Mean values of subtemperatures over level surfaces of Green functions

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1. Introduction

Let θ denote the heat operator $\sum_{i=1}^{n} D_i^2 - D_t$ in $\mathbf{R}^{n+1} = \{p=(x, t)\colon x\in\mathbf{R}^n, t\in\mathbf{R}\}$, and let θ^* denote its adjoint $\sum_{i=1}^{n} D_i^2 + D_t$. Concepts relative to the adjoint equation will carry the prefix θ^*. Let D be an open subset of $\mathbf{R}^{n+1}$, and let G_D denote its Green function unless $D=\mathbf{R}^{n+1}$, in which case the subscript is omitted. Then for each fixed $q\in D$, the function $G_D(\cdot, q)$ is a non-negative supertemperature on D and a temperature on $D\setminus\{q\}$, whereas $G_D(q, \cdot)$ is a non-negative θ^*-supertemperature on D and a θ^*-temperature on $D\setminus\{q\}$. In [14, 15], it was shown that the characteristic integral mean values of subtemperatures over surfaces defined by $G(q, \cdot)=(4\pi c)^{-n/2}$, are convex functions of $c^{-n/2}$. Here we extend this result to integral mean values over surfaces defined by $G_D(q, \cdot)=(4\pi c)^{-n/2}$, for any D which is θ^*-Dirichlet regular. Since it is possible that the gradient of $G_D(q, \cdot)$ may vanish at points where $G_D(q, \cdot)>0$, we cannot assume that the sets where $G_D(q, \cdot)=(4\pi c)^{-n/2}$ are smooth surfaces for all values of c, only for (Lebesgue) almost all values. Thus our integral mean values are not in general defined for all c, and what we prove is that there is a convex function φ such that they are equal to $\varphi(c^{-n/2})$ whenever they are defined. The means also have the other, more elementary properties that are well-known in the case where $D=\mathbf{R}^{n+1}$. Our methods are generalizations of those in [15], not least because we lack sufficient knowledge of the Dirichlet regularity of points on the surfaces to use the methods in [14]. The convexity theorem enables us to generalize the other results in [14], on volume means, the extension of the mean value theorem for temperatures, and unique thermic continuation of subtemperatures, from the case of $\mathbf{R}^{n+1}$ to a general D. Some of these results are then used to prove a necessary and sufficient condition for thermic majorization on D, analogous to the one given for $\mathbf{R}^n\times]-\infty, a[$ in [15].

In [2], Bauer proved that the measure on the surface where $G(q, \cdot)=(4\pi c)^{-n/2}$

that appears in the characteristic integral mean value of subtemperatures, is obtained by sweeping the unit mass at q onto the set where $G(q, \cdot)\leqq(4\pi c)^{-n/2}$. Here we are again able to extend the result from $\mathbf{R}^{n+1}$ to a general D. Bauer's proof uses knowledge of the Dirichlet regular boundary points on the surface, which is not available in the general case; ours avoids this by using the convexity theorem for the surface means. A more recent proof for the case of $\mathbf{R}^{n+1}$, given by Netuka [9], requires knowledge of the stable boundary points on the surface, which is again lacking in the general case. In outline, our proof follows that of Bauer.

Finally, we turn our attention from integrals over surfaces where $G_D(q, \cdot)=(4\pi c)^{-n/2}$, to suprema over sets where $G_E(\cdot, q)=(4\pi c)^{-n/2}$, for an arbitrary open set E. Here we establish an analogue of the classical three spheres theorem on subharmonic functions. Since we do not require any smoothness of the sets over which the suprema are taken, we need not assume any Dirichlet regularity of E, and we do not have to avoid exceptional values of c. Suitable versions of the standard consequences of the three spheres theorem [10, p. 131] are deduced, as well as a new maximum principle that involves only approach to ∂E along sequences $\{p_j\}$ such that $G_E(p_j, q)\to 0$.

A typical point of $\mathbf{R}^{n+1}$ will be denoted by p or (x, t), whichever is convenient. We put

$$\nabla_x u = (D_1 u, \ldots, D_n u), \quad \nabla u = (\nabla_x u, D_t u), \quad \langle x, y\rangle = \textstyle\sum_{i=1}^n x_i y_i,$$

and use $\|\cdot\|$ to denote the Euclidean norm in both $\mathbf{R}^n$ and $\mathbf{R}^{n+1}$. We use σ to denote surface area measure. When no measure is specified, the term 'almost everywhere' (or 'a.e.') is to be interpreted with respect to Lebesgue measure. The terms 'increasing' and 'decreasing' are used in the wide sense. Integrals with respect to $(n+1)$-dimensional Lebesgue measure are denoted by $\int dp$, and those with respect to n-dimensional Lebesgue measure by $\int dx$. A temperature is a solution of the heat equation $\theta u=0$, and supertemperatures and subtemperatures are the corresponding supersolutions and subsolutions (see [12, 13] for details; also [6], where they are called superparabolic and subparabolic functions). A subtemperature w is said to have a thermic majorant on a set E, if there is a temperature u, such that $w\leqq u$ on E.

2. Preliminary discussion of surfaces and means

Let D be θ^*-Dirichlet regular, and let $p_0\in D$. There is a positive, bounded θ^*-temperature h on D such that

$$G_D(p_0, \cdot) = G(p_0, \cdot)-h, \tag{1}$$

so that $G_D(p_0, \cdot)\in C^\infty(D\setminus\{p_0\})$. (The function h is the PWB solution of the θ^*-

Dirichlet problem on D with boundary values $G(p_0, \cdot)$ [13; 6, p. 331].) It therefore follows from Sard's theorem [11, p. 45] that, for almost every $c>0$, the set

$$\{p\in D\colon\ G_D(p_0, p) = (4\pi c)^{-n/2}\} \tag{2}$$

is a smooth regular n-dimensional manifold. We call such a value of c a *regular value*. (To avoid confusion, the potential-theoretic notion of regularity is always referred to as Dirichlet regularity.) For an arbitrary value of $c>0$, we put

$$\Omega_D(p_0, c) = \{p\in D\colon\ G_D(p_0, p) > (4\pi c)^{-n/2}\}$$

(omitting the subscripts D if $D=\mathbf{R}^{n+1}$). For any regular value of c, the union of $\{p_0\}$ with the set in (2) is $\partial\Omega_D(p_0, c)$. Since $G_D(p_0, \cdot)$ is lower semicontinuous, any $\Omega_D(p_0, c)$ is open. The assumption that D is θ^*-Dirichlet regular implies that $G_D(p_0, \cdot)$ can be continuously extended to zero on ∂D, so that $\bar{\Omega}_D(p_0, c)\subseteqq D$. In view of (1), there is a positive constant δ such that

$$G(p_0, \cdot)-\delta \leqq G_D(p_0, \cdot) \leqq G(p_0, \cdot) \tag{3}$$

on D, so that if d satisfies $(4\pi d)^{-n/2}=(4\pi c)^{-n/2}+\delta$, we have

$$\Omega(p_0, d) \subseteqq \Omega_D(p_0, c) \subseteqq \Omega(p_0, c). \tag{4}$$

Hence $\Omega_D(p_0, c)$ is bounded. Furthermore, $\Omega_D(p_0, c)$ is connected; for otherwise it would have a component K that did not contain $\Omega(p_0, d)$. Then we would have $G(p_0, p)\leqq(4\pi d)^{-n/2}$ for all $p\in K$, so that $G_D(p_0, \cdot)$, a θ^*-temperature on K, would be bounded on K because of (3). Since $G_D(p_0, \cdot)=(4\pi c)^{-n/2}$ on $\partial K\setminus\{p_0\}$, it would follow that $G_D(p_0, \cdot)=(4\pi c)^{-n/2}$ throughout K, contrary to the definition of $\Omega_D(p_0, c)$.

If c is a regular value, the outward unit normal $\nu=(\nu_x, \nu_t)$ to $\partial\Omega_D(p_0, c)$ is given by the standard formula

$$\nu = -\nabla G_D(p_0, \cdot)\,\|\nabla G_D(p_0, \cdot)\|^{-1}.$$

We shall need to integrate $\langle \nabla_x G_D(p_0, \cdot), \nu_x\rangle$ over $\partial\Omega_D(p_0, c)$ with respect to surface measure. Now,

$$\langle \nabla_x G_D(p_0, \cdot), \nu_x\rangle = -\|\nabla_x G_D(p_0, \cdot)\|^2\,\|\nabla G_D(p_0, \cdot)\|^{-1}$$

is obviously dominated by $\|\nabla G_D(p_0, \cdot)\|$, and so is bounded outside any neighbourhood of $p_0=(x_0, t_0)$ (relative to $\partial\Omega_D(p_0, c)$). It follows from (4) that

$$2n(t_0-t)\log\big(d/(t_0-t)\big) \leqq \|x_0-x\|^2 \leqq 2n(t_0-t)\log\big(c/(t_0-t)\big) \tag{5}$$

whenever $p=(x, t)\in\partial\Omega_D(p_0, c)\setminus\{p_0\}$, so that for such p we have

$$\begin{aligned}\|\nabla_x G(p_0, p)\|^2 &= G(p_0, p)^2\,\|x_0-x\|^2/4(t_0-t)^2\\ &\geqq (4\pi c)^{-n}\,n\log\big(d/(t_0-t)\big)2(t_0-t),\end{aligned}$$

which implies that $\|\nabla_x G(p_0, p)\| \to \infty$ as $p \to p_0$ through that set. Recalling (1), and noting that $\|\nabla h\|$ is bounded on $\partial\Omega_D(p_0, c)$, we deduce that $\|\nabla_x G_D(p_0, p)\| \sim \|\nabla_x G(p_0, p)\|$ and $\|\nabla G_D(p_0, p)\| \sim \|\nabla G(p_0, p)\|$ as $p \to p_0$ through $\partial\Omega_D(p_0, c) \setminus \{p_0\}$. Therefore

$$-\langle \nabla_x G_D(p_0, p), \nu_x \rangle \sim \|\nabla_x G(p_0, p)\|^2 \, \|\nabla G(p_0, p)\|^{-1} = Q(p_0, p) G(p_0, p),$$

say, where

$$(6) \qquad Q(p_0, p) = \|x_0 - x\|^2 \left\{4\|x_0 - x\|^2 (t_0 - t)^2 + \left(\|x_0 - x\|^2 - 2n(t_0 - t)\right)^2\right\}^{-1/2}$$

$$= \left\{\frac{4(t_0 - t)^2}{\|x_0 - x\|^2} + \left(1 - \frac{2n(t_0 - t)}{\|x_0 - x\|^2}\right)^2\right\}^{-1/2}$$

$$\leqq \left\{\frac{2(t_0 - t)}{n \log(c/(t_0 - t))} + \left(1 - \frac{1}{\log(d/(t_0 - t))}\right)^2\right\}^{-1/2},$$

in view of (5). Since the last expression tends to 1 as $t \to t_0-$, we deduce from (3) that $\langle \nabla_x G_D(p_0, \cdot), \nu_x \rangle$ is bounded on $\partial\Omega_D(p_0, c) \setminus \{p_0\}$, and is therefore surface integrable.

Our first theorem introduces the integral means that we shall study, via a property of temperatures. Given $p_0 \in D$ and any regular value of c, we put

$$\mathcal{M}_D(u, p_0, c) = \int_{\partial\Omega_D(p_0, c)} K_D(p_0, p) u(p) \, d\sigma(p)$$

whenever the integral exists, where

$$K_D(p_0, \cdot) = \|\nabla_x G_D(p_0, \cdot)\|^2 \, \|\nabla G_D(p_0, \cdot)\|^{-1}$$
$$= -\langle \nabla_x G_D(p_0, \cdot), \nu_x \rangle$$

on $\partial\Omega_D(p_0, c) \setminus \{p_0\}$.

Theorem 1. *Let E be an arbitrary open set, and let D be an open set which is θ^*-Dirichlet regular. If p_0 and c_0 are such that $\bar{\Omega}_D(p_0, c_0) \subseteq E$, and u is a temperature on E, then*

$$u(p_0) = \mathcal{M}_D(u, p_0, c)$$

for every regular value of $c \in]0, c_0]$.

Proof. Let A be any bounded domain whose boundary is smooth enough for the divergence theorem to be applicable, and whose closure lies in E. It follows from Green's formula for the heat equation that, whenever v is a θ^*-temperature on a neighbourhood of $\bar{A}$, we have

$$(7) \qquad \int_{\partial A} \left(\langle u \nabla_x v - v \nabla_x u, \nu_x \rangle + u v \nu_t\right) d\sigma = 0,$$

where (ν_x, ν_t) is the outward unit normal to ∂A. In particular, if $v=1$,

$$\int_{\partial A} (-\langle \nabla_x u, \nu_x\rangle + u\nu_t)\, d\sigma = 0. \tag{8}$$

Let c be a regular value in $]0, c_0]$, and put $\Omega = \Omega_D(p_0, c)$, $\Omega_t = \Omega \cap (\mathbf{R}^n \times]-\infty, t[)$, and $B_t = (\partial\Omega) \cap (\mathbf{R}^n \times]-\infty, t[)$ (whenever the intersections are non-empty). Applying (7) to the components of Ω_t, with $v = G_D = G_D(p_0, \cdot)$, we obtain

$$\int_{\partial\Omega_t} (\langle u\nabla_x G_D - G_D \nabla_x u, \nu_x\rangle + uG_D\nu_t)\, d\sigma = 0. \tag{9}$$

On $\Phi^t = \partial\Omega_t \setminus \bar{B}_t$ we have $\nu_x = 0$, $\nu_t = 1$, and $d\sigma = dx$, so that

$$\int_{\Phi^t} uG_D\, dx = -\int_{B_t} u\langle \nabla_x G_D, \nu_x\rangle\, d\sigma - (4\pi c)^{-n/2} \int_{B_t} (-\langle \nabla_x u, \nu_x\rangle + u\nu_t)\, d\sigma,$$

since $G_D = (4\pi c)^{-n/2}$ on B_t.

We now make $t \to t_0-$. In view of (4) and (8),

$$\int_{B_t} (-\langle \nabla_x u, \nu_x\rangle + u\nu_t)\, d\sigma \to \int_{\partial\Omega} (-\langle \nabla_x u, \nu_x\rangle + u\nu_t)\, d\sigma = 0.$$

Since $\langle \nabla_x G_D, \nu_x\rangle$ is bounded on $\partial\Omega$,

$$-\int_{B_t} u\langle \nabla_x G_D, \nu_x\rangle\, d\sigma \to \mathcal{M}_D(u, p_0, c);$$

so it remains to prove that

$$\int_{\Phi^t} uG_D\, dx \to u(p_0). \tag{10}$$

Now,

$$\int_{\Phi^t} u(p)\, G_D(p)\, dx - u(p_0)$$
$$= \int_{\Phi^t} (u(p) - u(p_0))\, G_D(p)\, dx + u(p_0) \left(\int_{\Phi^t} G_D(p)\, dx - 1\right),$$

and the first integral on the right tends to zero, because u is continuous and

$$\int_{\Phi^t} G_D(p)\, dx \leqq \int_{\mathbf{R}^n \times \{t\}} G(p)\, dx = 1.$$

Also, in view of (3),

$$0 \leqq 1 - \int_{\Phi^t} G_D(p)\, dx \leqq 1 - \int_{\Phi^t} G(p)\, dx + \int_{\Phi^t} \delta\, dx.$$

It follows from (4) that the Lebesgue measure of Φ^t tends to zero, and that

$$\int_{\Phi^t} G(p)\, dx \to 1.$$

Therefore (10) follows, and the theorem is proved.

Notation. For generality, we shall sometimes work with domains whose boundaries consist of two surfaces of the form $\partial\Omega_D(p_0, c)$. If $0 < c_1 < c_2$, and c_1, c_2 are regular values, we put

$$A_D(p_0, c_1, c_2) = \Omega_D(p_0, c_2) \setminus \bar{\Omega}_D(p_0, c_1).$$

It is sometimes convenient to write $A_D(p_0, 0, c_2)$ for $\Omega_D(p_0, c_2)$.

The constant $2^{n+1}\pi^{n/2}n^{-1}$ appears several times, and we denote it by $\varkappa_n$. Its occurrence is partly due to the following lemma on the transformation of integrals.

Lemma 1. *Let F be a measurable function on $A_D(p_0, c_1, c_2)$, where $0 \leqq c_1 < c_2$. If*

$$\int_{A_D(p_0, c_1, c_2)} F\, dp$$

exists, then it is equal to

$$\varkappa_n^{-1} \int_{c_1}^{c_2} \gamma^{-(n/2)-1}\, d\gamma \int_{\partial\Omega_D(p_0, \gamma)} F \|\nabla G_D(p_0, \cdot)\|^{-1}\, d\sigma.$$

Proof. Let $g = G_D(p_0, \cdot)$ and $p_0 = (x_0, t_0)$. Given any $\tau < t_0$, define g_τ on $\mathbf{R}^{n+1}$ by putting $g_\tau(x, t) = g(x, t)$ if $(x, t) \in A_D(p_0, c_1, c_2)$ and $t \leqq \tau$, $g_\tau(x, t) = (4\pi c_1)^{-n/2}$ if $(x, t) \in \bar{\Omega}_D(p_0, c_1)$ and $t \leqq \tau$ (if $c_1 > 0$), $g_\tau(x, t) = (4\pi c_2)^{-n/2}$ if $(x, t) \in \mathbf{R}^{n+1} \setminus \Omega_D(p_0, c_2)$ and $t \leqq \tau$, and $g_\tau(x, t) = g_\tau(x, \tau)$ if $t > \tau$. Then g_τ is a Lipschitz function on $\mathbf{R}^{n+1}$. Now define F_τ and f_τ on $\mathbf{R}^{n+1}$ by putting $F_\tau(x, t) = F(x, t)$ if $(x, t) \in A_D(p_0, c_1, c_2)$ and $t \leqq \tau$, $F_\tau(x, t) = 0$ otherwise, and $f_\tau = F_\tau \|\nabla g\|^{-1}$ on $\mathbf{R}^{n+1}$. Noting that Federer uses the term 'integrable' to mean 'has a well-defined integral', we can apply the coarea formula [7, p. 249] to the positive part of f_τ and obtain

$$\int_{\mathbf{R}^{n+1}} f_\tau^+ \|\nabla g_\tau\|\, dp = \int_{\mathbf{R}} d\alpha \int_{L(\tau, \alpha)} f_\tau^+\, d\sigma,$$

where $L(\tau, \alpha) = g_\tau^{-1}(\{\alpha\})$. Thus

$$\int_{A_D(p_0, c_1, c_2)} F_\tau^+\, dp = \int_{(4\pi c_2)^{-n/2}}^{(4\pi c_1)^{-n/2}} d\alpha \int_{L(\tau, \alpha)} f_\tau^+\, d\sigma,$$

$$= \varkappa_n^{-1} \int_{c_1}^{c_2} \gamma^{-(n/2)-1}\, d\gamma \int_{\partial\Omega_D(p_0, \gamma)} f_\tau^+\, d\sigma.$$

As $\tau \to t_0-$, the positive parts of F_τ and f_τ increase to those of F and $F\|\nabla g\|^{-1}$ respectively, so that the monotone convergence theorem yields

$$\int_{A_D(p_0, c_1, c_2)} F^+\, dp = \varkappa_n^{-1} \int_{c_1}^{c_2} \gamma^{-(n/2)-1}\, d\gamma \int_{\partial\Omega_D(p_0, \gamma)} F^+ \|\nabla g\|^{-1}\, d\sigma.$$

A similar argument can be applied to the negative parts, and the result follows.

3. An essential lemma

The following result is fundamental to the entire paper. We say that a function is *smooth* if the partial derivatives that occur in θ exist as continuous functions.

Lemma 2. *Let* $p_0 \in \mathbf{R}^{n+1}$, *let* $0 < c_1 < c_2$, *let* w *be a smooth function on an open superset* E *of* $\bar{A}_D(p_0, c_1, c_2)$, *and let* $\Omega_D(c) = \Omega_D(p_0, c)$ *and* $\mathcal{M}_D(c) = \mathcal{M}_D(w, p_0, c)$ *for all regular values of* $c \in [c_1, c_2]$. *Then there is an absolutely continuous function* f *on* $[c_1, c_2]$, *such that* $f(c) = \mathcal{M}_D(c)$ *for all regular values of* $c \in [c_1, c_2]$ *and*

$$\varkappa_n c^{(n/2)+1} f'(c) = \int_{\partial\Omega_D(c)} (\langle \nabla_x w, v_x \rangle - w v_t)\, d\sigma \tag{11}$$

for almost all $c \in [c_1, c_2]$. *If, in addition,* $\theta w \geqq 0$ *on* E, *then* f' *exists everywhere on* $[c_1, c_2]$, *is absolutely continuous there, and satisfies*

$$\varkappa_n^2 (c^{(n/2)+1} f'(c))' = c^{-(n/2)-1} \int_{\partial\Omega_D(c)} \|\nabla G_D(p_0, \cdot)\|^{-1} \theta w\, d\sigma \tag{12}$$

for almost all c.

Proof. We require certain Green identities, which can be found in [15]. If v is a smooth function on E, and A is a domain, with closure in E, for which the divergence theorem is applicable, then

$$\int_A (w\theta^* v + \langle \nabla_x w, \nabla_x v \rangle - w D_t v)\, dp = \int_{\partial A} w \langle \nabla_x v, v_x \rangle\, d\sigma \tag{13}$$

and

$$\int_A (v\theta w + \langle \nabla_x v, \nabla_x w \rangle - w D_t v)\, dp = \int_{\partial A} (v \langle \nabla_x w, v_x \rangle - v w v_t)\, d\sigma.$$

The latter formula will be used only in the case where $v = 1$, that is,

$$\int_A \theta w\, dp = \int_{\partial A} (\langle \nabla_x w, v_x \rangle - w v_t)\, d\sigma. \tag{14}$$

Let c be a regular value in $]c_1, c_2]$, and let $A_D = A_D(p_0, c_1, c)$. We want to use (13) with $A = A_D$, but with $v = (4\pi)^{-1} G_D(p_0, \cdot)^{-2/n}$, and the smoothness of this choice of v breaks down at p_0. We therefore use an approximation argument. For $t < t_0$, let

$$B_1(t) = \partial\Omega_D(c_1) \cap (\mathbf{R}^n \times]-\infty, t[),$$

$$B_2(t) = \partial\Omega_D(c) \cap (\mathbf{R}^n \times]-\infty, t[),$$

$$V(t) = A_D \cap (\mathbf{R}^n \times]-\infty, t[).$$

Applying (13) on $V(t)$, with $v=(4\pi)^{-1}G_D(p_0,\cdot)^{-2/n}$, we obtain

$$\int_{V(t)}\left(\frac{2}{n}\left(\frac{2}{n}+1\right)\frac{\|\nabla_x G_D\|^2}{G_D^2}vw-\frac{2v}{nG_D}\langle\nabla_x w,\nabla_x G_D\rangle+\frac{2D_t G_D}{nG_D}vw\right)dp$$

$$=\int_{\partial V(t)}-\frac{2vw}{nG_D}\langle\nabla_x G_D,\nu_x\rangle\,d\sigma=\frac{2}{n}\left(\int_{B_2(t)}-\int_{B_1(t)}\right)\frac{vw}{G_D}K_D(p_0,\cdot)\,d\sigma$$

$$=\varkappa_n\left(c^{(n/2)+1}\int_{B_2(t)}-c_1^{(n/2)+1}\int_{B_1(t)}\right)wK_D(p_0,\cdot)\,d\sigma$$

$$\to\varkappa_n\left(c^{(n/2)+1}\mathscr{M}_D(c)-c_1^{(n/2)+1}\mathscr{M}_D(c_1)\right)$$

as $t\to t_0-$, since $K_D(p_0,\cdot)$ is integrable. To show that the integral over $V(t)$ tends to the corresponding integral over A_D, we first note that, since $v\leqq c$ on A_D, the integrand is dominated by a multiple of

$$\|\nabla_x G_D\|^2G_D^{-2}+\|\nabla_x G_D\|\,G_D^{-1}+|D_tG_D|\,G_D^{-1}.$$

In view of (1) and (4) (with c replaced by c_1), there is $d_1>0$ such that

$$\|\nabla_x G_D\|\leqq\|\nabla_x G\|+\|\nabla_x h\|\leqq\|\nabla_x G\|\,G^{-1}(4\pi d_1)^{-n/2}+\|\nabla_x h\|\leqq C_1(\|\nabla_x G\|\,G^{-1}+1)$$

on A_D, for some constant C_1. Therefore

$$\|\nabla_x G_D\|\,G_D^{-1}\leqq C_1(\|\nabla_x G\|\,G^{-1}+1)(4\pi c)^{n/2},$$

and similarly

$$|D_tG_D|\,G_D^{-1}\leqq C_2(|D_tG|\,G^{-1}+1)$$

on A_D, for some constant C_2. Hence the integrand is dominated by a multiple of

$$\|\nabla_x G\|^2G^{-2}+\|\nabla_x G\|\,G^{-1}+|D_tG|\,G^{-1}+1,$$

whose value at (x,t) does not exceed a multiple of

$$\|x_0-x\|^2(t_0-t)^{-2}+2n(t_0-t)^{-1}+1. \tag{15}$$

This expression is obviously integrable over $V(t)$ for any $t<t_0$. Furthermore, in view of (4) there is $d_1>0$ such that $A_D\subseteqq A(p_0,d_1,c)$, and if $(x,t)\in A(p_0,d_1,c)$ we have $\|x_0-x\|^2\geqq 2n(t_0-t)\log\left(d_1/(t_0-t)\right)$; therefore, if $t_0-t<d_1e^{-1}$ also, then the expression in (15) is majorized by

$$2\,\|x_0-x\|^2(t_0-t)^{-2}+1,$$

which is integrable over $\Omega(p_0,c)$, by [12, Lemma 4]. We can therefore make $t\to t_0-$ and obtain the identity

$$\int_{A_D}\left(\frac{2}{n}\left(\frac{2}{n}+1\right)\frac{\|\nabla_x G_D\|^2}{G_D^2}vw-\frac{2v}{nG_D}\langle\nabla_x w,\nabla_x G_D\rangle+\frac{2D_tG_D}{nG_D}vw\right)dp \tag{16}$$

$$=\varkappa_n\left(c^{(n/2)+1}\mathscr{M}_D(c)-c_1^{(n/2)+1}\mathscr{M}_D(c_1)\right).$$

We now want to apply Lemma 1 to the left side of (16). If γ is a regular value in $[c_1, c]$, then on $\partial\Omega_D(\gamma)$ we have

$$\left(\frac{2}{n}\left(\frac{2}{n}+1\right)\frac{\|\nabla_x G_D\|^2}{G_D^2}vw-\frac{2v}{nG_D}\langle\nabla_x w, \nabla_x G_D\rangle+\frac{2D_t G_D}{nG_D}vw\right)\|\nabla G_D\|^{-1}$$
$$=\left(\frac{2}{n}\left(\frac{2}{n}+1\right)\frac{K_D}{G_D^2}\gamma w+\frac{2\gamma}{nG_D}(\langle\nabla_x w, \nu_x\rangle-w\nu_t)\right)$$
$$=\varkappa_n^2\left(1+\tfrac{n}{2}\right)\gamma^{n+1}K_D w+\varkappa_n\gamma^{(n/2)+1}(\langle\nabla_x w, \nu_x\rangle-w\nu_t),$$

since $v=\gamma$, $\nu=-\nabla G_D\|\nabla G_D\|^{-1}$, $K_D=\|\nabla_x G_D\|^2\|\nabla G_D\|^{-1}$ and $G_D=(4\pi\gamma)^{-n/2}$. The integral of this expression over $\partial\Omega_D(\gamma)$ is

$$\varkappa_n^2\left(1+\tfrac{n}{2}\right)\gamma^{n+1}\mathscr{M}_D(\gamma)+\varkappa_n\gamma^{(n/2)+1}\int_{\partial\Omega_D(\gamma)}(\langle\nabla_x w, \nu_x\rangle-w\nu_t)\,d\sigma,$$

so that from Lemma 1 and (16) we obtain

$$c^{(n/2)+1}\mathscr{M}_D(c)-c_1^{(n/2)+1}\mathscr{M}_D(c_1) \tag{17}$$
$$=\left(1+\tfrac{n}{2}\right)\int_{c_1}^{c}\gamma^{n/2}\mathscr{M}_D(\gamma)\,d\gamma+\varkappa_n^{-1}\int_{c_1}^{c}d\gamma\int_{\partial\Omega_D(\gamma)}(\langle\nabla_x w, \nu_x\rangle-w\nu_t)\,d\sigma.$$

Since (17) holds with $c=c_2$, the right side of (17) defines an absolutely continuous function of c on $[c_1, c_2]$. Therefore (17) enables us to extend $\mathscr{M}_D(c)$ to an absolutely continuous function f on $[c_1, c_2]$ such that

$$c^{(n/2)+1}f(c)-c_1^{(n/2)+1}f(c_1)$$
$$=\left(\tfrac{n}{2}+1\right)\int_{c_1}^{c}\gamma^{n/2}f(\gamma)\,d\gamma+\varkappa_n^{-1}\int_{c_1}^{c}d\gamma\int_{\partial\Omega_D(\gamma)}(\langle\nabla_x w, \nu_x\rangle-w\nu_t)\,d\sigma$$

for all $c\in[c_1, c_2]$. The function f is differentiable a.e., with

$$\left(c^{(n/2)+1}f(c)\right)'=\left(\tfrac{n}{2}+1\right)c^{n/2}f(c)+\varkappa_n^{-1}\int_{\partial\Omega_D(c)}(\langle\nabla_x w, \nu_x\rangle-w\nu_t)\,d\sigma,$$

which establishes (11).

We now take a regular value of $c\in]c_1, c_2]$, put $A_D=A_D(p_0, c_1, c)$ again, and apply (14) with $A=A_D$ to obtain

$$\int_{A_D}\theta w\,dp=\int_{\partial A_D}(\langle\nabla_x w, \nu_x\rangle-w\nu_t)\,d\sigma. \tag{18}$$

With the extra hypothesis that $\theta w\geqq 0$, we can apply Lemma 1 to the left side of (18) and obtain

$$\int_{A_D}\theta w\,dp=\varkappa_n^{-1}\int_{c_1}^{c}\gamma^{-(n/2)-1}d\gamma\int_{\partial\Omega_D(\gamma)}\|\nabla G_D\|^{-1}\theta w\,d\sigma.$$

It now follows from (11) and (18) that, for almost all $c\in[c_1, c_2]$,

$$(19)\quad \varkappa_n\left(c^{(n/2)+1}f'(c)-c_1^{(n/2)+1}f'(c_1)\right) = \varkappa_n^{-1}\int_{c_1}^{c} \gamma^{-(n/2)-1}\,d\gamma\int_{\partial\Omega_D(\gamma)}\|\nabla G_D\|^{-1}\theta w\,d\sigma.$$

(Since we can vary c_1 and c_2 slightly without affecting the statement of the lemma, we may assume that (11) holds when $c\in\{c_1, c_2\}$, and then (19) holds with $c=c_2$.) The right side of (19) defines an absolutely continuous function of c on $[c_1, c_2]$, so that f' coincides a.e. with an absolutely continuous function g on $[c_1, c_2]$. Then f is the indefinite integral of g, so that $f'(c)$ exists and (19) holds for all $c\in[c_1, c_2]$. Since $f'=g$, it is differentiable a.e., and (12) follows from (19).

4. Mean values of subtemperatures

In this section we establish various properties of the means $\mathcal{M}_D(w, p_0, c)$ when w is a subtemperature. We also extend these properties to volume means, and establish inequalities between the volume and surface means. The volume means are defined by

$$\mathcal{V}_D(u, p_0, c) = \left(1+(2/n)\right)(4\pi c)^{-(n/2)-1}\int_{\Omega_D(p_0, c)} G_D(p_0, \cdot)^{-(2/n)-2}\|\nabla_x G_D(p_0, \cdot)\|^2 u\,dp$$

for any function u on $\Omega_D(p_0, c)$ such that the integral exists. As before, the open set D must be θ^*-regular, but $\mathcal{V}_D(u, p_0, c)$ is defined whether c is a regular value or not.

Theorem 2. *If w is a subtemperature on an open set E and $p_0\in E$, then $\mathcal{M}_D(w, p_0, \cdot)$ is increasing and real-valued on the set of regular values of c such that $\bar{\Omega}_D(p_0, c)\subseteq E$, and $w(p_0)=\inf\mathcal{M}_D(w, p_0, \cdot)$. Furthermore, for every c such that $\bar{\Omega}_D(p_0, c)\subseteq E$, we have $w(p_0)\leqq\mathcal{V}_D(w, p_0, c)$.*

Proof. Let $I=\{c>0\colon \bar{\Omega}_D(p_0, c)\subseteq E\}$, and suppose first that w is smooth, so that $\theta w\geqq 0$. By Lemma 2, if c_1, c_2 are regular values in I with $c_1<c_2$, there is an absolutely continuous function f on $[c_1, c_2]$ such that $f(c)=\mathcal{M}_D(w, p_0, c)$ for all regular values of $c\in[c_1, c_2]$, and

$$\varkappa_n c^{(n/2)+1}f'(c) = \int_{\partial\Omega_D(c)} \left(\langle\nabla_x w, v_x\rangle - wv_t\right)d\sigma$$

for almost all $c\in[c_1, c_2]$. Applying (14) with $A=\Omega_D(c)$, we obtain

$$\varkappa_n c^{(n/2)+1}f'(c) = \int_{\Omega_D(c)} \theta w\,dp \geqq 0.$$

Therefore f is increasing on $[c_1, c_2]$, and it follows that $\mathcal{M}_D(w, p_0, \cdot)$ is increasing.

If now w is an arbitrary subtemperature, given any $c\in I$ we can find a decreasing sequence $\{w_j\}$ of smooth subtemperatures with limit w on a neighbourhood of

$\bar{\Omega}_D(p_0, c)$, by [6, p. 281]. For each j, the function $\mathscr{M}_D(w_j, p_0, \cdot)$ is increasing, so that the same is true of $\mathscr{M}_D(w, p_0, \cdot) = \lim_{j\to\infty} \mathscr{M}_D(w_j, p_0, \cdot)$.

The upper semicontinuity of w easily implies that $\mathscr{M}_D(w, p_0, c) \to w(p_0)$ as $c \to 0$ through regular values (cf. [12, p. 406]), so that $w(p_0) = \inf \mathscr{M}_D(w, p_0, \cdot)$.

The next step is to prove that $w(p_0) \leqq \mathscr{V}_D(w, p_0, c)$ for all $c \in I$, as this is used in the proof that $\mathscr{M}_D(w, p_0, \cdot)$ is real-valued. We begin by showing that $\mathscr{V}_D(1, p_0, c) = 1$ for all c. By Theorem 1, for almost all γ we have

$$1 = \int_{\partial\Omega_D(\gamma)} K_D(p_0, \cdot)\, d\sigma. \tag{20}$$

Multiplying by $\gamma^{n/2}$ and integrating over $[0, c]$, we obtain

$$\begin{aligned} 1 &= \left(\tfrac{n}{2}+1\right) c^{-(n/2)-1} \int_0^c \gamma^{n/2}\, d\gamma \int_{\partial\Omega_D(\gamma)} K_D\, d\sigma \\ &= \left(\tfrac{n}{2}+1\right) c^{-(n/2)-1} \int_0^c \gamma^{-(n/2)-1}\, d\gamma \int_{\partial\Omega_D(\gamma)} (4\pi)^{-n-1} G_D^{-2-(2/n)} K_D\, d\sigma. \end{aligned}$$

Since $K_D = \|\nabla_x G_D\|^2 \|\nabla G_D\|^{-1}$, it follows from Lemma 1 that

$$1 = \left(\tfrac{n}{2}+1\right) c^{-(n/2)-1} \varkappa_n \int_{\Omega_D(c)} (4\pi)^{-n-1} G_D^{-2-(2/n)} \|\nabla_x G_D\|^2\, dp = \mathscr{V}_D(1, p_0, c).$$

Since w is locally bounded above, it follows that $\mathscr{V}_D(w, p_0, c)$ is defined for every $c \in I$. Therefore, if we replace (20) by the inequality

$$w(p_0) \leqq \int_{\partial\Omega_D(\gamma)} K_D(p_0, \cdot)\, w\, d\sigma,$$

a similar calculation yields $w(p_0) \leqq \mathscr{V}_D(w, p_0, c)$ for all $c \in I$.

We now show that $\mathscr{V}_D(w, p_0, c) > -\infty$ for all sufficiently small c. Recall that there is a bounded θ^*-temperature h on D such that $G_D(p_0, \cdot) = G(p_0, \cdot) - h$. Choose c_0 such that $\bar{\Omega}(p_0, c_0) \subseteqq D$. Then $\|\nabla_x h\|$ is bounded on $\bar{\Omega}(p_0, c_0)$, so that

$$\|\nabla_x G_D\|^2 \leqq 2(\|\nabla_x G\|^2 + \|\nabla_x h\|^2) \leqq 2\|\nabla_x G\|^2 + K \tag{21}$$

on $\bar{\Omega}(p_0, c_0)$, for some constant K. Since h is bounded, there is $\delta > 0$ such that $G_D(p_0, \cdot) \geqq G(p_0, \cdot) - \delta$ on D. If c_1 is chosen such that $0 < c_1 \leqq c_0$ and $(4\pi c_1)^{-n/2} \geqq 2\delta$, then on $\bar{\Omega}(p_0, c_1)$ we have both (21) and $G(p_0, \cdot) \geqq 2\delta$, so that

$$G_D^{(2/n)+2} \geqq 2^{-(2/n)-1} G^{(2/n)+2} - \delta^{(2/n)+2} \geqq 2^{-(2/n)-2} G^{(2/n)+2},$$

and hence

$$\|\nabla_x G_D\|^2 G_D^{-(2/n)-2} \leqq K_1 \|\nabla_x G\|^2 G^{-(2/n)-2} + K_2$$

for some constants K_1, K_2. Since $\Omega_D(p_0, c) \subseteqq \Omega(p_0, c)$, it follows that

$$\mathscr{V}_D(w^-, p_0, c) \leqq K_1 \mathscr{V}(w^-, p_0, c) + K_3 c^{-(n/2)-1} \int_{\Omega_D(c)} w^-\, dp$$

for all $c \in]0, c_1[$, where K_3 is a constant and $\mathscr{V}$ denotes $\mathscr{V}_D$ with $D = \mathbf{R}^{n+1}$. Since

w is locally integrable on E by [12], and $\mathcal{V}(w, p_0, c)\in\mathbf{R}$ by [14], we deduce that $\mathcal{V}_D(w, p_0, c)\in\mathbf{R}$ whenever $c\in]0, c_1[$.

We can now show that $\mathcal{M}_D(w, p_0, \cdot)$ is real-valued. If $c\in]0, c_1[$, we have

$$(22)\qquad \left(\tfrac{n}{2}+1\right) c^{-(n/2)-1}\int_0^c \gamma^{n/2}\mathcal{M}_D(w, p_0, \gamma)\, d\gamma = \mathcal{V}_D(w, p_0, c)\in\mathbf{R},$$

so that $\mathcal{M}_D(w, p_0, \cdot)$ is finite a.e. on $]0, c_1[$. Since $\mathcal{M}_D(w, p_0, \cdot)$ is increasing, it is therefore finite at every regular value of $c\in]0, c_1[$. Since w is locally bounded above and (20) holds, $\mathcal{M}_D(w, p_0, c)\in\mathbf{R}$ whenever it is defined.

Theorem 3. *Let $p_0\in D$, let $0<c_1<c_2$, and let w be a subtemperature on an open superset E of $\bar{A}_D(p_0, c_1, c_2)$. Then there is a function φ, either finite and convex or identically $-\infty$, such that $\mathcal{M}_D(w, p_0, c)=\varphi(c^{-n/2})$ for all regular values of $c\in[c_1, c_2]$.*

Proof. Suppose first that w is smooth. By Lemma 2, there is an absolutely continuous function f on $[c_1, c_2]$ such that $f(c)=\mathcal{M}_D(w, p_0, c)$ for all regular values of $c\in[c_1, c_2]$, f' exists everywhere and is absolutely continuous on $[c_1, c_2]$ and

$$\varkappa_n^2\left(c^{(n/2)+1}f'(c)\right)' = c^{-(n/2)-1}\int_{\partial\Omega_D(c)} \|\nabla G_D(p_0, \cdot)\|^{-1}\theta w\, d\sigma$$

for almost all c. Since $\theta w\geqq 0$, for almost all c we have

$$(23)\qquad \left(c^{(n/2)+1}f'(c)\right)' \geqq 0.$$

The function λ on $[c_1, c_2]$, defined by $\lambda(c)=c^{n/2}f(c)$, is differentiable everywhere, with

$$\lambda'(c) = (n/2)\, c^{(n/2)-1}f(c)+c^{n/2}f'(c).$$

Hence λ' is absolutely continuous, so that for almost all c we have

$$\lambda''(c) = \tfrac{n}{2}\left(\tfrac{n}{2}-1\right) c^{(n/2)-2}f(c)+nc^{(n/2)-1}f'(c)+c^{n/2}f''(c).$$

If $\Lambda(\xi)=\lambda(\xi^{2/n})$, then $\Lambda'(\xi)=(2/n)\lambda'(\xi^{2/n})\xi^{(2/n)-1}$ for all ξ, and

$$\Lambda''(\xi) = \tfrac{4}{n^2}\,\xi^{(2/n)-2}\left(\left(\tfrac{n}{2}+1\right)\xi f'(\xi^{2/n})+\xi^{(2/n)+1}f''(\xi^{2/n})\right)$$

for almost all ξ. Writing $c=\xi^{2/n}$, this becomes

$$\Lambda''(\xi) = \tfrac{4}{n^2}\,\xi^{(2/n)-2}\left(\left(\tfrac{n}{2}+1\right)c^{n/2}f'(c)+c^{(n/2)+1}f''(c)\right),$$

so that $\Lambda''(\xi)\geqq 0$ for almost all ξ, by (23). Since λ' is absolutely continuous, so is Λ'. Therefore Λ' is the indefinite integral of Λ'', so that Λ' is increasing and Λ is convex. Thus $c^{n/2}f(c)$ is a convex function of $c^{n/2}$, which means that $f(c)$ is a convex function of $c^{-n/2}$. This establishes the result for smooth subtemperatures.

If w is not smooth, take a decreasing sequence $\{w_j\}$ of smooth subtemperatures which converges to w on a neighbourhood of $\bar{A}_D(p_0, c_1, c_2)$. Then for each j there is a convex function φ_j such that $\mathcal{M}_D(w_j, p_0, c) = \varphi_j(c^{-n/2})$ for all regular values of c. Therefore

$$\mathcal{M}_D(w, p_0, c) = \lim_{j\to\infty} \mathcal{M}_D(w_j, p_0, c) = \lim_{j\to\infty} \varphi_j(c^{-n/2})$$

at the regular values.

Corollary. *Let $p_0 \in D$, let $0 < c_1 < c_2$, and let w be a subtemperature on an open superset E of $\bar{A}_D(p_0, c_1, c_2)$. If v is defined on $\partial\Omega_D(p_0, c)$ for all regular values of $c \in [c_1, c_2]$ by*

$$v(p) = \mathcal{M}_D(w, p_0, G_D(p_0, p)^{-2/n}),$$

and v is not identically $-\infty$, then v can be extended to a θ^*-subtemperature on $A_D(p_0, c_1, c_2)$.

Proof. By Theorem 3, there is a convex function φ such that $v(p) = \varphi(G_D(p_0, p))$. Since $\varphi \circ G_D(p_0, \cdot)$ is defined on $A_D(p_0, c_1, c_2)$, and $G_D(p_0, \cdot)$ is a θ^*-temperature there, it follows from the dual of [12, Theorem 2] that $\varphi \circ G_D(p_0, \cdot)$ is a θ^*-subtemperature on $A_D(p_0, c_1, c_2)$.

We now establish that the volume means of subtemperatures have similar properties to those of the surface means, and derive inequalities between the two.

Theorem 4. *Let w be a subtemperature on an open superset E of $\bar{\Omega}_D(p_0, c_0)$. Then*

(i) *$\mathcal{V}_D(w, p_0, \cdot)$ is real-valued and increasing on $]0, c_0]$,*

(ii) *there is a convex function ψ such that $\mathcal{V}_D(w, p_0, c) = \psi(c^{-n/2})$ for all $c \in]0, c_0]$,*

(iii) *for all regular values of c in $]0, c_0]$,*

$$\mathcal{V}_D(w, p_0, c) \leqq \mathcal{M}_D(w, p_0, c),$$

(iv) *if $\varkappa = ((n/2)+1)^{-2/n}$, then for all regular values of $\varkappa c$ in $]0, \varkappa c_0]$,*

$$\mathcal{M}_D(w, p_0, \varkappa c) = \mathcal{V}_D(w, p_0, c).$$

The proof of Theorem 4 is similar to that of [14, Theorem 3], and is based upon the formula in (22). When $D = \mathbf{R}^{n+1}$, it is known that $\varkappa$ is the best possible constant in (iv), but we are unable to establish this in general.

5. Some consequences of the convexity theorem

We first use Theorem 3 to obtain the conclusion of Theorem 1 under weaker hypotheses.

Theorem 5. *Let w be a subtemperature on an open superset E of $\bar{\Omega}_D(p_0, c_0)$. If w is a temperature on $\Omega_D(p_0, c_0)$, then*

$$w(p_0) = \mathcal{M}_D(w, p_0, c)$$

for every regular value of $c \in]0, c_0]$.

We omit the proof of Theorem 5, since it differs from that of [14, Theorem 4] only in minor details.

The next result extends and, in a minor way, strengthens [14, Theorem 5]. The proof is necessarily different, and requires the concepts of upper and lower PWB solutions of the Dirichlet problem, and we refer to [6, p. 329] for a discussion of these.

Theorem 6. *Let w be a subtemperature on an open superset E of $\bar{\Omega}_D(p_0, c_0)$, where c_0 is a regular value. Then there is a unique subtemperature v on E such that v is a temperature on $\Omega_D(p_0, c_0)$ and $v=w$ on $E \setminus \bar{\Omega}_D(p_0, c_0)$. Furthermore, $v \geqq w$ on E, $\mathcal{M}_D(v, p_0, c_0) = \mathcal{M}(w, p_0, c_0)$, and on $\Omega_D(p_0, c_0)$ the function v is the PWB solution of the Dirichlet problem with boundary function the restriction of w to $\partial\Omega_D(p_0, c_0)$.*

Proof. Let $\Omega = \Omega_D(p_0, c_0)$. Let $\{c_j\}$ be a strictly decreasing sequence of regular values with limit c_0, and for each j let $\Omega_j = \Omega_D(p_0, c_j)$. Then $\{\Omega_j\}$ is a contracting sequence with intersection $\bar{\Omega} \setminus \{p_0\}$, and we can therefore suppose that $\bar{\Omega}_j \subseteq E$ for all j.

Given j, let Γ_j denote the class of all subtemperatures u on E such that $u \leqq w$ on $E \setminus \Omega_j$. If $u \in \Gamma_j$, then its restriction to Ω_j belongs to the lower PWB class on Ω_j for the restriction of w to $\partial\Omega_j$, so that $\sup \Gamma_j \leqq \underline{H}_w^j$ on Ω_j, where $\underline{H}_w^j$ denotes the corresponding lower PWB solution. Since w is locally bounded above, there is a constant λ such that $\underline{H}_w^j \leqq \bar{H}_w^j \leqq \lambda$ on Ω_j (where $\bar{H}_w^j$ denotes the corresponding upper PWB solution), so that w is resolutive, by [6, pp. 332, 115]. Therefore $\sup \Gamma_j \leqq H_w^j$, the PWB solution. On the other hand, if h belongs to the lower PWB class on Ω_j for w, then so does $\max\{h, w\}$, which can be extended by w to a subtemperature on E. This extension belongs to Γ_j, which implies that $H_w^j = \sup \Gamma_j$. By the parabolic fundamental convergence theorem [6, p. 314], the upper semicontinuous regularization w_j of $\sup \Gamma_j$ is a subtemperature on E. Clearly $w_j = w$ on $E \setminus \bar{\Omega}_j$, w_j is a temperature on Ω_j, and $w_j \geqq w$ on E. The sequence $\{w_j\}$ is decreasing, so that the function $w_\infty = \lim_{j\to\infty} w_j$ is a subtemperature on E that majorizes w on E, coincides with w on $E\setminus\bar{\Omega}$, and is a temperature on Ω by the Harnack convergence theorem [6, p. 276].

Now let v be an arbitrary subtemperature on E that is a temperature on Ω and equal to w on $E\setminus\bar{\Omega}$. For each j we have $v=w$ on $\partial\Omega_j\setminus\{p_0\}$, so that $H_v^j=H_w^j$. Therefore, if v_j bears the same relation to v as w_j does to w, then $v_j=w_j$ on $E\setminus\partial\Omega_j$. Since c_j is a regular value, $\partial\Omega_j$ has $(n+1)$-dimensional Lebesgue measure zero, so it follows that $v_j=w_j$ on E, by [13, p. 276]. Therefore $v\leqq\lim_{j\to\infty}v_j=w_\infty$ on E. By Theorem 5, for any regular value of $c\in]0, c_0]$, we have

$$\mathcal{M}_D(v, p_0, c) = v(p_0) = \mathcal{M}_D(v, p_0, c_0).$$

By Theorem 3,

$$\mathcal{M}_D(v, p_0, c_0) = \lim_{j\to\infty}\mathcal{M}_D(v, p_0, c_j). \tag{24}$$

Since $v=w=w_\infty$ on $E\setminus\bar{\Omega}$, we have

$$\mathcal{M}_D(v, p_0, c_j) = \mathcal{M}_D(w_\infty, p_0, c_j)$$

for all j, so that by Theorems 3 and 5,

$$\mathcal{M}_D(v, p_0, c) = \lim_{j\to\infty}\mathcal{M}_D(w_\infty, p_0, c_j) = \mathcal{M}_D(w_\infty, p_0, c_0) = w_\infty(p_0) = \mathcal{M}_D(w_\infty, p_0, c)$$

for every regular value of $c\in]0, c_0]$. Since $w_\infty-v\geqq 0$ on E, it follows that $w_\infty=v$ on $\partial\Omega_D(p_0, c)$ for all such c, and therefore $w_\infty=v$ on $\bar{\Omega}$ because both functions are continuous on Ω. Hence $v=w_\infty\geqq w$ on E. Since $v=w$ on $E\setminus\bar{\Omega}$, it follows from (24) and the corresponding formula for w, that $\mathcal{M}_D(v, p_0, c_0)=\mathcal{M}_D(w, p_0, c_0)$. Finally, by an argument similar to that used on Ω_j above, the upper semicontinuous regularization of the function that is equal to w on $E\setminus\Omega$ and to the PWB solution for w on Ω, is a subtemperature on E with the properties of v, and is therefore equal to v.

We are now in a position to generalize the results on thermic majorization in [15]. If $p_0\in D$, we denote by $\Lambda(p_0, D)$ the set of all points $p\in D\setminus\{p_0\}$ which can be joined to p_0 by a polygonal line in E along which the t-coordinate increases strictly from p to p_0. By the dual of [13, Theorem 14], we have

$$\Lambda(p_0, D) = \{p\in D\colon\ G_D(p_0, \cdot) > 0\} = \bigcup\nolimits_{c>0}\Omega_D(p_0, c).$$

Lemma 3. *Let w be a subtemperature on an open superset E of $\Lambda(p_0, D)\cup\{p_0\}$. If $\mathcal{M}_D(w, p_0, \cdot)$ is bounded above on the set of all positive regular values, then there is an increasing family $\{w_c\colon c$ is a regular value$\}$ of subtemperatures on E such that the function $u=\lim_{c\to\infty}w_c$ is the least thermic majorant of w on $\Lambda(p_0, D)$. Furthermore,*

$$u(p_0) = \lim_{c\to\infty}\mathcal{M}_D(w, p_0, c).$$

Theorem 7. *Let w be a subtemperature on D. Then w has a thermic majorant on D if and only if there is a sequence $\{p_j\}$ in D such that*

$$D = \bigcup_{j=1}^{\infty} \Lambda(p_j, D)$$

and $\mathcal{M}_D(w, p_j, \cdot)$ is bounded above on the set of all positive regular values for every j. If w has a thermic majorant on D, and u is the least one, then

$$u(p) = \sup \mathcal{M}_D(w, p, \cdot) = \lim_{c\to\infty} \mathcal{M}_D(w, p, c)$$

for every $p \in D$.

The proofs of these results follow those of [15, Lemma 2 and Theorem 3].

6. The relation between the integral means and swept measures

In this section we generalize a result of Bauer [2] by proving that the measure $K_D(p_0, \cdot)d\sigma$ is obtained by sweeping the unit mass at p_0 out to the complement of $\Omega_D(p_0, c)$. We use the notation $f-\lim$ to denote a fine limit, that is, a limit in the coarsest topology in which every subtemperature is continuous. We refer to [6] for details on sweeping and fine limits.

Lemma 4. *Let E be an open superset of $\bar{\Omega}_D(p_0, c_0)$, where c_0 is a regular value, let w be the difference of two subtemperatures on E, and let u be the PWB solution of the Dirichlet problem on $\Omega_D(p_0, c_0)$ corresponding to the boundary function w. Then*

$$\mathcal{M}_D(w, p_0, c_0) = f-\lim_{p\to p_0} u(p).$$

Proof. Let w_1 and w_2 be subtemperatures on E such that $w=w_1-w_2$, and let u_1 and u_2 be the PWB solutions of the Dirichlet problem on $\Omega_D(p_0, c_0)$ corresponding to the boundary functions w_1 and w_2, so that $u=u_1-u_2$. Using Theorem 6, we can extend the temperatures u_i to subtemperatures on E, which we also denote by u_i, in such a way that $\mathcal{M}_D(u_i, p_0, c_0)=\mathcal{M}_D(w_i, p_0, c_0)$. Let $u=u_1-u_2$ throughout E. Then Theorem 5 implies that

$$u(p_0) = \mathcal{M}_D(u_1, p_0, c_0) - \mathcal{M}_D(u_2, p_0, c_0) = \mathcal{M}_D(w, p_0, c_0).$$

By [6, p. 308], every $\Omega(p_0, d)$ is a deleted fine neighbourhood of p_0, so that the same is true of $\Omega_D(p_0, c_0)$, in view of (4). Since u is a difference of subtemperatures on E, we have $u(p_0)=f-\lim_{p\to p_0} u(p)$, and the result follows.

For Theorem 8, we need some notation. We let ε_0 denote the unit mass at p_0, and for any set A we let ε_0^{CA} denote ε_0 swept out to $\mathbf{R}^{n+1}\setminus A$.

Theorem 8. *Let $p_0 \in D$, let c_0 be a regular value, and let μ_0^D denote the probability measure on $\partial\Omega_D(p_0, c_0)$ given by $d\mu_0^D = K_D(p_0, \cdot)d\sigma$. Then $\mu_0^D = \varepsilon_0^{C\Omega}$, where $\Omega = \Omega_D(p_0, c_0)$.*

Proof. Let S denote the class of all continuous functions on $\partial\Omega$ that are the restrictions of subtemperatures defined on open supersets of $\bar{\Omega}$. By Lemma 4, given any $g \in S$ we can find a sequence $\{p_j\}$ in Ω such that $p_j \to p_0$ and $\mu_j^\Omega(g) \to \mu_0^D(g)$, where μ_j^Ω denotes the parabolic measure for Ω at p_j [6, p. 332]. From this sequence of probability measures, we can extract a subsequence which converges vaguely to a probability measure λ_g on $\partial\Omega$, by [1, p. 243]. Therefore $\mu_0^D(g) = \lambda_g(g)$. By a result of Boboc and Cornea [3] (for a simple proof, see [8]), there is $\beta_g \in [0, 1]$ such that

$$\lambda_g = \beta_g \varepsilon_0 + (1-\beta_g)\varepsilon_0^{C(\Omega \cup \{p_0\})}.$$

Since $\{p_0\}$ is polar, it follows that

$$\mu_0^D(g) = \beta_g \varepsilon_0(g) + (1-\beta_g)\varepsilon_0^{C\Omega}(g). \tag{25}$$

Let f be a continuous function on $\partial\Omega$. By [13, p. 290], there is a sequence $\{f_i\}$ in $S-S$ which converges uniformly to f on $\partial\Omega$. By (25), for each i there is $\beta_i = \beta_{f_i} \in [0, 1]$ such that

$$\mu_0^D(f_i) = \beta_i \varepsilon_0(f_i) + (1-\beta_i)\varepsilon_0^{C\Omega}(f_i),$$

that is,

$$\beta_i\big(\varepsilon_0^{C\Omega}(f_i) - \varepsilon_0(f_i)\big) = \varepsilon_0^{C\Omega}(f_i) - \mu_0^D(f_i).$$

Since $f_i \to f$ uniformly on $\partial\Omega$, we have $\nu(f_i) \to \nu(f)$ for every finite measure ν on $\partial\Omega$. If $\varepsilon_0^{C\Omega}(f_i) - \varepsilon_0(f_i) \to 0$ then $\varepsilon_0^{C\Omega}(f) = \mu_0^D(f)$, and we put $\alpha_f = 0$. Otherwise

$$\beta_i \to \frac{\varepsilon_0^{C\Omega}(f) - \mu_0^D(f)}{\varepsilon_0^{C\Omega}(f) - \varepsilon_0(f)},$$

and we label this quotient α_f. Then, in both cases, $\alpha_f \in [0, 1]$ and

$$\mu_0^D(f) = \alpha_f \varepsilon_0(f) + (1-\alpha_f)\varepsilon_0^{C\Omega}(f).$$

It now follows from a result of Bauer [2, p. 80] that there is $\alpha \in [0, 1]$ such that $\mu_0^D = \alpha\varepsilon_0 + (1-\alpha)\varepsilon_0^{C\Omega}$. Since μ_0^D is absolutely continuous with respect to σ, we have

$$0 = \mu_0^D(\{p_0\}) = \alpha + (1-\alpha)\varepsilon_0^{C\Omega}(\{p_0\}) \geqq \alpha,$$

so that $\alpha = 0$ and $\mu_0^D = \varepsilon_0^{C\Omega}$.

Remark. It follows from (4), and the fact that p_0 is a Dirichlet irregular boundary point of every $\Omega(p_0, d)$ (see [12, p. 399]), that p_0 is also Dirichlet irregular for $\Omega_D(p_0, c_0)$. We can therefore combine a result of Netuka [9, p. 7] with Theorem 8, and deduce that, for every lower bounded resolutive function f on $\partial\Omega_D(p_0, c_0)$, the integral $\mathcal{M}_D(f, p_0, c_0)$ exists and

$$f-\lim_{p\to p_0} H_f(p) = \mathcal{M}_D(f, p_0, c_0), \tag{26}$$

which greatly generalizes Lemma 4. Netuka also gives an example, for the case $D=\mathbf{R}^{n+1}$, of a resolutive boundary function f such that the fine limit in (26) does not exist.

7. The three sets theorem

Let E be an arbitrary open subset of $\mathbf{R}^{n+1}$, and let $p_0 \in E$. We shall prove a result analogous to the classical three spheres theorem on subharmonic functions, in which the spheres are effectively replaced by level sets of the Green function G_E. Such a result is not possible if we take the suprema of a subtemperature over $\partial\Omega_E(p_0, c)$ for $c>0$, since it would imply their continuity as a function of c, and the characteristic function of $\mathbf{R}^n \times]-\infty, 0]$ is a subtemperature which would create a discontinuity. We therefore replace $\partial\Omega_E(p_0, c)$ by $\partial\Omega_E^*(p_0, c)$, where for any $c>0$

$$\Omega_E^*(p_0, c) = \{p\in E\colon G_E(p, p_0) > (4\pi c)^{-n/2}\}.$$

Since $G_E(\cdot, p_0) \leqq G(\cdot, p_0)$, we have $\Omega_E^*(p_0, c) \subseteqq \Omega^*(p_0, c)$, so that $\Omega_E^*(p_0, c)$ is bounded. If $0<c_1<c_2$, we put

$$A_E^*(p_0, c_1, c_2) = \Omega_E^*(p_0, c_2) \setminus \bar{\Omega}_E^*(p_0, c_1).$$

The result seems to be new even in the case $E=\mathbf{R}^{n+1}$.

Theorem 9. *Let w be an upper bounded subtemperature on $A=A_E^*(p_0, c_1, c_2)$, and define w on $\partial A \setminus \{p_0\}$ by*

$$w(q) = \limsup_{p\to q,\ p\in A} w(p).$$

If the function $\mathcal{S}_E$ is defined on $[c_1, c_2]$ by

$$\mathcal{S}_E(c) = \sup\big\{w(p)\colon p\in\partial\Omega_E^*(p_0, c)\cap(E\setminus\{p_0\})\big\},$$

then there is a real-valued convex function φ such that $\mathcal{S}_E(c)=\varphi(c^{-n/2})$ for all $c\in[c_1, c_2]$.

Proof. Let $0\leqq d_1<d_2$, and put $R=A_E^*(p_0, d_1, d_2)$ if $d_1>0$, $R=\Omega_E^*(p_0, d_2)$ if $d_1=0$. If E is not Dirichlet regular, then $\partial R\cap\partial E$ may not be empty. However, we can show that $\partial R\cap\partial E$ is a parabolic measure null set relative to R. The proof is similar to one given for Laplace's equation by Brelot and Choquet in [5, p. 228], and again by Brelot in [4, p. 119]. That is, suppose that v is an upper bounded subtemperature on R, such that $\limsup v(p)\leqq 0$ as p approaches an arbitrary point of $E\cap\partial R$ from inside R. Choose $\varkappa>0$ such that

$$\varkappa(\sup_R v^+) \leqq (4\pi d_2)^{-n/2},$$

and put $f=-\varkappa v^+$ on R, $f=0$ on $E\setminus R$. Then f is lower semicontinuous on E, and since it satisfies locally the characteristic mean value inequality, it is a supertemperature on E, by [12, p. 408]. Whenever $p\in R$ we have

$$G_E(p, p_0)+f(p) \geqq (4\pi d_2)^{-n/2}-\varkappa(\sup_R v^+) \geqq 0,$$

so that $G_E(\cdot, p_0)+f\geqq 0$ on E. Since $G_E(\cdot, p_0)=G(\cdot, p_0)-\eta$ where η is a temperature on E, it follows from the minimality of $G_E(\cdot, p_0)$ [13, p. 267] that $G_E(\cdot, p_0)+f\geqq G_E(\cdot, p_0)$ on E. Therefore $v\leqq 0$ on R, so that $\partial R\cap\partial E$ is a parabolic measure null set.

It follows that $\mathscr{S}_E$ is real valued. For if $\mathscr{S}_E(d_2)=-\infty$, then $w(p)=-\infty$ for all $p\in\partial\Omega_E^*(p_0, d_2)\cap(E\setminus\{p_0\})$. Since $\partial R\cap\partial E$ is a parabolic measure null set, there is a negative subtemperature w_0 on R with limit $-\infty$ at every point thereof, by [6, pp. 329, 108], and therefore $w-\sup_R w+w_0$ is a negative subtemperature on R with limit $-\infty$ at every point of $\partial R\setminus(\partial\Omega_E^*(p_0, d_1)\cap E)$, making that a parabolic measure null set. But $G_E(\cdot, p_0)$ is a nonconstant, bounded temperature on R which is continuous and constant on $\partial\Omega_E^*(p_0, d_1)\cap E$, so that the remainder of ∂R cannot be a parabolic measure null set [6, pp. 329, 110]. A similar argument shows that $\mathscr{S}_E(d_1)>-\infty$.

Now suppose that $c_1\leqq d_1<d_2\leqq c_2$. The function u, defined for all $p\in E$ by

$$u(p) = \frac{(4\pi d_1)^{-n/2}\mathscr{S}_E(d_2)-(4\pi d_2)^{-n/2}\mathscr{S}_E(d_1)+\big(\mathscr{S}_E(d_1)-\mathscr{S}_E(d_2)\big)G_E(p, p_0)}{(4\pi d_1)^{-n/2}-(4\pi d_2)^{-n/2}}$$

is a temperature on $E\setminus\{p_0\}$ and is constant on $\partial\Omega_E^*(p_0, c)\cap(E\setminus\{p_0\})$ for every $c>0$. Therefore $w-u$ is an upper bounded subtemperature on A (since $G_E(\cdot, p_0)$ is bounded on A), and for all $q\in\partial\Omega_E^*(p_0, c_i)\cap(E\setminus\{p_0\})$, $i\in\{1, 2\}$, we have

$$w(q)-u(q) = w(q)-\mathscr{S}_E(c_i) \leqq 0.$$

Since $(\partial A\cap\partial E)\cup\{p_0\}$ is a parabolic measure null set, it follows that $w\leqq u$ on A, and hence on $(\bar{A}\cap E)\setminus\{p_0\}$. Therefore, whenever $d\in[d_1, d_2]$,

$$\mathscr{S}_E(d) \leqq \frac{(d^{-n/2}-d_2^{-n/2})\mathscr{S}_E(d_1)+(d_1^{-n/2}-d^{-n/2})\mathscr{S}_E(d_2)}{d_1^{-n/2}-d_2^{-n/2}} \tag{27}$$

as required.

We can deduce from Theorem 9 appropriate analogues of the standard consequences of the classical three spheres theorem [10, p. 131].

Corollary 1. *Let* $c_2>0$, *and let* w *be a subtemperature which is bounded above on* $A_E^*(p_0, c_1, c_2)$ *for each* $c_1\in]0, c_2[$. *Then* w *is bounded above on* $\Omega_E^*(p_0, c_2)$ *if* *and only if*

$$\liminf_{c_1\to 0} c_1^{n/2}\mathscr{S}_E(c_1) \leqq 0. \tag{28}$$

If w is not bounded above on $\Omega_E^*(p_0, c_2)$, then there exist a sequence $\{p_j\}$ and a positive constant δ such that

$$w(p_j) \geqq \delta G_E(p_j, p_0) \to \infty. \tag{29}$$

Proof. If w is bounded above on $\Omega_E^*(p_0, c_2)$, then (28) obviously holds. Conversely, if (28) holds and $0<d<d_2\leqq c_2$, it follows from (27) that

$$\mathscr{S}_E(d) \leqq (d^{-n/2}-d_2^{-n/2}) \liminf_{d_1\to 0} d_1^{n/2}\mathscr{S}_E(d_1)+\mathscr{S}_E(d_2) \leqq \mathscr{S}_E(d_2).$$

Therefore $\mathscr{S}_E$ is increasing, and hence $w\leqq\mathscr{S}_E(c_2)$ on $\Omega_E^*(p_0, c_2)$.

If w is not bounded above, then the lower limit in (28) is positive; denote it by $2\delta(4\pi)^{-n/2}$. Then $(4\pi c_1)^{n/2}\mathscr{S}_E(c_1)\geqq\delta$ for all sufficiently small c_1, so that

$$\sup\{w(p)G_E(p, p_0)^{-1}\colon p\in\partial\Omega_E^*(p_0, c_1)\cap(E\setminus\{p_0\})\} \geqq \delta$$

for such c_1, and (29) follows.

Corollary 2. *Let $c_1\geqq 0$, and let w be a subtemperature which is bounded above on $A_E^*(p_0, c_2, c_3)$ whenever $c_1<c_2<c_3$. If*

$$\liminf_{c_3\to\infty} \mathscr{S}_E(c_3) \leqq 0, \tag{30}$$

then the function $c\mapsto c^{n/2}\mathscr{S}_E(c)$ is decreasing on $]c_1, \infty[$. Furthermore, if (30) holds, $c_1=0$, and $(4\pi c)^{n/2}\mathscr{S}_E(c)\to\lambda$ as $c\to 0$, then $w\leqq\lambda G_E(\cdot, p_0)$ on the set

$$\Lambda^*(p_0, E) = \bigcup_{c>0}\Omega_E^*(p_0, c).$$

Proof. Suppose that $c_1<c_2<c<c_3$. Then, by (27),

$$\mathscr{S}_E(c) \leqq \frac{(c^{-n/2}-c_3^{-n/2})\mathscr{S}_E(c_2)+(c_2^{-n/2}-c^{-n/2})\mathscr{S}_E(c_3)}{c_2^{-n/2}-c_3^{-n/2}}.$$

Making $c_3\to\infty$, we obtain

$$c_2^{-n/2}\mathscr{S}_E(c) \leqq c^{-n/2}\mathscr{S}_E(c_2)+(c_2^{-n/2}-c^{-n/2})\liminf_{c_3\to\infty}\mathscr{S}_E(c_3) \leqq c^{-n/2}\mathscr{S}_E(c_2)$$

by (30), so that the function $c\mapsto c^{n/2}\mathscr{S}_E(c)$ is decreasing on $]c_1, \infty[$.

For the last part, $\lambda=\sup_{c>0}(4\pi c)^{n/2}\mathscr{S}_E(c)$, so that

$$\sup\{w(p)G_E(p, p_0)^{-1}\colon p\in\partial\Omega_E^*(p_0, c)\cap(E\setminus\{p_0\})\} \leqq \lambda$$

for all $c>0$, so that $w\leqq\lambda G_E(\cdot, p_0)$ on $\Lambda^*(p_0, E)$.

As a final corollary, we give a new maximum principle for subtemperatures, which is analogous to a theorem of Brelot and Choquet [5, p. 229] for subharmonic functions on manifolds. (The Euclidean case is also given in [4, p. 121].)

Corollary 3. *Suppose that w is an upper bounded subtemperature on $\Lambda^*(p_0, E)$. If* $\limsup w(p_j) \leqq 0$ *for every sequence* $\{p_j\}$ *such that* $G_E(p_j, p_0) \to 0$, *then* $w \leqq 0$ *on* $\Lambda^*(p_0, E)$.

Proof. Given $\varepsilon > 0$, there is c such that $w \leqq \varepsilon$ on $\partial\Omega_E^*(p_0, c)$; for otherwise, whatever the positive integer k there would be $q_k \in \partial\Omega_E^*(p_0, k)$ such that $w(q_k) > \varepsilon$, so we would have $G(q_k, p_0) \to 0$ and $\limsup w(q_k) \geqq \varepsilon$, contrary to hypothesis. Therefore (30) holds, and since w is bounded above on $\Lambda^*(p_0, E)$ we have $c^{n/2}\mathcal{S}_E(c) \to 0$ as $c \to 0$. Hence $w \leqq 0$, by Corollary 2.

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