Generalized Hardy and Nevanlinna classes

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Let D be a bounded domain in $\mathbf{R}^n$ which is given by a continuous subharmonic function $V(x)$. By this we mean that

$$(1) \qquad D = \{x;\ V(x) < 0\} \quad \text{and} \quad \lim_{x \to y,\, x \in D} V(x) = 0 \quad \text{for} \quad y \in \partial D.$$

(We do not require $V(x)$ to be defined outside of $\bar{D}$). Such a domain is regular for the solution of the Dirichlet problem, and for every regular domain D, there exist many continuous subharmonic functions which satisfy (1). (For all the information that we shall need on potential theory, the reader is referred to Helms [5].)

Let $K_\varepsilon = \{x \in D;\ V(x) \leqq -\varepsilon\}$, which is compact in D for $\varepsilon > 0$, and let $V^\varepsilon = \sup(V(x), -\varepsilon)$, which is subharmonic in D. Then ΔV^ε, taken in the sense of a distribution [13], defines a positive measure in D, which, when restricted to K_ε, has its support contained in ∂K_ε. If f is continuous in D. we set $A(p, \varepsilon, f) = \int_{K_\varepsilon} |f|^p \Delta V^\varepsilon \tau_n$, $1 \leqq p < \infty$.

This depends only on the values of f on ∂K_ε. We define $H^p(V, D)$ to be the class of harmonic functions for which $A(p, \varepsilon, f)$ is bounded. Similarly, if $D \subset \mathbf{C}^n$, we define $\mathscr{H}^p(V, D)$ to be the class of holomorphic functions such that $A(p, \varepsilon, f)$ is bounded. In this case, it is more natural to consider V plurisubharmonic, although in general, that is not necessary. If we choose $V(x) = \|x\| - 1$ (Euclidean norm), we obtain the familiar Hardy classes of the unit sphere in $\mathbf{R}^n$. We shall also see that the Hardy classes of a domain D with C^2 boundary as considered by Aronszajn and Smith [1] (see also Stein [15]) as well as those defined with respect to harmonic measures by Lumer-Naim [11] are special cases of the classes that we consider here.

We begin by establishing some of the properties of the class $H^p(V, D)$. By a simple adaptation of some results of Hunt and Wheeden [8], we develop a form of Fatou's theorem. Let Γ be a truncated cone with vertex in ∂D. We say that Γ is non-tangential at $y \in \partial D$ if there exists a truncated cone Γ' such that $\bar{\Gamma} \setminus \{y\} \subset \Gamma' \setminus \{y\} \subset D$. The function f has a non-tangential limit at $y \in \partial D$ if $\lim_{x \to y,\, x \in \Gamma} f(x)$

exists for every non-tangential cone Γ with vertex y (the limit need not be unique). The *non-tangential set* $Q\subset\partial D$ is the set of vertices of non-tangential cones in D. The *restricted non-tangential set* $Q'\subset Q$ is the set of $y\in Q$ for which there exists a truncated cone in $\complement D$ with vertex y. If $f\in H^p(V, D)$, $1\leqq p<\infty$, then f has non-tangential limit values in Q' except for a set of harmonic measure zero.

A domain D is said to be a *Lipschitz domain* if for each $y\in\bar{D}$, there exist a neighborhood N_y and a function b such that (after perhaps a rotation):

(i) $N\cap\partial D = N\cap\{(x, X);\ x_1 = b(X)\},\quad$ where $\quad X = (x_2, \dots, x_n)\in\mathbf{R}^{n-1}$,

(ii) $N\cap D = N\cap\{(x, X);\ x_1 > b(X)\}$,

(iii) $|b(X)-b(X')| \leqq c|X-X'|$.

We show, for $f\in H^p(V, D)$, $1<p<\infty$, and D a Lipschitz domain, that the limit values determine a function which is in L^p with respect to harmonic measure in D, and that f can be approximated in norm by functions harmonic in a neighborhood of $\bar{D}$.

Of special interest (and in fact the motivation for this approach) is the case where $D\subset\mathbf{C}^n$ and V is plurisubharmonic. One would then like to prove under suitable assumptions that $f\in\mathscr{H}^p$ can be approximated by functions holomorphic in a neighborhood of $\bar{D}$. We will come back to that problem in another publication.

If $D\subset\mathbf{C}^n$, we define the Nevanlinna class $N(V, D)$ to be the set of holomorphic functions such that $A(1, \varepsilon, \log^+|f|)$ is bounded. This includes $\mathscr{H}^p(V, D)$ for all p. We show that for $f\in N(V, D)$, f has non-tangential limit values in Q' except perhaps for a set of harmonic measure zero.

If f is a meromorphic function in D and D is a domain of holomorphy, then $f=g/h$ where both g and h are holomorphic and relatively prime at each point of D. Following Nevanlinna (cf. Hayman [4], or Taylor [16] for a presentation closer to this context), we define the characteristic of f by

$$T(f, \varepsilon) = \int_{K_\varepsilon} \sup(\log|g|, \log|h|)\,\Delta V^\varepsilon\tau_n. \tag{2}$$

If λ is an increasing convex function of r, we say that f is of finite λ-type if there exist constants A and B such that $T(f, \varepsilon)\leqq A\exp\lambda(-B\log\varepsilon)$. If ΔV has compact support, this definition is independent of the choice of g and h and depends only on $\log^+|f|$ and the poles of f. Using techniques of Skoda [14], we show that if D is Lipschitz and λ non-constant and if $H^2(D, \mathbf{Z})=0$, then f is of finite λ-type in D if and only if it is the quotient of two holomorphic functions of finite λ-type.

These results were in part announced in [3]. The proof that was announced there that in a domain with C^2 boundary, any function of bounded type is the quotient of two holomorphic functions of bounded type, was incorrect, so this is still an open question. We conjecture that this is true in strongly pseudoconvex domains. Since originally announcing these results, we have come across the work of Lumer-Naim [11] which contains a portion of them. Nonetheless, we feel that the spirit here is sufficiently different to merit the development from this point of view.

1. Preliminaries

We now proceed to develop some of the basic properties of the class $H^p(V, D)$. These will in general be simple applications of Green's formula and Gauss' formula.

Let $\alpha \in C_0^\infty(\mathbf{R}^n)$, $0 \leqq \alpha \leqq 1$ where α depends only on $\|x\|$ and $\int \alpha(x)\, d\lambda = 1$, where $d\lambda$ is $\mathbf{R}^n$ Lebesgue measure. We set $\alpha_\varrho(x) = \varrho^{-n}\alpha(x/\varrho)$. If $S(x)$ is subharmonic in D, then $S_\varrho(x) = S * \alpha_\varrho = \int S(x-y)\alpha_\varrho(y)\, d\lambda(y)$ is a C^∞ subharmonic function in $D_\varrho = \{x \in D;\, d(x, \complement D) > \varrho\}$. It follows from the properties of subharmonic functions that $S_\varrho(x) \geqq S(x)$, $S_\varrho(x)$ is decreasing as $\varrho \to 0$ and $\lim_{\varrho \to 0} S_\varrho(x) = S(x)$, the convergence being uniform on compact subsets if $S(x)$ is continuous.

Lemma 1. *If $s(x)$ is a positive subharmonic function in D, then $\int_{K_\varepsilon} s(x)\Delta V^\varepsilon \tau_n$ increases as $\varepsilon \to 0$ and*

$$\lim_{\varepsilon \to 0} \int_{K_\varepsilon} s(x)\Delta V^\varepsilon \tau_n = \int_D s(x)\Delta V \tau_n - \int_D \Delta s(x) V(x) \tau_n. \tag{3}$$

Thus, $f \in H^p(V, D)$ (resp. $\mathscr{H}^p(V, D)$) if and only if

$$\int_D |f|^p \Delta V \tau_n - \int_D \Delta |f|^p V \tau_n = \lim_{\varepsilon \to 0} A(p, \varepsilon, f) < \infty. \tag{4}$$

Proof. Let $\{\beta_m\}$ be a decreasing sequence of C^∞ functions with compact support in D which converges to χ_{K_ε}, the characteristic function of K_ε. Let $K_{\varepsilon,\varrho} = \{x;\, V_\varrho(x) \leqq -\varepsilon\} \subseteqq K_\varepsilon$ and set $T_{\varepsilon,\varrho} = \partial K_{\varepsilon,\varrho}$. Then $S(\varepsilon) = \int_{K_\varepsilon} s\Delta V^\varepsilon \tau_n = \lim_{m\to\infty} \int_D (\beta_m s)\Delta V^\varepsilon \tau_n$, so given $\delta > 0$, there exists M_0 such that for all $m \geqq M_0$, $\int_D (\beta_m s)\Delta V^\varepsilon \tau_n \leqq S(\varepsilon) + \delta/3$. If we set $s_\eta = s * \alpha_\eta$, then for η_0 sufficiently small, $\int_D (\beta_m s_\eta)\Delta V^\varepsilon \tau_n \leqq \int_D (\beta_m s)\Delta V^\varepsilon \tau_n + \delta/3$ for all $m \geqq M_0$ and all $\eta \leqq \eta_0$. Let us now fix η and m. Then

$$\int_D (\beta_m s_\eta)\Delta V^\varepsilon \tau_n = \int_D (\beta_m s_\eta)\Delta [V^\varepsilon + \varepsilon]\tau_n = \int_D \Delta(\beta_m s_\eta)[V^\varepsilon + \varepsilon]\, d\lambda.$$

For ϱ sufficiently small,

$$\left| \int_D \Delta(\beta_m s_\eta)[V^\varepsilon + \varepsilon]\, d\lambda - \int_D \Delta(\beta_m s_\eta)[V_\varrho^\varepsilon + \varepsilon]\, d\lambda \right|$$

can be made as small as we like. By Sard's Theorem [12], the set of ε for which $T_{\varepsilon,\varrho}$ is not a C^∞ manifold is of measure zero in $\mathbf{R}$, so by varying ε slightly if necessary, we may assume that $T_{\varepsilon,\varrho}$ is a C^∞ manifold. By Green's formula, we have (where $\vec{v}$ is the exterior normal to $T_{\varepsilon,\varrho}$)

$$\int_D \Delta(\beta_m s_\eta)[V_\varrho^\varepsilon+\varepsilon]\,d\lambda =$$

$$= \int_{\complement K_{\varepsilon,\varrho}} (\beta_m s_\eta)\Delta[V_\varrho+\varepsilon]\,d\lambda - \int_{T_{\varepsilon,\varrho}} \frac{d}{d\vec{v}} s_\eta[V_\varrho+\varepsilon]\,dS - \int_{T_{\varepsilon,\varrho}} \frac{d}{d\vec{v}}[V_\varrho+\varepsilon]s_\eta\,dS =$$

$$= \int_{\complement K_{\varepsilon,\varrho}} (\beta_m s_\eta)\Delta[V_\varrho+\varepsilon]\,d\lambda + \int_{K_{\varepsilon,\varrho}} s_\eta \Delta V_\varrho\,d\lambda - \int_{K_{\varepsilon,\varrho}} \Delta s_\eta[V_\varrho+\varepsilon]\,d\lambda.$$

Letting $\varrho\to 0$ and setting $D_\varepsilon = (\bigcup_\varrho K^\circ_{\varepsilon,\varrho})$, we get

$$\int \Delta(\beta_m s_\eta)[V^\varepsilon+\varepsilon]\,d\lambda = \int_{D_\varepsilon} \beta_m s_\eta \Delta V \tau_n - \int_{D_\varepsilon} \Delta s_\eta [V+\varepsilon]\,d\lambda.$$

If we now let $\eta\to 0$ and $m\to\infty$, we see that

$$S(\varepsilon) = \int_{D_\varepsilon} s\Delta V\tau_n - \int_{D_\varepsilon} \Delta s(V+\varepsilon)\tau_n,$$

which increases as $\varepsilon\to 0$ and proves (3). Applying this to the subharmonic function $|f|^p$, we get (4). Q.E.D.

Remark 1. ΔV cannot have empty support in D, since in that case V would be harmonic in D and hence identically zero.

Remark 2. It is useful to have the bounded functions in $H^p(V, D)$. Thus, we will always assume (without stating it explicitly) that $\int_D \Delta V\tau_n < \infty$.

Lemma 2. *Given $f\in H^p(V, D)$ and a compact set $K\subset D$, there exists a constant $C(K, p)$ such that $|f|\leqq C(K,p)\|f\|_p$ on K. Hence $H^p(V, D)$ is a Banach space.*

Proof. Let K_0 be a compact set so large that there exists a ball $B(r, x_0)\subset\subset K_0$ with $\int_{B(r/3,x_0)} \Delta V\tau_n \geqq C_0>0$. Then by (4), there exists a point $y_0\in B(r/3, x_0)$ such that $|f(y_0)|^p\leqq M/C_0$, where $M=\|f\|_p^p$. Let $\vec{\omega}$ be a unit vector in $\mathbf{R}^n$, ω_{n-1} the measure on the unit sphere and Ω_{n-1} its total measure. Then, by Gauss' formula

$$\frac{1}{\Omega_{n-1}}\int |f(2r\vec{\omega}/3+y_0)|^p d\omega_{n-1} - |f(y_0)|^p = \int_0^{2/3} \frac{1}{t^{n-1}}\Big(\int_{|x+y_0|\leqq t} \Delta|f|^p\tau_n\Big)\,dt \leqq C'\|f\|_p^p$$

by (4). Let $f_0(x)$ be the harmonic function in $B(2r/3, y_0)$ which has values $|f(x)|^p$ on the boundary of $B(2r/3, y_0)$. Then $f_0(x)\geqq|f(x)|^p$ in $B(2r/3, y_0)$, and by the Poisson integral formula, there exists a constant C'' such that $|f(x)|^p\leqq C''\|f\|_p^p$ in $B(r/3, y_0)$. Let K' be any compact connected set containing $K\cup K_0$ and let B_i, $i=1,\dots,N$ be an open covering of K' by open balls $B_i=B(r_i/2, x_i)$ each of which is relatively compact in D. Then every B_i can be attached to $B(r/3, y_0)$ by a chain B_{i_j} with $B_{i_0}=B(r/3, y_0)$, $B_{i_M}=B_i$, $B_{i_j}\cap B_{i_{j+1}}\neq\emptyset$. By adding a finite number of balls, if necessary, we may assume that $x_{i_{j+1}}$ is contained in $B(r_{i_j}/2, x_{i_j})$. By reasoning as above, we show successively that there exist constants C_j such that $|f|^p\leqq C_j\|f\|_p^p$ for $x\in B(r_{i_j}/2, x_{i_j})$. Thus there exists a constant C_i such that $|f|^p\leqq C_i\|f\|_p^p$ in B_i, which completes the proof. Q.E.D.

2. Harmonic and related measures

If E is a Borel measurable set in ∂D and χ_E is its characteristic function, we say that the subharmonic function $s(x)$ is a lower function for χ_E if $\overline{\lim}_{x'\to y\in\partial D} s(x')\leqq\leqq\chi_E(y)$. The supremum of all lower functions for χ_E is a harmonic function $h_E(x)$. Each point $x\in D$ thus determines a Borel measure ν_x on ∂D given by the formula $\nu_x(E)=h_E(x)$. Since $0\leqq h_E(x)\leqq 1$, $h_E(x)\in H^1(V, D)$. We introduce the Borel measure μ on ∂D given by $\mu_\nu(E)=\|h_E(x)\|_1=\int_D h_E(x)\Delta V\tau_n$. Clearly $\mu\ll\nu_x$ for all x. Furthermore, by Lemma 2, there exists a constant c_x (depending on x) such that $\nu_x(E)\leqq\leqq c_x\mu(E)$. For fixed $x\in D$, if $G_x(y)=[d\mu/d\nu_x]$, the Radon—Nikodym derivative of μ with respect to ν, then $G_x(y)\geqq c_x$ (a.e.).

Lemma 3. *Let f be harmonic in D and continuous in $\bar{D}$. Then*

$$\|f\|_p^p = \int_{\partial D}|f|^p\,d\mu = \int_{\partial D}|f(y)|^p G_x(y)\,d\nu_x(y).$$

Proof. Let $h(x) = \int_{\partial D}|f|^p\,d\nu_x$ be the harmonic function with boundary values $|f|^p$. Then $h(x)\geqq|f(x)|^p$ in D. Since D is a regular domain for the solution of the Dirichlet problem, given $\delta>0$, there exists $\varepsilon>0$ such that $0\leqq h(x)-|f(x)|^p\leqq\delta$ for $V(z)>-\varepsilon$. Thus $\int_{K_\varepsilon}(h(x)-|f(x)|^p)\Delta V^\varepsilon\tau_n\leqq\delta\int_{K_\varepsilon}\Delta V^\varepsilon\tau_n\leqq C\delta$ for some constant C, and so $\|f(x)\|_p^p=\|h(x)\|_1=\int_D h(y)G_x(y)\,d\nu_x(y)=\int_{\partial D}h(y)\,d\mu(y)$.
Q.E.D.

Lemma 4. *Let V_1 and V_2 be two continuous subharmonic functions in D which satisfy* (1). *We assume that $H^p(V_1, D)$ has norm $\|\cdot\|_p^{(1)}$ and $H^p(V_2, D)$ has norm $\|\cdot\|_p^{(2)}$. If ΔV_1 has compact support, then there exists a constant c such that $\|f\|_p^{(1)}\leqq\leqq c\|f\|_p^{(2)}$, so $H^p(V_2, D)\subset H^p(V_1, D)$.*

Proof. Let $\operatorname{supp}\Delta V_1\subset K_0\subset\subset D$ and let ε_0 be so small that $K_0\subset\subset\{x;\,V_2(x)\leqq-\varepsilon\}$ for $\varepsilon\leqq\varepsilon_0$. Given $\varepsilon'>0$, we choose $\varepsilon>0$ so small that $\varepsilon<\varepsilon_0$ and $D^1_{\varepsilon'}=$ $=\{x;\,V_1(x)<-\varepsilon'\}\subset\subset\{x;\,V_2(x)<-\varepsilon\}=D^2_\varepsilon$. Let $h(x)$ be the harmonic function in D^2_ε which has boundary values $|f|^p$. Then $h(x)\geqq|f|^p$ in D^2_ε so

$$\int_{D^1_{\varepsilon'}}|f|^p\Delta V_1^{\varepsilon'}\tau_n\leqq\int_{D^1_{\varepsilon'}}h(x)\Delta V_1^{\varepsilon'}\tau_n\leqq\int_{D^1_{\varepsilon'}}h(x)\Delta V_1\leqq c_1\sup_{K_0}h(x).$$

As in the proof of Lemma 2, there exists a c_2 (depending only on K_0) such that $\sup_{K_0} h(x) \leqq c_2 \int_{D_\varepsilon^2} h(x) \Delta V_2 \tau_n$ for ε sufficiently small. But by Lemma 3, $\int_{D_\varepsilon^2} h(x) \Delta V_2 \tau_n = \int_{D_\varepsilon^2} |f|^p \Delta V_2^\varepsilon \tau_n \leqq (\|f\|_p^{(2)})^p$ so

$$\int_{D_{\varepsilon'}^1} |f|^p \Delta V_1^{\varepsilon'} \tau_n \leqq c_1 c_2 (\|f\|_p^{(2)})^p.$$

Since ε' was arbitrary, this proves the lemma. Q.E.D.

We now form a special class of subharmonic functions which will be useful in what follows. Let $\beta(x)$ be a positive C^∞ function with compact support in D, and let $s_\beta(x) = -\int \|x-y\|^{2-n} \beta(y) d\lambda(y)$, which is subharmonic and C^∞ in $\mathbf{R}^n$ and harmonic in a neighborhood of ∂D. Let $h_\beta(x)$ be the harmonic function in D which has boundary values $s_\beta(x)$ on ∂D. We set $V_\beta^*(x) = -h_\beta(x) + s_\beta(x)$ in D. By Lemma 4, the space $H(V_\beta^*, D)$ is independent of the choice of β. (This remains true even if β is a more general measure with compact support.)

Let $A = \{D_\gamma\}_1^\infty$ be a nested sequence of domains in D each regular for the solution of the Dirichlet problem such that $\cup D_\gamma = D$. If supp $\beta(x) \subset\subset D_1$ and $\nu_x^{(\gamma)}$ is the harmonic measure with respect to D_γ, then there exists a constant c' (independent of γ) such that $G_x(y) = [d\mu^{(\gamma)}/d\nu_x^{(\gamma)}] \leqq c'$ (almost everywhere) by Harnack's principle. As in the proof of Lemma 2, there exists a c_{x_0} which is also independent of γ such that $0 < c_{x_0} \leqq [d\mu^{(\gamma)}/d\nu_x^{(\gamma)}]$. Thus, if B^p is the space of harmonic functions f such that $\int_{\partial D_\gamma} |f|^p d\nu_x^{(\gamma)}$ is uniformly bounded in γ, then B^p is isomorphic to $H^p(V_\beta, D)$. Thus, those classes studied by Lumer-Naim [11] and by Aronszajn and Smith [1] are included in the above Hardy classes. This also shows that B^p is independent of the sequence D_γ.

3. Non-tangential limits

We say that S is an oriented non-tangential region in D if $S = \cup \Gamma_\alpha$, where Γ_α is a non-tangential cone and all the Γ_α have similar size and orientation. If all of the vertices of the Γ_α are contained in a set E which is sufficiently small (depending on the size of the Γ_α), S is a starlike Lipschitz domain (cf. [8]), and hence S is a regular domain for the solution of the Dirichlet problem.

Remark 3. We can assume without loss of generality that supp $\Delta V \cap S \neq \emptyset$, for if not, we choose a C^∞ function $\eta(x)$ with compact support in S and set $V_\eta(x) = -\int \|x-a\|^{2-n} \eta(a) d\lambda(a)$. Then, if $h_\eta(x)$ is the harmonic function in D with boundary values $V_\eta(x)$ and $V_\eta'(x) = V_\eta(x) - h_\eta(x)$, the norm for $H^p(V + V_\eta', D)$ is the sum of the norms of $H^p(V, D)$ and $H^p(V_\eta', D)$ by (4), and by Lemma 4, these norms are equivalent, so the Banach spaces are isomorphic.

We now assume that $\operatorname{supp} \Delta V \cap S \neq \emptyset$. Since V is continuous, there exists a harmonic function $h(x)$ in S such that $h(x) = V(x)$ on ∂S. Let $V_s'(x) = V(x) - h(x)$ and consider $H^p(V_s', S)$.

Lemma 5. *If $f \in H^p(V, D)$, then $f \in H^p(V_s', S)$. If ν' is the harmonic measure in S, then f has limit values almost everywhere which, for $p > 1$, define a function $\tilde{f}$ on ∂S which is in $L^p(\mu')$, where μ' is the measure on ∂S determined by ΔV. Furthermore, $\mu'(E) \leqq \mu(E)$ for $E \subset \partial S \cap \partial D$.*

Proof. We assume that 0 is the star-center of S. Let $S_r = \{rx;\ x \in S\}$, $r \leqq 1$. Then S_r is a regular domain for the solution of the Dirichlet problem. We set $h_r(x)$ to be the harmonic function in S_r with boundary values $V(x)$ and $V_r(x) = V(x) - h_r(x)$. Then it follows from (4) that $f \in H^1(V_r, S_r)$ for $r \leqq 1$ and, if we let $\|f\|_p^{(r)}$ be the norm in $H^p(V_r, S_r)$, then $\|f\|_p^{(r)}$ increases and $\lim_{r\to 1} \|f\|_p^{(r)} = \|f\|_p$.

Let $f^+(x) = \sup(0, f(x))$ which is subharmonic and satisfies $\int_{K_\varepsilon} f^+(x) \Delta V^\varepsilon \tau_n \leqq \leqq \int_{K_\varepsilon} |f(x)| \Delta V^\varepsilon \tau_n$. If $k_r(x)$ is the harmonic function on S with boundary values $f^+(rx)$, then $k_r(x) \geqq 0$ in S and as in the proof of Lemma 2, we can show that $k_r(x)$ has uniformly bounded $L^1(d\nu_0')$ norm on ∂S. We can now apply *mutatis mutandis* the reasoning of Hunt and Wheeden [8, § 4, Lemma 1]. We have $f(rx) \leqq f^+(rx) \leqq \leqq \int_{\partial S} k_r(y)\, d\nu_x'(y)$. The measures $k_r(y)\, d\nu_x'(y)$ are uniformly bounded (for x fixed), so there is a sequence $r_n \to 1$ such that $k_{r_n}(y)\, d\nu_x(y)$ converges weakly to a positive Borel measure σ_x. The sequence r_n depends on x, but we assume that for all x, r_n is a subsequence of the sequence used for 0. If T is any open subset of ∂S, then $\int_T d\sigma_x(y) = \sup \int_{\partial S} g(y)\, d\sigma_x(y)$, where the supremum is taken over all continuous functions with $0 \leqq g(y) \leqq \chi_T(y)$. For such g, $\int_{\partial S} g(y)\, d\sigma_x(y) \leqq \{\operatorname{ess\,sup}_T [d\nu_x'/d\nu_0']\}$ $\int_T d\sigma_0(y)$, and applying the same reasoning as in [8], we conclude that f^+ (and hence f) is bounded above non-tangentially ν'-almost everywhere on ∂S, and hence has non-tangential limits ν'-almost everywhere in ∂S.

Let $f_r(x) = f(rx)$ and $V_r'(x) = V_r(x/r)$ in S. Then $f(rx)$ is of uniformly bounded norm in $L^p(d\nu_0')$, $p > 1$, and so converges to a function $\tilde{f} \in L^p(d\nu_0')$. Let μ_r be the measure on ∂S determined by $\Delta V_r'$. Then $G_r(y) = [d\mu_r/d\nu_0']$ is uniformly bounded in norm in $L^1(\nu_0')$, so we can choose a sequence $\{G_{r_n}\}$ which converges weakly to a measure σ on ∂S, and since for any non-negative continuous function $g(y)$ on ∂S which determines a harmonic function $h_g(x) = \int g(y)\, d\nu_x(y)$, $\int_{\partial S} g(y)\, d\sigma(y) = = \lim_{r_n \to 1} \int h_g(x) \Delta V_{r_n}' \tau_n = \int h_g(x) \Delta V \tau_n = \int_{\partial S} g(y)\, d\mu'(y)$, we have $\sigma = \mu'$ independent of the sequence chosen. Thus $\|f\|_p = \lim_{r\to 1} \|f_r\|_p^{(r)} = \lim_{r \to 1} \int |f_r|^p G_r(y)\, d\nu_0(y) \geqq \geqq \int |\tilde{f}|^p\, d\mu'(y)$, so $\tilde{f} \in L^p(\mu')$. Let $\{f_m\}$ be a Cauchy sequence of continuous functions in $L^p(\mu')$ with limit $\tilde{f}$. Then

$$\|f\|_p \leqq \lim_{m\to\infty} \|f_m\|_p = \lim_{m\to\infty} \int_{\partial S} |f_m|^p\, d\mu' = \int |\tilde{f}|^p\, d\mu'.$$

Finally, if $E \subset \partial S \cap \partial D$ is a measurable set and if h_E is its harmonic measure in D, then h'_E, its harmonic measure in S, satisfies $h'_E \leqq h_E$ in S and hence $\int_D h'_E \Delta V \tau_n \geqq \geqq \int_S h'_E \Delta V \tau_n$. Q.E.D.

As in [8], we have the global result:

Theorem 1. *Let $f \in H^p(V, D)$. Then f has non-tangential limits ν-almost everywhere in the restricted non-tangential set Q'.*

Theorem 2. *If $f \in N(V, D)$, then f has non-tangential limits ν-almost everywhere in the restricted non-tangential set Q'.*

Proof. If $f \in N(V, D)$, then $f \in N(V_{s'}, S)$ for any starlike Lipschitz domain contained in S. Reasoning as in Lemma 5, we see that $\log |f|$ is bounded above non-tangentially for ν-almost all $y \in \partial S$, so this holds for $\operatorname{Re} f$ and $\operatorname{Im} f$, which are harmonic. We then use [8] again to arrive at the conclusion. Q.E.D.

It is not necessary for the limit to be the same for every cone with common vertex even on a set of positive harmonic measure. Thus, in general, we cannot recuperate in a simple fashion the function $f \in H^p(V, D)$, $p > 1$, from its boundary values except in certain geometrically simple cases, such as Lipschitz domains. (See Lumer-Naim [11] where the function is recuperated in a systematic though rather complex manner.)

In what follows, we shall always assume that D is a bounded Lipschitz domain. It is clear from the definition that we can cover D by a finite number of starlike Lipschitz domains $S_i \subset D$ with star-centers P_i.

Lemma 6. *Let $E \subset\subset \partial S_i \cap D$ for some i. Then for $Q \subset E$ and $x \in D \cap \complement S_i$, there exists a constant c such that $h_Q(x) \leqq c h_Q(P_i)$.*

Proof. Let $K(x, x', y) = [d\nu_x / d\nu'_x]$. Then it follows as in [8, § 3, Lemma 5], that $\lim_{x \to \partial S_j \cap D} K(x, P_j, y) = 0$ uniformly for $x \in S_j$, $y \in E$. There, D is supposed to be starlike, but since the arguments are essentially local, with minor modifications, this result can be adapted to the present context. Then, by Harnack's principle, there exists a constant c_2 such that $K(x, P_i, y) \leqq c_2$, from which the lemma follows. Q.E.D.

Theorem 3. *Let D be a bounded Lipschitz domain. Then if $f \in H^p(V, D)$, $1 < p < \infty$, and if $\tilde{f}(y)$ are its non-tangential limit values (which exist almost everywhere), then $\tilde{f} \in L^p(\mu)$, $\|f\|_p^p = \int |\tilde{f}|^p d\mu$ and $\int_{\partial D} \tilde{f}(y) d\nu_x(y) = f(x)$.*

Proof. We assume without loss of generality that $\operatorname{supp} \Delta V \cap S_i \neq \emptyset$ for all i. As in Lemma 5, $f \in H^p(V'_i, S_i)$. Let $E_i = \partial D \cap S_i$, which we suppose open in ∂D. It follows from Lemmas 5 and 6 that there exist constants c_1 and c_2 such that

$$c_2 G_i(y) \leqq G_0(y) \leqq c_1 G_i(y) \quad \text{for} \quad y \in E_i, \tag{5}$$

where $G_i(y) = [d\mu_i'/dv_{P_i'}]$ where μ_i' is the measure determined on ∂S_i by ΔV, v_{P_i}' the harmonic measure at P_i in S_i, and $G_0(y) = [d\mu/dv_{x_0}]$ on ∂D, for some fixed $x_0 \in D$.

By Lemma 5, f has non-tangential limit values $\tilde{f}(y)$ on S_i such that $(\|f\|_p^{(i)})^p = \int |f|^p d\mu_i$, so by (5), $\tilde{f}(y) \in L^p(\mu)$. Let f_n be a continuous function on ∂D such that $\int |f_n - \tilde{f}|^p d\mu < 1/n$ and let $h_n(x)$ be the harmonic function in D with boundary values $|f_n - \tilde{f}|^p$. Then $\int h_n(x)\, d\mu = \int h_n(x) \Delta V \tau_n < 1/n$. Since $h_n(x) \geqq |f_n - f|^p$ in D, $1/n > \int h_n(x) \Delta V \tau_n \geqq \int_D |f_n - f|^p \Delta V \tau_n - \int_D \Delta |f_n - f|^p V \tau_n$ so $\|f_n - f\|_p \to 0$, and since $\|f_n\|_p^p = \int |f_n|^p d\mu$ by Lemma 2, $\|f\|_p^p = \lim_{n\to\infty} \int |f_n|^p d\mu = \int |\tilde{f}|^p d\mu$. Q.E.D.

Theorem 4. *Let $f \in H^p(V, D)$, $1 < p < \infty$ for a Lipschitz domain D. Then given $\varepsilon > 0$, there exists $\tilde{f}$ harmonic in a neighborhood $\bar{D}$ such that $\|f - \tilde{f}\|_p < \varepsilon$.*

Proof. It is clear from Theorem 3 that we can assume that f is continuous on $\bar{D}$. We shall find $\tilde{f}$ harmonic in a neighborhood of $\bar{D}$ such that $\sup_{\bar{D}} |\tilde{f} - f| < \varepsilon$. We can assume without loss of generality that there exists a starlike Lipschitz domain S_0 with star-center P_0 such that $0 \leqq f \leqq M$ and $d(\operatorname{supp} f \cap \partial D, \complement S_0 \cap \partial D) > 0$. There exist a finite number N of starlike Lipschitz domains S_i with star-centers P_i such that

(i) $\bigcup_i (\partial S_i \cap \partial D) = \partial D$,

(ii) $\min_i d(\partial S_i \cap \partial D_i, \operatorname{supp} f \cap \partial D) > 0$.

We let

$$S_i^\delta = \{P_i + (1+\delta)(x - P_i);\ x \in S_i\} \quad \text{and} \quad D^\delta = D \cup \bigcup_{i=0}^{N} S_i^\delta$$

for small δ.

Let f be the harmonic function in D^δ such that f has boundary values $f(y)$ for $(y - P_0)/(1+\delta) + P_0 \in \partial S_0$ and 0 elsewhere.

Let $s \in \partial D^\delta$. Then there exists a cone Γ and a truncated version Γ_s such that $\Gamma_s \subset \complement D^\delta$. We assume without loss of generality that Γ is oriented along the positive x_1-axis and s is the origin.

Let C_η be the cylinder of radius η about the x_1-axis. Let $B_\eta = C_\eta \cap \complement \Gamma_s \cap \{x;\ x_1 \geqq -\gamma\}$. Let $D_\eta = \partial D \cap B_\eta$ and let $A = \sup_{D_\eta} f(y)$, $B = \inf_{D_\eta} f(y)$. Let $\bar{f}$ be the harmonic function in B_η with limit values A in $\partial \Gamma_s \cap C_\eta$ and M in the rest of ∂B_η and let $\underline{f}$ be the harmonic function in B_η with boundary values B in $\partial \Gamma_s \cap C_\eta$ and 0 on the rest of $\partial B(\eta, \gamma)$. Then $\underline{f} \leqq f \leqq \bar{f}$ in $D \cap B_\eta$ if γ is sufficiently small. For η sufficiently small, $A - B < \varepsilon/3$. There exists $\xi > 0$ such that $|\bar{f}(x) - A| < \varepsilon/3$ and $|\underline{f}(x) - B| < \varepsilon/3$ for $x \in D_\gamma$, $|x - s| < \xi$ uniformly in δ, so $|f(x) - f(s)| < \varepsilon$ for $|x - s| < \xi$. These estimates are uniform in s and δ. Q.E.D.

4. Meromorphic functions

We assume in this section that D is pseudoconvex and that $H^2(D, \mathbf{Z})=0$ (cf. Hörmander [6]). For f meromorphic in D, we define the characteristic of f in the following way. Let V_β^* be defined in D as after Lemma 4, i.e., ΔV_β^* has compact support in D; and let $s(z)=\max(\log|g|, \log|h|)$ where $f=g/h$, g and h being two holomorphic functions in D relatively prime at each point. Then $s(x)$ is plurisubharmonic in D, (cf. Lelong [9]). We define

$$T(f, \varepsilon) = \int_{K_\varepsilon} s(z)\Delta V_\beta^{*\varepsilon} = \int_{D_\varepsilon} s(z)\Delta V_\beta^* \tau_n - \int_{D_\varepsilon} \Delta s(z) V_\beta^{*\varepsilon}\tau_n .$$

This is independent (up to a constant) of the choice of g and h since ΔV_β^* has compact support in D. Writing $\sup(\log|g|, \log|h|)=\log^+|f|+\log|h|$ and noting that $\Delta\log|h|$ is a measure of the zeros of h (hence the poles of f) (cf. Lelong [9]), we see that this definition depends only on $\log^+|f|$ and the poles of f. We have the relationships [4, 16]:

(i) $T(f+g, \varepsilon) \leqq T(f, \varepsilon)+T(g, \varepsilon)+O(1)$,

(ii) $T(fg, \varepsilon) \leqq T(f, \varepsilon)+T(g, \varepsilon)+O(1)$,

(iii) $T(1/f, \varepsilon) \leqq T(f, \varepsilon)+O(1)$.

Let λ be an increasing convex function of r. We say that f, meromorphic in D, is of finite λ type if

$$T(f, \varepsilon) \leqq A \exp \lambda(-B\log\varepsilon)$$

for some constants A and B. We begin by proving the following:

Lemma 7. *If D has C^2 boundary, then there exist constants c_1 and c_2 such that $c_1 d(z) \leqq -V_\beta^*(z) \leqq c_2 d(z)$ (where $d(z)=d(z, \complement D)$). If D is Lipschitz, then there exist constants c_1', c_2' and t_D such that*

$$c_1 d(z)^{t_D} \leqq -V_\beta^*(z) \leqq c_2 d(z)^{1/t_D}.$$

Proof. Suppose D has a C^2 boundary. Since $V_\beta^*(z)$ is harmonic in a neighborhood of ∂D, if we let $B(y, r)$ be the ball of radius r which is internally tangent to ∂D at y (which is possible for small r), then V_β^* is harmonic in $B(y, r)$ (for small r) and continuous on its closure. Since $V_\beta^*(z) \leqq -\varepsilon_0$ for some constant $\varepsilon_0>0$ for $d(z)>r/2$, it now follows from the Poisson integral formula for the ball that $c_1 d(z) \leqq -V_\beta^*(z)$ for some constant $c_1>0$. That $-V_\beta^*(z) \leqq c_2 d(z)$ follows from standard estimates of the Poisson kernel on D (cf. [1]).

Let $y \in \partial D$, D a Lipschitz domain, and let Γ_y be a truncated cone in D with vertex y. Let P_y be those points in D on the axis of Γ_y. We assume for convenience that y is the origin. Choose a_1 on Γ_y and let B_1 be the ball centered at a_1 and inscribed in Γ_y. We assume that $\|a_1\|$ is sufficiently small so that V_β^* is harmonic in B_1. There exists a constant $\varepsilon_0 > 0$ and a measurable set $E_1 \subset \partial B_1$ such that $-V_\beta^* \geqq \varepsilon_0$ on E_1 and $m(E_1) \geqq c_0 m(\partial B_1)$ (m can be the Lebesgue measure on the ball or the Hausdorff $(2n-1)$ dimensional measure). Let $a_2 = ka_1$ for $k < 1$ sufficiently large so that B_2, the ball centered at a_2 and inscribed in Γ_y, has the property that for $E_2 = \{kz;\, z \in E_1\}$, $E_2 \subset \partial B_2 \cap B_1$. Then, by the Poisson integral formula, there exists a constant c_0' such that $-V_\beta^* \geqq c_0' \varepsilon_0$ for $x \in E_2$. We construct a sequence $a_n = k^n a_1$ by induction such that $-V_\beta^*(a) \geqq (c_0 c_0')^n \varepsilon_0$ for $k^{n-1} a_1 \leqq a \leqq k^n a_1$. Since these estimates are locally uniform, we have proved the first half of the inequality.

Now let Γ_y' be a truncated cone contained in the complement of D with axis P_y. We again assume y to be the origin. If B_n is the ball centered at y of radius $r/2^n$, then for r sufficiently small, V_β^* is harmonic in $B_n \cap D$. Let $\varepsilon = \sup_{B_1 \cap D} (-V_\beta^*)$. If h_1 is the harmonic function in B_1 which has boundary values ε on $\partial B(y, r) \cap \complement \Gamma_y$ and 0 on $\partial B(y, r) \cap \Gamma_y$, then $h_1 \geqq -V_\beta^*$ in $B(y, a) \cap D$ and there exists a constant $k < 1$ such that $-V_\beta^*(z) \leqq k\varepsilon$ for $z \in B_2 \cap D$. Proceeding by induction, we let $h_n(z)$ be the harmonic function in B_n with boundary values $k^n \varepsilon$ on $\partial B_n \cap \Gamma_y$ and 0 on $\partial B_n \cap \Gamma_y$. Then $h_n(z) \geqq -V_\beta^*(z)$ in $B_n \cap D$ and so $-V_\beta^*(z) \leqq k^{n+1}\varepsilon$ on B_{n+1}. Thus, for $a \in P_y$, $r/2^n \leqq a \leqq r/2^{n+1}$, $-V_\beta^*(z) \leqq k^n \varepsilon$. Since this estimate is locally uniform in y, this proves the lemma. Q.E.D.

Remark 4. If D has a C^2-boundary defined by some function $\varphi(z)$, $\vec{\nabla}\varphi(z) \neq \vec{0}$ in a neighborhood of $\bar{D}$, and if we define $T'(f, \varepsilon) = \int_{\partial\Omega_\varepsilon} s(z)\, dS$, where $\Omega_\varepsilon = \{z;\, \varphi(z) \leqq -\varepsilon\}$, then it is clear from Lemmas 4 and 7 that the classes of meromorphic functions of finite λ-type are the same as those defined by $T(f, \varepsilon)$.

If a holomorphic function h is of finite λ-type, then

$$\int_{K_\varepsilon} \log^+ |h| \Delta V_\beta^{\varepsilon *} \geqq \int_{K_\varepsilon} \log |h| \Delta V_\beta^{\varepsilon *} = \int_{K_\varepsilon} \log |h| \Delta V_\beta^* \tau_n - \int_{K_\varepsilon} \Delta \log |h| V_\beta^{\varepsilon *} \tau_n$$

so

$$-\int_{K_\varepsilon} \Delta \log |h| V_\beta^{\varepsilon *} \tau_n \leqq A \exp \lambda(-B \log \varepsilon). \tag{6}$$

We shall show that if D is a Lipschitz domain and λ is non-constant, then if h is any function which satisfies (6), h has the same zero set as a holomorphic function of finite λ-type. This will imply that any meromorphic function of finite λ-type can be written as a quotient of holomorphic functions of finite λ-type. To do this, we will closely follow Skoda [14].

Lemma 8. *Let D be a bounded starlike domain with the origin as star-center and let $D_r=\{rz;\, z\in D\}$. Suppose that there exist constants A and B such that $\sigma(r)=$ $=\int_{D_r}\Delta\log|h|\tau_n\leqq A\exp\lambda(-B\log(1-r))$. Then there exists a holomorphic function h_1 in D of finite λ-type which has the same zeros as h (i.e. $h/h_1\neq 0$, $h_1/h\neq 0$ in D).*

Proof. Let $\alpha_\varrho(z)$ be defined as in § 1. Then $V_\varrho=\log|h|*\alpha_\varrho$ is plurisubharmonic in D_r for ϱ sufficiently small. Let $\theta^\varrho_{jk}=\partial^2 V_\varrho/\partial z_j\partial\bar z_k$. Then

$$\int_{D_r}|\theta^\varrho_{jk}|\,d\lambda \leqq \int_{D_r}\theta^\varrho_{jj}\,d\lambda+\int_{D_r}\theta^\varrho_{kk}\,d\lambda \leqq \int_{D_r}\Delta V_\varrho\,d\lambda$$

(cf. [14]), and by choosing ϱ sufficiently small, we have $\int_{D_r}|\theta^\varrho_{jk}|\,d\lambda\leqq$ $\leqq A'\exp\lambda(-B\log(1-r))$. Let $\omega^\varrho=i\partial\bar\partial V_\varrho$. Then $d\omega^\varrho=0$, so there exists a 1-form v^ϱ such that $idv^\varrho=\omega^\varrho$; v^ϱ splits into two terms $v^\varrho=v^\varrho_2-v^\varrho_1$, v^ϱ_2 of complex degree $(1,0)$ and v^ϱ_1 of complex degree $(0,1)$ such that $\bar\partial v^\varrho_2=\partial v^\varrho_1=0$. We can write v^ϱ_2 and v^ϱ_1 explicitly

$$v^\varrho_2=\sum_{j=1}^n\Big[\sum_{k=1}^n z_k\int_0^1 t\theta^\varrho_{kj}(tz)\,dt\Big]\,d\bar z_j,$$

$$v^\varrho_1=\sum_{j=1}^n\Big[\sum_{k=1}^n \bar z_k\int_0^1 t\theta^\varrho_{jk}(tz)\,dt\Big]\,dz_j.$$

Thus, using the notation of Hörmander [6], where we let $\varphi(z)$ be the plurisubharmonic function $\lambda(-t_D B\log d(z))$, $\int_{D_r}|v^\varrho_i|^2\exp(-2\varphi(z))\,d\lambda<A'$ for $i=1,2$, and A' is independent of ϱ. Thus, by Hörmander [7], we can find functions u^ϱ_1 and u^ϱ_2 such that $\bar\partial u^\varrho_2=v^\varrho_2$ and $\partial u^\varrho_1=v^\varrho_1$ and such that

$$\int_{D_r}|u^\varrho_i|^2\exp(-2\varphi(z))\,d\lambda< C\int_{D_r}|v^\varrho_i|^2\exp(-2\varphi(z))\,d\lambda,\quad i=1,2,$$

and C is independent of ϱ and r. Then $H=\mathrm{Re}\,(u^\varrho_1+u^\varrho_2)$ satisfies $i\partial\bar\partial H=\omega^\varrho$. We choose a sequence $r_n\to 1$ (so that $\varrho_n\to 0$) and repeat the above process for each r_n. Then the H_n are locally bounded uniformly, so we can choose a subsequence r_{n_j} such that H_{n_j} converges to a pluriharmonic function S on $D=\mathrm{supp}\,(\Delta\log|h|)$. Since S is locally bounded above on some neighborhood of every $z\in\mathrm{supp}\,(\Delta\log|h|)$, S extends uniquely to a plurisubharmonic function in D [cf. Lelong [10]), and $i\partial\bar\partial S=i\partial\bar\partial\log|h|$ in the sense of distributions. Then, as in Skoda [14], we can find a holomorphic function h_1 in D such that $\Delta\log|h_1|=\Delta\log|h|$ and

$$\int_D|\log|h_1||^2\exp(-2\varphi(z))\,d\lambda<\infty.$$

Thus (cf. [7]), there exists a B' such that $\log^+|h_1|<A\exp\lambda(-B'\log d(z))$, so for $z\in\partial K_\varepsilon$, $\log^+|h_1(z)|\leqq A\exp\lambda(-B''\log\varepsilon)$, by Lemma 7. Q.E.D.

Theorem 5. *Let D be a bounded Lipschitz domain of holomorphy such that $H^2(D,\mathbf{Z})=0$. Then if f is meromorphic and of finite λ-type in D, f is the quotient of two holomorphic functions of finite λ-type.*

Proof. We cover $\bar{D}$ with a finite number of balls B_j, $j=1, \dots, N$, such that

(i) $D_j = B_j \cap D$ is starlike, $D_i \cap D_j$ is simply connected,

(ii) if $z \in D$, then there exists a B_j such that $d(z, \complement B_j) > \eta$ (where η is independent of z).

We assume, without loss of generality that $\operatorname{supp} \Delta V_\beta^*$ is relatively compact in each D_j. If we let k_j be the harmonic function in D_j with boundary values V_β^* and set $V_j = V_\beta^* - k_j$, then f is of finite λ-type with respect to V_j in D_j for each j. Assume $f = g/h$ in D. Then, by Lemma 8, there exists a function h_j holomorphic in D_j with the same zeros as h in D_j such that h_j is of finite λ-type in D_j (with respect to V_j). We now solve the second problem of Cousin with bounds. Let $\varphi_j(z)$ be a partition of unity subordinate to B_j. Define $g_{kj} = \log h_k - \log h_j$ in $D_k \cap D_j$. Then if $\gamma_j(z) = \sum_k \varphi_k g_{jk}$, $\delta = \bar{\partial}\gamma_j$ is a globally defined (0, 1) form and $\int_D |\delta|^2 \exp(-2\varphi(z)) d\lambda < < \infty$, where $\varphi(z) = \lambda(-Bt_D \log d(z))$. Since D is a domain of holomorphy, we can find a function $\xi(z)$ such that $\bar{\partial}\xi = \delta$ and $\int |\xi(z)|^2 \exp(-2\varphi(z)) d\lambda < \infty$. Then $\xi - \gamma_j = \tau_j$ is defined and holomorphic in D_j. Since $H^2(D, \mathbf{Z}) = 0$, we can add constants $2\pi i m_j$ to τ_j (m_j an integer) such that for $\tau_j' = 2\pi i m_j + \tau_j$, $h' = h_j \exp(-\tau_j')$ defines a global holomorphic function in D, and h' is of finite λ-type.

If f is of finite λ-type, $1/f$ is also of finite λ-type, so we can find g' holomorphic in D of finite λ type with the same zeros as g. Then $k = h'f/g'$ is of finite λ-type and has no zeros or poles, and $g'' = g'k$ is of finite λ-type. But $f = g''/h'$. Q.E.D.

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