AVERAGES OF THE COUNTING FUNCTION OF A QUASIREGULAR MAPPING

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1. Introduction

The theory of quasiregular and quasimeromorphic mappings has turned out to form a natural real n-dimensional generalization of the theory of analytic and meromorphic functions of one complex variable. The study of these mappings was initiated by Rešetnjak in 1966 in a series of papers listed in [9]. Since then the theory has been developed in many directions by several authors. For basic parts of it we refer to [9–11]. Definitions are given in 2.1 of Section 2.

Large parts of the theory of analytic functions of one complex variable have their analogs for n-dimensional quasiregular mappings. The methods of proofs for $n \geqslant 3$ are for the most part completely different from the classical methods in the plane theory. This state of affairs has had its influence also on the classical theory. On one hand, new and sometimes simpler proofs have been found for known theorems. On the other hand, some interesting results are new discoveries for the value distribution theory in the plane.

In this paper we study value distribution of quasiregular mappings in Riemannian manifolds. Let us consider the basic case, a nonconstant quasimeromorphic mapping f of the Euclidean n-space $\mathbf{R}^n$ into $\bar{\mathbf{R}}^n = \mathbf{R}^n \cup \{\infty\}$. The fundamental question of value distribution of f is how $f^{-1}(y)$ is distributed and how this set varies with changing of y. A natural quantitative measurement of the behavior of $f^{-1}(y)$ is the *counting function* $n(r, y)$ which is the number of points of $f^{-1}(y)$ in the ball $|x| \leqslant r$ with multiplicity regarded. The spherical average $A(r)$ is the average of $n(r, y)$ with respect to the spherical n-measure on $\bar{\mathbf{R}}^n$ when y runs over $\bar{\mathbf{R}}^n$. The well-known covering theorems in Ahlfors's theory of covering surfaces [1, p. 164, 165] imply for $n = 2$ that the average of $n(r, y)$ when y runs over a subdomain or a "regular curve" in $\bar{\mathbf{R}}^2$, is close to $A(r)$ outside a set of radii r with finite logarithmic measure. This suggests that $n(r, y)$ is usually close to $A(r)$ and that "equidistribution" occurs to some general extent. The purpose of this paper is to study how strong such equidistribution is. Our main results are new also for the plane theory of meromorphic functions. We work all over on the "nonintegrated level" and do not use smoothed counting functions, obtained for example by integrating $n(r, y)$ logarithmically with respect to r as is typical in the Nevanlinna theory [13].

For an arbitrary point y, there need not be any bounded ratio between $n(r, y)$ and $A(r)$ outside a thin exceptional set of r-values. First, if y is omitted by f, then $n(r,y)=0$ for all r. In the other direction, it follows from Toppila's Theorem 4 in [23] that for any $k>1$ there exists a nonconstant meromorphic function of the plane for which $n(r, 0)/A(r)>k$ in a set of positive lower logarithmic density. For a modification of Toppila's result, see Example 6.1 in Section 6.

The study of the value distribution of quasimeromorphic mappings of $\mathbf{R}^n$ into $\bar{\mathbf{R}}^n$ was started in [20], where the main emphasis was on the relationship between the pointwise behavior of $n(r, y)$ and the spherical average. One of the problems treated in [20] is the question of the validity of an inequality

$$\limsup_{r\to\infty} n(r, y)/A(\theta r) \leqslant c \tag{1.1}$$

where $\theta, c>1$ are constants. Even for meromorphic functions (1.1) need not hold no matter how the constants θ and c are chosen. This follows from a slight modification of [23, Theorem 4]; see also Example 6.1. On the other hand, if the quasimeromorphic mapping has an asymptotic value a_0, then given any $c>1$, there exists a constant $\theta>1$ such that (1.1) holds for all $a \neq a_0$ [20, Theorem 5.11]. In the proofs of such theorems a good estimate is needed for comparing averages of the counting function over concentric $(n-1)$-dimensional spheres. If we denote by $\nu(r, s)$ the average of $n(r, y)$ when y runs over the sphere $|y|=s$, such an estimate is given by the inequality

$$c\nu(\theta r, t) \geqslant \nu(r, s) - \frac{K(f)|\log(t/s)|^{n-1}}{(1-1/c)(\log\theta)^{n-1}}, \tag{1.2}$$

valid for all $\theta, c>1$, $0<s, t<\infty$ [20, Theorem 4.1]. Here $K(f)$ is the maximal dilatation of f. The inequality (1.2) is proved by a special technique of path families where one combines the modulus inequality [26, Theorem 3.1] with a result on maximal path lifting given in [19]. The factor $(\log\theta)^{1-n}$ in the error term in (1.2) makes it possible to show that the stronger inequality $c\nu(r, t)\geqslant\nu(r, s)$ holds for all r outside a set of finite logarithmic measure, and in fact each average $\nu(r, s)$ is arbitrarily close to the spherical average $A(r)$ for all r outside such a set [20, Theorem 4.19].

The idea of the proof of (1.2) suggests that a similar inequality with respect to θ holds in a much wider sense, and as a consequence, averages of the counting function with respect to various measures are arbitrarily close to each other outside an exceptional set for the exhaustion parameter. On the other hand, the discussion of the pointwise case above and Example 6.1 shows that a regularity assumption on the measure is needed which prevents too strong singularities at points.

We shall establish an equidistribution theory for averages of the counting function of a quasiregular mapping with respect to measures with a regularity condition. More precisely, we are given a nonconstant quasiregular mapping $f\colon M\to N$ of a noncompact Riemannian n-manifold M into a compact Riemannian n-manifold N and the counting function $n(s, y)$, $0<a\leqslant s<b\leqslant\infty$, of f with respect to an admissible exhaustion function of M, i.e. an exhaustion function which is normalized by means of conformal capacity and which satisfies the condition in 2.16. Let μ be a measure in N such that Borel sets are μ-measurable and $0<\mu(N)<\infty$. Let $h\colon [0, \infty[\to[0, \infty[$ be increasing, continuous, and such that $h(0)=0$ and $h(r)>0$ for $r>0$. We call h a calibration function and μ *h-calibrated* if

$$\mu(B(x, r))\leqslant h(r) \tag{1.3}$$

for all balls $B(x, r)\subset N$. Our main result is that the average $\nu_\mu(s)$ of $n(s, y)$ with respect to μ is arbitrarily close to the average $A(s)$ of $n(s, y)$ with respect to the Lebesgue measure for all s outside an exceptional set A provided μ is h-calibrated with h satisfying

$$\int_0^1 \frac{h(r)^{1/pn}}{r}\,dr<\infty \tag{1.4}$$

for some $p>2$. This is expressed by the limit condition

$$\lim_{\substack{s\to b\\ s\notin A}} \frac{\nu_\mu(s)}{A(s)}=1. \tag{1.5}$$

The exceptional set A for the exhaustion parameter s has in the parabolic case $b=\infty$ finite logarithmic measure, whereas in the hyperbolic case $b<\infty$ the condition

$$\limsup_{s\to b}\,(b-s)A(s)^{1/p\lambda}=\infty \tag{1.6}$$

is needed to ensure that A is thin near b. Here $\lambda\geqslant n-1$ is a constant depending on the exhaustion. Our theory generalizes the covering theorems in [1, p. 164, 165].

The problem of comparing averages is unsymmetric in the sense that the inequality

$$\liminf_{\substack{s\to b\\ s\notin A}} \frac{\nu_\mu(s)}{A(s)}\geqslant 1 \tag{1.7}$$

is true already if $\limsup_{r\to 0}\mu(B(x,r))/h(r)\leqslant 1$ for μ almost every $x\in N$ with h satisfying (1.4) for some $p>2$. An example of such a measure μ is the restriction measure $F\mapsto\mathcal{H}^\alpha(F\cap E)$ of the α-dimensional Hausdorff measure $\mathcal{H}^\alpha$, $0<\alpha\leqslant n$, where E is any $\mathcal{H}^\alpha$-measurable set with $0<\mathcal{H}^\alpha(E)<\infty$; more generally, see 5.12.5. Example 6.1 shows that (1.5) need not hold for such measures.

After preliminary results we first prove in Section 3 a lemma which tells how much extreme values of the counting function in a set can, in terms of conformal capacity, deviate from averages over spheres lying in a chart. In Section 4 relationships between capacity and h-calibrated measures are used to establish inequalities of type (1.2) for averages (Theorem 4.8).

The integral condition (1.4) for h originates from the proof of [17, Theorem 8] in connection with a lower bound for capacity. This is presented in Lemma 4.2. The condition $p>2$ is essentially needed in the proof of (4.5) of Lemma 4.4 to obtain effective upper bounds for the μ-measure of sets in which the counting function exceeds an average value.

The main results are presented in Theorem 5.11 and are proved by means of the inequalities in Section 4 and lemmas on real functions. Our methods apply also to the study of the pointwise behavior of $n(s,y)$. In fact, we prove (Theorem 5.13), under a restriction for the hyperbolic case, that there exists a sequence $s_i\nearrow b$ and a set $E\subset N$ of capacity zero such that for $y\in N\setminus E$, $n(s_i,y)/A(s_i)$ tends to one. This result is known earlier for meromorphic functions in the plane with the standard exhaustion by disks. In fact, Miles proves in [12, Theorem 2] a stronger statement in the sense that the limit is obtained outside an exceptional set in the exhaustion parameter.

To prove the results in this paper for Riemannian n-manifolds instead of just $\mathbf{R}^n$ and $\bar{\mathbf{R}}^n$, does not require much extra work. Essentially all what is needed is the inequality 2.10 of moduli of path families, a discussion on exhaustions in Section 2, and some basic facts about Riemannian manifolds.

2. Preliminary results

2.1. *Quasiregular mappings in Riemannian manifolds.* We assume throughout the paper that Riemannian manifolds are always pure dimensional without boundary, C^∞, connected, paracompact, orientable, with a given C^∞ Riemannian metric, and with a given C^∞ volume form defining the orientation. Chart maps are always taken orientation preserving. In any Riemannian n-manifold M we denote the ball $\{y\in M\mid d(y,x)<r\}$ by $B(x,r)$ and the sphere $\{y\in M\mid d(y,x)=r\}$ by $S(x,r)$ where d is the Riemannian distance. If $M=\mathbf{R}^n$, we set $B(r)=B(0,r)$, $S(r)=S(0,r)$.

We assume throughout the paper that $n \geqslant 2$. Let G be a domain in $\mathbf{R}^n$. A continuous mapping $f\colon G \to \mathbf{R}^n$ is *quasiregular* if (1) f is ACL^n and (2) there exists K, $1 \leqslant K < \infty$, such that

$$\|f'(x)\|^n \leqslant K J_f(x) \tag{2.2}$$

holds a.e. in G. Here $f'(x)$ is the formal derivative of f at x, i.e. the linear map defined by means of the partial derivatives $D_i f(x)$ as $f'(x)e_i = D_i f(x)$, e_i being the ith standard basis vector in $\mathbf{R}^n$. $\|f'(x)\|$ is the supremum norm of $f'(x)$ and $J_f(x)$ the Jacobian determinant of f at x.

If $f\colon G \to \mathbf{R}^n$ is quasiregular, then it is either constant or discrete, open, and sense-preserving [17], [18]. It is also differentiable a.e. [17], and hence $f'(x)$ is the derivative at x a.e. If f is not constant, $J_f(x) > 0$ a.e. [9, 8.2].

Let M and N be Riemannian n-manifolds and $f\colon M \to N$. f is called *locally quasiregular* if at each point $x \in M$ there is a local expression of f which is quasiregular in the above sense. The tangent linear map $T_x f\colon T_x M \to T_{f(x)} N$ is then defined a.e. if f is locally quasiregular. The mapping f is called *quasiregular* if (1) f is locally quasiregular and (2) there exists K, $1 \leqslant K < \infty$, such that

$$\|T_x f\|^n \leqslant K J_f(x) \tag{2.3}$$

holds a.e. (cf. [5]). The smallest K in (2.3) is the *outer dilatation* $K_O(f)$ of f, and the smallest K for which

$$J_f(x) \leqslant K \inf_{\|h\|=1} \|T_x f h\|^n$$

holds a.e. is the *inner dilatation* $K_I(f)$ of f. $K(f) = \max(K_O(f), K_I(f))$ is the *maximal dilatation* of f. The term *quasimeromorphic* is reserved for the case where M is a domain in $\mathbf{R}^n$ or $\bar{\mathbf{R}}^n$ and $N = \bar{\mathbf{R}}^n$. $\bar{\mathbf{R}}^n$ is equipped with the spherical metric. A quasiregular homeomorphism is called a *quasiconformal* mapping.

2.4. *Inequalities for moduli of path families for quasiregular mappings.* We shall present two important inequalities for moduli of path families well known for quasiregular mappings in $\mathbf{R}^n$. These are of global nature in contrast to our definition of a quasiregular mapping.

We shall use the terminology of paths mainly from [25] modified to manifolds and also from [22]. Let $\alpha\colon I \to M$ be a path. The length of α is denoted by $l(\alpha)$ and the locus αI by $|\alpha|$. If α is rectifiable and closed, we denote by $\alpha^0\colon [0, l(\alpha)] \to M$ its parametrization by arc length, by s_α its length function $s_\alpha\colon I \to [0, l(\alpha)]$ such that $\alpha = \alpha^0 \circ s_\alpha$. A map $f\colon M \to N$ is called absolutely continuous on α if $f \circ \alpha^0$ is absolutely continuous.

Let Γ be a family of nonconstant paths in M and let $1 \leqslant p < \infty$. We denote by $F(\Gamma)$ the family of all Borel functions ϱ: $M \to [0, \infty[$ such that the line integral satisfies

$$\int_\gamma \varrho\, ds \geqslant 1 \tag{2.5}$$

for all locally rectifiable $\gamma \in \Gamma$. The number

$$M_p(\Gamma) = \inf_{\varrho \in F(\Gamma)} \int_M \varrho^p \, d\mathcal{L}^n$$

is called the *p-modulus* of Γ. We denote the Lebesgue measure on a Riemannian n-manifold defined by its volume form by $\mathcal{L}^n$. $M_n(\Gamma)$ is also denoted by $M(\Gamma)$ and called simply the *modulus* of Γ. Basic properties such as Theorems 6.2, 6.4, and 6.7 in [25] are also true here. However, even for $p = n$ one cannot replace $F(\Gamma)$ by the larger family of functions ϱ for which (2.5) holds whenever $\gamma \in \Gamma$ is rectifiable as in $\mathbf{R}^n$ [25, 6.9].

We need the following substitute for Fuglede's theorem (see [25, 28.2]):

2.6. Lemma. *Let f: $M \to N$ be quasiregular and let Γ_0 be the family of paths in M such that each $\gamma \in \Gamma_0$ has a closed subpath on which f is not absolutely continuous. Then $M(\Gamma_0) = 0$.*

Proof. We cover M and N by relatively compact charts (U_i, φ_i) and (V_i, ψ_i), $i = 1, 2, \ldots$, respectively, such that φ_i and ψ_i are bilipschitzian and such that for each i there exists j for which $fU_i \subset V_j$. For $i, k \geqslant 1$ set

$$\Gamma_i = \{\gamma \in \Gamma_0 \mid \gamma \text{ closed}, |\gamma| \subset U_i\},$$
$$V^k = \bigcup_{i \leqslant k} U_i.$$

If $\gamma \in \Gamma_0$, there exists a closed subpath β: $[a, b] \to M$ of γ on which f is not absolutely continuous, hence β is in V^k for some k. There exists a division of $[a, b]$ into a finite number of closed subintervals $\Delta_1, \ldots, \Delta_q$ such that each $\beta | \Delta_p$ is in some U_{i_p}, $i_p \leqslant k$. There exists p such that f is not absolutely continuous on $\beta | \Delta_p$, hence $\beta | \Delta_p \in \Gamma_{i_p}$. It follows that Γ_0 is minorized by $\bigcup_i \Gamma_i$, hence

$$M(\Gamma_0) \leqslant \sum_{i=1}^{\infty} M(\Gamma_i).$$

It thus suffices to show that $M(\Gamma_i) = 0$ for an arbitrary i.

Let $\gamma \in \Gamma_i$ and let j be such that $fU_i \subset V_j$. Since φ_i and ψ_j are bilipschitzian and f is not absolutely continuous on γ, the map $h = \psi_j \circ f \circ \varphi_i^{-1}$ is not absolutely continuous on $\varphi_i \circ \gamma$.

Furthermore, h is quasiregular since it is quasiregular locally and has bounded dilatation. By [25, 28.2] we have $M(\varphi_i \Gamma_i)=0$. By [22, 5.3] $M(\Gamma_i)=0$. The lemma is proved.

2.7. LEMMA. *Let $f: M \to N$ be quasiregular, let $v: N \to \mathbf{R}$ be a nonnegative Borel function, and let $A \subset X$ be a Borel set. Then*

$$\int_A (v \circ f) J_f \, d\mathcal{L}^n = \int_N v(y) N(y, f, A) \, d\mathcal{L}^n(y)$$

where $N(y, f, A) = \operatorname{card} A \cap f^{-1}(y)$.

If A and fA are contained in charts, Lemma 2.7 follows by [22, 3.8] and by the application of [16, Theorem 3, p. 364] to the functions $v_k = \min(k, v)$, $k = 1, 2, \ldots$. The general case is handled by the use of decompositions of M and N [22, 3.1].

For completeness we include the following analog of [9, 3.2] for manifolds although it is not used in this paper. We use the notation $N(f, A) = \sup_{y \in N} N(y, f, A)$ for $A \subset M$.

2.8. THEOREM. *Suppose that $f: M \to N$ is a quasiregular mapping and that A is a Borel set in M such that $N(f, A) < \infty$. If Γ is a family of paths in A,*

$$M(\Gamma) \leqslant N(f, A) K_O(f) M(f\Gamma).$$

This theorem is proved as in [9, 3.2] by the use of 2.6 and 2.7. Note, however, that in [9] (2.5) is required only for rectifiable paths.

For the other inequality we need a lemma of Poleckii [15, Lemma 6], see also [26, 2.6]. As in [26] we use the following terminology. Let $f: M \to N$ be continuous and light and let $\alpha: I \to M$ be a closed path. We say that f is absolutely precontinuous on α if the path $\beta = f \circ \alpha$ is rectifiable and the path $\alpha^*: [0, l(\beta)] \to M$ such that $\alpha = \alpha^* \circ s_\beta$, given by an analog of [26, 2.3] for manifolds, is absolutely continuous.

2.9. LEMMA. *Let $f: M \to N$ be nonconstant and quasiregular. Let Γ_0 be the family of all paths β in N such that either β is not locally rectifiable or there exists a closed path α in M such that $f \circ \alpha$ is a subpath of β and f is not absolutely precontinuous on α. Then $M(\Gamma_0) = 0$.*

Proof. The subfamily $\tilde{\Gamma}$ of Γ_0 consisting of paths which are not locally rectifiable has zero modulus. We cover M and N by charts (U_i, φ_i) and (V_i, ψ_i), $i = 1, 2, \ldots$, as in the proof of 2.6 and for $i \geqslant 1$ we let Γ_i be the set of all closed paths in U_i on which f is not absolutely precontinuous. Then Γ_0 is minorized by the union of $\bigcup_i f\Gamma_i$ and $\tilde{\Gamma}$, hence it suffices to show that $M(f\Gamma_i) = 0$ for all i. To prove this we use [15, Lemma 6] and a similar argument to that in the proof of 2.6. The lemma is proved.

Our second inequality is [26, 3.1] for manifolds. Its corollary 2.11 was proved for $\mathbf{R}^n$ by Poleckii [15].

2.10. THEOREM. *Suppose that $f: M\to N$ is a nonconstant quasiregular mapping, Γ is a path family in M, Γ' is a path family in N, and that m is a positive integer such that the following condition is satisfied:*

There is a set $E_0\subset M$ of measure zero such that for every path $\beta: I\to N$ in Γ' there are paths $\alpha_1, \dots, \alpha_m$ in Γ such that $f\circ\alpha_i$ is a subpath of β for all i and for every $x\in M\setminus E_0$ and $t\in I$ the relation $\alpha_i(t)=x$ holds for at most one i. Then

$$M(\Gamma')\leqslant\frac{K_I(f)}{m}M(\Gamma).$$

Proof. Let Γ_0 be the family of Lemma 2.9. We set $\Gamma_1=\Gamma'\setminus\Gamma_0$. Then $M(\Gamma_1)=M(\Gamma')$ and it suffices to prove

$$M(\Gamma_1)\leqslant\frac{K_I(f)}{m}M(\Gamma).$$

By only slight modifications and by the use of 2.7 to homeomorphisms we can follow the proof of [26, 3.1]. Note that here Γ_1 contains also paths which are only locally rectifiable. We point out that in the proof of [26, 3.1] the family $F(\Gamma)$ has the same meaning as in this paper.

2.11. COROLLARY. *If $f: M\to N$ is a nonconstant quasiregular mapping and if Γ is a path family in M, then*

$$M(f\Gamma)\leqslant K_I(f)M(\Gamma).$$

2.12. *Condensers and capacities.* A *condenser* in M is a pair (A, C) where $A\subset M$ is open with $M\setminus A\neq\emptyset$ and $C\subset A$ is compact and nonempty. The (conformal) *capacity* cap (A, C) of a condenser (A, C) is the modulus $M(\Delta(C, \partial A; A\setminus C))$ where we have used the notation $\Delta(E, F; H)$ for the family of paths γ in H such that $\overline{|\gamma|}\cap E\neq\emptyset\neq\overline{|\gamma|}\cap F$.

A compact subset K of M is said to be of *capacity zero* if the modulus of the family of paths in M with one endpoint in K is zero. An arbitrary subset E of M is said to be of capacity zero if all compact subsets of E are of capacity zero. If E is of capacity zero, we write cap $E=0$, otherwise cap $E>0$.

2.13. *Exhaustions.* We shall carry out our study of value distribution of a quasiregular mapping of a noncompact Riemannian n-manifold M into a compact Riemannian n-manifold N with respect to an exhaustion of M by compact subsets which will be parametrized as presented below. We assume now that M is noncompact.

By an *exhaustion function* of M we mean a function $D\colon [a, b[\to \mathcal{D}(M)$, where $-\infty < a < b \leqslant \infty$, such that each $D(t) = D_t \subset M$ is open, connected, the closure $\bar{D}_t$ is compact, $\bar{D}_t \subset D_u$ for $t < u$, and

$$M = \bigcup_{t \in [a, b[} D_t.$$

We shall use exhaustion functions $D\colon [a, b[\to \mathcal{D}(M)$ with $a > 0$, $D_a \neq \emptyset$, and parametrized via the equation

$$t = a \exp\left(\left(\frac{\omega_{n-1}}{M(\Gamma_{a,t})}\right)^{1/(n-1)}\right) \tag{2.14}$$

for $t > a$, where $\Gamma_{a,t}$ is the family of paths in $D_t \setminus \bar{D}_a$ which connect ∂D_t and $\bar{D}_a$ and ω_{n-1} is the $(n-1)$-dimensional measure on the unit sphere in $\mathbf{R}^n$. This could for $n = 2$ be called a parametrization by normalized harmonic module. Let $M = \mathbf{R}^n$ or $M = B(b)$. Then $t \mapsto B(t)$ is an exhaustion satisfying (2.14).

In order to obtain significant value distribution results with respect to a given exhaustion we need a measure of the deviation from an "extremal" exhaustion with respect to conformal capacity which is the substitute for harmonic exhaustion on a Riemann surface. Let $a < s < t < b$. Then $\Gamma_{a,t}$ is minorized by both $\Gamma_{a,s}$ and $\Gamma_{s,t}$ which are separate, hence

$$\left(\log \frac{t}{s}\right)^{n-1} \geqslant \frac{\omega_{n-1}}{M(\Gamma_{s,t})}. \tag{2.15}$$

We shall need an opposite inequality. More precisely, we give the following definition.

2.16. *Definition.* An exhaustion $D\colon [a, b[\to \mathcal{D}(M)$ satisfying (2.14) is called *admissible* if there exist constants $a_0 \in]a, b[$, $\theta_0 > 1$, $\varkappa > 0$, and $\lambda \geqslant n-1$ such that

$$\left(\log \frac{t}{s}\right)^{\lambda} \leqslant \varkappa \frac{\omega_{n-1}}{M(\Gamma_{s,t})} \tag{2.17}$$

holds for $a_0 \leqslant s < t < b$, $t/s \leqslant \theta_0$.

Note that in the case $b < \infty$ always $t/s \leqslant b/a_0$. The exhaustion of $\mathbf{R}^n$ or $B(b)$ by balls $B(t)$ is admissible and satisfies (2.17) with $\lambda = n-1$, $\varkappa = 1$ for $a < s < t < b$. $M = B(b)$ here is a special case of the exhaustion of a relatively compact domain U in $\mathbf{R}^n$ with a condition on the boundary as follows. Let $F \subset U$ be a nondegenerate continuum such that $U \setminus F$ is a domain. Let $U \setminus F$ satisfy Martio's condition $M_x = \infty$ [7] at each of its boundary points x. By [7, 5.9] there exists an extremal function $u\colon \overline{U \setminus F} \to \mathbf{R}$ in the definition

[6, 6.2] of the conformal capacity of the condenser $E=(U, F)$ with boundary values $u|\partial F=0$, $u|\partial U=1$. Then the level sets $D_t=\{x\in U\setminus F\,|\,u(x)<u_t\}\cup F$, where

$$u_t=\left(\frac{\operatorname{cap} E}{\omega_{n-1}}\right)^{1/(n-1)}\log\frac{t}{a},$$

give an admissible exhaustion for U which satisfies (2.17) with $\varkappa=1, \lambda=n-1$ for $a<s<t<b$. While this method takes partly care of the "hyperbolic" case $b<\infty$, no existence result for admissible exhaustions in the "parabolic" case $b=\infty$ is known if $n\geq 3$. For $n=2$ it is well known that parabolicity is equivalent to the existence of an Evans-Selberg potential which then can be used to produce a harmonic exhaustion. However, by using a preliminary discrete exhaustion (G_k) of M, it is possible with an idea of Ohtsuka to produce an exhaustion function of M which is "admissible on intervals" of $[a, b[$. Value distribution with respect to such partly admissible exhaustions can be established in the spirit of the present article, although formulation of the results becomes slightly more complicated.

One can prove that the class of admissible exhaustions of $\mathbf{R}^2$ contains every exhaustion which is obtained from the exhaustion by concentric disks by applying a quasiconformal self-map $h: \mathbf{R}^2\to\mathbf{R}^2$, i.e. $D_t=hB(t)$. The corresponding result for $\mathbf{R}^n$ is probably also true but there seems to be a lack of sufficiently sharp modulus estimates.

2.18. *Counting function.* Let $f: M\to N$ be a nonconstant quasiregular mapping of a noncompact Riemannian manifold M into a Riemannian manifold N. Assume that we are given a fixed exhaustion function D: $[a, b[\to\mathcal{D}(M)$ of M. The *counting function* of f with respect to D is then

$$n(t, y)=\sum_{x\in\bar{D}_t\cap f^{-1}(y)} i(x, f),$$

defined for $t\in[a, b[$, $y\in N$. Here $i(x, f)$ is the local index of f at x [24]. Since $\bar{D}_t$ is compact, $n(t, y)$ is finite. The function $y\mapsto n(t, y)$ is upper semicontinuous.

3. Comparison of extreme values and averages

3.1. In the rest of the paper let $f: M\to N$ be a nonconstant quasiregular mapping of a noncompact Riemannian n-manifold M into a compact Riemannian n-manifold N with inner dilatation $K_I=K_I(f)$. We assume that M has an admissible exhaustion D: $[a, b[\to\mathcal{D}(M)$ with constants a_0, θ_0, λ, and $\varkappa$ as in 2.16.

For small r we denote by $\nu(s, S(x,r))$ the average of $n(s,y)$ over the sphere $S(x,r)\subset N$ with respect to the $(n-1)$-dimensional (normalized) Hausdorff measure $\mathcal{H}^{n-1}$. For any nonempty set $E\subset N$ we define

$$\bar{n}(s, E)=\sup_{y\in E} n(s,y),$$

$$\underline{n}(s, E)=\inf_{y\in E} n(s,y).$$

Since N is compact, there exists $r_0>0$ such that for each $\zeta\in N$ there is a chart map φ_ζ: $B(\zeta, r_0)\to B(r_0)$ which is 2-bilipschitzian (i.e. the Lipschitz constants of φ_ζ and φ_ζ^{-1} are bounded by 2) and which has the property $\varphi_\zeta S(\zeta, r)=S(r)$ for all $r\in]0, r_0]$. We fix $\tau\in]0, \frac{1}{3}[$ such that $c_n \log 2 > \omega_{n-1} (\log (1/\tau))^{1-n}$ where $c_n>0$ is the positive constant in [25, (10.11)] depending only on n. Recall that $\omega_{n-1}=\mathcal{H}^{n-1}(S(1))$.

3.2. Lemma. *Let* $0<u<v<\infty$, $F_1\subset \bar{B}(u)$, $F_2\subset\partial B(v)$, $\Gamma_{12}=\Delta(F_1, F_2; \bar{B}(v))$, $\Gamma_1=\Delta(F_1, \partial B(v); \bar{B}(v))$, *and* $\Gamma_2=\Delta(F_2, \partial B(u); \bar{B}(v)\setminus B(u))$ (see 2.12 for notation). *Then*

$$M(\Gamma_{12})\geqslant 3^{-n}\min (M(\Gamma_1), M(\Gamma_2), c_n \log (v/u))$$

where $c_n>0$ *is the constant in* [25, (10.11)].

Proof. The proof is similar to the proof of [10, 3.11] and [14, 3.3]. Choose $\varrho\in F(\Gamma_{12})$. Consider first the case where

$$\int_{\gamma_1} \varrho\, ds \geqslant \tfrac{1}{3}$$

holds for every locally rectifiable path $\gamma_1\in\Gamma_1$ or

$$\int_{\gamma_2} \varrho\, ds \geqslant \tfrac{1}{3}$$

holds for every locally rectifiable path $\gamma_2\in\Gamma_2$. Then $3\varrho\in F(\Gamma_1)$ or $3\varrho\in F(\Gamma_2)$ which implies

$$\int_{\mathbf{R}^n} \varrho^n\, d\mathcal{L}^n \geqslant 3^{-n}\min (M(\Gamma_1), M(\Gamma_2)).$$

In the remaining case there exist paths $\gamma_1\in\Gamma_1$ and $\gamma_2\in\Gamma_2$ such that

$$\int_{\gamma} \varrho\, ds \geqslant \tfrac{1}{3}$$

for every locally rectifiable path $\gamma \in \Delta(|\gamma_1|, |\gamma_2|; B(v)\setminus\bar{B}(u)) = \Gamma$. Then $3\varrho \in F(\Gamma)$, and by [25, 10.12]

$$\int_{\mathbf{R}^n} \varrho^n d\mathcal{L}^n \geqslant 3^{-n} M(\Gamma) \geqslant 3^{-n} c_n \log(v/u).$$

The lemma is proved.

3.3. Lemma. *For each $c>1$ there exists $d>0$ such that the following holds. Let $2<q\leqslant 3$, $0<r<r_0$, $z\in N$, and let $F\subset B(z,\tau r)$ be a set with $M(\Delta(F,\partial B(z,r);\bar{B}(z,r)))\geqslant\delta>0$, where r_0 and τ are as in* 3.1. *Then*

$$cv(\theta s, S(z,r)) \geqslant \underline{n}(s,F) - \frac{d}{\delta(\log\theta)^{\lambda}}, \tag{3.4}$$

and

$$v(s, S(z,r)) \leqslant c\bar{n}(\theta s, F) + \frac{d}{\delta(q-2)(\log\theta)^{q\lambda}} \tag{3.5}$$

whenever $a_0 \leqslant s \leqslant \theta s < b$, $\theta \leqslant \theta_0$.

Proof. To prove (3.4) fix s and θ, set $Y = S(z,r)$ and

$$A = \{y \in Y \mid n(\theta s, y) < \underline{n}(s,F)/c\}.$$

Then

$$\begin{aligned} c\int_Y n(\theta s, y)\, d\mathcal{H}^{n-1}(y) &\geqslant \underline{n}(s,F)\,\mathcal{H}^{n-1}(Y\setminus A) \\ &= \underline{n}(s,F)\,\mathcal{H}^{n-1}(Y) - \underline{n}(s,F)\,\mathcal{H}^{n-1}(A). \end{aligned} \tag{3.6}$$

We may assume $\mathcal{H}^{n-1}(A)>0$ and $\underline{n}(s,F)>0$. Let $A'\subset A$ be compact such that $\mathcal{H}^{n-1}(A')>\mathcal{H}^{n-1}(A)/2$ and let Γ be the family of paths $\gamma\colon[0,1]\to\bar{B}(z,r)$ with $\gamma(0)\in F$, $\gamma(1)\in A'$. If $\gamma\in\Gamma$ and if $\{x_1,\dots,x_k\} = f^{-1}(\gamma(0))\cap\bar{D}_s$, then

$$m = \sum_{j=1}^{k} i(x_j, f) \geqslant \underline{n}(s,F).$$

By the analog of [19, Theorem 1] for manifolds there exists a maximal sequence $\alpha_1,\dots,\alpha_m$ of $f|D_{\theta s}$-liftings of γ starting at the points of $f^{-1}(\gamma(0))\cap\bar{D}_s$ in the terminology of [19]. Let j be the smallest integer such that $j\geqslant\underline{n}(s,F)(1-1/c)$. Since $n(\theta s,\gamma(1))<\underline{n}(s,F)/c$, at least j of the lifts $\alpha_1,\dots,\alpha_m$ must end in $\partial D_{\theta s}$. Let Γ^* be the family of all such lifts when γ runs through Γ. By 2.10 with E_0 equal to the branch set B_f of f, by 2.16, and by the fact that φ_z is 2-bilipschitzian we obtain

$$M(\varphi_z\Gamma) \leqslant \frac{2^{2n-2}K_I M(\Gamma^*)}{\underline{n}(s, F)(1-1/c)} \leqslant \frac{2^{2n-2}K_I \varkappa \omega_{n-1}}{\underline{n}(s, F)(1-1/c)(\log\theta)^{\lambda}}. \tag{3.7}$$

Set $\delta_1 = M(\Delta(\varphi_z F, \partial B(r); \bar{B}(r)))$, $\delta_2 = M(\Delta(\varphi_z A', \partial B(r/2); \bar{B}(r)\setminus B(r/2)))$. In the following we shall denote by $b_1, b_2, \ldots$ positive constants which depend only on n and by $d_1, d_2, \ldots$ positive constants which depend only on $n, K_I, \theta_0, \lambda$, and $\varkappa$. By 3.2 $M(\varphi_z\Gamma) \geqslant 3^{-n} \min(c_n \log 2, \delta_1, \delta_2)$. According to the choice of τ we have $c_n \log 2 \geqslant \omega_{n-1} (\log 1/\tau)^{1-n} \geqslant \delta_1$.

Assume $M(\varphi_z\Gamma) < 3^{-n}\delta_1$. Then $M(\varphi_z\Gamma) \geqslant 3^{-n}\delta_2$. Let $A'' = \bar{B}(re_n, \sigma) \cap S(r)$ be a spherical symmetrization of $\varphi_z A'$, $\sigma \in]0, 2r]$ being then defined by the condition $\mathcal{H}^{n-1}(A'') = \mathcal{H}^{n-1}(\varphi_z A')$. By [21, 7.5] $\operatorname{cap}(\bar{\mathbf{R}}^n \setminus \bar{B}(r/2), \varphi_z A') \geqslant \operatorname{cap}(\bar{\mathbf{R}}^n \setminus \bar{B}(r/2), A'')$. Assume first $\sigma < r/4$. By using an auxiliary quasiconformal mapping of $\bar{\mathbf{R}}^n$ onto itself we first obtain $\operatorname{cap}(\bar{\mathbf{R}}^n \setminus \bar{B}(r/2), A'') \geqslant b_1 \operatorname{cap} R_G(4r/\sigma)$ where $R_G(v)$, $v > 1$, denotes the Grötzsch ring. In condenser notation $R_G(v) = (\mathbf{R}^n \setminus \{x \in \mathbf{R}^n \mid x_1 \geqslant v,\ x_2 = \ldots = x_n = 0\}, \bar{B}(1))$. By the n-dimensional analog of [3, Lemma 8] we have $\operatorname{cap} R_G(v) \geqslant b_2 (\log v)^{1-n}$. It follows that $\operatorname{cap}(\bar{\mathbf{R}}^n \setminus \bar{B}(r/2), A'') \geqslant b_3 (\log(4r/\sigma))^{1-n}$. This is true also if $\sigma \geqslant r/4$. By [4, Lemma 1] $\delta_2 \geqslant 2^{-1} \operatorname{cap}(\bar{\mathbf{R}}^n \setminus \bar{B}(r/2), \varphi_z A')$. By putting the estimates together we get $M(\varphi_z\Gamma) \geqslant 3^{-n} 2^{-1} b_3 (\log(4r/\sigma))^{1-n}$ which combined with (3.7) gives

$$\exp\left((d_1 \underline{n}(s, F)(1-1/c)(\log\theta)^{\lambda})^{1/(n-1)}\right) \leqslant 4r/\sigma.$$

Since $\mathcal{H}^{n-1}(A) \leqslant b_4 \sigma^{n-1}$ and $r^{n-1} \leqslant b_4 \mathcal{H}^{n-1}(Y)$ for some b_4, we obtain

$$\mathcal{H}^{n-1}(A) \leqslant b_5 \mathcal{H}^{n-1}(Y) \left(\exp\left((d_1 \underline{n}(s, F)(1-1/c)(\log\theta)^{\lambda})^{1/(n-1)}\right)\right)^{1-n}. \tag{3.8}$$

By $\exp u > u$ we obtain

$$\mathcal{H}^{n-1}(A) \leqslant \frac{d_2 \mathcal{H}^{n-1}(Y)}{\underline{n}(s, F)(1-1/c)(\log\theta)^{\lambda}}. \tag{3.9}$$

If $M(\varphi_z\Gamma) \geqslant 3^{-n}\delta_1$, then

$$\delta_1 \leqslant 3^n M(\varphi_z\Gamma) \leqslant \frac{d_3}{\underline{n}(s, F)(1-1/c)(\log\theta)^{\lambda}}. \tag{3.10}$$

The substitution of (3.9) or (3.10) into (3.6) yields (3.4).

To prove (3.5) set $A_k = \{y \in Y \mid n(s, y) = k\}$, $B_k = \{y \in Y \mid n(s, y) \geqslant k\}$ for $k = 1, 2, \ldots$. We may assume $c < 2$ and $\nu(s, Y) > \max(c\bar{n}(\theta s, F), 4)$. Let $c' = \sqrt{c}$ and $k \geqslant \nu(s, Y)/c'$. We shall use a similar argument as for (3.8) and (3.10). Assume $\mathcal{H}^{n-1}(B_k) > 0$ and let $B_k' \subset B_k$ be compact such that $\mathcal{H}^{n-1}(B_k') > \mathcal{H}^{n-1}(B_k)/2$. Let Γ be the family of paths $\gamma\colon [0, 1] \to \bar{B}(z, r)$ with $\gamma(0) \in B_k'$, $\gamma(1) \in F$. Let $\gamma \in \Gamma$, $m = n(s, \gamma(0))$, and let $\alpha_1, \ldots, \alpha_m$ be a maximal sequence of $f\,|\,D_{\theta s}$-liftings of γ starting at the points of $f^{-1}(\gamma(0)) \cap \bar{D}_s$ given by [19, Theorem 1]. Let j be the smallest integer such that $j \geqslant k - \nu(s, Y)/c$. Since $m \geqslant k$ and $n(\theta s, \gamma(1)) \leqslant \bar{n}(\theta s, F) < \nu(s, Y)/c$, at least j of the lifts $\alpha_1, \ldots, \alpha_m$ must end in $\partial D_{\theta s}$. As (3.7) we now obtain

$$M(\varphi_z \Gamma) \leqslant \frac{2^{2n-2} K_I \varkappa \omega_{n-1}}{(k - \nu(s, Y)/c)(\log \theta)^{\lambda}}.$$

Let δ_1 be as before, i.e. $\delta_1 = M(\Delta(\varphi_z F,\ \partial B(r);\ \bar{B}(r)))$, and set $\delta_2 = M(\Delta(\varphi_z B_k',\ \partial B(r/2);\ \bar{B}(r) \setminus B(r/2)))$.

If $M(\varphi_z \Gamma) < 3^{-n}\delta_1$, we use the same argument as for (3.8) to get

$$\mathcal{H}^{n-1}(B_k) \leqslant b_5 \mathcal{H}^{n-1}(Y)(\exp((d_1(k - \nu(s, Y)/c)(\log \theta)^{\lambda})^{1/(n-1)}))^{1-n}. \tag{3.11}$$

If $M(\varphi_z \Gamma) \geqslant 3^{-n}\delta_1$, then

$$\delta_1 \leqslant \frac{d_3}{(k - \nu(s, Y)/c)(\log \theta)^{\lambda}} \leqslant \frac{d_3 c}{\nu(s, Y)(c' - 1)(\log \theta)^{\lambda}}. \tag{3.12}$$

From (3.12) we get

$$\nu(s, Y) \leqslant \frac{d_4 c (\log \theta_0)^{(q-1)\lambda}}{\delta (c' - 1)(\log \theta)^{q\lambda}}$$

which is of the required form (3.5). Thus it suffices to consider the case where (3.11) is true for all $k \geqslant \nu(s, Y)/c'$. We use $\exp u > u^q/6$ and obtain from (3.11)

$$k \mathcal{H}^{n-1}(B_k) \leqslant \frac{d_5 c^q \mathcal{H}^{n-1}(Y)}{k^{q-1}(c' - 1)^q (\log \theta)^{q\lambda}},$$

from which

$$\sum_{k \geqslant \nu(s, Y)/c'} k \mathcal{H}^{n-1}(A_k) \leqslant \sum_{k \geqslant \nu(s, Y)/c'} k \mathcal{H}^{n-1}(B_k) \leqslant \frac{d_5 c^q \mathcal{H}^{n-1}(Y)}{(q-2)(\nu(s, Y)/c' - 1)^{q-2}(c' - 1)^q (\log \theta)^{q\lambda}}.$$

Hence

$$\begin{aligned} \nu(s, Y) &\leqslant \mathcal{H}^{n-1}(Y)^{-1} \sum_{k < \nu(s, Y)/c'} k \mathcal{H}^{n-1}(A_k) + d_5 (q-2)^{-1} c^q (c' - 1)^{-q} (\log \theta)^{-q\lambda} \\ &\leqslant \nu(s, Y)/c' + d_5 (q-2)^{-1} c^q (c' - 1)^{-q} (\log \theta)^{-q\lambda}, \end{aligned}$$

and

$$\nu(s, Y) \leqslant d_6 (q-2)^{-1} c^{q+1} (c' - 1)^{-q-1} (\log \theta)^{-q\lambda}$$

which is also of the required form (3.5). The lemma is proved.

4. Averages with respect to h-calibrated measures

4.1. Let μ be a measure in N such that Borel sets are μ-measurable and $0<\mu(N)<\infty$. Recall from Introduction that μ is h-calibrated if $\mu(B(x, r)) \leqslant h(r)$ for all $x \in N$, $r>0$, where h is a calibration function. We shall prove our results on equidistribution of the counting function for averages with respect to an h-calibrated μ with h satisfying (1.4) for some $p>2$. In this section we shall establish a basic comparison result (Theorem 4.8) with error terms similar to those in 3.3. In Lemmas 4.2–4.4 we fix a calibration function h satisfying (1.4), an h-calibrated measure μ, and a number $p>2$ such that (1.4) is true.

The average of $n(s, y)$ with respect to μ over a μ-measurable set $E \subset N$ with $\mu(E)>0$ is denoted by $\nu_\mu(s, E)$, i.e.

$$\nu_\mu(s, E) = \mu(E)^{-1} \int_E n(s, y)\, d\mu(y).$$

We abbreviate $\nu_\mu(s, N) = \nu_\mu(s)$ and denote $A(s) = \nu_{\mathcal{L}^n}(s)$.

For $A \subset \mathbf{R}^n$ let $\gamma_h(A)$ be the infimum of the sums $\sum h(r_i)$ when A is covered by at most a countable number of balls $B(x_i, r_i)$. We need some connections between capacity and the outer measure γ_h. Recall the notation r_0 and τ introduced in 3.1.

4.2. Lemma. *There exists $L>0$ such that*

$$\gamma_h(A) \leqslant L\, (\operatorname{cap}\, (B(r), A))^p$$

whenever A is a compact set in $B(r)$ and $0<r<r_0$.

Proof. The proof is similar to that of [17, Theorem 8], cf. also the proof of [8, Theorem 3.1]. Define $h_1 = h^{1/p}$. Applying [17, Lemma 6] with $\lambda = 1$, $p = n$ we find positive constants K_1, K_2, and C such that if u is a nonnegative function in $L^n(\mathbf{R}^n)$ with $u \,|\, \complement B(r_0) = 0$ and

$$w(x) = \int_{B(r_0)} \frac{u(y)}{|x-y|^{n-1}}\, d\mathcal{L}^n(y),$$

then for all $\delta>0$

$$\gamma_{h_1} \{x \in \mathbf{R}^n \,|\, w(x) > K_1/\delta + K_2 \|u\|_n\} \leqslant C(\delta \|u\|_n)^n,$$

where $\|u\|_n$ is the L^n-norm of u. Here K_2 and C depend only on n and K_1 is of the form

$$K_1 = b_1 \int_0^{r_0} \frac{h_1(\varrho)^{1/n}}{\varrho}\, d\varrho$$

where b_1 depends only on n.

Suppose first that $\operatorname{cap}(B(r), A) < (2K_2)^{-n}\omega_{n-1}^n$. Let $\varepsilon > 0$ be so small that $\operatorname{cap}(B(r), A) + \varepsilon < (2K_2)^{-n}\omega_{n-1}^n$. Then there is a continuously differentiable function $v: \mathbf{R}^n \to [0, \infty[$ such that $v|\complement B(r_0) = 0$, $v(x) > 1$ for $x \in A$, and

$$\int_{\mathbf{R}^n} |\nabla v|^n d\mathcal{L}^n < \operatorname{cap}(B(r), A) + \varepsilon < (2K_2)^{-n}\omega_{n-1}^n.$$

We take $u = |\nabla v|/\omega_{n-1}$ and define w as above. Then $\|u\|_n < (2K_2)^{-1}$ and by [17, Lemma 3]

$$v(x) = \frac{1}{\omega_{n-1}} \int_{\mathbf{R}^n} \frac{\nabla v(y) \cdot (x-y)}{|x-y|^n} d\mathcal{L}^n(y) \leqslant w(x).$$

We choose $\delta = K_1(1 - K_2\|u\|_n)^{-1}$. Then $w(x) > 1 = K_1/\delta + K_2\|u\|_n$ for $x \in A$ and we obtain

$$\begin{aligned}\gamma_{h_1}(A) &\leqslant C(\delta\|u\|_n)^n = CK_1^n(1 - K_2\|u\|_n)^{-n}\|u\|_n^n \\ &\leqslant CK_1^n 2^n \omega_{n-1}^{-n} \int_{\mathbf{R}^n} |\nabla v|^n d\mathcal{L}^n < CK_1^n 2^n \omega_{n-1}^{-n} (\operatorname{cap}(B(r), A) + \varepsilon).\end{aligned}$$

Letting $\varepsilon \to 0$ we get

$$\gamma_{h_1}(A) \leqslant CK_1^n 2^n \omega_{n-1}^{-n} \operatorname{cap}(B(r), A).$$

If $\operatorname{cap}(B(r), A) \geqslant (2K_2)^{-n}\omega_{n-1}^n$, then

$$\gamma_{h_1}(A) \leqslant h_1(r_0) \leqslant h_1(r_0)(2K_2)^n \omega_{n-1}^{-n} \operatorname{cap}(B(r), A).$$

Hence there is a constant L_1 such that in both cases

$$\gamma_{h_1}(A) \leqslant L_1 \operatorname{cap}(B(r), A).$$

The result follows now from the inequality $\gamma_h(A) \leqslant \gamma_{h_1}(A)^p$ which is true because $\sum h(r_i) \leqslant (\sum h(r_i)^{1/p})^p$.

4.3 LEMMA. *There exists $Q > 0$ such that if $z \in N$, $0 < r < r_0$, and E is a Borel set in $\bar{B}(z, \tau r)$, then*

$$\mu(E) \leqslant Q(M(\Delta(E, \partial B(z, r); \bar{B}(z, r))))^p.$$

Proof. Let $F \subset E$ be compact such that $2\mu(F) \geqslant \mu(E)$. Let $\varepsilon > 0$ and let the balls $B(u_i, r_i)$, $i = 1, 2, \ldots$ cover $\varphi_z F$ such that $\gamma_h(\varphi_z F) + \varepsilon \geqslant \sum h(r_i)$. Since $\tau < \frac{1}{8}$, we may assume $B(u_i, r_i) \subset B(r_0)$ for all i. The balls $B(\varphi_z^{-1}(u_i), 2r_i)$ cover F. There exists an integer q_n depending only on n such that each $B(\varphi_z^{-1}(u_i), 2r_i)$ can be covered by at most q_n balls with radius r_i. Then, since μ is h-calibrated,

$$\mu(F) \leqslant q_n \sum h(r_i) \leqslant q_n \gamma_h(\varphi_z F) + q_n \varepsilon.$$

The result follows then by Lemma 4.2 and the 2^{2n-2}-quasiconformality of φ_z.

4.4. LEMMA. *Let* $2<q\leqslant 3$. *For each* $c>1$ *there exists* $d>0$ *such that the following holds. Let* $z\in N$, $0<r<r_0$, E *a Borel set in* $B(z,\tau r)$ *with* $\mu(E)>0$. *Then*

$$c\nu(\theta s, S(z,r))\geqslant \nu_\mu(s,E)-\frac{d}{\mu(E)(\log\theta)^{p\lambda}} \tag{4.5}$$

and

$$\nu(s, S(z,r))\leqslant c\nu_\mu(\theta s,E)+\frac{d}{\mu(E)(\log\theta)^{q\lambda}} \tag{4.6}$$

whenever $a_0\leqslant s<\theta s<b$, $\theta\leqslant\theta_0$.

Proof. To prove (4.5) fix s,θ, and set $Y=S(z,r)$, $c'=\sqrt{c}$,

$$E_k=\{w\in E\mid n(s,w)=k\},\quad k=1,2,\ldots,$$
$$A=\{y\in Y\mid n(\theta s,y)\leqslant c'\nu(\theta s, Y)\}.$$

For $k>c\nu(\theta s, Y)$ let Γ_k be the family of paths $\gamma:[0,1]\to\bar{B}(z,r)$ such that $\gamma(0)\in E_k$, $\gamma(1)\in A$. Then as in the proof of Lemma 3.3 we obtain by 2.10 and 2.16

$$M(\varphi_z\Gamma_k)\leqslant\frac{2^{2n-2}K_I\varkappa\omega_{n-1}}{(k-c'\nu(\theta s, Y))(\log\theta)^{\lambda}}. \tag{4.7}$$

Since $\nu(\theta s, Y)\mathcal{H}^{n-1}(Y)\geqslant c'\nu(\theta s, Y)\mathcal{H}^{n-1}(Y\setminus A)$, we have for $\nu(\theta s, Y)>0$ $\mathcal{H}^{n-1}(A)\geqslant \mathcal{H}^{n-1}(Y)(1-1/c')$. This holds trivially if $\nu(\theta s, Y)=0$. Then $M(\Delta(\varphi_z A, \partial B(r/2);\bar{B}(r)))\geqslant\alpha>0$ where α depends only on n and c'. From Lemma 3.2 we obtain $M(\varphi_z\Gamma_k)\geqslant 3^{-n}\min(M(\Delta(\varphi_z E_k,\partial B(r);\bar{B}(r))),\alpha, c_n\log 2)$. By the choice of τ $M(\Delta(\varphi_z E_k,\partial B(r);\bar{B}(r)))<c_n\log 2$. Hence $M(\varphi_z\Gamma_k)\geqslant 3^{-n}\min(1,\alpha/(c_n\log 2))M(\Delta(\varphi_z E_k,\partial B(r);\bar{B}(r)))$. With (4.7) this yields

$$M(\Delta(E_k,\partial B(z,r);\bar{B}(z,r)))\leqslant\frac{d_1}{(k-c'\nu(\theta s, Y))(\log\theta)^{\lambda}}.$$

Here we denote by $d_1, d_2, \ldots$ positive constants which are independent of s,θ,z,r,E, and k. By Lemma 4.3 we hence obtain for $k>c\nu(\theta s, Y)$

$$k\mu(E_k)\leqslant\frac{d_2}{k^{p-1}(1-1/c')^p(\log\theta)^{p\lambda}}.$$

The inequality (4.5) follows then from the estimate

$$\begin{aligned}\int_E n(s,y)\,d\mu(y)&=\sum_{k\leqslant c\nu(\theta s, Y)}k\mu(E_k)+\sum_{k>c\nu(\theta s, Y)}k\mu(E_k)\\&\leqslant c\nu(\theta s, Y)\mu(E)+d_3(\log\theta)^{-p\lambda}.\end{aligned}$$

To prove (4.6) we make use of (3.5) in Lemma 3.3. If

$$E' = \{w \in E \mid n(\theta s, w) \leqslant c' \nu_\mu(\theta s, E)\},$$

then $\mu(E)\nu_\mu(\theta s, E) \geqslant c'\nu_\mu(\theta s, E)\mu(E \setminus E')$, hence $\mu(E') \geqslant (1-1/c')\mu(E)$. Lemma 4.3 gives a constant Q_1 such that $\mu(E') \leqslant Q_1 M(\Delta(E', \partial B(z, r); \bar{B}(z, r)))$, and hence by Lemma 3.3

$$\nu(s, Y) \leqslant c' \bar{n}(\theta s, E') + \frac{d_4}{\mu(E)(\log \theta)^{q\lambda}}$$

$$\leqslant c\nu_\mu(\theta s, E) + \frac{d_4}{\mu(E)(\log \theta)^{q\lambda}}.$$

The lemma is proved.

4.8. THEOREM. *Let μ be an h-calibrated measure in N with h satisfying* (1.4) *for some $p > 2$. Then for each $c > 1$ there exists $d > 0$ such that*

$$cA(\theta s) \geqslant \nu_\mu(s) - d(\log \theta)^{-p\lambda} \tag{4.9}$$

and

$$A(s) \leqslant c\nu_\mu(\theta s) + d(\log \theta)^{-p\lambda} \tag{4.10}$$

whenever $a_0 \leqslant s < \theta s < b$, $\theta \leqslant \theta_0$.

Proof. We observe that the Lebesgue measure of N is h_0-calibrated with $h_0(r) = Cr^n$, where $C > 0$ is a constant, and the function h_0 satisfies (1.4) for any $p > 0$. We shall first prove (4.9). Let $p > 2$ be as in the theorem and set $q = \min(p, 3)$. Fix $c > 1$ and $r \in]0, r_0[$. We cover N by balls $V_i = B(z_i, \tau r)$, $i = 1, \ldots, l$. Let $E_i \subset V_i$ be disjoint Borel sets such that $\mathcal{L}^n(E_i) > 0$ and

$$N = \bigcup_{i=1}^{l} E_i.$$

Let α be the minimum of the numbers $\mathcal{L}^n(E_i)$, $\mathcal{L}^n(V_i \cap V_j)$ for $V_i \cap V_j \neq \emptyset$, $i, j = 1, \ldots, l$. Fix i, s and θ. We shall first estimate $\nu_\mu(s, E_i)$ from above provided $\mu(E_i) > 0$. Set $c' = c^{1/l}$, $\theta' = \theta^{1/l}$. Let $1 \leqslant j \leqslant l$. We can choose a chain $X_1, \ldots, X_m$, $m \leqslant l$, of the balls $V_1, \ldots, V_l$ such that $X_1 = V_i$, $Z_k = X_k \cap X_{k+1} \neq \emptyset$, $k = 1, \ldots, m-1$, and $X_m = V_j$. We apply (4.5) to μ and (4.6) to $\mathcal{L}^n$ and obtain

$$\nu_\mu(s, E_i) \leqslant c' \nu_{\mathcal{L}^n}(\theta' s, Z_1) + c_1\left(\frac{1}{\mu(E_i)(\log \theta)^{p\lambda}} + \frac{1}{\mathcal{L}^n(Z_1)(\log \theta)^{q\lambda}}\right) \tag{4.11}$$

where $c_1 > 0$ is independent of s, θ, and E_i. Similarly

$$\nu_{\mathcal{L}^n}(\theta'^k s, Z_k) \leqslant c' \nu_{\mathcal{L}^n}(\theta'^{k+1} s, Z_{k+1}) + 2c_1 \alpha^{-1} (\log \theta)^{-q\lambda} \tag{4.12}$$

for $k = 1, ..., m-2$, and

$$\nu_{\mathcal{L}^n}(\theta'^{m-1} s, Z_{m-1}) \leqslant c' \nu_{\mathcal{L}^n}(\theta'^m s, E_j) + 2c_1 \alpha^{-1} (\log \theta)^{-q\lambda}. \tag{4.13}$$

The inequalities (4.11)–(4.13) give

$$\nu_\mu(s, E_i) \leqslant c \nu_{\mathcal{L}^n}(\theta s, E_j) + d_1 (\log \theta)^{-q\lambda} + c_1 \mu(E_i)^{-1} (\log \theta)^{-p\lambda}$$

where $d_1 = 2lc_1 c/\alpha$. Multiplying by $\mu(E_i)$, summing over i, and dividing by $\mu(N)$, we obtain

$$\begin{aligned} \nu_\mu(s) &\leqslant c \nu_{\mathcal{L}^n}(\theta s, E_j) + d_1 (\log \theta)^{-q\lambda} + c_1 l \mu(N)^{-1} (\log \theta)^{-p\lambda} \\ &\leqslant c \nu_{\mathcal{L}^n}(\theta s, E_j) + (d_2 + c_1 l \mu(N)^{-1}) (\log \theta)^{-p\lambda}, \end{aligned} \tag{4.14}$$

where $d_2 = d_1 \max(1, (\log \theta_0)^{p\lambda})$. Multiplying (4.14) by $\mathcal{L}^n(E_j)$, summing over j, and dividing by $\mathcal{L}^n(N)$, we obtain

$$\nu_\mu(s) \leqslant c A(\theta s) + (d_2 + c_1 l \mu(N)^{-1}) (\log \theta)^{-p\lambda}.$$

The inequality (4.10) is proved similarly as follows. In place of (4.11) we obtain by applying (4.5) to $\mathcal{L}^n$ and (4.6) to μ the inequality

$$\nu_{\mathcal{L}^n}(\theta'^{m-1} s, Z_1) \leqslant c' \nu_\mu(\theta'^m s, E_i) + c_1 \left(\frac{1}{\mathcal{L}^n(Z_1)} + \frac{1}{\mu(E_i)} \right) (\log \theta)^{-q\lambda}, \tag{4.15}$$

The inequalities (4.12) and (4.13) are replaced by

$$\nu_{\mathcal{L}^n}(\theta'^{m-k-1} s, Z_{k+1}) \leqslant c' \nu_{\mathcal{L}^n}(\theta'^{m-k} s, Z_k) + 2c_1 \alpha^{-1} (\log \theta)^{-q\lambda}, \quad k = 1, ..., m-2, \tag{4.16}$$

$$\nu_{\mathcal{L}^n}(s, E_j) \leqslant c' \nu_{\mathcal{L}^n}(\theta' s, Z_{m-1}) + 2c_1 \alpha^{-1} (\log \theta)^{-q\lambda}, \tag{4.17}$$

respectively. The inequalities (4.15)–(4.17) give

$$\nu_{\mathcal{L}^n}(s, E_j) \leqslant c \nu_\mu(\theta s, E_i) + (d_1 + c_1 \mu(E_i)^{-1}) (\log \theta)^{-q\lambda}.$$

As in the end of the proof of (4.9) we obtain from this the inequality (4.10) in the form

$$A(s) \leqslant c \nu_\mu(\theta s) + (d_1 + c_1 l \mu(N)^{-1}) \max(1, (\log \theta_0)^{p\lambda}) (\log \theta)^{-p\lambda}.$$

The following theorem shows that a weaker assumption is enough to ensure a onesided estimate.

4.18. THEOREM. *Suppose that μ is a measure in N, $0<\mu(N)<\infty$, all Borel sets of N are μ-measurable, and there is a calibration function h satisfying* (1.4) *for some $p>2$ such that the condition*

$$\limsup_{r\to 0} \mu(B(x, r))/h(r) \leqslant 1$$

holds for μ almost every $x\in N$. Then for each $c>1$ there is $d>0$ such that

$$A(s) \leqslant c\nu_\mu(\theta s) + d\,(\log\theta)^{-p\lambda}$$

whenever $a_0 \leqslant s<\theta s<b$, $\theta\leqslant\theta_0$.

Proof. Set $c'=\sqrt{c}$. The function $x\mapsto\limsup_{r\to 0}\mu(B(x, r))/h(r)$ is a Borel function. In fact, for each $r>0$ the function $x\mapsto\mu(B(x, r))$ is lower semicontinuous and since $\mu(B(x, r))$ is increasing in r and h is continuous, the upper limit does not change if r is restricted to positive rational numbers. Hence there are a Borel set $E\subset N$ and $r_1>0$ such that $c'\mu(E)\geqslant\mu(N)$ and

$$\mu(B(x, r)) \leqslant 2h(r) \quad \text{for } x\in E,\ 0<r<r_1;$$

thus

$$\mu(E\cap B(x, r)) \leqslant 2h(2r) \quad \text{for } x\in N,\ 0<r<r_1/2.$$

It follows that the restriction measure $A\mapsto\mu(E\cap A)$ is h_1-calibrated with $h_1(r)=C\,h(2r)$ for some $C>0$. Clearly h_1 satisfies (1.4) for p. By Theorem 4.8 there is $d>0$ such that

$$c'\nu_\mu(\theta s, E) \geqslant A(s) - d\,(\log\theta)^{-p\lambda}$$

whenever $a_0\leqslant s<\theta s<b$, $\theta\leqslant\theta_0$. Hence

$$c\nu_\mu(\theta s) \geqslant c\mu(E)\mu(N)^{-1}\nu_\mu(\theta s, E) \geqslant c'\nu_\mu(\theta s, E) \geqslant A(s) - d\,(\log\theta)^{-p\lambda}.$$

5. Main results

In Section 4 we presented in Theorems 4.8 and 4.18 basic comparison estimates with a ratio $\theta>1$ in the exhaustion parameter and with error terms. We shall now turn to establish results without a difference in the exhaustion parameter. For this purpose we need two lemmas on real functions which are refinements of Lemma 4.14 in [20].

5.1. LEMMA. *Suppose that $1<c'<c$, c_1, $\sigma>0$, that ψ is a non-negative, continuous, and increasing function of $[a, b[$, and either $b=\infty$ and $\lim_{s\to\infty}\psi(s)=\infty$, or $b<\infty$ and* $\limsup_{s\to b}(b-s)\psi(s)^{1/\sigma}=\infty$. *Then there exists a set $A\subset[a, b[$ such that*

$$\int_A \frac{ds}{s} < \infty \quad \textit{if} \quad b=\infty, \tag{5.2}$$

(5.3) $$\liminf_{s\to b} \frac{\mathfrak{L}^1(A \cap [s, b[)}{b-s} = 0 \quad \textit{if} \quad b < \infty,$$

and the following holds:

(i) *If* $\varphi\colon [a, b[\to \mathbf{R}_+ = \{r \in \mathbf{R} \mid r \geqslant 0\}$ *is such that*

$$\psi(s) \leqslant c'\varphi(\theta s) + c_1(\log\theta)^{-\sigma}$$

for all s *and* $\theta \leqslant \theta_0$, $a_0 \leqslant s < \theta s < b$, *then*

(5.4) $$\psi(s) \leqslant c\varphi(s)$$

for all $s \in [a, b[\setminus A$.

(ii) *If* $\varphi\colon [a, b[\to \mathbf{R}_+$ *is such that*

$$c'^{-1}\varphi(s/\theta) - c_1(\log\theta)^{-\sigma} \leqslant \psi(s)$$

for all s *and* $\theta \leqslant \theta_0$, $a_0 \leqslant s/\theta < s < b$, *then*

(5.5) $$\varphi(s) \leqslant c\psi(s)$$

for all $s \in [a, b[\setminus A$.

Proof. We choose constants $M > 1$ and $c_2 > 0$ such that $c(1 - c_1 c_2^\sigma) \geqslant c'M$, $c'M(1 + c_1 c_2^\sigma) \leqslant c$. We may assume $\psi(a_0) \geqslant 1$. Set

$$\beta(s) = c_2\psi(s)^{1/\sigma}/ps$$

where $p > 1$ is chosen so that for $s \geqslant a_0$

$$(\log(1 + 1/s\beta(s)))^{-\sigma} \leqslant c_2^\sigma\psi(s).$$

Since $\psi(s) \to \infty$ as $s \to b$, we may assume that $1 + 1/s\beta(s) < (1 - 1/s\beta(s))^{-1} \leqslant \theta_0$ for $s \geqslant a_0$. Let F be the set of all $s \in [a_0, b[$ such that $s + 1/\beta(s) \geqslant b$ or the inequality

$$\psi(s + 1/\beta(s)) \leqslant M\psi(s)$$

does not hold. We denote $\theta_s = 1 + 1/s\beta(s)$.

We shall first prove (i). Let φ satisfy the hypothesis in (i) and let $s \in [a_0, b[\setminus F$. Then $c'\varphi(\theta_s s) \geqslant \psi(s) - c_1(\log\theta_s)^{-\sigma} \geqslant \psi(s)(1 - c_1c_2^\sigma) \geqslant \psi(\theta_s s)(1 - c_1 c_2^\sigma)/M \geqslant c'\psi(\theta_s s)/c$, hence

(5.6) $$\psi(\theta_s s) \leqslant c\varphi(\theta_s s).$$

We consider first the case $b=\infty$. This part is similar to the proof of [20, Lemma 4.14]. We define a sequence $a_0=r_0'\leqslant r_1<r_1'\leqslant r_2<r_2'\leqslant\ldots$ as follows. Let $r_k=\inf F\cap\,]r_{k-1}',\infty[$, and if $r_k<\infty$, set $r_k'=r_k+2/\beta(r_k)$. Consider then the union

$$E=\bigcup_{k\geqslant 1}[r_k,\varrho_k]$$

of intervals where $\varrho_k=r_k'+pr_k'/c_2\psi(r_k)^{1/\sigma}$. If $u\in\,]\theta_{a_0}a_0,\infty[\,\setminus E$, then since ψ is increasing, there exists $s\in\,]a_0,\infty[\,\setminus F$ such that $u=\theta_s s$ and (5.6) holds. It hence suffices to estimate the logarithmic measure of E. We obtain

$$\begin{aligned}\int_E\frac{dr}{r}&\leqslant\sum_{k\geqslant 1}\int_{r_k}^{\varrho_k}\frac{dr}{r}\leqslant\sum_{k\geqslant 1}(\varrho_k-r_k)/r_k\\&=\sum_{k\geqslant 1}\left((r_k'-r_k)+\frac{pr_k'}{c_2\psi(r_k)^{1/\sigma}}\right)\Big/r_k\\&\leqslant\sum_{k\geqslant 1}\left(\frac{2p}{c_2\psi(r_k)^{1/\sigma}}+\frac{p(1+2p/c_2)}{c_2\psi(r_k)^{1/\sigma}}\right).\end{aligned}$$

The last sum is finite because of

$$\psi(r_{k+1})\geqslant\psi(r_k')\geqslant M\psi(r_k).$$

Assume then $b<\infty$. Let $0<\varepsilon<\frac{1}{2}$ and $a_0\leqslant t_0<b$. By assumption there exists $t\in[t_0,b[$ such that

$$(b-t)\,\psi(t)^{1/\sigma}>\frac{4pb}{c_2\,\varepsilon(1-1/M^{1/\sigma})}. \tag{5.7}$$

Set $t_1=b-\varepsilon(b-t)$. It suffices to prove that (5.4) is true in $[t,t_1]$ outside a set independent of φ and of length $\leqslant 2\varepsilon(b-t)$. We consider two cases:

Case 1. $F\cap\,]t,t_1]=\emptyset$. If $s\in\,]t,t_1]$, we have $\theta_s s-s\leqslant bp/c_2\psi(s)^{1/\sigma}<\varepsilon(b-t)$. Hence $\{\theta_s s\,|\,s\in\,]t,t_1[\}$ covers the interval $]t+\varepsilon(b-t),t_1[$ and (5.4) holds by (5.6) in $[t,t_1]$ outside a set of length $\varepsilon(b-t)$.

Case 2. $F\cap\,]t,t_1]\neq\emptyset$. Now we define a sequence $t=r_0'\leqslant r_1<r_1'\leqslant r_2\ldots\leqslant r_q<r_q'$ of points in $[t,b[$ inductively by $r_k=\inf F\cap\,]r_{k-1}',b[$, $r_k'=r_k+2/\beta(r_k)$ such that q is the last index k for which $F\cap\,]r_{k-1}',b[\,\neq\emptyset$ and $r_k\leqslant t_1$. If now $u\in\,]t+\varepsilon(b-t),t_1]\setminus E$ where E is defined as before, then there exists $s\in\,]t,t_1]\setminus F$ such that $u=\theta_s s$ and (5.6) holds. For the length of E we get an estimate as follows:

$$\mathcal{L}^1(E) \leqslant \sum_{k=1}^{q} (\varrho_k - r_k) < \sum_{k=1}^{q} \frac{4pb}{c_2 \psi(r_k)^{1/\sigma}} \leqslant \frac{4pb}{c_2 \psi(t)^{1/\sigma}(1-1/M^{1/\sigma})} < \varepsilon(b-t).$$

This yields the desired result.

Next we consider (ii) in the case $b=\infty$. Let $t \geqslant a_0$ and let $s' \in [t, \infty[\setminus F$. Since ψ is assumed to be continuous, there exists $s \in [t, \infty[$ such that $s' = s/\zeta_s = s - 1/\beta(s)$. From the choice of p it follows that also $(\log \zeta_s)^{-\sigma} \leqslant c_2^\sigma \psi(s)$. Let φ satisfy the assumption in (ii). If $s \leqslant s' + 1/\beta(s')$, we get

$$\begin{aligned} c'^{-1}\varphi(s') &\leqslant \psi(s) + c_1 (\log \zeta_s)^{-\sigma} \leqslant \psi(s)(1 + c_1 c_2^\sigma) \\ &\leqslant \psi(s' + 1/\beta(s'))(1 + c_1 c_2^\sigma) \\ &\leqslant M(1 + c_1 c_2^\sigma)\psi(s') \leqslant cc'^{-1}\psi(s') \end{aligned}$$

which is the desired inequality for s'. On the other hand, if $s > s' + 1/\beta(s')$, we get $\beta(s') > \beta(s)$, hence

$$\left(\frac{\psi(s')}{\psi(s)}\right)^{1/\sigma} > 1 - \frac{p}{c_2 \psi(s)^{1/\sigma}}.$$

By choosing t larger if necessary, we obtain $M\psi(s') \geqslant \psi(s)$ which yields the desired inequality for s'. It thus suffices to estimate the logarithmic measure of $F \cap]t, \infty[$. This is done by a similar but simpler argument as used in the proof of (i), in fact $F \cap]t, \infty[\subset E$.

Finally, to prove (ii) for $b < \infty$ we let $0 < \varepsilon < \frac{1}{4}$, $a_0 \leqslant t_0 < b$, and choose $t \in [t_0, a[$ so that (5.7) holds. We can imitate the case $b = \infty$ if we require $s' \in]t, t_1[\setminus F$ and observe $F \cap]t, t_1] \subset E$ where t_1 and E are defined as in the proof of (*i*) for the case $b < \infty$.

5.8. Lemma. *Suppose that ψ is a function on $[a, b[$ satisfying the hypothesis of Lemma* 5.1. *Then there exists a set $A \subset [a, b[$ satisfying* (5.2) and (5.3) *and such that the following holds: If φ: $[a, b[\to \mathbf{R}_+$ is such that for every $c > 1$ there exists $c_1 > 0$ with*

$$\psi(s) \leqslant c\varphi(\theta s) + c_1 (\log \theta)^{-\sigma}$$

for all s and $\theta \leqslant \theta_0$, $a_0 \leqslant s < \theta s < b$, then

$$\liminf_{\substack{s \to b \\ s \notin A}} \varphi(s)/\psi(s) \geqslant 1.$$

If φ: $[a, b[\to \mathbf{R}_+$ is such that for every $c > 1$ there exists $c_1 > 0$ such that

$$c^{-1}\varphi(s/\theta) - c_1 (\log \theta)^{-\sigma} \leqslant \psi(s)$$

for all s *and* $\theta \leqslant \theta_0$, $a_0 \leqslant s/\theta < s < b$, *then*

$$\limsup_{\substack{s \to b \\ s \notin A}} \varphi(s)/\psi(s) \leqslant 1.$$

Proof. We shall first prove the first part in the case $b = \infty$. If E is a measurable subset of $[a, \infty[$, we denote

$$\tau E = \int_E \frac{ds}{s}.$$

First fix $c > 1$. For $m = 1, 2, \ldots$ let $\mathfrak{F}_m$ be the set of all those φ satisfying the hypothesis of the first part of the lemma for which the corresponding $c_1 < m$. By Lemma 5.1 there exists $A_m \subset [a, \infty[$ such that $\tau A_m < \infty$ and $\psi(s) \leqslant c\varphi(s)$ for all $\varphi \in \mathfrak{F}_m$ and $s \in [a, \infty[\setminus A_m$. Choose a sequence $\varrho_m \nearrow \infty$ such that $\tau A'_m < 2^{-m}$ where $A'_m = A_m \cap [\varrho_m, \infty[$. Let $A = \bigcup A'_m$. Then $\tau A < \infty$. Let φ satisfy the hypothesis of the first part of the lemma. Then there is an m such that $\varphi \in \mathfrak{F}_m$. If $s \in [a, \infty[\setminus A$ and $s > \varrho_m$, then $s \in [a, \infty[\setminus A_m$ and $\psi(s) \leqslant c\varphi(s)$. Hence

$$\liminf_{\substack{s \to \infty \\ s \notin A}} \frac{\varphi(s)}{\psi(s)} \geqslant 1/c.$$

Next choose a sequence $d_m \searrow 1$, denote by A^m the exceptional set corresponding to $c = d_m$, and apply a similar ϱ_m-method as above to the sets A^m to obtain a set $A \subset [a, \infty[$ such that $\tau A < \infty$ and

$$\liminf_{\substack{s \to \infty \\ s \notin A}} \frac{\varphi(s)}{\psi(s)} \geqslant 1.$$

In the case $b < \infty$ for a fixed $c > 1$ we choose the sets $A_m \subset [a, b[$ given by 5.1 so that

$$\frac{\mathcal{L}^1(A_m \cap [t_m, b[)}{b - t_m} < 1/m$$

for some sequence $t_m \nearrow b$ satisfying $b - t_{m+1} < (b - t_m)/m$. With $A'_m = A_m \cap [t_m, t_{m+1}]$ set $A = \bigcup A'_m$. Then clearly

$$\liminf_{t \to b} \frac{\mathcal{L}^1(A \cap [t, b[)}{b - t} = 0$$

and

$$\liminf_{\substack{s \to b \\ s \notin A}} \frac{\varphi(s)}{\psi(s)} \geqslant 1/c.$$

Repeating the procedure for a sequence $d_m \searrow 1$ we get as in the case $b = \infty$ the desired result.

The second part of the lemma follows similarly.

5.9. *Remark.* Observe that the continuity of ψ was used only in the proofs of the second parts in 5.1 and 5.8.

The following result takes care of the case where $A(s)$ is bounded in the case $b=\infty$.

5.10. LEMMA. *Let $b=\infty$ and $\lim_{s\to\infty} A(s)=d<\infty$. Then $\lim_{s\to\infty} n(s,y)\leqslant d$ for all $y\in N$ and $\lim_{s\to\infty} n(s,y)=d$ for $y\in N\setminus E$ where $E\subset N$ is a Borel set of capacity zero.*

Proof. Set

$$F=\{y\in N \mid \lim_{s\to\infty} n(s,y)\leqslant d\},$$

$$A_j=\{y\in N\setminus F \mid n(j,y)\geqslant d+1/j\}.$$

Suppose $\mathcal{L}^n(F)=0$. Then $\mathcal{L}^n(A_{j_0})>0$ for some j_0, and for $s\geqslant j_0$, l

$$A(s)\geqslant ((d+1/j_0)\,\mathcal{L}^n(A_{j_0})+d\mathcal{L}^n(B_l))/\mathcal{L}^n(N),$$

where $B_l=\{y\in N\setminus F\setminus A_{j_0} \mid n(l,y)>d\}$. The lower bound for $A(s)$ tends to $((d+1/j_0)\,\mathcal{L}^n(A_{j_0})+d\mathcal{L}^n(N\setminus A_{j_0}))/\mathcal{L}^n(N)>d$ as $l\to\infty$, which gives a contradiction. Hence $\mathcal{L}^n(F)>0$. This implies cap $F>0$.

Suppose now that cap $(N\setminus F)>0$. Then cap $A_j>0$ for some j. Let now Γ be the family of paths $\gamma\colon [0,1]\to N$ such that $\gamma(0)\in A_j$, $\gamma(1)\in F$. If $\gamma\in\Gamma$, there exists by the analog of [19, Theorem 1] for manifolds for $s>j$ a maximal $f\,|\,D_s$-lifting α of γ which starts at a point in $f^{-1}(\gamma(0))\cap \bar{D}_j$ and which ends in ∂D_s. Denote the family of these maximal lifts by Γ_s. Then $M(\Gamma_s)\to 0$ as $s\to\infty$. But $M(f\Gamma_s)\geqslant M(\Gamma)>0$ because cap F, cap $A_j>0$. This contradicts for large s the inequality $M(f\Gamma_s)\leqslant K_I M(\Gamma_s)$ in 2.11. We have proved cap $(N\setminus F)=0$. Let $y\in N\setminus F$. Then $n(s,y)>d$ for some s. Since the exhausting sets satisfy $\bar{D}_s\subset D_t$ for $s<t$, we also have $n(t,z)>d$ for z in a neighborhood of y for $t\geqslant s+1$. Therefore $N\setminus F$ is open and thus empty.

To prove the second statement set

$$H=\{y\in N \mid \lim_{s\to\infty} n(s,y)=d\}$$

and suppose cap $(N\setminus H)>0$. Set $C_j=\{y\in N \mid n(s,y)=d \text{ if } s\geqslant j\}$. Then H is the union of the sets C_j and since $F=N$, we have $\mathcal{L}^n(C_j)>0$ and hence cap $C_j>0$ for some j. Let now Γ' be the family of paths $\gamma\colon [0,1]\to N$ with $\gamma(0)\in C_j$ and $\gamma(1)\in N\setminus H$. If Γ_s' denotes the set of maximal $f\,|\,D_s$-liftings for $s>j$ similarly as above, we get again a contradiction with $M(f\Gamma_s')\leqslant K_I M(\Gamma_s')$ as $s\to\infty$. The lemma is proved.

We are now in a position to give our main result. Recall that f is a quasiregular mapping of a non-compact Riemannian n-manifold M into a compact Riemannian n-manifold N, $n(s, y)$ is the counting function of f with respect to the given admissible exhaustion of M, and $\lambda \geqslant n-1$ is related to this exhaustion by the inequality (2.17). Recall also that $\nu_\mu(s)$ and $A(s)$ are the averages of $n(s, y)$ with respect to a measure μ and the Lebesgue measure of N, respectively, and that μ is h-calibrated if $\mu B(x, r) \leqslant h(r)$ for all balls $B(x, r) \subset N$.

5.11. THEOREM. *Suppose either $b=\infty$, or $b<\infty$ and $\limsup_{s\to b}(b-s)A(s)^{1/p\lambda}=\infty$ for some $p>2$. Then there exists a measurable set $A \subset [a, b[$ such that*

$$\int_A \frac{ds}{s} < \infty \quad \textit{if} \quad b=\infty,$$

$$\liminf_{s\to b} \frac{\mathcal{L}^1(A \cap [s, b[)}{b-s} = 0 \quad \textit{if} \quad b<\infty$$

and the following holds. Let μ be a measure in N such that $0<\mu(N)<\infty$ and Borel sets of N are μ-measurable and let h be a calibration function satisfying (1.4) *for p.*

(1) *If μ is h-calibrated, then*

$$\lim_{\substack{s\to b\\ s\notin A}} \frac{\nu_\mu(s)}{A(s)} = 1.$$

(2) *If $\limsup_{r\to 0}\mu(B(x, r))/h(r) \leqslant 1$ holds for μ almost every $x \in N$, then*

$$\liminf_{\substack{s\to b\\ s\notin A}} \frac{\nu_\mu(s)}{A(s)} \geqslant 1.$$

Proof. If $b<\infty$ or $b=\infty$ and $\lim_{s\to\infty} A(s)=\infty$, the proof follows from 4.8, 4.18 and 5.8. Suppose then that $b=\infty$ and $\lim A(s)=d<\infty$. Consider (2). As in the proof of 4.18 we conclude that for each $\varepsilon>0$ there exists a Borel set $F \subset N$ such that $\mu(N \setminus F)<\varepsilon$ and $T \mapsto \mu(F \cap T)$ is h_1-calibrated for some h_1 satisfying (1.4) for p. Let E be the Borel set of capacity zero in 5.10 and let $E' \subset E$ be compact. By the application of 4.3 to sets $F \cap \bar{B}(z, \tau r) \cap E'$ we conclude $\mu(E' \cap F)=0$. Hence $\mu(E \cap F)=0$ which implies $\mu(E)=0$. If C_j is the Borel set $\{y \in N \mid n(s, y)=d \text{ if } s \geqslant j\}$ for $j=1, 2, \dots$, and $H=N \setminus E$, then by 5.10 for $s \geqslant j$

$$d\mu(H) \geqslant \int_H n(s, y)\, d\mu(y) \geqslant \int_{C_j} n(s, y)\, d\mu(y) = d\mu(C_j),$$

from which the assertion follows since $\mu(C_j) \to \mu(H)$, in fact we obtained the conclusion in (1). (1) follows from this.

5.12. *Remarks.* 1. The conclusion (1) is essentially included in [20, Theorem 4.19] for the following special case: $M=\mathbf{R}^n$ with the standard exhaustion by balls, $N=\bar{\mathbf{R}}^n$, $\mu(F)=\mathcal{H}^{n-1}(F\cap Y)$, Y an $(n-1)$-dimensional sphere.

2. If in Lemmas 5.1 and 5.8 we assume in the case $b<\infty$ that $\lim_{s\to b}(b-s)\psi(s)^{1/p\lambda}=\infty$ and in Theorem 5.11 the same for $\psi(s)=A(s)$, then the set A can be chosen so that

$$\lim_{s\to b}\frac{\mathcal{L}^1(A\cap[s,b[)}{b-s}=0.$$

This follows by direct inspection of the proofs of 5.1 and 5.8. It is possible to draw the conclusion in (1) in the hyperbolic case $b<\infty$ under the weaker condition $\limsup_{s\to b}(b-s)A(s)^{1/\lambda}=\infty$ for a smaller class of measures μ. This is for $n=2$ and $\lambda=1$ recognized as a condition which ensures regular exhaustibility of a covering surface in [1].

3. We shall show in Example 6.1 for $n=2$, $b=\infty$, by a meromorphic function that the assumption for μ in (2) is not sufficient to draw the conclusion in (1). In Example 6.5 we show that the condition of finiteness of logarithmic measure of A cannot be improved.

4. If X is a compact k-dimensional C^1 submanifold of N, $k\geqslant 1$, then the measure $E\mapsto\mathcal{H}^k(E\cap X)$ is h-calibrated with $h(r)=Cr^k$ for some constant $C>0$. The same conclusion holds also for $k=n$ if X is an $\mathcal{L}^n$-measurable subset with $\mathcal{L}^n(X)>0$ or for $n=2$, $k=1$ and X is a regular curve in the sense of Ahlfors [1].

5. Let h be a calibration function satisfying (1.4) for some $p>2$ and let μ^h be the Hausdorff measure generated by h. If $E\subset N$ is μ^h-measurable with $0<\mu^h(E)<\infty$, then $\limsup_{r\to 0}\mu^h(E\cap B(x,r))/h(2r)\leqslant 1$ for μ^h almost every $x\in E$. One can prove this by a method similar to that of [2, 2.10.18] by observing that any ball $B(x,5r)\subset N$ can be covered with k balls of radius r where k is independent of x and r. By the use of the calibration function $h_1(r)=h(2r)$ we obtain that the conclusions in 4.18 and 5.11(2) hold with $\mu(F)=\mu^h(E\cap F)$, $F\subset N$.

6. It is clear from the proof of 4.8 that if N is not assumed to be compact, averages with respect to measures supported in a compact subset of N are still similarly comparable.

As an application of Lemmas 3.3, 4.4 and 5.1 we are able to prove the following result on pointwise behavior of the counting function.

5.13. THEOREM. *Suppose that either $b=\infty$ or $b<\infty$ and $\lim_{s\to b}(b-s)A(s)^{1/p\lambda}=\infty$ for some $p>2$. Then there exist a sequence (s_i) and a set $E\subset N$ of capacity zero such that $\lim s_i=b$ and for all $y\in N\setminus E$*

$$\lim_{i\to\infty}\frac{n(s_i,y)}{A(s_i)}=1.$$

Proof. We shall present the proof for the case $b=\infty$. The case $b<\infty$ is handled similarly with regard of Remark 5.12.2. We cover N by balls $B_k = B(z_k, \tau r_0/2)$, $k=1, \dots, m$, where r_0 and τ are as in 3.1. Denote $C_k = B(z_k, r_0/2)$,

$$\nu_k(s) = \nu_{\mathcal{L}^n}(s, B_k),$$

and let A be the exceptional set of Theorem 5.11 with $\int_A ds/s < \infty$. Then each ν_k is continuous and

$$\lim_{\substack{s\to\infty \\ s\notin A}} \frac{\nu_k(s)}{A(s)} = 1. \tag{5.14}$$

Let $c>1$. Combining 3.3 and 4.4 for $q=\min(p,3)$ with μ replaced by $\mathcal{L}^n$ and E replaced by B_k, we find that given $\delta>0$ there is $d_\delta>0$ such that

$$\nu_k(s) \leqslant \sqrt{c}\,\bar{n}(\theta s, F) + d_\delta (\log\theta)^{-p\lambda}$$

$$\nu_k(s) \geqslant \underline{n}(s/\theta, F)/\sqrt{c} - d_\delta (\log\theta)^{-p\lambda}$$

whenever F is a set in B_k with $M(\Delta(F, \partial C_k; \bar{C}_k)) \geqslant \delta > 0$ and $a_0 \leqslant s/\theta < \theta s < b$, $\theta \leqslant \theta_0$. By Lemma 5.1 there is a set $A_\delta \subset [a_0, \infty[$ of finite logarithmic measure such that $A \subset A_\delta$ and

$$c^{-1}\underline{n}(s, F) < \nu_k(s) < c\bar{n}(s, F)$$

whenever F is a set in B_k with $M(\Delta(F, \partial C_k; \bar{C}_k)) \geqslant \delta > 0$ and $s \in [a_0, \infty[\setminus A_\delta$. We can choose A_δ independent of k by taking union. Set

$$E_{k,s} = \{y \in B_k \mid n(s,y) > c\nu_k(s)\} \cup \{y \in B_k \mid n(s,y) < \nu_k(s)/c\}.$$

Then for all $s \in [a_0, \infty[\setminus A_\delta$, $M(\Delta(E_{k,s}, \partial C_k; \bar{C}_k)) \leqslant 2\delta$. For each positive integer i choose $s_i = s_i(c, \delta) \in [a_0, \infty[\setminus A_{2^{-i-1}\delta}$ such that $\lim s_i = \infty$. Then

$$M\left(\Delta\left(\bigcup_{i=1}^{\infty} E_{k,s_i}, \partial C_k; \bar{C}_k\right)\right) \leqslant \sum_{i=1}^{\infty} M(\Delta(E_{k,s_i}, \partial C_k; \bar{C}_k)) \leqslant \sum_{i=1}^{\infty} 2^{-i}\delta = \delta. \tag{5.15}$$

Choose now sequences $c_j \searrow 1$, $\delta_j \searrow 0$ such that $\sum_{i>j} \delta_i \leqslant \delta_j$ and for each j a sequence $s_{j,i} = s_i(c_j, \delta_j)$ as above. Letting $s_i = s_{i,i}$ we will show that

$$\lim_{i\to\infty} \frac{n(s_i, y)}{\nu_k(s_i)} = 1 \tag{5.16}$$

for all $y \in B_k$ outside a set of capacity zero. Suppose this is not true for some k. Assume e.g. that

$$\operatorname{cap}\left\{y \in B_k \,\middle|\, \limsup_{i\to\infty} \frac{n(s_i, y)}{\nu_k(s_i)} > 1\right\} > 0.$$

Then for some j, $M(\Delta(D_j, \partial C_k; \bar{C}_k)) > \delta_j$ where

$$D_j = \left\{y \in B_k \,\middle|\, \limsup_{i\to\infty} \frac{n(s_i, y)}{\nu_k(s_i)} > c_j\right\}.$$

From (5.15) we obtain $M(\Delta(E_j, \partial C_k; \bar{C}_k)) \leqslant \delta_j$ where

$$E_j = \left\{y \in B_k \,\middle|\, \frac{n(s_j, y)}{\nu_k(s_{j,i})} > c_j \quad \text{for some } i\right\}.$$

If $\limsup_{i\to\infty} n(s_i, y)/\nu_k(s_i) > c_j$, then there is $i > j$ such that $n(s_{i,i}, y)/\nu_k(s_{i,i}) > c_j \geqslant c_i$, which yields $D_j \subset \bigcup_{i>j} E_i$ and

$$\delta_j < M(\Delta(D_j, \partial C_k; \bar{C}_k)) \leqslant \sum_{i>j} M(\Delta(E_i, \partial C_k; \bar{C}_k)) \leqslant \sum_{i>j} \delta_i \leqslant \delta_j.$$

This contradiction shows that (5.16) holds. The theorem follows now from (5.16) and (5.14).

5.17. *Remark*. In the plane Miles has proved for meromorphic functions a result which is stronger than 5.13 in the sense that the limit is obtained outside an exceptional set for the exhaustion parameter, see [12, Theorem 2].

6. Examples

In this section we shall present two examples of meromorphic functions in the plane refered to in Remark 5.12(3). Corresponding examples of quasiregular mappings for dimensions $n \geqslant 3$ of equal sharpness have not been constructed. In the following we shall denote by $\nu^i(s, E) = \nu_{\mathcal{H}^i}(s, E)$ the average over an $\mathcal{H}^i$-measurable set E with $0 < \mathcal{H}^i(E) < \infty$, where $\mathcal{H}^i$ is the normalized i-dimensional Hausdorff measure in $\mathbf{R}^2$.

6.1. *Example*. We shall construct a nonconstant meromorphic function $f\colon \mathbf{R}^2 \to \bar{\mathbf{R}}^2$ such that for each $c > 1$ there exist a set $E \subset \mathbf{R}^2$ which is a countable union of circles centered at the origin with $\mathcal{H}^1(E) < \infty$ and a measurable set $A \subset [1, \infty[$ such that

$$\nu^1(r, E) > cA(r) \quad \text{for} \quad r \in A, \tag{6.2}$$

and

$$\int_A \frac{dr}{r} = \infty, \tag{6.3}$$

where the exhaustion is the standard exhaustion by disks.

Our function f will be a slight modification of the meromorphic function in Theorem 4 in [23]. We define an increasing sequence $s_1, s_2, \ldots$ of integers by the condition $s_{4i-2} = s_{4i-1} = s_{4i} = s_{4i+1} = i + 10$ for $i = 1, 2, \ldots$ and set $s_1 = s_2$. We set $r_m = \exp((2s_m)^m)$ and

$$f(z) = \prod_{m=1}^{\infty} (1 - z/r_m)^{(-s_m)^m}.$$

Fix i and set $m = 4i$, $\sigma_m = r_m^{1+1/4s_m}$, $\varrho_m = r_m^{1+1/2s_m}$. Arguing as in [23] we obtain $A(\varrho_m) < 2s_m^{m-1}$, $n(r_m, 0) > s_m^m$. Similar calculations give the estimate

$$\lambda_i = r_m^{-s_m^{m-1}} < |f(z)| \quad \text{for} \quad \sigma_m \leqslant |z| \leqslant \varrho_m.$$

Set $d = 12c - s_1$ and $\mu_i = s_{m+4}/(s_{m+4} + d) - s_m/(s_m + d)$. The set E is constructed as follows. Let E_i be a union of p_i disjoint circles with center 0 and radii in the interval $[\lambda_i/2, \lambda_i]$ where $2\mu_i/(\pi\lambda_i) \leqslant p_i < 2\mu_i/(\pi\lambda_i) + 1$. Then set $E = \bigcup_{j=1}^{\infty} E_j$.

Suppose $\sigma_m \leqslant r \leqslant \varrho_m$. Then

$$\begin{aligned}\int_E n(r, y)\, d\mathcal{H}^1(y) &\geqslant \sum_{j=i}^{\infty} \int_{E_j} n(r, y)\, d\mathcal{H}^1(y) \geqslant n(r_m, 0) \sum_{j=i}^{\infty} \mu_j \\ &= n(r_m, 0)\left(1 - \frac{s_m}{s_m + d}\right) > \frac{d s_m^m}{s_m + d}.\end{aligned} \tag{6.4}$$

Since $\mathcal{H}^1(E) \leqslant \sum_{j=1}^{\infty} 3\mu_j = 3(1 - s_1/(s_1 + d))$, we get with (6.4) for $s_m \geqslant d$, $\nu^1(r, E) > cA(\varrho_m)$. Finally, let $m_0 = 4i_0$ be such that $s_{m_0} \geqslant d$ and set

$$A = \bigcup_{i=i_0}^{\infty} [\sigma_{4i}, \varrho_{4i}].$$

Then (6.2) holds for $r \in A$ and clearly A satisfies (6.3).

Denote by μ the restriction measure $C \mapsto \mathcal{H}^1(C \cap E)$ of $\mathcal{H}^1$. Then μ is a measure in $\bar{\mathbf{R}}^2$ and satisfies the condition in 5.11(2) for $h(r) = 2r$, μ almost everywhere, in fact in all points except 0. Hence the conclusion in 5.11(2) holds. On the other hand, (6.2) and (6.3) show that the conclusion in 5.11(1) is not true for μ. From the conctruction it is clear that such a μ is also obtained by giving the Lebesgue measure a weight which has a suitable singularity at the origin.

6.5. *Example.* We shall show that there exists a disk $E\subset\mathbf{R}^2$ and a number $c>1$ such that for a given decreasing positive function φ of the positive real axis with $\varphi(r)\to 0$ as $r\to\infty$ there exists a meromorphic function $f\colon \mathbf{R}^2\to\bar{\mathbf{R}}^2$ such that with respect to the standard exhaustion by disks we have

$$\nu^2(r, E) > cA(r) \quad \text{for} \quad r\in A,$$

where $A\subset[1, \infty[$ is a measurable set which for some $r_0>0$ satisfies

$$\int_{A\cap[r,\infty[}\frac{dr}{r} > \varphi(r) \quad \text{for} \quad r\in[r_0, \infty[.$$

In this example we take for μ the restriction measure $C\mapsto\mathcal{H}^2(C\cap E)$. Then μ is h-calibrated with $h(r)=Cr^2$ for some $C>0$.

The following construction was given by S. Toppila. For $1<r_i<\varrho_i<R_i$ with $r_iR_i=\varrho_i^2$ and t_i a positive integer consider the function

$$g_i(z)=(1-(z/r_i)^{t_i})(1-z/\varrho_i)^{-2t_i}(1-z/R_i)^{t_i}.$$

For small and large $|z|$, $g_i(z)$ is near 1, and if ϱ_i/r_i is large, the behavior of $g_i(z)$ is determined by the first factor near $|z|=r_i$. Set $h_i(z)=1-(z/r_i)^{t_i}$ and $\sigma_i=r_i/2t_i$. Then the counting function of h_i satisfies for $r_i\leqslant r\leqslant r_i+\sigma_i$

$$n_{h_i}(r, y)=t_i \quad \text{if} \quad y\in B^2(1)+1,$$
$$n_{h_i}(r, y)=0 \quad \text{if} \quad y\notin B^2(2)+1,$$

and we choose ϱ_i/r_i so large that for g_i we have

$$n_{g_i}(r, y)=t_i \quad \text{if} \quad y\in B^2(1-\delta)+1,$$
$$n_{g_i}(r, y)=0 \quad \text{if} \quad y\notin B^2(2+\delta)+1,$$

where δ is some number with $0<\delta<\frac{1}{3}$. Let $p\geqslant 1$ be an integer and set $t_i=(1+p)^i$. Then

$$p\sum_{i=1}^{k-1}t_i<t_k. \tag{6.6}$$

We may choose the ratios ϱ_i/r_i and r_{i+1}/R_i so large that the meromorphic function $f\colon\mathbf{R}^2\to\bar{\mathbf{R}}^2$,

$$f(z)=\prod_{i=1}^{\infty}g_i(z),$$

behaves up to a small error term as $h_i(z)$ near $|z| = r_i$. With suitable choices of these ratios we have then for the counting function of f for $r_k \leqslant r \leqslant r_k + \sigma_k$

$$n(r, y) = \sum_{i=1}^{k-1} 2t_i + t_k \quad \text{if} \quad y \in B^2(1/2) + 1,$$

$$n(r, y) \leqslant \sum_{i=1}^{k-1} 2t_i + t_k \quad \text{if} \quad y \in B^2(3) + 1,$$

$$n(r, y) \leqslant \sum_{i=1}^{k-1} 2t_i \qquad \text{if} \quad y \notin B^2(3) + 1.$$

Set $E = B^2(\frac{1}{2}) + 1$, $F = B^2(3) + 1$, and let β be the spherical measure of F divided by the total spherical measure π. Then for $r_k \leqslant r \leqslant r_k + \sigma_k$

$$A(r) \leqslant \beta\left(\sum_{i=1}^{k-1} 2t_i + t_k\right) + (1-\beta)\sum_{i=1}^{k-1} 2t_i,$$

$$\nu^2(r, E) = \sum_{i=1}^{k-1} 2t_i + t_k.$$

With regard of (6.6) we obtain

$$A(r)/\nu^2(r, E) \leqslant \beta + \frac{1}{1+p/2}(1-\beta) = 1/c < 1 \quad \text{for} \quad r_k \leqslant r \leqslant r_k + \sigma_k.$$

Let now φ be a decreasing positive function with $\varphi(r) \to 0$ as $r \to \infty$. Since

$$\int_{r_i}^{r_i+\sigma_i} \frac{dr}{r} = \log(1 + 1/2t_i)$$

is independent of r_i, we may choose the r_i's in addition so that

$$\varphi(r_{k-1}) < \sum_{i=k}^{\infty} \int_{r_i}^{r_i+\sigma_i} \frac{dr}{r}.$$

We can therefore take $\bigcup_{i=1}^{\infty} [r_i, r_i + \sigma_i]$ to be the required set A.

The problem of covering a disk more than $A(r)$ was also considered in Example 2 in [12].

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