SOME GEOMETRIC AND ANALYTIC PROPERTIES OF HOMOGENEOUS COMPLEX MANIFOLDS

PART I: SHEAVES AND COHOMOLOGY

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This is the first of two papers dealing with homogeneous complex manifolds; since the second work is a continuation of this one, we shall let the following introduction serve for both.

The general problem is to study the geometric, analytic, and function-theoretic properties of homogeneous complex manifolds. The present paper, referred to as Part I, is concerned mainly with sheaves and cohomology; the results here may be viewed as the linear part of the solutions to the questions discussed in the second paper (Part II). In fact, in Part II, using the results of Part I as a guide and first approximation, we utilize a variety of geometric, analytic, and algebraic constructions to treat the various problems which we have posed. A previous paper [11], cited as D. G., was concerned with the differential geometry of our spaces, and the results obtained there will be used from time to time.

The study alone of certain locally free sheaves on these manifolds is a rather interesting one and has been pursued in [4], [5], [16], and [21]. The situation is the following: Let $X = G/U = M/V$ be a homogeneous complex manifold written either as the coset space of complex Lie proups G, U or compact Lie groups M, V where M is semi-simple. Then M acts in any analytic vector bundle $\mathbf{E}^{\varrho}$ (1) associated to the principal fibering $U \to G \to X$ by a holomorphic representation $\varrho: U \to GL(E^{\varrho})$. (Such bundles are called homogeneous vector bundles.) The sheaf cohomology groups $H^*(X, \mathcal{E}^{\varrho})$ are then M-modules by an induced representation ϱ^*; these modules have been determined in [5] and [21] when ϱ is irreducible and X is algebraic, and in other special

(1) The notations used here are explained in § 1.

cases in [5] and [16], In §§ 2 and 5, we shall determine ϱ^* when X is arbitrary and ϱ is irreducible (thereby giving new proofs in the algebraic case) and shall give in § 3 an algorithm for finding ϱ^* when ϱ is arbitrary; this algorithm covers the known results (§ 4) and suffices for most of our purposes. In particular, it plus elementary Kähler geometry gives an explanation of the "strange equality" observed in [5]. For applications, we need not only the modules and their transformation rule under M, but also the explicit Dolbeault forms representing cohomology classes; this construction, which turns out to involve a connexion, is given in § 5.

In the remainder of Part I, we give the more immediate and simpler applications of §§ 2–5. In § 6, the group of line bundles $L[X]$ and function field $F[X]$ are determined, and in § 7 the characteristic ring and its relation to sheaf cohomology groups is discussed. In § 8 the endomorphisms and embedding of homogeneous vector bundles are treated. Also in this section we discuss some extrinsic geometry of C-spaces, and we give a projective-geometric proof of rigidity in the Kähler case.

In §§ 9 and 10 at the beginning of Part II, the variation of complex structure of our spaces in examined in some detail; here we come across a rather interesting mixture of techniques in differential geometry, representation theory, and partial differential equations, and we outline briefly our treatment of this problem.

It is known that, roughly speaking, the parameters of deformation of X turn up infinitesimally in $H^1(X, \Theta)$, and thus in § 9 we solve the linear part of the problem by determining completely the M-modules $H^q(X, \Theta)$. However, not every $\gamma \in H^1(X, \Theta)$ is suitable for a deformation parameter, and in the last part we determine those γ's which are "obstructed". Then in § 10 we use representation theory (primarily the Frobenius reciprocity law) and partial differential equations to construct local deformations through the unobstructed $\gamma \in H^1(X, \Theta)$; these new manifolds are generally non-homogeneous. Finally, using the fact that γ transforms in a certain way under M, we discuss which among the new manifolds are biregularly equivalent and in so doing encounter the phenomenon of "jumping of structures".

Paragraphs 11 and 12 are a discussion of various properties of homogeneous vector bundles such as the moduli of homogeneous bundles and the extension theory and automorphisms of these same bundles. For example, in § 11 we characterize the homogeneous bundles over a Kähler C-space as being those bundles which, with a suitable reduction of structure group, are locally rigid. In § 13 bundles over general homogeneous Kähler manifolds are treated, and § 14 is given to examples of the general theory and counter-examples to show why some results cannot be sharpened.

It may be well to show how the above applies to a specific manifold. Let

$X = SU\,(5)$ with any left-invariant complex structure; writing $X = G/U$, $G = SL\,(5, \mathbf{C})$ and U is a certain subgroup of the maximal solvable subgroup

$$\hat{U} = \left\{ \begin{pmatrix} a_{11} & \cdots & a_{15} \\ & \ddots & \\ 0 & & a_{55} \end{pmatrix} \right\} \quad (\det\,(a_{ij}) = 1).$$

Any representation of $\hat{U}$ induces one of U (but not conversely) and we denote the 1-dimensional representation $(a_{ij}) \to a_{kk}$ by θ_k. It turns out that, for any $\lambda \in \mathbf{C}^*$, $\lambda\theta_k$ defined on $\mathfrak{u}$ = complex Lie algebra of U by $\lambda\theta_k(a_{ij}) = \lambda a_{kk}((a_{ij}) \in \mathfrak{u})$ induces a representation of U and we may form the homogeneous line bundle $E^{\lambda\theta_k} \to \mathbf{E}^{\lambda\theta_k} \to X$. Then $L[X] \cong \mathbf{C}^2$ and the most general line bundle on X is of the form $E^{\lambda_1\theta_1} \ldots E^{\lambda_5\theta_5} = \mathbf{E}^{\lambda}(\lambda = (\lambda_1, \ldots, \lambda_5))$, there being three relations among the λ_j's. All these line bundles have non-zero $\bar{\partial}$-cohomology class but zero d-cohomology class. It will be seen that, in general, $H^*(X, \mathcal{E}^{\lambda}) = 0$ if some $\lambda_j \notin \mathbf{Z}$ and $H^0(X, \mathcal{E}^{\lambda}) \neq 0 \Leftrightarrow \lambda$ is integral and $\lambda_1 \geqslant \ldots \geqslant \lambda_5$; in this case, $H^0(X, \mathcal{E}^{\lambda})$ is the irreducible $SU\,(5)$ module given by the Young diagram

$1, 1$	$\ldots$	$\cdot$	$1, \lambda_1$
$2, 1$	$\ldots$	$2, \lambda_2$	
$\vdots$	$\ldots$ $\ldots$	$\cdot$	
$5, 1$	$\ldots$ $5, \lambda_5$		

Furthermore, $H^q(X, \mathcal{E}^{\lambda}) \cong H^0(X, \mathcal{E}^{\lambda}) \otimes \mathbf{C}^{\binom{2}{q}}$ and $SU\,(5)$ acts by $\lambda^* \otimes 1$.

The set of bundles $\mathbf{E}$ such that we have $0 \to \mathbf{E}^{\lambda_1\theta_1} \to \mathbf{E} \to \mathbf{E}^{\tau_1\theta_1} \to 0$ forms a vector space which is non-trivial $\Leftrightarrow \lambda_1 - \tau_1$ is integral and non-negative. If $\lambda_1 - \tau_1 = n > 0$, the bundles $\mathbf{E}$ are all non-homogeneous and form a space of dimension $\binom{4+n}{n-1}$; if $\lambda_1 = \tau_1$, these bundles $\mathbf{E}$ are all homogeneous and form a vector space of dimension 2.

The groups $H^q(X, \Theta) \cong \left\{sl\,(5, \mathbf{C}) \otimes \mathbf{C}^{\binom{2}{q}}\right\} \oplus \left\{\mathbf{C}^2 \otimes \mathbf{C}^{\binom{2}{q}}\right\}$ and M acts by $\{\mathrm{Ad} \otimes 1\} \oplus \{1 \otimes 1\}$. We have that $\dim H^1(X, \Theta) = 52$; of these 52 parameters, there are a maximum of 28 which parametrize a local deformation of the analytic structure of X, and, in fact, 28 such parameters exist. The remaining 24 parameters are obstructed. Of the 28 suitable parameters, 4 preserve the homogeneous structure on X and 24 do not; any two elements in $H^1(X, \Theta)$ differing by an action of M give equivalent manifolds.

This then is an outline of the contents of this paper. Throughout we have tried to maintain the dual attitudes of studying in some detail those properties arising from the homogeneity of our spaces while at the same time keeping an eye on those properties which seem to have wider applicability. The latter aim was especially in mind when studying the variation of manifold and bundle structures and its relation therewith to obstructions.

This paper grew out of the author's dissertation at Princeton University, and to D. C. Spencer and many others we express gratitude for generous help given. Some of the results appearing below were announced in *Proc. Nat. Acad. Sci.*, May 1962.

Table of contents

1. Review and Preparatory Discussion 119
   (i) Notations and Terminology 119
   (ii) Lie Algebras and Representation Theory 119
   (iii) The C-Spaces of Wang 120
   (iv) Sheaf Cohomology and Lie Algebra Cohomology 121
2. Homogeneous Bundles Defined by an Irreducible Representation 124
3. Homogeneous Bundles Defined by a Non-Irreducible Representation 129
4. Applications of §§ 2 and 3 132
5. Homogeneous Bundles in the Non-Kähler Case 136
6. Line Bundles and Functions on C-Spaces 141
7. Some Properties of the Characteristic Classes of Homogeneous Bundles 143
8. Some Properties of Homogeneous Vector Bundles 146
   (i) Endomorphisms of Homogeneous Vector Bundles 146
   (ii) Embedding of Homogeneous Vector Bundles 148
   (iii) Extrinsic Geometry of C-Spaces and a Geometric Proof of Rigidity in the Kähler Case 150
9. Deformation Theory—Part I 157
   (i) The Infinitesimal Theory 157
   (ii) Obstructions to Deformation 160
10. Deformation Theory—Part II 162
   (i) The Kähler Case 162
   (ii) The Non-Kähler Case 166
   (iii) The Question of Equivalence 172
11. Some General Results on Homogeneous Vector Bundles 174
   (i) The Equivalence Question for Homogeneous Vector Bundles 174
   (ii) Extension Theory of Homogeneous Vector Bundles 176
   (iii) The Deformation Theory of Homogeneous Vector Bundles 180
12. Some Applications of § 11 185
13. Bundles over Arbitrary Homogeneous Kähler Manifolds 193

14. Examples and Counterexamples 199
(i) An Illustration of the General Theory 199
(ii) An Example Concerning the Semi-Simplicity of Certain Representations . . . 204
(iii) Line Bundles over $P_r(\mathbf{C})$ 205
(iv) A New Type of Obstruction 206

1. Review and Preparatory Discussion

(i) Notations and Terminology

If V is a vector space over a field K, and if $V_1, V_2, \ldots$ are subsets of V, we denote by ${}_K(V_1, V_2, \ldots)$ the smallest linear subspace of V containing $V_1, V_2, \ldots$. As usual, $GL(V)$ is the Lie group of automorphisms of V and $gl(V)$ is the Lie algebra of endomorphisms of V. The symbols $\mathbf{Z}, \mathbf{Q}, \mathbf{C}, \mathbf{R}$ represent the integers, rationals, complex numbers, and real numbers respectively. The dual of a vector space V is denoted by V'; if V is defined over $\mathbf{Q}$ or $\mathbf{R}$, its complexification $V \otimes_{\mathbf{Q}} \mathbf{C}$ or $V \otimes_{\mathbf{R}} \mathbf{C}$ is denoted by $\tilde{V}$. If A is a Lie group, $\mathfrak{a}^0$ is its real Lie algebra, $\tilde{\mathfrak{a}}^0 = \mathfrak{a}^0 \otimes_{\mathbf{R}} \mathbf{C}$; if A is a complex Lie group, $\mathfrak{a}$ is its complex Lie algebra.

For a manifold $X, T(X)$ denotes its tangent bundle; if X is complex, $\widetilde{T(X)} = T(X) \otimes_{\mathbf{R}} \mathbf{C}$ splits $T(X) \cong L(X) \oplus \overline{L(X)}$ into vectors of type $(1,0)$ and $(0,1)$ respectively. The symbol $E \to \mathbf{E} \to X$ will denote a vector bundle over X with fibre E; $E' \to \mathbf{E}' \to X$ is its dual. The usual operations $\oplus, \otimes, \Lambda^q, \ldots$ among vector bundles will be used freely. If $E \to \mathbf{E} \to X$ is an analytic vector bundle over a complex manifold $X, \mathcal{E}$ is the sheaf of germs of holomorphic cross-sections of $\mathbf{E}$ ([14]); in this case, $H^q(X, \mathcal{E})$ denotes sheaf cohomology. The symbol $\mathbf{1}$ denotes the trivial line bundle and we set $\mathfrak{l} = \Omega_X (= \Omega$ if there is no confusion). Also, we write $\Theta = \mathfrak{L}(X)$ and $\Omega^q = \Lambda^q \mathfrak{L}(X)'$. The notations and terminology concerning differential geometry are those used in D. G.; they shall be used without explicit reference.

(ii) Lie Algebras and Representation Theory

We review the structure theory of complex semisimple Lie algebras and some facts from representation theory ([25]). Let $\mathfrak{g}$ be a complex semi-simple Lie algebra, $\mathfrak{h} \subset \mathfrak{g}$ a Cartan sub-algebra, $(\ ,\ )$ the Cartan-Killing form on $\mathfrak{h}$ and on $\mathfrak{h}'$. Then, if Σ is the system of roots of $(\mathfrak{g}, \mathfrak{h})$, we may write $\mathfrak{g} = \mathfrak{h} \oplus (\oplus_{\alpha \in \Sigma} v_\alpha)$ where the v_α are one dimensional and, for $h \in \mathfrak{h}$, $v \in v_\alpha$, $[h, v] = \langle \alpha, h \rangle v$. As usual, we set $h_\alpha = [e_\alpha, e_{-\alpha}]$. One may choose an ordering in Σ which defines the positive roots Σ^+ and the negative roots $\Sigma^- = -(\Sigma^+)$; furthermore, there exists a minimal set of generators (over $\mathbf{Z}$) $\prod \subset \Sigma^+$; $\prod = \{\alpha_1, \ldots, \alpha_l\}$ ($l = \dim \mathfrak{h} = \text{rank } \mathfrak{g}$) is a system of simple roots.

The Weyl group $W(\mathfrak{g})$ acts as a finite group on $\mathfrak{h}$ and on $\mathfrak{h}'$; one speaks of singular and regular elements as usual. Having chosen an ordering in Σ, we have the Weyl chamber $D(\mathfrak{g}) = \{\lambda \in \mathfrak{h}' : (\lambda, \varphi) = \langle \lambda, h_\varphi \rangle \geqslant 0 \text{ for all } \varphi \in \Sigma^+\}$; the interior $D^0(\mathfrak{g}) = \{\lambda \in \mathfrak{h}' : (\lambda, \varphi) > 0 \; \varphi \in \Sigma^+\}$. Then $D(\mathfrak{g})$ is a fundamental domain for $W(\mathfrak{g})$. The element $g = \frac{1}{2} \sum_{\varphi \in \Sigma^+} \varphi$ lies in $D^0(\mathfrak{g})$; $2(g, \alpha_j)/(\alpha_j, \alpha_j) = (\alpha_j, \alpha_j)$ for all $\alpha_j \in \prod$. In $W(\mathfrak{g})$, there is the involution δ satisfying $\delta(g) - g = \sum_{\varphi \in \Sigma^-} \varphi$.

An element $\lambda \in \mathfrak{h}'$ is integral if $2(\lambda, \varphi)/(\varphi, \varphi) \in \mathbf{Z}(\varphi \in \Sigma)$; we denote the integral elements in $\mathfrak{h}'$ by $Z(\mathfrak{g})$. A complex finite-dimensional representation space E^ϱ decomposes into weight spaces:

$$E^\varrho = \oplus_{\lambda \in \Sigma(\varrho)} E^{\varrho(\lambda)}$$

($\Sigma(\varrho) \subset Z(\mathfrak{g}) = $ weights of ϱ); an irreducible representation is uniquely determined by its highest weight. We set $\mathfrak{h}^{\#} = {}_{\mathbf{Q}}(Z(\mathfrak{g}))$ and also define the fundamental weights $\tilde{\omega}_1, \ldots, \tilde{\omega}_l$ by $2(\tilde{\omega}_i, \alpha_j)/(\alpha_j, \alpha_j) = \delta^i_j$; these $\tilde{\omega}_j$ form a minimal basis for $Z(\mathfrak{g})$. If $\lambda = \sum \lambda_j \tilde{\omega}_j \in Z(\mathfrak{g})$, $\lambda \in D(\mathfrak{g}) \Leftrightarrow \lambda_j \geqslant 0 (j = 1, \ldots, l)$ and, for $\lambda \in D(\mathfrak{g})$, we denote by E^λ the irreducible $\mathfrak{g}$-module with highest weight λ.

For any $\mathfrak{g}$-module E^ϱ, $E^{-\varrho}$ is the contragredient $\mathfrak{g}$-module with highest weight $\delta(-\varrho)$. We may write $E^\varrho = \sum_{\lambda \in D(\mathfrak{g})} m_\lambda(\varrho) E^\lambda$ where $m_\lambda(\varrho) = $ multiplicity of λ in ϱ. Schur's lemma then reads:

$$\dim \operatorname{Hom}_{\mathfrak{g}} (E^\lambda, E^\varrho) = m_\lambda(\varrho);$$

this simple equation will be used time and again.

(iii) The C-spaces of Wang

We recall the structure of C-spaces as given in [13] and [24]. A C-space X may be written as $X = G/U$ or $X = M/V$ where G, U are complex Lie groups, M, V are compact groups. Furthermore, we may assume that G, M are semi-simple and that G is the complexification of M; then $\mathfrak{g} = \tilde{\mathfrak{m}}^0$. One has a holomorphic principal fibering $U \to G \to G/U$ and if ϱ: $U \to GL(E^\varrho)$ is a holomorphic representation, we form the homogeneous vector bundle $E^\varrho \to \mathbf{E}^\varrho \to G/U$ where $\mathbf{E}^\varrho = G \times_U E^\varrho$ (see [5]). The sheaf of germs of holomorphic cross-sections of $\mathbf{E}^\varrho$ is denoted by $\mathcal{E}^\varrho$; the sheaf-cohomology group by $H^*(X, \mathcal{E}^\varrho)$. Since M acts holomorphically on $\mathbf{E}^\varrho$, $H^*(X, \mathcal{E}^\varrho)$ is a finite dimensional M-module and it is this action we are interested in.

We describe G, U, M, V by giving their complex Lie algebras. If $\mathfrak{g} = \tilde{\mathfrak{m}}^0 = \mathfrak{h} \oplus$

$(\oplus_{\alpha\in\Sigma} v_\alpha)$ as above, then there exists a closed subsystem $\Psi\subset\Sigma$ such that Ψ is a root-system for V. Furthermore, there exists a rational splitting $\mathfrak{h}=\mathfrak{c}\oplus\mathfrak{h}_V$ such that $\mathfrak{h}_V\supset\mathfrak{c}(h_\alpha\colon \alpha\in\Psi)$ and a splitting of $\mathfrak{c}$ into complex spaces: $\mathfrak{c}=\mathfrak{p}\oplus\bar{\mathfrak{p}}$ such that $\tilde{\mathfrak{v}}^0=\mathfrak{h}_V\oplus(\oplus_{\alpha\in\Psi} v_\alpha)$ and, setting $\mathfrak{n}=\mathfrak{c}(e_{-\alpha}\colon \alpha\in\Sigma^+-\Psi^+)$, $\mathfrak{u}=\bar{\mathfrak{p}}\oplus\tilde{\mathfrak{v}}^0\oplus\mathfrak{n}$. The complex vector space $\bar{\mathfrak{p}}\oplus\mathfrak{h}_V$ will lie on no rational hyperplane; $\alpha\in\mathfrak{h}^\#$ and $\langle\alpha,h\rangle=0$ for all $h\in\bar{\mathfrak{p}}\oplus\mathfrak{h}_V\Rightarrow\alpha=0$. We denote by an * (or alternatively by a $^-$) the conjugation in $\tilde{\mathfrak{m}}^0$; thus, e.g., $\mathfrak{n}^*=\mathfrak{c}(e_\alpha\colon \alpha\in\Sigma^+-\Psi^+)$.

Let now $X=G/U$ be an arbitrary C-space where U is solvable; if $T^{2a}\to G/U\to G/\hat{U}$ is the fundamental fibering, $\hat{U}$ will be maximal solvable and $\hat{X}=G/\hat{U}$ will be a flag manifold. Let $\dim{}_{\mathbf{C}}\hat{X}=n$ so that $\dim{}_{\mathbf{C}}X=n+a$. A homogeneous line bundle $E^\varrho\to\mathbf{E}^\varrho\to X$ is given by a linear form ϱ on $\mathfrak{h}\cap\mathfrak{u}$; we recall Theorem 6′ of D.G. where it was shown that if the characteristic class $c_1(E^\varrho)$ was negative semi-definite of index k, then

$$H^q(X,\mathcal{E}^\varrho)=0\quad(q<n-k).\tag{1.1}$$

From this and from the argument in Proposition 8.2 of D.G. it follows that, if $\langle\varrho,h_\alpha\rangle<0$ for all $\alpha\in\Sigma^+$, then

$$H^q(X,\mathcal{E}^\varrho)=0\quad(q<n-a).\tag{1.2}$$

If $\mathbf{K}$ is the canonical bundle on X, then (D.G., Proposition 5.2, or directly) $\mathbf{K}=\mathbf{E}^{-2g}$ $(g=\frac{1}{2}\sum_{\alpha\in\Sigma^+}\alpha)$. Since for an integral form ϱ on $\mathfrak{h}\cap\mathfrak{u}$, $-\varrho-2g$ is strictly negative $\Leftrightarrow\varrho+g$ is non-negative, we find that, using Serre duality,

$$H^q(X,\mathcal{E}^\varrho)=0\quad(a<q)\tag{1.3}$$

if $\varrho+g$ is non-negative on $\mathfrak{h}\cap\mathfrak{u}$.

(iv) Sheaf Cohomology and Lie Algebra Cohomology

Let X be a C-space (arbitrary) and $E^\varrho\to\mathcal{E}^\varrho\to X$ a homogeneous vector bundle. It is due to Bott that $H^*(X,\mathcal{E}^\varrho)$ may be written in terms of Lie algebra; we shall constantly use this and a similar result which we now describe.

Let M,V with $V\subset M$ be arbitrary compact connected Lie groups and such that $X=M/V$ is simply connected. Given a representation $\varrho\colon V\to GL(E^\varrho)$, we may form the differentiable homogeneous vector bundle $E^\varrho\to\mathbf{E}^\varrho\to X$ where $\mathbf{E}^\varrho=M\times_V\mathbf{E}^\varrho$. In particular, if $E^\varrho=\tilde{\mathfrak{m}}^0/\tilde{\mathfrak{v}}^0$ and $\varrho=\mathrm{Ad}$ (induced action), then $\mathbf{E}^\varrho\cong\widetilde{T(X)}=\tilde{\mathcal{T}}$. If $E^\varrho=E^{\varrho'}\oplus E^{\varrho''}$ (as V-modules), then $\mathbf{E}^\varrho\cong\mathbf{E}^{\varrho'}\oplus\mathbf{E}^{\varrho''}$ and we may speak of the cross-sections of $\mathbf{E}^\varrho$ as being of a certain type. For example, if we take a $\tilde{\mathfrak{v}}^0$-reductive splitting $\tilde{\mathfrak{m}} = \tilde{\mathfrak{v}}^0 \oplus \mathfrak{k}$ and if $\mathfrak{k}^\# \subseteq \mathfrak{k}$ is $\tilde{\mathfrak{v}}^0$-stable, then $\mathfrak{k} = \mathfrak{k}^b \oplus \mathfrak{k}^\#$ ($\tilde{\mathfrak{v}}^0$-decomposition) and $\tilde{T} \cong T^b \oplus T^\#$. This induces a type decomposition on differential forms. Since M acts on $\mathbf{E}^\varrho$, the vector space $\Delta(\mathbf{E}^\varrho)$ of C^∞ cross-sections of $\mathbf{E}^\varrho$ is an M-module with induced representation ϱ^*: $M \to GL(\Delta(\mathbf{E}^\varrho))$.

THEOREM 0. *Let $D(M)$ be an index set for the irreducible representations of M and let $E^\tau \to \mathbf{E}^\tau \to X$ be a homogeneous differentiable bundle. Then we have an M-isomorphism*

$$\Delta(\mathbf{E}^\tau) \sim \sum_{\lambda \in D(M)} V^\lambda \otimes (E^\tau \otimes V^{-\lambda})^{\tilde{\mathfrak{v}}^0}, \tag{1.4}$$

where
$$\varrho^* \,|\, V^\lambda \otimes (E^\tau \otimes V^{-\lambda})^{\tilde{\mathfrak{v}}^0} = \lambda \otimes 1.$$

Proof. Let $C^\infty(M) = C^\infty$ complex valued functions on M; M acts on $C^\infty(M)$ in two ways:

(i) $R: M \to GL(C^\infty(M))$ defined by
$(R(m)f)(m') = f(m'm) \quad (f \in C^\infty(M);\ m, m' \in M)$;

(ii) $L: M \to GL(C^\infty(M))$ defined by
$(L(m)f)(m') = f(m^{-1}m')$.

There are induced representations r: $\tilde{\mathfrak{m}}^0 \to gl(C^\infty(M))$ and l: $\tilde{\mathfrak{m}}^0 \to gl(C^\infty(M))$. In the fibering $V \to M \xrightarrow{\pi} M/V$, in order that $f \in C^\infty(M)$ be of the form $\tilde{f} \circ \pi$ ($\tilde{f} \in C^\infty(M/V)$), it is necessary and sufficient that f be constant along the fibres. This is expressed analytically by $R(v)f = f$ ($v \in V$) or, since V is connected, $r(v)f = 0$ ($v \in \tilde{\mathfrak{v}}^0$). Thus $C^\infty(M/V) \cong$ (as a vector space) $(C^\infty(M))^{\tilde{\mathfrak{v}}^0} = \{f \in C^\infty(M) : r(v)f = 0, v \in \tilde{\mathfrak{v}}^0\}$. Since M acts on $C^\infty(M/V)$ by $\pi \circ L$, the Frobenius reciprocity law together with the Peter-Weyl decomposition of $C^\infty(M)$ gives (1.4) for $\tau = 0$ ($=$ trivial representation).

In general, we have the fibre bundle diagram

$$\begin{array}{ccc} E^\tau \times M & \to & \mathbf{E}^\tau \\ \downarrow & & \downarrow \\ M & \xrightarrow{\pi} & M/V \end{array}$$

(Proposition 5.1 below) and the C^∞ cross-sections of $E^\tau \times M$ are given by $C^\infty(M) \otimes E^\tau$. The same argument as given above shows that, as vector spaces,

$$\Delta(\mathbf{E}^\tau) \cong (C^\infty(M) \otimes E^\tau)^{\tilde{\mathfrak{v}}}.$$

Applying the Peter-Weyl theorem again gives (1.4). Q.E.D.

COROLLARY. *Let $\mathfrak{k}^{\#} \subseteq \mathfrak{k}$ and $E^{\varrho} \to \mathbf{E}^{\varrho} \to X$ be as given above the statement of Theorem 0. Then*

$$\Delta(\mathbf{E}^{\varrho} \otimes \Lambda^{q}(T^{\#})') \sim \sum_{\lambda \in D(M)} V^{\lambda} \otimes (E^{\varrho} \otimes \Lambda^{q}(\mathfrak{k}^{\#})' \otimes V^{-\lambda})^{\tilde{\mathfrak{v}}^{0}}.$$

Let now $X = G/U$ (complex form) $= M/V$ (compact form) be a C-space. We may write $\tilde{\mathfrak{m}}^{0} = \tilde{\mathfrak{v}}^{0} \oplus \mathfrak{n} \oplus \mathfrak{n}^{*}$ and since $[\tilde{\mathfrak{v}}^{0}, \mathfrak{n}] \subseteq \mathfrak{n}$, $[\tilde{\mathfrak{v}}^{0}, \mathfrak{n}^{*}] \subseteq \mathfrak{n}^{*}$, we may write

$$\Lambda^{r}(\widetilde{T(X)})' = \sum_{p+q=r} (\Lambda^{p}(\mathfrak{n}^{*}_{\mathrm{Ad}})') \otimes (\Lambda^{q}(\mathfrak{n}_{\mathrm{Ad}})');$$

this is simply the decomposition of the complex r-forms on X into type components.(1) If σ: $U \to GL(E^{\sigma})$ gives a holomorphic bundle $E^{\sigma} \to \mathbf{E}^{\sigma} \to X$, then it is a priori a differentiable homogeneous bundle and we may apply (1.4) to conclude that

$$\Delta(\mathbf{E}^{\sigma} \otimes \Lambda^{q}(\mathfrak{n}_{\mathrm{Ad}})') = \Delta(\mathbf{E}^{\sigma} \otimes \Lambda^{q}(\overline{L(X)}')) = \sum_{\lambda \in D(M)} V^{\lambda} \otimes (E^{\sigma} \Lambda^{q}(\mathfrak{n})' \otimes V^{-\lambda})^{\tilde{\mathfrak{v}}^{0}}. \tag{1.5}$$

Here we write the bundle of $(0, q)$ vectors on X as $\Lambda^{q}(\mathfrak{n}_{\mathrm{Ad}})'$ or $\Lambda^{q}(\overline{L(X)}')$ (using the decomposition $\widetilde{T(X} \cong L(X) \oplus \overline{L(X)})$).

We have $\tilde{\mathfrak{g}}^{0} \cong \mathfrak{g}_{h} \oplus \mathfrak{g}_{\bar{h}}$, $\tilde{\mathfrak{u}}^{0} = \mathfrak{u}_{h} \oplus \mathfrak{u}_{\bar{h}}$ and, in D.G. § 2, Definition (2.4), we described explicitly an isomorphism $\varrho : \tilde{\mathfrak{m}}^{0} \to \mathfrak{g}_{h}$; it is easily checked that $\varrho(\mathfrak{n}) \subseteq \mathfrak{u}_{h}$ and thus $\mathfrak{n}$ acts on E^{σ} by $\sigma \circ \varrho$ or just σ. Thus the expressions $C^{q}(\mathfrak{n}, E^{\sigma} \otimes V^{-\lambda})$ and $C^{q}(\mathfrak{n}, E^{\sigma} \otimes V^{-\lambda})^{\tilde{\mathfrak{v}}^{0}}$ as defined in the sense of Lie algebra cohomology make sense and

$$C^{q}(\mathfrak{n}, E^{\sigma} \otimes V^{-\lambda})^{\tilde{\mathfrak{v}}^{0}} \cong (\Lambda^{q}(\mathfrak{n})' \otimes E^{\sigma} \otimes V^{-\lambda})^{\tilde{\mathfrak{v}}^{0}}. \tag{1.6}$$

Thus the cohomology module

$$H^{q}(\mathfrak{n}, E^{\sigma} \otimes V^{-\lambda})^{\tilde{\mathfrak{v}}^{0}} \tag{1.7}$$

is well-defined.

On the other hand, we have a well-defined mapping

$$\bar{\partial} : \Delta(\Lambda^{q}(\overline{L(X)}')) \otimes \mathbf{E}^{\sigma} \to \Delta(\Lambda^{q+1}(\overline{L(X)}') \otimes \mathbf{E}^{\sigma}),$$

$\bar{\partial}^{2} = 0$, and the cohomology groups are simply the Dolbeault groups $H^{0,q}(X, \mathbf{E}^{\varrho})$ (see [14]); the Dolbeault isomorphism reads: $H^{0,q}(X, \mathbf{E}^{\varrho}) \cong H^{q}(X, \mathcal{E}^{\varrho})$. A calculation in local coordinates gives the following commutative diagram:

$$\begin{array}{ccc} C^{q}(\mathfrak{n}, E^{\sigma} \otimes V^{-\lambda})^{\tilde{\mathfrak{v}}^{0}} & \xrightarrow{d} & C^{q+1}(\mathfrak{n}, E^{\sigma} \otimes V^{-\lambda})^{\tilde{\mathfrak{v}}^{0}}, \\ \downarrow & & \downarrow \\ C^{0,q}(X, \mathbf{E}^{\sigma}) & \xrightarrow{\bar{\partial}} & C^{0,q+1}(X, \mathbf{E}^{\sigma}), \end{array}$$

(1) For convenience, we write $\mathfrak{n} = \bigoplus_{\alpha \in \Sigma^{-} - \Psi^{-}} \mathfrak{v}_{-\alpha} \oplus \mathfrak{o}$.

where $C^{\sigma,q}(X,\mathbf{E}^\sigma)=\Delta(\Lambda^q(\overline{L(X)'}))\otimes\mathbf{E}^\sigma$ and we thus get the M-isomorphisms

$$C^{0,q}(X,\mathbf{E}^\sigma)\sim\sum_{\lambda\in D(\mathfrak{g})}V^\lambda\otimes C^q(\mathfrak{n},E^\sigma\otimes V^{-\lambda})^{\tilde{\mathfrak{v}}^\sigma}, \tag{1.8}$$

$$H^{0,q}(X,\mathbf{E}^\sigma)\cong\sum_{\lambda\in D(\mathfrak{g})}V^\lambda\otimes H^q(\mathfrak{n},E^\sigma\otimes V^{-\lambda})^{\tilde{\mathfrak{v}}^\sigma}. \tag{1.9}$$

((1.9) is equation (1.6) in [5].)

We return now to the situation described at the end of section (iii) of this §; $X=G/U$ where U is solvable and $T^{2a}\to X\to\hat{X}=G/\hat{U}$ is the fundamental fibering. Referring to (1.9), we have that

$$H^0(X,\mathcal{E}^\varrho)=\sum_{\lambda\in D(\mathfrak{g})}V^\lambda\otimes(E^\sigma\otimes V^{-\lambda})^{\mathfrak{u}} \tag{1.10}$$

(σ is now 1-dimensional) and since $\mathfrak{u}\supseteq\mathfrak{c}(e_{-\alpha}\colon\ \alpha\in\Sigma^+)$, we conclude (as in [5], § 4) using (1.10) the following:

PROPOSITION. *$H^0(X,\mathcal{E}^\varrho)=0$ unless $\varrho\in D(\mathfrak{g})$ in which case*

$$H^0(X,\mathcal{E}^\varrho)\cong V^\varrho\quad(\textit{as } M\textit{-modules}) \tag{1.11}$$

and

$$H^q(X,\mathcal{E}^\varrho)=0\quad(q>a). \tag{1.12}$$

Remark. The complete proof of this Proposition was given in D.G., only for $a=0$. (However, this "vanishing theorem" for arbitrary a is true for general compact complex manifolds.) Thus, in order to have completeness, we shall use (1.12) only when $a=0$. The general statement would allow us to assimilate §§ 2 and 5 into a single theorem.

2. Homogeneous Bundles Defined by an Irreducible Representation

In this section we shall determine the M-module structure of $H^*(G/U,\mathcal{E}^\varrho)$ when ϱ is irreducible and G/U is Kähler. These results, for $H^0(G/U,\mathcal{E}^\varrho)$ are due to Borel-Weil [4] and for $H^q(G/U,\mathcal{E}^\varrho)$ $(q>0)$ to Bott [5]. Also the same result has been obtained in a purely algebraic manner by Kostant [21]. Our method uses (1.11) and (1.12) above together with a spectral sequence in Lie algebra cohomology. In [5] the Leray spectral sequence (which is *not* the geometric counterpart of the spectral sequence given here) was used, however for us the use of the Lie algebra spectral sequence has two advantages. First, the spectral sequence used here carries the M-module structure of $H^*(G/U,\mathcal{E}^\varrho)$ (for arbitrary G/U and ϱ) allong with it and secondly, and more important, this same spectral sequence allows us to obtain information when G/U is non-Kähler and/or ϱ is not completely reducible. In fact, by successively applying the same spectral sequence in Lie algebra cohomology, we obtain (i) the main theorem in [5], (ii) the M-module structure of $H^*(G/U, \mathcal{E}^\varrho)$ when ϱ is irreducible and G/U is non-Kähler, and (iii) information on the M-module $H^*(G/U, \mathcal{E}^\varrho)$ when G/U and ϱ are both arbitrary.

Let now $X = G/U = M/V$ be Kähler and let $\varrho: U \to GL(E^\varrho)$ be irreducible. We observe that since $\mathfrak{u} = \mathfrak{n} \oplus \tilde{\mathfrak{v}}^0$ where $\mathfrak{n}$ is a nilpotent ideal, $\varrho|\mathfrak{n} = 0$ and thus ϱ is essentially the complexification of an irreducible representation of $\tilde{\mathfrak{v}}^0$. For each $\sigma \in W(\mathfrak{g})$, we define a mapping I_σ: $Z(\mathfrak{g}) \to Z(\mathfrak{g})$ by

$$I_\sigma(\lambda) = \sigma(\lambda + g) - g \quad (\text{for } \lambda \in Z(\mathfrak{g})). \tag{2.1}$$

Furthermore, define $I: D(\tilde{\mathfrak{v}}^0) \to D(\mathfrak{g}) \cup \{0\}$ (0 giving the zero M-module) as follows: if $\lambda + g$ is singular in $\mathfrak{h}^\#$, $I(\lambda + g) = 0$; if $\lambda + g$ is regular, there exists a unique $\sigma \in W(\mathfrak{g})$ such that $\sigma(\lambda + g) \in D^0(\mathfrak{g})$ and we define

$$I(\lambda) = I_\sigma(\lambda) \in D(\mathfrak{g}) \tag{2.2}$$

(g is a "minimal" element in $D(\mathfrak{g})$). Finally, we recall that the *index* $|\eta|$ of $\eta \in \mathfrak{h}^\#$ is defined to be the number of roots $\varphi \in \Sigma^+$ such that $(\eta, \varphi) < 0$. If $\sigma \in W(\mathfrak{g})$, we define the index $|\sigma|$ of σ as follows: $|\sigma|$ = cardinality of the set $\sigma(\Sigma^+) \cap \Sigma^-$ = number of roots which "change sign" under σ (recall that $\sigma(\Sigma) = \Sigma$). The connection between these two is the following: if η is regular, there exists a unique $\sigma_\eta \in W(\mathfrak{g})$ such that $\sigma_\eta(\eta) \in D^0(\mathfrak{g})$ and then $|\eta| = |\sigma_\eta|$. Finally, if $\lambda \in D(\mathfrak{g})$ ($\varrho \in D(\tilde{\mathfrak{v}}^0)$), we denote by $V^\lambda(E^\varrho)$ the irreducible representation space for the irreducible representation of $M(V)$ with highest weight $\lambda(\varrho)$.

THEOREM B [5]. *To each V-module E^ϱ, there is associated an irreducible M-module $H^*(X, \mathcal{E}_\varrho)$.*

This transformation takes irreducible V-modules into irreducible M-modules and the transformation of $D(\tilde{\mathfrak{v}}^0)$ into $D(\mathfrak{g}) \cup \{0\}$ is simply I given in (2.2). Thus $H^q(X, \mathcal{E}^\varrho) \neq 0$ for at most one q and in fact $q = |\varrho + g| = |\sigma|$ where $I(\varrho) = I_\sigma(\varrho)$ or $H^*(X, \mathcal{E}^\varrho) = 0$ if $\varrho + g$ is singular. Restated:

$$\left.\begin{aligned} &V^\lambda \otimes H^q(\mathfrak{n}, E^\varrho \otimes V^{-\lambda})^{\tilde{\mathfrak{v}}^0} = 0 \quad q \neq |\sigma| \text{ or } \lambda \neq I(\varrho), \\ &V^{I(\varrho)} \otimes H^{|\sigma|}(\mathfrak{n}, E^\varrho \otimes V^{-I(\varrho)})^{\tilde{\mathfrak{v}}^0} = V^{I(\varrho)}. \end{aligned}\right\} \tag{2.3}$$

Now we turn Theorem B around. Since ϱ is irreducible, $\mathfrak{n} \circ E^\varrho = 0$ and $H^q(\mathfrak{n}, E^\varrho \otimes V^{-\lambda}) = H^q(\mathfrak{n}, V^{-\lambda}) \otimes E^\varrho$. On the other hand, if $\lambda = I(\varrho) = \sigma(\varrho + g) - g$, $\sigma^{-1}(\lambda + g) - g = \varrho$ and applying Schur's lemma to (2.3), we have

THEOREM K [21]. *As a* $\tilde{\mathfrak{v}}^0$*-module,*

$$H^q(\mathfrak{n}, V^{-\lambda}) = \sum_{\sigma \in \{W(\mathfrak{g})/W(\mathfrak{v})^0\}^q} V^{q, \lambda(\sigma)}, \tag{2.4}$$

where $V^{q,\lambda(\sigma)}$ *is the representation space for the irreducible representation of* V *with lowest weight* $-\{\sigma^{-1}(\lambda+g)-g\}$ *and* $\{W(\mathfrak{g})/W(\tilde{\mathfrak{v}}^0)\}^q = \{\sigma \in W(\mathfrak{g}) : |\sigma| = q$ *and* $\sigma^{-1}(D(\mathfrak{g})) \subseteq D(\tilde{\mathfrak{v}}^0)\}$.

Since (2.4) implies (2.3), it will suffice to prove either. Note that for a homogeneous line bundle $\mathbf{E}^\varrho$ where $\varrho \in D(\mathfrak{g})$, we have already proven (2.3). The spectral sequences used now were motivated by those in [5]. We proceed in a sequence of steps. First we treat line bundles over a flag manifold M/T.

(i) Let $E^\varrho \to \mathbf{E}^\varrho \to M/T$ be given by a character ϱ of T such that $\varrho + g \in D(\mathfrak{g})$. Then

$$\left.\begin{array}{l} H^q(X, \mathcal{E}^\varrho) = 0 \quad \text{if } q > 0 \text{ or } q = 0 \quad \text{and} \quad \varrho + g \text{ is singular,} \\ H^0(X, \mathcal{E}^\varrho) = V^\varrho \quad (\text{as an } M\text{-module}) \quad \varrho \in D(\mathfrak{g}), \end{array}\right\} \tag{2.5}$$

$$H^q(M/T, \mathcal{E}^\varrho) \quad \text{and} \quad H^{n-q}(M/T, \mathcal{E}^{-\varrho} \otimes \mathcal{E}^{-2g}) \tag{2.6}$$

are dual M-modules where $n = \dim_{\mathbf{C}} M/T$. This is just Serre duality where $\mathbf{K} = \mathbf{E}^{-2g}$.

$$H^q(M/T, \mathcal{E}^\varrho) \cong H^{n+q}(M/T, \mathcal{E}^{I_\delta(\varrho)}) \tag{2.7}$$

as an M-module if $\varrho + g \in D(\mathfrak{g})$. Indeed, since

$$-I_\delta(\varrho) - 2g = -\delta(\varrho - 2g + g) - g = -\delta(\varrho), \ H^n(M/T, \mathcal{E}^{I_\delta(\varrho)})$$

is dual to $H^0(M/T, \mathcal{E}^{-\delta(\varrho)})$ which in turn is dual to $H^0(M/T, \mathcal{E}^\varrho)$. Now use (2.5):

$$(2.3) \text{ is true for } M/T \Leftrightarrow \varrho + g \in D(\mathfrak{g}) \Rightarrow H^q(M/T, \mathcal{E}^\varrho) \cong H^{q+|\sigma|}(M/T, \mathcal{E}^{I_\sigma(\varrho)}) \tag{2.8}$$

as M-modules.

If $\alpha \in \Sigma^+$, we set $D(\alpha) = \{\eta \in Z(\mathfrak{g}) : (\eta, \alpha) \geqslant 0\}$; then $D(\mathfrak{g}) = \bigcap_{\alpha \in \Sigma^+} D(\alpha)$ and (2.8) will be true if we can prove

$$\varrho \in D(\alpha_j) \Rightarrow H^q(M/T, \mathcal{E}^\varrho) \cong H^{q+1}(M/T, \mathcal{E}^{\tau_{\alpha_j}(\varrho)}) \tag{2.9}$$

as M-modules (here $\alpha_j \in \pi$). Indeed, we may write $\sigma^{-1} = \tau_{\alpha_{i_1}} \tau_{\alpha_{i_2}} \cdots \tau_{\alpha_{i_q}}$ (1) $(\alpha_{i_j} \in \pi)$ and if (2.9) holds, we may proceed inductively to (2.8) since $\tau_{\alpha_j}(\Sigma^+) \cap \Sigma^- = -\alpha_j$.

$$\text{If we consider } M/T \text{ where } \mathfrak{g} = v_{-\alpha} \oplus \mathfrak{h} \oplus v_\alpha \tag{2.10}$$

(1) Here $\tau_{\alpha_{i_j}} \in W(\mathfrak{g})$ is the reflection across the root plane of the simple root α_{i_j}.

(i.e., $\dim_{\mathbf{C}} \mathfrak{g} = 3$), then since $\delta = \tau_\alpha$, (2.7) $\Rightarrow$ (2.9) $\Rightarrow$ (2.8). In other words, Serre duality on $P_1(\mathbf{C})$ gives the theorem.

Remark. If $|\sigma| \leq 2$, (2.7) may be proven using the Nakano inequality only (Proposition 8.1 in D.G.).

(ii) For a flag manifold M/T, $\mathfrak{n} = \mathfrak{c}(e_{-\alpha} : \alpha \in \Sigma^+)$ and $\mathfrak{n}^* = \mathfrak{c}(e_\alpha : \alpha \in \Sigma^+)$. For $\alpha_i \in \prod$, we set $\mathfrak{n}_{\alpha_i} = \mathfrak{c}(e_\alpha : \alpha \in \Sigma^+ - \{\alpha_i\})$; then $\mathfrak{n}_{\alpha_i}$ is an ideal in $\mathfrak{n}$. Thus, for any $\mathfrak{n}$-module F, there exists a spectral sequence $\{E_r\}$ such that E_∞ is associated to $H^*(\mathfrak{n}, F)$ and $E_2^{p,q} = H^p(\mathfrak{n}/\mathfrak{n}_{\alpha_i}, H^q(\mathfrak{n}_{\alpha_i}, F))$. Using this spectral sequence and (2.10), we shall prove (2.9). Indeed, if $\varrho \in D(\alpha_i)$, we have $\mathfrak{n}$-modules $F_1 = V^{-\lambda} \otimes E^\varrho$ and $F_2 = V^{-\lambda} \otimes E^{\tau_{\alpha_i}(\varrho)}$ and to prove (2.9), we must show:

$$H^q(\mathfrak{n}, F_1)^{\mathfrak{h}} = H^{q+1}(\mathfrak{n}, F_2)^{\mathfrak{h}}. \tag{2.11}$$

There are two spectral sequences $\{^1E_r\}$ and $\{^2E_r\}$ corresponding to the $\mathfrak{n}$-modules F_1 and F_2. Here

$$^1E_2^{p,q} = H^p(\mathfrak{n}/\mathfrak{n}_{\alpha_i}, H^q(\mathfrak{n}_{\alpha_i}, F_1)) = H^p(\mathfrak{n}/\mathfrak{n}_{\alpha_i}, H^q(\mathfrak{n}_{\alpha_i}, V^{-\lambda}) \otimes E^\varrho).$$

If we set $\tilde{\mathfrak{v}}_{\alpha_i}^0 = v_{-\alpha_i} \oplus \mathfrak{h} \oplus v_{\alpha_i}$, then $H^q(\mathfrak{n}_{\alpha_i}, V^{-\lambda})$ is a completely reducible $\tilde{\mathfrak{v}}_{\alpha_i}^0$-module and we write

$$H^q(\mathfrak{n}_{\alpha_i}, V^{-\lambda}) = \sum_{\xi \in D(\alpha_i)} V_\xi^{-\lambda, q}$$

since $D(\tilde{\mathfrak{v}}_{\alpha_i}^0) = D(\alpha_i)$ Thus

$$^1E_2^{p,q} = \sum_{\xi \in D(\alpha_i)} H^p(\mathfrak{n}/\mathfrak{n}_{\alpha_i}, V_\xi^{-\lambda, q}) \otimes E^\varrho$$

and similarly,
$$^2E_2^{p,q} = \sum_{\xi \in D(\alpha_i)} H^p(\mathfrak{n}, \mathfrak{n}_{\alpha_i}, V_\xi^{-\lambda, q}) \otimes E^{\tau_{\alpha_i}(\varrho)}.$$

We may derive both spectral sequences throughout by $\mathfrak{h}$ to get new spectral sequences $\{^1E_r'\}$, $\{^2E_r'\}$ with $^1E_\infty'$ associated to $H^*(\mathfrak{n}, V^{-\lambda} \otimes E^\varrho)^{\mathfrak{h}}$, $^2E_\infty'$ associated to $H^*(\mathfrak{n}, V^{-\lambda} \otimes E^{\tau_{\alpha_i}(\varrho)})^{\mathfrak{h}}$. But

$$H^p(\mathfrak{n}/\mathfrak{n}_{\alpha_i}, V_\xi^{-\lambda, q} \otimes E^\varrho)^{\mathfrak{h}} = H^{p+1}(\mathfrak{n}/\mathfrak{n}_{\alpha_i}, V_\xi^{-\lambda, q} \otimes E^{\tau_{\alpha_i}(\varrho)})^{\mathfrak{h}} = 0$$

unless $p = 0$ by (2.10), Thus both spectral sequences are trivial and

$$H^q(\mathfrak{n}, V^{-\lambda} \otimes E^\varrho)^{\mathfrak{h}} = {}^1E_2'^{0,q} = {}^2E_2'^{1,q} = H^{q+1}(\mathfrak{n}, V^{-\lambda} \otimes E^{\tau_{\alpha_i}(\varrho)})^{\mathfrak{h}}$$

and Theorems B and K are established for flags.

(iii) For a general Kähler C-space M/V, we use the fiberings $V/T \to M/T \to M/V$; knowing the theorem for the total space and the fibre, we shall deduce it for the base space. Let $E^{\varrho} \to \mathbf{E}^{\varrho} \to M/V$ be given by an irreducible V-module E^{ϱ} with highest weight $\hat{\varrho}$; $E^{\hat{\varrho}}$ is the irreducible T-module with character $\hat{\varrho}$. In the notation of § 0 (iii), we set $\mathfrak{n} = \mathfrak{c}(e_{-\alpha} : \alpha \in \Sigma^{+})$, $\hat{\mathfrak{n}} = \mathfrak{c}(e_{-\alpha} : \alpha \in \Sigma^{+} - \Psi^{+})$ so that $\hat{\mathfrak{n}}$ is an ideal in $\mathfrak{n}$. Then, for an $\mathfrak{n}$-module F, there exists an $\{E_r\}$ with E_∞ associated to $H^*(\mathfrak{n}, F)$ and $E_2^{p,q} = H^p(\mathfrak{n}/\hat{\mathfrak{n}}, H^q(\hat{\mathfrak{n}}, F))$. If $F = V^{-\lambda}$, then

$$H^q(\hat{\mathfrak{n}}, V^{-\lambda}) = \sum_{\xi \in D(\tilde{\mathfrak{v}})^0} V_{\xi}^{-\lambda, q} \tag{2.12}$$

and $$H^p(\mathfrak{n}/\hat{\mathfrak{n}}, H^q(\hat{\mathfrak{n}}, V^{-\lambda}) = \sum_{\xi \in D(\tilde{\mathfrak{v}}^0)} H^p(\mathfrak{n}/\hat{\mathfrak{n}}, V_{\xi}^{-\lambda, q}) = \sum_{\xi \in D(\tilde{\mathfrak{v}}^0)} \sum_{\sigma \in \{W(\tilde{\mathfrak{v}}^0)\}^p} V_{p, \xi(\sigma)}^{-\lambda, q}$$

(applying Theorem K to $\mathfrak{n}/\hat{\mathfrak{n}}$, i.e. V/T). On the other hand,

$$H^q(\mathfrak{n}, V^{-\lambda}) = \sum_{\theta \in \{W(\mathfrak{g})\}^q} V^{q, \lambda(\theta)}$$

(applying Theorem K to M/T). Now we tensor the spectral sequence throughout by $E^{\hat{\varrho}}$ and derive by $\mathfrak{h}$ as above to get $\{E_r'\}$ where

$$\begin{aligned} E_2'^{p,q} &= \sum_{\xi \in D(\tilde{\mathfrak{v}}^0)} \sum_{\sigma \in \{W(\tilde{\mathfrak{v}}^0)\}^p} (V_{p, \xi(\sigma)}^{-\lambda, q} \otimes E^{\hat{\varrho}})^{\mathfrak{h}} \\ &= \begin{cases} (V_{0, -\hat{\varrho}}^{-\lambda, q} \otimes E^{\hat{\varrho}})^{\mathfrak{h}} & \text{if } p = 0 \\ 0 & \text{if } p \neq 0 \quad \text{since } \hat{\varrho} \in D(\tilde{\mathfrak{v}}^0). \end{cases} \end{aligned} \tag{2.13}$$

On the other hand,

$$\begin{aligned} H^q(\mathfrak{n}, V^{-\lambda} \otimes E^{\hat{\varrho}})^{\mathfrak{h}} &= \sum_{\theta \in \{W(\mathfrak{g})\}^q} (V^{q, \lambda(\theta)} \otimes E^{\hat{\varrho}})^{\mathfrak{h}} \\ &= \begin{cases} 0 & I_{\theta^{-1}}(\lambda) \neq \hat{\varrho} \quad \text{or } q \neq |\theta|, \\ (V^{q, \lambda(\theta)} \otimes E^{\hat{\varrho}})^{\mathfrak{h}} & \text{if } I_\theta(\hat{\varrho}) = \lambda, \ |\theta| = q. \end{cases} \end{aligned}$$

Thus the spectral sequence $\{E_r'\}$ is trivial and

$$\begin{cases} (V_{0, -\hat{\varrho}}^{-\lambda, q} \otimes E^{\varrho})^{\mathfrak{h}} \neq 0 & \text{if } I_\theta(\hat{\varrho}) = \lambda, \ |\theta| = q, \\ 0 & \text{otherwise}. \end{cases}$$

Then by Schur's lemma,

$$H^q(\hat{\mathfrak{n}}, V^{-\lambda}) = \sum_{\theta \in \{W(\mathfrak{g})/W(\tilde{\mathfrak{v}}^0)\}^q} V^{q, \lambda(\theta)}$$

which is just the statement of Theorem K.

Needless to say, the essential point in the above derivation was Schur's lemma which allows us to pass from the dimension of a vector space to the multiplicity of a representation.

3. Homogeneous Bundles Defined by Non-Irreducible Representations

In this section, we treat the question of determining the cohomology of a homogeneous vector bundle $E^{\varrho}\to\mathbf{E}^{\varrho}\to X$ over a Kähler C-space $X=G/U$ where $\varrho\colon U\to GL(E^{\varrho})$ may not be irreducible. There are several ways of doing this; one may use a spectral sequence in a fashion similar to above; this is a method slightly different from the one used here. We have chosen this one solely because of its applications.

THEOREM 1. *Let $E^{\varrho}\to\mathbf{E}^{\varrho}\to X=G/U=M/T$ be a homogeneous vector bundle over a flag manifold M/T. If the weights of $\varrho\colon \mathfrak{u}\to H(E^{\varrho})$ are $\lambda_1,\dots,\lambda_{n+1}$, each λ_i gives a homogeneous line bundle $\mathbf{E}^{\lambda_i}\to\mathbf{E}^{\lambda_i}\to M/T$. If any of the conditions α, β, γ given below are satisfied, then*

$$H^*(X,\mathcal{E}^{\varrho})=\sum_{j=1}^{n+1}H^*(X,\mathcal{E}^{\lambda_j})\quad (\textit{as } M\textit{-modules}) \tag{3.1}$$

(α) $\qquad I(\lambda_i)\neq I(\lambda_j)\quad (i\neq j)$

(β) $\qquad ||\lambda_i+g|-|\lambda_j+g||>2\quad (i\neq j)$

(γ) $\qquad |\lambda_i+g|=|\lambda_j+g|\quad (\textit{for all } i,j).$

Furthermore, $H^i(X,\mathcal{E}^{\varrho})=0$ if

(τ) $\qquad i<\min|\lambda_j+g|,\ i>\max|\lambda_j+g|\quad or\quad i\neq|\lambda_j+g|\quad (\textit{for all } j).$

Proof. We are considering the flag manifold $X=G/U=M/T$; $\mathfrak{u}$ is thus a (maximal) solvable subalgebra over an algebraically closed field and hence there is a simultaneous eigenvector e_1 for $\varrho(\mathfrak{u})$; i.e., $\varrho(u)=\lambda_1(u)\,e_1$ for all $u\in\mathfrak{u}$. Note that

(i) $\qquad \varrho([u,u'])\,e_1=\lambda_1(u)\,\lambda_1(u')\,e_1-\lambda_1(u')\,\lambda_1(u)\,e_1=0;$

(ii) $\qquad \varrho(u+u')\,e_1=\lambda_1(u)\,e_1+\lambda_1(u')\,e_1\quad u,u'\in\mathfrak{u};$

and thus, by restricting ϱ to $\mathfrak{h}$, λ_1 lies in $\mathfrak{h}'$. Set $E^{\varrho,1}=\mathbf{C}(e_1)=\mathfrak{c}(e_1)$ so that $\varrho(\mathfrak{u})\,E^{\varrho,1}\subseteq E^{\varrho,1}$.

LEMMA 3.1. *There exists an exact sequence of homogeneous vector bundles $0\to\mathbf{E}^{\varrho,1}\to\mathbf{E}^{\varrho}\to\mathbf{F}^{\varrho,1}\to 0$ over X.*

Proof. Let $F^{\varrho,1}=E^{\varrho}/E^{\varrho,1}$; then $F^{\varrho,1}$ is a $\mathfrak{u}$-module via ϱ and we may form $G\times_U E^{\varrho,1}=\mathbf{E}^{\varrho,1}$, $G\times_U E^{\varrho}=\mathbf{E}^{\varrho}$, and $G\times_U F^{\varrho,1}=\mathbf{F}^{\varrho,1}$. Exactness is easily verified by taking the obvious maps.

Now, still denoting by ϱ the induced representation on $F^{\varrho,1}$, we see that $\varrho(\mathfrak{u})\subseteq H(F^{\varrho,1},\mathbf{C})$ is a solvable subalgebra and there is a common eigenvector $e_2\in F^{\varrho,1}$; setting

$c(e_2) = E^{\varrho,2}$ and $F^{\varrho,2} = F^{\varrho,1}/E^{\varrho,2}$, we get as above the exact sequence of homogeneous vector bundles $0 \to \mathbf{E}^{\varrho,2} \to \mathbf{F}^{\varrho,1} \to \mathbf{F}^{\varrho,2} \to 0$. Continuing this process until $\dim \mathbf{F}^{\varrho,n} = 1$, we end up with

$$\left.\begin{array}{l} 0 \to \mathbf{E}^{\varrho,1} \to \mathbf{E}^{\varrho} \to \mathbf{F}^{\varrho,1} \to 0, \\ 0 \to \mathbf{E}^{\varrho,2} \to \mathbf{F}^{\varrho,1} \to \mathbf{F}^{\varrho,2} \to 0, \\ \vdots \\ 0 \to \mathbf{E}^{\varrho,n-1} \to \mathbf{F}^{\varrho,n-2} \to \mathbf{F}^{\varrho,n-1} \to 0, \\ 0 \to \mathbf{E}^{\varrho,n} \to \mathbf{F}^{\varrho,n-1} \to \mathbf{F}^{\varrho,n} \to 0. \end{array}\right\} \tag{3.2}$$

Now (i) above implies that

$$\varrho(e_\alpha)\, e_1 = \frac{1}{(\alpha, \alpha)}\, \varrho([h_\alpha, e_\alpha])\, e_1 = 0;$$

i.e., λ_1 is the weight of ϱ on $E^{\varrho,1}$ and thus $\mathbf{E}^{\varrho,1}$ is the homogeneous line bundle $\mathbf{E}^{\lambda_1}$. Similarly, $\mathbf{E}^{\varrho,2}, \ldots, \mathbf{E}^{\varrho,n}, \mathbf{F}^{\varrho,n}$ are the homogeneous line bundles $\mathbf{E}^{\lambda_2}, \mathbf{E}^{\lambda_{n+1}}$ where $\lambda_1, \ldots, \lambda_{n+1}$ are precisely the weights of the original representation ϱ. From the exact cohomology sequences (these being exact sequences of M-modules) we get

$$\left.\begin{array}{l} \ldots \to H^p(\mathcal{E}^{\lambda_1}) \to H^p(\mathcal{E}^{\varrho}) \to H^p(\mathcal{F}^{\varrho,1}) \to H^{p+1}(\mathcal{E}^{\lambda_1}) \to \ldots \\ \ldots \to H^p(\mathcal{E}^{\lambda_2}) \to H^p(\mathcal{F}^{\varrho,1}) \to H^p(\mathcal{F}^{\varrho,2}) \to H^{p+1}(\mathcal{E}^{\lambda_2}) \to \ldots \\ \vdots \\ \ldots \to H^p(\mathcal{E}^{\lambda_{n-1}}) \to H^p(\mathcal{F}^{\varrho,n-2}) \to H^p(\mathcal{F}^{\varrho,n-1}) \to H^{p+1}(\mathcal{E}^{\lambda_{n-1}}) \to \ldots \\ \ldots \to H^p(\mathcal{E}^{\lambda_n}) \to H^p(\mathcal{F}^{\varrho,n-1}) \to H^p(\mathcal{E}^{\lambda_{n+1}}) \to H^{p+1}(\mathcal{E}^{\lambda_n}) \to \ldots \end{array}\right\} \tag{3.3}$$

(α) If (α) is satisfied, then

$$H^*(\mathcal{F}^{\varrho,n-1}) = H^*(\mathcal{E}^{\lambda_n}) \oplus H^*(\mathcal{E}^{\lambda_{n+1}}).$$

Indeed, there are groups in $H^*(\mathcal{E}^{\lambda_i})$ in at most one dimension for each λ_i (Theorem B). Our assertion is clearly true unless $|\lambda_{n+1} + g| + 1 = |\lambda_n + g|$ in which case the non-trivial piece of the exact sequence is (setting $|\lambda_{n+1} + g| = p$)

$$0 \to H^p(\mathcal{F}^{\varrho,n-1}) \to V^{I(\lambda_{n+1})} \to V^{I(\lambda_n)} \to H^{p+1}(\mathcal{F}^{\varrho,n-1}) \to 0$$

and since $V^{I(\lambda_j)}$ are irreducible $\mathfrak{g}$-modules, we see that

$$H^p(\mathcal{F}^{\varrho,n-1}) = V^{I(\lambda_{n+1})}$$

$$H^{p+1}(\mathcal{F}^{\varrho,n-1}) = V^{I(\lambda_n)}$$

and we are done. This same reasoning allows us to proceed inductively up through the above system of exact sequences to get our conclusion.

β) The argument is similar where now our hypothesis serves to sever the exact sequences into disjoint sequences of the form $0 \to A \to B \to 0$.

(γ) The exact sequences become, reading from bottom to top and setting $p = |\lambda_i + g|$ (for all i)

$$\left.\begin{array}{l} 0 \to V^{I(\lambda_n)} \to H^p(\mathfrak{F}^{\varrho, n-1}) \to V^{I(\lambda_n+1)} \to 0 \\ 0 \to V^{I(\lambda_n-1)} \to H^p(\mathfrak{F}^{\varrho, n-2}) \to H^p(\mathfrak{F}^{\varrho, n-1}) \to 0 \\ \vdots \\ 0 \to V^{I(\lambda_2)} \to H^p(\mathfrak{F}^{\varrho,1}) \to H^p(\mathfrak{F}^{\varrho,2}) \to 0 \\ 0 \to V^{I(\lambda_1)} \to H^p(\mathcal{E}^{\varrho}) \to H^p(\mathfrak{F}^{\varrho,1}) \to 0 \end{array}\right\}$$

From this we again get Theorem 1.

(τ) If i is in the specified range, then

$$H^i(\mathfrak{F}^{\varrho, n-1}) = 0, \quad H^i(\mathfrak{F}^{\varrho, n-2}) = 0, \ldots, H^i(\mathfrak{F}^{\varrho,1}) = 0,$$

and $H^i(\mathcal{E}^{\varrho}) = 0$ which was required. Q.E.D.

There is one difference from the above discussion when we consider $E^{\varrho} \to \mathbf{E}^{\varrho} \to M/V = X = G/U$ where X is Kähler but where V may be non-abelian so that $\mathfrak{u}$ is not solvable. In this case, we consider the nilpotent radical $\mathfrak{n} = \mathfrak{c}(\mathfrak{e}_{-\alpha} : \alpha \in \Sigma^+ - \Psi^+) \subset \mathfrak{u}$. Then $\varrho(\mathfrak{n}) \subset gl(E^{\varrho})$ is nilpotent and annihilates some non-trivial subspace $E^{\varrho,1} \subset E^{\varrho}$ (i.e., $\varrho(n)\, e_1 = 0$ for all $n \in \mathfrak{n}$, $e_1 \in E^{\varrho,1}$). Since $\mathfrak{n}$ is an ideal in $\mathfrak{u}$, $\varrho(\mathfrak{u})\, E^{\varrho,1} \subset E^{\varrho,1}$; as above we have the sequence of U-modules

$$0 \to E^{\varrho,1} \to E^{\varrho} \to F^{\varrho,1} \to 0 \quad (F^{\varrho,1} = E^{\varrho}/E^{\varrho,1})$$

and the associated exact sequence of homogeneous vector bundles

$$0 \to \mathbf{E}^{\varrho,1} \to \mathbf{E}^{\varrho} \to \mathbf{F}^{\varrho,1} \to 0.$$

Here we assume that any $\mathfrak{u}$-module F such that $\mathfrak{n} \circ F = 0$ (i.e., $F^{\mathfrak{n}} = F$) is a semi-simple $\tilde{\mathfrak{v}}^0$-module. Continuing the above process, we end up with a semi-simple $\tilde{\mathfrak{v}}^0$-module $F^{\varrho, n}$ described by the following sequences:

$$\left.\begin{array}{l} 0 \to \mathbf{E}^{\varrho,1} \to \mathbf{E}^{\varrho} \to \mathbf{F}^{\varrho,1} \to 0 \\ 0 \to \mathbf{E}^{\varrho,2} \to \mathbf{F}^{\varrho,1} \to \mathbf{F}^{\varrho,2} \to 0 \\ \vdots \\ 0 \to \mathbf{E}^{\varrho,n} \to \mathbf{F}^{\varrho,n-1} \to \mathbf{F}^{\varrho,n} \to 0. \end{array}\right\} \tag{3.4}$$

Now $E^{\varrho, j}$ $(j = 1, \ldots, n)$ and $F^{\varrho, n} = E^{\varrho, n+1}$ are by assumption semi-simple $\tilde{\mathfrak{v}}^0$-modules, and by theorem B we know $H^*(X, \mathcal{E}^{\varrho, j})$ $(j = 1, \ldots, n+1)$ as M-modules. By reasoning as in Theorem 1, we may derive information on the modules $H^*(X, \mathcal{E}^\varrho)$. Thus, letting $\varrho_1, \ldots, \varrho_m$ be the weights of ϱ, if

(α') $\quad I(\varrho_i) \neq I(\varrho_j) \quad (i \neq j)$ or

(β') $\quad \big||\varrho_i + g| - |\varrho_j + g|\big| > 2 \quad (i \neq j)$ or

(γ') $\quad |\varrho_i + g| = |\varrho_j + g|$ (for all i, j) or

(τ') $\quad i < \min |\varrho_j + g|$, $i > \max |\varrho_j + g|$, or $i \neq |\varrho_j + g|$ (all j),

then
$$H^*(X, \mathcal{E}^\varrho) = \bigoplus_{j=1}^{n+1} H^*(X, \mathcal{E}^{\varrho, j}). \tag{3.5}$$

4. Applications of §§ 2 and 3

We shall now give some applications of (3.1) and (3.5). In general, we shall use Theorem 1 to prove results for M/T and only make the statement of the corresponding result for M/V. In all cases, the proofs will be easy from (3.5).

We give a preliminary proposition which will be quite useful later.

Let X be a C-space G/U and let $\hat{U} \supset U$ be such that $G/\hat{U}$ is again a C-space (all groups involved are connected) and $\hat{U}/U$ is a homogeneous complex manifold. In general $\hat{U}/U$ may not have a finite fundamental group; however, it will be compact. There is the usual analytic fibre-space diagram:

$$\begin{array}{ccc} G & \xrightarrow{U} & G/U \\ & {\scriptstyle \hat{U}}\searrow \quad \swarrow {\scriptstyle \hat{U}/U} & \\ & G/\hat{U} & \end{array}$$

Suppose now that $\hat{\varrho}: \hat{U} \to GL(E^{\hat{\varrho}})$ is a holomorphic representation of $\hat{U}$; then $\hat{\varrho}\,|\,U = \varrho : U \to GL(E^\varrho) = GL(E^{\hat{\varrho}})$ is a holomorphic representation of U and we may form the homogeneous vector bundles

$$E^{\hat{\varrho}} \to \mathbf{E}^{\hat{\varrho}} \xrightarrow{\hat{\pi}} G/\hat{U}$$
$$E^\varrho \to \mathbf{E}^\varrho \to G/U.$$

On the other hand, denoting by σ the projection in the fibering $\hat{U}/U \to G/U \to G/\hat{U}$, we may form the analytic vector bundle $\sigma^{-1}(\mathbf{E}^{\hat{\varrho}})$ over

$$G/U: \begin{array}{ccc} \sigma^{-1}(\mathbf{E}^{\hat{\varrho}}) & \xrightarrow{\pi} & G/U \\ {\scriptstyle \sigma^{-1}}\uparrow & & \downarrow{\scriptstyle \sigma} \\ \mathbf{E}^{\hat{\varrho}} & \xrightarrow{\hat{\pi}} & G/\hat{U} \end{array} \tag{4.1}$$

PROPOSITION 4.1. *In the above notation, $\sigma^{-1}(\mathbf{E}^{\hat{\varrho}})$ is a homogeneous vector bundle and indeed $\sigma^{-1}(\mathbf{E}^{\hat{\varrho}}) = \mathbf{E}^{\varrho}$.*

Proof. We first recall the construction of $\sigma^{-1}(\mathbf{E}^{\hat{\varrho}})$. Setting $X = G/U$, $\hat{X} = G/\hat{U}$, consider the product $X \times \mathbf{E}^{\hat{\varrho}}$; then $\sigma^{-1}(\mathbf{E}^{\hat{\varrho}})$ $X \times \mathbf{E}^{\hat{\varrho}}$ consists of those pairs $(x, \hat{e})$ such that $\sigma(x) = \hat{\pi}(\hat{e})$. By defining $\pi(x, \hat{e}) = x$ we get a projection map $\pi : \sigma^{-1}(\mathbf{E}^{\hat{\varrho}}) \to X$ which gives rise to the analytic vector bundle $\sigma^{-1}(\mathbf{E}^{\hat{\varrho}})$. Writing points of $X(\hat{X})$ in the form gU (respectively $g\hat{U}$), the map σ is given by $\sigma(gU) = g\hat{U}$. On the other hand, we denote points of $\mathbf{E}^{\varrho}(\mathbf{E}^{\hat{\varrho}})$ by $[g, e]_{\varrho}$ $([g, e]_{\hat{\varrho}})$ where, by definition, $[g, e]_{\varrho} = [g', e']_{\varrho} \Leftrightarrow$ there exists $u \in U$ such that $g' = gu$, $e = \varrho(u)\, e'$ $([g, e]_{\hat{\varrho}} = [g', e']_{\hat{\varrho}} \Leftrightarrow$ there exists $\hat{u} \in \hat{U}$ such that $g' = g\hat{u}$, $e = \hat{\varrho}(\hat{u})\, e')$. With this clearly understood, $\sigma^{-1}(\mathbf{E}^{\hat{\varrho}})$ consists of those pairs $(gU, [\hat{g}, \hat{e}]_{\hat{\varrho}})$ such that $g\hat{U} = \hat{\pi}[\hat{g}, \hat{e}] = \hat{g}\hat{U}$. Thus in order that $(gU, [\hat{g}, \hat{e}]_{\hat{\varrho}}) \in X \times \mathbf{E}^{\hat{\varrho}}$ lie in $\sigma^{-1}(\mathbf{E}^{\hat{\varrho}})$, it is necessary and sufficient that there exist a $\hat{u} \in \hat{U}$ such that $g\hat{u} = \hat{g}$. We define a mapping $f : \mathbf{E}^{\varrho} \to \sigma^{-1}(\mathbf{E}^{\hat{\varrho}})$ by $f([g, e]_{\varrho}) = (gU, [g, e]_{\hat{\varrho}})$; f is thus a mapping of $\mathbf{E}^{\varrho}$ into $X \times \mathbf{E}^{\varrho}$ whose image clearly lies in $\sigma^{-1}(\mathbf{E}^{\hat{\varrho}})$.

(i) f is surjective: indeed let $(gU, [\hat{g}, \hat{e}]_{\hat{\varrho}})$ lie in $\sigma^{-1}(\mathbf{E}^{\hat{\varrho}})$; then there exists $\hat{u} \in \hat{U}$ such that $g\hat{u} = \hat{g}$ and since $[\hat{g}, \hat{e}]_{\hat{\varrho}} = [g, \varrho(\hat{u}^{-1})\, \hat{e}]_{\hat{\varrho}}$, $(gU, [\hat{g}, \hat{e}]_{\hat{\varrho}}) = (gU, [g, \varrho(\hat{u}^{-1})\, \hat{e}]_{\hat{\varrho}}) = f([g, \varrho(\hat{u}^{-1})\, \hat{e}]_{\varrho})$.

(ii) f is injective: suppose that $f([g, e]_{\varrho}) = (gU, [g, e]_{\varrho}) = (g'U, [g', e']_{\varrho}) = f([g', e']_{\varrho})$; this implies first of all that $g = gu$ and hence $[g', e']_{\hat{\varrho}} = [g, \varrho(u^{-1})\, e']_{\hat{\varrho}} = [g, e]_{\hat{\varrho}}$ which in turn implies that $e = \varrho(u^{-1})\, e'$ and thus $[g', e']_{\varrho} = [g, e]_{\varrho}$. Q.E.D.

COROLLARY. *Let $X = G/U$ be an arbitrary C-space and let $\hat{\varrho} : G \to GL(E^{\hat{\varrho}})$ be a holomorphic representation. Then upon restricting $\hat{\varrho}$ to U we get a holomophic representation $\varrho : U \to GL(E^{\varrho})$ $(E^{\varrho} = E^{\hat{\varrho}})$ and the homogeneous vector bundle $E \to \mathbf{E}^{\varrho} \to G/U$ is analytically trivial.*

Proof. Take $\hat{U} = G$ in (4.1) and apply Proposition 4.1.

In the applications to be given, we shall need a property of the Weyl group $W(\mathfrak{g})$ which is found in [3]. For any subset $\Phi \subset \Sigma$, we set $\langle \Phi \rangle = \sum_{\varphi \in \Phi} \varphi$. If $\sigma \in W(\mathfrak{g})$, we set $\Phi_{\sigma} = \sigma(\Sigma^{-}) \cap \Sigma^{+}$; then it follows that

$$\sigma(g) = g - \langle \Phi_{\sigma} \rangle. \tag{4.2}$$

Thus, for example, if $\alpha \in \prod$, then $\tau_{\alpha}(g) = g - \alpha$ since $(g, \alpha) = \frac{1}{2}(\alpha, \alpha)$.

PROPOSITION 4.2. *If $\Phi \subset \Sigma^{-}$, then $\langle \Phi \rangle + g$ is M-regular $\Leftrightarrow \langle \Phi \rangle = -\langle \Phi_{\sigma} \rangle$ for some $\sigma \in W(\mathfrak{g})$ in which case $\Phi = \Phi_{\sigma}$. Thus, $I\langle \Phi \rangle = 0$ unless $\Phi = -\Phi_{\sigma}$ and then*

$$\left.\begin{array}{l}\text{(i)}\ \ I(-\langle\Phi_\sigma\rangle)=0\ \ (=0 \textit{ element in } \mathfrak{h}^{\#})\\ \text{(ii)}\ \ |-\langle\Phi_\sigma\rangle+g|=|\sigma|.\end{array}\right\} \tag{4.3}$$

In particular there exists a unique $\delta \in W(\mathfrak{g})$ *such that* $\Sigma^+ = \delta(\Sigma^-) \cap \Sigma^+ = \Phi_\delta$*; this is the same* δ *as discussed in § 1.*

We now give our first application. The exact sequence of U-modules

$$0 \to \mathfrak{u} \to \mathfrak{g} \to \mathfrak{g}/\mathfrak{u} \to 0 \tag{4.4}$$

gives an exact sequence of homogeneous vector bundles

$$0 \to \mathbf{L} \to \mathbf{Q} \to L \to 0, \tag{4.5}$$

where $L = L(X)$ is the holomorphic tangent bundle of X and $\mathbf{L} = \operatorname{End}(L) = \operatorname{Hom}(L, L)$ is the bundle of endomorphisms of L. This is the Atiyah sequence; see D.G., section 7.

PROPOSITION 4.3. *The bundle* $\mathbf{Q}$ *is analytically isomorphic to* $X \times \mathfrak{g}$ *(i.e.* $\mathbf{Q}$ *is analytically trivial).*

Proof. Corollary to Proposition 4.1.

Letting Ω = sheaf of germs of holomorphic functions on X and $\Theta = \mathfrak{L}$, we have from (4.5)

$$\left.\begin{array}{l}0 \to H^0(X, \mathfrak{L}) \to H^0(X, \Omega) \otimes \mathfrak{g} \to H^0(X, \Theta) \to H^1(X, \mathfrak{L})\\ \to \dots \to H^q(X, \mathfrak{L}) \to H^q(X, \Omega) \otimes \mathfrak{g} \to H^q(X, \Theta) \to \dots\end{array}\right\} \tag{4.6}$$

Now we assume that $X = G/U$ is Kähler; then we will see in Theorem 3 below that $H^q(X, \Omega) = 0$ $(q > 0)$. Thus we have from (4.6)

$$0 \to H^0(X, \mathfrak{L}) \to \mathfrak{g} \to H^0(X, \Theta) \to H^1(X, \mathfrak{L}) \to 0$$

and
$$H^q(X, \Theta) \cong H^{q+1}(X, \mathfrak{L}) \quad (q > 0) \tag{4.7}$$

THEOREM 2. $H^q(X, \mathfrak{L}) = 0$ *for all* q.

COROLLARY 1 (Bott). $H^q(X, \Theta) = 0$ $(q > 0)$.

COROLLARY 2. $H^0(X, \Theta) \cong \mathfrak{g}$.

Thus the connected component of the group of analytic automorphisms of X is G.

We prove Theorem 2 for M/T; the general case is the same. The weights of the U-module $\mathfrak{u}$ are the 0-weight with multiplicity l and the negative roots $\alpha \in \Sigma^-$.

Furthermore, if $\alpha \in \Sigma^-$, then $\alpha + g$ is M-regular $\Leftrightarrow \alpha = \langle \Phi_{\tau_{\alpha_j}} \rangle = -\alpha_j$ for some $\alpha_j \in \prod$ by Proposition 4.2. Thus for $\alpha \in \Sigma^-$, $|\alpha + g| \leqslant 1$ and τ) of Theorem 2 tells us that $H^q(X, \mathfrak{L}) = 0$ $(q \geqslant 2)$ which gives Corollary 1.

Referring now to the proof of Theorem 1, there are l bundles $\mathbf{E}^{\varrho, j}$ such that

$$\left.\begin{array}{ll} H^0(X, \mathcal{E}^{\varrho, j}) & \text{is the trivial one-dimensional } M\text{-module} \\ H^q(X, \mathcal{E}^{\varrho, j}) = 0 & (q > 0) \quad \text{(these are the 0-weights)} \end{array}\right\}$$

and there are l bundles $\mathbf{E}^{\varrho, k} = \mathbf{E}^{-\alpha_k}$ such that

$$\left.\begin{array}{ll} H^q(X, \mathcal{E}^{-\alpha_k}) = 0 & (q \neq 1) \quad \text{and} \\ H^1(X, \mathcal{E}^{-\alpha_k}) & \text{is the trivial one-dimensional } M\text{-module.} \end{array}\right\}$$

From this one checks without too much trouble that the coboundary maps applied to $H^0(X, \mathcal{E}^{\varrho, j})$ knock out the terms $H^1(X, \mathcal{E}^{-\alpha_k})$ (one looks into the exact sequences of Lie algebra cohomology modules). Thus $H^0(X, \mathfrak{L}) = H^1(X, \mathfrak{L}) = 0$. Q.E.D.(1)

From Theorem K, it follows that

$$\dim H^q(\mathfrak{n}, V^\lambda) = \{\text{number of } \sigma \in \{W(\mathfrak{g})/W(\mathfrak{v}^0)\}^q. \tag{4.7}$$

In particular, if $\mathfrak{n} = \mathfrak{c}(e_{-\alpha} : \alpha \in \Sigma^+)$, Bott observed the "strange equality"

$$\dim H^q(\mathfrak{n}, V^\lambda) = \dim H^{2q}(M/T, \mathbf{C}) = \{\text{number of } \sigma \in \{W(\mathfrak{g})\}^q\}. \tag{4.8}$$

We explain this inequality by applying Theorem 1 coupled with the Dolbeault Theorem (in the Kähler case). If $\Omega^s =$ sheaf of germs of $(s, 0)$-forms on X, then (see [14])

$$H^q(M/T, \mathbf{C}) = \sum_{r+s=q} H^r(M/T, \Omega^s). \tag{4.9}$$

Theorem 3. *$H^p(M/T, \Omega^q) = 0$ unless $p = q$ and*

$$\dim H^q(M/T, \Omega^q) = \dim H^{2q}(M/T, \mathbf{C}) = \{\textit{number of } \sigma \in \{W(\mathfrak{g})\}^q\}.$$

Proof. $\Lambda^q L'(X)$ is the homogeneous vector bundle derived from the U-module $\Lambda^q(\mathfrak{g}/\mathfrak{u})'$ (here the prime signifies contragredient action). There roots of $\Lambda^q(\mathfrak{g}/\mathfrak{u})'$ are the elements $\langle \Phi \rangle \in \mathfrak{h}^\#$ where $\Phi \subset \Sigma^-$ and Φ contains q roots. Thus in (3.2) the bundles $\mathbf{E}^{\varrho, j}$ are of the form $\mathbf{E}^{-\alpha_{j_1}} \otimes \mathbf{E}^{-\alpha_{j_2}} \otimes \ldots \otimes \mathbf{E}^{\alpha_{j_q}}$ where $\alpha_{j_k} \in \Sigma^+$ and $\alpha_{j_k} \neq \alpha_{j_l}$ $(k \neq l)$. For $\Phi = \{-\alpha_{j_1}, \ldots, -\alpha_{j_q}\}$, $\langle \Phi \rangle + g$ is regular $\Leftrightarrow \Phi = -\Phi_\sigma$ for some $\sigma \in \{W(\mathfrak{g})\}^q$ and then $|\langle -\Phi_\sigma \rangle + g| = q$. Thus (γ) of Theorem 1 is satisfied and we are done.

If $q = 1$, $H^1(X, \Omega^1) \cong H^2(X, \mathbf{C})$ and $\dim H^1(X, \Omega^1) = l = \operatorname{rank} \mathfrak{g}$. Thus the elements in π are paired to $H^2(M/T, \mathbf{C})$ and one checks that for $\alpha_j \in \prod$, $\alpha_j \to \sum_{\varphi \in \Sigma^+} (\alpha_j, \alpha)\, \omega^\alpha \wedge \bar{\omega}^\alpha$.

(1) We have proven that, for a flag magnifold G/U, G is the connected automorphism group. In general, we write $X = G/U$ where G is the connected automorphism group; G is semi-simple by what we have just shown.

From D.G., section 6, we see that $\sum_{\varphi \in \Sigma^+}(\alpha_j, \alpha)\,\omega^\alpha \wedge \bar{\omega}^\alpha = c_1(\mathbf{E}^\lambda)$, where $\lambda \in \mathfrak{h}^{\#}$ is defined by $\langle \lambda, h_\alpha \rangle = (\alpha_j, \alpha)$. Since (over $\mathbf{C}$) the 2-classes generate $H^*(M/T, \mathbf{C})$, $H^*(M/T, \mathbf{C})$ coincides with its characteristic sub-algebra.

Similar statements hold for an arbitrary Kähler C-space $X = M/V$.

THEOREM 3'. *$H^p(M/V, \Omega^q) = 0$ unless $p = q$ and $H^q(M/V, \Omega^q) \cong H^2(M/V, \mathbf{C})$ and*

$$\dim H^q(M/V, \Omega^q) = \{\text{number of } \sigma \in \{W(\mathfrak{g})/W(\tilde{\mathfrak{v}}^0)\}^q\}.$$

COROLLARY. *$\chi(M/T) =$ order of $W(\mathfrak{g})$, $\chi(M/V) =$ order $W(\mathfrak{g})/$order $W(\tilde{\mathfrak{v}}^0)$.*

5. Homogeneous Bundles in the Non-Kähler Case

We shall now obtain the M-modules $H^q(X, \mathcal{E}^\varrho)$ where $X = G/U$ is a non-Kähler C-space and ϱ: $U \to GL(E^\varrho)$ is a holomorphic representation. We shall do this first when ϱ is irreducible and then proceed as in § 3. To a non-Kähler C-space $X = G/U = M/V$, we may associate a Kähler C-space $\hat{X} = G/\hat{U} = M/\hat{V}$ by increasing V to the full centralizer of a torus. We then have a *fundamental fibering* $T^{2a} \to X \to \hat{X}$ where T^{2a} is a complex a-torus. To this fibering, we shall apply the spectral sequence of § 2. Indeed, our calculations will be based upon the following:

LEMMA 5.1. *Let $\mathfrak{g}$ be a complex Lie algebra and $\mathfrak{e} \subset \mathfrak{g}$ be a complex sub-algebra. Suppose that we are given an ideal $\mathfrak{f} \subset \mathfrak{e}$ such that $\mathfrak{e} = \mathfrak{f} \oplus \mathfrak{b}$ where $\mathfrak{b}$ is a sub-algebra. Furthermore let $\mathfrak{a} \subset \mathfrak{g}$ be a sub-algebra such that $[\mathfrak{a}, \mathfrak{f}] \subseteq \mathfrak{f}$, $[\mathfrak{a}, \mathfrak{e}] \subseteq \mathfrak{e}$ and this latter action is reductive. Finally let M be a $\mathfrak{c}(\mathfrak{a} \cup \mathfrak{e})$-module which is a semi-simple $\mathfrak{a}$-module. Then there exists a spectral sequence $\{'E_r\}$ such that*

$$\left.\begin{aligned} &\text{(i) } {}'E_\infty \text{ is associated to } H^*(\mathfrak{e}, M)^{\mathfrak{a}}, \\ &\text{(ii) } {}'E_2^{p,q} \cong H^p(\mathfrak{b}, H^q(\mathfrak{f}, M))^{\mathfrak{a}}. \end{aligned}\right\} \tag{5.1}$$

Proof. There exists a spectral sequence $\{E_r\}$ such that E_∞ is associated to $H^*(\mathfrak{e}, M)$ and $E_2^{p,q} \cong H^p(\mathfrak{b}, H^q(\mathfrak{f}, M))$. Now by assumption, the action of $\mathfrak{a}$ on the chain group $C(\mathfrak{e}, M)$ is semi-simple as is the action of $\mathfrak{a}$ on $C(\mathfrak{b}, C(\mathfrak{f}, M))$; hence the process of taking $\mathfrak{a}$-invariant factors commutes with derivations. Q.E.D.

Let $E^\varrho \to \mathbf{E}^\varrho \to G/U$ be a homogeneous vector bundle defined by an irreducible U-module E^ϱ. We denote by $\hat{}$ everything associated to $\hat{X}$; e.g., $\hat{\mathfrak{n}} = \mathfrak{c}(e_{-\alpha} : \alpha \in \Sigma^+ - \Psi^+)$ and $\mathfrak{n} = \hat{\mathfrak{n}} \oplus \mathfrak{p}$ (c.f. section 1). By irreducibility, $\hat{\mathfrak{n}} \circ E^\varrho = 0$ and E^ϱ is an irreducible $\tilde{\mathfrak{v}}^0 \oplus \mathfrak{p}$-module. By Schur's lemma, $\varrho(p)e = \gamma(p)e$ for $p \in \mathfrak{p}$, $e \in E^\varrho$, and some $\gamma \in \mathfrak{p}'$, E^ϱ is an irreducible $\tilde{\mathfrak{v}}^0$-module. We must describe briefly this representation $\varrho : \mathfrak{u} \to gl(E^\varrho)$.

Now $\mathfrak{p} \oplus \mathfrak{c} \oplus \mathfrak{h}_\psi$(1) is a maximal abelian sub-algebra of $\mathfrak{u}$ and upon restricting ϱ to $\mathfrak{p} \oplus \mathfrak{c} \oplus \mathfrak{h}_\psi$, we get a decomposition of E^ϱ into "weight" spaces; $E^\varrho = \oplus_\lambda E^\varrho_\lambda$ where the λ's are linear forms on $\mathfrak{p} \oplus \mathfrak{c} \oplus \mathfrak{h}_\psi = \mathfrak{h} \cap \mathfrak{u}$. One of two things happens:

(i) Each λ such that $E^\varrho_\lambda \neq 0$ is the restriction to $\mathfrak{p} \oplus \mathfrak{c} \oplus \mathfrak{h}_\psi$ of a unique weight form $\hat{\lambda}$ on $\bar{\mathfrak{p}} \oplus \mathfrak{p} \oplus \mathfrak{c} \oplus \mathfrak{h}_\psi = \mathfrak{h}$;

(ii) Some λ for which $E^\varrho_\lambda \neq 0$ is not the restriction of a weight form on $\mathfrak{h}$.

Clearly (i) is a necessary and sufficient condition that ϱ on $\mathfrak{u}$ be the restriction of a representation $\hat{\varrho}$ on $\hat{\mathfrak{u}}$ arising from $\hat{\varrho}: \hat{U} \to GL(E^{\hat{\varrho}})$, $E^{\hat{\varrho}} = E^\varrho$. In this case if E^ϱ is an irreducible $\mathfrak{u}$-module, $E^{\hat{\varrho}}$ is an irreducible $\hat{\mathfrak{u}}$-module. Case (i) is, by Proposition 4.1, the group theoretic analogue of the geometrical situation described by the following diagram of analytic fibrations:

$$\begin{array}{ccc} \mathbf{E}^\varrho & \to & G/U \\ \uparrow \sigma^{-1} & & \downarrow \sigma \\ \mathbf{E}^{\hat{\varrho}} & \to & G/\hat{U} \end{array}$$

Definition. If ϱ satisfies (i), we call ϱ a *rational* representation; in the alternative case satisfies (ii) and we call ϱ an *irrational* representation.

THEOREM 4. *Let $E^\varrho \to \mathbf{E}^\varrho \to X$ be a homogeneous vector bundle over a non-Kähler C-space $X = G/U$ given by an irreducible U-module E^ϱ. Then*:

(i) *If ϱ is irrational, $H^*(X, \mathcal{E}^\varrho) = 0$;*

(ii) *If ϱ is rational, then as above we have $E^{\hat{\varrho}} \to \mathbf{E}^{\hat{\varrho}} \to \hat{X}$ and, if $\tilde{\hat{\varrho}} \in Z(\mathfrak{h})$ is the highest weight of $\hat{\varrho}$ and $I(\tilde{\hat{\varrho}}) = I_\sigma(\tilde{\hat{\varrho}})$, $|\sigma| = q$, then as M-modules, $H^{p+q}(X, \mathcal{E}^\varrho) \cong H^p(\hat{U}/U, \Omega) \otimes V^{I(\tilde{\hat{\varrho}})}$ where $H^p(\hat{U}/U, \Omega)$ is acted upon trivially by M and by convention $H^p(\hat{U}/U, \Omega) = 0$ for $p < 0$.*

Proof. To determine $H^*(X, \mathcal{E}^\varrho)$ as an M-module, it will suffice to know $H^*(\mathfrak{n}, V^{-\lambda} \otimes E^\varrho)^{\tilde{\mathfrak{v}}^0}$ since we are assuming that E^ϱ is a semi-simple $\tilde{\mathfrak{v}}^0$-module. But $V^{-\lambda}$, as the representation space of the compact group M and hence of the compact subgroup V, is a completely irreducible $\tilde{\mathfrak{v}}^0$-module; since $\mathfrak{n}$ is also a semi-simple $\tilde{\mathfrak{v}}^0$-module, the conditions of Lemma 5.1 are met by taking $\mathfrak{g} = \mathfrak{g}$, $\mathfrak{e} = \mathfrak{n}$, $\mathfrak{f} = \hat{\mathfrak{n}}$, $\mathfrak{b} = \mathfrak{p}$, $\mathfrak{a} = \tilde{\mathfrak{v}}^0$, and $M = V^{-\lambda} \otimes E^\varrho$. Thus there is a spectral sequence $\{'E_r\}$ such that

(i) $'E_\infty$ is associated to $H^*(\mathfrak{n}, V^{-\lambda} \otimes E^\varrho)^{\tilde{\mathfrak{v}}^0}$,

(ii) $'E_2^{p,q} = H^p(\mathfrak{p}, H^q(\hat{\mathfrak{n}}, V^{-\lambda} \otimes E^\varrho))^{\tilde{\mathfrak{v}}^0}$.

(1) Here we write $\mathfrak{h}_\gamma = \mathfrak{c} \oplus \mathfrak{h}_\psi$ where $\mathfrak{c}$ = center of $\tilde{\mathfrak{v}}^0$ and $\mathfrak{h}_\psi$ is the complement of $\mathfrak{c}$ under the Cartan-Killing form.

Returning to the spectral sequence (5.1) with $'E_\infty$ associated to $H^*(\mathfrak{n}, V^{-\lambda}\otimes E^\varrho)^{\tilde{\mathfrak{v}}^o}$ and whose $'E_2$ term is given by $'E_2^{p,q} = H^p(\mathfrak{p}, H^q(\hat{\mathfrak{n}}, V^{-\lambda}\otimes E^\varrho))^{\tilde{\mathfrak{v}}^o}$, we have from Theorem K that

$$H^q(\hat{\mathfrak{n}}, V^{-\lambda}\otimes E^\varrho) = H^q(\hat{\mathfrak{n}}, V^{-\lambda})\otimes E^\varrho = \sum_{\sigma\in\{W(\mathfrak{g})/W(\tilde{\mathfrak{v}}^o)\}^q} V^{q,\lambda(\sigma)}\otimes E^\varrho.$$

Thus
$$'E_2^{p,q} = \sum_{\sigma\in\{W(\mathfrak{g})/W(\tilde{\mathfrak{v}}^o)\}^q} H^p(\mathfrak{p}, V^{q,\lambda(\sigma)}\otimes E^\varrho)^{\tilde{\mathfrak{v}}^o};$$

but $\mathfrak{p}\subset Z(\mathfrak{v}^0)$ and furhermore $\mathfrak{p}$ itself is abelian and thus

$$'E_2^{p,q} = \sum_{\sigma\in\{W(\mathfrak{g})/W(\tilde{\mathfrak{v}}^o)\}^q} H^p(\mathfrak{p}, \mathbf{C})\otimes (V^{q,\lambda(\sigma)}\otimes E^\varrho)^{\tilde{\mathfrak{v}}^o\oplus\mathfrak{p}}.\text{(1)}$$

We consider cases:

Case (ii). In this situation, $(V^{q,\lambda(\sigma)}\otimes E^\varrho)^{\tilde{\mathfrak{v}}^o\oplus\mathfrak{p}} = 0$ because the form γ defined by $p\cdot e = \gamma(p)\cdot e$ for $p\in\mathfrak{p}$, $e\in E^\varrho$ is not the restriction of a weight form on $\mathfrak{h}$ to $\mathfrak{u}\cap\mathfrak{h}$.

Case (i). Here $(V^{q,\lambda(\sigma)}\otimes E^\varrho)^{\tilde{\mathfrak{v}}^o\oplus\mathfrak{p}} = (V^{q,\lambda(\sigma)}\otimes E^\varrho)^{\tilde{\mathfrak{v}}^o\oplus\mathfrak{p}\oplus\bar{\mathfrak{p}}} = (V^{q,\lambda(\sigma)}\otimes E^\varrho)^{\tilde{\tilde{\mathfrak{v}}}^o}$ since $\mathfrak{p}\oplus\mathfrak{c}\oplus\mathfrak{h}_\psi$ lies on no rational hyperplane. In this case, if $\tilde{\hat{\varrho}}$ is the highest weight of $\hat{\varrho}:\hat{\mathfrak{u}}\to H(E^{\hat{\varrho}})$, and if $I(\tilde{\hat{\varrho}}) = I_\sigma(\tilde{\hat{\varrho}})$ with $|\sigma| = q$, then $'E_2^{i,j} = 0$ unless $j = q$ and the spectral sequence is trivial in either Case (ii) or Case (i). Finally, in Case (i), $'E_2^{p,q} = H^p(\mathfrak{p}, \mathbf{C})$ and since the action of M is trivial here, we are done.

COROLLARY 1 (Bott). *For any homogeneous vector bundle $E^\varrho\to \mathrm{E}^\varrho\to X$ over a non-Kähler C-space,*

$$\chi(X, \mathcal{E}^\varrho) = 0, \tag{5.2}$$

where χ is the sheaf Euler characteristic.

Proof. One simply applies the Euler–Poincaré principle to the spectral sequence given in the proof of Theorem 4 where ϱ may be arbitrary.

Remark. Let $E^{\varrho(p)} = \Lambda^p(\mathfrak{g}/\mathfrak{u})'$ where the representation $\varrho(p)$ is Ad^p of U on $\Lambda^p(\mathfrak{g}/\mathfrak{u})'$. In standard notation, $\mathcal{E}^\varrho(p) = \Omega^p(X)$. Since $\mathfrak{u}$ is an ideal in $\hat{\mathfrak{u}}$, it follows that $\chi(X, \Omega^p(X)) = \sum_q(-1)^q \dim H^q(X, \Omega^p(X)) = \sum_q(-1)^q h^{p,q} = 0$ and thus $\sum_{p,q}(-1)^q h^{p,q} = 0$. This says that the index $\tau(X) = \sum_{p,q}(-1)^q h^{p,q}$ as defined by Hodge of a non-Kähler C-space is zero.

(1) See § 14, section (ii) for a discussion of this point.

COROLLARY 2. *Let $E^{\varrho} \to \mathbf{E}^{\varrho} \to X$ be a rational homogeneous vector bundle and assume that $H^*(\hat{X}, \mathcal{E}^{\varrho})$ is a trivial M-module. Then $H^*(X, \mathcal{E}^{\varrho})$ is a trivial M-module.*

A special case of this corollary is Theorem 2 in [5].

The following proposition follows from the proof of Theorem 4:

PROPOSITION 5.1. *Let X be a non-Kähler C-space with basic fibering $T^{2a} \to X \xrightarrow{\sigma} \hat{X}$. Take $\mathbf{E}^{\varrho}$ to be a rational homogeneous vector bundle over X so that we have a homogeneous vector bundle $\mathbf{E}^{\hat{\varrho}}$ over $\hat{X}$ and the following diagram:*

$$\begin{array}{ccc} \mathbf{E}^{\varrho} & \to & X \\ \uparrow \sigma^{-1} & & \downarrow \sigma \\ \mathbf{E}^{\hat{\varrho}} & \to & \hat{X} \end{array}$$

Suppose furthermore that E^{ϱ} is a semi-simple $\tilde{\mathfrak{v}}^0$-module and that $H^p(\hat{X}, \mathcal{E}^{\hat{\varrho}}) = 0$ unless $p = q$ and $H^p(\hat{X}, \mathcal{E}^{\hat{\varrho}})$ is an M-module by a representation $\hat{\varrho}^*$ (in general we will denote the induced representation by *). Then $H^{p+q}(X, \mathcal{E}^{\varrho}) \cong H^p(\hat{U}/U, \Omega) \otimes H^q(\hat{X}, \mathcal{E}^{\hat{\varrho}})$ and $\varrho^* = 1 \otimes \hat{\varrho}^*$.

COROLLARY. *If our fundamental fibering is $T^{2a} \to X \to \hat{X}$, then*

$$\left.\begin{array}{l} H^p(X, \Omega^q) = 0 \quad (p < q) \\ H^{q+r}(X, \Omega^q) \ \textit{is a trivial } M\textit{-module of dimension} \\ \binom{a}{r} b_{\alpha q}(M/\hat{V}, \ \textit{where } \chi(M/\hat{V}) \textit{ is given by Theorem } 3'. \end{array}\right\} \tag{5.3}$$

In particular $$H^r(X, \Omega) \cong C^{\binom{a}{r}} \quad (\textit{trivial } M\textit{-module}). \tag{5.4}$$

We shall now give a geometric interpretation of the Dolbeault forms representing classes in $H^q(X, \mathcal{E}^{\varrho}) \cong H^{0,q}(X, \mathbf{E}^{\varrho})$; this interpretation depends upon results obtained in D.G., §§ VI and VIII, concerning the *canonical complex connexion* in the fibering $U \to G \to G/U$. We recall briefly the construction. Writing $X = G/U = M/V$, there is in the fibering $U \to G \xrightarrow{\pi} G/U$ a canonical M-invariant connexion which respects the complex structures involved. To give this connexion, we must define a splitting $L_g(G) = V(g) \oplus H(g)$ $(g \in G)$ which is M-invariant (on the left) and V-invariant (on the right in $U \to G \to G/U$); here $V(g) \subset L_g(G) =$ vertical space at $g \in G = \ker \pi_*$ where $\pi_* : L_g(G) \to L_{gU}(X)$. At $g = e$, the splitting of the canonical complex connexion is given by $\mathfrak{g} = \mathfrak{u} \oplus \mathfrak{n}^*$; we refer to § VI in D.G. for a further discussion of this connexion.

PROPOSITION 5.2. *Let $T^{2a} \to X \xrightarrow{\sigma} \hat{X}$ be the fundamental fibering of X where*

$X = G/U$, $\hat{X} = G/\hat{U}$. Then there exists a holomorphic representation $\varrho : \hat{U} \to T^{2a}$ such that $\ker \varrho = U$, *$T^{2a} \cong \hat{U}/U$, and X is the homogeneous principal bundle $G \times_{\hat{U}} T^{2a}$.*

Proof. Let $T^{2a} = \hat{U}/U$ and ϱ = projection homomorphism; $\varrho(u) = \hat{u}U$ for $\hat{u} \in \hat{U}$. Then $G \times_{\hat{U}} T^{2a}$ is the set of pairs $[g, (\hat{u}U)]$ factored by the equivalence relation $[g, (\hat{u}'U)] \sim [g\hat{u}^{-1}, \hat{u} \circ (\hat{u}'U)] = [g\hat{u}^{-1}, (\hat{u}\hat{u}'U)]$. We define a mapping $\tau : X \to G \times_{\hat{U}} T^{2a}$ by $\tau(gU) = [g, (U)]$; τ is clearly surjective. Furthermore, $\tau(gU) = \tau(g'U) \Leftrightarrow [g, (U)] \sim [g', (U)] \Leftrightarrow g = g'u$ for some $u \in U$ and thus τ is injective. Q.E.D.

The canonical complex connexion $\hat{U} \to G \to G/\hat{U}$ induces a complex connexion in $T^{2a} \to X \xrightarrow{\sigma} \hat{X}$ in the usual way. Letting $\mathfrak{p}$ = complex Lie algebra of T^{2a}, the connexion form ω in $T^{2a} \to X \to \hat{X}$ is an M-invariant $\mathfrak{p}$-valued $(1,0)$ form on X. Since $\mathfrak{p}$ is abelian, we may choose an isomorphism $\mathfrak{p} \cong \mathbf{C}^a$ and then write $\omega = \omega_1 + \ldots + \omega_a$ where the ω_j are global scalar M-invariant $(1,0)$ forms on X. The Cartan structure equation giving the curvature Ξ of ω is

$$\Xi = d\omega + \tfrac{1}{2}[\omega, \omega] = d\omega \quad (\text{since } \mathfrak{p} \text{ is abelian}).$$

On the other hand, it is given in D.G., § VI, eq. (6.2), that Ξ is of type $(1,1)$ and is given by

$$\Xi(n, \bar{n}') = -\tfrac{1}{2}\varrho([n, \bar{n}']_{\hat{\mathfrak{u}}}) \quad (\mathrm{n}, \mathrm{n}' \in \mathfrak{n}^*).$$

It follows from this that Ξ is non-zero and thus $\partial\omega = 0$, $\Xi = \bar{\partial}\omega \not\equiv 0$, and the connexion in $T^{2a} \to X \to \hat{X}$ is not holomorphic. Using again the isomorphism $\mathfrak{p} \cong \mathbf{C}^a$, we may write $\Xi = \Xi_1 + \ldots + \Xi_a$, where $\Xi_j = \bar{\partial}(\omega_j) \not\equiv 0$.

If we consider the forms $\bar{\omega}_j (j = 1, \ldots, a)$, they are global M-invariant $(0,1)$ forms with the following properties:

(i) $\bar{\partial}\bar{\omega}_j = \overline{\partial\omega_j} = 0$,

(ii) $\partial\bar{\omega}_j = \overline{\bar{\partial}\omega_j} \not\equiv 0$, and thus

(iii) $d\bar{\omega}_j \not\equiv 0$.

We introduce the notation $\bar{\omega}_{j_1 \ldots j_p} = \bar{\omega}_{j_1} \wedge \ldots \wedge \bar{\omega}_{j_p}$.

Let $\hat{\varrho} : \hat{U} \to GL(E^{\hat{\varrho}})$ give rise to $\mathbf{E}^{\hat{\varrho}} \to \hat{X}$ and suppose that ξ is an $\mathbf{E}^{\hat{\varrho}}$-valued form on $\hat{X}$ representing a class $[\xi] \in H^{0,q}(\hat{X}, \mathbf{E}^{\hat{\varrho}}) \cong H^q(\hat{X}, \mathcal{E}^{\hat{\varrho}})$. Then $\hat{\varrho}|U$ induces $\varrho : U \to GL(E^{\varrho})$ $(E^{\varrho} = E^{\hat{\varrho}})$ and applying propositions 4.1 and 5.2, we see that $\sigma^*(\xi)$ is a well-defined $\bar{\partial}$-closed $\mathbf{E}^{\varrho}$-valued form on X giving rise to a class $[\sigma^*(\xi)] \in H^{0,q}(X, \mathbf{E}^{\varrho}) \cong H^q(X, \mathcal{E}^{\varrho})$. From the explicit calculations in Theorem 4 coupled with the explicit form of the canonical connexion given above we conclude:

PROPOSITION 5.3. *In the above notations, assume that* $H^q(\hat{X}, \mathcal{E}^{\hat{\varrho}}) \neq 0$ *for at most* $q = q_0$ *and let the induced representation of* G *on* $H^{q_0}(\hat{X}, \mathcal{E}^{\hat{\varrho}})$ *be* $\hat{\varrho}^*$. *Then*

$$\left.\begin{aligned} &H^q(X, \mathcal{E}^{\varrho}) = 0 \quad q < q_0, \\ &H^{q_0+p}(X, \mathcal{E}^{\varrho}) \cong H^{q_0}(\hat{X}, \mathcal{E}^{\hat{\varrho}}) \otimes H^p(X, \Omega), \end{aligned}\right\} \tag{5.5}$$

the induced representation is $(\hat{\varrho}^*) \otimes 1$, *and the Dolbeault forms representing* $H^{0,\, q_0+p}(X, \mathcal{E}^{\varrho})$ *may be chosen to be* $\sigma^*(\xi) \otimes \bar{\omega}_{j_1 \ldots j_p}$.

6. Line Bundles and Functions on C-Spaces

If X is a compact complex manifold, we denote by $F[X]$ the field of meromorphic functions on X. As an application of Theorem 4, we determine $F[X]$ when $X = G/U$ is a non-Kähler C-space. Recall that a rational algebraic variety is by definition an n-complex dimensional submanifold X of a complex projective space $P_N(\mathbf{C})$ $(N \geqslant n)$ such that the meromorphic function field $F[X]$ is isomorphic (qua abstract fields) to $F[P_n(\mathbf{C})]$. Now the Kählerian C-spaces are algebraic varieties (a positive line bundle was exhibited in D.G); that they are moreover rational varieties was proven by Goto [9].

THEOREM 5. *The non-Kähler C-spaces are rational non-algebraic varieties. If* X *is one such of complex dimension* n *with basic fibering* $T^{2a} \rightarrow X \xrightarrow{\sigma} \hat{X}$, *then*

$$F(X) \cong F[P_{n-a}(\mathbf{C})].$$

Proof. The proof will be done in three steps.

(i) Every line bundle $E \rightarrow \mathbf{E} \rightarrow \hat{X}$ is homogeneous.

There is a different proof of this in [16]. Let $\mathfrak{L}(\hat{X})$ denote the group of complex line bundles on $\hat{X}$ and set $\mathfrak{P}(\hat{X})$ equal to the Picard variety of $\hat{X}$. Then ([18]) $\mathfrak{L}(\hat{X})/\mathfrak{P}(\hat{X}) \cong H^2_{(1,1)}(\hat{X}, \mathbf{Z})$, where $H^2_{(1,1)}(\hat{X}, \mathbf{Z})$ are the integral classes whose harmonic representatives are of type $(1, 1)$. In our case, $H^2_{(1,1)}(\hat{X}, \mathbf{Z}) = H^2(\hat{X}, \mathbf{Z})$ and $\mathfrak{P}(\hat{X}) = 0$, i.e., a line bundle $\mathbf{E}$ is uniquely determined by its characteristic class $c_1(\mathbf{E})$. The result now follows from the discussion following Theorem 3.

(ii) Every line bundle $E \rightarrow \mathbf{E} \rightarrow X$ is homogeneous. From the exact sheaf sequence $0 \rightarrow \mathbf{Z} \rightarrow \Omega \xrightarrow{j} \Omega^* \rightarrow 0$ we get $0 \rightarrow H^1(X, \Omega) \xrightarrow{j_*} H^1(X, \Omega^*) \xrightarrow{\delta_*} H^2(X, \mathbf{Z}) \rightarrow \ldots$ (since $\pi_1(X)$ is finite) and thus any line bundle $\mathbf{E}$ over X is determined by $c_1(\mathbf{E})$ modulo $j_* H^1(X, \Omega)$. But from Theorem 4, $H^1(X, \Omega) = H^0(\hat{X}, \Omega) \otimes H^1(\hat{U}/U, \Omega) \cong \mathfrak{p}'$. The mapping

$\eta : \mathfrak{p}' \to \mathfrak{L}(X) \cong H^1(X, \Omega^*)$ given by $\eta(\lambda) = \mathbf{E}^\lambda$ is the counterpart of j_*, is clearly injective, and since dimensions check out, (ii) is proven.

For a divisor D on X, we denote $H^0(X, [D])$ by $L(D)$ where $[D]$ = line bundle determined by $-D$.

(iii) For every divisor D on X for which $L(D) \neq 0$ the associated line bundle $[D]$ is rational. Indeed $[D]$ is homogeneous and since $H^0(X, [D]) \neq 0$ we see from Theorem 4 that $[D]$ must be rational since if $[D]$ is irrational, $H^i(X, [D]) = 0$ for all i. Thus there is a divisor $\hat{D}$ such that $\sigma^{-1}[\hat{D}] = [D]$; from Theorem 4 again we have that $L(\hat{D}) = H^0(X, [\hat{D}]) \cong H^0(X, [\sigma^{-1}[\hat{D}]) = H^0(X, [\hat{D}]) = L(D)$. This all says that for any divisor D on X such that $L(D) \neq 0$, there exists a divisor $\hat{D}$ on $\hat{X}$ with $\sigma^{-1}(\hat{D}) = D$ and $L(\hat{D}) \cong L(D)$ which proves Theorem 5.

Remark. The above situation seems to be general in the following sense. Let $T^{2a} \to B \xrightarrow{\pi} V$ be an analytic fibration where T^{2a} is homologous to zero. Then one would like every subvariety of B to be π^{-1} of a subvariety of V so that $F[V] \cong F[B]$. If $W \subset B$ is one such subvariety, and if $x \in W$, then we want $\pi^{-1}(\pi(x)) \cap W = \pi^{-1}(\pi(x))$. Setting $T_x = \pi^{-1}(\pi(x))$, if $\pi^{-1}(\pi(x)) \cap W \neq T_x$, then one may argue that the intersection number $T_x \circ W > 0$, which is impossible since $T_x \sim 0$. We have given the above proof because it is more explicit and we hope shows the undefinitive role the irrational bundles play.

We take this opportunity to record and give a geometric proof of the result on line bundles used in Theorem 5.

PROPOSITION 6.1. *Let X be a C-space and let $E \to \mathbf{E} \to X$ be a line bundle; then $\mathbf{E}$ is homogeneous.*

Proof. By D.G., § 9, it will suffice to show: let $\theta \in H^0(X, \Theta)$ be a holomorphic vector field induced by a 1-parameter subgroup $g_t \subset G$ and let Ξ be a scalar-valued (1, 1) form representing the characteristic class of $\mathbf{E}$; then $i(\theta)\Xi = \partial f_\theta$ for some function f_θ on X. If $T^{2a} \to X \xrightarrow{\pi} \hat{X}$ is the fundamental fibering, then (see § 7 below) $\Xi = \pi^* \hat{\Xi}$ for some d-closed (1, 1) form $\hat{\Xi}$ on $\hat{X}$; by Proposition 5.2 $i(\theta)\Xi = i(\theta)\pi^*\hat{\Xi} = \pi^* i(\theta)\hat{\Xi}$. However, $i(\theta)\hat{\Xi}$ is a $\bar{\partial}$-closed (0, 1) form on $\hat{X}$; by Theorem 3, $i(\theta)\hat{\Xi} = \bar{\partial}\hat{f}_\theta$ for some function $\hat{f}_\theta$ on $\hat{X}$ and, setting $f_\theta = \pi^* \hat{f}_\theta$, $i(\theta)\Xi = \bar{\partial} f_\theta$. Q.E.D.

7. Some Properties of the Characteristic Classes of Homogeneous Bundles

We now use the results of D.G. to discuss the position of the characteristic subring in the complex cohomology ring of a C-space X and we also prove a theorem stated in D.G. (Theorem 7) giving a geometric interpretation of the Chern class of a line bundle as defined by Atiyah in [1].

We recall here a few definitions from [14]. Let X be a compact complex manifold of complex dimension n and suppose that $V \to \mathbf{V} \to X$ is an analytic fibre bundle with an r-dimensional vector space V as fibre. Let $c_0 = 1,\ c_1, \ldots, c_n$ be the Chern characteristic classes of X (i.e., the characteristic classes of the fibering $\mathbf{C}^n \to L(X) \to X$) and let $d_0 = 1,\ d_1, \ldots, d_r$ be the characteristic classes of $V \to \mathbf{V} \to X$. Writing formally

$$1 + c_1 \chi + \ldots + c_n \chi^n = \sum_{j=1}^{n} (1 + \gamma_j \chi)$$

$$1 + d_1 \chi + \ldots + d_r \chi^r = \sum_{k=1}^{r} (1 + \delta_k \chi),$$

the *Todd genus* $T(X, \mathbf{V})$ is defined by

$$T(X, \mathbf{V}) = \left(e^{\delta_1} + \ldots + e^{\delta_r} \prod_{i=1}^{n} \frac{\gamma_i}{1 - e^{-\gamma_i}} \right) [X] \tag{7.1}$$

($[X]$ means to evaluate a cohomology class on the fundamental cycle determined by the orientation of X). Then the Hirzebruch–Riemann–Roch (hereafter written H–R–R) identity reads

$$\chi(X, \mathfrak{V}) = T(X, \mathbf{V}). \tag{7.2}$$

In view of (5.2) and the fact that (7.2) is true for algebraic manifolds, to prove (7.2) for homogeneous vector bundles over C-spaces, we must show

THEOREM 6. *If $E^{\varrho} \to \mathbf{E}^{\varrho} \to X$ is a homogeneous vector bundle over a non-Kähler C-space $X = G/U = M/V$, then*

$$T(X, \mathbf{E}^{\varrho}) = 0. \tag{7.3}$$

The proof will be done by writing down the Chern classes $c_1, \ldots, c_n$ of X and $d_1, \ldots, d_r$ of $\mathbf{E}^{\varrho}$as invariant differential forms at the origin and then observing that (7.1) is zero. Since X is non-Kähler, we must choose a complex connexion (see D.G.) to write down the c_j and d_k; by the Theorem of Weil (see [10] for a discussion of these points), we need not be restricted to a metric connexion and we shall actually use the canonical complex connexion discussed in D.G. With these remarks in mind, the rest is computational and we do not belabor the details.

From section 1, we may choose a complex subspace $\mathfrak{p} \subset \mathfrak{h}$ and a subsystem $\Psi \subset \Sigma$ such that

$$\mathfrak{g} = \mathfrak{h} \oplus (\underset{\alpha \in \Sigma}{\oplus} v_\alpha),$$

$$\mathfrak{h} = \mathfrak{p} \oplus \bar{\mathfrak{p}} \oplus \mathfrak{h}_\gamma,$$

$$\mathfrak{u} = \bar{\mathfrak{p}} \oplus \mathfrak{h}_\lambda \oplus (\underset{\alpha \in \Sigma - \Psi}{\oplus} v_\alpha),$$

and then $$L_0(X) \cong \mathfrak{n}^* \oplus \mathfrak{p}, \quad \bar{L}_0(X) \cong \mathfrak{n} \oplus \bar{\mathfrak{p}},$$

where $$\mathfrak{n}^* = \mathfrak{c}(e_\alpha : \alpha \in \Sigma^+ - \Psi^+), \quad \mathfrak{n} = \mathfrak{c}(e_{-\alpha} : \alpha \in \Sigma^+ - \Psi^+).$$

Letting $e_\alpha \in v_\alpha$ be root vectors, $\omega^\alpha \in v'_\alpha$ dual to e_α, and ξ^j $(j = 1, \ldots, a)$ be a basis for $\mathfrak{p}'$, we claim that to prove (7.3), it will suffice to show that for any ϱ,

$$c_k(\mathbf{E}^\varrho) = P_k(\omega^\alpha, \bar{\omega}^\alpha) \quad (\bar{\omega}^\alpha = \omega^{-\alpha}, \alpha \in \Sigma^+ - \Psi^+), \tag{7.4}$$

where P_k is an exterior polynomial of degree $2k$ involving only the ω^α and $\bar{\omega}^\alpha$ and not ξ^j and $\bar{\xi}^j$. This is clear since the component of degree $2n$ in (7.1) evaluated at the origin is of the form

$$\lambda \big(\underset{\alpha \in \Sigma^+ - \Psi^+}{\Lambda} (\omega^\alpha \wedge \bar{\omega}^\alpha) \underset{j}{\Lambda} (\xi^j \wedge \bar{\xi}^j) \big)$$

and if (7.4) holds, then $\lambda = 0$.

From the form of the Chern-Weil theorem as given in [10] together with equation (6.2) in D.G. giving the curvature of the canonical complex connexion in $\mathbf{E}^\varrho$, it will suffice to show: if $p \in \mathfrak{p}$, $v \in \bar{\mathfrak{p}} \oplus \mathfrak{n}$, then

$$\sigma_k(\varrho[p, v]_{\mathfrak{u}}) \equiv 0, \tag{7.5}$$

where σ_k denotes the kth elementary symmetric function of the operator $\varrho[p, v]_{\mathfrak{u}} \in gl(E^\varrho)$. This will imply (7.4). However, since $[\mathfrak{p}, \mathfrak{p}] = 0$, $[\mathfrak{p}, \mathfrak{n}] \subseteq \mathfrak{n}$, and $\varrho \mid \mathfrak{n}$ is nilpotent, it is clear that (7.5) is true. Q.E.D.

COROLLARY. *$\tau(X) = 0$ for non-Kähler X where τ is the topological index* [14].

Proof. Same as Theorem 6 together with the fact that $(1 - p_1 + p_2 + \ldots) = (1 + c_1 + \ldots)(1 - c_1 - \ldots)$ where the p_j are the Pontrjagin classes of X.

This corollary coupled with the corollary to Proposition 5.1 says that the Hodge index theorem holds for C-spaces.

We refer to D.G. (section 7) for a discussion of the Atiyah definition of the first Chern class $c_1(\mathbf{E}^{\varrho})$ of a homogeneous line bundle. The theorem stated there without proof is:

THEOREM 7. *Let X be a non-Kähler C-space and let $T^{2a} \to X \to \hat{X}$ be the fibering of X over a Kähler C-space $\hat{X}$. Then there are independent line bundles $\mathbf{E}^{\varrho_1}, \dots, \mathbf{E}^{\varrho_a}$ (see (ii) in the proof of Theorem 5) whose Atiyah Chern class is $\neq 0$ but whose usual Chern class is 0.*

Remark. If $X = SU(3)$, $a = 1$ and we have the example given in [5].

Proof. We keep the notation used in the proof of Theorem 6. Then, if $X = M/V$, $\hat{X} = M/\hat{V}$,

$$\tilde{\mathfrak{v}}^0 = \mathfrak{h}_\gamma \oplus (\bigoplus_{\alpha \in \Sigma - \psi} v_\alpha)$$

$$\tilde{\hat{\mathfrak{v}}}^0 = \mathfrak{h} \oplus (\bigoplus_{\alpha \in \Sigma - \psi} v_\alpha) = \tilde{\mathfrak{v}}^0 \oplus \mathfrak{p} \oplus \bar{\mathfrak{p}}.$$

We also set $\mathfrak{h}_\gamma^\# = \mathfrak{c}(h_\alpha : \alpha \in \Sigma - \Psi)$; then $\mathfrak{h} = \mathfrak{z} \oplus \mathfrak{h}_\gamma^\#$, where $\mathfrak{z} = Z(\tilde{\hat{\mathfrak{v}}}^0)$.

PROPOSITION 7.1. *Every invariant closed form ω on $M/\hat{V}$ may be written at the origin as*

$$\sum_{\alpha \in \Sigma^+ - \Psi^+} \langle \eta, h_\alpha \rangle \, \omega^\alpha \wedge \bar{\omega}^\alpha, \tag{7.6}$$

where $\eta \in \mathfrak{h}'$ and η is orthogonal to $\mathfrak{h}_\gamma^\#$.

Proof. It is easily checked that ω must be of type (1, 1); $\omega = \sum_{\alpha,\beta} h_{\alpha\bar{\beta}} \omega^\alpha \wedge \bar{\omega}^\beta$. By invariance (under M), $\sum_{\alpha,\beta} h_{\alpha\bar{\beta}} \langle \alpha - \beta, h \rangle \omega^\alpha \wedge \bar{\omega}^\beta = 0$ $(h \in \mathfrak{h})$ and thus $h_{\alpha\bar{\beta}} = 0$ $(\alpha \neq \beta)$. Setting $h_{\alpha\bar{\alpha}} = \lambda_\alpha$, we have $\omega = \sum_{\alpha \in \Sigma^+ - \Psi^+} \lambda_\alpha \omega^\alpha \wedge \bar{\omega}^\alpha$; we would like to define a linear form λ on $\mathfrak{h}$ by $\langle \lambda, h_\alpha \rangle = \lambda_\alpha$. We must show that if $\alpha + \beta = \tau$, $\lambda_\alpha + \lambda_\beta = \lambda_\tau$; and this is so since $d\omega = 0$. Indeed,

$$\begin{aligned} 0 &= d\omega(e_\alpha, \bar{e}_\tau, e_\beta) \\ &= \omega([e_\alpha, e_{-\tau}], e_\beta) - \omega([e_\alpha, e_\beta], \bar{e}_\tau) + \omega([e_\tau, e_\beta], e_\alpha) \\ &= \pm N_{\alpha,\beta}(\lambda_\beta - \lambda_\tau + \lambda_\alpha) \end{aligned}$$

and since $N_{\alpha,\beta} \neq 0$, $\lambda_\tau = \lambda_\alpha + \lambda_\beta$. Q.E.D.

From (7.6), the Dolbeault theorem (see Theorem 3 above), and (5.3), it follows that a basis for $H^1(X, \Omega^1)$ consist of forms $\omega_\eta = \sum_{\alpha \in \Sigma^+ - \Psi^+} \langle \eta, h_\alpha \rangle \omega^\alpha \otimes \bar{\omega}^\alpha$ where $\eta \in h'$

is orthogonal to $\mathfrak{h}_\gamma^\#$. It is easily checked that $d\omega_\eta = 0$, $\bar\partial\omega_\eta = 0$, and ω_η is $\bar\partial \sim 0 \Leftrightarrow \eta = 0$. Now if $\eta \in \mathfrak{p}'$ (then $\langle \eta, \mathfrak{h}_\gamma \rangle = 0$), it follows from D.G., eq. (6.1), that $c_1(\mathbf{E}^\eta) = \omega_\eta$. On the other hand, η is an invariant $(1, 0)$ form on M/V and $d\eta = \sum_{\alpha \in \Sigma^+ - \Psi^+} \eta([e_\alpha, e_{-\alpha}]) = \sum_{\alpha \in \Sigma^+ - \Psi^+} \langle \eta, h_\alpha \rangle \omega^\alpha \otimes \bar\omega^\alpha$. Theorem 7 follows from this.

8. Some Properties of Homogeneous Vector Bundles

(i) Endomorphisms of Homogeneous Vector Bundles

Let Y be a compact complex manifold and let $E \to \mathbf{E} \to Y$ be an analytic vector bundle. Then $H^0(Y, \mathrm{Hom}\,(\mathbf{E}, \mathbf{E}))$ is not only a vector space but also a finite dimensional algebra, the (associative) algebra of endomorphisms of $\mathbf{E}$ denoted $\mathfrak{a}(\mathbf{E})$. As usual, $\mathbf{E}$ is termed indecomposable if no exact sequence $0 \to \mathbf{E}' \to \mathbf{E} \to \mathbf{E}'' \to 0$ splits analytically. Now $\mathfrak{a}(\mathbf{E})$ is related to the indecomposability of $\mathbf{E}$ as follows: If $\mathbf{E}$ is indecomposable and $\varphi \in \mathfrak{a}(\mathbf{E})$, the characteristic equation of φ has holomorphic, hence constant, coefficients. Thus the eigenvalues of φ are constant, and, if φ has two distinct eigenvalues, this would give a splitting $\mathbf{E} = \mathbf{E}' \oplus \mathbf{E}''$. Thus $\mathfrak{a}(\mathbf{E})$ is an algebra consisting of multiples of the identity (1-dimensional subspace) plus a nilpotent ideal; i.e., $\mathfrak{a}(\mathbf{E})$ is a *special algebra* ([1]). The converse is clearly true and we have

PROPOSITION 8.1 (Atiyah.). $\mathbf{E}$ *is indecomposable* $\Leftrightarrow \mathfrak{a}(\mathbf{E})$ *is a special alqebra.*

Let now $E^\varrho \to \mathbf{E}^\varrho \to X$ be a homogeneous vector bundle over a C-space $X = M/V = G/U$. Then if $\varrho : U \to GL(E^\varrho)$ is reducible, $\mathbf{E}^\varrho$ is decomposable; on the other hand, ϱ may not be irreducible and $\mathbf{E}^\varrho$ may still be indecomposable $(L(U(n)/T^n))$. Thus we may ask: if $\varrho : U \to GL(E^\varrho)$ is irreducible, then is $\mathbf{E}^\varrho$ indecomposable?

THEOREM 8. *If* $E^\varrho \to \mathbf{E}^\varrho \to X$ *is a homogeneous vector bundle over a C-space* $X = G/U = M/V$, *then, if* ϱ *is irreducible,* $\mathbf{E}^\varrho$ *is indecomposable.*

Proof. The proof is done in five steps.

(i) We shall show that $\mathfrak{a}(\mathbf{E}^\varrho)$ is a special algebra; by Theorem 5, it will suffice to do this when X is Kähler. If then $\mathfrak{u} = \mathfrak{n} \oplus \tilde{\mathfrak{v}}^0$, $\varrho \,|\, \mathfrak{n} = 0$ and ϱ is the complexification of an irreducible representation of the compact group V. If ϱ is 1-dimensional, the result is trivial and thus we may assume that $\varrho \,|\, Z(\tilde{\mathfrak{v}}^0) = 0$ where $Z(\tilde{\mathfrak{v}}^0)$ = center of $\tilde{\mathfrak{v}}^0$. Thus ϱ is essentially the complexification of an irreducible representation of a compact semi-simple group and we may use the theory of weights ([26]).

(ii) We denote by V^λ, $V^\sigma, \ldots$ irreducible $\mathfrak{g}$-modules with highest weights $\lambda, \sigma, \ldots$ and by $E^\varrho, E^\tau, \ldots$ irreducible $\tilde{\mathfrak{v}}^0$-modules with highest weights $\varrho, \tau, \ldots$. Then, by Theorem K (§ 2),

$$\begin{aligned}\mathfrak{a}(\mathbf{E}^\varrho) &= \sum_{\lambda \in D(\mathfrak{g})} V^{-\lambda} \otimes H^0(\mathfrak{n}, V^\lambda \otimes \operatorname{Hom}(E^\varrho, E^\varrho))^{\tilde{\mathfrak{v}}^0} \\ &= \sum_{\lambda \in D(\mathfrak{g})} V^{-\lambda} \otimes (H^0(\mathfrak{n}, V^\lambda) \otimes \operatorname{Hom}(E^\varrho, E^\varrho))^{\tilde{\mathfrak{v}}^0} \\ &= \sum_{\lambda \in D(\mathfrak{g})} V^{-\lambda} \otimes (E^\lambda \otimes \operatorname{Hom}(E^\varrho, E^\varrho))^{\tilde{\mathfrak{v}}^0}.\end{aligned}$$

Now those $\varphi \in \mathfrak{a}(\mathbf{E})$ such that $\varphi \in V^0 \otimes (E^0 \otimes \operatorname{Hom}(E^\varrho, E^\varrho))^{\tilde{\mathfrak{v}}^0} \cong \operatorname{Hom}(E^\varrho, E^\varrho)^{\tilde{\mathfrak{v}}^0}$ are simply the multiples of the identity automorphism (Schur's Lemma). It will suffice to show: if

$$\sum_{j,k} v_{-\lambda_j} \otimes (e_{\lambda_k} \otimes g_k) \in V^{-\lambda} \otimes (E^\lambda \otimes \operatorname{Hom}(E^\varrho, E^\varrho))^{\tilde{\mathfrak{v}}^0} \quad (\lambda \neq 0),$$

then each g_k is nilpotent. For this we may prove: if

$$\sum_k (e_{\lambda_k} \otimes g_k) \in (E^\lambda \otimes \operatorname{Hom}(E^\varrho, E^\varrho))^{\tilde{\mathfrak{v}}^0} \quad (\lambda \neq 0),$$

then each g_k is nilpotent. We prove this latter statement assuming that $\tilde{\mathfrak{v}}^0$ is semi-simple ((i) above).

(iii) Let
$$E^\lambda = \sum_{\lambda_j \in \Sigma(\lambda)} E^{\lambda_j}, \quad E^\varrho = \sum_{\varrho_k \in \Sigma(\varrho)} E^{\varrho_k}$$

be decompositions into weight spaces relative to a Cartan sub-algebra $\mathfrak{h} \subset \tilde{\mathfrak{v}}^0$ and let $e_{\lambda_j} \in E^{\lambda_j}$, $e_{\varrho_k} \in E^{\varrho_k}$ be weight vectors. For an E^{λ_j}, there will be in general several e_{λ_j}'s If $\varphi \in (E^\lambda \otimes \operatorname{Hom}(E^\varrho, E^\varrho))^{\tilde{\mathfrak{v}}^0}$, then by Schur's lemma

$$\varphi = \sum_j e_{\lambda_j} \otimes g_j \quad (g_j \in \operatorname{Hom}(E^\varrho, E^\varrho)).$$

Since $h \circ \varphi = 0$, we have for $h \in \mathfrak{h}$,

$$0 = \sum_j \langle \lambda_j, h \rangle e_{\lambda_j} \otimes g_j(e_{\varrho_k}) + e_{\lambda_j} \otimes \varrho(h) g_j(e_{\varrho_k}) - e_{\lambda_j} \otimes \langle \varrho_k, h \rangle g_j(e_{\varrho_k}),$$

which implies that, for each j and all k, either

(α_1) $g_j(e_{\varrho_k}) = 0$ or

(α_2) $g_j(e_{\varrho_k}) \neq 0$ and $\varrho(h) g_j(e_{\varrho_k}) = \langle \varrho_k - \lambda_j, h \rangle g_j(e\varrho_k)$.

We may examine (α_2), and, in this case, $g_j(e_{\varrho_k}) \in E^{\varrho_k - \lambda_j}$ and either

(α_3) $g_j^N(e_{\varrho_k}) = 0$ for all k and some N) or

(α_4) $\lambda_j = 0$.

It will suffice to examine (α_4); i.e., g_0.

(iv) Now $e_\alpha \circ \varphi = 0$ (for all $\alpha \in \Psi'$) and thus

$$0 = e_\alpha \circ \left(\sum_j e_{\lambda_j} \otimes g_j\right)$$
$$= \sum \lambda(e_\alpha) e_{\lambda_j} \otimes g_j + e_{\lambda_j} \otimes \varrho(e_\alpha) g_j - e_{\lambda_j} \otimes g_j \varrho(e_\alpha).$$

The terms with an e_0 (0-weight) occurring here are

$$\sum \lambda(e_\alpha) e_{-\alpha} \otimes g_\alpha + \sum e_0 \otimes \varrho(e_\alpha) g_0 - e_0 \otimes g_0 \varrho(e_\alpha)$$

and since the e_{λ_j} are a basis for E^λ, this term must $=0$. Thus having picked out a particular term $e_0 \otimes g_0$, there exists $e \in E^{-\lambda}$ (e may $=0$) and $g_{-\alpha} \in \operatorname{Hom}(E^\varrho, E^\varrho)$ such that $\lambda(e_{-\alpha}) e = e_0$ and then

$$\lambda(e_\alpha) e \otimes g_{-\alpha} + e_0 \otimes \varrho(e_\alpha) g_0 - e_0 \otimes g_0 \varrho(e_\alpha) = 0; \text{ or}$$

(α_5) $g_{-\alpha} = \varrho(e_\alpha) g_0 - g_0 \varrho(e_\alpha)$.

(v) From (α_3) in (iii), it follows that, for $\alpha \neq 0$, $(g_{-\alpha})^N = 0$ (large N) and thus the mapping $g_0^\# : \tilde{\mathfrak{v}}^0 \to gl(E^\varrho)$ defined by $g_0^\#(e_\alpha) = [g_0, \varrho(e_\alpha)]$ $(= (\alpha_5))$ gives a nilpotent representation of $\tilde{\mathfrak{v}}^0$. But then $g_0^\# = 0$ and $g_0 \in \operatorname{Hom}(E^\varrho, E^\varrho)^{\tilde{\mathfrak{v}}^0}$ and either $g_0 = 0$ or all $g_\alpha = 0$ ((α_5)); in either case, we are done.

(ii) The Embedding Theorem for Homogeneous Bundles

Let X be a complex manifold and $L \to \mathbf{L} \to X$ a holomorphic line bundle. For a suitable covering $\{U_i\}$ of X, we may take a local nowhere zero section σ_i of $\mathbf{L} \mid U_i$; then any section of $\mathbf{L} \mid U_i$ is given by $Z \to \xi^i(z) \sigma_i(z)$ ($z \in U_i$). Let $H^0(\mathbf{L}) = H^0(X, \mathfrak{L})$; if for each $x \in X$, there exists a $\sigma \in H^0(\mathbf{L})$ such that $\sigma(x) \neq 0$, then we may classically define Σ: $X \to P_{N-1}(\mathbf{C})$, where $\mathbf{N} = \dim H^0(\mathbf{L})$. Indeed, if $\xi_1, \dots, \xi_N$ give a basis of $H^0(\mathbf{L})$, then $\Sigma \mid U_i$ is given by the mapping $Z \to [\xi_1^i(z), \dots, \xi_N^i(z)]$ where $[\xi_0, \dots, \xi_{N-1}]$ are homogeneous coordinates in $P_{N-1}(\mathbf{C})$. If $\mathbf{H} \to P_{N-1}(\mathbf{C})$ is the hyperplane bundle, then $\Sigma^{-1}(\mathbf{H}) = \mathbf{L}$. The same remarks hold for a vector bundle, provided that the global sections generate the fibre at each point.

Now if $X = G/U$ is a C-space, then $L \to \mathbf{L} \to X$ is a homogeneous line bundle, and thus the canonical complex connection (§ 5 and [11]) is defined in $\mathbf{L}$.

THEOREM 8. *Let* $\mathbf{L}$ *be such that there exists a non-zero* σ *in* $H^0(\mathbf{L})$. *Then the above mapping* $\Sigma : X \to P_{N-1}(\mathbf{C})$ *is defined. Furthermore,* Σ *gives a projective embedding of* X *if and only if* $c_1(\mathbf{L})$, *when computed from the canonical complex connexion, is positive definite.*

Proof. Since $\mathbf{L}$ is homogeneous, we may write $\mathbf{L} = \mathbf{L}^\varrho$ for same holomorphic $\varrho : U \to GL(L^\varrho)$ ($\dim L^\varrho = 1$). Let $\sigma : X \to \mathbf{L}^\varrho$ be such that $\sigma(x) \not\equiv 0$; if $x' \in X$, and if $g \in G$

is such that $g(x)=x'$, then $(g\circ\sigma)(x')=g(\sigma(x))\neq 0$ in $(\mathbf{L}^\varrho)_x$. (Similarly, for homogeneous vector bundles if the global sections generate the fibre at one point, they do so at all points.) From this, the first statement in the theorem is clear.

We now define $\tau: H^0(\mathbf{L}^\varrho)\to L^\varrho\to 0$ as follows: for $\sigma\in H^0(\mathbf{L}^\varrho)$, then $\tau(\sigma)=\sigma(e)$ where we consider σ as a holomorphic function from G to L^ϱ satisfying $\sigma(gu)=\varrho(u^{-1})\,\sigma(g)$ $(g\in G, u\in U)$. Then, for $u\in U$, $\sigma\in H^0(\mathbf{L}^\varrho)$, $\varrho(u)\,\tau(\sigma)=\tau\,\varrho^*(u)\,\sigma$ (ϱ^* = induced representation on cohomology). Indeed, $\tau\,\varrho^*(u)\,\sigma=(\varrho^*(u)\,\sigma)(e)=\sigma(u^{-1}e)=\varrho(u)\,\sigma(e)=\varrho(u)\,\tau(\sigma)$. Thus, if $K^\varrho=\ker\,\tau$, the sequence $0\to K^\varrho\to H^0(\mathbf{L}^\varrho)\to L^\varrho\to 0$ is an exact sequence of U-modules. Let $\mathcal{G}=GL(H^0(\mathbf{L}^\varrho))$ and $V=\{\gamma\in\mathcal{G}\,|\,\gamma(K^\varrho)\subseteq K^\varrho\}$; then $\mathcal{G}/V=P_{N-1}(C)$. Now the holomorphic representation $\varrho^*: G\to\mathcal{G}$ satisfies $\varrho^*(U)\subseteq V$ the induced mapping of G/U to $\mathcal{G}/V$ is just $\Sigma: G/U\to P_{N-1}(\mathbf{C})$. If $U'=(\varrho^{*-1})(V)$, then $U'\supseteq U$ and Σ is injective if and only if $U'=U$. But $\varrho: U\to GL(L^\varrho)$ extends to $\varrho': U'\to GL(L^\varrho)$, and thus to prove the Theorem, we must show:

LEMMA. *Let $X=M/V$ be a C-space, let $\varrho: V\to GL(L^\varrho)$ be a 1-dimensional representation. Let Ξ^ϱ be the curvature of the canonical complex connexion in the bundle $\mathbf{L}^\varrho\to M/V$. Then $\Omega_\varrho=(2\pi\sqrt{-1})^{-1}\,\Xi^\varrho$ represents $c_1(\mathbf{L}^\varrho)$ and Ω_ϱ is positive-definite if and only if ϱ does not extend to a C-subgroup $\hat{V}\underset{\neq}{\supset} V$.*

Proof. In the notations of § 1, $\Omega_\varrho=(2\pi\sqrt{-1})^{-1}\sum_{\alpha\Sigma^+-\Psi^+}\langle\varrho,\alpha\rangle\,\omega^\alpha\wedge\bar{\omega}^\alpha$. Since $H^0(L^\varrho)\neq 0$, we have that $\langle\varrho,\alpha\rangle\geqslant 0$ for $\alpha\in\Sigma^+-\Psi^+$; if, for some α, $\langle\varrho,\alpha\rangle=0$, then we may extend ϱ to $\hat{V}$ where $\tilde{\mathfrak{v}}^0=\mathfrak{v}^0\oplus v_\alpha\oplus\mathfrak{v}_{-\alpha}$. If, conversely, we may extend ϱ to $\hat{V}$, then, for some $\alpha\in\Sigma^+-\Psi^+$, $e_\alpha\in\tilde{\mathfrak{v}}^0-\mathfrak{v}^0$ and then $\varrho[e_\alpha,e_{-\alpha}]=\varrho(h_\alpha)=\langle\varrho,\alpha\rangle=0$. Q.E.D.

DEFINITION 8.1. A holomorphic mapping $\Sigma: G/U\to P_{N-1}(\mathbf{C})$ is called *equivariant* if there exists $\varrho^*: G\to SL(N,\mathbf{C})$ such that, for any $g\in G$, $\varrho^*(g)\,\Sigma(x)=\Sigma(gx)$ $(x\in G/U)$.

PROPOSITION 8.1. *Any mapping $\Sigma: G/U\to P_{N-1}(\mathbf{C})$ is analytically equivalent to an equivariant mapping.*

Proof. If Σ is defined by means of global sections of a homogeneous line bundle $\mathbf{L}^\varrho\to G/U$, then ϱ^* may be taken to be the induced representation on $H^0(\mathbf{L}^\varrho)$ and then, from the proof of Theorem 8, Σ is equivariant. But any mapping $\Sigma: G/U\to P_{N-1}(\mathbf{C})$ is by global sections of $\Sigma^{-1}(\mathbf{H})$, and any line bundle is analytically equivalent to a homogeneous line bundle. Q.E.D.

The following theorem was given without proof in § 1, where it was stated that a differential-geometric proof could be given. Using the proof of Theorem 8, we give a direct proof.

THEOREM. *Let $X = G/U$ be a C-space and let $L^\varrho \to \mathbf{L}^\varrho \to X$ be a homogeneous line bundle. Let Ω_ϱ be as in the proof of Theorem 8 ($\Omega_\varrho = c_1(\mathbf{L}^\varrho)$), and assume that Ω_ϱ has $n-r$ negative eigenvalues and r 0-eigenvalues ($n = \dim_{\mathbf{C}} X$). Then*

$$H^q(X, \mathcal{L}^\varrho) = 0 \quad \text{for} \quad q < n - r.$$

Proof. By the remarks in the proof of Theorem 8, we may find a C-subgroup $\hat{V} \supset V$ such that ϱ extends to $\hat{V}$ and furthermore $\dim \hat{V}/V = 2r$ and $\hat{V}/V$ is a C-space. Then we have a holomorphic fibering $\hat{V}/V \to M/V \to M/\hat{V}$. As in § 2, there is an $\{E^r\}$ such that $E^\infty \approx H^*(M/V, \mathcal{L}^\varrho)$ and $E_2^{p,q} = H^p(M/\hat{V}, \mathcal{L}^\varrho) \otimes H^q(\hat{V}/V, \Omega)$. Thus $E_2^{p,q} = 0$ ($q \neq 0$) and $H^p(M/V, \mathcal{L}^\varrho) = H^p(M/\hat{V}, \mathcal{L}^\varrho) = 0$ for $0 \leq p < n - r = \dim_{\mathbf{C}} M/\hat{V}$, since $\mathbf{L}^\varrho \to M/\hat{V}$ is a negative line bundle. Q.E.D.

Remark. This theorem seems to hold, in some extent, for general compact, complex manifolds. We can prove it for $r = n-1, n-2$, and for all r provided that we replace L by L^μ for a suitable $\mu > 0$.

(iii) Extrinsic Geometry of *C*-Spaces and a Geometric Proof of Rigidity in the Kähler Case

We shall now use the above proposition about equivariant embeddings to have a look at some homogeneous sub-manifolds of $P_N(\mathbf{C})$, Let X and X' be compact homogeneous complex manifolds and write $X = G/U$, $X' = G'/U'$, where G, U, G', and U' are connected complex Lie groups. Furthermore, let $\varrho: G \to G'$ be a holomorphic homomorphism, and let $f: X \to X'$ be a proper holomorphic mapping.

DEFINITION 8.2. The mapping f is said to be *equivariant* with respect to ϱ if, for any $x \in X$ and $g \in G$,

$$f(g \cdot x) = \varrho(g) \cdot f(x).\text{[1]} \tag{8.1}$$

Since f is proper, $f(X)$ is a sub-variety of X', and equivariance implies that $f(X)$ is in fact a non-singular sub-variety. Thus we may define the *normal bundle* $\mathbf{N}_f$ of $f(X)$ in X'. Indeed, we have over $f(X)$ the exact sequence

$$0 \to L_{f(X)} \to L_{X'} \mid f(X) \to \mathbf{N}_f \to 0. \tag{8.2}$$

We remark now that we may assume in the sequel that f is an embedding. In fact, $f(X)$ is clearly a homogeneous complex manifold, and the mapping $f: X \to f(X)$ is a homogeneous fibration. More precisely, in the cases we shall be considering, G and G' will be semi-simple, ϱ may be assumed faithful, and hence we may write $f(X) = G/\hat{U}$ for some analytic subgroup $\hat{U} \supseteq U$.

[1] It is due to Blanchard that any surjective f with connected fibres is equivariant.

Then f is just the projection in the fibration

$$\hat{U}/U \to G/U \xrightarrow{f} G/\hat{U}$$

$$X \xrightarrow{f} f(X).$$

All statements we shall make about injections $f(X) \to G'/U'$ will "lift" to $X = G/U$.

We may describe an equivariant f as follows. Let $x_0 \in X$ be the origin; then any $x \in X$ may be written as $gx_0 (g \in G)$ and $f(x) = f(gx_0) = \varrho(g)\, f(x_0)$. In particular, for $u \in U$, $f(x_0) = f(ux_0) = \varrho(u)\, f(x_0)$, which implies that, taking $x_0' = f(x_0)$ to be the origin in X', $\varrho(U) \subseteq U'$. Thus the equivariant mappings are given by the representations $\varrho : G \to G'$ such that $\varrho(U) \subseteq U'$, and this mapping is an embedding if and only if $\varrho(U) = \varrho(G) \cap U'$.

PROPOSITION 8.2. *The normal bundle $\mathbf{N}_f$ of an equivariant embedding is a homogeneous vector bundle.*

Proof. $L_{f(X)} \cong L_X$ is the homogeneous bundle obtained from the adjoint representation Ad of U on $\mathfrak{g}/\mathfrak{u}$, and $L_{X'} \mid f(X) = f^{-1}(L_{X'})$ is the homogeneous bundle obtained by the representation $\mathrm{Ad} \circ \varrho$ of U on $\mathfrak{g}'/\mathfrak{u}'$. Now since f is an embedding, the injection $\varrho : \mathfrak{g} \to \mathfrak{g}'$ induces an injection of U-modules $\mathfrak{g}/\mathfrak{u} \to g'/\mathfrak{u}'$ and we have the exact sequence of U-modules

$$0 \to \mathfrak{g}/\mathfrak{u} \to \mathfrak{g}'/\mathfrak{u}' \to \mathfrak{q} \to 0, \tag{8.3}$$

and $\mathbf{N}_f$ is just the homogeneous bundle obtained by the action U on $\mathfrak{q}$. Q.E.D.

For a U-module $\mathfrak{r}$, we denote by $(\mathfrak{r})$ the corresponding homogeneous bundle and by $H^q(\mathfrak{r})$ the groups $H^q(X, (\mathfrak{r}))$. We have an exact diagram of U-modules

$$\left.\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \to & \mathfrak{g}/\mathfrak{u} & \to & \mathfrak{g}'/\mathfrak{u}' & \to & \mathfrak{q} & \to & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \to & \mathfrak{g} & \to & \mathfrak{g}' & \to & \mathfrak{g}'/\mathfrak{g} & \to & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \to & \mathfrak{u} & \to & \mathfrak{u}' & \to & \mathfrak{u}'/\mathfrak{u} & \to & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}\right\} \tag{8.4}$$

From § IV, we get the two diagrams of M-modules:

$$\left.\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \uparrow & & \uparrow & & \\
 & & 0 & \to & H^1(\mathfrak{u}') & \to & H^1(\mathfrak{u}'/\mathfrak{u}) & \to & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \to & \mathfrak{g} & \to & H^0(\mathfrak{g}'/\mathfrak{g}) & \to & H^0(\mathfrak{q}) & \to & 0 \\
 & & \uparrow & & \uparrow & & \uparrow \tau & & \\
0 & \to & \mathfrak{g} & \to & \mathfrak{g}' & \to & \mathfrak{g}'/\mathfrak{g} & \to & 0 \\
 & & \uparrow & & \uparrow \gamma & & \uparrow & & \\
 & & 0 & \to & H^0(\mathfrak{u}') & \to & H^0(\mathfrak{u}'/\mathfrak{u}) & \to & 0 \\
 & & & & \uparrow & & \uparrow & & \\
 & & & & 0 & & 0 & &
\end{array}\right\} \tag{8.5}$$

and

$$\begin{array}{ccccccc}
0 & \to & H^2(\mathfrak{u}') & \to & H^2(\mathfrak{u}'/\mathfrak{u}) & \to & 0 \\
 & & & & \uparrow & & \\
 & & & & H^1(\mathfrak{q}) & \to & 0 \\
 & & & & \uparrow & & \\
 & & & & 0 & &
\end{array} \tag{8.6}$$

We want to use (8.5) and (8.6) to obtain geometric information about the position of $f(X)$ in X'. Clearly the key to the situation lies in the groups $H^q(\mathfrak{u}')$ $(q=0,1,2,\ldots)$, and we have, unfortunately, been able to treat these groups only in very special cases.

Case I. We assume that $[\mathfrak{g},\mathfrak{u}']\subseteq\mathfrak{u}'$. Then $H^0(\mathfrak{u}')=\mathfrak{u}'$ and $H^q(\mathfrak{u}')=0$ $(q>0)$. We let $\hat{\mathfrak{u}}\subset\mathfrak{g}'$ be the sub-space spanned by $\mathfrak{g}$ and $\mathfrak{u}'$.

PROPOSITION 8.3. *$G'/\hat{U}$ is a C-space and the injection $f: G/U\to G'/U'$ is the injection of a fibre in the homogeneous fibration*

$$G/U \xrightarrow{f} G'/U' \xrightarrow{\pi} G'/\hat{U}. \tag{8.7}$$

Proof. The fact that $\hat{\mathfrak{u}}$ is a complex sub-algebra of $\mathfrak{g}'$ follows from the relation $[\mathfrak{g},\mathfrak{u}']\subseteq\mathfrak{u}'$. The rest of the Proposition is then clear.

Remarks. $\mathfrak{q}=\mathfrak{g}'/\hat{\mathfrak{u}}$ and the normal bundle $\mathbf{N}_f$ is just the restriction of $\pi^{-1}(L_{G'/\hat{U}})$ to a fibre in (8.7). This is the homogeneous bundle given by the action of $U\subseteq\hat{U}$ on $\mathfrak{g}'/\hat{\mathfrak{u}}$. For example, if we let $F(n)=U(n)/T^n$ then the inclusion $U(n-1)\to U(n)$ induces a fibering

$$F(n-1)\to F(n)\to P_{n-1}(\mathbf{C}) \quad \text{as given in (8.7).}$$

Let Y be a compact non-singular sub-manifold of a complex manifold X, and let $\mathbf{N}_Y$ be the normal bundle of Y in X. One may consider the *deformations* of Y in X; a 1-parameter family of such is given by a family $Y_t (t \in \mathbf{C}, |t| < \varepsilon)$ of compact complex sub-manifolds of X such that $Y_0 = Y$. The manifolds Y_t as abstract manifolds will not in general have the same complex structure as that on X, although they are all differentially equivalent. It was shown in [18] that, if $H^1(Y, \mathcal{N}_Y) = 0$, then a neighborhood of 0 in $H^0(Y, \mathcal{N}_Y)$ parametrizes a complete local family of sub-manifolds varying $Y \subset X$; we shall use this fact to give a geometric proof of the rigidity of Kähler C-spaces.

Case II. Let $X = G/U = M/V$ be a Kähler C-space. We shall prove:

THEOREM. *The complex structure on X is locally rigid.*

The proof is done in two steps:

(i) Let $f: X \to P_N = P_N(\mathbf{C})$ be an equivariant projective embedding of X (§8, (ii)), and let $X_t (X_0 = X)$ be a 1-parameter variation of the complex structure on X. Then we shall show:

PROPOSITION 8.4. *There exist projective embeddings $f_t: X \to P_N$ such that the family $(f_t(X_t))$ gives a variation of the sub-manifold $f(X)$ of P_N.*

PROPOSITION 8.5. *For a suitable projective embedding $f: X \to P_N$, there exists a complex curve $g_t \subset SL(N+1, \mathbf{C}) = G'$ such that $f_t(X_t) = g_t(f_0(X_0))$.*

Remark. Intuitively we shall show that any deformation X_t of X can be "covered" by projective embeddings f_t of X_t, and then we shall show that the variations of the equivariantly embedded sub-manifold $f(X) \subset P_N$ are given by the orbits of $f(X)$ under G', and this shows that the manifolds all have the same complex structure.

Proof of Proposition 8.4. Let X be a compact complex manifold with $H^1(X, \Omega) = 0 = H^2(X, \Omega)$ and let $E \to \mathbf{E} \to X$ be a positive line bundle such that the global sections of $\mathbf{E}$ give a projective embedding. If X_t is a variation of $X = X_0$, then we know:

(i) $H^1(X_t, \Omega_t) = 0 = H^2(X_t, \Omega_t)$ (upper-semi-continuity, see [16]),

(ii) $H^2(X_t, \Omega^*) = H^2(X_t, \mathbf{Z})$ by (i),

(iii) there are positive line bundles $E_t \to \mathbf{E}_t \to X_t$ by (ii) and since the X_t are differentiably equivalent,

(iv) $\dim H^0(X_t, \mathcal{E}_t) = \dim H^0(X, \mathcal{E}_0)$ (by upper-semi-continuity and since the sheaf Euler-characteristic $\chi(X, \mathcal{E}_t)$ is constant) and

(v) the global sections of $\mathbf{E}_t \to X_t$ give projective embeddings $f_t: X_t \to P_N$ and the sub-manifolds $f_t(X_t)$ give a deformation of $f_0(X_0)$ by (i)–(v) (for more details, see [16], § 13). Finally, by § IV, if X is a Kähler C-space, then

$$H^1(X, \Omega) = 0 = H^2(X, \Omega). \quad \text{Q.E.D.}$$

Proof of Proposition 8.5. We first remark that, for a suitable equivariant embedding $f: X \to P_N$ given by global sections of a positive line bundle $E^\varrho \to \mathbf{E}^\varrho \to G/U = X$, it will suffice to prove that

$$H^q(X, \mathcal{N}_f) = 0 \ (q > 0) \quad \text{and} \quad H^0(X, \mathcal{N}_f) \cong \mathfrak{g}'/\varrho(\mathfrak{g}) \ (\mathfrak{g}' = sl(N+1, \mathbf{C})).$$

This is so, since by Kodaira's theorem, this will prove that all variations of $f(X)$ in P_N are given locally by the action of $G' = SL(N+1, \mathbf{C})$ on $f(X)$ (the stability group being $\varrho(G) \subseteq G'$). From (8.5) and (8.6), it will suffice to find a $\varrho: U \to GL(E^\varrho)$ such that $\varrho \in D^0(\mathfrak{g})$ (i.e., $\mathbf{E}^\varrho$ is positive) and $H^q(\mathfrak{u}') = 0$ for $q = 0, 1, 2$. We suspect that this is in fact true for *all* $\varrho \in D^0(\mathfrak{g})$, but we do not know a proof. However, if

$$\mathfrak{g} = \mathfrak{h} \oplus (\bigoplus_{\alpha \in \Sigma} \mathfrak{v}_\alpha), \quad \tilde{\mathfrak{v}}^0 = h \oplus (\bigoplus_{\alpha \in \Psi} \mathfrak{v}_\alpha),$$

and if we take

$$\varrho = g_1 = \tfrac{1}{2} \sum_{\alpha \in \Sigma^+ - \Psi^+} \alpha,$$

then we may use Lemma 5.9 of [21] and a calculation just as in the proof of Theorem 4 to prove that $H^q(\mathfrak{u}') = 0$ for all q. (Then the bundle $\mathbf{E}^\varrho = \mathbf{K}^{-1}$ where $\mathbf{K}$ is the canonical bundle—the embedding is classically called the *canonical embedding.*) We shall not go into the details now, and thus we conclude the proof of the theorem.

Remarks. (i) In Case I above, the normal bundle is trivial; $\mathbf{N}_f \cong X \times \mathfrak{g}'/\hat{\mathfrak{u}}$ and $H^0(X, \mathcal{N}_f) = \mathfrak{g}/\hat{\mathfrak{u}}$, which is just as it should be.

(ii) We may give in any case a geometric proof of the fact that $H^0(\mathfrak{u}') = 0$. Indeed, we make a $\mathfrak{g}$-reductive decomposition $\mathfrak{g}' = \mathfrak{g} \oplus \mathfrak{k}$ and in (8.5) it is seen that $\gamma(H^0(\mathfrak{u}')) \subseteq \mathfrak{k}$. Let $\mathfrak{r} = H^0(\mathfrak{u}')$ and let $\mathfrak{s} = \mathfrak{g} \oplus \mathfrak{r} \subseteq \mathfrak{g}'$. Now $H^0(\mathfrak{u}')$ is a $\mathfrak{g}$-module and thus $\mathfrak{s}$ is a $\mathfrak{g}$-module; since $\mathfrak{r}$ is a sub-algebra, $\mathfrak{s}$ is a sub-algebra and in (8.5), $\ker \tau = \mathfrak{g}'/\mathfrak{s}$. Geometrically, this means that S (with Lie algebra $\mathfrak{s}$) is the stability group of the variety $f(X) \subseteq P_N$. Since G = automorphism group of X (see § IV), there is an analytic onto homomorphism $\sigma: S \to G$ and if we let $K = \ker \sigma$, K is a closed analytic subgroup of $SL(N+1, \mathbf{C})$ which leaves $f(X) \subseteq P_N$ pointwise fixed. However, this is impossible unless $\dim X = 0$ or $\dim K = 0$, provided that $f(X) \subset P_N$ is in general position. Q.E.D.

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