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Harmonic measure on simply connected domains of fixed inradius

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Abstract. Let $D\subset\mathbf{C}$ be a simply connected domain that contains 0 and does not contain any disk of radius larger than 1. For $R>0$, let $\omega_D(R)$ denote the harmonic measure at 0 of the set $\{z:|z|\geq R\}\cap\partial D$. Then it is shown that *there exist* $\beta>0$ *and* $C>0$ *such that for each such* D, $\omega_D(R)\leq Ce^{-\beta R}$, *for every* $R>0$. Thus a natural question is: What is the supremum of all β's, call it β_0, for which the above inequality holds for every such D? Another formulation of the problem involves hyperbolic metric instead of harmonic measure. Using this formulation a lower bound for β_0 is found. Upper bounds for β_0 can be obtained by constructing examples of domains D. It is shown that a certain domain whose boundary consists of an infinite number of vertical half-lines, i.e. a comb domain, gives a good upper bound. This bound disproves a conjecture of C. Bishop which asserted that the strips of width 2 are extremal domains. Harmonic measures on comb domains are also studied.

1. Introduction

The inradius $R(D)$ of a domain D is the radius of the largest disk contained in D. More precisely

$$R(D)=\sup_{z\in D}\operatorname{dist}(z,\partial D). \tag{1.1}$$

Let $\mathcal{B}$ be the class of all simply connected domains that contain the origin and have inradius 1. Several extremal problems for domains in $\mathcal{B}$ have been studied. The most famous is the problem of determining the univalent Bloch constant U. This problem can be formulated as follows: Let $\sigma(z,D)$ be the density of the hyperbolic metric on D with curvature -4, that is, $\sigma(z,D)=|f'(z)|$, where f is a function that maps D conformally onto the unit disk $\mathbf{D}$, with $f(z)=0$. It follows from Koebe's $\frac{1}{4}$-theorem that $U:=\inf_{D\in\mathcal{B}}\inf_{z\in D}\sigma(z,D)\geq\frac{1}{4}$. The univalent Bloch constant U remains unknown. For a brief history of the work on U we refer to [BC] (which also reviews some other problems involving inradius). Here we mention only the following lower bound due to Zhang [Z]: $U>0.57088$.

We will study a similar problem for harmonic measure. For a domain $D \in \mathcal{B}$ and for $R>0$, let $\omega_D(R)$ denote the harmonic measure at 0 of the set $\partial D \cap \{z:|z| \geq R\}$ with respect to D. It is obvious that ω_D is a decreasing function of R. In fact, one can prove that ω_D decreases exponentially. This follows, at least intuitively, from the probabilistic interpretation of harmonic measure as hitting probability of Brownian motion in D: A Brownian particle starting from the circle $|z|=r$ and stopping when it hits the boundary of D has small probability to reach the circle $|z|=r+2$, because of the inradius condition $R(D)=1$. Now repeated applications of the Markov property shows that ω_D decays exponentially. Of course, this argument can be made rigorous, see Proposition 3.4. Our purpose is to study more precisely the exponential decay of ω_D.

For a domain D, let $\beta(D)$ be the exponent of decay of ω_D, that is

$$\beta(D) = \sup\{\beta > 0 : \text{ for some } C > 0, \ \omega_D(R) \leq C e^{-\beta R} \text{ for all } R > 0\}.$$

Our problem is to determine or estimate the exact value of the number

$$\beta_0 = \inf\{\beta(D) : D \in \mathcal{B}\}.$$

Thus β_0 is the smallest possible (in the sense of infimum) exponent of decay of ω_D for some $D \in \mathcal{B}$.

C. Bishop conjectured that $\beta_0 = \beta(S) = \frac{1}{2}\pi$, where S is a strip of inradius 1, i.e. of width 2. We will disprove Bishop's conjecture by presenting a domain D^* for which $\beta(D^*) \approx 0.428\pi$. Thus (Theorem 9.14)

$$\beta_0 \leq 0.4285\pi.$$

The domain D^* is a comb domain, i.e. its boundary consists of an infinite number of vertical half-lines. Certain extremal lengths on comb domains can be computed explicitly. These computations lead to estimates of harmonic measure via Beurling's inequalities relating extremal length and harmonic measure. We will study harmonic measures on several types of comb domains: parasymmetric, periodic and symmetric comb domains (see Figure 1).

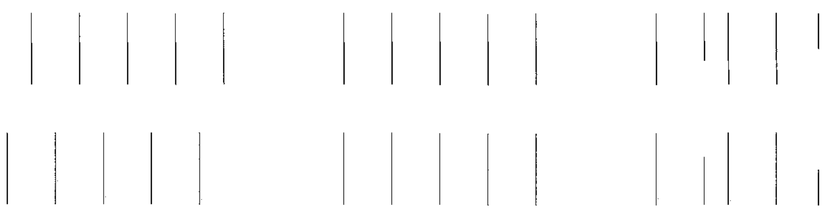

Figure 1. A parasymmetric comb domain, a periodic comb domain and a symmetric comb domain, respectively.

A lower bound for β_0 is found in Section 4 (Theorem 4.16):

$$\beta_0 \geq 2U,$$

where U is the univalent Bloch constant. This bound follows from a characterization of β_0 in terms of hyperbolic distance instead of harmonic measure.

The exact value of β_0 remains unknown. We can determine, however, the value of a related constant. Let $\mathcal{B}_c$ be the subset of $\mathcal{B}$ consisting of all domains D symmetric with respect to the real axis and convex in the y-direction. The latter condition means that each vertical line intersects D in a single vertical *interval.* We will prove that the class of all symmetric comb domains is dense in $\mathcal{B}_c$ in the sense of Carathéodory convergence. Then we will show (Theorem 8.4) that a certain periodic comb domain D_0 has the smallest exponent of decay $\beta(D_0)$ among all domains in $\mathcal{B}_c$:

$$\min\{\beta(D) : D \in \mathcal{B}_c\} = \beta(D_0) \approx 0.457\pi.$$

As we mention above, we will use extremal length to prove estimates for harmonic measure. In the next section we review some results on extremal length.

2. Extremal length and Beurling's inequalities for harmonic measure

Let D be a plane domain and E_0, E_1 be two disjoint closed sets on ∂D. Let $\mathcal{F}$ be the family of all rectifiable curves in D joining E_0 to E_1. We consider nonnegative Borel functions $\varrho(z)$ in D and define

$$L(\varrho, \mathcal{F}) = \inf_{\gamma \in \mathcal{F}} \int_\gamma \varrho\,|dz| \quad \text{and} \quad A(\varrho, D) = \iint_D \varrho^2\,dx\,dy.$$

The extremal distance $\lambda(E_0, E_1, D)$ between E_0 and E_1 with respect to D is

$$\lambda(E_0, E_1, D) = \sup_\varrho \frac{L(\varrho, \mathcal{F})^2}{A(\varrho, D)}, \tag{2.1}$$

where the supremum is taken over all ϱ with $0 < A(\varrho, D) < \infty$.

Extremal distances on the upper half-plane $\mathbf{C}_+$ can be computed explicitly. Let a, b, c be positive numbers. The extremal distance $\lambda([-a, 0], [b, b+c], \mathbf{C}_+)$ can be expressed in terms of elliptic integrals. Precisely, we have (see [O, §2.26])

$$\lambda([-a, 0], [b, b+c], \mathbf{C}_+) = 4\nu\left(\sqrt{\frac{ac}{(a+b)(b+c)}}\right), \tag{2.2}$$

where

$$\nu(r)=\frac{1}{4}\frac{K(\sqrt{1-r^2})}{K(r)}, \quad \text{and} \quad K(r)=\int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-r^2x^2)}} \tag{2.3}$$

is the complete elliptic integral of modulus $r\in(0,1)$.

The function $\nu(r)$ plays an important role in the theory of quasiconformal mappings because it is equal to the modulus of the Grötzsch ring $\mathbf{D}\setminus[0,r]$, i.e.

$$\nu(r)=\lambda([0,r],\partial\mathbf{D},\mathbf{D}). \tag{2.4}$$

It follows from (2.4) that ν is a decreasing function. Also, the expression (2.3) of ν in terms of elliptic integrals implies

$$\nu(r)\nu(\sqrt{1-r^2})=\tfrac{1}{16}. \tag{2.5}$$

We mention some more formulae for ν taken from [O]:

$$\nu(r)\nu\left(\frac{1-r}{1+r}\right)=\frac{1}{8}, \tag{2.6}$$

$$\frac{1}{2}\nu(r)=\nu\left(\frac{2\sqrt{r}}{1+r}\right), \tag{2.7}$$

$$\nu(s)=\frac{1}{2\pi}\log\frac{4}{s}+o(1), \quad \text{as } s\to 0, \tag{2.8}$$

$$\nu(s)^{-1}=\frac{4}{\pi}\log\frac{8}{1-s}+o(1), \quad \text{as } s\to 1. \tag{2.9}$$

Using (2.2) and the conformal invariance of extremal length we can compute extremal distances on some simply connected domains. We do two such computations: for the strip S and for the unit disk $\mathbf{D}$.

Proposition 2.10. *Let $S=\{z:x<\operatorname{Re} z<y\}$ with $x,y\in\mathbf{R}$. Consider the sets $A=\{z:\operatorname{Re} z=x,\ |\operatorname{Im} z|\le a\}$, $a>0$, and $B=\{z:\operatorname{Re} z=x,\ |\operatorname{Im} z|\le b\}$, $b>0$. Then*

$$\lambda(A,B,S)=\left[4\nu\left(\frac{X+Y}{1+XY}\right)\right]^{-1}, \tag{2.11}$$

where

$$X=\exp\left(\frac{-\pi a}{y-x}\right), \quad Y=\exp\left(\frac{-\pi b}{y-x}\right).$$

Proof. The function

$$f_1(z)=i\exp\left(\frac{i\pi}{y-x}\left(z-\frac{x+y}{2}\right)\right)$$

maps S onto $\mathbf{C}_+$ and the function

$$f_2(z) = z + \exp\left(\frac{-\pi b}{y-x}\right)$$

maps $\mathbf{C}_+$ onto itself so that $f_2 \circ f_1(A) = [Y+X, Y+X^{-1}]$ and $f_2 \circ f_1(B) = [Y-Y^{-1}, 0]$. Using (2.2) and (2.5) we obtain

$$\begin{aligned}\lambda(A,B,S) &= 4\nu\left(\sqrt{\frac{(Y^{-1}-Y)(X^{-1}-X)}{(Y^{-1}+X)(X^{-1}+Y)}}\right) = 4\nu\left(\sqrt{1-\left(\frac{X+Y}{1+XY}\right)^2}\right) \\ &= \frac{4}{16}\frac{1}{\nu\left(\dfrac{X+Y}{1+XY}\right)} = \frac{1}{4\nu\left(\dfrac{X+Y}{1+XY}\right)},\end{aligned}$$

and (2.11) is proven.

Proposition 2.12. *Let* $A = \{e^{it} : t \in [-\theta, \theta]\}$ *and* $B = -A = \{-\zeta : \zeta \in A\}$. *Then*

$$\lambda(A,B,\mathbf{D}) = \frac{1}{2\nu(g(\theta)^{-2})}, \tag{2.13}$$

where $g(\theta) = (1+\sin\theta)/\cos\theta$.

Proof. The function $f(z) = -i(z+i)(z-i)^{-1}$ maps $\mathbf{D}$ onto $\mathbf{C}_+$ with $f(A) = [g(\theta)^{-1}, g(\theta)]$ and $f(B) = [-g(\theta), -g(\theta)^{-1}]$. So (2.2) and (2.6) give

$$\begin{aligned}\lambda(A,B,\mathbf{D}) &= 4\nu\left(\frac{g(\theta)-g(\theta)^{-1}}{g(\theta)+g(\theta)^{-1}}\right) = 4\nu\left(\frac{1-g(\theta)^{-2}}{1+g(\theta)^{-2}}\right) \\ &= 4\frac{1}{8}\frac{1}{\nu(g(\theta)^{-2})} = \frac{1}{2\nu(g(\theta)^{-2})}\end{aligned} \tag{2.14}$$

and the proposition is proven.

Now we present some inequalities (due mainly to Beurling) that relate harmonic measure and extremal length. These inequalities will be used several times in the subsequent sections.

Let D be a simply connected domain in $\mathbf{C}$ and let E consist of a finite number of arcs lying on ∂D. Fix $z_0 \in D$ and choose a crosscut γ_0 of D that contains z_0 and joins two points of ∂D. Then

$$\omega(z_0, E, D) \le Ce^{-\pi\lambda(\gamma_0, E, D)}, \tag{2.15}$$

where C is an absolute constant. The inequality (2.15) is a special case of Theorem 3 in [Be, p. 372]. Some related results appear in [K].

Next we investigate the possibility of an inequality opposite to (2.15): Let D, z_0 and E be as above and assume in addition that E is an arc (of prime ends) on ∂D. We map D onto $\mathbf{D}$ by the conformal mapping f so that $f(z_0)=0$ and $f(E)=\{e^{i\theta}:\theta\in[-t,t]\}$ for some $t\in[0,\pi]$. Let $\gamma_E=f^{-1}([-1,0])$ and $\Gamma_E=f^{-1}([-i,i])$. We will refer to γ_E as "the geodesic of D opposite to E" and to Γ_E as "the geodesic of D perpendicular to γ_E at z_0". It follows from [K, p. 100] that

$$\omega(z_0,E,D)\geq Ce^{-\pi\lambda(\gamma_E,E,D)}, \tag{2.16}$$

with an absolute constant $C>0$.

We will now prove a similar inequality.

Lemma 2.17. *Let D be a simply connected domain, $z_0\in D$ and A be an arc on ∂D such that $\omega(0,A,D)\leq\frac{1}{4}$. Let γ_A be the geodesic of D opposite to A and Γ_A be the geodesic of D perpendicular to γ_A at z_0. If E is a subarc of A, then*

$$\omega(z_0,E,D)\geq Ce^{-\pi\lambda(\Gamma_A,E,D)}, \tag{2.18}$$

where C is an absolute positive constant.

Proof. By conformal invariance we may assume that $D=\mathbf{D}$, $\Gamma_A=[-i,i]$, $A=\{e^{i\theta}:\theta\in[-\frac{1}{8}\pi,\frac{1}{8}\pi]\}$.

If $|E|\geq\delta$ for some fixed $\delta>0$, then we have nothing to prove. So we assume $|E|<\delta$. The exact value of δ will be determined later.

Because of (2.16) it suffices to prove two estimates:

(i) $\lambda(\Gamma_A,E,\mathbf{D})\geq\lambda(\Gamma_A,E^*,\mathbf{D})-C$, where $E^*=\{e^{i\theta}:\theta\in[-\frac{1}{2}|E|,\frac{1}{2}|E|]\}$ is the circular symmetrization of E, and C is an absolute positive constant,

(ii) $\lambda(\Gamma_A,E^*,\mathbf{D})\geq\lambda(\gamma_A,E^*,\mathbf{D})-C$, where C is an absolute positive constant.

Proof of (ii). Let $|E|=2s$. A square root transformation (see [K, p. 98]) shows that

$$\lambda(\gamma_A,E^*,\mathbf{D})=\lambda(\Gamma_A,E_1^*,\mathbf{D}), \tag{2.19}$$

where $E_1^*=\{e^{i\theta}:\theta\in[-\frac{1}{2}s,\frac{1}{2}s]\}$. Also by symmetry, $2\lambda(\Gamma_A,E_1^*,\mathbf{D})=\lambda(-E_1^*,E_1^*,\mathbf{D})$ and $2\lambda(\Gamma_A,E^*,\mathbf{D})=\lambda(-E^*,E^*,\mathbf{D})$. By Proposition 2.12, the extremal lengths $\lambda(-E_1^*,E_1^*,\mathbf{D})$ and $\lambda(-E^*,E^*,\mathbf{D})$ can be computed in terms of the function ν:

$$\lambda(-E_1^*,E_1^*,\mathbf{D})=\frac{1}{2\nu\big(1/g(\frac{1}{2}s)^2\big)}, \tag{2.20}$$

$$\lambda(-E^*,E^*,\mathbf{D})=\frac{1}{2\nu(1/g(s)^2)}, \tag{2.21}$$

where $g(t)=(\sin t+1)/\cos t$.

Now we use the remark involving δ at the beginning of the proof. Given $\varepsilon>0$, we choose δ small enough so that the asymptotic formula (2.9) for the function ν gives, for $s\le\delta$,

$$\lambda(-E_1^*,E_1^*,\mathbf{D})\le\frac{1}{2}\frac{4}{2\pi}\log\frac{8}{1-1/g\left(\frac{1}{2}s\right)^2}+\varepsilon, \tag{2.22}$$

and

$$\lambda(-E^*,E^*,\mathbf{D})\ge\frac{1}{2}\frac{4}{2\pi}\log\frac{8}{1-1/g(s)^2}-\varepsilon. \tag{2.23}$$

The estimate (ii) follows from (2.22) and (2.23) by elementary calculus.

The proof of (i) is very similar: we express $\lambda(\Gamma_A,E_1,\mathbf{D})$ and $\lambda(\Gamma_A,E^*,\mathbf{D})$ in terms of the function ν and then we use the asymptotic formula for ν.

3. An extremal problem for harmonic measure

We will use the following notation for harmonic measure: If $D\subset\widehat{\mathbf{C}}$ is an open set and $K\subset\widehat{\mathbf{C}}$, $\omega(z,K,D)$ is the harmonic measure at z of the set $\operatorname{clos}K\cap\operatorname{clos}D$ with respect to the component of $D\setminus\operatorname{clos}K$ that contains z.

We consider the following class of domains:

$$\mathcal{B}=\{D\subset\mathbf{C}:D\text{ is simply connected, }R(D)=1\text{ and }0\in D\}. \tag{3.1}$$

For a domain $D\in\mathcal{B}$ and for $R>0$, let

$$\omega_D(R)=\omega(0,\partial D\cap\{z:|z|\ge R\},D), \tag{3.2}$$

$$\widetilde{\omega}_D(R)=\omega(0,\{z:|z|=R\},D). \tag{3.3}$$

Every $D\in\mathcal{B}$ is a BMO domain, i.e. the boundary function $f(e^{i\theta})$ of any analytic function $f:\mathbf{D}\to D$ is a function of bounded mean oscillation. This follows from work of Baernstein, Hayman, Pommerenke, Stegenga and Stephenson, see [B1] and references therein. Thus $\omega_D(R)$, as a function of R, is expected to decrease exponentially (the John–Nirenberg phenomenon). This is actually proved in the next proposition (cf. [B1, p. 22]).

Proposition 3.4. *There exist positive constants β and C with the property*

$$\omega_D(R)\le Ce^{-\beta R},\quad D\in\mathcal{B},\ R>0. \tag{3.5}$$

For the proof of the proposition we need two lemmas.

Lemma 3.6. *There exists a constant $\delta\in(0,1)$ such that for all $D\in\mathcal{B}$, $s>0$, and $z_0\in D\cap\{|z|=s\}$, $\omega(z_0,\{|z|=s+2\},D)<\delta$.*

Proof. By the maximum principle

$$\begin{aligned}\omega(z_0,\{|z|=s+2\},D)&\le\omega(z_0,\{|z-z_0|=2\},D)\\&=1-\omega(z_0,\partial D\cap\{|z-z_0|<2\},D\cap D(z_0,2)).\end{aligned}$$

Now by the Beurling–Nevanlinna projection theorem (see [N])

$$\omega(z_0,\partial D\cap\{|z-z_0|<2\},D\cap D(z_0,2))\ge\omega(0,[1,2],D(0,2)):=\eta.$$

Then $\omega(z_0,\{|z|=s+2\},D)\le1-\eta<1$, since $\eta>0$. Choose any δ in the open interval $(1-\eta,1)$. For such a δ we have $\omega(z_0,\{|z|=s+2\},D)<\delta$.

The next lemma states the strong Markov property for harmonic measure. This property follows from the probabilistic interpretation of harmonic measure. We will use only a special case of the Markov property. One can actually prove it using the potential-theoretic definition of harmonic measure, see [HK, p. 114].

Lemma 3.7. (The strong Markov property for harmonic measure.) *Let Ω_1 and Ω_2 be two domains in $\mathbf{C}$. Assume that $\Omega_1\subset\Omega_2$ and let $F\subset\partial\Omega_2$ be a closed set. Let $\sigma=\partial\Omega_1\setminus\partial\Omega_2$. Then for $z\in\Omega_1$,*

$$\omega(z,F,\Omega_2)=\omega(z,F,\Omega_1)+\int_\sigma\omega(z,ds,\Omega_1)\omega(s,F,\Omega_2). \tag{3.8}$$

We explain the notation $\omega(z,ds,\Omega_1)$ that appears in (3.8): The harmonic measure $\omega(z,\cdot,\Omega_1)$ is a measure for fixed $z\in\Omega_1$. Call this measure $\mu_z^{\Omega_1}$. In integrals the usual notation is $d\mu_z^{\Omega_1}(s)$ where s is the variable of integration. Instead of this notation we will use the notation $\omega(z,ds,\Omega_1)$, i.e. $d\mu_z^{\Omega_1}(s)=\omega(z,ds,\Omega_1)$.

Proof of Proposition 3.4. Let $D\in\mathcal{B}$ and $s>0$. By the Markov property there exists $z_1\in\{|z|=s\}$ such that

$$\widetilde{\omega}_D(s+2)\le\omega(z_1,\{|z|=s+2\},D)\widetilde{\omega}_D(s). \tag{3.9}$$

By the lemma above, $\omega(z_1,\{|z|=s+2\},D)<\delta\in(0,1)$. Hence (3.9) implies

$$\widetilde{\omega}_D(s+2)\le\delta\widetilde{\omega}_D(s). \tag{3.10}$$

Now let $R>4$ (if $R\in(0,4]$ the theorem holds trivially). Let $R=2k+q$, where $k\in\mathbf{Z}^+$ and $q\in[0,2)$. By iterating (3.10) we obtain

$$\widetilde{\omega}_D(R)\le\delta\widetilde{\omega}_D(R-2)\le\delta^2\widetilde{\omega}(R-4)\le\ldots\le\delta^k\widetilde{\omega}_D(R-2k)=\delta^k\widetilde{\omega}_D(q)\le\delta^{R/2-q/2}.$$

Therefore

$$\widetilde{\omega}_D(R) \le Ce^{(\log \delta)R/2} = Ce^{-\beta R}, \tag{3.11}$$

where $\beta = 1/2 \log(1/\delta) > 0$ and $C = \delta^{-1}$. By the maximum principle $\omega_D(R) \le \widetilde{\omega}_D(R)$. Hence (3.11) implies $\omega_D(R) \le Ce^{-\beta R}$.

The above proof is similar to a proof in [HP].

Definition 3.12. Let $D \in \mathcal{B}$. The *β-exponent $\beta(D)$ of D* is defined by

$$\beta(D) = \sup\{\beta > 0 : \text{ for some } C > 0,\ \omega_D(R) \le Ce^{-\beta R} \text{ for all } R > 0\}.$$

The β-exponent of a domain $D \in \mathcal{B}$ indicates how fast $\omega_D(R)$ decays as R increases to ∞. Proposition 3.4 shows that for all $D \in \mathcal{B}$, $\beta(D) > C$ for an absolute constant $C > 0$.

The strip $S = \{z = x + iy \in \mathbf{C} : -1 < y < 1\}$ of width 2 has β-exponent $\beta(S) = \frac{1}{2}\pi$. This can be proved by a direct calculation of $\omega_S(R)$ using the conformal mapping

$$f(z) = \frac{1 - e^{\pi z/2}}{1 + e^{\pi z/2}}$$

that maps S onto $\mathbf{D}$.

Now we consider the number

$$\beta_0 = \inf_{D \in \mathcal{B}} \beta(D). \tag{3.13}$$

Problem 3.14. Find the exact value of β_0.

C. Bishop [Bi, p. 296] conjectured that $\beta_0 = \beta(S) = \frac{1}{2}\pi$, where S is a strip of width 2. In Section 9 we disprove Bishop's conjecture. We do not give a complete solution to Problem 3.14 but we find a lower and an upper bound for β_0. The problem of the existence of a domain $D \in \mathcal{B}$ for which $\beta_0 = \beta(D)$ also remains open.

4. Lower bound for β_0

We give two additional characterizations of β_0, in terms of Green function and hyperbolic distance. A lower bound for β_0 will then come from an estimate of the hyperbolic density.

Proposition 4.1. *There exists $\beta>0$ and $C>0$ with the following property: For all $D\in\mathcal{B}$,*

$$\sup_{|z|=R} g(0,z,D)\le Ce^{-\beta R},\quad R\ge 1. \tag{4.2}$$

Proof. Let $\zeta\in\{|z|=R\}\cap D$. We may assume that $\zeta=R$. Note that $g(0,z,D)$ is subharmonic in $\{z:0<|z|\le\infty\}$. Hence the maximum principle, the Poisson integral representation of harmonic functions and a standard inequality for the Poisson kernel give

$$g(0,R,D)\le \frac{R}{\pi}\int_0^{2\pi} g(0,(R-1)e^{it},D)\,dt. \tag{4.3}$$

We use the following identity of Baernstein [B3]:

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} g(0,Re^{i\theta},D)\,d\theta=\int_R^{\infty}\frac{\omega_D(t)}{t}\,dt,\quad R>0. \tag{4.4}$$

This and Proposition 3.4 give

$$\frac{1}{2\pi}\int_0^{2\pi} g(0,(R-1)e^{i\theta},D)\,d\theta\le\int_{R-1}^{\infty}\frac{e^{-\beta t}}{t}\,dt\le\frac{e^{-\beta(R-1)}}{(R-1)\beta}. \tag{4.5}$$

Inequalities (4.3) and (4.5) give

$$g(0,R,D)\le Ce^{-\beta R}, \tag{4.6}$$

with an absolute constant C, and so the proposition is proven.

Based on this proposition, we define, for $D\in\mathcal{B}$,

$$\beta_1(D)=\sup\Big\{\beta>0:\ \text{for some } C>0,\ \sup_{|z|=R} g(0,z,D)\le Ce^{-\beta R}\ \text{for all } R>1\Big\}.$$

The proof of Proposition 4.1 implies

$$\beta_1(D)\ge\beta(D). \tag{4.7}$$

Actually equality holds:

Proposition 4.8. *For all $D\in\mathcal{B}$, $\beta_1(D)=\beta(D)$.*

Proof. We use again Baernstein's identity (4.4). Since $\omega_D(t)$ is a decreasing function, (4.4) implies

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} g(0,Re^{i\theta},D)\,d\theta \geq \int_R^{R+1} \frac{\omega_D(t)}{t}\,dt \geq \frac{\omega_D(R+1)}{R+1}, \quad R>0.$$

Hence

$$\omega_D(R+1) \leq (R+1)\max_{|z|=R} g(0,z,D). \tag{4.9}$$

Let $\varepsilon>0$. The inequality (4.9) and the definition of β_1 give

$$\omega_D(R+1) \leq RCe^{-(\beta_1(D)-\varepsilon)R}, \quad R>0. \tag{4.10}$$

Therefore

$$\omega_D(R+1) \leq Ce^{-(\beta_1(D)-2\varepsilon)R}, \quad R>0, \tag{4.11}$$

for a constant C that depends on ε but not on R.

The inequality (4.11) implies $\beta(D)\geq\beta_1(D)-2\varepsilon$. Now letting $\varepsilon\to 0$ and using (4.7) we conclude $\beta_1(D)=\beta(D)$.

In Section 1 we defined the hyperbolic density $\sigma(z,D)$ on D. The hyperbolic distance $d(z_1,z_2,D)$ between z_1 and z_2 in D is

$$d(z_1,z_2,D) = \inf_{\gamma\in\Gamma}\int_\gamma \sigma(z,D)\,|dz|, \tag{4.12}$$

where Γ is the family of all curves in D that join z_1 to z_2.

Proposition 4.13. *Let $D\in\mathcal{B}$ and $\beta>0$. The following are equivalent:*

(i) *There exists $C_2>0$ such that for $R>1$ and $z\in\{|z|=R\}\cap D$, $g(0,z,D)\leq C_2e^{-\beta R}$.*

(ii) *There exists $C_3>0$ such that for $R>0$ and $z\in\{|z|=R\}\cap D$, $d(0,z,D)\geq \frac{1}{2}\beta R-C_3$.*

Proof. Let $R>1$ and $z\in\{|z|=R\}\cap D$. Then

$$g(0,z,D) = \log\frac{1+e^{-2d(0,z,D)}}{1-e^{-2d(0,z,D)}}. \tag{4.14}$$

If $z\in(0,1)$ and $D=\mathbf{D}$, this identity follows at once from the formulae

$$d(0,z,\mathbf{D})=\frac{1}{2}\log\frac{1+|z|}{1-|z|}\quad\text{and}\quad g(0,z,\mathbf{D})=-\log|z|.$$

In general it holds by conformal invariance. Now with a little calculus one shows that (i) is equivalent to (ii).

For $D\in\mathcal{B}$, let

$$\beta_2(D)=2\sup\Bigl\{b>0:\ \text{for some } C>0,\ \inf_{|z|=R} d(0,z,D)\geq bR-C \text{ for all } R>0\Bigr\}.$$

Propositions 4.8 and 4.13 imply

$$\beta(D)=\beta_1(D)=\beta_2(D),\quad D\in\mathcal{B}. \tag{4.15}$$

We use (4.15) to get a lower bound for β_0.

Theorem 4.16. *We have*

$$\beta_0\geq 2U>1.14176, \tag{4.17}$$

where U is the univalent Bloch constant.

Proof. Let $D\in\mathcal{B}$, $R>1$ and $z\in\{|z|=R\}\cap D$. Then

$$d(0,z,D)=\inf_{\gamma\in\Gamma}\int_\gamma\sigma(z,D)\,|dz|\geq\inf_\gamma\{\sigma_D l(\gamma)\}\geq\sigma_D|z|=\sigma_D R\geq UR, \tag{4.18}$$

where Γ is the class of all curves in D joining 0 and z, $\sigma_D=\inf_{z\in D}\sigma(z,D)$, $\sigma(z,D)$ is the hyperbolic density on D and $l(\gamma)$ is the length of γ.

Hence, by (4.15), $\beta_0\geq 2U$. As noted in Section 1, the bound $U>0.57088$ is due to Zhang [Z].

The inequalities in (4.18) are rather crude and it is unlikely that $\beta_0=2U$.

5. An extremal problem for extremal length

In this section we formulate and solve an extremal problem:

Let $x\in[-1,0]$, $y\in(x,1]$ and $S=\{z:x<\operatorname{Re} z<y\}$. Consider two boundary sets of S:

$$A\subset\partial S\cap\mathbf{D}\cap\{\operatorname{Re} z=x\}, \tag{5.1}$$

$$B\subset\partial S\cap\mathbf{D}\cap\{\operatorname{Re} z=y\}, \tag{5.2}$$

and let $\lambda(x,y,A,B):=\lambda(A,B,S)$.

Problem 5.3. Find

$$\inf_{x,y,A,B}\frac{\lambda(x,y,A,B)}{y-x}.$$

We start with some reductions of the problem:

Since $\lambda(x,y,A,B)\geq\lambda(x,y,A',B')$, where $A'=\partial S\cap\mathbf{D}\cap\{\operatorname{Re} z=x\}$ and $B'=\partial S\cap\mathbf{D}\cap\{\operatorname{Re} z=y\}$, we may assume that $A=A'$ and $B=B'$, and write $\lambda(x,y,A,B)=\lambda(x,y)$.

If $y<0$ then $\lambda(x,y)>\lambda(x-y,0)$. So, without loss of generality, from now on we assume that $y\geq 0$.

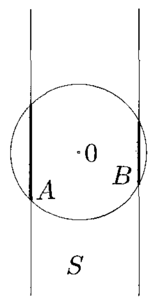

Figure 2. The vertical segments A and B on ∂S.

Applying (2.11) we obtain

$$\lambda(x,y)=\left[4\nu\left(\frac{X+Y}{1+XY}\right)\right]^{-1}, \tag{5.4}$$

where

$$X=\exp\left(\frac{-\pi\sqrt{1-x^2}}{y-x}\right),\quad Y=\exp\left(\frac{-\pi\sqrt{1-y^2}}{y-x}\right).$$

Claim 5.5. *We have*

$$\lim_{\substack{x\to 0\\ y\to 0}}\frac{\lambda(x,y)}{y-x}=\frac{1}{2}. \tag{5.6}$$

Proof. When $x\to 0$ and $y\to 0$, $(X+Y)/(1+XY)\to 0$. We use the asymptotic formula (2.8)

$$\nu(s)=\frac{1}{2\pi}\log\frac{4}{s}+o(1),\quad \text{as } s\to 0. \tag{5.7}$$

We have

$$\begin{aligned}
\lim_{\substack{x\to 0\\ y\to 0}}(y-x)\nu\left(\frac{X+Y}{1+XY}\right) &= \lim_{\substack{x\to 0\\ y\to 0}}(y-x)\left[\frac{1}{2\pi}\log\frac{4(1+XY)}{X+Y}\right]\\
&= \lim_{\substack{x\to 0\\ y\to 0}}(y-x)\left[-\frac{1}{2\pi}\log(X+Y)\right]\\
&= \lim_{\substack{x\to 0\\ y\to 0}} -\frac{1}{2\pi}(y-x)\log\left[e^{-\pi\sqrt{1-x^2}\,/(y-x)}+e^{-\pi\sqrt{1-y^2}\,/(y-x)}\right]\\
&= \lim_{\substack{x\to 0\\ y\to 0}} \frac{-(y-x)}{2\pi}\,\frac{-\pi\sqrt{1-x^2}}{y-x}\\
&\quad + \lim_{\substack{x\to 0\\ y\to 0}} \frac{-(y-x)}{2\pi}\log\left[1+e^{(-\pi\sqrt{1-y^2}+\pi\sqrt{1-x^2})/(y-x)}\right]\\
&= \tfrac{1}{2}+0=\tfrac{1}{2}.
\end{aligned}$$

So

$$\lim_{\substack{x\to 0\\ y\to 0}}\frac{\lambda(x,y)}{y-x}=\frac{1}{4\cdot\frac{1}{2}}=\frac{1}{2} \tag{5.8}$$

and the claim is proven.

We extend the function $\lambda(x,y)$ to $[-1,0]\times[0,1]$ by setting $\lambda(0,0)=\frac{1}{2}$. Continuity implies that the infimum in (5.4) is attained for a pair (x_0,y_0), where $(x_0,y_0)\in[-1,0]\times[0,1]$. A numerical computation shows that the value of the infimum in (5.4) is approximately equal to 0.457443. We will return later to this numerical result. For now, we will use only the fact that $\lambda(-0.4,0.4)/0.8\approx 0.45<0.5$.

Claim 5.9. *There exist* $x_0\in(-1,0)$ *and* $y_0\in(0,1)$ *such that*

$$\min_{x,y}\frac{\lambda(x,y)}{y-x}=\frac{\lambda(x_0,y_0)}{y_0-x_0}, \tag{5.10}$$

where the minimum here and below is taken over all $x\in[-1,0]$ *and* $y\in[0,1]$.

Proof. If $x=-1$ or $y=1$ then $\lambda(x,y)=+\infty$.

If $-1<x<0$ and $y=0$, then, because of symmetry,

$$\frac{\lambda(x,y)}{y-x}=\frac{\lambda(x,-x)}{-2x}.$$

If $x=0$ and $0<y<1$, then similarly

$$\frac{\lambda(x,y)}{y-x}=\frac{\lambda(-y,y)}{2y}.$$

The minimum of the function $\lambda(x,y)/(y-x)$ cannot be attained at $(0,0)$ because as we remarked above $\lambda(-0.4,0.4)/0.8\approx0.45<0.5$. The above remarks show that the minimum is attained for a point in the interior of the square $[-1,0]\times[0,1]$ and the claim is proven.

Let $(x_0,y_0)\in(-1,0)\times(0,1)$ be a minimizing pair whose existence is asserted by Claim 5.9. We write $\lambda_0=\lambda(x_0,y_0)$ and $\alpha_0=y_0-x_0$ so that $\alpha_0\in(0,2)$ and

$$\min_{x,y}\frac{\lambda(x,y)}{y-x}=\frac{\lambda_0}{\alpha_0}. \tag{5.11}$$

Claim 5.12. *We have* $x_0=-y_0$.

Proof. We have

$$\min_{x,y}\frac{\lambda(x,y)}{y-x}=\frac{\lambda(x_0,y_0)}{y_0-x_0}=\frac{\lambda_0}{\alpha_0}. \tag{5.13}$$

In particular

$$\min_{y}\frac{\lambda(y-\alpha_0,y)}{\alpha_0}=\frac{\lambda_0}{\alpha_0}, \tag{5.14}$$

where the minimum is taken over all $y\in[\max(0,\alpha_0-1),\min(\alpha_0,1))$.

Let $g(y)=\lambda(y-\alpha_0,y)/\alpha_0$. The function g attains its minimum for $y=y_0$. By (5.4) we have

$$g(y)=\left[4\nu\left(\frac{X+Y}{1+XY}\right)\right]^{-1}, \tag{5.15}$$

where

$$X=\exp\left(\frac{-\pi\sqrt{1-(y-\alpha_0)^2}}{\alpha_0}\right),\quad Y=\exp\left(\frac{-\pi\sqrt{1-y^2}}{\alpha_0}\right).$$

Since ν is a decreasing function, g is minimal when $F(y):=(X+Y)/(1+XY)$ is minimal. So $F'(y_0)=0$. We differentiate and obtain

$$X'+Y'=X^2Y'+Y^2X'. \tag{5.16}$$

Because of symmetry, we may assume that $y_0 \geq \frac{1}{2}\alpha_0$. We set $y_0 = \frac{1}{2}\alpha_0 + \varepsilon$ and $E := 1 - \varepsilon^2 - \frac{1}{4}\alpha_0^2$ so that

$$1 - (y_0 - \alpha_0)^2 = E + \varepsilon\alpha_0 \quad \text{and} \quad 1 - y_0^2 = E - \varepsilon\alpha_0. \tag{5.17}$$

With this notation (5.16) becomes

$$\begin{aligned} &\frac{(\varepsilon - \frac{1}{2}\alpha_0)e^{-\pi\sqrt{E+\varepsilon\alpha_0}/\alpha_0}}{\sqrt{E+\varepsilon\alpha_0}} + \frac{(\varepsilon + \frac{1}{2}\alpha_0)e^{-\pi\sqrt{E-\varepsilon\alpha_0}/\alpha_0}}{\sqrt{E-\varepsilon\alpha_0}} \\ &\quad = \frac{(\varepsilon + \frac{1}{2}\alpha_0)e^{-2\pi\sqrt{E+\varepsilon\alpha_0}/\alpha_0}e^{-\pi\sqrt{E-\varepsilon\alpha_0}}}{\sqrt{E-\varepsilon\alpha_0}} + \frac{(\varepsilon - \frac{1}{2}\alpha_0)e^{-2\pi\sqrt{E-\varepsilon\alpha_0}/\alpha_0}e^{-\pi\sqrt{E+\varepsilon\alpha_0}}}{\sqrt{E+\varepsilon\alpha_0}}. \end{aligned} \tag{5.18}$$

After some algebraic calculations (5.18) becomes

$$\begin{aligned} &\sqrt{E+\varepsilon\alpha_0}\,\big(\tfrac{1}{2}\alpha_0 + \varepsilon\big)e^{-\pi\sqrt{E-\varepsilon\alpha_0}/\alpha_0}\big[1 - e^{-2\pi\sqrt{E+\varepsilon\alpha_0}/\alpha_0}\big] \\ &\qquad = \sqrt{E-\varepsilon\alpha_0}\,\big(\tfrac{1}{2}\alpha_0 - \varepsilon\big)e^{-\pi\sqrt{E+\varepsilon\alpha_0}/\alpha_0}\big[1 - e^{-2\pi\sqrt{E-\varepsilon\alpha_0}/\alpha_0}\big]. \end{aligned} \tag{5.19}$$

Each of the four factors in the left-hand side of (5.19) is positive and at least as large as the corresponding factor in the right-hand side, with equality if and only if $\varepsilon = 0$. Hence (5.19) implies $\varepsilon = 0$ and therefore $y_0 = \frac{1}{2}\alpha_0$. So $x_0 = -\frac{1}{2}\alpha_0 = -y_0$ and the claim is proven.

Using (5.4) we can find a numerical solution of Problem 5.3. The identity (5.4) expresses $\lambda(x, y)$ in terms of the function ν. Recall from Section 2 that $\nu(s) = K'(s)/4K(s)$ where K' and K are the complete elliptic integrals of modulus $\sqrt{1-s^2}$ and s, respectively. These integrals are built into *Mathematica*, which is thus able to give the following result:

$$\min_{x,y} \frac{\lambda(x,y)}{y-x} \approx 0.457443. \tag{5.20}$$

This minimum is attained for $y = -x \approx 0.403$ and $r = 1$.

We summarize our results on Problem 5.3 in the following proposition.

Proposition 5.21. *Let x, y, A, B be as in the beginning of this section. There exists a number $y_0 \in (0,1)$ such that*

$$\frac{\lambda(x,y,A,B)}{y-x} \geq \frac{\lambda(-y_0, y_0)}{2y_0} = \frac{\lambda_0}{\alpha_0}. \tag{5.22}$$

The following approximate equalities hold: $y_0 \approx 0.403$, $\lambda_0/\alpha_0 \approx 0.457443$, $\alpha_0 \approx 0.806$.

Remark (1). We observed that

$$\Lambda\Big(\frac{\pi}{e}\Big) := \frac{\lambda\Big(-\cos\dfrac{\pi}{e}, \cos\dfrac{\pi}{e}\Big)}{2\cos\dfrac{\pi}{e}} \approx 0.457443, \tag{5.23}$$

that is, the numerical solution of the extremal problem agrees with the number $\Lambda(\pi/e)$ in its first six decimal digits. It would be interesting if one could prove that $y=\cos(\pi/e)$ is indeed the minimizing value in (5.22).

(2) Is the number y_0 in Proposition 5.21 unique?

(3) Proposition 5.21 will play an important role in Section 8.

6. Periodic comb domains

A periodic comb domain is a domain of the form

$$D=D(z_0)=\mathbf{C}\setminus\bigcup_{j\in\mathbf{Z}}\{(2j-1)x_0+iy:|y|\geq y_0\}, \tag{6.1}$$

where $z_0=x_0+iy_0$ with $x_0>0$, $y_0>0$.

If $D\in\mathcal{B}$, then $x_0^2+y_0^2=1$, and conversely. We denote by $\mathcal{B}_p$ the class of all periodic comb domains in $\mathcal{B}$. We write $D=D(x_0)$ for $D(z_0)$, if $D\in\mathcal{B}_p$.

Let D be a periodic comb domain. For $j\in\mathbf{Z}$, we define the crosscuts Γ_j of D by $\Gamma_j=\{(2j-1)x_0+iy:|y|\leq y_0\}$. Let $S=\{z:-x_0<\operatorname{Re} z<x_0\}$. There exists a unique conformal mapping f and a unique number $\lambda>0$ such that f maps S onto the rectangle $G=\left(-\frac{1}{2}\lambda,\frac{1}{2}\lambda\right)\times\left(-\frac{1}{2}i,\frac{1}{2}i\right)$ with $f(\Gamma_0)=\left[-\frac{1}{2}\lambda-\frac{1}{2}i,-\frac{1}{2}\lambda+\frac{1}{2}i\right]$ and $f(\Gamma_1)=\left[\frac{1}{2}\lambda-\frac{1}{2}i,\frac{1}{2}\lambda+\frac{1}{2}i\right]$. Notice that $\lambda=\lambda(\Gamma_0,\Gamma_1,S)$. By repeated reflections f extends to a conformal mapping of D onto the strip $G_1=\left\{z:|\operatorname{Im} z|<\frac{1}{2}\right\}$. Using the mapping f one can easily estimate harmonic measures at 0 on D.

Proposition 6.2. *Let $D\in\mathcal{B}_p$ and x_0, λ be as above. Then for all $R>0$,*

$$\omega_D(R)\geq Ce^{-\beta R}, \tag{6.3}$$

where $\beta=\min(\pi/2x_0,\pi\lambda/2x_0)$ and C is an absolute constant.

We study now the conformal radius $R(0,D)$ of a domain $D\in\mathcal{B}_p$. Note that for a simply connected domain D and for $z\in D$, $R(z,D)=\sigma(z,D)^{-1}$. Let $D=D(x_0)\in\mathcal{B}_p$. We will compute $R(0,D)$ as a function of x_0.

Recall that $S=\{z:-x_0<\operatorname{Re} z<x_0\}$ and $\lambda=\lambda(\Gamma_0,\Gamma_1,S)$. The number λ can be explicitly computed using Proposition 2.10 and (2.6),

$$\lambda=4\nu\left(\frac{1-e^{-\pi y_0/x_0}}{1+e^{-\pi y_0/x_0}}\right)=\frac{1}{2\nu(e^{-\pi y_0/x_0})},\quad y_0=\sqrt{1-x_0^2}. \tag{6.4}$$

We will use elliptic functions (see [A]). Set $k=e^{-\pi\sqrt{1-x_0^2}/x_0}$ and $k'=\sqrt{1-k^2}$. The function

$$f_1(z)=i\exp\left[-\frac{i\pi\left(z+i\sqrt{1-x_0^2}\right)}{2x_0}\right] \tag{6.5}$$

maps S onto $\mathbf{C}_+$ so that $f_1(0)=i/\sqrt{k}$, $f_1\big(x_0-i\sqrt{1-x_0^2}\,\big)=1$, $f_1\big(x_0+i\sqrt{1-x_0^2}\,\big)=1/k$, $f_1\big(-x_0-i\sqrt{1-x_0^2}\,\big)=-1$, $f_1\big(-x_0+i\sqrt{1-x_0^2}\,\big)=-1/k$.

The Jacobi elliptic function $g(z):=\operatorname{sn}(z,k)$ maps the rectangle $\Pi=(-K,K)\times(0,K')$ onto $\mathbf{C}_+$ with $g(-K)=-1$, $g(K)=1$ and $g\big(\frac{1}{2}iK'\big)=i/\sqrt{k}$. Here

$$K=K(k)=\int_0^1(1-x)^{-1/2}(1-k^2x^2)^{-1/2}\,dx, \tag{6.6}$$

$$K'=K\big(\sqrt{1-k^2}\,\big)=\int_0^1(1-x)^{-1/2}(1-(k')^2x^2)^{-1/2}\,dx. \tag{6.7}$$

Hence the function $F=g^{-1}\circ f_1$ maps S onto Π. By repeated reflections we extend F to a function that maps $D(x_0)$ onto the strip $S_1=\{0<\operatorname{Im} z<K'\}$ which has conformal radius $2K'/\pi$.

So, for the conformal radius $R(0,D):=\sigma(0,D)^{-1}$ of D at 0, we have

$$\begin{aligned} R(0,D)&=\frac{2K'}{\pi}\frac{1}{|F'(0)|}=\frac{2K'}{\pi}\frac{1}{|(g^{-1}\circ f_1)'(0)|}=\frac{2K'}{\pi}\frac{\big|g'\big(i/\sqrt{k}\,\big)\big|}{|f_1'(0)|}\\ &=\frac{2K'}{\pi}\frac{(1+k)/\sqrt{k}}{\pi/2x_0\sqrt{k}}=\frac{4x_0(1+k)K'}{\pi^2}. \end{aligned} \tag{6.8}$$

In the computation above we used some formulae for the derivative of the function $\operatorname{sn}(z,k)$, see [A, p. 208]. Since k and K' are known functions of x_0, (6.8) is the expression we sought.

Using *Mathematica* we found the following result.

Let $R(x_0)=R(0,D(x_0))$. Then

$$\max_{x_0\in(0,1)} R(x_0)=1.39304. \tag{6.9}$$

The computations of *Mathematica* suggest that this maximum is attained uniquely for $x_0=0.4227$. The computation of $R(x_0)$ gives an upper bound $U\le R(x_0)^{-1}$ for the univalent Bloch constant U. The best (approximately) upper bound we obtain is $U\le 1.39304^{-1}\approx 0.718$ and this is worse (larger) than the upper bound $U\le 0.6566$ obtained by Goodman [Go].

7. Harmonic measure and convergence of domains in $\mathcal{B}$

Let $\{D_n\}$ be a sequence of domains in $\mathcal{B}$ and assume that $D_n\to D\in\mathcal{B}$, as $n\to\infty$, with respect to 0, in the sense of Carathéodory. This means (see [P, p. 13]) that:

(i) For every $z\in D$ there exists an open set O that contains z and lies in D_n for all $n>n_0$, n_0 may depend on z and O.

(ii) For $\zeta\in\partial D$, and each n, there exists $\zeta_n\in\partial D_n$ such that $\zeta_n\to\zeta$, as $n\to\infty$.

In this section we will study some problems related to the following question: *Let* $\mathcal{B}\ni D_n\to D\in\mathcal{B}$. *Is it true that* $\omega_{D_n}(R)\to\omega_D(R)$?

First we need a result of Baernstein [B2]. Let S be the family of all univalent functions $f:\mathbf{D}\to\mathbf{C}$ with $f(0)=0$ and $f'(0)=1$. Let H^p, $0<p\le\infty$, be the Hardy space on the unit disk (see [D]). For $0<p<1$, H^p is a complete, separable metric space with distance function

$$d(f,g)=\int_0^{2\pi}|f(e^{i\theta})-g(e^{i\theta})|^p\,d\theta. \tag{7.1}$$

From the subharmonicity of f_n-f in $\mathbf{D}$, one sees that if $f_n\to f$ in H^p, then $f_n\to f$ locally uniformly in $\mathbf{D}$. Baernstein's result asserts that for $0<p<\frac{1}{2}$ the converse holds, too. This follows from the following theorem.

Theorem 7.2. (Baernstein) *For* $0<p<\frac{1}{2}$, *S is a compact subset of H^p.*

Since this theorem is not published we include a proof taken from [B2].

Proof. By Hölder's inequality, it suffices to consider $\frac{2}{5}<p<\frac{1}{2}$. For these p, by a theorem of Feng and MacGregor [FM], we have

$$\int_0^{2\pi}|f'(re^{i\theta})|^p\,d\theta\le\frac{A_p}{(1-r)^{3p-1}},\quad 0<r<1, \tag{7.3}$$

where the constant A_p depends only on p.

Now $3-1/p<1$, if $p<\frac{1}{2}$. So, by an argument of Gwilliam [Gw],

$$\int_0^{2\pi}|f(re^{i\theta})-f(e^{i\theta})|^p\,d\theta\le A_p(1-r)^{1-2p},\quad \tfrac{2}{5}<p<\tfrac{1}{2}. \tag{7.4}$$

Since S is a normal family, compactness of H^p follows easily.

Corollary 7.5. *Let f_n be a sequence in S and assume that $f_n\to f$ locally uniformly on $\mathbf{D}$. Then*

$$\lim_{n\to\infty}\int_0^{2\pi}|f_n(e^{i\theta})-f(e^{i\theta})|^p\,d\theta=0,\quad 0<p<\tfrac{1}{2}. \tag{7.6}$$

We will use this corollary to prove the following theorem. Its proof is also due to Baernstein.

Theorem 7.7. *Let $\{D_n\}_{n=1}^{\infty}$ be a sequence in $\mathcal{B}$ and assume that $D_n \to D_0 \in \mathcal{B}$ with respect to 0. Then for all $\varepsilon>0$ and $R>0$*

$$\overline{\lim_{n\to\infty}}\, \omega_{D_n}(R) \le \omega_{D_0}(R) \le \underline{\lim_{n\to\infty}}\, \omega_{D_n}(R-\varepsilon). \tag{7.8}$$

Proof. Let f_n be the function that maps $\mathbf{D}$ conformally onto D_n, $n=0,1,2,\dots$, with $f_n(0)=0$ and $f_n'(0)>0$. By Carathéodory's convergence theorem (see [P, p. 13])

$$\lim_{n\to\infty} f_n = f_0 \quad \text{locally uniformly on } \mathbf{D}. \tag{7.9}$$

So, by Corollary 7.5,

$$\lim_{n\to\infty} \int_0^{2\pi} |f_n(e^{i\theta})-f_0(e^{i\theta})|^p \, d\theta = 0, \quad 0<p<\tfrac{1}{2}. \tag{7.10}$$

Fix $0<p<\frac{1}{2}$ and let m denote the Lebesgue measure on $\partial\mathbf{D}$. Since L^1 convergence implies convergence in measure, (7.10) implies that for every $\varepsilon>0$,

$$\lim_{n\to\infty} m(\{\theta : |f_n(e^{i\theta})-f_0(e^{i\theta})|^p \ge \varepsilon\}) = 0. \tag{7.11}$$

Now since $(a+b)^p \le a^p+b^p$ for $a>0$, $b>0$, we have

$$\{|f_n|^p \ge \alpha+\varepsilon\} \subset \{|f_n-f_0|^p \ge \varepsilon\} \cup \{|f_0|^p \ge \alpha\} \tag{7.12}$$

and

$$\{|f_0|^p \ge \alpha\} \subset \{|f_n-f_0|^p \ge \varepsilon\} \cup \{|f_0|^p \ge \alpha-\varepsilon\}, \tag{7.13}$$

for all $\alpha>0$ and all $\varepsilon>0$. Since $m(\{|f_0|^p \ge \alpha\}) = \omega_{D_0}(\alpha^{1/p})$, (7.11) and (7.12) imply

$$\overline{\lim_{n\to\infty}}\, \omega_{D_n}((\alpha+\varepsilon)^{1/p}) \le \omega_{D_0}(\alpha^{1/p}). \tag{7.14}$$

Similarly, (7.11) and (7.13) imply

$$\underline{\lim_{n\to\infty}}\, \omega_{D_n}((\alpha-\varepsilon)^{1/p}) \ge \omega_{D_0}(\alpha^{1/p}). \tag{7.15}$$

Setting $\alpha^{1/p}=R$, we see easily that (7.14) and (7.15) imply

$$\overline{\lim_{n\to\infty}}\, \omega_{D_n}(R+\varepsilon) \le \omega_{D_0}(R) \le \underline{\lim_{n\to\infty}}\, \omega_{D_n}(R-\varepsilon). \tag{7.16}$$

The left-hand side inequality can be written as

$$\varlimsup_{n\to\infty} \omega_{D_n}(R) \le \omega_{D_0}(R-\varepsilon). \tag{7.17}$$

Now letting $\varepsilon\to 0$, and using the fact that ω_D is continuous from the left, we obtain

$$\varlimsup_{n\to\infty} \omega_{D_n}(R) \le \omega_{D_0}(R), \tag{7.18}$$

and (7.8) is proven.

Remark. For $D\in\mathcal{B}$, $\omega_D(R)$ is not in general a continuous function of R. It is, however, a decreasing function and so it can have at most a countable number of discontinuities. It is easy to see that if D is a comb domain then $\omega_D(R)$ is a continuous function.

Before stating the next theorem we need some definitions.

Definition 7.19. A symmetric comb domain is a domain D of the form

$$D=\mathbf{C}\setminus\bigcup_{j\in\mathbf{Z}}\{x_j+iy:|y|>y_j\}, \tag{7.20}$$

where $\{x_j\}_{j\in\mathbf{Z}}$ is an increasing sequence of real numbers and $\{y_j\}_{j\in\mathbf{Z}}$ is a sequence of positive numbers.

If, in addition, there exists $d=d(D)>0$ such that $x_{j+1}-x_j>d$ for all $j\in\mathbf{Z}$, then D will be called a symmetric a-comb domain.

Definition 7.21. A domain $D\subset\mathbf{C}$ will be called convex in the y-direction, if for all $x\in\mathbf{R}$ the set $D_x=\{x+iy:y\in D\}$ is connected.

We will use the following notation:

$\mathcal{B}_s$ is the class of all symmetric comb domains that belong to $\mathcal{B}$,

$\mathcal{B}_a$ is the class of all symmetric a-comb domains that belong to $\mathcal{B}$,

$\mathcal{B}_c$ is the class of all domains that are symmetric with respect to the real axis, convex in the y-direction and belong to $\mathcal{B}$.

Recall that $\mathcal{B}_p$ is the class of all periodic comb domains in $\mathcal{B}$. With the above notation we have $\mathcal{B}_p\subset\mathcal{B}_a\subset\mathcal{B}_s\subset\mathcal{B}_c$.

In the next section we will study the harmonic measure $\omega_D(R)$, $R>0$, and the β-exponent $\beta(D)$ of domains $D\in\mathcal{B}_c$. Here we prove that any $D\in\mathcal{B}_c$ can be approximated in the sense of Carathéodory by a sequence of domains in $\mathcal{B}_a$. More precisely we will prove the following proposition.

Proposition 7.22. *Let $D\in\mathcal{B}_c$. There exists a sequence $\{D_n\}\subset\mathcal{B}_a$ such that $D_n\to D$, with respect to 0, in the sense of Carathéodory.*

Proof. For $x\in D$, let $f(x)=\sup\{y:x+iy\in D_x\}$. It is easy to see that if I is a closed interval lying in $\mathbf{R}\cap D$, then either $f(x)=+\infty$ for all $x\in I$, or there exists at least one point $x_I\in I$ such that $f(x_I)=\min_{x\in I} f(x)<\infty$.

Fix an integer $n>10$ and consider the intervals $I_{n,k}=[k/n,(k+1)/n]$, $k\in\mathbf{Z}$. If $I_{n,k}\subset\mathbf{R}\cap D$ and $f\neq+\infty$ on $I_{n,k}$, let $x_{n,k}$ be a point of minimum whose existence was asserted above. Let also $a=\inf(D\cap\mathbf{R})$, $b=\sup(D\cap\mathbf{R})$.

The domain G_n is the symmetric comb domain whose boundary is defined as follows:

Let $S_{n,k}=\{x_{n,k}+iy:|y|\geq f(x_{n,k})\}$, $k=\dots-4,-2,0,2,4,\dots$.

If $a=-\infty$ and $b=+\infty$, then $\partial G_n=\bigcup_k S_{n,k}$.

If $a=-\infty$ and $b<+\infty$, then $\partial G_n=\bigcup_k S_{n,k}\cup\{z:\operatorname{Re} z=b+1/n\}$.

If $a>-\infty$ and $b=+\infty$, then $\partial G_n=\bigcup_k S_{n,k}\cup\{z:\operatorname{Re} z=a-1/n\}$.

If $a>-\infty$ and $b<+\infty$, then $\partial G_n=\bigcup_k S_{n,k}\cup\{z:\operatorname{Re} z=a-1/n$ or $\operatorname{Re} z=b+1/n\}$.

So we have constructed a sequence G_n of symmetric a-comb domains. It is easy to see that (i) $D\subset G_n$ for all n and (ii) for all $\zeta\in\partial D$ there exists $\zeta_n\in\partial G_n$ such that $\lim_{n\to\infty}\zeta_n=\zeta$. Hence $G_n\to D$, as $n\to\infty$.

The inradius of G_n may be larger than 1, but the triangle inequality implies that

$$R(G_n)\leq 1+\frac{2}{n}. \tag{7.23}$$

Let $D_n=G_n/R(G_n)$. Then $\{D_n\}\subset\mathcal{B}_a$ and $D_n\to D$, as $n\to\infty$.

8. Domains which are symmetric and convex in the y-direction

We will use the results of the previous section to study the harmonic measure and $\omega_D(R)$ and the β-exponent $\beta(D)$ for $D\in\mathcal{B}_c$. We will show that a certain domain $D_0\in\mathcal{B}_a$ has the smallest β-exponent among all $D\in\mathcal{B}_c$. This extremal comb domain D_0 has an additional symmetry: it is a periodic comb domain defined as follows:

In Section 5 we defined the numbers α_0 and λ_0 and we proved that (see Proposition 5.21)

$$\pi\lambda_0/\alpha_0:=\delta<\tfrac{1}{2}\pi. \tag{8.1}$$

Let $D_0=D\big(\tfrac{1}{2}\alpha_0\big)$, i.e.

$$D_0=\mathbf{C}\setminus\bigcup_{j\in\mathbf{Z}}\Big\{z:\operatorname{Re} z=\tfrac{1}{2}(2j-1)\alpha_0,\ |\operatorname{Im} z|\geq\sqrt{1-\alpha_0^2/4}\Big\}. \tag{8.2}$$

From Proposition 6.2 it follows that

$$Ce^{-\delta R} \le \omega_{D_0}(R). \tag{8.3}$$

Theorem 8.4. *For all $D \in \mathcal{B}_c$,*

$$\omega_D(R) \le C\omega_{D_0}(R), \quad R > 0, \tag{8.5}$$

where C is an absolute constant.

From Theorem 8.4 and Proposition 6.2 we obtain the following corollary.

Corollary 8.6. *For all $D \in \mathcal{B}_c$, $\beta(D) \ge \beta(D_0) = \pi\lambda_0/\alpha_0$.*

Remark. Since $\lambda_0/\alpha_0 \approx 0.457443$, Corollary 8.6 implies $\beta(D_0) < \frac{1}{2}\pi$. This fact disproves Bishop's conjecture. A smaller upper bound for β_0 will be obtained in Section 9.

The rest of this section is devoted to the proof of Theorem 8.4.

Proof. The proof has three steps.

Step 1. In Steps 1 and 2 we prove (8.5) with the additional assumption $D \in \mathcal{B}_a$. The class $\mathcal{B}_a$ was defined in Section 7. Let

$$D = \mathbf{C} \setminus \bigcup_{j \in \mathbf{Z}} \{x_j + iy : |y| \ge y_j\} \in \mathcal{B}_a. \tag{8.7}$$

We may assume that $y_j < 1$ for all $j \in \mathbf{Z}$. Otherwise, we replace D by the domain D' obtained from D by deleting the half-lines $\{x_j + iy_j : |y| \ge y_j\}$ with $y_j \ge 1$. Then $D \subset D'$ and $R(D') = 1$.

We will construct a domain $D^* \in \mathcal{B}_a$ that contains D and has additional properties. We may assume that $x_j > 0$ for $j \in \mathbf{Z}^+$ and $x_j \le 0$ for $j \in \mathbf{Z}^- \cup \{0\}$. The domain D^* is obtained from D by deleting certain half-lines of ∂D. It has the form

$$D^* = \mathbf{C} \setminus \bigcup_{j \in \mathbf{Z}} \{x_j^* + iy : |y| \ge y_j^*\}, \tag{8.8}$$

where $\{x_j^*\}_{j \in \mathbf{Z}}$ and $\{y_j^*\}_{j \in \mathbf{Z}}$ are subsequences of $\{x_j\}_{j \in \mathbf{Z}}$ and $\{y_j\}_{j \in \mathbf{Z}}$, respectively. They are defined inductively as follows:

Set $x_1^* = x_1$ and $y_1^* = y_1$. Let $\Delta_1 = D(\xi, 1)$ be the disk satisfying

(i) $x_1 + iy_1 \in \partial\Delta_1$,

(ii) $\xi \ge x_1$.

By a simple argument (involving a renumbering of the sequence $\{x_j\}_{j\in\mathbf{Z}}$) we may assume that $\Delta_1\cap\{z:\operatorname{Re} z<\xi\}\subset D$. Then, since $R(D)=1$, there exists a finite number of points $x_1^1<x_1^2<...<x_1^{k_1}$ in $\{x_j\}_{j=2}^{\infty}$ with $\xi<x_1^1$ such that $x_1^j+iy_1^j\in\operatorname{clos}\Delta_1$, $j=1,2,\dots,k_1$. We set $x_2^*=x_1^{k_1}$ and $y_2^*=y_1^{k_1}$.

Applying the same construction starting from x_2^* and y_2^* we define the sequences $\{x_1^*,x_2^*,\dots,x_n^*,...\}\subset\{x_j\}_{j=1}^{\infty}$, $\{y_1^*,y_2^*,\dots,y_n^*,...\}\subset\{y_j\}_{j=1}^{\infty}$ and the sequence of disks $\{\Delta_1,\Delta_2,\dots,\Delta_n,...\}$.

Similarly, working from right to left we define the sequences

$$\{x_0^*,x_{-1}^*,\dots,x_{-n}^*,...\}\subset\{x_j\}_{j=0}^{-\infty},\quad \{y_0^*,y_{-1}^*,,\dots,y_{-n}^*,...\}\subset\{y_j\}_{j=0}^{-\infty}$$

and the sequence of disks $\{\Delta_0,\Delta_{-1},\dots,\Delta_{-n},...\}$.

By the construction the domain

$$D^*=\mathbf{C}\setminus\bigcup_{j\in\mathbf{Z}}\{x_j^*+iy:|y|\geq y_j^*\} \tag{8.9}$$

is a comb domain in $\mathcal{B}_a$, contains D and satisfies

$$\{x_j^*+iy_j^*,x_{j+1}^*+iy_{j+1}^*\}\subset\operatorname{clos}\Delta_j,\quad j\in\mathbf{Z}. \tag{8.10}$$

Step 2. We continue to assume that $D\in\mathcal{B}_a$. Let

$$D=\mathbf{C}\setminus\bigcup_{j\in\mathbf{Z}}\{x_j+iy:|y|\geq y_j\}. \tag{8.11}$$

We again assume that $x_j>0$ for $j\in\mathbf{Z}^+$ and $x_j\leq 0$ for $j\in\mathbf{Z}^-\cup\{0\}$. Set $d_j=x_{j+1}-x_j$, $\Gamma_j=\{\operatorname{Re} z=x_j\}\cap D$, $S_j=\{z:x_j<\operatorname{Re} z<x_{j+1}\}$, $j\in\mathbf{Z}$.

By Step 1 we may assume (by replacing D by D^*) that for all $j\in\mathbf{Z}$, the points x_j+iy_j and $x_{j+1}+iy_{j+1}$ lie in the closure of a disk Δ_j of radius 1 centered on $\mathbf{R}$. Now we may apply the solution of the extremal problem of Section 5 (Proposition 5.21) to get

$$\lambda(\Gamma_j,\Gamma_{j+1},S_j)\geq\frac{\lambda_0}{\alpha_0}(x_{j+1}-x_j),\quad j\in\mathbf{Z}. \tag{8.12}$$

Let $R>10$ and $D_R=D\cap\{|z|<R\}$. Let E_1 be the component of $D\cap\{|z|=R\}$ that intersects $\mathbf{R}^+$ and let E_2 be the component of $D\cap\{|z|=R\}$ that intersects $\mathbf{R}^-$.

Let k be the largest positive integer with the properties $x_k\leq R-1$ and

$$\operatorname{clos} E_1\cap\{x_k+iy:|y|\geq y_k\}\neq\emptyset. \tag{8.13}$$

Since $R(D)=1$, it follows that

$$R-3<x_k\leq R-1. \tag{8.14}$$

Then, by the maximum principle,

$$\omega(0,E_1,D)\leq\omega(0,\Gamma_k,D). \tag{8.15}$$

Similarly we have

$$\omega(0,E_2,D)\leq\omega(0,\Gamma_{k'},D), \tag{8.16}$$

where k' is the smallest negative integer with the property

$$\operatorname{clos}E_2\cap\{x_{k'}+iy:|y|\geq y_{k'}\}\neq\emptyset. \tag{8.17}$$

Next let $A_j=\{|z|=R\}\cap S_j$, $k'\leq j\leq k-1$. The set A_j has two components.

Claim 8.18. *The harmonic measure* $\omega(z,A_j,D)\leq Ce^{-\pi(u_j-1)/d_j}$, $1\leq j\leq k-1$, *where C is an absolute constant,* $z\in\Gamma_j$ *and* $u_j=\min\{\operatorname{Im}z:z\in A_j^+\}=(R^2-x_{j+1}^2)^{1/2}$.

Proof. By the maximum principle it suffices to prove that

$$\omega\big(\tfrac{1}{2}d_j+i,A_j^*,\Omega\big)\leq Ce^{-\pi(u_j-1)/d_j},$$

where $\Omega=\mathbf{C}\setminus\{iy:y\geq1\}\setminus\{d_j+iy:y\geq1\}$ and $A_j^*=\{x+iu_j:x\in[0,d_j]\}$. By Beurling's inequality (2.15)

$$\omega\big(\tfrac{1}{2}d_j+i,A_j^*,\Omega\big)\leq Ce^{-\pi\lambda(\gamma,A_j^*,\Omega)},$$

where $\gamma=\{x+i:x\in[0,d_j]\}$. But $\lambda(\gamma,A_j^*,\Omega)=(u_j-1)/d_j$ and hence the claim is proven.

Next, let

$$\omega_j=\omega(0,A_j,D),\quad k'+1\leq j\leq k-1. \tag{8.19}$$

By the strong Markov property (Lemma 3.7)

$$\omega_j\leq\max_{z\in\Gamma_j}\omega(z,A_j,D)\omega(0,\Gamma_j,D). \tag{8.20}$$

By Beurling's inequality (2.15) and the subadditivity property of extremal distance (see [O, Theorem 2.10])

$$\omega(0,\Gamma_j,D)\leq Ce^{-\pi\lambda(\Gamma_0,\Gamma_j,D)}\leq Ce^{-\pi\sum_{n=0}^{j-1}\lambda(\Gamma_n,\Gamma_{n+1},S_n)}. \tag{8.21}$$

Using (8.12) and (8.21) we obtain

$$\omega(0,\Gamma_j,D)\le Ce^{-\pi\lambda_0(x_j-x_0)/\alpha_0}. \tag{8.22}$$

Claim 8.18, (8.29) and (8.22) yield

$$\omega_j\le C\exp\left(-\frac{\pi(u_j-1)}{d_j}-\frac{\pi\lambda_0 x_j}{\alpha_0}\right), \tag{8.23}$$

with an absolute constant C. Recall that $\delta=\pi\lambda_0/\alpha_0$. Then (8.23) and the fact $x_{j+1}-x_j<2$ give for $1\le j\le k-1$,

$$\begin{aligned}\omega_j&\le C\exp\left(-\delta x_j-\frac{\pi(u_j-1)}{d_j}\right)\le C\exp\left(-\delta x_{j+1}-\frac{\pi(u_j-1)}{d_j}\right)\\&=C\exp\left(\delta(x_k-x_{j+1}-x_k)-\frac{\pi u_j}{d_j}\right)=C\exp(-\delta x_k)\exp\left(\delta(x_k-x_{j+1})-\frac{\pi(u_j-1)}{d_j}\right)\\&\le C\exp(-\delta R)\exp\left(\delta(x_k-x_{j+1})-\frac{\pi(u_j-1)}{d_j}\right),\end{aligned}$$

where we used (8.14).

Now $x_{j+1}^2+u_j^2=R^2$. So $x_{j+1}+u_j-1\ge R-1>x_k$. Hence

$$\omega_j\le Ce^{-\delta R}\exp\left(-(x_k-x_{j+1})\left(\frac{\pi}{d_j}-\delta\right)\right),\quad 1\le j\le k-1. \tag{8.24}$$

Similarly we show

$$\omega_j\le Ce^{-\delta R}\exp\left(-(x_{k'}-x_{j-1})\left(\frac{\pi}{d_j}-\delta\right)\right),\quad k'+1\le j\le -1. \tag{8.25}$$

For $l=0,1,2,\dots$, let

$$J_l=\{j\ge 1: l\le x_{k-1}-x_{j+1}<l+1\}. \tag{8.26}$$

Then

$$\sum_{j=1}^{k-1}\omega_j\le\sum_{j\in J_0}\omega_j+\sum_{j\in J_1}\omega_j+\sum_{l=2}^{\infty}\sum_{j\in J_l}\omega_j. \tag{8.27}$$

Let $m\in\mathbf{Z}^+$ be such that $R-7<x_m<R-5$. By Beurling's inequality (2.15), the subadditivity property of extremal distance (see [O]), and (8.12)

$$\sum_{j\in J_0}\omega_j+\sum_{j\in J_1}\omega_j\le\omega(0,\Gamma_m,D)\le Ce^{-\pi\lambda(\Gamma_0,\Gamma_m,D)}\le Ce^{-\pi\lambda_0 R/\alpha_0}=Ce^{-\delta R}. \tag{8.28}$$

By (8.27), (8.28) and (8.24) we get

$$\begin{aligned}\sum_{j=1}^{k-1}\omega_j &\le Ce^{-\delta R}\sum_{l=2}^{\infty}\sum_{j\in J_l}\exp\Big(-(x_{k-1}-x_{j+1})\Big(\frac{\pi}{d_j}-\delta\Big)\Big)+Ce^{-\delta R}\\ &\le Ce^{-\delta R}\sum_{l=2}^{\infty}\sum_{j\in J_l}e^{-l(\pi/d_j-\delta)}+Ce^{-\delta R}.\end{aligned}\tag{8.29}$$

Now we will estimate $\sum_{l=2}^{\infty}\sum_{j\in J_l}e^{-l(\pi/d_j-\delta)}$ by an absolute constant. We need the following lemma.

Lemma 8.30. *Let f be a positive function, convex, increasing on $[0,2]$ and smooth on $(0,2)$. Assume also that $f(0)=0$. Let*

$$g(d_1,d_2,\dots,d_N)=\sum_{j=1}^{N}f(d_j).\tag{8.31}$$

Then the maximum of g under the conditions $d_1+d_2+\dots+d_N\le 2$ and $d_j\ge 0$ for all $j=1,2,\dots,N$ is attained when $d_j=2$ for some j.

The proof of the lemma follows easily from the theorem on Lagrange multipliers.

We apply the lemma to $f(x)=\exp[-l(\pi/x-\delta)]$. Note that $\sum_{j\in J_l}d_j\le 2$. It is easy to check that f satisfies the other conditions of the lemma if $l\in\{2,3,\dots\}$. So we have

$$\sum_{j\in J_l}e^{-l(\pi/d_j-\delta)}\le e^{-l(\pi/2-\delta)},\quad l=2,3,\dots\,.\tag{8.32}$$

Therefore, since $\delta<\frac{1}{2}\pi$,

$$\sum_{l=2}^{\infty}\sum_{j\in J_l}e^{-l(\pi/d_j-\delta)}\le\sum_{l=2}^{\infty}e^{-l(\pi/2-\delta)}<C,\tag{8.33}$$

for an absolute constant C.

Now (8.29) and (8.33) give

$$\sum_{j=1}^{k-1}\omega_j\le Ce^{-\delta R}.\tag{8.34}$$

Similarly we use (8.25) to find the estimate

$$\sum_{j=k'}^{k-1}\omega_j\le Ce^{-\delta R}.\tag{8.35}$$

Also $\omega_0 \le Ce^{-\delta R}$. This follows from Claim 8.18 and the fact $\pi/d_0 \ge \frac{1}{2}\pi > \delta$. Hence

$$\widetilde{\omega}_D(R) \le \omega(0, E_1, D) + \omega(0, E_2, D) + \sum_{j=k'}^{k-1} \omega_j \le Ce^{-\delta R}. \tag{8.36}$$

Recall now that

$$D_0 = \mathbf{C} \setminus \bigcup_{j \in \mathbf{Z}} \left\{ z : \operatorname{Re} z = \tfrac{1}{2}(2j-1)\alpha_0,\ |\operatorname{Im} z| \ge \sqrt{1-\alpha_0^2/4} \right\}. \tag{8.37}$$

By Proposition 6.2

$$\omega_{D_0}(R) \ge Ce^{-\pi\lambda_0 R/\alpha_0} = Ce^{-\delta R}. \tag{8.38}$$

The inequalities (8.36) and (8.38) imply

$$\omega_D(R) \le C\widetilde{\omega}_D(R) \le C\omega_{D_0}(R), \quad R > 10. \tag{8.39}$$

This holds actually for all $R>0$.

Step 3. In Steps 1 and 2 we have assumed that $D \in \mathcal{B}_a$. Here we drop this assumption.

Let $D \in \mathcal{B}_c$ and consider a sequence D_n in $\mathcal{B}_a$ which converges to D, in the sense of Carathéodory. The existence of such a sequence was proved in Proposition 7.22. By Step 2 we have for all n and R

$$\omega_{D_n}(R) \le C\omega_{D_0}(R). \tag{8.40}$$

By Theorem 7.7 and (8.40) for each $\varepsilon>0$, we have

$$\omega_D(R) \le \varliminf_{n \to \infty} \omega_{D_n}(R-\varepsilon) \le C\omega_{D_0}(R-\varepsilon). \tag{8.41}$$

Now since D_0 is a comb domain, ω_{D_0} is continuous. Hence, letting $\varepsilon \to 0$ we obtain (8.5) and the theorem is proved.

9. Parasymmetric comb domains. Upper bound for β_0

As we saw in Corollary 8.6, $\beta_0 \le \pi\lambda_0/\alpha_0 < 0.46\pi$. Now we find a better upper bound for β_0.

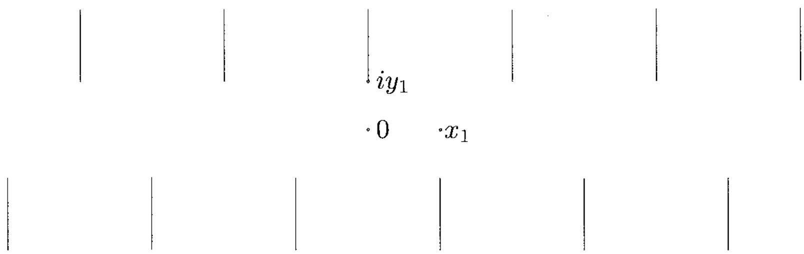

Figure 3. The parasymmetric comb domain $D(x_1)$.

Definition 9.1. A *parasymmetric comb domain* is a domain $D=D(x_1,y_1)$ of the form

$$D=\mathbf{C}\setminus\biggl(\bigcup_{k\in\mathbf{Z}}\{2kx_1+iy:y\ge y_1\}\cup\bigcup_{k\in\mathbf{Z}}\{(2k+1)x_1+iy:y\le -y_1\}\biggr),$$

where x_1, y_1 are positive numbers.

A parasymmetric comb domain $D(x_1,y_1)$ belongs to $\mathcal{B}$ if and only if $x_1\in(0,1)$ and $y_1=\frac{1}{2}\bigl(1+\sqrt{1-x_1^2}\bigr)$. Thus the parasymmetric comb domains in $\mathcal{B}$ form a one-parameter family. We write $D=D(x_1)$ when $D\in\mathcal{B}$.

We extend each half-line of $\partial D(x_1)$ and obtain a decomposition of $D(x_1)$ into an infinite number of vertical strips. Let $S=\{z:0<\operatorname{Re} z<x_1\}$, $A=\{iy:y\le y_1\}$ and $B=\{x_1+iy:y\ge -y_1\}$. We compute the extremal distance $\lambda=\lambda(x_1)=\lambda(A,B,S)$.

Lemma 9.2. *Let A, B, S be as above. Then $\lambda=4\nu(\Phi)$, where $\Phi=\Phi(x_1)=\bigl(1+e^{-\pi(\sqrt{1-x_1^2}+1)/x_1}\bigr)^{-1/2}$.*

Proof. We map S conformally onto the upper half plane $\mathbf{C}_+$. A function that does this mapping is $f(z)=e^{-i\pi(z-x_1)/x_1}$. Then $f(A)=[-e^{\pi y_1/x_1},0]$ and $f(B)=[e^{-\pi y_1/x_1},+\infty)$. By conformal invariance of the extremal distance we have (see 2.3)

$$\begin{aligned}\lambda(x_1):=\lambda(f(A),f(B),\mathbf{C}_+)&=4\nu\biggl(\sqrt{\frac{e^{\pi y_1/x_1}}{e^{-\pi y_1/x_1}+e^{\pi y_1/x_1}}}\biggr)\\&=4\nu((1+e^{-2\pi y_1/x_1})^{-1})=4\nu(\Phi(x_1)).\end{aligned}\tag{9.3}$$

Next we consider the following extremal problem: Find

$$\min_{x_1\in(0,1)}\frac{\lambda(x_1)}{x_1}.\tag{9.4}$$

Mathematica gives

$$\min_{x_1\in(0,1)} \frac{\lambda(x_1)}{x_1} = 0.428517, \tag{9.5}$$

and this minimum is attained only for $x_1 \approx 0.660895 =: x^*$. We will not provide a proof of the existence of the minimum. Below we will only use the fact that $\lambda(0.66)/0.66 \approx 0.428517$.

For $k\in\mathbf{Z}$, let $A_k = \{2kx^* + iy : y \le y^*\}$ and $B_k = \{(2k+1)x^* + iy : y \ge -y^*\}$, where $y^* = \frac{1}{2}\left(1+\sqrt{1-(x^*)^2}\right) = 0.875239$. The sets A_k and B_k are vertical crosscuts of the parasymmetric comb domain $D^* = D(x^*)$.

Claim 9.6. *We have*

$$\beta(D^*) \le \pi \frac{\lambda(x^*)}{x^*}. \tag{9.7}$$

Proof. We could prove this claim by using a conformal mapping obtained by repeated reflections. Instead we give a proof based on Lemma 2.17.

Let $R>100$. Then $x^*k < R \le (k+1)x^*$ for some $k\in\mathbf{Z}^+$. We assume that k is even. The case k odd is treated similarly.

Let E_R be the component of $D^* \cap \{|z|=R\}$ that intersects $\mathbf{R}^+$. It is obvious that for all $\zeta\in E_R$,

$$\omega(\zeta, \{|z|\ge R\}\cap\partial D^*, D^*) \ge \delta \tag{9.8}$$

for some $\delta>0$. So the strong Markov property (Lemma 3.7) gives

$$\begin{aligned} \omega_D(R) &\ge \int_{E_R} \omega(0, d\zeta, D\backslash E_R)\omega(\zeta, \{|z|\ge R\}\cap\partial D^*, D^*) \\ &\ge \delta \int_{E_R} \omega(0, d\zeta, D\backslash E_R) = \delta\omega(0, E_R, D). \end{aligned} \tag{9.9}$$

By the maximum principle

$$\omega(0, E_R, D) \ge \omega(0, B_k, D) \ge \omega(0, B_k, D_k), \tag{9.10}$$

where D_k is the component of $D^*\backslash B_k\backslash B_{-k}$ that contains 0.

Now we apply Lemma 2.17. We map D_k onto $\mathbf{D}$ so that 0 goes to 0 and A_0 goes to $[-i, i]$. Since $R>100$, B_k is mapped into $\left\{e^{it} : t\in\left[-\frac{1}{8}\pi, \frac{1}{8}\pi\right]\right\}$. So we may apply the lemma with $\Gamma_A = A_0$ and obtain

$$\omega(0, B_k, D_k) \ge Ce^{-\pi\lambda(A_0, B_k, D^*)} \ge Ce^{-\pi\lambda(B_{-1}, B_k, D^*)}. \tag{9.11}$$

By symmetry we have

$$\lambda(B_{-1}, B_k, D^*) = (k+2)\lambda(x^*). \tag{9.12}$$

We combine (9.9), (9.10), (9.11) and (9.12) to obtain

$$\omega_D(R) \geq C_1 e^{-\pi(k+2)\lambda(x^*)} \geq C_2 e^{-k\pi\lambda(x^*)} \geq C_3 e^{-\pi\lambda(x^*)R/x^*}, \tag{9.13}$$

with absolute constants, which implies the claim.

Thus we obtain the following upper bound for β_0.

Theorem 9.14. *We have* $\beta_0 \leq 1.34622 \leq 0.4286\pi$.

Conjecture 9.15. *It is true that* $\beta_0 = \beta(D(x^*))$.

Remark. (1) R. Goodman [Go] constructed a domain $G \in \mathcal{B}$ which is important for some extremal problems involving conformal radius, the first eigenvalue of the Laplacian and the expected lifetime of Brownian motion, see [Go] and [BC]. We have proved that the β-exponent of G satisfies the inequality $\beta(G) \geq \pi \log 2 \approx 0.693\pi > \frac{1}{2}\pi$.

(2) By disproving the conjecture of Bishop, we showed that the strip S of width 2 is not an extremal domain. However, S *is* the extremal domain for the following problem: Find $\inf\{\beta(D) : D \in \mathcal{B}$ and D is convex$\}$. This fact can easily be proved by using an old theorem of Szegő (see [BC]).

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