Mean values of subharmonic functions

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1. Introduction

Let u be a subharmonic function in $\mathbf{R}^n$. We introduce the maximum modulus

$$M(r) = M(r, u) = \max \{u(x) : |x| = r\},$$

the lower order

$$\lambda = \lambda(u) = \liminf_{r\to\infty} \frac{\log M(r)}{\log r},$$

and the mean value

$$T(r) = T(r, u) = \sigma_n^{-1} \int\limits_{|x|=1} u^+(rx)d\sigma(x),$$

where $d\sigma$ denotes the $(n-1)$-dimensional Hausdorff-measure, σ_n is the area of the unit sphere, $\sigma_n = \int_{|x|=1} d\sigma$, and $u^+ = \max \{u, 0\}$.

We shall study the relationship between the quantity

$$A(u) = \limsup_{r\to\infty} \frac{T(r, u)}{M(r, u)}$$

and the lower order of u.

Suppose $\lambda \in (0, \infty)$ is given. The Gegenabuer functions C_λ^γ are given as solutions of the differential equation

$$(1-x^2)\frac{d^2u}{dx^2} - (2\gamma+1)x\frac{du}{dx} + \lambda(\lambda+2\gamma)u = 0, \quad -1 < x < 1,$$

with the normalization $C_\lambda^\gamma(1) = \Gamma(\lambda+2\gamma)/\Gamma(2\gamma)\Gamma(\lambda+1)$. Put

$$a_\lambda = \sup \{t : C_\lambda^{\frac{n-2}{2}}(t) = 0\}$$

and define the function u_λ in $\mathbf{R}^n$, $n \geq 3$, by

$$u_\lambda(x) = \begin{cases} 0 \text{ if } x_1 \leq a_\lambda r \\ r^\lambda C_\lambda^{\frac{n-2}{2}}(x_1/r) \text{ if } x_1 > a_\lambda r, \end{cases}$$

where $x = (x_1, \ldots, x_n)$ and $r = |x|$.

Since u_λ is harmonic in $\{x \in \mathbf{R}^n : x_1 > a_\lambda |x|\} = K$ and has boundary values zero on ∂K, u_λ is subharmonic in $\mathbf{R}^n$ and the lower order of u_λ is λ. We define

$$C(\lambda, n) = A(u_\lambda) \tag{1.1}$$

We are now in a position to formulate our main result.

THEOREM 1.2. *Let u be a subharmonic function in $\mathbf{R}^n$, $n \geq 3$, of lower order λ, $0 < \lambda < \infty$. Then we have that*

$$\limsup_{r\to\infty} \frac{T(r, u)}{M(r, u)} \geq C(\lambda, n).$$

Hayman [4] has shown that for the set of subharmonic functions of finite lower order λ, $A(u)$ has a lower bound; his bounds are not best possible but of the right magnitudes as $\lambda \to \infty$. By the construction of $C(\lambda, n)$, it is clear that our bounds are best possible.

For subharmonic functions in higher dimensions Theorem 1.2 may be considered as an analogue of the following result by Petrenko [10] on the Paley conjecture:

Let f be a meromorphic function in $\mathbf{C}$ and put $\mu(r, f) = \sup_\theta |f(re^{i\theta})|$ and let $T(r, f)$ be the Nevanlinna characteristic of f. If the lower order of f is

$$\lambda = \liminf_{r\to\infty} \frac{\log T(r)}{\log r},$$

then

$$\limsup_{r\to\infty} \frac{T(r, f)}{\log \mu(r, f)} \geq \begin{cases} \dfrac{\sin \pi\lambda}{\pi\lambda} \text{ if } \lambda \leq \frac{1}{2} \\ \dfrac{1}{\pi\lambda} \text{ if } \frac{1}{2} < \lambda < \infty. \end{cases}$$

The plan of the paper is now as follows. In section 2 we derive some properties of the Neumann function for a cone. In section 3 these are used to establish an inequality for subharmonic functions. The proof of Theorem 1.2 is given in section 4 and we proceed in section 5 to some applications, which complete the paper.

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2. Some properties of the Neumann function

If $\Omega \subset \mathbf{R}^n$, $n \geq 3$, is an unbounded domain and $y \in \Omega$, then the Neumann function of Ω with pole at $y, N(\cdot, y)$, is a harmonic function in $\Omega - \{y\}$ such that

(i) $d/d\nu N(x, y) = 0$ for all $x \in \partial\Omega$, where $\partial\Omega$ is the boundary of Ω and $d/d\nu$ denotes directional derivative in the direction of the unit inner normal.

(ii) $N(\cdot, y) - r_y$ can be extended to a harmonic function in Ω where $r_y(x) = |x - y|^{2-n}$.

In the rest of this section we will use the following notation. Suppose $-1 < a < 1$ and put

$$K = \{x \in \mathbf{R}^n : x = (x_1, \ldots, x_n),\ \ x_1 > a|x|\}.$$

We let $D = \{x \in K : |x| = 1\}$ and $\partial' D = \{x \in \partial K : |x| = 1\}$. If $x \in \mathbf{R}^n$, then we introduce polar coordinates by putting $|x| = r$, $\theta = \arccos(x_1/r)$ and $x^* = x/r$. The Neumann function of K is denoted by N. If δ is the Laplace-Beltrami operator on the unit sphere and Δ is the Laplace operator in $\mathbf{R}^n$ then the following relation holds:

$$\Delta = \frac{d^2}{dr^2} + \frac{n-1}{r}\frac{d}{dr} + r^{-2}\delta.$$

Denote by $\{\lambda_i\}_{i=0}^{\infty}$, $0 = \lambda_0 < \lambda_1 < \lambda_2 < \ldots$, the sequence of eigenvalues of δ in D, where the corresponding eigenfunctions φ_i are assumed to be symmetric around the x_1-axis and satisfy the relation

$$\delta\varphi_i + \lambda_i\varphi_i = 0,\ \frac{d\varphi_i}{d\nu} = 0 \ \text{ on } \ \partial' D. \tag{2.1}$$

Let α_i, β_i, $\alpha_i \geq 0 > \beta_i$, be the roots of the equation

$$t(t + n - 2) = \lambda_i. \tag{2.2}$$

If $r \in \mathbf{R}$, then we identify r with $(r, 0, \ldots, 0) \in \mathbf{R}^n$. We observe that the function $x \to N(\varrho, x)$ is symmetric around the x_1-axis if $\varrho > 0$. Hence, following Bouligand [2], we have, if $\varrho > 0$ and $|x| = r \neq \varrho$, that

$$N(\varrho, x) = \sigma_n \sum_{i=0}^{\infty} \frac{s^{\alpha_i} R^{\beta_i} \varphi_i(x^*)\varphi_i(1)}{\sqrt{4\lambda_i + (n-2)^2}} \tag{2.3}$$

where $s = \min(r, \varrho)$ and $R = \max(r, \varrho)$ and φ_i are normalized so that

$$\int_D |\varphi_i|^2 d\sigma = 1$$

and N is normalized by $\lim_{|x|\to\infty} N(\varrho, x) = 0$.

It is well known that there exists an $\alpha \in (0, \infty)$ such that

$$\lim_{i=\infty} \alpha_i i^{-1} = \alpha. \tag{2.4}$$

In the sequel, the letter C will denote constants which will not necessarily be the same at each occurrence, and which may depend on the cone K or the dimension n.

We need some estimates of $\{\varphi_i\}$.

LEMMA 2.5. *There exists to each* $M > 1$ *a number* $C > 0$ *such that*
(I) $|\varphi_i(p)| \leq CM^{\alpha_i}$ *for all* $p \in D$,
(II) $|d\varphi_i/d\theta(p)| \leq CM^{\alpha_i}$ *for all* $p \in D$,
Here φ_i *is normalized by* $\int_D |\varphi_i|^2 d\sigma = 1$

Proof. Since φ_i are assumed to be symmetric with respect to the x_1-axis we have,

$$\varphi_i(p) = d_i C_{\alpha_i}^{\frac{n-2}{2}}(p_1), \quad p = (p_1, \ldots, p_n) \in D,$$

where C_α^γ are the Gegenbauer functions and $d_i > 0$ is chosen so that

$$\int_D |\varphi_i|^2 d\sigma = 1.$$

From the representation formula (22) in [3], p. 178, we have for $\gamma > 0$ and $0 \leq \theta < \pi/2$:

$$C_\alpha^\gamma(\cos\theta) = \pi^{-\frac{1}{2}}\Gamma(\alpha + 2\gamma)\Gamma(\gamma + \tfrac{1}{2})\{\Gamma(\gamma)\Gamma(2\gamma)\Gamma(\alpha + 1)\}^{-1} \times$$
$$\times \int_0^\pi \{\cos\theta + \sqrt{-1}\sin\theta\cos t\}^\alpha (\sin t)^{2\gamma-1} dt$$

This gives easily that for $\gamma > 0$ and $0 \leq \theta < \pi/2$

$$|C_\alpha^\gamma(\cos\theta)| \leq \Gamma(\alpha + 2\gamma)/\Gamma(2\gamma)\Gamma(\alpha + 1) = C_\alpha^\gamma(1). \tag{2.6}$$

To estimate $C_\alpha^\gamma(\cos\theta)$ for $\theta \geq \pi/2$ we use representation formula (23) in [3] p. 178, which gives

$$C_\alpha^\gamma(\cos\theta) = 2^\gamma \pi^{-\frac{1}{2}}\Gamma(\alpha + 2\gamma)\Gamma(\gamma + \tfrac{1}{2})\{\Gamma(\gamma)\Gamma(2\gamma)\Gamma(\alpha + 1)\}^{-1} \times$$
$$\times (\sin\theta)^{1-2\gamma} \int_0^\theta \cos[(\gamma + \alpha)t](\cos t - \cos\theta)^{\gamma-1} dt,$$

which is valid if $\gamma > 0$ and $0 < \theta < \pi$. Consequently

$$|C_\alpha^\gamma(\cos\theta)| \leq 2^{2\gamma}\pi^{\frac{1}{2}}(\sin\theta)^{1-2\gamma}\Gamma(\gamma+\tfrac{1}{2})(\Gamma\gamma))^{-1}C_\alpha^\gamma(1) \tag{2.7}$$

if $\gamma \geq 1$ and $0 < \theta < \pi$. If $\gamma = \frac{1}{2}$, then it is known that

$$|C_\alpha^{\frac{1}{2}}(\cos\theta)| \leq 2\alpha^{-\frac{1}{2}}\pi^{-\frac{1}{2}}(\sin\theta)^{-\frac{1}{2}}C_\alpha^{\frac{1}{2}}(1)$$

for $\alpha \geq 1$ and $0 < \theta < \pi$, see Hobson [6], § 200. From (2.6) and (2.7) it follows that there exists a number $C > 0$ such that $|\varphi_i(p)| \leq C\varphi_i(1)$ if $p \in D$. From formula (30), [3] page 178, we have that $d/dxC_\alpha^\gamma(x) = 2\gamma C_\alpha^{\gamma+1}(x)$, and hence, there exists a number $C > 0$ such that

$$\left|\frac{d\varphi_i}{d\theta}(p)\right| \leq C\, C_{\alpha_i}^{n/2}(1)\{C_{\alpha_i}^{\frac{n-2}{2}}(1)\}^{-1}\varphi(1).$$

But $C_{\alpha_i}^{n/2}(1)\{C_{\alpha_i}^{\frac{n-2}{2}}(1)\}^{-1} = (\alpha_i + n)(\alpha_i + n - 1)(n^2 - n)^{-1}$, so to prove Lemma 2.5 it is now sufficient to prove (I) for $p = 1$. An application of Green·s formula to the harmonic function $x \to r^{\alpha_i}\varphi_i(x^*)$ and $N(1, \cdot)$ yields:

$$\varphi_i(1) = \sigma_n^{-1}(n-2)^{-1} \int\limits_{\{x\in:K|x|=M\}} \left\{M^{\alpha_i-1}\varphi_i(x^*)N(1,x) - M^{\alpha_i}\varphi_2(x^*)\frac{d}{dr}N(1,x)\right\}d\sigma(x)$$

Hence there exists a number $C > 0$, such that

$$\varphi_i(1) \leq CM^{\alpha_i}\int\limits_D |\varphi_i(x)|d\sigma(x) \leq CM^{\alpha_i}\left(\text{since} \int\limits_D |\varphi_i|^2 = 1\right)$$

and this completes the proof of Lemma 2.5.

We need to know where the Neumann function assumes its smallest value.

Lemma 2.8. *Take any point $e \in \partial K$ with $|e| = 1$. Then for all $\varrho > 0$ and all $x \in K$ we have*

$$N(\varrho, x) \geq N(\varrho, |x|e).$$

Proof. If u is a function, which only depends on r and θ, then

$$\Delta u = \frac{d^2u}{dr^2} + \frac{n-1}{r}\frac{du}{dr} + r^{-2}\frac{d^2u}{d\theta^2} + (n-2)r^{-2}\cot\theta\frac{du}{d\theta}.$$

For a harmonic function u we have that for $0 < \theta < \pi$

$$\Delta\frac{du}{d\theta} = (n-2)r^{-2}(\sin\theta)^{-2}\frac{du}{d\theta}. \tag{2.9}$$

Let $\Omega = \{x \in K: \theta > 0$ and $d/d\theta N(\varrho, x) > 0\}$. Lemma 2.8 follows, if we can show that Ω is empty. Assume that $\Omega \neq \emptyset$. From relation (2.9) it follows that the function $d/d\theta N(\varrho, \cdot)$ is subharmonic in Ω and has boundary values zero on all of $\partial\Omega$ with the possible exception of $\partial\Omega \cap \mathbf{R}$. From inequality (2.6) and the expansion (2.3) it follows that $\lim_{z\to r} d/d\theta N(\varrho, z) \leq 0$ for all $r \neq \varrho$. Let h_ϱ be the harmonic function in K such that $N(\varrho, x) = |x - \varrho|^{2-n} + h_\varrho(x)$. We have that $d/d\theta|x - \varrho|^{2-n} = -(n-2)|x-\varrho)^{-n}\varrho|x|\sin\theta \leq 0$ if $x \neq \varrho$. Since $d/d\theta|x-\varrho|^{2-n} \to 0$ when $x \to r \neq \varrho$ we must have that $\lim_{x\to r} d/d\theta h_0(x) \leq 0$ for all $r > 0$, and hence $\limsup_{x\to\varrho} d/d\theta N(\varrho, x) \leq 0$. Recalling (2.4) and Lemma 2.5 we have $\lim_{|x|\to\infty} d/d\theta N(\varrho, x) = 0$. The maximum principle now gives that $d/d\theta N(\varrho, x) \leq 0$ in Ω, and this contradiction completes the proof of Lemma 2.8.

Now we shall prove a result concerning the boundary values of the Neumann function.

LEMMA 2.10. *Take any* $e \in \partial K$ *with* $|e| = 1$. *Given* $\varrho > 0$, *define* $\psi(\varrho, x) = N(\varrho, |x|e)$. *Then* ψ *is independent of the particular* e *chosen and* $\psi(\varrho, \cdot)$ *is superharmonic in* $\mathbf{R}^n - \{0\}$.

Proof. From Bouligand [2] it follows that $\psi(\varrho, \cdot)$ is two times continuously differentiable in $\mathbf{R}^n - \{0\}$. Suppose that there exists r_1, r_2, such that $\Delta\psi(\varrho, \cdot) \geq 0$ in $B = \{x: r_1 < |x| < r_2\}$. Then $\varepsilon(\varrho, \cdot) = \psi(\varrho, \cdot) - N(\varrho, \cdot)$ is subharmonic in $B \cap K = E$. From Lemma 2.8 $\varepsilon(\varrho, \cdot) \leq 0$ and $\varepsilon(\varrho, \cdot)$ is 0 on $\partial E \cap \partial K$ and has normal derivatives zero on $\partial E \cap \partial K \cap B = F$. But each $y \in F$ is a regular boundary point and a nonconstant subharmonic function has its normal derivatives different from zero at a point where it assumes it maximum, see Protter and Weinberger [11], p. 67. This contradiction establishes the lemma.

Given $x \in K$ we define

$$d(x) = \operatorname{dist}\{x, \partial K\}. \tag{2.11}$$

LEMMA 2.12. *Define* $\varepsilon(\varrho, \cdot) = \psi(\varrho, \cdot) - N(\varrho, \cdot)$, *with* ψ *as in Lemma* 2.10. *Given* $M > 1$, *there exists a number* $C > 0$ *such that if* $|x| > M\varrho$, $x \in K$, *then*

$$\text{(I)} \qquad -Cd(x)\varrho^{\alpha_1}r^{\beta_1-1} \leq \varepsilon(\varrho, x) \leq 0$$

and

$$\text{(II)} \qquad \left|\frac{d\varepsilon(\varrho, x)}{dr}\right| \leq C\varrho^{\alpha_1}r^{\beta_1-1}$$

Here $|x| = r$ *and* α_1, β_1 *are defined in* (2.2).

Proof. Take any $v \in \partial K$ with $|v| = 1$. Since $\varphi_0 = \text{const.}$, we have from (2.3) if $\varrho < r$, then

$$\varepsilon(\varrho, x) = \sigma_n \sum_{i=1}^{\infty} \frac{\varrho^{\alpha_i}r^{\beta_i}(\varphi(v) - \varphi_i(x^*))\varphi_i(1)}{\sqrt{4\lambda_i + (n-2)^2}}$$

Lemma 2.5 now yields the second inequality of the lemma. Extend $\varepsilon(\varrho, \cdot)$ to $\mathbf{R}^n$ by putting it equal to 0 in $\mathbf{R}^n - K$. Then $\varepsilon(\varrho, \cdot)$ is superharmonic in $\mathbf{R}^n - \{\varrho\}$. If we define $h(s) = \inf_{|x|\geq s}\varepsilon(\varrho, x)$, then to all $m > 1$, there exists a $C > 0$, such that if $s > m\varrho$ then $h(s) \geq - C\varrho^{\alpha_1}s^{\beta_1}$. Pick a number $e > 0$, so small that $M_1 = (1 - e)M > 1$. Fix $x \in K$ with $|x| > M\varrho$ and let $x_0 \in \partial K$ be a point with $|x - x_0| = d(x)$. To prove (I) we need only to consider the case when $\delta(x) \leq \frac{1}{2}e|x|$. Choose $b \in \mathbf{R}$ and $z \in \mathbf{R}^n$ such that $(z, x_0) = b$ and z is the outward normal of ∂K at x_0. Let $E = \{y \in \mathbf{R}^n\colon |y - x_0| < \frac{1}{2}er, \ (y, z) < b\}$ and $B = \{y\colon |y - x_0| = \frac{1}{2}er, \ (y, z) \leq b\}$ and let ω be the harmonic measure of B with respect to E. There exists a number $C > 0$ only depending on the dimension, such that $\omega(y) \leq Ce^{-1}r^{-1}|y - x_0|$ for all $y \in E$. Since $\varepsilon(\varrho, \cdot)$ is superharmonic and has boundary values 0 on $\partial E - B$ and the boundary values are $\geq h((1 - e)r)$ on B, the minimum principle gives $\varepsilon(\varrho, x) \geq Ce^{-1}r^{-1}|x - x_0|h((1 - e)r) \geq - Cr^{-1}d(x)\varrho^{\alpha_1}r^{\beta_1 - 1}$ for some number $C > 0$, and Lemma 2.12 is proved.

For a domain Ω on the unit sphere with boundary $\partial'\Omega$ let $\lambda = \lambda(\Omega)$ be the first eigenvalue to the problem $\delta u + \lambda(\lambda + n - 2)u = 0$, $u = 0$ on $\partial'\Omega$ and let $\varphi = \varphi_\Omega$ be the corresponding eigenfunction, normalized so that $\varphi > 0$.

LEMMA 2.13. *Let λ be the first eigenvalue of D and let $\varphi = \varphi_D$ be an eigenfunction. Then we have that $\lambda < \alpha_1$ and $\varphi(p) \leq \varphi(1)$ for all $p \in D$. Here α_1 is given by* (2.2).

Proof. Suppose $\alpha_1 < \lambda$. Pick $z \in \partial K$ with $|z| = 1$ and let $e = \operatorname{sign} \varphi_1(z)$. The Phragmén-Lindelöf theorem (see Lelong-Ferrand [9]) applied to $x \to r^{\alpha_1}e\varphi_1(x^*)$ yields that $e\varphi_1 > 0$ in D. But this contradicts the fact that $\int_D \varphi_1 = 0$. Since φ_1 and φ are given by Gegenbauer functions we cannot have $\alpha_1 = \lambda$. For the second half of the proposition, suppose that $d\varphi/d\theta(p) = 0$ for some $p \in C - \{1\}$. Let $D_1 = \{q \in D\colon q_1 > p_1\}$. For D_1, let α_1 be given by (2.2). Since $C_1 \subset C$ we have $\lambda_1 = \lambda(C_1) \geq \lambda$ and we have also $\alpha_1 \leq \lambda \leq \lambda_1$. But this contradicts the first half of the proposition, applied to D_1.

3. An inequality for subharmonic functions

We continue the notation of section 2. In addition we introduce

$$K_R = K \cap \{|x| < R\} \text{ and } D_R = K \cap \{|x| = R\}.$$

We take as our starting point the following lemma, which gives a relation between the values on the symmetry axis of K_R and the averages over D_R, $0 < r < R$, of a smooth function in K_R.

LEMMA 3.1. *Suppose* u *is two times continuously differentiable in* $\overline{K_R}$. *If* $0 < \varrho < R$, *then we have that*

$$u(\varrho) = V(u, \varrho, R) + \sigma_n^{-1}(n-2)^{-1} \int_{K_R} \Delta u(z)\varepsilon(\varrho, z)dz + S(u, \varrho, R).$$

Here $\varepsilon(\varrho, \cdot)$ *is given Lemma* 2.12, $\psi(\varrho, \cdot)$ *by Lemma* 2.10,

$$V(u, \varrho, R) = -\sigma_n^{-1}(n-2)^{-1} \int_{K_R} u(z)\Delta\psi(\varrho, z)dz,$$

and

$$S(u, \varrho, R) = \sigma_n^{-1}(n-2)^{-1} \int_{D_R} \left\{ u(x) \frac{d\varepsilon(\varrho, x)}{dr} - \frac{du(x)}{dr} \varepsilon(\varrho, x) \right\} d\sigma(x).$$

Proof. Observing that $\varepsilon(\varrho, x) = d/d\nu\varepsilon(\varrho, x) = 0$ for all $x \in \partial K \cap \partial K_R$, an application of Green's formula to $\varepsilon(\varrho, \cdot)$ and u gives Lemma 3.1.

In order to make use of Lemma 3.1 we need a preliminary result on the Green function.

LEMMA 3.2. *Let* G *and* G_R *be the Green functions of* K *and* K_{3R}. *Then, with the notation of Lemma* 3.1, *we have for all* $\varrho > 0$ *and all* $y \in K$

$$G(\varrho, y) \geq -\sigma_n^{-1}(n-2)^{-1} \int_K G(z, y)\Delta\psi(\varrho, z)dz.$$

There exists a number $C > 0$, *only depending on* K, *such that if* $0 < \varrho < R/2$ *and* $y \in K_{3R}$, *then*

$$E(\varrho, R, y) = G_R(\varrho, y) - V(G_R(\cdot, y), \varrho, R) \geq C\varrho^{\alpha_1}R^{\beta_1}.$$

Proof. Since the function $F(\varrho, \cdot) = \varepsilon(\varrho, \cdot) + G(\varrho, \cdot)$ is superharmonic in K, has boundary values zero, and $\lim_{|x|\to\infty} F(\varrho, x) = 0$, the largest harmonic minorant of $F(\varrho, \cdot)$ in K is 0, and hence $F(\varrho, \cdot)$ is a potential (by Helms [5], p. 117).

Hence, by Lemma 2.12 and Riesz decomposition theorem, we have for $y \in K$,

$$G(\varrho, x) \geq F(\varrho, x) = \sigma_n^{-1}(n-2)^{-1} \int_K G(z, x)\Delta\psi(\varrho, z)dz.$$

If $y, z \in K_{3R}$ and $z^* = (3R)^2|z|^{-2}z$, then

$$G_R(z, y) = G(z, y) - h(z, y),$$

where $h(z, y) = (3R/|z|)^{n-2}G(z^*, y)$. Now we have

$$E(\varrho, R, y) = G(\varrho, y) - h(\varrho, y) - V(G(\cdot, y), \varrho, R) + V(h(\cdot, y), \varrho\ R).$$

By the first part of the lemma and by Lemma 3.1 applied to the harmonic function $h(\cdot, y)$ we get

$$E(\varrho, R, y) \geq V(h(\cdot, y), \varrho, R) - h(\varrho, y) = - S(h(\cdot, y), \varrho, R).$$

We record the following fact for later use (cf. Protter-Weinberger [11]): If u is harmonic in a domain $\Omega \subset \mathbf{R}^n$ and ∇u denotes the gradient of u, then for all $x \in \Omega$

$$|\nabla u(x)| \leq CM[\text{dist.}\{x, \partial\Omega\}]^{-1} \tag{3.3}$$

where $M = \sup\{|u(x)|: x \in \Omega\}$ and C is a number only depending on n.

Since the boundary values of $h(\cdot, y)$ are zero on $\partial K \cap \partial K_{3R}$ and $h(\cdot, y) \geq 0$, we have that $m(y) = \sup\{|h(z, y)|: z \in K_{2R}\} = \sup\{h(z, y): y \in D_{2R}\}$ and consequently $m(y) = (3/2)^{n-2} \sup\{G(z, y): z \in D_{9R/2}\}$. If we put

$$A = (3/2) \max\{G(z, x\}: x \in \overline{K_2},\quad z \in D_{9/2}\},$$

then $A < \infty$ and $m(y) \leq R^{2-n}A$. There exists a number $c > 0$ such that $\text{dist}\{z, \partial K_{2R}\} \geq cd(z)$ for all $z \in D_R$, where d is given in (2.11). From (3.3) and Lemma 2.12 it follows that

$$E(\varrho, R, y) \geq - S(h(\cdot, y), \varrho, R) \geq - \int_{D_R} \left|\frac{dh(x, y)}{dr}\right| \varepsilon(\varrho, x) d\sigma(x) -$$

$$- \int_{D_R} h(x, y) \left|\frac{d\varepsilon(\varrho, x)}{dr}\right| d\sigma(x) \geq - C\varrho^{\alpha_1}R^{\beta_1},$$

and Lemma 3.2 is proved.

The next lemma is the main result of this section.

LEMMA 3.4. *Suppose* u *is a two times continuously differentiable nonnegative subharmonic function in* $\mathbf{R}^n$ *and suppose further that* $\Delta u = 0$ *in* $\{|x| < e\}$ *for some* $e > 0$. *Then there is a number* $C > 0$, *only depending on* K, *such that if* $0 < \varrho < R/2$, *then*

$$u(\varrho) \leq V(u, \varrho, R) + CM(6R, u)(\varrho/R)^{\alpha_1}.$$

Here V *is given in Lemma* 3.1 *and* α_1 in (2.2).

Proof. Let h be the harmonic majorant of u in K_{3R}. Then $u = h - p$ in K_{3R}, where

$$p(y) = \sigma_n^{-1}(n-2)^{-1} \int_{K_{3R}} (G_R(y, z)\Delta u(z) dz,\quad y \in K_{3R},$$

and G_R is the Green function of K_{3R}.

From Lemma 3.2 we have

$$u(\varrho) = V(u, \varrho, R) + \sigma_n^{-1}(n-2)^{-1} \int_{K_R} \Delta u(z)\varepsilon(\varrho, z)dz + S(u, \varrho, R).$$

It remains to estimate the last two terms in this equality. We write $S(u, \varrho, R) = S(h, \varrho, R) - S(p, \varrho, R)$. An application of (3.3) and Lemma 2.12 yields

$$|S(h, \varrho, r)| \leq C \int_{D_R} M(3R)\varrho^{\alpha_1}R^{\beta_1 - 1}d\sigma(x) = CM(3R)(\varrho/R)^{\alpha_1}, \tag{3.5}$$

remembering that $\beta_1 = -\alpha_1 - (n-2)$.

It remains to estimate $\sigma_n^{-1}(n-2)^{-1} \int_{K_R} \Delta u(z)\varepsilon(\varrho, z)dz - S(p, \varrho, R) = H$. An application of Lemma 3.1 gives (since $\Delta p = -\Delta u$) $H = V(p, \varrho, R) - p(\varrho)$. If E is as Lemma 3.2, then a change of the order of integration gives

$$H = -\int_{K_{3R}} E(\varrho, R, y)\Delta u(y).$$

If we put $\mu(t) = \int_{|y|<t} \Delta u(y)dy$, then Lemma 3.2 yields

$$H \leq C\varrho^{\alpha_1}R^{\beta_1}\mu(3R). \tag{3.6}$$

To estimate μ we argue as follows: From the Riesz representation formula we have

$$u(0) = T(2R, u) - \sigma_n^{-1}(n-2)^{-1} \int_{|y|<2R} (|y|^{2-n} - (2R)^{2-n})\Delta u(y)dy.$$

Since we have assumed that $\Delta u = 0$ for $|y| < e$, the integral above is convergent. Since $u \geq 0$ we have

$$T(2R, u) \geq \sigma_n^{-1}(n-2)^{-1} \int_0^{2R} \{t^{2-n} - (2R)^{2-n}\}d\mu(t).$$

But $\int_0^{2R} \{t^{2-n} - (2R)^{2-n}\}d\mu(t) = (n-2) \int_0^{2R} \mu(t)t^{1-n}dt \geq \mu(R)(1 - 2^{2-n})R^{2-n}$.

This implies that there exists a number $C > 0$, depending only on n, such that $\mu(R) \leq CM(2R)R^{2-n}$. If we use this inequality in (3.6) we have that

$$H \leq C(\varrho/R)^{\alpha_1}M(6R, u). \tag{3.7}$$

Combining (3.5) and (3.7) we find that

$$u(\varrho) = V(\mu, \varrho, R) + H + S(h, \varrho, R) \leq V(u, \varrho, R) + CM(6R, u)(\varrho/R)^{\alpha_1},$$

and this completes the proof of Lemma 3.4.

4. The main result

The proof of Theorem 1.2 will be based on the following result, which is interesting in itself. We continue the notation of section 1.

THEOREM 4.1. *Suppose u is subharmonic in $\mathbf{R}^n$, $n \geq 3$ and there exists a number $r_0 > 0$, such that*

$$T(r, u) \leq C(\lambda, n) M(r, u) \quad \textit{for all} \quad r > r_0. \tag{4.2}$$

Then either u is bounded from above or $\lim_{r\to\infty} M(r, u) r^{-\lambda} = A$ exists and $0 < A \leq \infty$.

We remark that by the construction of $C(\lambda, n)$, λ is the best possible choice for the growth of functions satisfying (4.2).

Proof of Theorem 4.1. Let a_λ be given as in the beginning of section 1. Put $K = \{x = (x_1, \ldots, x_n) \in \mathbf{R}^n\colon x_1 > a_\lambda |x|\}$, $D = \{x \in K\colon |x| = 1\}$.

Let us make the assumption that u is not bounded from above and that r_0 is so large that $M(r_0, u) > 0$. Define

$$v = (u^+ - M(r_0, u))^+. \tag{4.3}$$

Then v has the following properties:

$$v \geq 0, \ v(x) = 0 \ \text{if} \ |x| \leq r_0, \ \text{and} \ M(r, v) = M(r, u) - M(r_0, u) \ \text{for} \ r > r_0 \tag{4.4}$$

$$T(r, v) \leq C(\lambda, n) M(r, v) \quad \text{for all} \quad r > 0. \tag{4.5}$$

The relation (4.4) follows from the maximum principle. To prove (4.5), fix $r > 0$ and put $\Omega = \{|x| = 1\colon u^+(rx) > M(r_0, u)\}$. If $\int_\Omega d\sigma \leq \sigma_n C(\lambda, n)$, then (4.5) follows easily. For the case when $\omega = \int_\Omega d\sigma > \sigma_n C(\lambda, n)$, we have that

$$T(r, v) = \sigma_n^{-1} \int\limits_\Omega \{u(rx) - M(r_0, u)\} d\sigma(x) \leq T(r, u) - \sigma_n^{-1} \omega M(r_0, u).$$

By (4.2) and (4.4) we find

$$\begin{aligned} T(r, v) \leq C(\lambda, n) M(r, u) - \sigma_n^{-1} \omega M(r_0, u) = C(\lambda, n) M(r, v) + \\ + (C(\lambda, n) - \sigma_n^{-1} \omega) M(r, u) \leq C(\lambda, n) M(r, v), \end{aligned}$$

and (4.5) is proved.

Now by Helms [5], p. 71, there exists a sequence $\{v_m\}_{m=1}^\infty$ of two times continuously differentiable subharmonic functions in $\mathbf{R}^n$, such that $v_m \downarrow v$ as $m \to \infty$. Moreover, since $v = 0$ for $|x| < r_0$, all v_m may be taken to be 0 for $|x| < r_0/2$. If we fix $\varrho > 0$, then we have after a rotation that $v(\varrho) = M(\varrho, v)$.

A rotation does not change any of our assumptions. We now apply Lemma 3.4 to all v_m, and then let $m \to \infty$. Then we have for $0 < \varrho < R/2$

$$M(\varrho, v) \leq V(v, \varrho, R) + CM(6R, v)(\varrho/R)^{\alpha_1}. \tag{4.6}$$

Define $P(\varrho, r) = -(n-2)^{-1}r^{n-1}\Delta\psi(\varrho, x)$, with $|x| = r$. Then we get from (4.6) when $0 < \varrho < R/2$:

$$M(\varrho, v) \leq \int_0^R T(r, v)P(\varrho, r)dr + CM(6R, v)(\varrho/R)^{\alpha_1} \tag{4.7}$$

Let φ be the first eigenfunction of D (which is 0 on the boundary of D) normalized so that $\varphi(1) = 1$. Then by the construction of K, φ corresponds to the eigenvalue λ and Φ: $x \mapsto r^\lambda\varphi(x^*)$, $x \in K$, is equal to $u_\lambda(1)^{-1}u_\lambda|K$, where u_λ is as in section 1. From Lemma 2.13 and the definition of $C(\lambda, n)$ we have

$$\sigma_n^{-1}\int_D \Phi(rx)d\sigma(x) = C(\lambda, n)r^\lambda \quad \text{for all} \quad r > 0 \tag{4.8}$$

From Lemma 3.1 applied to Φ we have $\varrho^\lambda = \int_0^R C(\lambda)r/P(\varrho, r)dr + S(\Phi, \varrho, R)$.

It is known (see Azarin [1]) that there exists a number $C > 0$, such that $\varphi(z) \leq Cd(z)$ for all $z \in C$, where $d(x) = \operatorname{dist}\{x, \partial K\}$. Hence, if $0 < \varrho < R/2$, then it is easy to see that $|S(\Phi, \varrho, R)| \leq C\varrho^{\alpha_1}R^{\lambda-\alpha_1}$ and Lemma 2.13 gives when $R \to \infty$

$$\varrho^\lambda = \int_0^\infty C(\lambda, n)r^\lambda P(\varrho, r)dr. \tag{4.9}$$

Define the function $H: r \mapsto r^{-\lambda}M(r, v)$. Then H is upper semicontinuous in $[0, \infty[$ and is 0 in $[0, r_0]$. We want to show that there exists a number $C > 0$ such that if $0 < r < R$, then

$$H(r) \leq CH(R). \tag{4.10}$$

Put $m(R) = \max\{H(r): 0 \leq r \leq 6R\}$. There exists a $\varrho, 0 \leq \varrho < 6R$, such that $m(R) = H(\varrho)$. If $R/2 \leq \varrho \leq 6R$, then

$$m(R) = H(\varrho) = M(\varrho)\varrho^{-\lambda} \leq M(6R)(6R)^{-\lambda}(6R/\varrho)^\lambda \leq 12^\lambda H(6R). \tag{4.11}$$

If $0 < \varrho \leq R/2$, then we have from (4.7)

$$\varrho^\lambda m(R) \leq m(R)\int_R^\infty C(\lambda)r^\lambda P(\varrho, r)dr + CM(6R, v)(\varrho/R)^{\alpha_1}.$$

Using (4.9) we have

$$m(R)\int_0^R C(\lambda)r^\lambda P(\varrho, r)dr \leq CM(6R, v)(\varrho/R)^{\alpha_1}. \tag{4.12}$$

From (2.3) we have that if $\varrho < r$, and $e \in \partial K$, $|e| = 1$, then

$$P(\varrho, r) = -(n-2)^{-1}\sigma_n \sum_{i=1}^{\infty} \frac{\varrho^{\alpha_i} r^{\beta_i+n-3}\varphi_i(e)\varphi_i(1)\beta_i(\beta_i + n - 2)}{\sqrt{4\lambda_i + (n-2)^2}}$$

Using that $\varphi_i(e) < 0$, (2.4) and Lemma 2.5 we see that there exists a $\gamma > 1$ and a number $k > 0$ such that $r \geq \gamma\varrho$ implies

$$P(\varrho, r) \geq k(\varrho/r)^{\alpha_1} r^{-1}. \tag{4.13}$$

Hence $\int_R^\infty r^\lambda P(\varrho, r)dr \geq \int_{\gamma R}^\infty \geq k_1(\varrho/R)^{\alpha_1}R^\lambda$.

Inserting this in (4.12) we find $m(R) \leq CH(6R)$ and this taken together with (4.11) proves (4.10). If we put $A = \liminf_{r\to\infty} H(r)$, $B = \limsup_{r\to\infty} H(r)$ and $L = \sup_{r>0} H(r)$, then relation (4.10) gives that

$$0 < A \leq B \leq L \leq CA. \tag{4.14}$$

We will now prove that $A = B$, i.e. $\lim r^{-\lambda}M(r, v)$ exists. If $A = \infty$, then this is clear, so we assume that $A < \infty$. If we let $R \to \infty$ in (4.7), then $\varrho > 0$ implies

$$M(\varrho, v) \leq C(\lambda)\int_0^\infty M(r, v)P(\varrho, r)dr, \quad C(\lambda) = C(\lambda, n)\,. \tag{4.15}$$

To prove that $A = B$, we use a technique similar to Kjellberg [7]. We start by showing that $B = L$. If $B < L$, then the upper semicontinuity of H implies the existence of a $\varrho > 0$, such that $H(s) = L$. From (4.15) we have that

$$Ls^\lambda \leq \int_{r_0}^\infty C(\lambda)r^\lambda P(s, r)dr,$$

since $\psi(r) = 0$ for $0 \leq r \leq r_0$. But $\int_{r_0}^\infty C(\lambda)r^\lambda P(s, r)dr < \int_0^\infty C(\lambda)r^\lambda P(s, r)dr = s^\lambda$, by using (2.10) and (4.9). This contradiction establishes that $B = L$. If we put $L(R) = \max_{0\leq r\leq R} H(r)$, then $L(R) < L$ and $\lim_{R\to\infty} L(R) = B$. Assume that $A < B$. Pick an R such that $H(R) \approx A$ and so large that $L(R) \approx B$. Take ϱ, $0 < \varrho \leq R$, such that $L(R) = H(\varrho)$ and put $t = R(H(R)/L(R))^{1/2\lambda}$. If $t \leq r \leq R$, then

$$H(r) = r^{-\lambda}M(r, v) \leq M(R, v)R^{-\lambda}(R/r)^\lambda \leq \sqrt{H(R)L(R)}.$$

We have therefore the following estimate of H:

$$H(r) \le \begin{cases} L(R) & \text{if } 0 \le r \le t \\ \sqrt{H(R)L(R)} & \text{if } t \le r \le R \\ B & \text{if } R \le r. \end{cases}$$

This implies that $\varrho < t$. From (4.15) we get

$$L(R)\varrho^\lambda \le L(R) \int_0^t C(\lambda) r^\lambda P(\varrho, r) dr + \sqrt{H(R)L(R)} \int_t^R C(\lambda) r^\lambda P(\varrho, r) dr +$$
$$+ B \int_R^\infty C(\lambda) r^\lambda P(\varrho, r) dr.$$

We subtract $L(R)\varrho^\lambda = L(R) \int_0^\infty C(\lambda) r^\lambda P(\varrho, r) dr$ from both sides of the inequality. This yields

$$(L(R) - \sqrt{L(R)H(R)}) \int_t^R r^\lambda P(\varrho, r) dr \le (B - L(R)) \int_R^\infty r^\lambda P(\varrho, r) dr. \tag{4.16}$$

There exists a number $C > 0$ such that $\varrho \le t$ implies that $P(\varrho, r) \le C(\varrho/r)^{\alpha_1} r^{-1}$ and hence

$$\int_R^\infty r^\lambda P(\varrho, r) d\varrho \le C(\varrho/R)^{\alpha_1} R^\lambda. \tag{4.17}$$

We now want to show that there exists a number $c > 0$, only depending on the ratio t/R, such that

$$\int_t^R r^\lambda P(\varrho, r) dr \ge c(\varrho/R)^{\alpha_1} R^\lambda. \tag{4.18}$$

It is easy to see that it is sufficient to consider the case when $R = 1$. From (4.13) it follows that $\varrho \le \gamma^{-1}$ and $0 < h < 1$ implies that

$$\int_h^1 r^\lambda P(\varrho, r) dr \ge k\varrho^{\alpha_1}(\alpha_1 - \lambda)^{-1}\{h^{\lambda - \alpha_1} - 1\}$$

The function $\varrho \to \int_1^h r^\lambda P(\varrho, r) dr$ is continuous and strictly positive in $[\gamma^{-1}, h]$ and hence there exists a number $c > 0$ depending on h such that $\int_1^h r^\lambda P(\varrho, r) dr \ge C\varrho^{\alpha_1}$. This proves (4.18), and combining (4.18) and (4.17) with (4.16) we find that there exists a number $C > 0$ such that $(L(R) - \sqrt{H(R)L(R)}) \le C(B - L(R))$. But this gives a contradiction, since the right hand side of the inequality tends to 0 as $R \to \infty$ and the left side tends to $B - \sqrt{AB}$ as $R \to \infty$. This contradiction arose from the assumption that $A < B$, and hence Theorem 4.1 is proved, since from (4.4) $M(r, v) = M(r, u) - M(r_0, u)$ for $r \geq r_0$.

We are now in a position to prove Theorem 1.2.

Proof of Theorem 1.2. Suppose u is subharmonic in $\mathbf{R}^n$, $n \geq 3$, and of lower order λ, $0 < \lambda < \infty$. Take any $\varepsilon > 0$. Then $\liminf_{r\to\infty} r^{-\lambda-\varepsilon} M(r, u) = 0$ and from Theorem 4.1 it follows that there must exist a sequence $\{r_m\}_1^\infty$, $r_m \to \infty$ as $m \to \infty$, such that $T(r_m, u) \geq C(\lambda + \varepsilon, n) M(r_m, u)$. Hence $\limsup_{r\to\infty} T(r, u)/M(r, u) \geq C(\lambda + \varepsilon, n)$ for all $\varepsilon > 0$, and letting $\varepsilon \to 0$ we find that $\limsup_{r\to\infty} T(r, u)/M(r, u) \geq C(\lambda, n)$.

5. Applications

We will as a first application give a result on the eigenfunctions of the Laplace-Beltramioperator.

THEOREM 5.1. *Suppose* Ω *is a domain in* $S^{n-1} = \{x \in \mathbf{R}^n\colon |x| = 1\}$, *where* $n \geq 3$. *Let* λ *be the first eigenvalue of*

$$\delta u + \lambda(\lambda + n - 2)u = 0, \quad u = 0 \quad on \quad \partial'\Omega,$$

and let φ *be the corresponding eigenfunction, normalized so that* $\max_{p\in\Omega} \varphi(p) = 1$. *Then*

$$\int_\Omega \varphi(p) d\sigma(p) \geq C(\lambda, n).$$

Let $\Omega' = \{rx\colon r > 0, x \in \Omega\}$ and define

$$u(x) = \begin{cases} 0 & \text{if } x \notin \Omega' \\ r^\lambda \varphi(x/r) & \text{if } x \in \Omega, \quad r = |x| \end{cases}$$

Then u is subharmonic in $\mathbf{R}^n$, since $u \geq 0$ in Ω' and $u|\Omega'$ has boundary values 0 on $\partial\Omega$. Clearly $M(r, u) = r^\lambda$ and from Theorem 1.2 we have

$$\limsup_{r\to\infty} \frac{T(r, u)}{M(r, u)} = \int_\Omega \varphi d\sigma \geq C(\lambda, n)$$

Remark. Theorem 5.1 may be interpreted as follows: among all domains Ω on the unit sphere with first eigenvalue λ the quantity $\int_\Omega \varphi d\sigma$ is minimized for geodesic balls.

The next result should be considered as a mean value anlogue of Hall's lemma.

THEOREM 5.2. *Let u be a positive superharmonic function in a cone*

$$K = \{x = (x_1, \ldots, x_n): x_1 > a|x|\}$$

where $a \in (-1, 1)$ and $n \geq 3$. *Put* $D = \{x \in K: |x| = 1\}$ *and* $\omega = \int_D d\sigma$. *Suppose*

$$\int_D \omega^{-1} \, u(rx)d\sigma(x) \geq 1 \text{ for all } r > 0.$$

Then $u(r) \geq 1$ *for all* $r > 0$.

Proof. Let G and P be the Greenfunction and the Poisson kernel of K. Let φ be the Martin function of K with pole at infinity. There exists a number $\alpha \geq 0$, a nonnegative measure λ on ∂K and a nonnegative measure μ on K such that for all $x \in K$ we have

$$u(x) = \alpha\varphi(x) + \int_{\partial K} P(y, x)d\lambda(y) + \int_K G(z, x)d\mu(z). \tag{5.3}$$

For any function $h \geq 0$ in K define,

$$V(h, \varrho) = -\sigma_n^{-1}(n-2)^{-1} \int_K h(z)\Delta\psi(\varrho, z)dz, \psi \text{ as in Lemma 2.10.}$$

If we put $t(r, h) = \sigma_n^{-1} \int_D h(rx)d\sigma(x)$ and $Q(\varrho, r) = -(n-2)^{-1}r^{n-1}\Delta\psi(\varrho, r)$, where $|x| = r$

$$V(h, \varrho) = \int_0^\infty t(r, h)Q(\varrho, r)d\varrho.$$

From the proof of Theorem 4.1 we have $V(\varphi, \varrho) = \varphi(\varrho)$ for all $\varrho > 0$. Lemma 3.2 says that $V(G(z, \cdot), \varrho) \leq G(\varrho, z)$ for all $z \in K$. Take any point $y \in \partial K$ and let ν be the inward unit normal of ∂K at y. Then

$$V(P(y, \cdot), \varrho) = -\sigma_n^{-1}(n-2)^{-1} \int \lim_{h \downarrow 0} h^{-1}G(y + h\nu, z)\Delta\psi(\varrho, z)dz \leq$$
$$\leq \liminf_{h \downarrow 0} h^{-1}V(G(y + h\nu, \cdot), \varrho) \leq \liminf_{h \downarrow 0} h^{-1}G(y + h\nu, \varrho) = P(y, \varrho),$$

by Fatou·s lemma and (3.2). We now find from (5.3) that $u(\varrho) \geq V(u, \varrho)$ for all ϱ and Lemma 3.1 yields that $1 = V(1, \varrho)$ for all $\varrho > 0$. We see that from the assumption on u we have $u(\varrho) \geq V(1, \varrho) = 1$ for all $\varrho > 0$, and this finishes the proof of Theorem 5.2.

We can also prove the following result by Huber [7].

THEOREM 5.4. *Let u be subharmonic in R^n, $n \geq 3$ and put*

$$E = \{x \in \mathbf{R}^n\colon u(x) \leq 0\}.$$

Suppose there exists number $c > 0$ and $r_0 > 0$ such that $\int_{E \cap \{|x| = r\}} d\sigma \geq cr^{n-1}$ for all $r > r_0$. Then there exists a $\mu > 0$, such that either u is bounded from above or $\lim_{r \to \infty} r^{-\mu} M(r) > 0$.

Proof. The assumptions on u implies that $T(r, u) \leq \sigma_n^{-1}(\sigma_n - C)M(r, u)$ for all $r > r_0$, and an application of Theorem 4.1 fulfills the proof.

We remark that our method of proof goes through without change for $n = 2$, $\lambda \geq \frac{1}{2}$. If $\lambda < \frac{1}{2}$, then we use as an extremal function $\operatorname{Re} z^\lambda$. We summarize this in

THEOREM 5.5. *Suppose u is subharmonic in $\mathbf{C}$ and is of lower order λ. Then we have*

$$\limsup_{r \to \infty} T(r, u)/M(r, u) \geq \begin{cases} \sin \pi\lambda/\pi\lambda & \text{if } \lambda \leq \frac{1}{2} \\ 1/\pi\lambda & \text{if } \lambda < \frac{1}{2}. \end{cases}$$

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