The Levi problem in nuclear Silva spaces

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1. Introduction

As far back as 1911 Levi [15] posed the problem whether every pseudoconvex domain in $\mathbf{C}^n$ is the domain of existence of a holomorphic function. In 1942 Oka [22] solved the problem for $n=2$ and it was only in 1953—1954 that Oka [23], Bremermann [2] and Norguet [18] independently solved the problem for every n.

Since its solution, new versions of the Levi problem have naturally arisen. During the last decade an infinite dimensional version has attracted the attention of several mathematicians and is perhaps the most studied problem in infinite dimensional holomorphy. Significant results in this direction were obtained in 1972 by Gruman [10] and Gruman—Kiselman [11], who solved the Levi problem in Banach spaces with a Schauder basis. Afterwards these results have been generalized to Fréchet spaces, and to other classes of locally convex spaces, like e.g. Silva spaces, but always under the assumption that the space has a Schauder basis, or, in the case of Fréchet spaces, under the weaker assumption that the space has the Banach approximation property (i.e. there exists a sequence of continuous linear operators of finite rank which converges to the identity pointwise). See Noverraz [19], [21], Pomes [24], Dineen [4], Schottenloher [26] and Dineen—Noverraz—Schottenloher [5]. On the other hand, it soon became clear, after a counterexample of Josefson [14], that a separability condition on the space is essential.

In this paper we solve the Levi problem on a class of spaces that includes all nuclear Silva spaces (i.e. strong duals of nuclear Fréchet spaces), without assuming the existence of a Schauder basis. This problem had been explicitly mentioned by Boland [1]. Our method is largely based on Gruman and Kiselman's original idea and actually the nuclear structure of the space plays the role of the Schauder basis.

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We refer to Noverraz [19] for the theory of holomorphic functions on infinite dimensional spaces, and to Schaefer [25] for the theory of topological vector spaces.

Let us mention that among the references at the end of this article we have not included those papers dealing with that special case of the Levi problem where the domain is assumed to be finitely Runge.

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2. Statement of the theorem

Let us consider the following conditions on a complex Hausdorff locally convex space E.

(i) The space E is nuclear.

(ii) Given a sequence of 0-neighbourhoods (W_j) there exists a sequence of scalars (λ_j), with $\lambda_j > 0$, such that $\cap \lambda_j W_j$ is again a 0-neighbourhood.

(iii) The space E is hereditarily Lindelöf.

(iv) The space E is hereditarily separable.

Of course (iii) (respectively (iv)) means that every subset of E is a Lindelöf topological space (respectively a separable topological space) under the induced topology.

If U is an open set in E then $\mathcal{H}(U)$ will denote the space of all holomorphic functions on U.

Theorem 2.1. *Let E be a space satisfying conditions* (i) *through* (iv). *Then every pseudoconvex domain in E is the domain of existence of a holomorphic function.*

Remark 2.2. Let E be a space satisfying conditions (i) through (iii). Then a slight modification of the proof of Theorem 2.1 shows that every pseudoconvex domain U in E is a domain of holomorphy. Furthermore, for each finite dimensional vector subspace S of E, the restriction mapping $\mathcal{H}(U) \to \mathcal{H}(U \cap S)$ is surjective.

Remark 2.3. Let E be a space satisfying conditions (ii) and (iii). Then an adaptation of the first part of the proof of Theorem 2.1 shows that every open set U in E is uniformly open and that every function $f \in \mathcal{H}(U)$ is uniformly holomorphic in the sense of Nachbin [17], i.e. there exists a continuous seminorm α on E such that U is open in the seminormed space (E, α) and f is holomorphic on U when U is regarded as an open set in (E, α).

3. Examples and applications

Every DF-space satisfies condition (ii): see Grothendieck [8, p. 64, Lemma 2]. on the other hand every Suslin space satisfies conditions (iii) and (iv): see Schwartz [27, Ch. II]. We recall that a Polish space is a separable complete metric space, and that a Suslin space is a Hausdorff topological space which is the continuous image of a Polish space. Thus we get

Example 3.1. The strong dual of a Fréchet—Montel space and any countable inductive limit of separable Banach spaces are both DF-spaces and Suslin spaces, and hence satisfy conditions (ii), (iii) and (iv). As a consequence we get

Example 3.2. Every nuclear Silva space satisfies conditions (i) through (iv). The space of holomorphic germs $\mathscr{H}(K)$, with $K \subset \mathbf{C}^n$ compact, and the spaces of distributions $\mathscr{D}'(K)$, $\mathscr{E}'(\Omega)$ and $\mathscr{S}'(\mathbf{R}^n)$, with $K \subset \mathbf{R}^n$ compact and $\Omega \subset \mathbf{R}^n$ open, are examples of nuclear Silva spaces: see Grothendieck [9, Ch. II, pp. 55—57].

Remark 3.3. E. Dubinsky, H. Jarchow, B. Perrot and F. Villamarin have kindly pointed out to us that there are examples of nuclear Silva spaces without a Schauder basis. Indeed Mitiagin—Zobin [16] have constructed an example of a nuclear Fréchet space without a Schauder basis and it is easy to see that a nuclear Fréchet space has a Schauder basis if and only if its strong dual has one: if (e_i) is a Schauder basis in one of these spaces, then it follows from the Banach—Steinhaus theorem that the biorthogonal sequence (f_j) (i.e. defined by $f_j(e_i)=\delta_{ij}$) is a Schauder basis in its strong dual. Moreover, a similar procedure yields an example of a nuclear Silva space without the Banach approximation property, for Dubinsky [6] has recently constructed an example of a nuclear Fréchet space without the Banach approximation property.

Example 3.4. It is clear that if a space E satisfies any of the conditions (i), (ii), (iii) or (iv) then every vector subspace of E satisfies the same condition. In particular every vector subspace of a nuclear Silva space satisfies conditions (i) through (iv).

Example 3.5. A sequence (x_j) in a Banach space is said to be **rapidly decreasing** if for every $n \in \mathbf{N}$, $j^n \|x_j\| \to 0$ as $j \to \infty$. Let E be a separable Banach space, let E' denote its strong dual, and let E_s denote the vector space E endowed with the topology of uniform convergence on the rapidly decreasing sequences of E'. Then E_s satisfies conditions (i) through (iv). Indeed, by Hogbe-Nlend [12, Prop. 4.7] E_s is nuclear. By Colombeau [3, Lemma 1] E_s satisfies condition (ii). And since the identity mapping $E \to E_s$ is continuous, we see that E_s satisfies conditions (iii) and (iv). We remark finally that by Colombeau [3, Lemma 2] E_s is not a Silva space when E is infinite dimensional.

Application 3.6. Every pseudoconvex domain in the space of distributions $\mathcal{D}'(\Omega)$, with $\Omega \subset \mathbf{R}^n$ open, is the domain of existence of a holomorphic function. Indeed by Dineen [4, Ex. 2.9] $\mathcal{D}'(\Omega)$ is the open surjective limit of the spaces $\mathcal{D}'(K)$, with $K \subset \Omega$ compact. Since Theorem 2.1 applies to each $\mathcal{D}'(K)$, by Example 3.2, the conclusion follows from Dineen [4, Prop. 4.5].

4. Proof of the theorem

A slight modification of the proof of the Cartan—Thullen Theorem (see Grauert—Frizsche [7, pp. 47—50]) yields the following lemma.

Lemma 4.1. *Let U be a holomorphically convex open set in $\mathbf{C}^n$. Then given a function $g \in \mathcal{H}(U)$, a compact set $K \subset U$ and $\varepsilon > 0$, there exists a function $f \in \mathcal{H}(U)$ of the form $f = g + x_n h$, where $h \in \mathcal{H}(U)$ and x_n denotes the n-th complex coordinate, and such that:*

(a) $\sup_K |f-g| < \varepsilon$.

(b) *If V is any convex open set in $\mathbf{C}^n$ such that $V \cap \partial U \neq \emptyset$ then f is unbounded on each connected component of $V \cap U$. In particular, U is the natural domain of f.*

Let H be a separable inner-product space and let (e_n) be a complete orthonormal sequence in H. Let H_n denote the vector subspace spanned by $e_1, \ldots, e_n$ and let $\pi_n \colon H \to H_n$ denote the projection. If U is a fixed open set in H then we set

$$K_n = \left\{ x \in U \cap H_n \colon \|x\| \leqq n,\ d(x, \complement U) \geqq \frac{1}{n},\ \|x - \pi_{n-1}(x)\| \leqq \frac{1}{2} d(x, \complement U) \right\}$$

Then K_n is a compact subset of $U \cap H_n$ and $\pi_{n-1}(K_n) \subset U \cap H_{n-1}$. If K is any compact subset of U then $\pi_n(K) \subset K_n$ for all sufficiently large n. Finally, if U is pseudoconvex then it follows from Hörmander [13, Th. 4.3.2] that K_n is Runge in $U \cap H_n$. With this notation we get the lemma below, which is nothing but Gruman—Kiselman [11, Lemma] and Noverraz [20, Prop. 2], slightly strengthtened with the help of Lemma 4.1.

Lemma 4.2. *Let U be a pseudoconvex open set in a separable inner-product space H. Then there exists a sequence (f_n), with $f_n \in \mathcal{H}(U \cap H_n)$, with the following properties:*

(a) $f_n | U \cap H_{n-1} = f_{n-1}$

(b) $\sup_{K_n} |f_n - f_{n-1} \circ \pi_{n-1}| \leqq 2^{-n}$

(c) *If V is any convex open set in H_n such that $V\cap\partial(U\cap H_n)\neq\emptyset$ then f_n is unbounded on each connected component of $V\cap U\cap H_n$. In particular $U\cap H_n$ is the natural domain of f_n.*

(d) *The sequence $(f_n\circ\pi_n)$ converges uniformly on compact subsets of U to a function $f\in\mathscr{H}(U)$. Moreover $f|U\cap H_n=f_n$ for every n.*

Proof of Theorem 2.1. We first show the existence of a continuous seminorm α on E such that:

(a) U is α-open.

(b) $H=(E,\alpha)/\alpha^{-1}(0)$ is a separable inner-product space.

By condition (iii) U is a Lindelöf space, and hence we may find a sequence $(x_j)\subset U$ and a sequence (W_j) of convex, balanced 0-neighbourhoods such that

$$U=\bigcup_{j=1}^{\infty}(x_j+W_j)\quad\text{and}\quad x_j+2W_j\subset U$$

By condition (ii) there exists a sequence (λ_j), with $\lambda_j>0$, such that $\cap\lambda_j W_j$ is again a 0-neighbourhood. Since E is nuclear, we may find a convex, balanced 0-neighbourhood $W\subset\cap\lambda_j W_j$ such that $H=(E,\alpha)/\alpha^{-1}(0)$ is a separable inner-product space, where α denotes the Minkowski functional of W: see Schaefer [25, p. 101, Prop. 7.3]. Since

$$U=\bigcup_{j=1}^{\infty}(x_j+W_j)\quad\text{and}\quad x_j+W_j+\lambda_j^{-1}W\subset U$$

we see that U is α-open, and so (a) and (b) are verified. On the other hand, by condition (iv) we may find countable dense sets $D_1\subset U$ and $D_2\subset\complement U$.

Let us assume, for the moment, that E has a continuous norm. Then we may assume that α is a norm. Then $H=(E,\alpha)$ and U is open in H. After applying the Grahm—Schmidt orthogonalization process to $D_1\cup D_2$ we may find a complete orthonormal sequence (e_n) in H such that $D_1\cup D_2\subset\cup H_n$. Let $f\in\mathscr{H}(U_H)$ denote the function given by Lemma 4.2, where U_H denotes the set U regarded as an open set in H. Then certainly $f\in\mathscr{H}(U)$ also. We claim that if V is any convex open set in E such that $V\cap\partial U\neq\emptyset$ then f is unbounded on each connected component of $V\cap U$. In particular, U is the domain of existence of f. Indeed the convex set V contains points both from D_1 and D_2 and since $D_1\cup D_2\subset\cup H_n$ we get that $(V\cap H_n)\cap\partial(U\cap H_n)\neq\emptyset$ for all sufficiently large n. Then by Lemma 4.2 f is unbounded on each connected component of $V\cap U\cap H_n$. Let ω be any connected component of $V\cap U$. Then each connected component of $\omega\cap H_n$ is a connected component of $V\cap U\cap H_n$. Hence f is unbounded on $\omega\cap H_n$, hence on ω.

The general case may be reduced to the case where E has a continuous norm by applying the preceding argument to the quotient space $E/\alpha^{-1}(0)$. Indeed, let $\sigma\colon E\to E/\alpha^{-1}(0)$ denote the quotient mapping. Then by Noverraz [19, p. 43, Th. 2.1.7] $U=\sigma^{-1}(\sigma(U))$ and $\sigma(U)$ is a pseudoconvex domain in $E/\alpha^{-1}(0)$. It is also clear that $\sigma(D_1)$ and $\sigma(D_2)$ are countable dense subsets of $\sigma(U)$ and $\complement\sigma(U)$ respectively. Then the preceding argument applied to $E/\alpha^{-1}(0)$ shows that $\sigma(U)$ is the domain of existence of a function $g\in\mathscr{H}(\sigma(U))$. Then the proof of Dineen [4, Prop. 4.3] shows that U is the domain of existence of $f=g\circ\sigma$.

M. Schottenloher has kindly informed us that he already knew the main result in this paper, namely Theorem 2.1 in the case where E is a nuclear Silva space, but he never published the result. He says that the result can be derived from his abstract theorem [26. Th. 3.1] if one knows that every open set in a Silva space is uniformly open.

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