $\tilde{\Sigma}^{m+2}={}_1M$. $(\Sigma^{m+4}, \tilde{\Sigma}^{m+2})$ is called the "suspended knot", and is also denoted by $\omega(\Sigma^{m+2}, F^m)$. Thus, $\omega^i(\Sigma^{m+2}, F^m)$ is the knot $(({}_{i+1}M)/O(1), {}_iM)$. If $n \geq 2$ and $(-1)^{k+1}=\varepsilon$, then $_{n+1}(M)$ is a homotopy sphere; hence, its orbit knot under $O(n)$, $(\Sigma^{m+4}, \tilde{\Sigma}^{m+2})$, is a $(-\varepsilon)$-knot. Thus, ω takes ε-knots to $(-\varepsilon)$-knots. Although the construction depends on the choices of B and of f, it is clearly well-defined up to concordance and therefore, defines a homomorphism $\omega: HC_m^{\varepsilon} \to HC_{m+2}^{-\varepsilon}$.

Recall that in Theorem 5.2 we constructed a regular $O(n)$-manifold $_nV \subset {}_{n+1}M$ with $\partial({}_nV)={}_nM$. The image of $_0V$ in $_1M/O(1)=\Sigma^{m+2}$ is a Seifert surface for F^m and the image of $_1V$ in $_2M/O(1)=\Sigma^{m+4}$ is a Seifert surface for $\tilde{\Sigma}^{m+2}$ in Σ^{m+4}. The manifold $_1V$ is the "suspended Seifert surface". Let $A=({}_nV)/O(n)$. Then $_1V$ (which is $E_1(V)$) is just the double branched cover of $A_1 \cup_p A_0$ along $_0V$ (which is A_0). In 5.2 we showed that A is homeomorphic to $B \times I$ with $\partial B \times I$ collapsed to $\partial B \times \{0\}$ and that the union of the singular strata of A is the image of $B \times \{1\}$. Hence, in this case $A_1 \cup_p A_0$ is homeomorphic to B. Thus, $_1V$ is the double branched cover of B along the Seifert surface $_0V$ (which has been pushed into the interior of B).

There is an alternative description of such double branched covers. We shall now use the notation $V^{m+1}={}_0V$ and $V^{m+3}={}_1V$. Let U^{m+2} denote $V^{m+1} \times I$ with $\partial V^{m+1} \times I$ pinched to $\partial V^{m+1} \times \{0\}$. Embed U^{m+2} in B^{m+3} so that it meets ∂B^{m+3} transversely in $V^{m+1} \times \{0\}$. Cut B^{m+3} open along U^{m+2} to obtain a new contractible manifold B'. Let U' denote the inverse image of U in $\partial B'$. Then V^{m+3} can be thought of as the union of two copies of B' glued along U' by switching the copies of U. The involution on V^{m+3} is given by switching the two copies of B. Since B' is contractible, it is clear from this description that V^{m+3} is $\mathbf{Z}/2$-equivariantly homotopy equivalent to the suspension of V^{m+1}. In particular, $H_*(V^{m+3}) \cong H_{*-1}(V^{m+1})$.

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(1) If $m=1$, then there are two ways to replace a tubular neighborhood of F by $F \times B(2)$. Only one of the resulting stratified spaces admits a stratified map to $B(2)$.

Remark 13.4. If $m+1$ is even, say $m+1=2t$, then there is a relationship between the Seifert form on $H_t(V)$ and an intersection form on $H_{t+1}(B', V\times\{1\})\cong H_t(V)$. Namely, let ζ_i^{t+1} and ζ_j^{t+1} be relative cycles in $(B', V\times\{1\})$ where the boundaries represent α_i and α_j in $H_t(V)$. Keep $\partial\zeta_i$ in $V\times\{1\}$ and deform ζ_j (keeping $\partial\zeta_j$ in U) until $\partial\zeta_j$ is contained in $V\times\{\frac{1}{2}\}$ embedded into the side of the cut corresponding to the positive normal direction on V in Σ. Then, relative to these constraints on the boundaries, push the cycles transverse and take their intersection number. It is $\theta(\alpha_i, \alpha_j)$. Using this fact one can prove the following theorem:

THEOREM 13.5 (*Bredon* [6]). *There is an identification of $H_*(V^{2l}; \mathbf{Z})$ with $H_{*+1}(V^{2l+2}; \mathbf{Z})$ under which θ: $F_l(V^{2l})\otimes F_l(V^{2l})\to\mathbf{Z}$ becomes identified with $\omega(\theta)$: $F_{l+1}(V^{2l+2})\otimes F_{l+1}(V^{2l+2})\to\mathbf{Z}$.*

COROLLARY 13.6. *Suppose that $(\Sigma^{2l+1}, F^{2l-1})$ is an ε-knot. Then the associated Seifert form is an $\varepsilon\cdot(-1)^l$-form.*

The proof of the next lemma is the same as the proof of Lemma 2 in [25].

LEMMA 13.7. *If an ε-knot is ε-knot cobordant to a trivial knot, then any Seifert form for it is null-cobordant.*

It follows that the map

$$\varphi_-: HC^-_{2l-1}\to G_{-\delta}, \quad \delta=(-1)^l,$$

is a well-defined homomorphism. As further corollaries to Theorem 13.5, we have the following results.

COROLLARY 13.8. *The following diagram commutes, where $\varepsilon=\pm$ and $\delta=(-1)^l$*

$$\begin{tikzcd} HC^{\varepsilon}_{2l-1} \arrow[rr, "\omega"] \arrow[dr, "\varphi_\varepsilon"'] & & HC^{-\varepsilon}_{2l+1} \arrow[dl, "\varphi_{-\varepsilon}"] \\ & G_{\varepsilon\delta} & \end{tikzcd}$$

COROLLARY 13.9. *For $l>1$ the map $\varphi_-: HC^-_{2l-1}\to G_{-\delta}$ is an epimorphism. For $l=1$ the image of φ_- is $G_+\cap G_-$.*

Proof. For $l>1$, this follows from the fact that φ_+ is an epimorphism and the fact that the following diagram commutes

$$\begin{tikzcd} HC^{+}_{2l-3} \arrow[rr, "\omega"] \arrow[dr, "\varphi_+"'] & & HC^{-}_{2l-1} \arrow[dl, "\varphi_-"] \\ & G_{-\delta} & \end{tikzcd}$$

For $l=1$, it follows from the well-known fact (see [13]) that the double branched cover of a knotted circle with Seifert form θ is an integral homology sphere if and only if $\theta+{}^t\theta$ is non-singular.

The remainder of this section is devoted to the proof that φ_- is a monomorphism when $l>2$.

Definition 13.10. An ε-knot $(\Sigma^{2l+1}, F^{2l-1})$ is *simple* if $\Sigma^{2l+1}\cong S^{2l+1}$ and if F^{2l-1} has an $(l-1)$-connected Seifert surface.

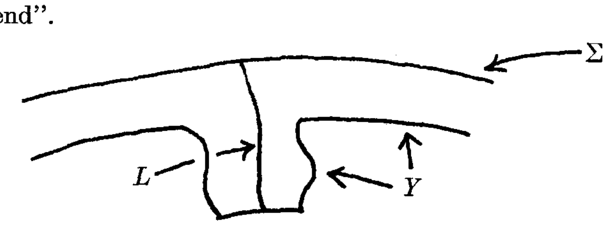
Suppose that V^{2l} is an $(l-1)$ connected Seifert surface for a simple minus-knot $(\Sigma^{2l+1}, F^{2l-1})$. Then the suspended Seifert surface V^{2l+2} is l-connected. Since ∂V^{2l+2} is an integral homology sphere, duality implies that the homology of V^{2l+2} also vanishes above the middle dimension, and that $H_{l+1}(V^{2l+2}; \mathbf{Z})$ is free abelian. Hence, $H_*(V^{2l}; \mathbf{Z})$ vanishes except in the middle dimension and $H_l(V^{2l}; \mathbf{Z})$ is free abelian.(1) The exact sequence of the pair (V^{2l}, F^{2l-1}) then shows that $H_i(F^{2l-1}; \mathbf{Z})$ vanishes for $i\neq l-1$ and is a finite odd torsion group when $i=l-1$. This shows that if the minus-knot $(\Sigma^{2l+1}, F^{2l-1})$ is simple, then F^{2l-1} is homology $(l-2)$-connected. It follows from this fact and Theorem 2 in [23], that $(\Sigma^{2l+1}, F^{2l-1})$ is simple if and only if $\Sigma-F$ has the $(l-1)$-type of a circle.

LEMMA 13.11. *Any minus-knot is cobordant to a simple minus-knot.*

Proof. Let $(\Sigma^{2l+1}, F^{2l-1})$ be a minus-knot. Do surgery on F to construct a parallelizable manifold L^{2l} such that:

(1) $\partial L=F\cup(\partial F\times I)\cup F'$,
(2) $H_*(L, F)\to H_{*-1}(F)$ is an isomorphism for $*<l$ and,
(3) $H_*(L, F)=0$ for $*\geqslant l$.

It follows that F' is $(l-2)$-connected. Form a bordism W by gluing $L\times D^2$ to $\Sigma\times I$. Let Y be the "other end".



Let $\tilde{W}$ be the double branched cover of W along L, let $\tilde{Y}$ be the double branched cover of Y along F', and let $X=Y-(F'\times D^2)$. We claim:

(1) We can not argue this fact directly using F since F is only a $\mathbf{Z}_{(2)}$-homology sphere.

(a) $H_*(\hat{W}; \mathbf{Z}) = \begin{cases} \text{odd torsion;} & \text{for } * \leqslant l-1 \\ 0; & \text{for } * \geqslant l, \end{cases}$

(b) $H_*(\hat{W}, \tilde{Y}) = 0$ for $* \leqslant l+1$, and

(c) $H_*(\tilde{X}; \mathbf{Z}) \to H_*(\tilde{Y}; \mathbf{Z})$ is an isomorphism for $* \leqslant l-1$ and onto for $* = l$.

Part (a) follows from the fact that $H_*(\hat{W}) = H_*(L, F)$. Part (b) results from the duality between $H_*(\hat{W}, \tilde{Y})$ and $H_*(\hat{W}, \tilde{\Sigma})$, and the fact that $\tilde{\Sigma}$ is a homology sphere. Part (c) follows, as in Lemma 12.3, from the fact that $H_*(F') = 0$ for $* \leqslant l-2$.

From (a), (b), and (c) above we see that it is possible to add handles of dimension $\leqslant (l+1)$ to X so as to construct a bordism W' with:

(1) $\partial W' = X \cup X'$,

(2) X' the $(l-1)$-type of the circle,

(3) $H_*(W', X; \mathbf{Z}[\mathbf{Z}/2]) \cong H_{*-1}(X; \mathbf{Z}[\mathbf{Z}/2])$ for $* \leqslant l$, and

(4) $H_*(W', X; \mathbf{Z}[\mathbf{Z}/2]) = 0$ for $* > l$.

This means that $H_*(W \cup W', \tilde{\Sigma}; \mathbf{Z}) = 0$ and hence, $(W \cup W', L)$ is a minus-knot cobordism from (Σ, F) to $(X' \cup (F' \times D^2), F')$. From (2) above, we see that this latter pair is a simple minus-knot.

The next result completes the proof of Theorem 13.1.

LEMMA 13.12. *Suppose that* $(\Sigma^{2l+1}, F^{2l-1})$ *is a simple minus-knot, that* V^{2l} *is an* $(l-1)$-*connected Seifert surface, and that the Seifert form* $\theta: H_l(V^{2l}) \otimes H_l(V^{2l}) \to \mathbf{Z}$ *is null-cobordant. If* $l > 2$, *then* $(\Sigma^{2l+1}, F^{2l-1})$ *is cobordant to the unknot.*

Proof. The idea is to do $\mathbf{Z}/2$-equivariant surgery on the suspended Seifert surface V^{2l+2} to a contractible manifold W^{2l+2}. This will have the effect of changing the fixed point set V^{2l} into a $\mathbf{Z}/2$-homology disk. The complement of an invariant open disk about a fixed point in the interior of W will then be the double branched cover of the desired minus-knot cobordism.

Since θ is null-cobordant, there is a basis $\{\alpha_1, ..., \alpha_r, \beta_1, ..., \beta_r\}$ for $H_l(V^{2l}; \mathbf{Z})$ so that $\theta(\alpha_i, \alpha_j) = 0$. Since V^{2l} is $(l-1)$ connected, we can represent the α_i by $f_i: (D_i^{l+1}, S_i^l) \to (B', V \times \{1\})$. Since $\theta(\alpha_i, \alpha_j) = 0$, we can take the f_i to be disjoint embeddings (see the discussion in 13.4). Form spheres S_i^{l+1} in V^{2l+2} by taking $f_i(D_i) \cup \gamma f_i(D_i)$, where γ denotes the involution. This collection of disjoint, invariant spheres forms a basis for a subkernel of $H_{l+1}(V^{l+2}; \mathbf{Z})$. Let T_i be an invariant tubular neighborhood of S_i^{l+1} in V^{2l+2}. From the fact that $\theta(\alpha_i, \alpha_i) = 0$, it follows that $T_i = S_i \times D^{l+1}$ and that $\gamma | T_i$ has the form $\gamma(x, y) = (rx, sy)$, where r and s are linear reflections through hyperplanes in $\mathbf{R}^{l+1}$. Let $\bar{\gamma}$ be the involution on $D^{l+1}\times S^l$ defined by $\bar{\gamma}(x, y)=(rx, sy)$. Do surgery on the S_i's by replacing each T_i by $D^{l+1}\times S^l$ with involution $\bar{\gamma}$. The resulting manifold with involution, W^{2l+2}. is clearly a contractible. This completes the proof.

14. Further remarks

First we deal with the special cases which were previously omitted.

The case $m=4$. In this case, the proof of 11.4 still shows that (σ_0, σ_1): $\Theta^1(k, n, 4)\to L_4(R_\varepsilon)+L_4(R_{-\varepsilon})$ is onto the kernel of $c+c$; however, in trying to prove that (σ_0, σ_1) is injective, one encounters a 4-dimensional surgery problem. Recall that in the case under consideration, B_0 is an R_ε-homology 3-sphere, $\varepsilon=(-1)^n$, and σ_0 is the surgery obstruction for f_0: $(A_0, B_0)\to(D^4, S^3)$ in $L_0(R_\varepsilon)$. Even if this obstruction vanishes, it does not follow that f_0 is normally bordant (relative to $f_0|B_0$) to an R_ε-homology equivalence. However, if we know, in addition, that B_0 is the boundary of some R_ε-homology disk C, then $f_0|B_0$ extends to a normal map h: $(C, B_0)\to(D^4, S^3)$ and there is a normal bordism (relative to the boundary) from f_0 to h. If we also have that $\sigma_1=0$, then there is no further problem to completing surgery. Thus, we are led to consider $\Theta_3^{\mathbf{Z}}$ and $\Theta_3^{\mathbf{Z}/2}$, where Θ_3^R denotes the group of R-homology h-cobordism classes of R-homology spheres. We shall also use the notation $\Theta_3^+=\Theta_3^{\mathbf{Z}}$ and $\Theta_3^-=\Theta_3^{\mathbf{Z}/2}$. Let $P\subset L_4(\mathbf{Z})\subset L_4(\mathbf{Z}_{(2)})$ be the subgroup of obstructions which can be represented by framed 4-manifolds with standard 3-sphere as boundary, i.e., P consists of classes of unimodular forms of index divisible by 16. Any R_ε-homology 3-sphere bounds a framed 4-manifold with surgery obstruction well-defined modulo P. Hence, there is a map μ^ε: $\Theta_3^\varepsilon\to L_4(R_\varepsilon)/P$, where $\varepsilon=\pm$. It is easily seen to be onto. Let $\tilde{\Theta}_3^\varepsilon$ be the kernel of μ^ε. (We know absolutely nothing about this group.) Then with the above notation we have the following result. (Compare 11.4.)

THEOREM 14.1. *Suppose $n\geqslant k>2$ and let $\varepsilon=(-1)^n$.*

(I) *If k is odd, then*

$$\Theta^1(k, n, 4)\cong\tilde{\Theta}_3^\varepsilon.$$

(II) *If k is even then the following sequence is exact*

$$0\longrightarrow\tilde{\Theta}_3^\varepsilon\longrightarrow\Theta^1(k, n, 4)\longrightarrow L_4(R_\varepsilon)\oplus L_{4+k}(R_{-\varepsilon})\xrightarrow{c+c}\mathbf{Z}/2.$$

(III) *Similarly, we have a short exact sequence*

$$0\to\tilde{\Theta}_3^\varepsilon\to\Theta^1(2, n, 4)\to G_\varepsilon\to 0.$$

The case $m=0$. By 8.10, there is an equivariant, stratified map $F: \Sigma^{kn-1} \to S^{kn-1}$ which is of positive degree on the odd strata. Such a map is unique up to equivariant, stratified homotopy. Let deg F denote the degree of F on any even stratum. By 5.2, Σ is the boundary of V^{kn}. Let $\sigma_1(\Sigma)$ be the Witt class of the intersection form on ${}_1V$ if $k \equiv 0(4)$, or the Arf-Kervaire invariant of ${}_1V$ if $k \equiv 2(4)$. Note that $\Theta^1(k, n, 0)$ does not have a natural group structure.

THEOREM 14.2. (a) *For n even and k even, $k \neq 4$, the map*

$$\Theta^1(k, n, 0) \xrightarrow{(\deg, \sigma_1)} (\{\pm 1\}, L_k(\mathbf{Z}_{(2)}))$$

is one-to-one. Its image is all pairs $(\pm 1, \sigma_1)$ such that the Kervaire invariant of σ_1, $c(\sigma_1)$, is zero.

(b) *For k odd the map*

$$\Theta^1(k, n, 0) \xrightarrow{\deg} \{\pm 1\}$$

is a bijection.

(c) *For n odd and k even, $k \neq 4$, the map*

$$\Theta^1(k, n, 0) \xrightarrow{(\deg, \sigma_1)} (\{\text{odd integers}\}, L_k(\mathbf{Z}))$$

is one-to-one. Its image is all pairs (d, σ_1) such that $c(d) = c(\sigma_1)$. (Here, $c(d)$ denotes the Arf-Kervaire invariant of the normal map of d point to 1 point. That is, $c(d) \neq 0$ if and only if $d \equiv \pm 3(8)$.)

Thus, for k odd or for $k \equiv 2(4)$ and n even, all actions are concordant to the linear action with one of its two orientations. When $k \equiv 2(4)$, and n is odd, all actions are concordant to some Brieskorn example:

$$(z_0^d + z_1^2 + \ldots + z_{kn/2}^2 = 0) \cap \sum_i |z_i|^2 = 1.$$

When $k \equiv 0(4)$ there are actions for which we know of no naturally arising model.

Proof. Let us begin by showing that $(\deg, \sigma_1)$ is injective. If $F: \Sigma^{kn-1} \to S^{kn-1}$ has degree d, then there is an extension as in 5.2, $F: (V, \Sigma) \to (D, S)$, so that the action of $O(n)$ on V has exactly $|d|$ fixed points. Hence, if $\deg(\Sigma) = \deg(\Sigma')$, we can take (V, Σ) and (V', Σ') as above, cut out neighborhoods of the fixed points, obtaining $\bar{V}$ and $\bar{V}'$, and then form $W = \bar{V}' \cup (-\bar{V})$, a bordism from Σ to Σ'. Let C be the qoutient of the action on W. There is a degree d map $G: (W, \Sigma', -\Sigma) \to (S \times I, S \times \{1\}, -S \times \{0\})$ covering $g: (C, B', -B) \to$

$(L\times I, L\times\{1\}, -L\times\{0\})$. We claim that if $\sigma({}_1W)=\sigma({}_1V')-\sigma({}_1V)$ is zero, then we can do surgery on $g\colon C\to L\times I$, relative to $B'\cup B$, to make W a concordance from Σ to Σ'. This will prove the one-to-one statement.

The idea is to work one stratum at a time identifying obstructions to making:

(1) $K_*(C_i;\, \mathbf{Z}_{(2)}[\mathbf{Z}/2])=0$ for $i\not\equiv n(2)$,

(2) $K_*(E_i(W);\, \mathbf{Z})=0$ for $i\equiv n(2)$ and $i>1$(1) and,

(3) $H_*(E_1(W);\, \mathbf{Z})\to H_*(E_1(S)\times I;\, \mathbf{Z})$ an isomorphism for $*\leqslant k-2$, whenever $1\equiv n(2)$.

If we mange to achieve (1), (2), and (3), then, by 8.9, $\Sigma\to W$ and $\Sigma'\to W$ will induce integral homology equivalence on the E_i's for $i\equiv n(2)$. Hence, by 7.1, W will be homologically a product. If, in addition, we make $\pi_1(C_k)=0$ (this presents no problem), then W will be a concordance from Σ to Σ', (see 3.5).

Of course, the fact that $C_i\to L_i\times I$ is not degree 1, but only of odd degree, has no effect on the problem of making $K_*(C_i;\, \mathbf{Z}_{(2)}[\mathbf{Z}/2])=0$ for $i\not\equiv n(2)$.

We claim that in the case $i\equiv n(2)$, the surgery obstruction group for making $K_*(E_i(W);\, \mathbf{Z})=0$ is exactly the same as in Section 12. The only point that is new here is that even though $E_i(W)\to E_i(S\times I)$ is *not* necessarily of degree 1 and is *not* necessarily an integral equivalence on the boundary, it is still the case that, once we have done the surgery on the lower strata, the $K_*(E_i(W);\, \mathbf{Z})$ have non-singular intersection and linking pairings.

If $i=1$ and $1\equiv n(2)$, then $E_1(\Sigma)$ and $E_1(\Sigma')$ have the integral homology of S^{k-1} whereas $E_1(S\times I)=S^{k+1}\times I$. Thus, $H_*(E_1(W);\, \mathbf{Z})\to H_*(S\times I;\, \mathbf{Z})$ is onto for $*\leqslant k-2$ and the kernels of these maps are dually paired for $0<*\leqslant k-2$.

If $i>1$, $i\equiv n(2)$, and surgery has been done through level $(i-1)$, then let $\delta E_i(W)=\bigcup_{j\leqslant i-2}\partial_j E_i(W)$. We have $\partial E_i(W)=E_i(\Sigma)\cup\delta E_i(W)\cup E_i(\Sigma')$. From the inductive hypothesis and 7.2, it follows that $\partial E_i(\Sigma)\to\delta E_i(W)$ and $\partial E_i(\Sigma')\to\delta E_i(W)$ induce isomorphisms in integral homology. Thus, $H_*(E_i(W), E_i(\Sigma))$ is dually paired with $H_*(E_i(W), E_i(\Sigma'))$. By 8.9, $H_*(E_i(\Sigma))\to H_*(E_i(S))$ and $H_*(E_i(\Sigma'))\to H_*(E_i(S))$ are isomorphisms. Hence, $K_*(E_i(W))=K_*(E_i(W), E_i(\Sigma))=H_*(E_i(W), E_i(\Sigma))$. Likewise, $K_*(E_i(W))=H_*(E_i(W), E_i(\Sigma'))$. This proves that $K_*(E_i(W))$ is dually paired with itself for $i\equiv n(2)$ and $i>1$.

Now that we have established this duality, the argument proceeds exactly as in Section 12. This means that the same surgery obstruction groups arise for the various strata, and all the obstructions, except for $\sigma(f_1)$, cancel out. This obstruction is identified with $\sigma({}_1W)$, which vanishes by assumption.

---

(1) If $F\colon W\to S\times I$ does not have degree ± 1, then $K_*(E_i(W);\, \mathbf{Z})$ is to be interpreted as the relative homology group $H_{*+1}(E_i(S\times I), E_i(W);\, \mathbf{Z})$.

We turn now to the image of (deg, σ_1). By 8.9, the degree must be ± 1 if k is odd or if n is even. By 15.5, $c(\deg)=c(\sigma_1)$. These relations show that the image of (deg, σ_1) is no larger than that claimed in 14.2.

To prove that the image is at least as large as claimed we shall show:

(1) There are homotopy spheres of all degrees allowed by the above relations.

(2) Given a homotopy sphere of degree d, it is possible to vary it to another homotopy sphere of degree d, changing σ_1 by any element (in the appropriate surgery group) with trivial Arf-Kervaire invariant.

For n even, the only possible degrees are ± 1 and these occur for the linear model with its two different orientations. If n is odd and k is odd, then again the only possible degrees are ± 1 and these are achieved by the linear model. If n is odd and k is even, let $d>0$ be a possible degree. Take M^{kn-1} to be d copies of S^{kn-1} and $F\colon M\to S$ to be d copies of the identity map. We want to do $O(n)$-equivariant surgery on $F\colon M\to S$ until M satisfies the conditions of 8.9 and hence, becomes a homotopy sphere. To prove that this is possible, we need to know that:

PROPOSITION 14.3. (a) *$H_*(E_1(M); \mathbf{Z})\to H_*(E_1(S); \mathbf{Z})$ is onto for $*\leqslant k-2$ and the kernels of these maps satisfy duality for $0<*\leqslant k-2$.*

(b) *If the conditions of (8.9) hold through level $(i-1)$, $i\equiv n(2)$, and $i>1$, then $K_*(E_i(M); \mathbf{Z})$ satisfies duality.*

Once we have these duality statements, the obstruction groups are the same as those in Section 12, and the arguments in 12 can be carried over to show that it is possible to complete surgery to make M a homotopy sphere.

Statement 14.3 (a) is obvious from the fact that $E_1(S^{kn-1})=S^{k-1}$. Statement 14.3 (b) is much more subtle. It requires the following lemma.

LEMMA 14.4. *Let $S=S^{kn-1}$ and let Y be the fiber of $\partial_1 S\to \mathbf{RP}^{k-1}$, i.e., let $Y=O(n)\times_{O(n-1)}S^{(k-1)(n-1)-1}$. The natural inclusion $Y\hookrightarrow S$ induces a $\mathbf{Z}/2$-equivariant map $E_i(Y)\hookrightarrow E_i(S)$. If $i\equiv 1(2)$, then*

(a) *the involution on $E_i(S)$ acts trivially on $H_*(E_i(S); \mathbf{Z}[1/2])$, and*

(b) *the induced map*

$$H_*(E_i(Y); \mathbf{Z}[1/2])^{\mathbf{Z}/2}\to H_*(E_i(S); \mathbf{Z}[1/2])$$

is an isomorphism.

Proof. Let $G(p, q)$ be the Grassmann of p-planes in q-space. Recall that $B_i(S)\sim G(i, k)$ (see 2.5). Also, $E_i(S)-B_{i-1}(S)=\tilde{B}_i(S)$ is homotopy equivalent to $\tilde{G}(i, k)$, the Grassman of oriented i-planes. For this proof all homology is taken with $\mathbf{Z}[1/2]$ coefficients. The proof consists of establishing the following:

(1) The inclusion of the fixed point set of the involution $B_{i-1}(S) \to E_i(S)$ induces an isomorphism on homology. (This implies (a).)

(2) The $\mathbf{Z}/2$-action on $G(i-1, k)$ induced by the involution $(x_1, x_2, \ldots, x_k) \to (-x_1, x_2, \ldots, x_k)$ acts trivially on $H_*(G(i-1, k))$ and hence,

$$p_*: H_*(G(i-1, k))^{\mathbf{Z}/2} \to H_*(G(i-1, k)/(\mathbf{Z}/2))$$

is an isomorphism.

(3) There is a homotopy equivalence

$$h: E_i(Y)/(\mathbf{Z}/2) \to G(i-1, k)/(\mathbf{Z}/2)$$

so that the following diagram commutes:

$$\begin{tikzcd}
H_*(E_i(S)) & H_*(B_{i-1}(S)) \arrow[l, "\cong"'] & H_*(G(i-1, k)) \arrow[l, "\cong"'] \\
H_*(E_i(Y))^{\mathbf{Z}/2} \arrow[u] \arrow[d] & & \\
H_*(E_i(Y)/(\mathbf{Z}/2)) \arrow[rr, "h_*", "\cong"'] & & H_*(G(i-1, k)/(\mathbf{Z}/2)) . \arrow[uu, "(p_*)^{-1}"']
\end{tikzcd}$$

We have that $E_i(S)$ is homotopy equivalent to the double mapping cylinder $M_\varphi \cup M_\theta$, where:

$$\begin{tikzcd}
\dfrac{O(k)}{S(O(i-1)\times O(1))\times O(k-i)} \arrow[r, "\varphi"] \arrow[d, "\theta"] & G(i-1, k) \\
\tilde{G}(i, k) &
\end{tikzcd}$$

Since the fiber of θ is $\mathbf{RP}^{i-1}$ and $(i-1)$ is even, θ is a $\mathbf{Z}[1/2]$-homology equivalence. This proves (1).

Consider the natural action of $O(k-1) \subset O(k)$ on $G(i-1, k)$. The orbit space is clearly an interval. Therefore, $G(i-1, k)$ is homeomorphic to the double mapping cylinder $M_\alpha \cup M_\beta$,

$$\begin{tikzcd}
\dfrac{O(k-1)}{O(i-2)\times O(k-i)} \arrow[r, "\alpha"] \arrow[d, "\beta"] & G(i-2, k-1) \\
G(i-1, k-1) , &
\end{tikzcd}$$

where α is the canonical S^{k-i}-bundle and β is the canonical S^{i-2}-bundle. The involution on $G(i-1, k)$ has fixed point set $G(i-2, k) \coprod G(i-1, k-1)$ and it acts by the fiberwise antipodal map on each sphere bundle. Since $i-2$ is odd, the involution is trivial on the homology of the total space of β and hence, on $H_*(G(i-1, k))$. This proves (2). Clearly, $G(i-1, k)/(\mathbf{Z}/2)$ is homeomorphic to the double mapping cylinder given by

$$\begin{array}{ccc} \dfrac{O(k-1)}{O(i-2)\times O(1)\times O(k-i)} & \longrightarrow & G(i-2, k-1) \\ \Big\downarrow & & \\ G(i-1, k-1) & & \end{array}$$

and this double mapping cylinder is also clearly homotopy equivalent to $E_{i-1}(k-1)/(\mathbf{Z}/2) \sim E_i(Y)/(\mathbf{Z}/2)$, proving (3).

In order to prove that $K_*(E_i(M); \mathbf{Z})$ is dually paired with itself, it clearly suffices to prove that the pairings on $K_*(E_i(M); \mathbf{Z}_{(2)})$, $K_*(E_i(M); \mathbf{Z}[1/2])^+$ and $K_*(E_i(M); \mathbf{Z}/1/2])^-$ are non-singular. Here $K_*(E_i(M); \mathbf{Z}[1/2])^\pm$ denotes the (± 1)-eigenspace of the involution. (Since $(k-i-1)$ is even, the involution on $E_i(M)$ is orientation preserving.) The case of $\mathbf{Z}_{(2)}$ coefficients presents no problem, since $E_i(M) \to E_i(S)$ is of odd degree and a $\mathbf{Z}_{(2)}$-equivalence on the boundary. According the above lemma, $H_*(E_i(S); \mathbf{Z}[1/2])^- = 0$; hence, $H_*(\partial E_i(S); \mathbf{Z}[1/2])^- = 0$. Consider the bundle $\partial_1 E_i(M) \to B_1$ with fiber $E_1(Y)$. Since the natural involution and the involution induced by $\pi_1(B_1)$ are the same on $H_*(E_1(Y))$ and since $H_*(B_1; \mathbf{Z}[1/2]^-) = H_*(\mathbf{RP}^{k-1}; \mathbf{Z}[1/2]^-) = 0$, it follows from the Serre spectral sequence that $H_*(\partial_1 E_i(M); \mathbf{Z}[1/2])^- \cong H_*(\partial E_i(S); \mathbf{Z}[1/2])^- = 0$. From 7.2, we know that

$$H_*(\overline{\partial E_i(M) - \partial_1 E_i(M)}) \to H_*(\overline{\partial E_i(S) - \partial_1 E_i(S)})$$

is an isomorphism. Thus, $K_*(E_i(M); \mathbf{Z}[1/2])^- = H_*(E_i(M); \mathbf{Z}[1/2])^-$ is dually paired with itself.

Finally, we consider $K_*(E_i(M); \mathbf{Z}[1/2])^+$. We know that $f_1\colon B_1 \to \mathbf{RP}^{k-1}$ is an isomorphism on integral homology in dimensions $\leqslant k-2$ and is of degree d. Arrange that $f_1^{-1}(y_0) = x_0$. (Of course, the local degree of f_1 and x_0 must be d.) Let $F_{x_0} \subset E_i(M)$ be the fiber over x_0 in $\partial_1 E_i(M)$ and $F_{y_0} \subset E_i(S)$ be the fiber over y_0 in $\partial_1 E_i(S)$. Clearly, $F_{x_0} \to F_{y_0}$ is a diffeomorphism and $F_{y_0} \cong E_i(Y)$. Since $H_*(E_i(S), F_{y_0}; \mathbf{Z}[1/2])^+ = 0$, it follows that $K_*(E_i(M); \mathbf{Z}[1/2])^+ = H_*(E_i(M), F_{x_0}; \mathbf{Z}[1/2])^+$. By duality, $H_*(E_i(S), \partial E_i(S) - F_{y_0}; \mathbf{Z}[1/2])^+ = 0$. We claim that $K_*(\partial E_i(M) - F_{x_0}; \mathbf{Z}) = 0$. The reason is that $\partial E_i(M) - F_{x_0}$ is made up of two pieces, to wit: $\partial E_i(M) - \partial_1 E_i(M)$ and $\partial_1 E_i(M) | (B_i \times \{x_0\})$. Both pieces, as well of their intersection, are mapped by homology isomorphisms to the corresponding pieces in

$\partial E_i(S) - F_{y_0}$. Consequently, $K_*(E_i(M); \mathbf{Z}[1/2])^+ = H_*(E_i(M), \partial E_i(M) - F_{x_0}; \mathbf{Z}[1/2])^+$. Hence, we have identified $K_*(E_i(M): \mathbf{Z}[1/2])^+$ with two different relative homology groups which are dually paired by Poincaré duality.

This completes the proof that the groups $K_*(E_i(M); \mathbf{Z})$ have dual pairings and hence, that the surgery groups are the same as those in Section 12. Hence, all degrees specified by 14.2 occur, and we can vary σ_1 by any element of trivial Arf-Kervaire invariant (in the appropriate surgery group).

Restricting actions. Suppose ϱ is a linear action of G on a sphere S^q and that $H \subset G$ is a closed subgroup. Also, suppose that there is a smooth H-action on a homotopy sphere Σ^q, which is modeled on $\varrho | H$. It makes sense to ask if the H-action on Σ^q is the restriction of some smooth G-action on Σ^q which is modeled on ϱ. If such a G-action exists, then we can ask if it is unique. Here, we shall be concerned with the case where G and H are chosen so that we remain within the category of regular actions of the orthogonal, unitary or symplectic groups. In general, such questions appear to be very difficult; however, up to concordance they are all easily answered (except for the usual difficulty with low dimensions).

Let $G^d(n)$ stand for $O(n)$, $U(n)$, or $Sp(n)$ as $d = 1, 2,$ or 4 and consider the embedding

$$\alpha = r\varrho_n^d + l \colon G^d(n) \hookrightarrow O(drn + l),$$

where ϱ_n^d denotes the standard representation and l the trivial l-dimensional representation. Then the process of restricting on $O(drn+l)$-action to $\alpha(G^d(n))$ defines a homomorphism $\alpha_* \colon \Theta^1(k', n', m') \to \Theta^d(k, n, m)$, where $k = k'r$, $m = tl + m'$, $n' = drn + l$. By using the results of Sections 11, 12 and 13 and those in [11], we can always compute the map α_* (provided $n \geq k$ and we stay away from the exceptional cases). Rather than trying to write down the answer in all cases, we shall be content with considering the most interesting ones.

THEOREM 14.5. *Suppose that* $n \geq k$, $m \neq 0, 4$ *and that* $m + dk \neq 4$. *Then the map* $\omega_* \colon \Theta^d(k, n+1, m) \to \Theta^d(k, n, m+dk)$, *induced by* $\omega = \varrho_n^d + 1 \colon G^d(n) \to G^d(n+1)$, *is an isomorphism.*

Proof. For $d = 2$ or 4, this follows from [11]. Suppose $d = 1$. If k is odd or if m is odd, then both groups are trivial. If k is even and $k \neq 2$, then we have the following diagram,

$$\begin{tikzcd}
\Theta^1(k, n+1, m) \arrow[rr, "\omega_*"] \arrow[dr, "(\sigma_0, \sigma_1)"'] & & \Theta^1(k, n, m+k) \arrow[dl, "(\sigma_1, \sigma_0)"] \\
& L_m(R_\varepsilon) + L_{m+k}(R_{-\varepsilon}) &
\end{tikzcd}$$

where $\varepsilon=(-1)^{n+1}$. By definition, $\sigma_0(\omega_*(\Sigma))=\sigma_1(\Sigma)$ and $\sigma_1(\omega_*(\Sigma))=\sigma_2(\Sigma)$. By 11.3, $\sigma_2(\Sigma)=\sigma_0(\Sigma)$. Hence, the diagram commutes. The result now follows since (σ_0, σ_1) and (σ_1, σ_0) are monomorphisms with the same image. Similarly, if $k=2$ and $m=2l$, we have the following commutative diagram,

$$\begin{tikzcd}
\Theta^1(2, n+1, 2l) \arrow[rr, "\omega_*"] & & \Theta^1(2, n, 2l+2) \\
HC^{\varepsilon}_{2l-1} \arrow[u, phantom, "\cong"] \arrow[dr, "\varphi^{\varepsilon}"', "\cong"] & & HC^{-\varepsilon}_{2l+1} \arrow[u, phantom, "\cong"] \arrow[dl, "\varphi^{-\varepsilon}", "\cong"'] \\
& G_{\varepsilon\delta} &
\end{tikzcd}$$

where $\varepsilon=(-1)^{n+1}$ and $\delta=(-1)^{l}$.

Remark. It is interesting to speculate about the map ω_*: $\Theta^1(2, n+1, 2)\to\Theta^1(2, n, 4)$.

THEOREM 14.6. *Consider the map*

$$\alpha_*: \Theta^1(2, n', m')\to\Theta^d(k, n, m)$$

where $\alpha=r\varrho^d_n+l$: $G^d(n)\to O(n')$, $n'=drn+l$, $k=2r$ *and* $m=2l+m'$. *Suppose* $n\geqslant k$.

(I) α_* *is onto except in the case* $d=1$, $k\equiv 0(4)$ *and* $m\equiv 0(4)$.

(II) *Suppose that* $d=1$, $r=2q$ *(so that* $k=4q$*), and* $m=4s$ *and that* $\varepsilon=(-1)^n$. *Then* $(-1)^{n'}(-1)^{m'/2}=+1$ *and the following diagram commutes*:

$$\begin{tikzcd}
\Theta^1(2, n', m') \arrow[rr, "\alpha_*"] \arrow[d, "\varphi_+"'] & & \Theta^1(k, n, m) \arrow[d, "(\sigma_0, \sigma_1)"] \\
G_+ \arrow[dr, "\beta"'] & & L_0(R_\varepsilon)+L_0(R_{-\varepsilon}) \\
& L_0(\mathbf{Z}) . \arrow[ur, "\Delta"'] &
\end{tikzcd}$$

The map β *associates to a Seifert form* θ, *the integer* $\frac{1}{8}$ *(the index of* $(\theta+{}^t\theta)$*). The map* Δ *is the composition* $\mathbf{Z}\overset{\text{diag}}{\hookrightarrow}\mathbf{Z}+\mathbf{Z}\hookrightarrow\mathbf{Z}+W$. *It follows that in this case the cokernel of* α_* *maps onto* $\overline{W}$ *and that it is equal to* $\overline{W}$ *provided* $m\neq 4$.

This theorem says that for $k\equiv 0(4)$ and $m\equiv 0(4)$, there are k-axial $O(n)$-actions on homotopy spheres which are not the restriction of any bi-axial action even up to concordance; but that in all other cases every k-axial $G^d(n)$-action is concordant to the restriction of some bi-axial action of the orthogonal group.

The proof is completely straightforward and is left for the reader.

Remark. Not every element of coker $(\alpha_*\colon \Theta^1(2, n', m') \to \Theta^1(k, n, m))$ actually represents a new example of an action. For, suppose that Σ represents an element of $\Theta^1(2, n', m')$. An orientation for Σ (as a manifold) determines one for $\{\alpha_*(\Sigma)\}^{O(1)} = \Sigma^{O(2q)}$, where $2q = r$. Let Σ' denote $\alpha_*(\Sigma)$ with the orientation of $\{\alpha_*(\Sigma)\}^{O(1)}$ reversed. If $\alpha_*(\Sigma)$ represents $(x, x) \in \mathbf{Z} + \mathbf{Z} \subset \mathbf{Z} + \overline{W} \cong \Theta^1(k, n, m)$, then Σ' represents $(x, -x)$. It follows that every element in the subgroup $\{(x, y) \in \mathbf{Z} + \mathbf{Z} \mid x + y \equiv 0(2)\}$ can be represented as the connected sum of the restriction of two bi-axial actions (possibly after changing an orientation).

15. Surgery lemmas

In Sections 12 and 14, we used several surgery lemmas. In this section we shall prove them. There are three types of results. Those of the first type, 15.1, 15.2, and 15.3, are product formulae for $\mathbf{CP}^{2l}$- and $\mathbf{RP}^{2l}$-bundles. Next, 15.4 and 15.5 relate surgery obstructions of fixed points of semi-free S^1- or $\mathbf{Z}/2$-actions with the surgery obstruction for the whole manifold. Lastly, there is a technical lemma about surgery on even dimensional, non-orientable manifolds which was used in 12.2.

Usually a product formula means a determination of the surgery obstruction of a normal map crossed with a closed manifold in terms of the surgery obstruction of the original normal map and invariants (often homological) of the closed manifold. Here, we have a slightly broader notion in mind. We shall begin with a normal map and a fiber bundle over the range with fiber a closed manifold. We pull back the bundle over the domain and take the induced normal map between the total spaces. We want a formula for the obstruction of this normal map in terms of the surgery obstruction of the map between the bases, the fibers, and the action of the fundamental group of the base on the fiber. It is a general principle that the only information about the bundle which is needed is the action of the fundamental group of the base. Thus, for example, if the group acts trivially on the fiber, then the formula for the bundle is the same as for the trivial bundle.

Recall that $\partial_i A_{i+1}$ is an $\mathbf{RP}^{k-i-1}$ bundle over A_i. Also, $\partial_i E_{i+2}$ is a $\mathbf{CP}^{k-i-1}$ bundle over A_i with the action of $\pi_1(A_i)$ on $\mathbf{CP}^{k-i-1}$ given by complex conjugation (see 3.3). We begin our study of the product formulae with bundles of the second type.

THEOREM 15.1. *Let $f\colon (M^{4m}, \partial M) \to (N^{4m}, \partial N)$ be a normal map between orientable manifolds with $\pi_1(N) = \mathbf{Z}/2$. Suppose $f \mid \partial M$ induces an isomorphism on $R[\mathbf{Z}/2]$-homology, where R*

is a subring of $\mathbf{Q}$. *Let* $E \to N$ *be a fiber bundle with fiber* $\mathbf{CP}^{2l}$ *and with* $\pi_1(N)$ *acting on* $H_*(\mathbf{CP}^{2l})$ *via complex conjugation. Let* $g: f^*E \to E$ *be the induced normal map between total spaces. If* $(\alpha_{ij}+\beta_{ij}\gamma)$ *represents* $\sigma(f)$ *in* $L_0(R[\mathbf{Z}/2], +)$, *then* $(\alpha_{ij}+(-1)^l\beta_{ij}\gamma)$ *represents* $\sigma(g)$.

Proof. We can always vary f by a normal bordism relative to ∂M without changing $\sigma(g)$ or $\sigma(f)$. Surgery below the middle dimension on f allows us to make $K_i(f; \mathbf{Z}[\mathbf{Z}/2])=0$ for $i<2m$. Additional surgery in dimensions $2m-1$ and $2m$ allows us to make the intersection form on $K_{2m}(f; R[\mathbf{Z}/2])$ equal to $\alpha_{ij}+\beta_{ij}\gamma$.

The kernel modules for g are

$$K_{2m+i}(g; R[\mathbf{Z}/2]) = K_{2m}(f; R[\mathbf{Z}/2]) \underset{\mathbf{Z}}{\otimes} H_i(\mathbf{CP}^{2l}; \mathbf{Z}).$$

We can do surgery below the middle dimension to kill all the kernel modules for g except $K_{2m+2l}(g; R[\mathbf{Z}/2])$. Since all the modules are free, this process will leave $K_{2m+2l}(g; R[\mathbf{Z}/2]) = K_{2m}(f; R[\mathbf{Z}/2]) \otimes_{\mathbf{Z}} H_{2l}(\mathbf{CP}^{2l}; \mathbf{Z})$, and will not change the intersection form on this module. If $x \in K_{2m}(f)$ is represented by an immersion $i: S_x^{2m} \to M$, then i^*f^*E is the trivial bundle over S_x^{2m}. Hence, the class $x \otimes \omega \in K_{2m+2l}(g)$ (with $\omega \in H_{2l}(\mathbf{CP}^{2l}; \mathbf{Z})$ a generator) is represented by the immersion

$$S_x^{2m} \times \mathbf{CP}^l \hookrightarrow S_x^{2m} \times \mathbf{CP}^{2l} \to f^*E$$

Given two such classes $x_1 \otimes \omega$ and $x_2 \otimes \omega$, we can arrange that the immersions $S_i \to M$ have only transverse points of intersection. Above a point with intersection number δe we find two copies of $\mathbf{CP}^l$, one from each cycle. Their intersection number is $+1$. Hence, such a point contributes δe to the intersection number of $(x_1 \otimes \omega)\cdot(x_2 \otimes \omega)$. Above a point of intersection of S_1 and S_2 with $\delta\gamma$ as intersection number, we can find two copies of $\mathbf{CP}^l$; but, this time, one of them represents ω while the other represents the result of complex conjugation on ω, i.e. $(-1)^l\omega$. Hence, such points contribute $(-1)^l\delta\gamma$ to $(x_1 \otimes \omega)\cdot(x_2 \otimes \omega)$. Summing over the points of intersection gives $(x_1 \otimes \omega)\cdot(x_2 \otimes \omega) = a+(-1)^l b\gamma$ if $x_1 \cdot x_2 = a+b\gamma$. The result follows easily.

THEOREM 15.2. *Let* $f: (M^{4m}, \partial M) \to (N^{4m}, \partial N)$ *and* $E \to N$ *be as in* 15.1 *except that* $\pi_1(N)$ *acts trivially on* $H_*(\mathbf{CP}^{2l})$. *Then* $\sigma(f)=\sigma(g)$ *in* $L_0(R[\mathbf{Z}/2], +)$.

Proof. The proof is exactly the same.

There is an analogous theorem for $\mathbf{RP}^{2l}$.

THEOREM 15.3. *Let* $f: (M^{2m}, \partial M) \to (N^{2m}, \partial N)$ *be a normal map with* $f|(\partial M)$ *inducing an isomorphism in* $\mathbf{Z}/2$*-homology. Let* $E \to N$ *be a fiber bundle with fiber* $\mathbf{RP}^{2l}$. *The Arf-Kervaire invariant of* $g: f^*E \to E$ *equals that of* f.

Proof. Once again it suffices to consider only normal maps f such that $K_i(f;\mathbf{Z}[\pi_1[N]])=0$ for $i<m$. The kernel groups $K_{m+i}(g;\mathbf{Z}/2)$ are then equal to $K_m(f;\mathbf{Z}/2)\otimes H_i(\mathbf{RP}^{2l},\mathbf{Z}/2)$. To calculate the Arf-Kervaire invariant of g we must consider the intersection and self-intersection form on $K_{m+l}(g;\mathbf{Z}/2)$. We claim that under the identification

$$K_{m+l}(g;\mathbf{Z}/2)=K_m(f;\mathbf{Z}/2)\otimes H_l(\mathbf{RP}^{2l};\mathbf{Z}/2)=K_m(f;\mathbf{Z}/2)$$

the self-intersection forms for f and g agree. The argument is similar to the $\mathbf{CP}^{2l}$-case. Namely, $x\otimes\omega\in K_{m+l}(g;\mathbf{Z}/2)$ is represented by

$$S_x\times\mathbf{RP}^l\hookrightarrow S_x\times\mathbf{RP}^{2l}\to E$$

where $S_x\to M$ represents x and $S_x\times\mathbf{RP}^{2l}\to E$ is just a trivialization of $E\,|\,S_x$. (This time ω represents the non-trivial class in $H_l(\mathbf{RP}^{2l};\mathbf{Z}/2)$.) Above each double point of $S_x\to M$ we have a doubled copy of $\mathbf{RP}^l$ in $\mathbf{RP}^{2l}$. Shifting one copy transverse to the other leaves a single point of self-intersection of $S_x\times\mathbf{RP}^l$. Thus, the number of double points of $S_x\times\mathbf{RP}^l\to E$ equal that of $S_x\to M$. If we begin with an immersion of S_x^m into M whose normal bundle, thought of as a reduction of the stable normal bundle of S_x in M, extends to a reduction over D^{m+1}, then the number of double points of $S_x\to M$ is the value of the self-intersection for f on x, see [31] and page 46 in [34]. The resulting immersion of $S_x\times\mathbf{RP}^l$ into E has a normal bundle which extends to a reduction over $D^{m+1}\times\mathbf{RP}^l$. Hence, the number of double points (mod 2) is the value of the self-intersection form for g on $x\otimes\omega$. This shows that the forms for g and f agree, and hence, that their Arf-Kervaire invariants agree.

We turn now to the two theorems required in the proof of 11.3.

THEOREM 15.4. *Let M^{2l} and N^{2l} be oriented manifolds with semi-free S^1-actions. Let $F^{2p}\subset M$ and $F'\subset N$ be the fixed point sets. Suppose $g\colon(M,\partial M)\to(N,\partial N)$ is an equivariant, stratified normal map with $g\,|\,\partial M$ inducing an isomorphism on R-homology. ($R\subset\mathbf{Q}$.) Then $\sigma(g\,|\,F)\in L_{2p}(R)$ and $\sigma(g)\in L_{2l}(R)$ are equal. This is interpreted to mean that both are 0 if $2l\not\equiv 2p(4)$.*

Proof. We shall use the notation of Section 1. Thus $\hat{M}_F$ means M blown up along F. Let $\nu\subset M$ and $\nu'\subset N$ be equivariant normal bundles with $\partial\nu$ and $\partial\nu'$ the associated sphere bundles and with $g\colon(\nu,\partial\nu)\to(\nu',\partial\nu')$. Things are simplified somewhat by assuming that F' is connected, though the proof given can be modified slightly to work in the general case. Also, since we are calculating simply connected obstructions, we can assume that F' and N are simply connected. Since $g\,|\,\partial M$ is an R-homology equivalence, so is $g\,|\,\partial F$ (by Smith Theory). Note that:

(a) the map induced by g, $\hat{g}$: $\partial\widehat{M}_{\partial F}/S^1 \to \partial\widehat{N}_{\partial F'}/S^1$ is an isomorphism on R-homology,

(b) h: $\partial\nu/S^1 \to \partial\nu'/S^1$ is a bundle map covering $g\,|\,F$, and

(c) the fibers of $\partial\nu'/S^1$ are $\mathbf{CP}^{l-p-1}$'s.

Let us dispose of the case $2l-2p\equiv 2(4)$. In this case the fiber is $\mathbf{CP}^{2l}$. Hence, 15.2 says that $\sigma(g\,|\,F)=\sigma(h)$. By (a), we see that $\sigma(h)=\sigma(h\cup\hat{g})$ in $L_{2l-2}(R)$. This last obstruction is zero since $h\cup\hat{g}$ is the boundary of the normal map induced by g from $\hat{M}_F/S^1$ to $\hat{N}_F/S^1$. This proves that $\sigma(g\,|\,F)=0$ in $L_{2p}(R)$. Thus, we can do surgery to make $g\,|\,F$ an R-homology equivalence. Then $\hat{M}_F/S^1 \to \hat{N}_{F'}/S^1$ is an R-homology equivalence on the boundary and of odd dimension. Hence, we can do surgery on this map, relative to its boundary, to make it an R-homology isomorphism. After we do this, the resulting map on total spaces is an R-homology isomorphism. Hence, $\sigma(g)\in L_{2l}(R)$ is zero.

We turn now to the more interesting case—$2l\equiv 2p(4)$. This time we can do surgery below the middle dimension on $(g\,|\,F)$: $F\to F'$ until $K_i(F;\,R)=0$ for $i<p$ and $K_p(F;\,R)$ is a free R-module with intersection form λ and self-intersection form μ. Now we do surgery on $\hat{h}$: $\hat{M}'_F/S^1 \to \hat{N}_{F'}/S^1$ until $K_i(\hat{M}_F/S^1;\,R)=0$ for $i<l-1$. Since $K_{l-1}(\partial(\hat{M}_F/S^1);\,R)=0$, the intersection pairing

$$K_{l-1}(\hat{M}_F/S^1;\,R)\otimes K_l(\hat{M}_F/S^1;\,R)\to R$$

is non-singular over $\mathbf{Q}$. Hence, surgery on a basis for the free part of $K_{l-1}(\hat{M}_F/S^1;\,R)$ produces a new equivariant stratified map g': $M'\to N$ such that

(a) $K_i(\hat{M}'_F/S^1;\,R)=0$, $i<l-1$ and

(b) $K_{l-1}(\hat{M}'_F/S^1;\,R)$ is torsion.

It follows by duality that $K_i(\hat{M}'_F/S^1,\,\partial\nu/S^1;\,R)=0$ for $i\geqslant l$. Also, since $l>p$, $K_l(M';\,R)=K_l(\hat{M}'_F;\,R)$. Consider the Gysin sequence for the circle bundle $\hat{M}'_F\to\hat{M}'_F/S^1$:

$$0\to K_{l-1}(\hat{M}'_F/S^1;\,R)\to K_l(\hat{M}'_F;\,R)\to K_l(\hat{M}'_F/S^1;\,R)\to 0.$$

The first term is torsion and the last is equal to (via the inclusion map) $K_l(\partial\nu/S^1;\,R)$. Thus

$$\begin{aligned} K_l(M';\,R)/\text{torsion} &= K_l(\hat{M}'_F;\,R)/\text{torsion} = K_l(\hat{M}'_F/S^1;\,R) \\ &= K_l(\partial\nu/S^1;\,R) = K_p(F;\,R)\underset{\mathbf{Z}}{\otimes} H_{l-p}(\mathbf{CP}^{l-p-1};\,\mathbf{Z}). \end{aligned}$$

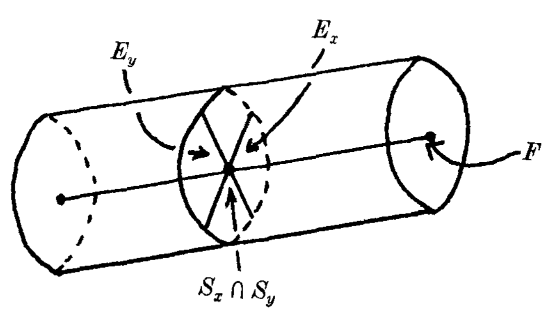
We claim that under these identifications the intersection and self-intersection forms on $K_l(M';\,R)$/torsion and $K_p(F;\,R)$ agree. To prove this non-obvious fact, we need a geometric description of the identification of $K_l(M';\,R)$/torsion and $K_p(F;\,R)$. Given

$x \in K_p(F; R)$ represent it by an immersed sphere $S_x^p \to F$. The $\mathbf{CP}^{l-p-1}$ bundle over F is actually the projective bundle associated to the complex structure on ν induced by the S^1-action. If we restrict this bundle to S_x^p, then it is trivial. Set $q = \frac{1}{2}(l-p)$ and consider $S_x^p \times \mathbf{CP}^{q-1} \to (\partial\nu)/S^1$. Over each double point of S_x^p we have two copies of $\mathbf{CP}^{q-1} \hookrightarrow \mathbf{CP}^{l-p-1}$. General position allows us to assume that they are disjoint standard linear sub-projective spaces. The cycle $S_x^p \times \mathbf{CP}^{q-1} \to (\partial\nu)/S^1$ bounds a relative cycle $Z_x^{l-1} \to \hat{M}'_F/S^1$, which represents a class in $K_{l-1}(\hat{M}'_F/S^1, \partial\nu/S^1; R)$. The class in $K_l(M; R)$ which corresponds under our identification to $x \in K_p(F; R)$ is represented by

$$\zeta_x^l = \pi^{-1}(Z_x^{l-1}) \cup E_x^{l-p}$$

where $\pi: \hat{M}'_F \to (\hat{M}'_F)/S^1$ is the projection map and E_x^{l-p} is the $2(l-p)$-dimensional disk bundle associated to $S_x^p \times \mathbf{CP}^{q-1} \to (\partial\nu \,|\, S_x)/S^1$. Clearly, if we are given 2 such classes $x, y \in K_p(F; R)$ we can take the resulting Z_x^{l-1} and Z_y^{l-1} to be disjoint. Hence, the geometric intersection of the cycles ζ_x and ζ_y will occur exactly and the points of intersection of S_x and S_y in F.

Near a point of intersection ζ_x and ζ_y will be $E_x \cdot E_y$ where E_x and E_y are complex sub-bundles of $\nu \,|\, S_x$ and $\nu \,|\, S_y$. If S_x and S_y are transverse, then we choose these bundles to have transverse fibers over each point of intersection. These two linear subspaces have intersection $+1$. Hence, $\zeta_x \cdot \zeta_y = S_x \cdot S_y$.



This proves that the intersection forms on $K_p(F; R)$ and $K_l(M'; R)$ agree. If we begin with $S_x^p \to F$ consistent bundle data, then $S_x^p \times \mathbf{CP}^{q-1} \to \partial\nu/S^1$ is also consistent with the bundle data. We take Z_x^{l-1} (or some odd multiple) to be an embedded manifold whose normal bundle is consistent with the bundle data. Then $\zeta_x^l \to M$ is consistent with the bundle data. Hence, the self-intersection forms take the same values of $[\zeta_x]$ and x.

This proves that $\sigma(g \,|\, F) = \sigma(g)$ in $L_{2l}(R)$.

THEOREM 15.5. *Let M^l and N^l have smooth $\mathbf{Z}/2$-actions with $F^p \subset M$ and $F' \subset N$ as fixed point sets. Suppose g: $(M, \partial M) \to (N, \partial N)$ is an equivariant stratified normal map with*

$g|\partial M$ inducing an isomorphism on $\mathbf{Z}/2$-homology. Then the Arf-Kervaire invariant of $g|F$ and the Arf-Kervaire invariant of g are equal. This is interpreted to mean that both are 0 if either p or l is odd.

Proof. First note that if $g|\partial M$ is a $\mathbf{Z}/2$-homology equivalence, then (by Smith Theory) so is $g|\partial F$. Hence, the Arf-Kervaire invariant of $g|F$ is defined. In this proof we use the notation established in 15.4. All coefficients are $\mathbf{Z}/2$.

The case $l \not\equiv p(2)$ is proved using Theorem 15.3.

Let us consider $l \equiv p \equiv 0(2)$. By surgery, below the middle dimension, we can assume that $K_i(F)=0$ for $i<p/2$, that $K_i(\hat{M}_F)=0$ for $i<(l/2)-1$, and that $K_{(l/2)-1}(\partial\nu/(\mathbf{Z}/2)) \to K_{(l/2)-1}(\hat{M}_F/(\mathbf{Z}/2))$ is trivial. We have a diagram of long exact sequences:

$$\begin{array}{ccccccc} & & & & \downarrow & & \\ & & & & K_{l/2}(\partial\nu/(\mathbf{Z}/2)) & & \\ & & & & \downarrow{\scriptstyle j_*} & & \\ \dots \longrightarrow & K_{l/2}(\hat{M}_F/(\mathbf{Z}/2)) & \xrightarrow{\mathrm{tr}} & K_{l/2}(\hat{M}_F) & \xrightarrow{\pi_*} & K_{l/2}(\hat{M}_F/(\mathbf{Z}/2)) & \longrightarrow \dots \\ & & & & \downarrow{\scriptstyle i_*} & & \\ & & & & K_{l/2}(\hat{M}_F/(\mathbf{Z}/2), \partial\nu/(\mathbf{Z}/2)) & & \\ & & & & \downarrow & & \end{array}$$

The transfer map, tr, assigns to a cycle in $\hat{M}_F/(\mathbf{Z}/2)$ its total inverse image in $\hat{M}_F$. If the original cycle is immersed consistent with the bundle data of the normal map, then so is its double cover. On the other hand, the number of points of self-intersection of the double cover is even. Thus, the self-intersection form vanishes identically on the image of tr. The transfer is dual to the map $i_* \circ \pi_*$. Hence, performing surgery on a basis for the image of tr, leaves a subquotient of $K_{l/2}(\hat{M}_F)$ which is $(\mathrm{Im}\,(\mathrm{tr}))^\perp/\mathrm{Im}\,(\mathrm{tr})$. (If $l\equiv 0(4)$, then it may not be possible to do this surgery geometrically but algebraically the form on $K_{l/2}(\hat{M}_F)$ and the one on $(\mathrm{Im}\,(\mathrm{tr}))^\perp/\mathrm{Im}(\mathrm{tr})$ differ by a hyperbolic form.)

Since $K_{(l/2)-1}(\partial\nu/(\mathbf{Z}/2)) \to K_{(l/2)-1}(\hat{M}_F/(\mathbf{Z}/2))$ is trivial, duality tells us that $j_*: K_{l/2}(\partial\nu/(\mathbf{Z}/2)) \to K_{l/2}(\hat{M}_F/(\mathbf{Z}/2))$ is an injection. Thus the subquotient of $K_{l/2}(\hat{M}_F; \mathbf{Z}/2)$, $(\mathrm{Im}\,(\mathrm{tr}))^\perp/\mathrm{Im}\,(\mathrm{tr})$ is identified via $j_*^{-1} \circ \pi_*$ with $K_{l/2}(\partial\nu/(\mathbf{Z}/2)) = K_p(F)$. We claim that under this identification the intersection and self-intersection forms agree. If we can show this, then we will have proved that the Arf-Kervaire invariants of g and $g|F$ agree.

The argument is similar to the one in the S^1-case. If we begin with an immersed cycle

$S_x^{p/2} \to F$ representing x in $K_{p/2}(F)$, then we construct an immersion representing the corresponding class in $K_{l/2}(M)$, as follows. Set $q = \frac{1}{2}(l-p)$ and associate to $S_x^{p/2} \to F$ the immersion

$$S_x^{p/2} \times \mathbf{RP}^{q-1} \hookrightarrow S_x^{p/2} \times \mathbf{RP}^{l-p-1} \to (\partial\nu)/(\mathbf{Z}/2)$$

(where the second term represents a trivialization of $\partial\nu/(\mathbf{Z}/2) \to F$ pulled back to $S_x^{p/2}$). We choose the two copies of $\mathbf{RP}^{q-1}$ in $\mathbf{RP}^{l-p-1}$ over each double point of $S_x \to F$ to be disjoint linear subspaces. We extend the resulting embedding of $S^{p/2} \times \mathbf{RP}^{q-1}$ to a immersed manifold $Z^{l/2} \to \hat{M}_F/(\mathbf{Z}/2)$ which represents a relative class in $K_{l/2}(\hat{M}_F/(\mathbf{Z}/2), \partial\nu/(\mathbf{Z}/2))$. A cycle representative for the class in $K_{l/2}(M)$ corresponding to $x \in K_{p/2}(F)$ is $\zeta_x = \pi^{-1}(Z_x^{l/2}) \cup E_x^q$ where E_x^q is the disk bundle associated to the $S_x^{p/2} \times \mathbf{RP}^{q-1}$. Clearly, the number of double points of this immersion is twice the number of double points of $Z^{l/2}$ plus the number of double points of $S_x^{p/2}$.

If both the original immersion of $S_x^{p/2} \to F$ and the embedding $Z_x^{l/2} \hookrightarrow \hat{M}_F/(\mathbf{Z}/2)$ have normal bundles which are consistent with the bundle data covering the normal map, then the same will be true for the resulting immersion of $\zeta_x \to M$. Hence, the self-intersection form for g evaluated on $[\zeta_x]$ equals that for $g \mid F$ evaluated on x. This proves that, modulo hyperbolic forms, the form on $K_{p/2}(F; \mathbf{Z}/2)$ and the one on $K_{l/2}(M; \mathbf{Z}/2)$ are equivalent. Thus, the Arf-Kervaire invariants of g and $g \mid F$ are equal.

LEMMA 15.6. *Let ψ: $(W^{2m}, \partial W) \to (Z^{2m}, \partial Z)$ be a normal map with $\pi_1(Z) = \mathbf{Z}/2$, Z non-orientable, and $2m > 4$. Suppose $K_i(\psi \mid \partial W; \mathbf{Z}_{(2)}[\mathbf{Z}/2]) = 0$ for all i and that the Arf-Kervaire invariant of ψ is 0. Then ψ is normally bordant, relative to $\psi \mid \partial W$ to ψ' such that:*

(a) $K_i(\psi'; \mathbf{Z}_{(2)}[\mathbf{Z}/2]) = 0$ *for all i,*
(b) $K_i(\psi'; \mathbf{Z}[\mathbf{Z}/2]) = 0$ *for $i < m-1$, and*
(c) $K_{m-1}(\psi'; \mathbf{Z}^-) = 0$.

Proof. Surgery below the middle dimension can be performed so as to make $K_i(\psi; \mathbf{Z}[\mathbf{Z}/2]) = 0$ for $i < m$. Consider $K_m(\psi; \mathbf{Z}[\mathbf{Z}/2])/T$ where T is the sub-module of elements of finite order. It is a free $\mathbf{Z}[\mathbf{Z}/2]$-module. Its intersection form is non-singular when tensored with $\mathbf{Z}_{(2)}$. The Arf-Kervaire invariant of this form is 0 by hypothesis. Since the Arf-Kervaire invariant induces an isomorphism $L_{2m}(\mathbf{Z}_{(2)}[\mathbf{Z}/2], -) \cong \mathbf{Z}/2$, it follows that the form on $K_m(\psi; \mathbf{Z}[\mathbf{Z}/2])/T$ becomes hyperbolic when tensored over $\mathbf{Z}$ with $\mathbf{Z}_{(2)}$. Let

$$\tilde{H} \oplus \tilde{H}^* \xrightarrow[\cong]{\varphi} (K_m(\psi; \mathbf{Z}[\mathbf{Z}/2]/T) \underset{\mathbf{Z}}{\otimes} \mathbf{Z}_{(2)}$$

be an isomorphism between a hyperbolic form and the geometrically defined forms on $K_m(\psi)$. This map φ defines a splitting $H \oplus H'$ of $K_m(\psi; \mathbf{Z}[\mathbf{Z}/2])/T$; namely

$$H = \varphi(\tilde{H}) \cap (K_m(\psi; \mathbf{Z}[\mathbf{Z}/2])/T) \quad \text{and} \quad H' = \varphi(\tilde{H}^*) \cap (K_m(\psi; \mathbf{Z}[\mathbf{Z}/2])/T).$$

In this splitting the matrix for the intersection form is

$$\begin{pmatrix} 0 & A \\ (-1)^k({}^t\bar{A}) & 0 \end{pmatrix}$$

and $\mu | H$ is 0. The matrix A is a $\mathbf{Z}[\mathbf{Z}/2]$-matrix, which is non-singular over $\mathbf{Z}_{(2)}[\mathbf{Z}/2]$. We will show that there is a splitting $\varphi: \tilde{H} \oplus \tilde{H}^* \cong K_m(\psi; \mathbf{Z}_{(2)}[\mathbf{Z}/2])$ of this type so that the matrix A becomes non-singular when reduced to $\mathbf{Z}^-$ (i.e., if in each entry of A we set $\gamma = -1$, then A becomes a non-singular integral matrix). If we have such a decomposition, then surgery on a basis for H produces a normal bordism from ψ to a normal map ψ' such that $K_{m-1}(\psi'; \mathbf{Z}^-) = 0$. To see this let $U \xrightarrow{\tilde{\psi}} Z \times I$ be the normal bordism created by adding $(m+1)$-handles along embedded spheres representing a basis for H. Let $\psi': W' \to Z$ be the "other end". Then $K_i(U, W; \mathbf{Z}[\mathbf{Z}/2]) = 0$ for $i \neq m+1$ and $K_{m+1}(U, W; \mathbf{Z}[\mathbf{Z}/2]) = H$. Furthermore, $K_m(U, W'; \mathbf{Z}[\mathbf{Z}/2]) = K_{m+1}(U, W; \mathbf{Z}[\mathbf{Z}/2])^*$. The map

$$\begin{array}{ccc} K_m(U; \mathbf{Z}[\mathbf{Z}/2]) & \xrightarrow{j_*} & K_m(U, W'; \mathbf{Z}[\mathbf{Z}/2]) \\ \wr\| & & \|\wr \\ H' & \longrightarrow & H^* \end{array}$$

sends $h' \in H'$ to the homomorphism whose value on $h \in H$ is $(-1)^m \cdot \lambda(h', h)$. It follows that the map j_* is identified with $(-1)^m \cdot A$.

If A^- is non-singular over $\mathbf{Z}^-$, then looking at the long exact sequence of the pair (U, W'), we see that $K_{m-1}(\psi'; \mathbf{Z}^-) = 0$.

Thus, to complete the proof of 15.6, it remains only to find the required splitting of $K_m(\psi; \mathbf{Z}[\mathbf{Z}/2])/T$. Given one splitting $H \oplus H' \cong K_m(\psi; \mathbf{Z}[\mathbf{Z}/2])/T$ in which A^- is not necessarily non-singular, then changing bases for H and H' corresponds to performing row and column operations on A. Since the Whitehead group of $\mathbf{Z}_{(2)}[\mathbf{Z}/2]$ is 0, this allows us to assume that A is a diagonal matrix. Now, we can inductively treat the problem of finding a new splitting in which A^- is non-singular. Hence, it suffices to consider a 2×2 subspace of $H \oplus H'$ with bases h and h', with $\mu(h) = \mu(h') = 0$ and with intersection pairing given by

$$\begin{pmatrix} 0 & a + b\gamma \\ (-1)^m(a - b\gamma) & 0 \end{pmatrix}, \quad a + b \equiv 1(2).$$

It is necessary to enlarge this space by forming the orthogonal sum with a hyperbolic form with basis $\{e, f\}$. Geometrically this can be accomplished by doing a trivial surgery in dimension $(m-1)$. Let $2s+1=a+b$. If $s\equiv 0(2)$ define a new basis to be

$$\left\{h(1-\gamma)-e(s(1-\gamma)+1),\, h'+f(1+\gamma),\, h'\left(\frac{-s}{2}\right)(1+\gamma)-f(s(1+\gamma)+1),\, h-e\left(\frac{s}{2}\right)(1-\gamma)\right\}.$$

If $s\equiv 1(2)$, then we take the new basis to be

$$\left\{h(1-\gamma)-e(s(1-\gamma)+1), h'+f(1+\gamma), h\left(\frac{s+1}{2}\right)(1+\gamma)+f(s(1+\gamma)+1), h-e\left(\frac{s+1}{2}\right)(1-\gamma)\right\}.$$

In either case, μ vanishes on all the given basis elements, and the matrix for the intersection pairing is

$$\left[\begin{array}{cc|cc} 0 & 0 & s(1+\gamma)+1 & 0 \\ 0 & 0 & 0 & (-1)^m[(a-b\gamma-(s+\varepsilon)(1-\gamma)] \\ \hline (-1)^m(s(1-\gamma)+1) & 0 & 0 & 0 \\ 0 & (a+b\gamma)-(s+\varepsilon)(1+\gamma) & 0 & 0 \end{array}\right]$$

where $\varepsilon=0$ in the first case and $\varepsilon=1$ in the second. One sees immediately that in either case, the matrix A is non-singular when reduced into $\mathbf{Z}^{-}$.

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